MEC001P1M
Numerical Methods in Engineering
(Solution of Linear Equations-2)

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Review

- ❑ Began the Solution of Linear equations
  - ❑ The motivation was from several engineering applications
  - ❑ Understood the definitions of diagonal, tri-diagonal, upper triangular and lower triangular matrices
  - ❑ Understood the logic for solution of equations when the coefficient matrix is either diagonal, upper triangular or lower triangular

Agenda for Today

- ❑ Understand Direct Solvers
- ❑ Non-Iterative
- ❑ Subject to Round-off errors
- ❑ Used for a smaller system (N<10), unless the Matrix is conditioned
- ❑ Involves reduction of the coefficient matrix to a diagonal matrix or an upper triangluar matrix, or a lower triangular matrix.
- ❑ This followed by the sweeps discussed earlier

Cramer's Rule-I

$$\begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 3 \\ 2 & -2 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 12 \\ 11 \\ 2 \end{Bmatrix} \qquad Sol = \begin{Bmatrix} 3 \\ 1 \\ 2 \end{Bmatrix}$$

Solution

$$x_1 = \frac{\begin{vmatrix} 12 & -1 & 2 \\ 11 & 2 & 3 \\ 2 & -2 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & -1 & 2 \\ 1 & 2 & 3 \\ 2 & -2 & -1 \end{vmatrix}} \qquad x_2 = \frac{\begin{vmatrix} 3 & 12 & 2 \\ 1 & 11 & 3 \\ 2 & 2 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & -1 & 2 \\ 1 & 2 & 3 \\ 2 & -2 & -1 \end{vmatrix}}$$

Cramer's Rule-II

$$x_3 = \frac{\begin{vmatrix} 3 & -1 & 12 \\ 1 & 2 & 11 \\ 2 & -2 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & -1 & 2 \\ 1 & 2 & 3 \\ 2 & -2 & -1 \end{vmatrix}}$$

- ❑ Number of multiplication operations ~ (n-1) (n+1)!
- ❑ For n = 10, M ~ $3.6*10^8$
- ❑ For n = 100, M ~ 10^{157}

Severe Round Off Error

Gauss Elimination-I

$$\begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 3 \\ 2 & -2 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 12 \\ 11 \\ 2 \end{Bmatrix} \quad Sol = \begin{Bmatrix} 3 \\ 1 \\ 2 \end{Bmatrix}$$

Augmented Matrix

$$\begin{bmatrix} 3 & -1 & 2 & 12 \\ 1 & 2 & 3 & 11 \\ 2 & -2 & -1 & 2 \end{bmatrix}$$

Gauss Operations

Augmented Matrix

$$\begin{bmatrix} 3 & -1 & 2 & 12 \\ 1 & 2 & 3 & 11 \\ 2 & -2 & -1 & 2 \end{bmatrix} \Rightarrow \begin{matrix} \\ 3R_2\text{-}R_1 \\ 3R_3\text{-}2R_1 \end{matrix} \begin{bmatrix} 3 & -1 & 2 & 12 \\ 0 & 7 & 7 & 21 \\ 0 & -4 & -7 & -18 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} \\ \\ 7R_3+4R_2 \end{matrix} \begin{bmatrix} 3 & -1 & 2 & 12 \\ 0 & 7 & 7 & 21 \\ 0 & 0 & -21 & -42 \end{bmatrix} \Rightarrow Sol = \begin{Bmatrix} 3 \\ 1 \\ 2 \end{Bmatrix}$$

Comments on Gauss Elimination

- ❑ Involves $1/3\ N^3 - N^2 - 1/3\ N$ (approx) Multiplicative operations
- ❑ For N=10, This is eq ~ 430
- ❑ Errors propagate and hence, not used for N > 10
- ❑ One should always check for Residues
- ❑ Pivoting and Scaling reduces error propagation

Pivoting and Scaling

- Diagonal dominance reduces error propagation
- Rearranging the coefficient to get diagonal dominance is called pivoting
- Switching of only rows to improve diagonal dominance is called partial pivoting
- Dividing all the coefficients by the maximum value in a row is called scaling
- These can easily be incorporated and is always recommended

Gauss Jordan Elimination

Augmented Matrix

$$\begin{bmatrix} 3 & -1 & 2 & 12 \\ 1 & 2 & 3 & 11 \\ 2 & -2 & -1 & 2 \end{bmatrix} \Rightarrow \begin{array}{l} R_1^* = R1/3 \\ R_2 - R_1^* \\ R_3 - 2R_1^* \end{array} \begin{bmatrix} 1 & -1/3 & 2/3 & 4 \\ 0 & 7/3 & 7/3 & 7 \\ 0 & -4/3 & -7/3 & -6 \end{bmatrix}$$

$R_2^{**} = 3R_2/7$

$$\begin{bmatrix} 1 & -1/3 & 2/3 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & -4/3 & -7/3 & -6 \end{bmatrix} \begin{array}{l} R_1 - R_2^{**}/(-3) \\ \\ R_3 - 4R_2^{**}/(-3) \end{array} \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

Comments on Gauss Jordan Elimination

$$\begin{array}{l} R_1 - R_3^* \\ R_2 = R_3^* \\ R_3^* = R_3/-1 \end{array} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

- $M_{GJ} = 0.5N^3 + N^2 - 0.5N$, For $N=10$, $M_{GJ} = 595$
- Requires 50% more effort than that for Gauss procedure
- Procedure is useful for getting inverse

Towards Solution of Linear Equations-III

$$\begin{bmatrix} 3 & -1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & -2 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -0.5714 & 0.7143 & 1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0.8571 & -0.5714 & -1 \end{bmatrix}$$

L-U Decomposition-I

- If the problem has to be repeated with several source vectors for the same coefficient vector, L-U decomposition is recommended

$$[A]=[L][U]$$

- Such a decomposition speeds up calculation
- In general two methods are available
  - Crout's Decomposition
  - Dolittle's Decomposition

Crout's Decomposition

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{array}{c|ccc} & 1 & 3 & 5 \\ \hline 2 & a_{11} & a_{12} & a_{13} \\ 4 & a_{21} & a_{22} & a_{23} \\ & a_{31} & a_{32} & a_{33} \end{array}$$

- $a_{11} = l_{11}$, $a_{21} = l_{21}$, $a_{31} = l_{31}$
- $a_{12} = l_{11} u_{12}$, $a_{13} = l_{11} u_{13}$
- $a_{22} = l_{21} u_{12} + l_{22}$, $a_{32} = l_{31} u_{12} + l_{32}$
- $a_{23} = l_{21} u_{13} + l_{22} u_{23}$
- $a_{33} = l_{31} u_{13} + l_{32} u_{23} + l_{33}$

Logic for Crout's Method

$l_{i1} = a_{i1}$, for $i = 1, n$

$u_{1j} = a_{1j} / l_{11}$, for $j = 2,n$

for $j = 2$, $n-1$

$$l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \quad (\text{ for } i = j , n)$$

$$u_{ji} = \left(a_{ji} - \sum_{k=1}^{j-1} l_{jk} u_{ki} \right) / l_{jj} \quad (\text{ for } i = j+1, n)$$

$$l_{nn} = a_{nn} - \sum_{k=1}^{n} l_{nk} u_{kn}$$

Solution for Crout's Method

$$[A]\{x\} = \{b\} \Rightarrow [L][U]\{x\} = \{b\}$$

- Introducing $[U]\{x\} = \{d\}$ 1
- This implies $[L]\{d\} = \{b\}$ 2
- Since [L] and {b} are known, {d} can be found from Eq.(2)by forward sweep
- Once {d} is found out, {x} can be found from Eq. (1) by backward sweep

Comments on Crout's Method-I

- $M_{crout} = M_{Gauss}$
- But, back substitution $M = n^2 - n$
- Therefore for a large set one may save substantial effort (n^3 vs n^2)
- It is possible to store the coefficients of [L] and [U] in [A] itself as [A] is no longer required. This saves memory
- Other decompositions are similar

Comments on Crout's Method-II

- It is possible to store L and U in A itself and conserve memory and logic written accordingly

$$\begin{bmatrix} l_{11} & u_{12} & u_{13} \\ l_{21} & l_{22} & u_{23} \\ l_{31} & l_{32} & u_{33} \end{bmatrix}$$

- Indices have to be carefully addressed
- Since memory is cheap, this no longer may be required