Lab-3 Session
Set of Linear Equations
Crout Decomposition and TDMA

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Crout's Decomposition

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{array}{c|ccc} & 1 & 3 & 5 \\ \hline 2 & a_{11} & a_{12} & a_{13} \\ 4 & a_{21} & a_{22} & a_{23} \\ & a_{31} & a_{32} & a_{33} \end{array}$$

- ❑ $a_{11} = l_{11}$, $a_{21} = l_{21}$, $a_{31} = l_{31}$
- ❑ $a_{12} = l_{11}\, u_{12}$, $a_{13} = l_{11}\, u_{13}$
- ❑ $a_{22} = l_{21}\, u_{12} + l_{22}$, $a_{32} = l_{31}\, u_{12} + l_{32}$
- ❑ $a_{23} = l_{21}\, u_{13} + l_{22}\, u_{23}$
- ❑ $a_{33} = l_{31}\, u_{13} + l_{32}\, u_{23} + l_{33}$

Logic for Crout's Method

$l_{i1} = a_{i1}$, for $i = 1, n$

$u_{1j} = a_{1j} / l_{11}$, for $j = 2,n$

for $j = 2$, $n-1$

$$l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \qquad (\text{for } i = j, n)$$

$$u_{ji} = \left(a_{ji} - \sum_{k=1}^{j-1} l_{jk} u_{ki} \right) / l_{jj} \qquad (\text{for } i = j+1, n)$$

$$l_{nn} = a_{nn} - \sum_{k=1}^{n} l_{nk} u_{kn}$$

- Step-1 Input [a(NxN)] and b{N}
- Step- 2 Decompose into L and U
  - Refer Ass-3 file for logic and compare it with the previous slide.
- Step-3 Solution for {d} Vector.
  - Refer Lecture 1
- Step-4 Solution for {x} Vector.

Solution for {x} by Crout’s Method

$$[A]\{x\}=\{b\}\Rightarrow[L][U]\{x\}=\{b\}$$

- Introducing $[U]\{x\}=\{d\}$ (1)
- This implies $[L]\{d\}=\{b\}$ (2)
- Since [L] and {b} are known, {d} can be found from Eq.(2)by forward sweep
- Once {d} is found out, {x} can be found from Eq. (1) by backward sweep

Solution of {d} Vector

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}=\begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

$$\Rightarrow \begin{aligned} x_1 &= \frac{b_1}{l_{11}} \\ x_2 &= \frac{b_2 - l_{21}x_1}{l_{22}} \\ x_3 &= \frac{b_3 - l_{31}x_1 - l_{32}x_2}{l_{33}} \end{aligned}$$

$$Logic \Rightarrow$$

$$d_1 = \frac{b_1}{l_{11}}$$
$$For\ i = 2 - n$$
$$d_i = \frac{b_i - \sum_{j=1}^{i-1} l_{ij}d_j}{l_{ii}}$$

Logic for {d} Vector

```
C  forward substitution)
      d(1)=b(1)/el(1,1)
      do 50 i = 2,n
      sum=0.
         do 55 J = 1, i-1
55          sum=sum+el(i,j)*d(j)
50    d(i)=(b(i)-sum)/el(i,i)
```


$$\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}=\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}$$

$$\begin{aligned} x_3 &= d_3 \\ x_2 &= b_2 - u_{23}x_3 \\ x_1 &= b_1 - u_{12}x_2 - u_{13}x_3 \end{aligned}$$

$$Logic$$

$$x_N = b_N$$
$$For\ i = N-1,\quad 1$$
$$x_i = b_i - \sum_{j=i+1}^{N} u_{ij}x_j$$

Solution of Upper Triangular Matrix

```
x(n)=d(n)/u(n,n)
 Do I= n-1,1,-1
     sum=0.
          Do J = i+1,n
               sum=sum+u(i,j)*x(j)
          Enddo
     x(i)=(d(i)-sum)/u(i,i)
 Enddo
Write x(I)
```


Tridiagonal Matrix

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{Bmatrix}$$

Thomas Algorithm (TDMA)-I

$$a_{12}^* = a_{12} / a_{11} \qquad b_1^* = b_1 / a_{11}$$

For I = 2 to N

$$\Rightarrow a_{i,i+1}^* = \frac{a_{i,i+1}}{a_{i,i} - a_{i-1,i}^* a_{i,i-1}}$$

Skip for I = N

$$b_i^* = \frac{b_i - b_{i-1}^* a_{i,i-1}}{a_{i,i} - a_{i-1,i}^* a_{i,i-1}}$$

Thomas Algorithm (TDMA)-II

- Back Substitution

$$x_N = b_N^*$$

- For I = N-1, N-2, …,1

$$x_i = b_i^* - x_{i+1} a_{i,i+1}^*$$

Thomas Algorithm (TDMA)-III

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \quad \begin{bmatrix} 0 & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} \\ a_{32} & a_{33} & a_{34} \\ a_{43} & a_{44} & 0 \end{bmatrix}$$

It is possible to store A as (N,3) to conserve memory and logic written accordingly

$$a_{i,i-1} = a_{i,1} \quad a_{i,i} = a_{i,2} \quad a_{i,i+1} = a_{i,3}$$