ALJABAR LINIER

Vector – Cross Product

Muhammad Afif Hendrawan, S.Kom., M.T.

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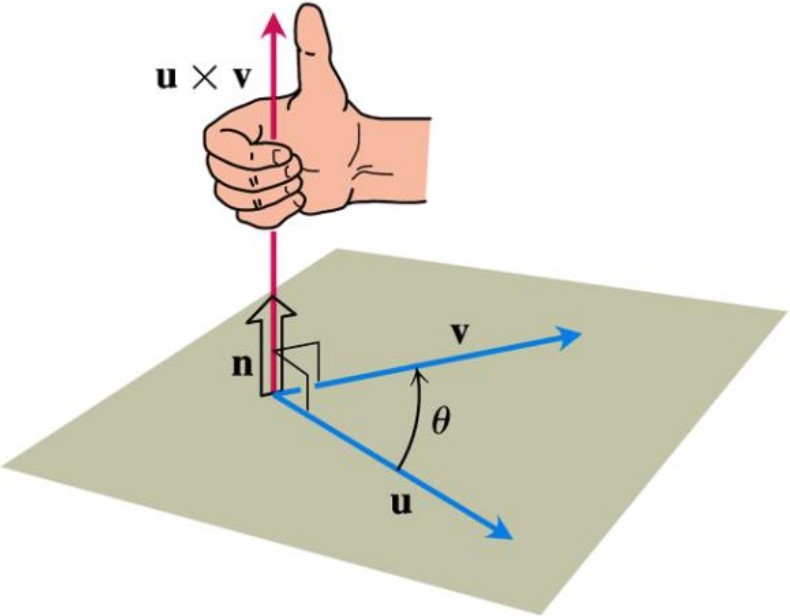
Cross Product

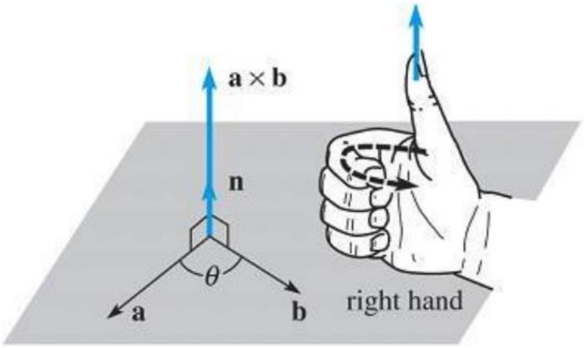
Deskripsi Geometri dari *Cross Product*

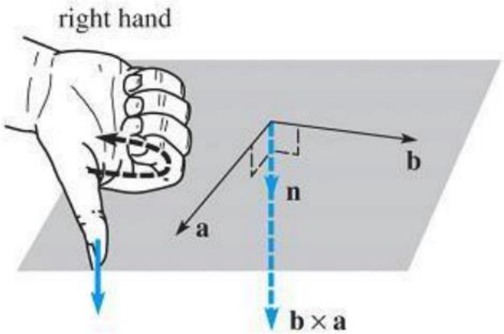
- $\boldsymbol{u} \times \boldsymbol{v}$ tegak lurus terhadap $\boldsymbol{u}$ dan $\boldsymbol{v}$
- Panjangya dari $\boldsymbol{u} \times \boldsymbol{v}$ adalah $|\boldsymbol{u} \times \boldsymbol{v}| = |\boldsymbol{u}||\boldsymbol{v}| sin\ \theta$
- Arah ditunjukkan dengan aturan tangan kanan



Aturan Tangan Kanan

- Letakkan 4 jari pada arah vektor pertama.
- Tekuk jari tersebut pada arah vektor kedua.
- Ibu jari menjukkan arah dari *cross product*





Definisi Aljabar dari Cross Product

Cross product dari $\boldsymbol{u} = \langle u_1, u_2, u_3 \rangle$ dan $\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle$ adalah

$$\boldsymbol{u} \times \boldsymbol{v} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle$$

Sehingga:

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{u} = \langle u_2v_3 - u_3v_2, u_3, v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle \cdot \langle u_1, u_2, u_3 \rangle$$
$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{u} = u_2v_3u_1 - u_3v_2u_1 + u_3v_1u_2 - u_1v_3u_2 + u_1v_2u_3 - u_2v_1u_3$$
$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{u} = 0$$

Begitu juga:

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{v} = 0$$

Nilai Cross Product Berdasarkan Determinan Vektor #1

Diberikan vektor $\boldsymbol{u} = \langle u_1, u_2, u_3 \rangle$ dan $\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle$, nilai cross product di definisikan dengan determinan vektor,

$$\boldsymbol{u} \times \boldsymbol{v} = \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle$$

Contoh, hitung $\boldsymbol{u} \times \boldsymbol{v}$ dan $\boldsymbol{v} \times \boldsymbol{u}$ untuk $\boldsymbol{u} = \langle 2, 3, 5 \rangle$ dan $\boldsymbol{v} = \langle 6, 7, 9 \rangle$.

$$\boldsymbol{u} \times \boldsymbol{v} = \left\langle \begin{vmatrix} 3 & 5 \\ 7 & 9 \end{vmatrix}, \begin{vmatrix} 5 & 2 \\ 9 & 6 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 6 & 7 \end{vmatrix} \right\rangle$$

$$\boldsymbol{u} \times \boldsymbol{v} = \langle 3 \cdot 9 - 7 \cdot 5, 5 \cdot 6 - 9 \cdot 2, 2 \cdot 7 - 6 \cdot 3 \rangle = \langle -8, 12, -4 \rangle$$

Nilai Cross Product Berdasarkan Determinan Vektor #2

Selanjutnya, jika, $\boldsymbol{v} \times \boldsymbol{u}$

$$\boldsymbol{v} \times \boldsymbol{u} = \left\langle \begin{vmatrix} 7 & 9 \\ 3 & 5 \end{vmatrix}, \begin{vmatrix} 9 & 6 \\ 5 & 2 \end{vmatrix}, \begin{vmatrix} 6 & 7 \\ 2 & 3 \end{vmatrix} \right\rangle = \langle 8, -12, 4 \rangle$$

Maka,

$$\boldsymbol{u} \times \boldsymbol{v} = -(\boldsymbol{v} \times \boldsymbol{u})$$

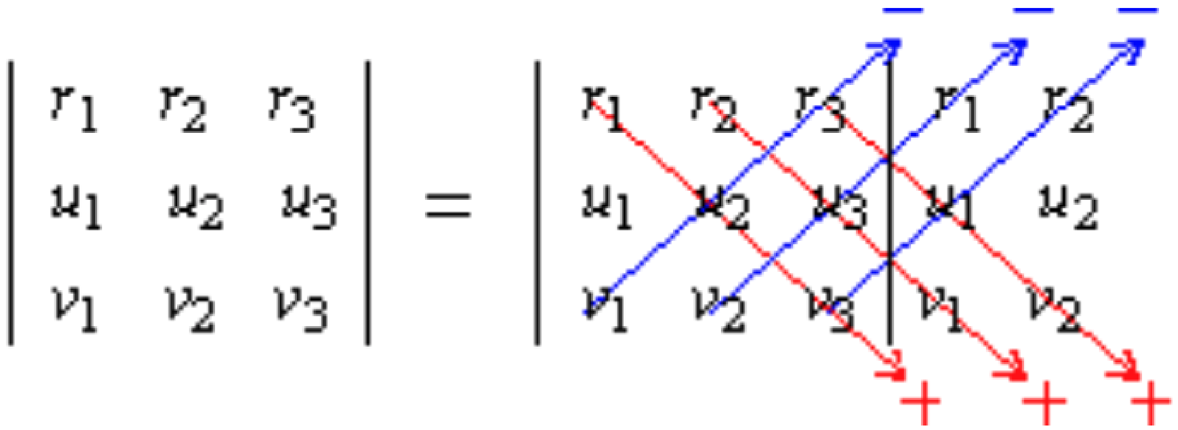
Sifat-sifat

Cross product di definisikan hanya dengan vektor 3 dimensi, vector $\boldsymbol{u}$ dan $\boldsymbol{v}$. Sehingga berlaku,

- $\boldsymbol{v} \times \boldsymbol{u} = -(\boldsymbol{u} \times \boldsymbol{v})$
- $\boldsymbol{u} \times \boldsymbol{u} = 0$
- $(k\boldsymbol{u}) \times \boldsymbol{v} = k(\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{u} \times k\boldsymbol{v}$
- $\boldsymbol{a} \times (\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{a} \times \boldsymbol{u} + \boldsymbol{a} \times \boldsymbol{v}$

Bagaimana jika menggunakan nilai determinan 3×3? #1

$$\begin{vmatrix} r_1 & r_2 & r_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = r_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} + r_2 \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix} + r_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$



Bagaimana jika menggunakan nilai determinan 3×3? #2

Jika kita letakkan, **i**, **j**, dan **k** pada baris pertama,

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} + \mathbf{j} \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

Bagaimana jika menggunakan nilai determinan 3×3? Contoh

Dengan determinan 3 dimensi, hitung $\boldsymbol{u} \times \boldsymbol{v}$ dan $\boldsymbol{v} \times \boldsymbol{u}$ untuk $\boldsymbol{u} = \langle 2, 1, 2 \rangle$ and $\boldsymbol{v} = \langle 3, 4, 5 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 3 & 4 & 5 \end{vmatrix} = \left| \begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ 2 & 1 & 2 & 2 & 1 \\ 3 & 4 & 5 & 3 & 4 \end{array} \right. = 5\mathbf{i} + 6\mathbf{j} + 8\mathbf{k} - 3\mathbf{k} - 8\mathbf{i} - 10\mathbf{j}$$

$$\boldsymbol{u} \times \boldsymbol{v} = -3\boldsymbol{i}, -4\boldsymbol{j} + 5\boldsymbol{k}$$

The Triple Scalar Product

If $\mathbf{r} = \langle r_1, r_2, r_3 \rangle$, $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

$$\mathbf{r} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} r_1 & r_2 & r_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\mathbf{r} \cdot (\mathbf{u} \times \mathbf{v}) = \left| \begin{array}{ccc|cc} r_1 & r_2 & r_3 & r_1 & r_2 \\ u_1 & u_2 & u_3 & u_1 & u_2 \\ v_1 & v_2 & v_3 & v_1 & v_2 \end{array} \right. = \left| \begin{array}{ccc|cc} v_1 & v_2 & v_3 & v_1 & v_2 \\ r_1 & r_2 & r_3 & r_1 & r_2 \\ u_1 & u_2 & u_3 & u_1 & u_2 \end{array} \right. = (\mathbf{r} \times \mathbf{u}) \cdot \mathbf{v}$$

We call this identity the *triple scalar product:*

$$\mathbf{r} \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{r} \times \mathbf{u}) \cdot \mathbf{v}$$

Latihan

Compute the cross product of $\mathbf{u} \times \mathbf{v}$ *and then compute the cross product of* $\mathbf{v} \times \mathbf{u}$. *Also, show that* $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal to* $\mathbf{u} \times \mathbf{v}$.

1. $\mathbf{u} = \langle 2, 1, 0 \rangle$, $\mathbf{v} = \langle 3, 1, 0 \rangle$
2. $\mathbf{u} = \langle 2, 1, 0 \rangle$, $\mathbf{v} = \langle -1, 3, 0 \rangle$
3. $\mathbf{u} = \langle 3, 3, 0 \rangle$, $\mathbf{v} = \langle 2, 0, 0 \rangle$
4. $\mathbf{u} = \langle 0, 1, 0 \rangle$, $\mathbf{v} = \langle 0, 0, 1 \rangle$
5. $\mathbf{u} = \langle 1, 0, 0 \rangle$, $\mathbf{v} = \langle 0, 1, 0 \rangle$
6. $\mathbf{u} = \langle 1, 0, 0 \rangle$, $\mathbf{v} = \langle 1, 0, 0 \rangle$
7. $\mathbf{u} = \langle 2, 3, 7 \rangle$, $\mathbf{v} = \langle 7, 3, 5 \rangle$
8. $\mathbf{u} = \langle 6, 2, 9 \rangle$, $\mathbf{v} = \langle 1, 0, 3 \rangle$
9. $\mathbf{u} = \langle 3, 4, 2 \rangle$, $\mathbf{v} = \langle 9, 12, 6 \rangle$
10. $\mathbf{u} = \langle 1, 1, 1 \rangle$, $\mathbf{v} = \langle -1, -1, -1 \rangle$


thank you
danke
謝謝
ngiyabonga
شكراً جزيلاً
teşekkür ederim
gracias
merci
obrigado
bedankt
dziękuję
спасибо
go raibh maith agat
mochchakkeram
sukriya
kop khun krap
terima kasih
grazie
arigatô
dakujem
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ありがとう
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dank je
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Referensi

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- https://math.etsu.edu/multicalc/prealpha/Chap1/Chap1-3/printversion.pdf