CH-2-16(MO)
Numerical Methods
Hyperbolic Equations

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Treatment of Hyperbolic Equations-I

- We have seen that the Parabolic and elliptic equations posed no special difficulty.
- Both FTCS and BTCS were very well behaved, though there was a time step restriction in FTCS.
- However, hyperbolic equations have to be carefully handled.
- Using schemes that violate the domain of dependence, can produce severe errors.
- We had already seen earlier that convection equation was a hyperbolic equation. Let us Recall.

Forward-Time Centered-Space Method

$$\left.\frac{\partial T}{\partial t}\right|_i^n = \frac{T_i^{n+1} - T_i^n}{\Delta t} \qquad \left.\frac{\partial T}{\partial x}\right|_i^n = \frac{T_{i+1}^n - T_{i-1}^n}{2\Delta x}$$

- Nodal Equation becomes

$$T_i^{n+1} = T_i^n - \frac{C}{2}(T_{i+1}^n - T_{i-1}^n) \quad \text{where} \quad C = \frac{u\Delta t}{\Delta x}$$

- Consistency Analysis gives

$$T_t + uT_x = \underbrace{-\frac{1}{2}u^2\Delta t T_{xx}}_{\text{Amplifying}} \underbrace{-\left(\frac{1}{6}u\Delta x^2 + \frac{1}{3}u^3\Delta t^2\right) T_{xxx}}_{\text{Dispersive}} + HOT$$

FTCS (Cont'd)

- von Neumann stability gives
  $G = 1 - IC\sin\theta$

→ Unconditionally unstable as $|G| \geq 1$

- This can also be judged by looking at the coefficient of T_{xx}. The condition for stability is the coefficient should be positive
- The above criterion is attributed to Hirt and is called Hirt's heuristic stability criterion. For convection equation this method is sufficient

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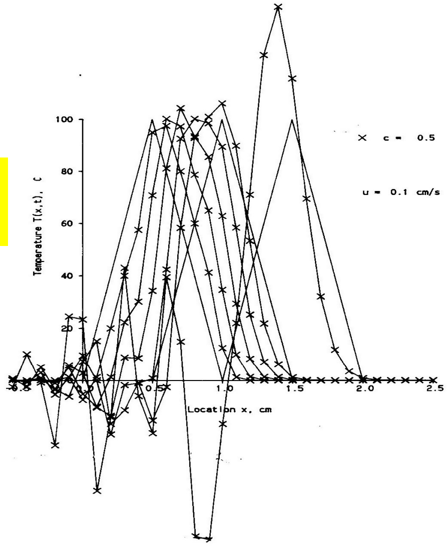
FTCS Scheme



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Lax Scheme

$$\left.\frac{\partial T}{\partial t}\right|_i^n = \frac{T_i^{n+1} - 0.5(T_{i+1}^n + T_{i-1}^n)}{\Delta t} \qquad \left.\frac{\partial T}{\partial x}\right|_i^n = \frac{T_{i+1}^n - T_{i-1}^n}{2\Delta x}$$

- Nodal Equation becomes

$$T_i^{n+1} = 0.5(T_{i+1}^n + T_{i-1}^n) - \frac{u\Delta t}{2\Delta x}(T_{i+1}^n - T_{i-1}^n)$$

- Consistency Analysis gives

$$T_t + uT_x = \left(0.5\frac{\Delta x^2}{\Delta t} - 0.5u^2\Delta t\right)T_{xx} + \left(\frac{u\Delta x^2}{3} - \frac{u^3\Delta t^2}{3}\right)T_{xxx} + HOT$$

Inconsistent

7:07 AM 7/35

Lax Scheme (Cont'd)

- MPDE can be rewritten as

$$T_t + uT_x = 0.5u\Delta x\left(\frac{1}{C} - C\right)T_{xx} + \frac{u\Delta x^2}{3}\left(1 - C^2\right)T_{xxx} + HOT$$

- Method is $O(\Delta t, \frac{\Delta x^2}{\Delta t})$
- Conditionally stable for $C \le 1$ $G = \cos(\theta_m) + I\,C\sin(\theta_m)$
- Exact for $C = 1$
- Diffusive for $C < 1$
- As C approaches 0 dissipation would increase

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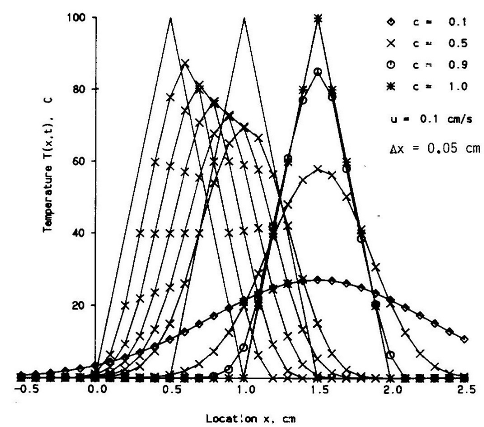
Lax Scheme



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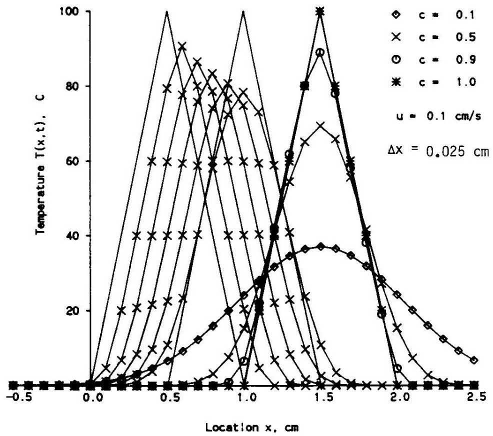
Lax Scheme



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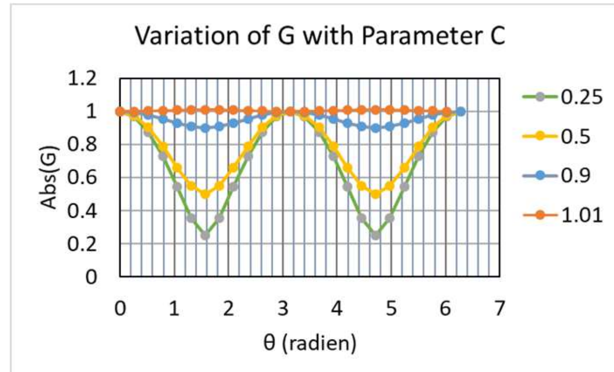

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First Order Upwind Scheme

$$\left.\frac{\partial T}{\partial t}\right|_i^n = \frac{T_i^{n+1}-T_i^n}{\Delta t} \qquad \left.\frac{\partial T}{\partial x}\right|_i^n = \frac{T_i^n-T_{i-1}^n}{\Delta x} \quad \text{for } u>0$$

- Nodal Equation becomes

$$T_i^{n+1} = T_i^n - C(T_i^n - T_{i-1}^n)$$

- Consistency Analysis gives

$$T_t + uT_x = \left(0.5u\Delta x - 0.5u^2\Delta t\right)T_{xx} + \left(\frac{u^3\Delta t^2}{3}\ \frac{u^2\Delta x\Delta t}{2} - \frac{u^3\Delta x^2}{6}\right)T_{xxx} + HOT$$

Consistent

7:07 AM 12/35

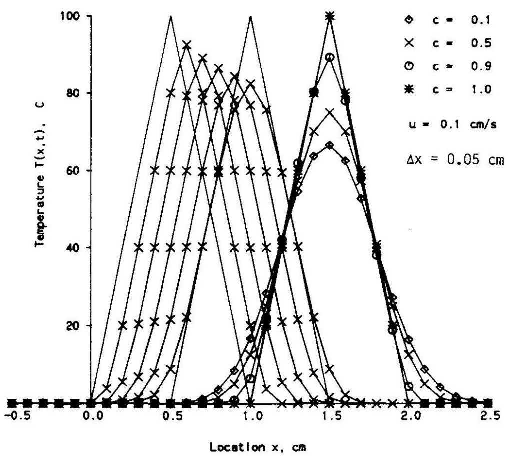
First Order Upwind (Cont'd)

- MPDE can be rewritten as

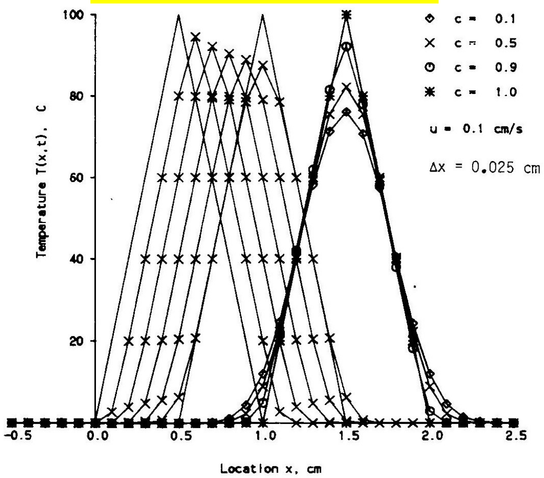
$$T_t + uT_x = 0.5u\Delta x(1-C)T_{xx} + \frac{u\Delta x^2}{6}\left(-2C^2+3C-1\right)T_{xxx} + HOT$$

- Method is $O(\Delta t, \Delta x)$
- Conditionally stable for $C \le 1$
- Exact for $C = 1$
- Diffusive for $C < 1$
- As C approaches 0 dissipation would increase but is bounded unlike in Lax

First Order Upwind



First Order Upwind



Second Order Upwind Scheme -I

- $\frac{\partial T}{\partial t} = \frac{T_i^{n+1} - T_i^n}{\Delta t}$; $\frac{\partial T}{\partial x} = \frac{3T_i^n - 4T_{i-1}^n + T_{i-2}^n}{2\Delta x}$
- Nodal Equation becomes

  $$T_i^{n+1} = T_i^n - \frac{C}{2}(3T_i^n - 4T_{i-1}^n + T_{i-2}^n)$$

- Consistency Analysis gives

  $$T_t + uT_x = -\frac{1}{2}u^2\Delta t T_{xx} + \left(-\frac{1}{3}u\Delta x^2 + \frac{1}{6}u^3\Delta t^3\right)T_{xxx}$$

- Unconditionally Unstable from Hirt's stability criterion

Lax-Wendroff Scheme

- Taylor Series can be used to devise a second order scheme for Convection Equation

  $$T_i^{n+1} = T_i^n + T_t\big|_i^n \Delta t + T_{tt}\big|_i^n \frac{\Delta t^2}{2} + O(\Delta t^3)$$

  $$\Rightarrow \quad T_i^{n+1} = T_i^n + \left(-uT_x\big|_i^n\right)\Delta t + u^2 T_{xx}\big|_i^n \frac{\Delta t^2}{2} + O(\Delta t^3)$$

- This can be expressed in finite difference form as

  $$T_i^{n+1} = T_i^n - \frac{C}{2}(T_{i+1}^n - T_{i-1}^n) + \frac{C^2}{2}(T_{i+1}^n - 2T_i^n + T_{i-1}^n) + HOT$$

Lax-Wendroff Scheme (Cont'd)

- Consistency Analysis gives

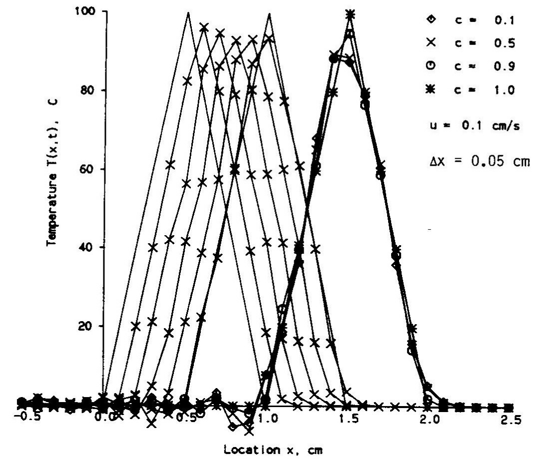
Consistent

$$T_t + uT_x = -\frac{1}{6}\left(u\Delta x^2 - u^3\Delta t^2\right) T_{xxx} - \frac{1}{8}\left(u^2\Delta x^2\Delta t - u^4\Delta t^3\right) T_{xxxx} + HOT$$

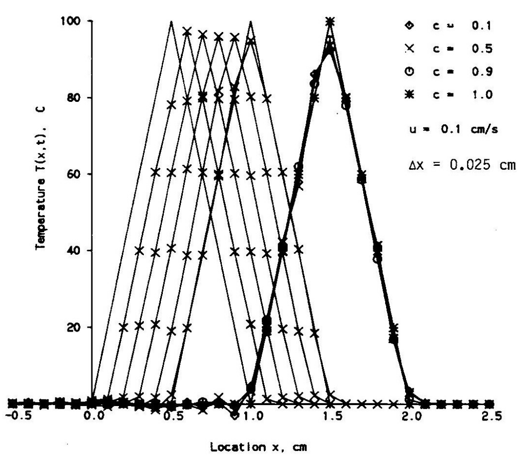
$$\Rightarrow \quad T_t + uT_x = -\frac{1}{6}u\Delta x^2\left(1 - C^2\right) T_{xxx} - \frac{1}{8}u\Delta x^3 C\left(C^2 - 1\right) T_{xxxx} + HOT$$

- Method is $O(\Delta t^2, \Delta x^2)$
- Conditionally stable for $C \le 1$
- Is less diffusive as leading diffusion term is of fourth order
- Exact for $C = 1$

Lax-Wendroff



Lax-Wendroff



The General Observations

- We generally noticed that in most explicit schemes the solution is exact for C = 1.
- There is numerical diffusion introduced in these schemes for C other than 1.
- Implicit Schemes were unconditionally stable but gave poor results.
- Though we are able to do consistency analysis and understand the nature of the schemes, no physical explanation was foreseeable.
- A lot of insight can be obtained by considering the method of characteristics.
- We shall look at the method in the following slides

Backward-Time Centered-Space Method - I

$$\left.\frac{\partial T}{\partial t}\right|_i^{n+1} = \frac{T_i^{n+1} - T_i^n}{\Delta t} \qquad \left.\frac{\partial T}{\partial x}\right|_i^{n+1} = \frac{T_{i+1}^{n+1} - T_{i-1}^{n+1}}{2\Delta x}$$

- Nodal Equation becomes

$$T_i^{n+1} - T_i^n + \frac{C}{2}(T_{i+1}^{n+1} - T_{i-1}^{n+1}) = 0$$ where $$\mathrm{C} = \frac{\mathrm{u}\Delta \mathrm{t}}{\Delta \mathrm{x}}$$

$$\frac{C}{2}T_{i+1}^{n+1} + T_i^{n+1} - \frac{C}{2}T_{i-1}^{n+1} = T_i^n$$

- Can Solve by TDMA

Backward-Time Centered-Space Method - II

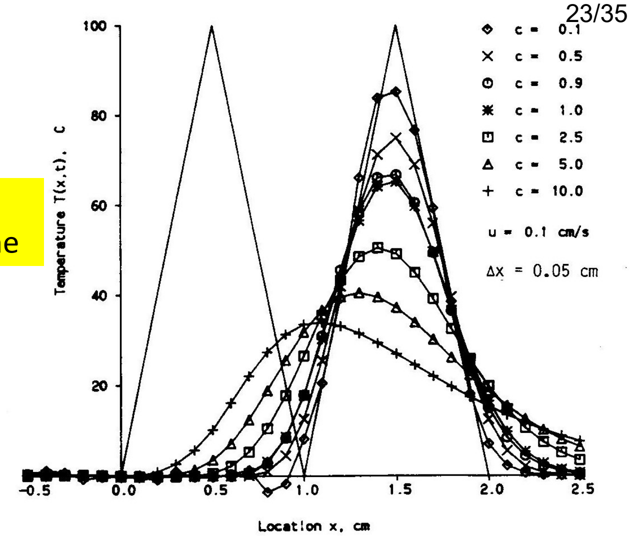
- Consistency Analysis gives

$$T_t + uT_x = \frac{1}{2}u^2\Delta t T_{xx} - \left(\frac{1}{6}u\Delta x^2 + \frac{1}{3}u^3\Delta t^2\right)T_{xxx} + HOT$$

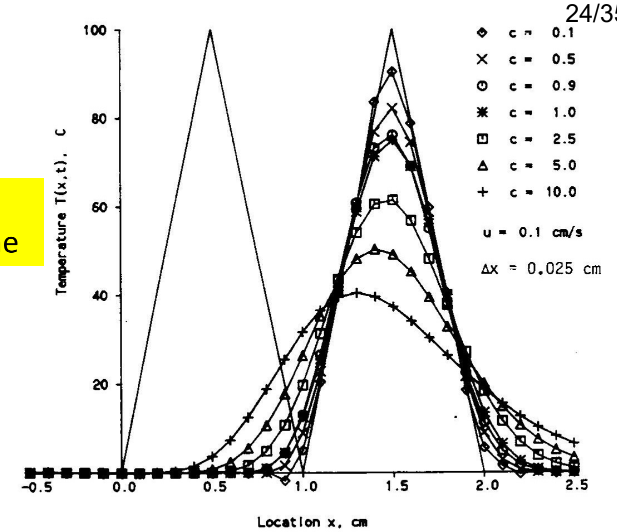
Dissipative Dispersive

- Method is $O(\Delta t, \Delta x^2)$
- Von Neumann analysis gives $G = \dfrac{1}{1 + ISin\,\theta}$
- Unconditionally stable
- For a given Δx, as Δt increases (C increases), the diffusion as well as dispersion will increase

BTCS Scheme



BTCS Scheme



Method of Characteristics - I

- MOC is a technique by which a PDE is reduced by one independent coordinate
- By this method, 1-D transient PDE can be reduced to an ODE along the characteristic directions
- Since T = T(x,t), using chain rule assuming continuity of T, we can write

$$dT = \frac{\partial T}{\partial t}dt + \frac{\partial T}{\partial x}dx \qquad \Rightarrow \frac{dT}{dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx}{dt}$$

- The governing equation is

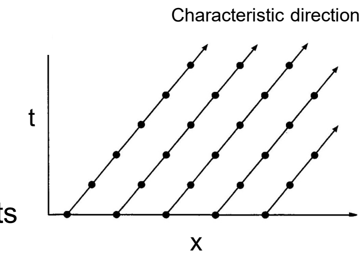
$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = 0$$

- From the above two equations, we can write

$$\frac{dT}{dt} = 0 \;\; along \;\; \frac{dx}{dt} = u$$

Method of Characteristics - II

- The first equation describes the spatial variation of field variable T along the characteristic direction
- Thus, the PDE has been split into two ODEs, one being characteristic direction and the other the compatibility condition
- For linear convection equation the point on the down stream of characteristic can only be influenced by the state of upstream points along the direction



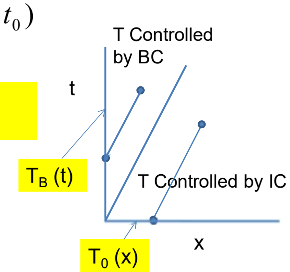
Method of Characteristics - III

- The analytical solution using MOC technique can be visualized as follows
- Integration of the characteristic equation with x = x_0 at t = t_0 gives

$$\int_{x_0}^{x} dx = u\int_{t_0}^{t} dt \quad \Rightarrow x = x_0 + u(t - t_0)$$

$$\frac{dT}{dt} = 0 \Rightarrow T = T_0$$

Along the Characteristic



Method of Characteristics - IV

- The analytical procedure will first need x_0. This can be obtained by putting t_0 = 0, for the point of interest (x,t) in the equation of path line

$$x_0 = x - u(t - t_0)$$

- If x_0 is greater than 0, then it is in IC controlled region, else it is in BC controlled region. If in IC controlled region

$$T(x,t) = T_0(x_0)$$

- If (x,t) is in BC controlled region, get t_0 by putting x_0 = 0 and then T (x,t) can be obtained as

$$t_0 = t - \frac{x}{u} \quad \text{and} \quad T(x,t) = T_B(t_0)$$

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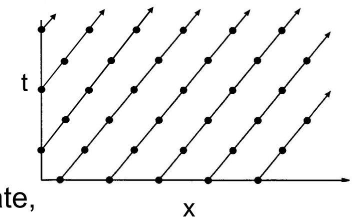
Numerical MOC - I

29/35

Forward Marching

- Originate points at the initial and boundary axes
- March along path line generating interior nodes
- The grids may not be equidistant if u is not a constant,
- The temperatures are computed using Compatibility equation
- Considered most accurate, but difficult to program for complex cases. Not popular.



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Numerical MOC - II

30/35

Backward Marching

- This method is to help formulating the method for a structured grid
- By its nature, it involves interpolations across the characteristic if the slopes of the characteristic lines are not same.
- This is the main cause of numerical diffusion.
- This method establishes connections with the schemes already described.

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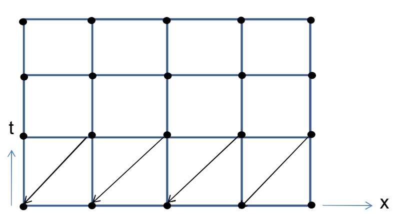
Numerical MOC - III

31/35

- First let us consider the uniform velocity case
- If we choose $u\Delta t = \Delta x$, or C = 1, then, the characteristic passes from (i-1, n) to (I, n+1) exactly.
- This implies that T = constant along these and we get exact solution.



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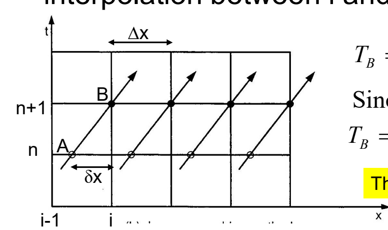
Numerical MOC - IV

32/35

- If the velocity is not uniform, then the characteristic, when back projected does not exactly pass through a grid point
- Compatibility implies that $T_B = T_A$
- Since point A does not exist in computational grid, we need to interpolate. For linear interpolation between i and i-1, we get



$$T_B = T_i^{n+1} = T_A = T_i^n - \left(T_i^n - T_{i-1}^n\right)\frac{\delta x}{\Delta x}$$

Since $\delta x = u\Delta t$

$$T_B = T_i^{n+1} = T_A = T_i^n - \left(T_i^n - T_{i-1}^n\right)\frac{u\Delta t}{\Delta x}$$

This is first order upwind scheme

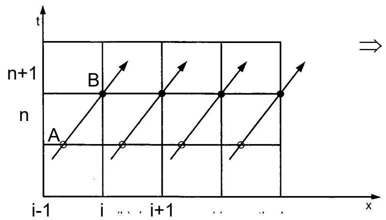
Numerical MOC - IV

- If we interpolate linearly using i+1 and i-1, we get the following

$$T_B = T_i^{n+1} = T_A$$

$$\frac{T_{i+1}^n - T_A}{T_{i+1}^n - T_{i-1}^n} = \frac{x_{i+1} - x_A}{x_{i+1} - x_{i-1}} = \frac{x_{i+1} - x_i + x_i - x_A}{x_{i+1} - x_{i-1}} = \frac{\Delta x + u\Delta t}{2\Delta x} = \frac{1+C}{2}$$

$$\Rightarrow T_{i+1}^n - T_i^{n+1} = \frac{1+C}{2}\left(T_{i+1}^n - T_{i-1}^n\right) \qquad \Rightarrow T_i^{n+1} = T_{i+1}^n - \frac{1+C}{2}\left(T_{i+1}^n - T_{i-1}^n\right)$$

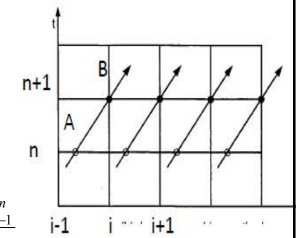


$$\Rightarrow T_i^{n+1} = \frac{T_{i+1}^n + T_{i-1}^n}{2} - \frac{C}{2}\left(T_{i+1}^n - T_{i-1}^n\right)$$

This is Lax scheme

Numerical MOC - IV

- If we interpolate Second order curve using i+1, i and i-1, we get the following



Let $T(x) = a + bx + cx^2$

If origin is taken at x(i,n)

$$\Rightarrow a = T_i^n, \quad b = \frac{T_{i+1}^n - T_{i-1}^n}{2\Delta x}, \quad c = \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{2\Delta x^2}$$

$$\Rightarrow T(x) = T_i^n + \frac{T_{i+1}^n - T_{i-1}^n}{2\Delta x}x + \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{2\Delta x^2}x^2$$

Putting $x = x_A = -u\Delta t$, $T(x) = T_A = T_B = T_i^{n+1}$

$$\Rightarrow T_i^{n+1} = T_i^n - \frac{T_{i+1}^n - T_{i-1}^n}{2\Delta x}u\Delta t + \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{2\Delta x^2}(u\Delta t)^2$$

$$\Rightarrow T_i^{n+1} = T_i^n - \frac{T_{i+1}^n - T_{i-1}^n}{2}C + \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{2}C^2$$

This is Lax-Wendroff scheme

Summary

- We have seen that handling of hyperbolic equations need certain care.
- We saw method of characteristics that laid the foundation for the analytical solution.
- Digested that the forward marching numerical method of characteristics is difficult to implement effectively.
- Backward methods involve interpolations and we could connect some of the finite difference schemes.