CH-2-16(MO)
Numerical Methods
(Solution of a Set of Non-Linear Equations)

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Review

- Looked at the principle of TDMA
- Got a glimpse of iterative methods
- Understood a few termination criteria.

- Today we shall look at methods for the solution of a set of non-linear solutions

General

Consider a set of linear equation:

$$f_1(x_1, x_2, ..x_n) = 0$$
$$f_2(x_1, x_2, ..x_n) = 0$$
$$\dots\dots\dots\dots$$
$$f_n(x_1, x_2, ..x_n) = 0$$

Simple example in two variables is:

$$x^2 + xy - 10 = 0$$
$$y + 3xy^2 - 57 = 0$$

Solution $\Longrightarrow$ $x = 2$, $y = 3$

Iteration Method-I

- We can rewrite the above equations as:

$$x = \frac{10 - x^2}{y} = g_1(x, y)$$
$$y = 57 - 3xy^2 = g_2(x, y)$$

- The above equations with an initial guess of x=1.5 and y=3.5 diverges

Iteration Method-II

- If we rewrite the equations as:

$$x = \sqrt{10 - xy} = g_1(x, y)$$

$$y = \sqrt{\frac{57 - y}{3x}} = g_2(x, y)$$

- The above equations converge to the correct solution with an initial guess of x=1.5 and y=3.5
- The condition for convergence is $\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_2}{\partial x}\right| \le 1 \quad and \quad \left|\frac{\partial g_1}{\partial y}\right| + \left|\frac{\partial g_2}{\partial y}\right| \le 1$

Newton-Raphson Method-I

- Principle

Expanding the functions linearly in the neighbourhood of (x,y)

$$f_1(x + \Delta x, y + \Delta y) = f_1(x, y) + \left.\frac{\partial f_1}{\partial x}\right|_{(x,y)} \Delta x + \left.\frac{\partial f_1}{\partial y}\right|_{(x,y)} \Delta y$$

For an arbitrary (x,y) we seek Δx and Δy such that

$$f_1(x + \Delta x, y + \Delta y) = 0$$

Newton-Raphson Method-II

$$\Rightarrow \left.\frac{\partial f_1}{\partial x}\right|_{(x,y)} \Delta x + \left.\frac{\partial f_1}{\partial y}\right|_{(x,y)} \Delta y = -f_1(x, y)$$

By similar reasoning we can write

$$\left.\frac{\partial f_2}{\partial x}\right|_{(x,y)} \Delta x + \left.\frac{\partial f_2}{\partial y}\right|_{(x,y)} \Delta y = -f_2(x, y)$$

- The above equations can be solved using a linear solver
- The derivatives can be found by finite difference

Newton-Raphson Method-III

- The above concepts can be extended for a set of N equations

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_1} & .... & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & .... & \frac{\partial f_2}{\partial x_n} \\ .... & .... & .... & .... \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & .... & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{Bmatrix} \Delta x_1 \\ \Delta x_2 \\ ... \\ \Delta x_n \end{Bmatrix} = \begin{Bmatrix} -f_1 \\ -f_2 \\ .... \\ -f_n \end{Bmatrix}$$

Jacobian Matrix

Newton-Raphson Method-IV

- The method converges with good initial guesses.
- The problem is to give good guesses.
- Continuation method is one powerful method to overcome this problem.

Continuation Method-I

- Here a set whose roots are known is considered
- The simplest way is to generate from the original function with an arbitrary initial guess

$$F_1(x,y) = f_1(x,y) - \theta\ f_1(x_0,y_0)$$

$$F_2(x,y) = f_2(x,y) - \theta\ f_2(x_0,y_0)$$

- For Theta = 1, (x_0,y_0) are the roots for F_1 and F_2
- The value of Theta is gradually reduced from 1 to 0 and the set is solved every time with the previous roots as the guess.

Continuation Method-II

Consider the example in Slide 3 is:

$$f_1(x,y) = x^2 + xy - 10 = 0,$$
$$f_2(x,y) = y + 3xy^2 - 57 = 0$$

Let guess be $x_0 = 0$, $y_0 = 0$

$$f_1(x_0,y_0) = -10,$$
$$f_2(x_0,y_0) = -57$$

Continuation Method-III

- Define a new set

$$F_1(x,y) = x^2 + xy - 10 - \theta(-10) = 0,$$
$$F_2(x,y) = y + 3xy^2 - 57 - \theta(-57) = 0$$

Note that for $\theta = 1$, (0, 0) is the root

For $\theta = 0.9$ we assume (0,0) is close to the root and we solve for the roots with this initial guess.

We proceed similarly with diminishing θ till $\theta = 0$. These roots are the solution of our Original set.