LINEAR ALGEBRA

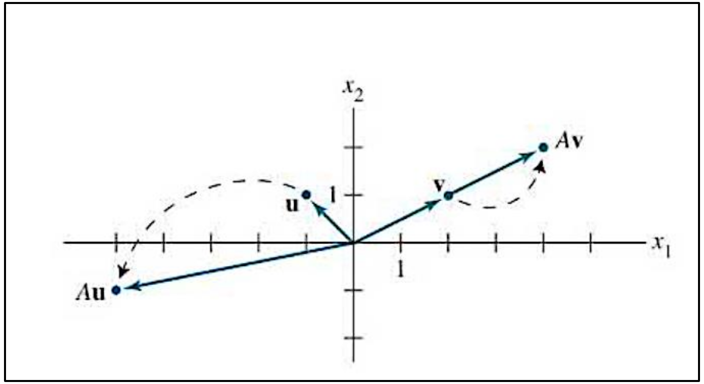
Eigenvectors and Eigenvalues

Muhammad Afif Hendrawan, S.Kom., M.T.

Intro to Eigenvectors and Eigenvalues

The Illustration #1

- The transformation of $x \mapsto Ax$ can be move vectors in a variety of direction
- However, it often happened that there are special vectors on which the action of A is quite simple.
- Example
  - Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\boldsymbol{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
  - The multiplication of u and v by A shown in the figure
  - Check! $A\boldsymbol{v}$ is just $2\boldsymbol{v}$

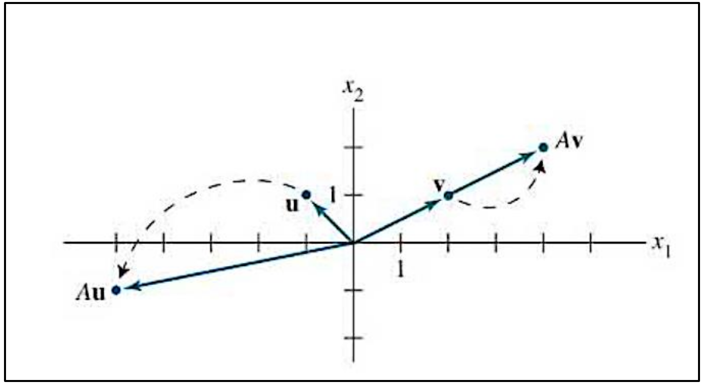


The Illustration #2

- Therefore, we try to find the equations such as,

$$Ax = 2x \text{ or } Ax = -4x$$

- Where ***specials vectors are transformed by A into scalar multiples of themself***



Formal Definition

An **eigenvectors** of an $n \times n$ matrix A is nonzero vector x such that,

$$Ax = \lambda x$$

For some scalar λ

A scalar λ is called **eigenvalue** of A

If there is a nontrivial solution $\boldsymbol{x}$ of $Ax = \lambda x$

Such an x is called an eigenvector corresponding of λ

Formal Definition – Example 1 #1

Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\boldsymbol{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, and $\boldsymbol{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are $\boldsymbol{u}$ and $\boldsymbol{v}$ eigenvectors of A?

Solution

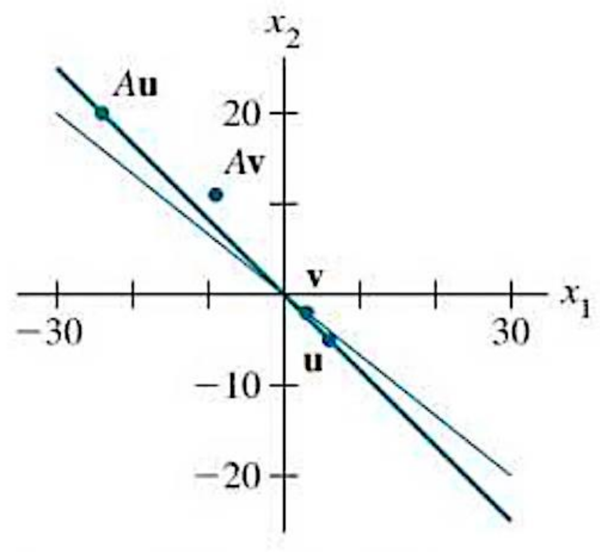
$$A\boldsymbol{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\boldsymbol{u}$$

$$A\boldsymbol{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

So, $\boldsymbol{u}$ is an eigenvector of A corresponding to an eigenvalue (-4); but $\boldsymbol{v}$ is not an eigenvector of A because $A\boldsymbol{v}$ is not a multiple of $\boldsymbol{v}$

Formal Definition – Example 1 #2

If you try to plot the projection of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, you will get,



$A\mathbf{u} = -4\mathbf{u}$, but $A\mathbf{v} \neq \lambda\mathbf{v}$.

Formal Definition – Example 2 #1

Show that 7 is an eigenvalue of matrix A from previous example and find the corresponding eigenvectors.

Solution → The scalar 7 is an eigenvalues of A **if and only if** the equation

$$A\boldsymbol{x} = 7\boldsymbol{x}$$

Has a nontrivial solution. That equation about equivalent to,

$$A\boldsymbol{x} - 7\boldsymbol{x} = 0$$

Or

$$(A - 7I)\boldsymbol{x} = 0$$

Where I is an identity matrix

Formal Definition – Example 2 #2

Solve that homogeneous equation → Form the matrix

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

The column of $A - 7I$ are **linearly dependent**. Therefore $(A - 7I)\boldsymbol{x} = 0$ has nontrivial solution.

The corresponding eigenvector can be found by row operation:

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Formal Definition – Example 2 #3

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution has a form,

$$x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Each vector of this form with $x_2 \neq 0$ is an eigenvector corresponding to $\lambda = 7$

Formal Definition – The Generalization of The Equation

The equation of,

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

Can be written as,

$$A\boldsymbol{x} = \lambda I\boldsymbol{x}$$

Also can be written as,

$$A\boldsymbol{x} - \lambda I\boldsymbol{x} = 0$$

Or can be simplified as,

$$(A - \lambda I)\boldsymbol{x} = 0$$

Determine The Eigenvalues and Eigenvectors

Determine The Eigenvalues

- To make λ the eigenvalue of $A\boldsymbol{x} = \lambda\boldsymbol{x}$, the solution of that equation should be nontrivial (at least there is one solution)
- The $A\boldsymbol{x} = \lambda\boldsymbol{x}$ is nontrivial if and only if,

$$\det(A - \lambda I) = 0$$

It also can be written as,

$$\det(\lambda I - A) = 0$$

Determine The Eigenvalues Example #1

Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$, find the eigenvalue of A

Compute the $A - \lambda I$

$$A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & -1 & 0 \\ 0 & -\lambda & -1 \\ 4 & -17 & 8 - \lambda \end{bmatrix}$$

Determine The Eigenvalues Example #2

Calculate $\det(A - \lambda I)$ → You can use any method!

$$\det\begin{bmatrix} -\lambda & -1 & 0 \\ 0 & -\lambda & -1 \\ 4 & -17 & 8-\lambda \end{bmatrix} = -\lambda^3 + 8\lambda^2 - 17\lambda + 4$$

By using $\det(A - \lambda I) = 0$,

$$-\lambda^3 + 8\lambda^2 - 17\lambda + 4 = 0$$
$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$
$$(\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0$$

By quadratic equation, the solution of $(\lambda^2 - 4\lambda + 1) = 0$ is $2 + \sqrt{3}$ and $2 - \sqrt{3}$

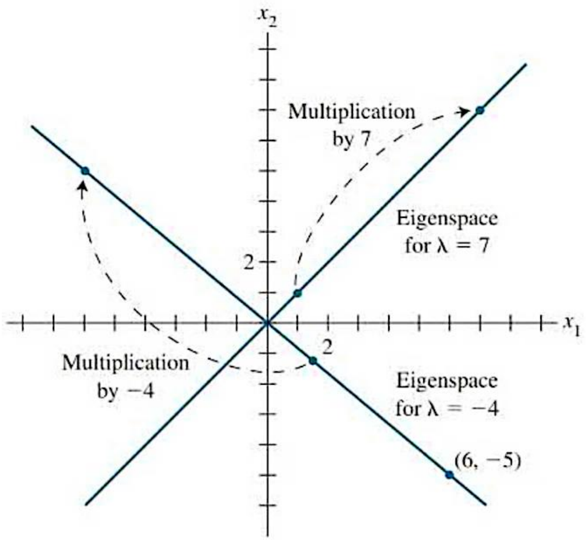
So, the eigenvalues is 4, $2 + \sqrt{3}$, and $2 - \sqrt{3}$

Intro to Eigenspace #1

- Row reduction was used in Formal Definition – Example 2 to find eigenvectors, it cannot to be used to find eigenvalues
- An Echelon form of a matrix A usually does not display the eigenvalues of A
- The equivalence of equation $A\boldsymbol{x} = 7x$ and $(A - 7I)\boldsymbol{x} = 0$ holds for any λ in place of $\lambda = 7$
- Thus, λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation
$$(A - \lambda I)\boldsymbol{x} = 0$$
Has a nontrivial solution
- The set of ***all solution*** of $(A - \lambda I)\boldsymbol{x} = 0$ is just the null space of the matrix $A - \lambda I$
- So, this set is a subspace of $\mathbb{R}^n$ and is called the ***eigenspace*** of A corresponding to λ
- **The eigenspace consist of the zero vector and all eigenvectors corresponding to λ**

Intro to Eigenspace #2

- From Formal Definition – Example 2 shows that for matrix A in Formal Definition Example 1, the eigenspace corresponding to λ consist of all multiples of $\begin{bmatrix}1\\1\end{bmatrix}$ → The line through $\begin{bmatrix}1\\1\end{bmatrix}$ and the origin
- From Formal Definition – Example 1, we can check that the eigenspace corresponding to $\lambda = -4$ is line through $\begin{bmatrix}6\\-5\end{bmatrix}$
- These eigenspace depict in the figure beside



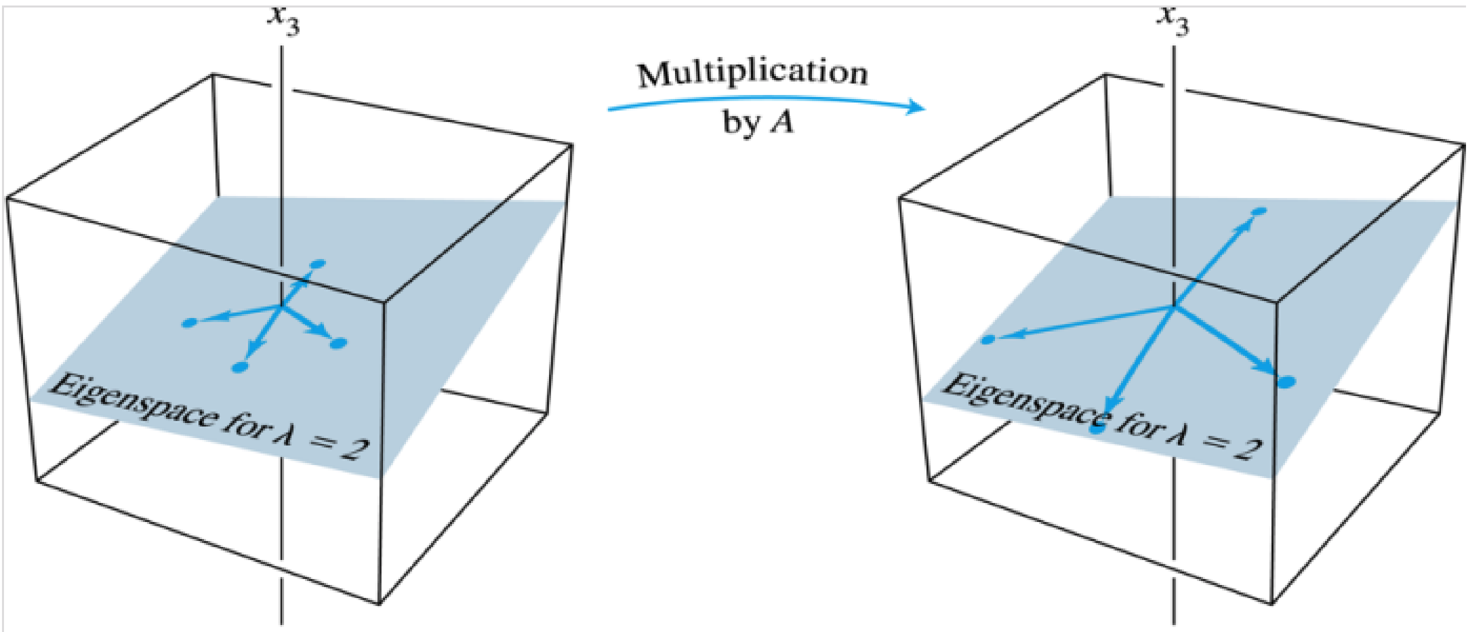
So, How to determine the eigenvector using basis of eigenspaces? #1

- Let we call $\mathbb{E}(\lambda)$ the λ-eigenspace for matrix A when λ is an eigenvalue for A
- So, we need to seek a basis for the eigenspace for $\mathbb{E}(\lambda) = \mathrm{A} - \lambda I$
- For example,
  - $\lambda = 2$ is an eigenvalue for $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. Find all the basis for the corresponding eigenvalue
- Solution,
  - We try to find $\mathbb{E}(2) = \mathrm{A} - 2I$, so
  - $\mathrm{A} - 2I = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ → x_2 and x_3 are free → let say $x_2 = 2s$ and $x_3 = t$
  - We get $2x_1 - 2s + 6t = 0$

So, How to determine the eigenvector using basis of eigenspaces? #2

- Solution,
  - From → $2x_1 - 2s + 6t = 0$
  - The general solution of $(A - 2I)x = 0$ has the form of $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s - 3t \\ 2s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$
  - So, $\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right\}$ is the basis for $\mathbb{E}(\lambda)$ and we can see that $\mathbb{E}(\lambda)$ is the plane in $\mathbb{R}^3$ spanned by two eigenvector $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

So, How to determine the eigenvector using basis of eigenspaces? #3



A acts as a dilation on the eigenspace.

Theorems

The 1st Theorem → Eigenvalues

The Eigenvalues of a triangular matrix are the entries of its main diagonal

Examples,

Let $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{bmatrix}$

The eigenvalue of A are 3,0,2 and the eigenvalues of B are 4 and 1

NB: You can proof it by using $(A - \lambda I)x = 0$

The 2nd Theorem → Eigenvectors

If $\boldsymbol{v}_1, \dots, \boldsymbol{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_r\}$ is linearly independent

Exercise!

Find the eigenvalue and eigenvector for this following matrices,

$$\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}$$


thank you
danke
рахмат
謝謝
ngiyabonga
شكراً جزيلاً
Баярлалаа
спасибо
faafetai lava
mersi
kia ora
barka
welalin
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teşekkür ederim
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tapadh leat
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спасибі
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감사합니다
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References

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