Numerical Methods
(Solution of Non-Linear Equations)

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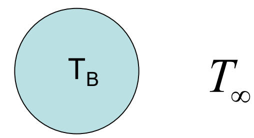
General-1

- There are many applications in TFE that requires solution of Non-linear equation

Heat generating sphere cooled by convection and radiation

$$\dot{Q} - hA\left(T_B - T_\infty\right) - \sigma\varepsilon A\left(T_B{}^4 - T_\infty{}^4\right) = 0$$



General-2

- Van-der-Waals Equation

$$\left(p + \frac{a}{v^2}\right)(v - b) - RT = 0$$

- Colebrook Friction Factor Relation

$$\frac{1}{\sqrt{f}} = -2.0 \log\left(\frac{\varepsilon/d}{3.7} + \frac{2.51}{\mathrm{Re}\sqrt{f}}\right)$$

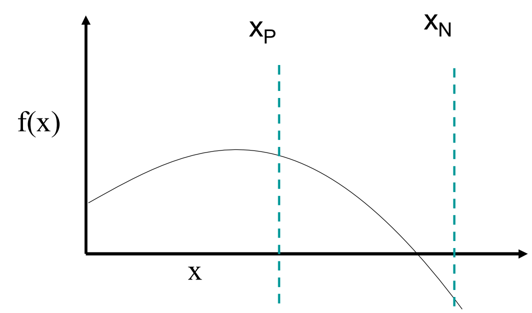
Some Observations-I

- While one linear equation will have a unique solution, a non-linear equation may have several solutions e.g. Sin(x) = 0.
- Some equations may have no solution at all? e.g. $x-e^x = 0$ for $x > 0$.
- Some equations may have no real solutions e.g. $x^2 + 1 = 0$.
- In most physical problems we will be looking for real solutions.
- No method is foolproof to guarantee a solution
- However, it is not very difficult to get a solution. Often these are called roots.

Some Observations-II

- A non-linear equation is also called a transcendental equation
- We shall discuss some of the most popular methods
- Broadly the methods can be divided in two-groups
  - Those that need bracketing of roots
  - Those that do not need bracketing

Bracketing

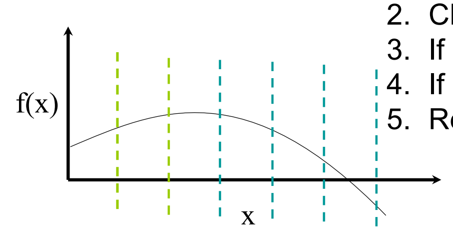


Bracketing of Roots-I

- Experience
  - Speed of an automobile may be 0-120 km/hr
  - Temperature of a furnace may be from 200-1000 ^{o}C
- Common sense
  - In open channel flow height of free liquid will be 0<h<D
- Incremental search
  - Will be discussed in next slide

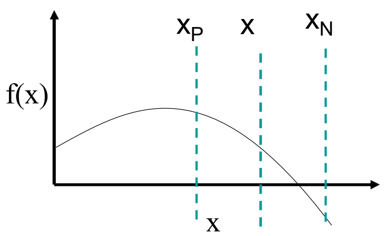
Bracketing of Roots-II

- Incremental Search



1. Start from $x = x_{min}$
2. Check if f(x)*f(x+h)<0
3. If (yes) roots bracketed
4. If (no) increment x by h
5. Repeat until $x > x_{max}$

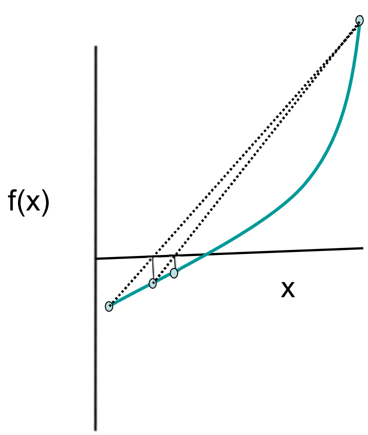
Bisection Method



- Needs bracketing
- Guaranteed solution unless there is discontinuity
- Slow convergence rate

1. Bracket the root and identify x_p and x_n
2. Do Until I < Imax
   1. $X_{new} = 0.5*(x_p+x_n)$
   2. If f(xnew) < Tol then
   3. Root = x_{new} and Get out
   4. ElseIf $f(x_{new}) < 0$ Then
      $x_n = x_{new}$
      Else $x_p = x_{new}$
      Endif
3. I = I +1
4. Enddo

Method of False Position-I



- Needs bracketing
- Guaranteed solution unless there is discontinuity
- Slow convergence rate but faster than bisection method

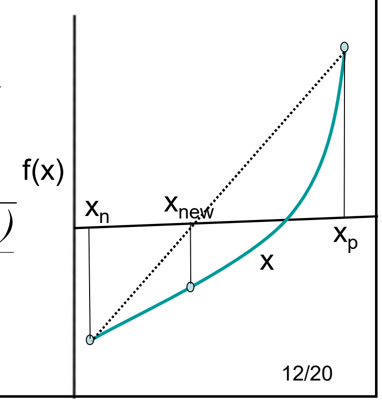
Method of False Position-II

- The guess for the new point can be obtained as follows

$$\frac{\overbrace{f(x_{new})}^{=0}-f(x_n)}{f(x_p)-f(x_n)} = \frac{x_{new}-x_n}{x_p-x_n}$$

$$\Rightarrow x_{new} = x_n - \frac{f(x_n)}{\frac{f(x_p)-f(x_n)}{x_p-x_n}}$$

$$x_{new} = x_n - \frac{f(x_n)}{m}$$



Logic

1. Bracket the root and identify x_p and x_n
2. Do Until I < Imax
    1. $m = (f(x_p)-f(x_n))/(x_p-x_n)$
    2. $X_{new} = x_n - f(x_n)/m$
    3. If f(xnew) < Tol then
    4. Root = x_{new} and Get out
    5. ElseIf $f(x_{new}) < 0$ Then

       $x_n = x_{new}$

       Else $x_p = x_{new}$

       Endif
3. I = I +1
4. Enddo

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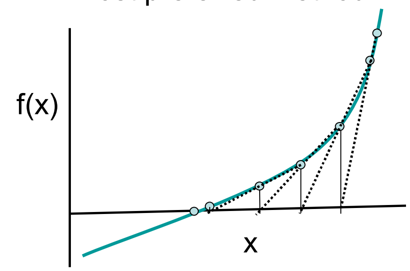
Secant Method-I

- Does not need bracketing
- Modification of false position method
- The new point replaces the last but one point
- Good Convergence
- Most preferred method



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Logic

1. Take any two points x_1 and x_2
2. Do Until I < Imax
    1. $m = (f(x_2)-f(x_1))/(x_2-x_1)$
    2. $X_{new} = x_2 - f(x_2)/m$ → One can under-relax here
    3. If f(xnew) < Tol then
    4. Root = x_{new} and Get out
    5. Else

       $x_1 = x_2$

       $x_2 = x_{new}$

       Endif
3. I = I +1
4. Enddo

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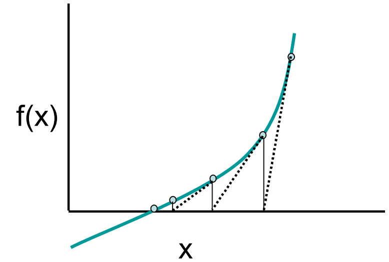
Newton's Method-I

- Does not need bracketing
- Modification of false position method
- The new point replaces the last but one point
- Good Convergence
- Also a preferred method



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Newton's Method-II

- The basis arises from linearised Taylor Series

$$f(x_{n+1}) = f(x_n + \Delta x_n) = f(x_n) + f'(x_n)\Delta x_n$$

- The new value of x_{n+1} is arrived by setting $f(x_{n+1}) = 0$

$$f(x_{n+1}) = 0 = f(x_n) + f'(x_n)(x_{n+1} - x_n)$$

$$\Rightarrow x_{n+1} = x_n - f(x_n)/f'(x_n)$$

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Logic

1. Take any point x
2. Do Until I < Imax
    1. m = f'(x)
    2. X_{new} = x- f(x)/ f'(x) → One can under-relax here
    3. If f(xnew) < Tol then
    4. Root = x_{new} and Get out
    5. Else x = x_{new}

       Endif
3. I = I +1
4. Enddo

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Fixed Point Iteration Method - I

- In this method, the equation $f(x) = 0$ is first transformed to the form g(x) = x
- This can be done in several ways. For e.g. x^2-2x+1=0 can be written as,

  $x = (x^2 + 1)/2$ Or $x = \sqrt{2x - 1}$
- The choice will be determined from the convergence rate of the method, which can be determined only after the problem is solved!
- Usually, it is done in a ad-hoc manner , as it takes little time to check if the method works or not

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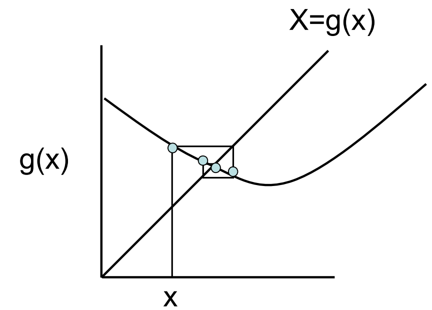
Fixed Point Iteration Method - II

- The recursive algorithm used is

$$x_{n+1} = g(x_n)$$



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Fixed Point Iteration Method - I

- Simplest to program
- Does not converge for every function. We shall derive the criterion for convergence later

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