CH-2-16(MO)
Numerical Methods
Solution of Elliptic Equations

Kannan Iyer
Kannan.iyer@iitjammu.ac.in

विद्याधनं सर्वधन प्रधानम्

Department of Mechanical Engineering
Indian Institute of Technology Jammu

Common Elliptic Equations-I

- One of the most common elliptic equation is the Steady Heat Equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=-\frac{q'''}{k}$$

- If the source term is 0, then above equation reduces to

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0$$ Laplace Eq.

- If the source term is not 0, then above equation in general can be written as

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=S(x,y,z)$$ Poisson Eq.

Common Elliptic Equations-III

- Comparing with our standard form

$$Af_{xx}+\cancel{Bf_{xy}}+Cf_{yy}+Df_{x}+Ef_{y}+F=0$$

$$B=0$$

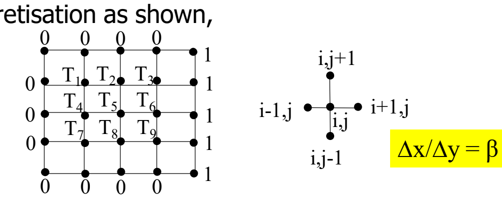
- For positive A and C, B^2-4AC < 0
- The equations are elliptic
  - No characteristic directions
  - No discontinuities
  - Information spreads in all directions
  - Every point affects every other point

Solution of Elliptic Equations-I

- Consider Poisson Equation

$$\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right]=S$$

- For a discretisation as shown,



- We can write it in finite difference form as

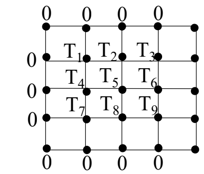
$$\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\Delta x^2}+\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\Delta y^2}=S_{i,j}$$

Solution of Elliptic Equations-II

- Multiplying by Δx^2, the Finite Difference Equation can be written as
$$T_{i+1,j} - 2T_{i,j} + T_{i-1,j} + \left(T_{i,j+1} - 2T_{i,j} + T_{i,j-1}\right)\beta^2 = S_{i,j}\Delta x^2$$
- The same can be rearranged as
$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1}\beta^2 + T_{i,j-1}\beta^2 - 2(1+\beta^2)T_{i,j} = S_{i,j}\Delta x^2$$
- For the simple case of S = 0, we get
$$T_{i+1,j} + T_{i-1,j} + \beta^2 T_{i,j+1} + \beta^2 T_{i,j-1} - 2(1+\beta^2)T_{i,j} = 0$$
- We can write the finite difference equation for each interior nodes and the temperature of these can be solved using one of the following methods.

Solution of Elliptic Equations-III

- For the case shown in the grid (β=1), the relations can be described by the following Matrix problem



$$\begin{bmatrix} -4 & 1 & & 1 & & & & & \\ 1 & -4 & 1 & & 1 & & & & \\ & 1 & -4 & & & 1 & & & \\ 1 & & & -4 & 1 & & 1 & & \\ & 1 & & 1 & -4 & 1 & & 1 & \\ & & 1 & & 1 & -4 & & & 1 \\ & & & 1 & & & -4 & 1 & \\ & & & & 1 & & 1 & -4 & 1 \\ & & & & & 1 & & 1 & -4 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$$

Banded Matrix

Solution of Elliptic Equations-IV

- Direct solution is not generally used.
- Several Iterative methods are common. These are:
  - Gauss-Siedel, Jacobi
  - Point Successive Over Relaxation (PSOR)
  - Line
  - Line-SOR
  - ADI
  - Accelerated ADI
  - Multigrid

Jacobi and Gauss Siedel (Point Methods)

- Jacobi
$$T_{i,j}^{k+1} = \frac{T_{i+1,j}^{k} + T_{i-1,j}^{k} + \left(T_{i,j+1}^{k} + T_{i,j-1}^{k}\right)\beta^2}{2\left(1+\beta^2\right)}$$
- Assume Ts and iterate using the above relation
- If the new values available are immediately used for other points, it is called Gauss Siedel
$$T_{i,j}^{k+1} = \frac{T_{i+1,j}^{k} + T_{i-1,j}^{k+1} + \left(T_{i,j+1}^{k} + T_{i,j-1}^{k+1}\right)\beta^2}{2\left(1+\beta^2\right)}$$
- The necessary and sufficient condition for convergence is the Scarborough Criterion (Diagonal ≥ sum of off-diagonal and Diagonal > sum of off-diagonal at least in one)

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Acceleration Procedures-I

- Point-SOR
  - The essence is that for a linear system it has been seen that we can accelerate by over relaxation as discussed earlier in the course.

$$T_{i,j}^{k+1} = T_{i,j}^{k} + \omega \left(T_{i,j}^{k+1} \Big|_{GS} - T_{i,j}^{k} \right)$$

  - The value of this over relaxation parameter, $\omega > 1$
  - Substituting and rearranging gives

$$T_{i,j}^{k+1} = T_{i,j}^{k} + \frac{\omega}{2(1+\beta^2)} \left(T_{i+1,j}^{k} + T_{i-1,j}^{k+1} + \left(T_{i,j+1}^{k} + T_{i,j-1}^{k+1} \right) \beta^2 - 2(1+\beta^2) T_{i,j}^{k} \right)$$

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Line by Line Method-I

- In point method, information flows from point to point
- Hence, the information from one boundary to another takes more time and it takes more effort to get convergence
- If we can facilitate exchange of information quickly, we can converge faster.
- Line methods accomplish this effectively. These methods are called block iterative methods.

$$T_{i-1,j}^{k+1} - 2\left(1+\beta^2\right) T_{i,j}^{k+1} + T_{i+1,j}^{k+1} = -\beta^2 \left(T_{i,j-1}^{k+1} + T_{i,j+1}^{k} \right)$$

- The idea is to construct the Tri-diagonal matrix and solve them by TDMA

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Line by Line Method-II

- If we could call the previous one as X-Sweep, we cal perform the next iteration in y direction and call it as Y-sweep. Here we use,

$$T_{i,j-1}^{k+1} - 2(1+\beta^2) T_{i,j}^{k+1} + T_{i,j+1}^{k+1} = -\beta^2 \left(T_{i-1,j}^{k+1} + T_{i+1,j}^{k} \right)$$

- If the boundary conditions are such that there is lot more variation in a given direction say y, then we can perform only y-sweeps.
- If there is no preferential direction then we can perform alternate x and y sweeps to improve convergence rates.

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ADI Method

- In line by line the information from one set of boundaries moves fast but not as fast from the other
- This can be accelerated, if we can alternate the direction
- This method is called Alternate Direction Implicit
- The iteration cycle has two parts.

X-Sweep

$$T_{i-1,j}^{k+0.5} - 2(1+\beta^2) T_{i,j}^{k+0.5} + T_{i+1,j}^{k+0.5} = -\beta^2 \left(T_{i,j-1}^{k+0.5} + T_{i,j+1}^{k} \right)$$

Y-Sweep

$$\beta^2 T_{i,j-1}^{k+1} - 2(1+\beta^2) T_{i,j}^{k+1} + \beta^2 T_{i,j+1}^{k+1} = -T_{i+1,j}^{k+0.5} - T_{i-1,j}^{k+1}$$

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Acceleration Procedures-II

- Line methods can also be accelerated -LSOR
  - The equation derived in the previous equation can be rearranged as

Same Eq.

$$T_{i,j}^{k+1} = T_{i,j}^{k} + \frac{\omega}{2(1+\beta^2)}\left(T_{i+1,j}^{k} + T_{i-1,j}^{k+1} + \left(T_{i,j+1}^{k} + T_{i,j-1}^{k+1}\right)\beta^2 - 2(1+\beta^2)T_{i,j}^{k}\right)$$

$$2(1+\beta^2)T_{i,j}^{k+1} = 2(1+\beta^2)T_{i,j}^{k} + \omega\left(T_{i+1,j}^{k} + T_{i-1,j}^{k+1} + \left(T_{i,j+1}^{k} + T_{i,j-1}^{k+1}\right)\beta^2 - 2(1+\beta^2)T_{i,j}^{k}\right)$$

$$\omega T_{i+1,j}^{k+1} - 2(1+\beta^2)T_{i,j}^{k+1} + \omega T_{i-1,j}^{k+1} = -2(1+\beta^2)(1-\omega)\, T_{i,j}^{k} - \omega\left(T_{i,j+1}^{k} + T_{i,j-1}^{k+1}\right)\beta^2$$

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Acceleration Procedures-III

- Accelerated ADI
  - The equation derived for LSOR can be rearranged as

X-Sweep

$$\omega T_{i+1,j}^{k+0.5} - 2(1+\beta^2)T_{i,j}^{k+0.5} + \omega T_{i-1,j}^{k+0.5} = -2(1+\beta^2)(1-\omega)T_{i,j}^{k} - \omega\left(T_{i,j+1}^{k} + T_{i,j-1}^{k+0.5}\right)\beta^2$$

Y-Sweep

$$\omega\beta^2 T_{i,j+1}^{k+1} - 2(1+\beta^2)T_{i,j}^{k+1} + \omega\beta^2 T_{i,j-1}^{k+1} = -2(1+\beta^2)(1-\omega)T_{i,j}^{k+0.5} - \omega\left(T_{i+1,j}^{k+0.5} + T_{i-1,j}^{k+1}\right)$$

- Relative Comparison
  - For a 21X41 nodes for a slab of 1X2 with one of the smaller side held at 1, and for the same accuracy, results have been given in Hoffman and Chiang
  - These are reproduced in the following slides

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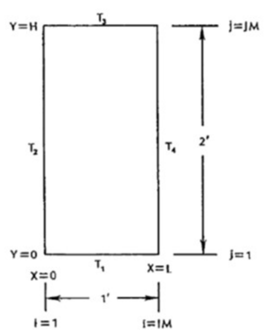
Sample Problem



- Rectangular Slab 1 x 2
- Nodes 21 x 41 (equispaced)
- $T_1 = T_2 = T_4 = 0$; $T_3 = 100$
- Convergence condition = summation of all Abs ($T^{k+1} - T^{k}$) is less than 0.01
- Studied all iteration schemes

Taken from Hoffman and Chiang

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Acceleration Procedures-IV

Relative Comparison

Method	Iterations	Time (CPU)	ω_{opt}
GS (point)	574	5.524	
GS(Line)	308	7.196	
PSOR	52	1.082	1.78
LSOR	36	1.410	1.265
ADI	157	6.693	
AADI	23	1.535	1.27

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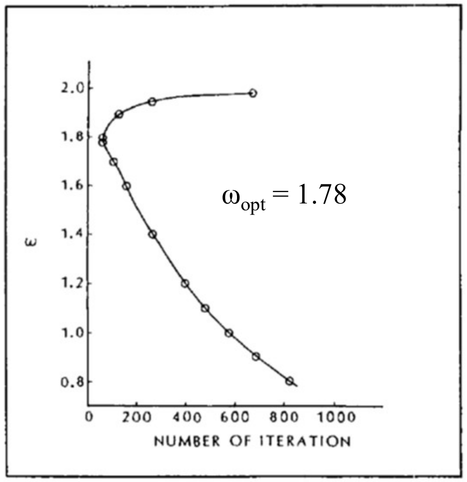
Influence of ω on Point SOR



$\omega_{opt} = 1.78$

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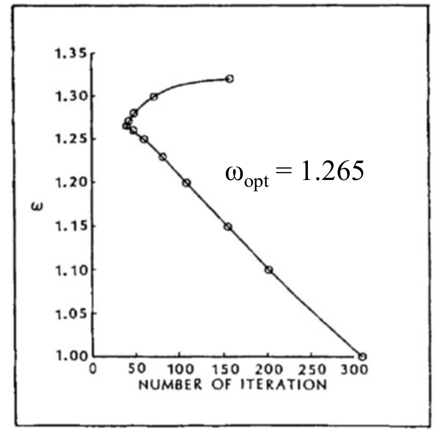
Influence of ω on Line SOR



$\omega_{opt} = 1.265$

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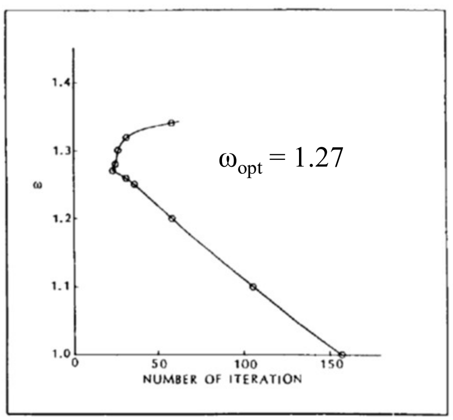
Influence of ω on Accelerated ADI



$\omega_{opt} = 1.27$

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Relative Convergence Rates