# CH-2-16(MO)
# Numerical Methods
# (Solution of Non-Linear Equations-2)


**Kannan Iyer**
**kannan.iyer@iitjammu.ac.in**

**Department of Mechanical Engineering**
**Indian Institute of Technology, Jammu**






## Review

- ❑ We had seen the following methods
  - ❑ Bisection Method
  - ❑ Method of False Position
  - ❑ Secant Method
  - ❑ Newton's Method
  - ❑ Fixed Point Iteration





## Agenda for Today

- ❑ Today We shall discuss the rate of convergence of the methods.
- ❑ First we will digest the meaning of the order of a method.
- ❑ Then we shall derive the order of the three popular methods we have discussed in the last class.

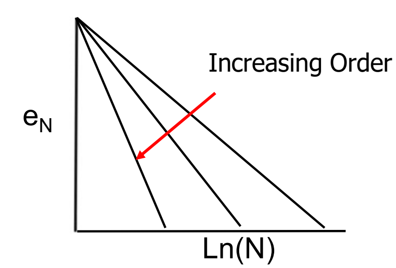




## Order of Convergence

A method is said to converge with order p, if the error decays as given by the expression

$$e_{N+1} = a(e_N)^p$$



## Fixed Point Iteration-I

- Our recursive relation was

$$x_{n+1} = g(x_n) \qquad (1)$$

- If $\alpha$ is our root then

$$\alpha = g(\alpha) \qquad (2)$$

- Eqs. (1) and (2) imply that

$$x_{n+1} - \alpha = g(x_n) - g(\alpha) \qquad (3)$$

- Defining the error at any level i as

$$e_i = x_i - \alpha \qquad (4)$$





## Fixed Point Iteration-II

- Eq. (3) can be written as

$$e_{n+1} = g(\alpha + e_n) - g(\alpha)$$

$$= \cancel{g(\alpha)} + e_n g'(\alpha) + \frac{e_n^2}{2} g''(\alpha) + \ldots - \cancel{g(\alpha)}$$

$$= e_n g'(\xi)$$ Using Mean Value Theorem

Where $\xi$ is such that it lies between $x_n$ and $\alpha$

$$\Rightarrow \frac{e_{n+1}}{e_n} = g'(\xi)$$





## Fixed Point Iteration-III

- For the method to converge,

$$\Rightarrow \left|\frac{e_{n+1}}{e_n}\right| < 1 \ \text{or} \ |g'(\alpha)| < 1$$

- In fact this must be true in its entire path of initial guess all the way to the route, as otherwise, it can be thrown out anywhere
- Since $e_{n+1} = c\, e_n$ the method is said to have linear convergence near the root.
- It implies that the error will decrease linearly in the error-number of iteration plot





## Newton's Method-I

- In this case our recursive relation was

$$\Rightarrow x_{n+1} = x_n - f(x_n)/f'(x_n) \qquad (1)$$

$$e_{n+1} + \alpha = e_n + \alpha - \frac{f(x_n) - f(\alpha)}{f'(x_n)}$$

Note that f(α) = 0 by definition

$$e_{n+1} + \cancel{\alpha} = e_n + \cancel{\alpha} - \frac{f(x_n) - f(x_n - e_n)}{f'(x_n)}$$

# Bracketing of Roots-II

$$\Rightarrow e_{n+1} = \cancel{e_n} - \frac{\cancel{f(x_n)} - (\cancel{f(x_n)} - \cancel{e_n} f'(x_n) + (e_n^2/2) f''(x_n))}{\cancel{f'(x_n)}}$$

$$\Rightarrow e_{n+1} = -\frac{e_n^2 (f''(\xi))}{2f'(\xi)}$$

- For the method to converge,

$$\Rightarrow \left|\frac{e_{n+1}}{e_n}\right| < 1 \Rightarrow \left|\frac{e_n (f''(\xi))}{2f'(\xi)}\right| < 1$$





# Continuation Method

- Many times, the equation may be difficult to solve as the root is not known and the function is difficult.
- A method called continuation method is very useful
- For an arbitrary $x_0$, we can say that $x_0$ is the root of the function $f(x)-f(x_0)$
- If we now define our function as
- $F(x) = f(x) - \beta f(x_0)$, and use $x_0$ as the guess for $\beta = 0.9$, we can find the root because the guess is good
- We can proceed in this manner successively by reducing $\beta$ to 0, root of f(x) can be found





# Closure

- We understood the meaning of the order of convergence.
- Fixed point iteration has linear convergence.
- Newton's method has quadratic convergence.
- Secant method has an order of convergence =1.6. (see slides 12-18 for those who are curious)
- We also understood the concept of the continuation method.





# Secant Method-I

- The error analysis for this method is tedious but very illustrative of the power law technique
- In this case our recursive relation was

$$\Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{\dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}$$

## Secant Method-II

$$\Rightarrow e_{n+1}+\cancel{\alpha}=e_n+\cancel{\alpha}-\frac{(x_n-x_{n-1})f(\alpha+e_n)}{f(\alpha+e_n)-f(\alpha+e_{n-1})}$$

= 0

$$\Rightarrow e_{n+1}=e_n-\frac{(e_n-e_{n-1})(f(\cancel{\alpha})+e_nf'(\alpha)+(e_n^2/2)f''(\alpha))}{f(\cancel{\alpha})+e_nf'(\alpha)+(e_n^2/2)f''(\alpha)-\left(f(\cancel{\alpha})+e_{n-1}f'(\alpha)+(e_{n-1}^2/2)f''(\alpha)\right)}$$





## Secant Method-III

$$\Rightarrow e_{n+1}=e_n-\frac{\cancel{(e_n-e_{n-1})}(e_nf'(\alpha)+(e_n^2/2)f''(\alpha))}{\cancel{(e_n-e_{n-1})}f'(\alpha)+\frac{e_n^2-e_{n-1}^2}{2}f''(\alpha)}$$

$e_n+e_{n+1}$

$$\Rightarrow e_{n+1}=e_n-\frac{e_n+\frac{e_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)}}{1+\frac{e_n+e_{n-1}}{2}\frac{f''(\alpha)}{f'(\alpha)}}$$





## Secant Method-IV

- As x approaches the root,

$$\frac{e_n+e_{n-1}}{2}\frac{f''(\alpha)}{f'(\alpha)}<<1$$

$$\Rightarrow e_{n+1}=e_n-\left(e_n+\frac{e_n^2}{2}\frac{f''(\alpha)}{f''(\alpha)}\right)\left(1-\frac{e_n+e_{n-1}}{2}\frac{f''(\alpha)}{f'(\alpha)}\right)$$

$$=\cancel{e_n}-\left(\cancel{e_n}-e_n\frac{\cancel{e_n}+e_{n-1}}{2}\frac{f''(\alpha)}{f'(\alpha)}+\cancel{\frac{e_n^2}{2}\frac{f''(\alpha)}{f''(\alpha)}}+O(e_n^3)\right)$$





## Secant Method-V

$$\Rightarrow e_{n+1}=\frac{e_ne_{n-1}}{2}\frac{f''(\alpha)}{f'(\alpha)} \qquad (1)$$

- If we assume that the method is of order p, we can write

$$e_{n+1}=ae_n^p \text{ and } e_n=ae_{n-1}^p \Rightarrow e_{n-1}=\left(\frac{e_n}{a}\right)^{\frac{1}{p}}$$

- Eq.(1) can now be written as

$$\Rightarrow ae_n^p=\frac{e_n}{2}\left(\frac{e_n}{a}\right)^{\frac{1}{p}}\frac{f''(\alpha)}{f'(\alpha)}$$

## Secant Method-VI

- By reorganising terms, we get

$$\Rightarrow a^{1+\frac{1}{p}} e_n^{\;p} = \frac{1}{2} e_n^{\;1+\frac{1}{p}} \frac{f''(\alpha)}{f'(\alpha)}$$

- Since the power of n has to be homogeneous, we can write

$$p = 1 + \frac{1}{p} \qquad \Rightarrow p^2 - p - 1 = 0$$

$$\Rightarrow p = \frac{1 \pm \sqrt{5}}{2}$$





## Secant Method-VII

- If p < 1, the method will diverge. Thus when the method converges, p > 1, which leads to

$$p = \frac{1 + \sqrt{5}}{2} = 1.62$$

- Thus the method is inferior to Newton's method, but needs only one function evaluation at a step and hence is competitive