CH-2-16(MO)
Numerical Methods
(Solution of Linear Equations-3)

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Review

- Began with the direct solution of linear equations
  - Studied Gauss Elimination
  - Understood that Pivoting improves accuracy. Concrete example will be studied later during analysis of error propagation
  - Studied Gauss Jordan method and understood that it can be used to compute inverse
  - Studied Crout's Decomposition and understood that it is an efficient method for repetitive solution for different RHS

Agenda for Today

- Understand the method to solve Tridiagonal System of equations.
- Study Iterative Methods
  - Jacobi Method
  - Gauss-Siedel Method
  - Successive Over Relaxation Method
- Discuss possible termination Criteria.

Tridiagonal Matrix Solution

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & a_{12}/a_{11} & 0 & 0 & b_1/a_{11} \\ a_{21} & a_{22} & a_{23} & 0 & b_2 \\ 0 & a_{32} & a_{33} & a_{34} & b_3 \\ 0 & 0 & a_{43} & a_{44} & b_4 \end{bmatrix}$$

Tridiagonal Matrix Solution

$R_2 - a_{21} R_1$

$$\begin{bmatrix} 1 & a_{12}^* & 0 & 0 & b_1^* \\ a_{21} & a_{22} & a_{23} & 0 & b_2 \\ 0 & a_{32} & a_{33} & a_{34} & b_3 \\ 0 & 0 & a_{43} & a_{44} & b_4 \end{bmatrix}$$

Note that $a_{11}^* = \dfrac{a_{12}}{a_{11}}, b_1^* = \dfrac{b_1}{a_{11}}$

$$\begin{bmatrix} 1 & a_{12}^* & 0 & 0 & b_1^* \\ 0 & a_{22} - a_{21}a_{12}^* & a_{23} & 0 & b_2 - a_{21}b_1^* \\ 0 & a_{32} & a_{33} & a_{34} & b_3 \\ 0 & 0 & a_{43} & a_{44} & b_4 \end{bmatrix}$$

Tridiagonal Matrix Solution

$$\begin{bmatrix} 1 & a_{12}^* & 0 & 0 & b_1^* \\ 0 & a_{22} - a_{21}a_{12}^* & a_{23} & 0 & b_2 - a_{21}b_1^* \\ 0 & a_{32} & a_{33} & a_{34} & b_3 \\ 0 & 0 & a_{43} & a_{44} & b_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & a_{12}^* & 0 & 0 & b_1^* \\ 0 & 1 & \dfrac{a_{23}}{a_{22} - a_{21}a_{12}^*} & 0 & \dfrac{b_2 - a_{21}b_1^*}{a_{22} - a_{21}a_{12}^*} \\ 0 & a_{32} & a_{33} & a_{34} & b_3 \\ 0 & 0 & a_{43} & a_{44} & b_4 \end{bmatrix}$$

Tridiagonal Matrix Solution

$$\begin{bmatrix} 1 & a_{12}^* & 0 & 0 & b_1^* \\ 0 & 1 & a_{23}^* & 0 & b_2^* \\ 0 & a_{32} & a_{33} & a_{34} & b_3 \\ 0 & 0 & a_{43} & a_{44} & b_4 \end{bmatrix}$$

$$a_{23}^* = \frac{a_{23}}{a_{22} - a_{21}a_{12}^*} \qquad b_2^* = \frac{b_2 - a_{21}b_1^*}{a_{22} - a_{21}a_{12}^*}$$

One can proceed this way and show that the final matrix shall be

$$\begin{bmatrix} 1 & a_{12}^* & 0 & 0 & b_1^* \\ 0 & 1 & a_{23}^* & 0 & b_2^* \\ 0 & 0 & 1 & a_{34}^* & b_3^* \\ 0 & 0 & 0 & 1 & b_4^* \end{bmatrix}$$

Thomas Algorithm (TDMA)-I

$$a_{12}^* = a_{12} / a_{11} \qquad b_1^* = b_1 / a_{11}$$

For I = 2 to N-1

$$\Rightarrow a_{i,i+1}^* = \frac{a_{i,i+1}}{a_{i,i} - a_{i-1,i}^* a_{i,i-1}}$$

$$b_i^* = \frac{b_i - b_{i-1}^* a_{i,i-1}}{a_{i,i} - a_{i-1,i}^* a_{i,i-1}}$$

Thomas Algorithm (TDMA)-II

$$\begin{bmatrix} 1 & a_{12}^* & 0 & 0 \\ 0 & 1 & a_{23}^* & 0 \\ 0 & 0 & 1 & a_{34}^* \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_{n-1} \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1^* \\ b_2^* \\ b_{n-1}^* \\ b_n^* \end{Bmatrix}$$

$$\Rightarrow x_N = b_N^* \quad x_{n-1} = b_{n-1}^* - x_n a_{n-1,n}^*$$

Back Substitution Logic

$$x_N = b_N^*$$

For I = N-1, N-2, …,1

$$x_i = b_i^* - x_{i+1} a_{i,i+1}^*$$

Thomas Algorithm (TDMA)-III

- It is possible to store A as (N,3) to conserve memory and logic written accordingly

$$a_{i,i-1} = a_{i,1} \qquad a_{i,i} = a_{i,2} \qquad a_{i,i+1} = a_{i,3}$$

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} \\ a_{32} & a_{33} & a_{34} \\ a_{43} & a_{44} & 0 \end{bmatrix}$$

Closing Remarks on Direct Solvers

- These methods are subject to error propagation
- The error propagation can be indicated by a term called condition number
- Ill conditioned systems are difficult to solve
- Several specialised methods exist
- Refer your book and advanced Linear Algebra Texts.
- This exposure is sufficient for most general problems

Iterative Methods-I

- For large systems, which are sparse Iterative methods are most widely used
- These naturally occur during the solution of ODE's and PDE's.
- These methods do not suffer from propagation of round-off errors
- The set of equations have to be diagonal dominant to obtain convergence
- This is generally a limitation but where they are used, it can be achieved by some techniques

Iterative Methods-II

- Consider

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$\Rightarrow \quad x_1 = \left(b_1 - \left(a_{12}x_2 + a_{13}x_3\right)\right)/\,a_{11}$$
$$x_2 = \left(b_2 - \left(a_{21}x_1 + a_{23}x_3\right)\right)/\,a_{22}$$
$$x_3 = \left(b_3 - \left(a_{31}x_1 + a_{32}x_2\right)\right)/\,a_{33}$$

- One can start with a guess and iterate

Jacobi Iteration

$$x_1^N = \left(b_1 - \left(a_{12}x_2^{N-1} + a_{13}x_3^{N-1}\right)\right)/\,a_{11}$$
$$x_2^N = \left(b_2 - \left(a_{21}x_1^{N-1} + a_{23}x_3^{N-1}\right)\right)/\,a_{22}$$
$$x_3^N = \left(b_3 - \left(a_{31}x_1^{N-1} + a_{32}x_2^{N-1}\right)\right)/\,a_{33}$$

- By adding and subtracting x_i^{N-1} on both sides

$$x_1^N = x_1^{N-1} + \left(b_1 - \left(a_{11}x_1^{N-1} + a_{12}x_2^{N-1} + a_{13}x_3^{N-1}\right)\right)/\,a_{11}$$
$$x_2^N = x_2^{N-1} + \left(b_2 - \left(a_{21}x_1^{N-1} + a_{22}x_2^{N-1} + a_{23}x_3^{N-1}\right)\right)/\,a_{22}$$
$$x_3^N = x_3^{N-1} + \left(b_3 - \left(a_{31}x_1^{N-1} + a_{32}x_2^{N-1} + a_{33}x_3^{N-1}\right)\right)/\,a_{33}$$

Gauss Siedel Iteration

- Here new values are used as soon as they are available

$$x_1^N = x_1^{N-1} + \left(b_1 - \left(a_{11}x_1^{N-1} + a_{12}x_2^{N-1} + a_{13}x_3^{N-1}\right)\right)/\,a_{11}$$
$$x_2^N = x_2^{N-1} + \left(b_2 - \left(a_{21}x_1^{N} + a_{22}x_2^{N-1} + a_{23}x_3^{N-1}\right)\right)/\,a_{22}$$
$$x_3^N = x_3^{N-1} + \left(b_3 - \left(a_{31}x_1^{N} + a_{32}x_2^{N} + a_{33}x_3^{N-1}\right)\right)/\,a_{33}$$

- Where Jacobi converges, Gauss-Siedel converges faster

Successive Over Relaxation (SOR)

- Sufficient Condition for convergence for both Jacobi and Gauss-Siedel Iterations is

$$|a_{ii}| \geq \sum_{j=1}^{N} |a_{ij}| \quad \text{for i } = 1, \text{N}$$

- Where the above criteria is satisfied it is possible to accelerate it further by introducing over-relaxation factor
- The value of the over-relaxation factor lies between 1-2. Optimum values are available for some specific form of the coefficient matrix. In general it should be found by trials

Logic for SOR

- For i = 1, N

$$x_i = x_i + \omega \left(b_i - \left(\sum_{j=1}^{N} a_{ij} x_j \right) \right) / a_{ii}$$

Norms of Vectors

- p norm of a vector is defined as

$$\|x\|_p = \left[\sum_{j=1}^{N} |x_j|^p \right]^{\frac{1}{p}}$$

$\|x\|_1 \Rightarrow$ Sum of absolute values of components

$\|x\|_\infty \Rightarrow$ Absolute value of the largest component

$\|x\|_2 \Rightarrow$ Euclidean Norm

Termination Criterion

$$\left\| x^{N+1} - x^N \right\|_\infty \leq \varepsilon$$

$$\frac{\left\| x^{N+1} - x^N \right\|_\infty}{\left\| x^N \right\|_\infty} \leq \varepsilon$$

$$\left\| r^N \right\|_\infty \leq \varepsilon$$