LINEAR ALGEBRA

Orthogonality

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Outlines

- (Recall) Dot Product / Inner Product
- (Recall) Magnitude / Length
- (Recall) Unit Vector
- (Recall) Distance Between Vectors
- Orthogonality

Inner Product

(Recall) Inner Product

- If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{R}^n$ → $\boldsymbol{u}$ and $\boldsymbol{v}$ as $n \times 1$ matrices
- If we transpose $\boldsymbol{u}^T$ is a $1 \times n$ matrix → the matrix product $\boldsymbol{u}^T\boldsymbol{v}$ is a 1×1 matrix → we write it as a single real number (a scalar) without brackets.
- The number $\boldsymbol{u}^T\boldsymbol{v}$ called inner product of $\boldsymbol{u}$ and $\boldsymbol{v}$ → $\boldsymbol{u} \cdot \boldsymbol{v}$
- Inner product == Dot product!
- Suppose we have,

$$\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_n \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_n \end{bmatrix} \text{ the inner product is } [\boldsymbol{u}_1 \quad \boldsymbol{u}_2 \quad \cdots \quad \boldsymbol{u}_n] \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_n \end{bmatrix} = \boldsymbol{u}_1\boldsymbol{v}_1 + \boldsymbol{u}_2\boldsymbol{v}_2 + \cdots + \boldsymbol{u}_n\boldsymbol{v}_n$$

(Recall) Inner Product Example

- Computer $\boldsymbol{u} \cdot \boldsymbol{v}$ and $\boldsymbol{v} \cdot \boldsymbol{u}$ for $\boldsymbol{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$
- Solution

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{v} = \begin{bmatrix} 2 & -5 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix} = (2)(3) + (-5)(2) + (-1)(-3) = -1$$

$$\boldsymbol{v} \cdot \boldsymbol{u} = \boldsymbol{v}^T \boldsymbol{u} = \begin{bmatrix} 3 & 2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} = (3)(2) + (2)(-5) + (-3)(-1) = -1$$

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$$

(Recall) Inner Product Theorem

Let u, v, and w be vectors in $\mathbb{R}^n$, and let c be a scalar. Then,

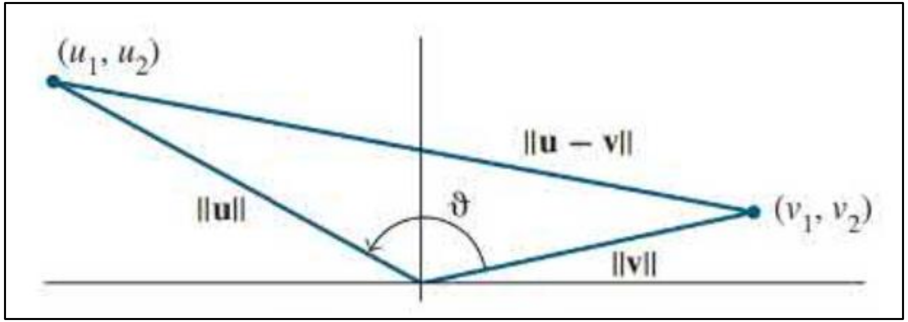
- $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$
- $(\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}$
- $(c\boldsymbol{u}) \cdot v = c(\boldsymbol{u} \cdot \boldsymbol{v}) = \boldsymbol{u} \cdot (c\boldsymbol{v})$
- $\boldsymbol{u} \cdot u \geq 0$, and $\boldsymbol{u} \cdot \boldsymbol{u} = 0$ if and only if $\boldsymbol{u} = 0$

(Recall) Angles in $\mathbb{R}^2$ and $\mathbb{R}^3$ #1

- If u and v are nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ there is a connection between inner product and the angle θ between the two lines segments from the origin to the points identified with u and v.

$$\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\| \|\boldsymbol{v}\| \cos \theta$$

- To verify the formula in $\mathbb{R}^2$ consider the figure beside
- It contain $\|\boldsymbol{u}\|$, $\|\boldsymbol{v}\|$, and $\|\boldsymbol{u} - \boldsymbol{v}\|$



(Recall) Angles in $\mathbb{R}^2$ and $\mathbb{R}^3$ #2

- By the law of cosines

$$\|\boldsymbol{u}-\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - 2\|\boldsymbol{u}\|\|\boldsymbol{v}\| \cos\theta$$

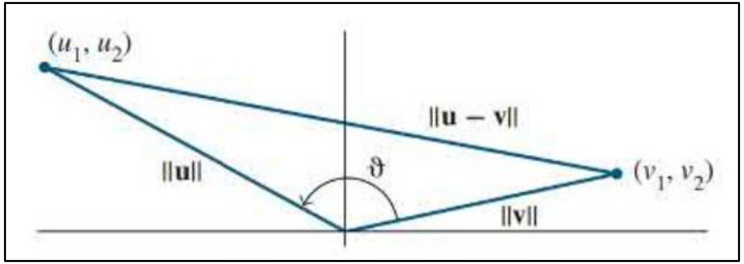
- You can rearrange the formula, therefore,

$$\|\boldsymbol{u}\|\|\boldsymbol{v}\| \cos\theta = \frac{1}{2}[\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - \|\boldsymbol{u}-\boldsymbol{v}\|^2]$$

$$\|\boldsymbol{u}\|\|\boldsymbol{v}\| \cos\theta = \frac{1}{2}[u_1^2 + u_2^2 + v_1^2 + v_2^2 - (u_1 - v_2)^2 - (u_2 - v_2)]$$

$$\|\boldsymbol{u}\|\|\boldsymbol{v}\| \cos\theta = u_1v_1 + u_2v_2$$

$$\|\boldsymbol{u}\|\|\boldsymbol{v}\| \cos\theta = \boldsymbol{u}\cdot\boldsymbol{v}$$



The Length of Vectors

(Recall) The Length of a Vector (Magnitude) #1

- If $\boldsymbol{v}$ is in $\mathbb{R}^n$ with the entries $v_1 \dots v_n$, then the square root of $\boldsymbol{v} \cdot \boldsymbol{v}$ is defined because is nonnegative
- Formal definition → The length (or norm or magnitude) of $\boldsymbol{v}$ is the nonnegative scalar $||\boldsymbol{v}||$ defined by

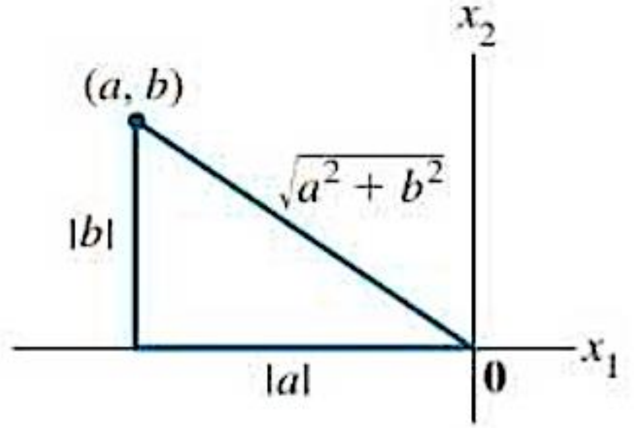
$$||\boldsymbol{v}|| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

And

$$||\boldsymbol{v}||^2 = \boldsymbol{v} \cdot \boldsymbol{v}$$

(Recall) The Length of a Vector (Magnitude) #2

- Suppose $\boldsymbol{v}$ is in $\mathbb{R}^2$, let say $\boldsymbol{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ with geometric point in the plane
- $||\boldsymbol{v}||$ coincides with the standard notion of the length of the line segment from origin to $\boldsymbol{v}$
- It will follow the Pythagorean theorem



Unit Vector

(Recall) Unit Vector

- A vector whose length is 1 called a **unit vector**.
- If we divide a nonzero vector $\boldsymbol{v}$ by its length → or multiply it by $\frac{1}{\|v\|}$ → we obtain a unit vector $\boldsymbol{u}$ because the length of $\boldsymbol{u}$ is $\frac{1}{\|v\|}\|\boldsymbol{v}\|$
- The process of creating $\boldsymbol{u}$ from $\boldsymbol{v}$ is called normalizing $\boldsymbol{v}$
- Or, we say that $\boldsymbol{u}$ ***in the same direction*** as $\boldsymbol{v}$

(Recall) Unit Vector Example

Let $\boldsymbol{v} = (1, -2, 2, 0)$. Find a unit vector $\boldsymbol{u}$ in the same direction as $\boldsymbol{v}$

Solution

Compute $||\boldsymbol{v}||$

$$||\boldsymbol{v}||^2 = \boldsymbol{v} \cdot \boldsymbol{v} = (1)^2 + (-2)^2 + (2)^2 + (0)^2 = 9$$
$$||\boldsymbol{v}|| = \sqrt{9} = 3$$

Then, multiply $\boldsymbol{v}$ by $\frac{1}{||\boldsymbol{v}||}$ to obtain

$$\boldsymbol{u} = \frac{1}{||\boldsymbol{v}||}\boldsymbol{v} = \frac{1}{3}\boldsymbol{v} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}$$

To check that $||\boldsymbol{u}|| = 1$ it suffices to show that $||\boldsymbol{u}||^2 = 1$

Try proof it by your self ☺

The Distance Between Vectors

(Recall) Distance Between Vectors #1

Formal Definition,

For $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$, the distance between $\boldsymbol{u}$ and $\boldsymbol{v}$, written as $dist(\boldsymbol{u}, \boldsymbol{v})$, **is the length of the vector $\boldsymbol{u} - \boldsymbol{v}$**

$$dist(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\|$$

Example #1

Compute the distance between the vectors $\boldsymbol{u} = (7,1)$ and $\boldsymbol{v} = (3,2)$

Solution

$$\boldsymbol{u} - \boldsymbol{v} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{4^2 + (-1)^2} = \sqrt{17}$$

(Recall) Distance Between Vectors #2

Example #2

if $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$

Solution

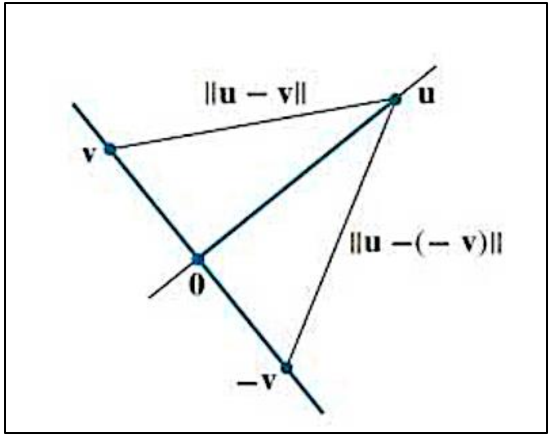
$$dist(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{(\boldsymbol{u} - \boldsymbol{v}) \cdot (\boldsymbol{v} - \boldsymbol{u})}$$

$$= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$$

Orthogonality

Orthogonal Vectors

- Consider $\mathbb{R}^2$ or $\mathbb{R}^3$ and two lines through the origin determined by $\boldsymbol{u}$ and $\boldsymbol{v}$
- **The two lines** show in the figure are **geometrically perpendicular** (*tegak lurus*) **if and only if *the distance* from u to v is *the same as* the *distance* from u to $-v$**



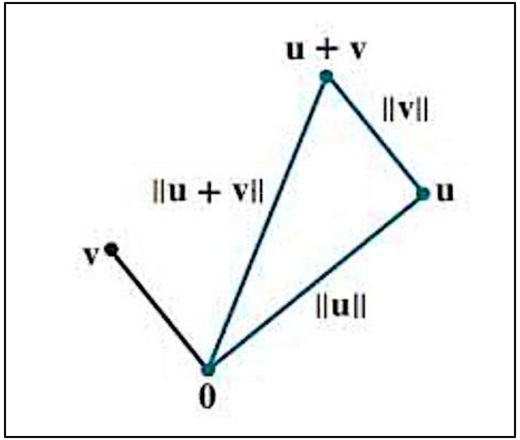
Formal Definition

Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$ are orthogonal (to each other) if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$

Orthogonal Vectors – The Pythagorean Theorem

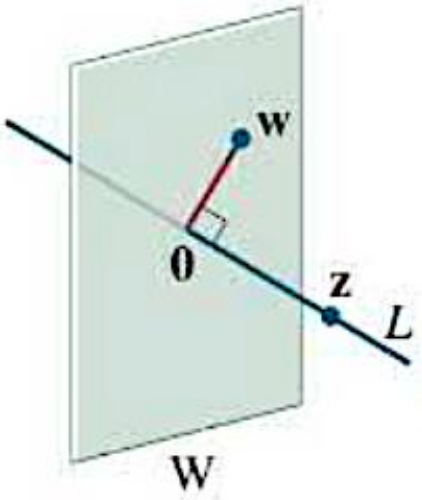
Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$ are orthogonal
If and only if,

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$$



Orthogonal Complements

- If a vector z is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then z is said to be **orthogonal** to W
- The set of all vectors z that are orthogonal to W is called **the orthogonal complements** of W → denoted as $W^{\perp}$ → read as W perpendicular or W perp



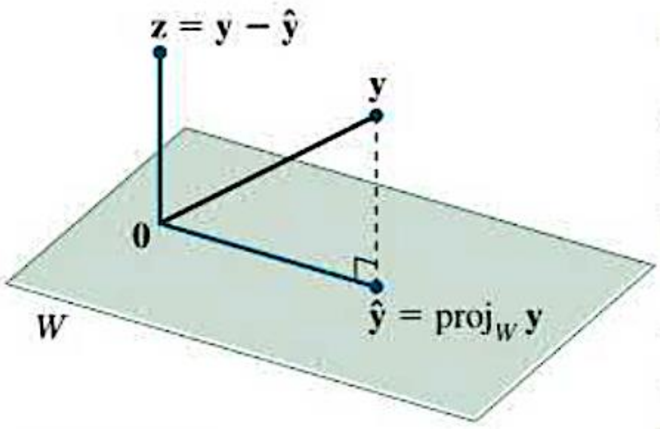
An Orthogonal Projection

An Orthogonal Projection #1

- Given a nonzero vector $\boldsymbol{u}$ in $\mathbb{R}^n$ → Consider the problem of decomposing a vector $\boldsymbol{y}$ in $\mathbb{R}^n$ in the sum of two vectors, one a multiple of $\boldsymbol{u}$ and the other orthogonal to $\boldsymbol{u}$. We write it as,

$$\boldsymbol{y} = \widehat{\boldsymbol{y}} + \boldsymbol{z}$$

- Where $\widehat{\boldsymbol{y}} = \alpha\boldsymbol{u}$ for some scalar α and $\boldsymbol{z}$ is some vector orthogonal to $\boldsymbol{u}$
- Check the figure!



An Orthogonal Projection #2

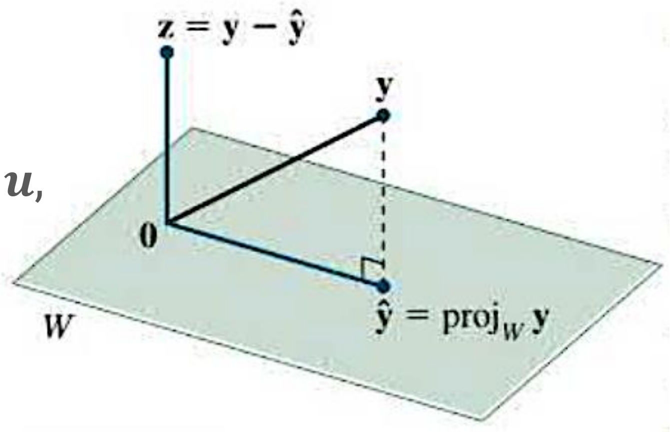
- Given any scalar α → let $\boldsymbol{z} = \boldsymbol{y} - \alpha\boldsymbol{u}$ so $\boldsymbol{y} = \hat{\boldsymbol{y}} + \boldsymbol{z}$
- Then $\boldsymbol{y} - \hat{\boldsymbol{y}}$ is orthogonal to $\boldsymbol{u}$ ***if and only if***,

$$0 = (\boldsymbol{y} - \alpha\boldsymbol{u}) \cdot \boldsymbol{u} = \boldsymbol{y} \cdot \boldsymbol{u} - (\alpha\boldsymbol{u}) \cdot \boldsymbol{u} = \boldsymbol{y} \cdot \boldsymbol{u} - \alpha(\boldsymbol{u} \cdot \boldsymbol{u})$$

- That is $\boldsymbol{y} = \hat{\boldsymbol{y}} + \boldsymbol{z}$ is satisfied with $\boldsymbol{z}$ orthogonal to $\boldsymbol{u}$, ***if and only if***

$$\alpha = \frac{\boldsymbol{y}\cdot\boldsymbol{u}}{\boldsymbol{u}\cdot\boldsymbol{u}} \text{ and } \hat{\boldsymbol{y}} = \frac{\boldsymbol{y}\cdot\boldsymbol{u}}{\boldsymbol{u}\cdot\boldsymbol{u}}\boldsymbol{u}$$

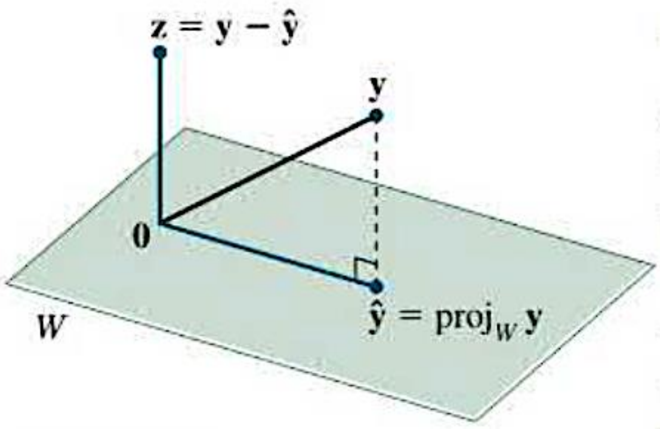
- $\hat{\boldsymbol{y}}$ → orthogonal projection of $\boldsymbol{y}$ to $\boldsymbol{u}$
- $\boldsymbol{z}$ → component of $\boldsymbol{y}$ orthogonal to $\boldsymbol{u}$



An Orthogonal Projection #3

- Hence this projection is determined by the subspace L spanned by $\boldsymbol{u}$
- $\hat{\boldsymbol{y}}$ → Also denoted as $proj_L\boldsymbol{y}$ → called the orthogonal projection of $\boldsymbol{y}$ onto L
- So,

$$\hat{\boldsymbol{y}} = proj_L\boldsymbol{y} = \frac{\boldsymbol{y} \cdot \boldsymbol{u}}{\boldsymbol{u} \cdot \boldsymbol{u}}\boldsymbol{u}$$



An Orthogonal Projection – Example

- Let $\boldsymbol{y} = (7,6)$ and $\boldsymbol{u} = (4,2)$. Find the orthogonal projection $\boldsymbol{y}$ to $\boldsymbol{u}$
- Solution,

Compute,

$$\boldsymbol{y} \cdot \boldsymbol{u} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 40$$

$$\boldsymbol{u} \cdot \boldsymbol{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 20$$

$$\hat{\boldsymbol{y}} = \frac{\boldsymbol{y} \cdot \boldsymbol{u}}{\boldsymbol{u} \cdot \boldsymbol{u}} \boldsymbol{u} = \frac{40}{20} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$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References

- https://www2.math.upenn.edu/~wziller/math114f13/ch12-4+5-1.pdf
- https://math.etsu.edu/multicalc/prealpha/Chap1/Chap1-3/printversion.pdf