An Augmented Lagrangian Method for Identifying Discontinuous Parameters in Elliptic Systems

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Jun Zou
Chinese University of Hong Kong

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AN AUGMENTED LAGRANGIAN METHOD FOR IDENTIFYING DISCONTINUOUS PARAMETERS IN ELLIPTIC SYSTEMS*

ZHIMING CHEN† AND JUN ZOU‡

Abstract. The identification of discontinuous parameters in elliptic systems is formulated as a constrained minimization problem combining the output least squares and the equation error method. The minimization problem is then proved to be equivalent to the saddle-point problem of an augmented Lagrangian. The finite element method is used to discretize the saddle-point problem, and the convergence of the discretization is also proved. Finally, an Uzawa algorithm is suggested for solving the discrete saddle-point problem and is shown to be globally convergent.

Key words. parameter identification, elliptic system, augmented Lagrangian, finite element method

AMS subject classifications. 65N30, 35R30

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1. Introduction. The main purpose of this paper is to propose a numerical approach and conduct convergence analyses on each approximation process in the identification of the unknown coefficient q in the elliptic problem

$$-\nabla \cdot (q\nabla u) = f \quad \text{in} \quad \Omega; \quad u = 0 \quad \text{on} \quad \Gamma.$$

The identifying process is carried out so that the solution u matches its observation data z optimally in a certain sense. Here Ω can be any bounded domain in R^d, $d = 1, 2$, or 3, with piecewise smooth boundary Γ and $f \in H^{-1}(\Omega)$ as given. The problem may describe the flow of a fluid (e.g., groundwater) through some medium with permeability $q(x)$, or the heat transfer in a material with conductivity $q(x)$; we refer to the books by Bank and Kunisch [1] and Engl, Hanke, and Neubauer [7]. Practically, it is often easier to measure the solution u at various points in the medium than to measure the parameter $q(x)$ itself [11]. Then the measured data of u (often the interpolated function of the data) are utilized to estimate the parameter $q(x)$ through the above boundary value problem. We study a hybrid method proposed in [13, 14] that combines the output least squares and the equation error formulation within the mathematical framework given by the augmented Lagrangian technique. The augmented Lagrangian methods have been widely used earlier in nonlinear constrained optimization problems and nonlinear boundary value problems to relax some complicated constraints or difficult couplings among some nonlinear and nonsmooth terms or to enhance convexities of the objective functions (cf. [10, 2]). Ito and Kunisch [13, 14] applied the augmented Lagrangian method for parameter identifying problems, incorporated with a regularization term of the H^2 seminorm of the parameters to be estimated. Their methods appear to be very efficient and successful in recovering the smooth parameters. The major novelty of this paper is to generalize the

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†Institute of Mathematics, Academia Sinica, Beijing 100080, People's Republic of China (zmchen@math03.math.ac.cn). The research of this author was partially supported by the China National Natural Science Foundation.
‡Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong (zou@math.cuhk.edu.hk). The research of this author was partially supported by Hong Kong RGC grant CUHK 338/96E.

aforementioned method so that we can identify even nonsmooth parameters. To this aim, we propose to search for the coefficients in the space of functions with bounded variation (BV), namely, in the space

$$BV(\Omega) = \Big\{ q \in L^1(\Omega); \quad \|q\|_{BV(\Omega)} < \infty \Big\}.$$

Here $\|q\|_{BV(\Omega)} = \|q\|_{L^1(\Omega)} + \int_\Omega |Dq|$ with the notation $\int_\Omega |Dq|$ defined by

$$\int_\Omega |Dq| = \sup \Big\{ \int_\Omega q \text{ div } g \; dx; \quad g \in \big(C_0^1(\Omega)\big)^d \quad \text{and} \quad |g(x)| \le 1 \quad \text{in} \quad \Omega \Big\},$$

which allows us to identify the discontinuous parameters in elliptic systems.

Because of the involvement of the $BV(\Omega)$ norm in the cost function and because there is not as much regularity as in [13, 14], we cannot apply the techniques of Ito and Kunisch to show the existence of the saddle-points of the augmented Lagrangian and the convergence of the discrete saddle-points to the continuous ones. Instead, our crucial tool for the convergence analyses will be an appropriate application of the Hahn–Banach convex separating theorem. This enables us to have a clear and simple convergence theory without making any a priori assumptions on cost functional or constraint functionals. We note that quite a different approach was used in [12] for the identification of discontinuous parameters.

We now formulate the aforementioned parameter identifying problem as the following constrained minimization problem:

$$\text{minimize} \quad J(q,v) = \frac{1}{2}\int_\Omega q|\nabla v - \nabla z|^2 dx + \beta \int_\Omega |Dq| \tag{1.1}$$

$$\text{subject to} \quad (q,v) \in K \times V \quad \text{and} \tag{1.2}$$

$$e(q,v) = (-\Delta)^{-1}(\nabla \cdot (q\nabla v) + f) = 0, \tag{1.3}$$

where $V = H_0^1(\Omega)$ and K is a subset of the function space $BV(\Omega)$ of BVs defined by

$$K = \{ q \in BV(\Omega); \quad \alpha_1 \le q(x) \le \alpha_2 \quad \text{almost everywhere (a.e.)} \quad \text{in } \Omega \}.$$

Here α_1 and α_2 are two positive constants and $\beta > 0$ is a regularization parameter. $-\Delta$ is the Laplace operator from $H_0^1(\Omega)$ to its dual space $H^{-1}(\Omega)$, so $e(q,v)$ is understood as an operator from $K \times V$ into V defined by

$$(\nabla e(q,v), \nabla\phi) = (q\nabla v, \nabla \phi) - (f, \phi) \quad \forall\, (q,v) \in K \times V, \qquad \phi \in V, \tag{1.4}$$

where $(\cdot,\cdot)$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, which is the extension of the inner product in $L^2(\Omega)$. It is useful to remark that $e(q,v)$ is convex with respect to each variable.

The problem (1.1)–(1.3) will be solved by the augmented Lagrangian method. Thus we introduce the augmented Lagrangian functional $\mathcal{L}_r : K \times V \times V \to R$ by

$$\mathcal{L}_r(q,v;\mu) = J(q,v) + (\nabla\mu, \nabla e(q,v)) + \frac{r}{2}\|\nabla e(q,v)\|^2_{L^2(\Omega)}, \tag{1.5}$$

where $r \ge 0$ is some given constant. The first main result of the paper states that the minimization problem (1.1)–(1.3) is equivalent to the saddle-point problem associated with the Lagrangian functional $\mathcal{L}_r$ in (1.5). To solve the saddle-point problem, we propose a finite element discretization of the problem and show that the saddle-points of the discrete problem converge to those of the continuous problem. Finally, we propose an Uzawa algorithm to solve the discrete saddle-point problem and prove the global convergence of the algorithm. We note that recently Chan and Tai have performed many numerical experiments on a local convergent Uzawa algorithm and its combination with domain decomposition and multigrid methods [4, 16].

Throughout the paper, the constant C is a generic constant that might be different at each occurrence but is independent of the mesh parameter h and of the various functions involved.

2. The continuous saddle-point problem. We start this section with the existence of the solutions to the minimization problem (1.1)–(1.3) and then prove that the minimization problem is equivalent to the saddle-point problem of the augmented Lagrangian $\mathcal{L}_r$ defined in (1.5).

LEMMA 2.1. *There exists at least one solution to the minimization problem* (1.1)–(1.3).

Proof. Let

$$A = \Big\{(q,v) \in K \times V; \quad e(q,v) = 0\Big\}$$

be the admissible set of the minimization problem (1.1)–(1.3). It is clear that $A \neq \emptyset$ and $J(q,v) \geq 0$ on A. Thus there exists a minimizing sequence $(q_n, v_n) \in A$ such that

$$\lim_{n\to\infty} J(q_n, v_n) = \inf_{(q,v)\in A} J(q,v). \tag{2.1}$$

Hence $J(q_n, v_n) \leq C$ for each $n > 0$, which implies by definition of J and K that

$$\|v_n\|_{H^1(\Omega)} + \|q_n\|_{BV(\Omega)} \leq C.$$

Therefore, by possibly extracting a subsequence, there exists a pair $(q^*, v^*) \in BV(\Omega) \times V$ satisfying

$$v_n \rightharpoonup v^* \quad \text{in} \quad H_0^1(\Omega), \qquad q_n \to q^* \quad \text{in} \quad L^1(\Omega). \tag{2.2}$$

Since $q_n \in K$, we also have $q^* \in K$. To show that $e(q^*, v^*) = 0$, we first note that $e(q_n, v_n) = 0$ as $(q_n, v_n) \in A$; therefore,

$$(q_n \nabla v_n, \nabla \phi) = (f, \phi) \quad \forall\, \phi \in V. \tag{2.3}$$

However,

$$\begin{aligned}
&\Big|(q_n \nabla v_n, \nabla\phi) - (q^*\nabla v^*, \nabla\phi)\Big| \\
&\leq \Big|((q_n - q^*)\nabla v_n, \nabla\phi)\Big| + \Big|(q^*\nabla(v_n - v^*), \nabla\phi)\Big| \\
&\leq \Big\{\int_\Omega |q_n - q^*|\,|\nabla\phi|^2 dx\Big\}^{1/2}\Big\{\int_\Omega |q_n - q^*|\,|\nabla v_n|^2 dx\Big\}^{1/2} + \Big|(q^*\nabla(v_n - v^*), \nabla\phi)\Big| \\
&\leq C\Big\{\int_\Omega |q_n - q^*|\,|\nabla\phi|^2 dx\Big\}^{1/2} + \Big|(q^*\nabla(v_n - v^*), \nabla\phi)\Big|,
\end{aligned} \tag{2.4}$$

where we have used the fact that $\alpha_1 \leq q_n, q \leq \alpha_2$ and $\|v_n\|_{H^1(\Omega)} \leq C$. Now letting $n \to \infty$ in (2.4), we obtain

$$(q_n \nabla v_n, \nabla\phi) \to (q^*\nabla v^*, \nabla\phi) \quad \forall\, \phi \in V$$

by means of the Lebesgue dominant convergence theorem and the weak convergence in (2.2). Thus we see that $e(q^*, v^*) = 0$ by (2.3) and the definition of $e(\cdot, \cdot)$. Now using (2.2), we have (cf. [9])

$$\int_\Omega |Dq^*| \le \liminf_{n\to\infty} \int_\Omega |Dq_n|.$$

On the other hand, by $e(q_n, v_n) = 0$, we have

$$(q_n \nabla(v_n - z),\ \nabla\phi) = (f, \phi) - (q_n \nabla z,\ \nabla\phi) \quad \forall\, \phi \in V.$$

Taking $\phi = v_n - z$ gives

$$\int_\Omega q_n |\nabla(v_n - z)|^2 dx = (f, v_n - z) - (q_n \nabla z,\ \nabla(v_n - z)).$$

Similarly, using $e(q^*, v^*) = 0$, we get

$$\int_\Omega q^* |\nabla(v^* - z)|^2 dx = (f, v^* - z) - (q^* \nabla z,\ \nabla(v^* - z)).$$

Then using the last two relations, (2.2), and the Lebesgue dominant convergence theorem, we can immediately derive

$$\int_\Omega q^* |\nabla(v^* - z)|^2 dx = \lim_{n\to\infty} \int_\Omega q_n |\nabla(v_n - z)|^2 dx, \tag{2.5}$$

which with (2.1) yields

$$\begin{aligned} J(q^*, v^*) &\le \liminf_{n\to\infty} \frac{1}{2} \int_\Omega q_n |\nabla v_n - \nabla z|^2 dx + \liminf_{n\to\infty} \int_\Omega |Dq_n| \\ &\le \liminf_{n\to\infty} J(q_n, v_n) = \inf_{(q,v)\in A} J(q, v). \end{aligned}$$

This completes the proof of Lemma 2.1 as $(q^*, v^*) \in A$. ∎

The following theorem is the main result of this section.

THEOREM 2.2. *$(q^*, v^*) \in K \times V$ is a solution of the minimization problem* (1.1)–(1.3) *if and only if there exists a $\lambda^* \in V$ such that $(q^*, v^*, \lambda^*) \in K \times V \times V$ is a saddle-point of the augmented Lagrangian $\mathcal{L}_r : K \times V \times V \to R$, namely,*

$$\mathcal{L}_r(q^*, v^*; \mu) \le \mathcal{L}_r(q^*, v^*; \lambda^*) \le \mathcal{L}_r(q, v; \lambda^*) \quad \forall\, (q, v, \mu) \in K \times V \times V. \tag{2.6}$$

The key step in proving Theorem 2.2 is an appropriate application of the Hahn–Banach convex set separating theorem. To do so, we introduce two subsets in $R \times V$:

$$S = \Big\{ (J(q, v) - J(q^*, v^*) + s,\ e(q, v)) \in R \times V; \quad (q, v) \in K \times V,\ s \ge 0 \Big\}, \tag{2.7}$$

$$T = \Big\{ (-t, 0) \in R \times V; \quad t > 0 \Big\}, \tag{2.8}$$

where $(q^*, v^*) \in K \times V$ is some minimal point of the problem (1.1)–(1.3). The following three lemmas provide the properties of two subsets required by the Hahn–Banach theorem.

LEMMA 2.3. *S and T are two convex subsets in $R \times V$.*

Proof. It is obvious that T is a convex subset in $R \times V$. To see that S is also a convex subset, we let

$$P_i = (J(q_i, v_i) - J(q^*, v^*) + s_i,\ e(q_i, v_i)), \qquad i = 1, 2,$$

be two points in S, where $(q_i, v_i) \in K \times V$ and $s_i \geq 0$. We let $0 < \alpha < 1$, and we have to show that

$$P_\alpha = \alpha P_1 + (1-\alpha) P_2 \equiv (p_\alpha, w_\alpha)$$

with

$$\begin{aligned} p_\alpha &= \alpha J(q_1, v_1) + (1-\alpha) J(q_2, v_2) - J(q^*, v^*) + \alpha s_1 + (1-\alpha) s_2, \\ w_\alpha &= \alpha e(q_1, v_1) + (1-\alpha) e(q_2, v_2) \end{aligned}$$

is also a point in S. Let us now define $q_\alpha \in K$ as

$$q_\alpha = \alpha q_1 + (1-\alpha) q_2$$

and $v_\alpha \in V$ as the solution of the variational problem

$$(q_\alpha \nabla v_\alpha, \nabla \phi) = (\alpha q_1 \nabla v_1 + (1-\alpha) q_2 \nabla v_2,\ \nabla \phi) \quad \forall\, \phi \in V. \tag{2.9}$$

Clearly, $(q_\alpha, v_\alpha) \in K \times V$ is well defined. By (2.9) and the definition of $e(\cdot,\cdot)$, we have

$$\begin{aligned} (\nabla e(q_\alpha, v_\alpha),\ \nabla\phi) &= (q_\alpha \nabla v_\alpha, \nabla \phi) - (f, \phi) \\ &= (\alpha q_1 \nabla v_1 + (1-\alpha) q_2 \nabla v_2,\ \nabla\phi) - (f,\ \phi) \\ &= \alpha\{(q_1 \nabla v_1,\ \nabla\phi) - (f,\ \phi)\} + (1-\alpha)\{(q_2 \nabla v_2,\ \nabla \phi) - (f,\ \phi)\} \\ &= (\alpha \nabla e(q_1, v_1) + (1-\alpha) \nabla e(q_2, v_2),\ \nabla\phi) \quad \forall\, \phi \in V, \end{aligned}$$

which implies that

$$e(q_\alpha, v_\alpha) = \alpha e(q_1, v_1) + (1-\alpha) e(q_2, v_2). \tag{2.10}$$

On the other hand, by the convexity of the BV-seminorm we have

$$\int_\Omega |Dq_\alpha| \leq \alpha \int_\Omega |Dq_1| + (1-\alpha) \int_\Omega |Dq_2|, \tag{2.11}$$

and we know from (2.9) that

$$(q_\alpha \nabla(v_\alpha - z),\ \nabla z) = (\alpha q_1 \nabla(v_1 - z) + (1-\alpha) q_2 \nabla(v_2 - z),\ \nabla\phi).$$

Then letting $\phi = v_\alpha - z$ and using Schwarz's inequality give

$$\begin{aligned} &\int_\Omega q_\alpha |\nabla(v_\alpha - z)|^2 dx \\ &\leq \int_\Omega q_\alpha^{-1} |\alpha q_1 \nabla(v_1 - z) + (1-\alpha) q_2 \nabla(v_2 - z)|^2 dx \\ &\leq \int_\Omega q_\alpha \left| \frac{\alpha q_1}{q_\alpha} \nabla(v_1 - z) + \frac{(1-\alpha) q_2}{q_\alpha} \nabla(v_2 - z) \right|^2 dx \\ &\leq \int_\Omega q_\alpha \left\{ \frac{\alpha q_1}{q_\alpha} |\nabla(v_1 - z)|^2 + \frac{(1-\alpha) q_2}{q_\alpha} |\nabla(v_2 - z)|^2 \right\} dx \\ &= \alpha \int_\Omega q_1 |\nabla(v_1 - z)|^2 dx + (1-\alpha) \int_\Omega q_2 |\nabla(q_2 - z)|^2 dx, \end{aligned}$$

where we have used the fact that $(\alpha q_1 + (1-\alpha)q_2)/q_\alpha = 1$ and the convexity of the function $|\cdot|^2$. Now combining this bound with (2.11) we obtain

$$J(q_\alpha, v_\alpha) \le \alpha J(q_1, v_1) + (1-\alpha)J(q_2, v_2), \tag{2.12}$$

and so (2.10) and (2.12) imply that

$$P_\alpha = (J(q_\alpha, v_\alpha) - J(q^*, v^*) + s_\alpha,\ e(q_\alpha, v_\alpha)\ ) \in S$$

since $(q_\alpha, v_\alpha) \in K \times V$ and

$$s_\alpha = \alpha s_1 + (1-\alpha)s_2 + \alpha J(q_1, v_1) + (1-\alpha)J(q_2, v_2) - J(q_\alpha, v_\alpha) \ge 0.$$

This completes the proof of Lemma 2.3. ◻

LEMMA 2.4. *We have $S \cap T = \emptyset$.*

Proof. Assume that $(a, w) \in S \cap T$; then there exists $(q, v) \in K \times V$ and $s \ge 0$ such that

$$a = J(q, v) - J(q^*, v^*) + s, \qquad w = e(q, v).$$

But $(a, w) \in T$ implies that $a < 0$ and $w = e(q, v) = 0$. Thus

$$J(q, v) + s < J(q^*, v^*),$$

which contradicts the assumption that (q^*, v^*) is a minimal point of the problem (1.1)–(1.3). ◻

LEMMA 2.5. *The subset S has at least one interior point.*

Proof. It is easy to see that for any $s_0 > 0$, $(s_0, 0) = (J(q^*, v^*) - J(q^*, v^*) + s_0,\ e(q^*, v^*))$ is a point in S. We will show that $(s_0, 0) \in R \times V$ is also an interior point of S. For any $\varepsilon \in (0, 1)$, let (s, w) belong to the ε-neighborhood of $(s_0, 0)$ in $R \times V$, that is,

$$|s - s_0| + \|w\|_{H^1(\Omega)} \le \varepsilon. \tag{2.13}$$

Let $q = q^*$ and $v \in V$ be the solution to the equation

$$(q\nabla v,\ \nabla\phi) = (f, \phi) + (\nabla w,\ \nabla\phi) \quad \forall\, \phi \in V. \tag{2.14}$$

Then we have $e(q, v) = w$. Let

$$\begin{aligned} s' &= s + J(q^*, v^*) - J(q, v) \\ &= s + \frac{1}{2}\int_\Omega q^*|\nabla(v^* - z)|^2 dx - \frac{1}{2}\int_\Omega q^*|\nabla(v - z)|^2 dx \\ &= s - \frac{1}{2}\int_\Omega q^*\nabla(v - v^*)\cdot\nabla(v + v^* - 2z)dx. \end{aligned} \tag{2.15}$$

From (2.14) and $e(q^*, v^*) = 0$, we derive that $\|\nabla v^*\|_{L^2(\Omega)} \le \|f\|_{H^{-1}(\Omega)}/\alpha_1$ and

$$(q^*\nabla(v - v^*),\ \nabla\phi) = (\nabla w,\ \nabla\phi) \quad \forall\, \phi \in V,$$

which yields $\|\nabla(v - v^*)\|_{L^2(\Omega)} \le \varepsilon/\alpha_1$ by (2.13). Also, (2.14) implies that $\|\nabla v\|_{L^2(\Omega)} \le (\|f\|_{H^{-1}(\Omega)} + \varepsilon)/\alpha_1$, thus we deduce from (2.15) that

$$\begin{aligned} s' &\ge s_0 - \varepsilon - \frac{1}{2}\alpha_2\|\nabla(v - v^*)\|_{L^2(\Omega)}\,\|\nabla(v + v^* - 2z)\|_{L^2(\Omega)} \\ &\ge s_0 - \varepsilon - \frac{\alpha_2}{2\alpha_1^2}\varepsilon\ \{\varepsilon + 2\|f\|_{H^{-1}(\Omega)} + 2\alpha_1\|\nabla z\|_{L^2(\Omega)}\}. \end{aligned}$$

Now if ε is sufficiently small, then $s' \geq 0$. Therefore

$$(s, w) = (J(q, v) - J(q^*, v^*) + s',\ e(q, v)\ ) \in K \times V$$

for any (s, w) in the ε-neighborhood of $(s_0, 0)$. This completes the proof. ∎

Now we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. First, assume that $(q^*, v^*, \lambda^*) \in K \times V \times V$ is a saddle-point of $\mathcal{L}_r$. Then the first inequality in (2.6) immediately gives $e(q^*, v^*) = 0$, and the fact that (q^*, v^*) is a minimal point of the problem (1.1)–(1.3) follows readily from the second inequality in (2.6).

Next we prove the remaining part of the theorem. Let (q^*, v^*) be a minimal point of the problem (1.1)–(1.3), so we have

$$J(q^*, v^*) \leq J(q, v) \quad \forall\, (q, v) \in K \times V \quad \text{satisfying} \quad e(q, v) = 0. \tag{2.16}$$

By Lemmas 2.3–2.5, we can apply the Hahn–Banach theorem (see, e.g., [3, 4, 5, 6]) to separate the two convex subsets S and T defined in (2.7) and (2.8). Thus there exists a pair $(\alpha_0, \lambda_0) \in R \times V$, but $(\alpha_0, \lambda_0) \neq (0, 0) \in R \times V$ such that

$$\alpha_0(J(q, v) - J(q^*, v^*) + s) + (\nabla\lambda_0,\ \nabla e(q, v)) \geq \alpha_0(-t)$$

for any $(q, v) \in K \times V$, $s \geq 0$, and $t > 0$. Taking $(q, v) = (q^*, v^*)$, $s = t = 1$, we get $\alpha_0 \geq 0$, while taking $s = 0$ and letting $t \to 0^+$, we obtain

$$\alpha_0(J(q, v) - J(q^*, v^*)) + (\nabla\lambda_0,\ \nabla e(q, v)) \geq 0 \quad \forall\, (q, v) \in K \times V. \tag{2.17}$$

We now claim that $\alpha_0 > 0$. Otherwise, if $\alpha_0 = 0$ we have from (2.17) that

$$(\nabla\lambda_0,\ \nabla e(q, v)) = (q\nabla v,\ \nabla\lambda_0) - (f, \lambda_0) \geq 0 \quad \forall\, (q, v) \in K \times V, \tag{2.18}$$

which implies that $\lambda_0 = 0$. In fact, taking $q = q^* \in K$ and $v \in V$ to be the solution of the equation

$$(q^*\nabla v,\ \nabla\phi) = (f - \lambda_0,\ \phi) \quad \forall\, \phi \in V, \tag{2.19}$$

we know from (2.18), (2.19) that $-\|\lambda_0\|^2_{L^2(\Omega)} \geq 0$. Thus we have $(\alpha_0, \lambda_0) = (0, 0)$, which is a contradiction. Therefore $\alpha_0 > 0$. Then taking $\lambda^* = \lambda_0/\alpha_0$ and dividing both sides of (2.17) by α_0, we get

$$J(q^*, v^*) \leq J(q, v) + (\nabla\lambda^*,\ \nabla e(q, v)) \quad \forall\, (q, v) \in K \times V,$$

which, combined with (2.16) indicates that $(q^*, v^*, \lambda^*) \in K \times V \times V$ is a saddle-point of the augmented Lagrangian $\mathcal{L}_r$. So we have proved Theorem 2.2. ∎

3. The discrete saddle-point problem. Theorem 2.2 tells us that the minimization problem (1.1)–(1.3) is equivalent to finding the saddle-points of the functional $\mathcal{L}_r$ defined in (1.5). In this section, we will consider how to discretize the augmented Lagrangian $\mathcal{L}_r$ and derive a discrete saddle-point problem.

Let Ω be a polyhedral domain in R^d, $d = 1, 2$, or 3, and $\{\mathcal{T}^h\}_{h>0}$ be a family of regular triangulations (cf. Ciarlet [5]) of the domain Ω, with simplicial elements. Denote by V_h the standard piecewise linear finite element space over the triangulation $\mathcal{T}^h$ and

$$\overset{\circ}{V}_h = V_h \cap H^1_0(\Omega), \qquad K_h = K \cap V_h.$$

We now introduce a discrete version of the operator $e(q,v) : K \times V \to V$ defined in (1.4): for any $(q_h, v_h) \in K_h \times \overset{\circ}{V}_h$, $e_h(q_h, v_h) \in \overset{\circ}{V}_h$ is the solution of the system

$$(3.1) \qquad (\nabla e_h(q_h, v_h),\ \nabla\phi) = (q_h \nabla v_h,\ \nabla\phi) - (f, \phi) \quad \forall\, \phi \in \overset{\circ}{V}_h\ .$$

It is clear that the operator $e_h : K_h \times \overset{\circ}{V}_h \to \overset{\circ}{V}_h$ is well defined. Moreover, the following estimate holds:

$$(3.2) \quad \|\nabla e_h(q_h, v_h)\|_{L^2(\Omega)} \le \{\alpha_2 \|\nabla v_h\|_{L^2(\Omega)} + C\|f\|_{H^{-1}(\Omega)}\} \quad \forall\, (q_h, v_h) \in K_h \times V_h,$$

where the constant C comes from the Poincaré inequality.

Now for any given $r \ge 0$, we define the discrete augmented Lagrangian L_r: $K_h \times \overset{\circ}{V}_h \times \overset{\circ}{V}_h \to R$ as follows:

$$(3.3) \quad L_r(q_h, v_h; \mu_h) = J_h(q_h, v_h) + (\nabla\mu_h,\ \nabla e_h(q_h, v_h)) + \frac{r}{2}\|\nabla e_h(q_h, v_h)\|^2_{L^2(\Omega)}$$

with

$$J_h(q_h, v_h) = \frac{1}{2}\int_\Omega q_h |\nabla(v_h - z)|^2 dx + \beta \int_\Omega \sqrt{|\nabla q_h|^2 + \delta(h)}\ dx,$$

where $\delta(h)$ above is any given positive function satisfying $\lim_{h\to 0} \delta(h) = \delta(0) = 0$.

With the above preparations, we can state the following theorem.

THEOREM 3.1. *For any $r \ge 0$, there exists at least one saddle-point for the discrete augmented Lagrangian $L_r : K_h \times \overset{\circ}{V}_h \times \overset{\circ}{V}_h \to R$. Moreover, each saddle-point $(q_h^*, v_h^*, \lambda_h^*)$ of L_0 is a saddle-point of L_r for any $r > 0$.*

Proof. It is obvious that each saddle-point of L_0 is a saddle-point of L_r for any $r > 0$. Then it suffices to show that $L_0 : K_h \times \overset{\circ}{V}_h \times \overset{\circ}{V}_h \to R$ has a saddle-point, which we can argue in exactly the same way as in the proof for the continuous saddle-point problem of the last section by showing first the existence of the solutions to the discrete minimization problem

$$(3.4) \qquad \min_{(q_h, v_h) \in A_h} J_h(q_h, v_h)$$

with

$$A_h = \{(q_h, v_h) \in K_h \times \overset{\circ}{V}_h; \quad e_h(q_h, v_h) = 0\},$$

and then the existence of the Lagrangian multiplier $\lambda_h^* \in \overset{\circ}{V}_h$ satisfying

$$J_h(q_h^*, v_h^*) \le J_h(q_h, v_h) + (\nabla\lambda_h^*,\ \nabla e_h(q_h, v_h)) \quad \forall\, (q_h, v_h) \in K_h \times \overset{\circ}{V}_h$$

for some minimal point (q_h^*, v_h^*) of the problem (3.4). We omit the details. ☐

The following theorem is the main result of this section.

THEOREM 3.2. *Each subsequence of the saddle-points $\{(q_h^*, v_h^*; \lambda_h^*)\}_{h>0}$ of the discrete augmented Lagrangian $L_r : K_h \times \overset{\circ}{V}_h \times \overset{\circ}{V}_h \to R$ defined in (3.3) has a subsequence that converges to some saddle-point $(q^*, v^*; \lambda^*)$ of the augmented Lagrangian $\mathcal{L}_r : K \times V \times V \to R$ defined in (1.5) strongly in $L^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$.*

The proof of Theorem 3.2 depends on the following three lemmas.

LEMMA 3.3. *Let $g \in BV(\Omega)$. Then for any $\varepsilon > 0$, there exists a function $g_\varepsilon \in C^\infty(\bar{\Omega})$ such that*

$$\int_\Omega |g - g_\varepsilon| dx < \varepsilon, \qquad \Big| \int_\Omega |\nabla g_\varepsilon| dx - \int_\Omega |Dg| \Big| < \varepsilon.$$

Proof. By the approximation property of functions with BVs (cf. p. 172 of [8]), there exists $\tilde{g}_\varepsilon \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$ satisfying

$$\int_\Omega |g - \tilde{g}_\varepsilon| dx < \varepsilon/2, \qquad \Big| \int_\Omega |\nabla \tilde{g}_\varepsilon| dx - \int_\Omega |Dg| \Big| < \varepsilon/2.$$

Then the lemma follows from the density of $C^\infty(\bar{\Omega})$ in $W^{1,1}(\Omega)$ as $\partial\Omega$ is Lipschitz continuous (cf. page 127 of [8]). ◻

In what follows we will make use of the standard nodal value interpolant $I_h : C(\bar{\Omega}) \to V_h$ and the projection operator $R_h : V \to \overset{\circ}{V}_h$ defined by

$$(\nabla R_h v,\ \nabla \phi) = (\nabla v,\ \nabla \phi) \quad \forall\, v \in V, \qquad \phi \in \overset{\circ}{V}_h\ . \tag{3.5}$$

It is well known (cf. [5]) that for any $p > d = \dim(\Omega)$,

$$\lim_{h\to 0} \|v - I_h v\|_{W^{1,p}(\Omega)} = 0 \quad \forall\, v \in W^{1,p}(\Omega), \tag{3.6}$$

$$\lim_{h\to 0} \|v - R_h v\|_{H_0^1(\Omega)} = 0 \quad \forall\, v \in V. \tag{3.7}$$

LEMMA 3.4. *Assume that $(q, v) \in K \times V$ and $(q_h, v_h) \in K_h \times \overset{\circ}{V}_h$. Then $\lim_{h\to 0} q_h = q$ in $L^1(\Omega)$ and $\lim_{h\to 0} v_h = v$ in $H_0^1(\Omega)$ imply $\lim_{h\to 0} e_h(q_h, v_h) = e(q, v)$ in $H_0^1(\Omega)$.*

Proof. By the definitions of $e(\cdot,\cdot)$ and $e_h(\cdot,\cdot)$ we have

$$(\nabla\{e_h(q_h, v_h) - e(q, v)\},\ \nabla\phi) = ((q_h - q)\nabla v,\ \nabla\phi) + (q_h \nabla(v_h - v), \nabla\phi) \quad \forall\, \phi \in \overset{\circ}{V}_h\ .$$

By taking $\phi = e_h(q_h, v_h) - R_h e(q, v) \in \overset{\circ}{V}_h$ above and using (3.5) we obtain

$$\begin{aligned}\|\nabla\{e_h(q_h, v_h) - R_h e(q, v)\}\|^2_{L^2(\Omega)} &\le 2\int_\Omega |q_h - q|^2\, |\nabla v|^2 dx + 2\int_\Omega q_h^2 |\nabla(v_h - v)|^2 dx \\ &\le 2\int_\Omega |q_h - q|^2\, |\nabla v|^2 dx + 2(\alpha_2)^2 \int_\Omega |\nabla(v_h - v)|^2 dx.\end{aligned}$$

Then the Lebesgue dominant convergence theorem and the fact that $\lim_{h\to 0} v_h = v$ in $H_0^1(\Omega)$ show that

$$\lim_{h\to 0} \|\nabla\{e_h(q_h, v_h) - R_h e(q, v)\}\|_{L^2(\Omega)} = 0.$$

Lemma 3.4 now follows from (3.7). ◻

LEMMA 3.5. *Assume that $(q, v) \in K \times V$ and $(q_h, v_h) \in K_h \times \overset{\circ}{V}_h$. Then $\lim_{h\to 0} q_h = q$ in $L^1(\Omega)$ and $\lim_{h\to 0} v_h = v$ weakly in $H_0^1(\Omega)$ imply that $\lim_{h\to 0} e_h(q_h, v_h) = e(q, v)$ weakly in $H_0^1(\Omega)$.*

Proof. For any $\phi \in V$, let $\phi_h = R_h\phi$. By the definition of R_h and $e_h(\cdot,\cdot)$ we have

$$\begin{aligned}
&(\nabla e_h(q_h, v_h),\ \nabla\phi)\\
&= (\nabla e_h(q_h, v_h),\ \nabla\phi_h)\\
&= (q\nabla v_h,\ \nabla\phi_h) + ((q_h - q)\nabla v_h,\ \nabla\phi_h) - (f, \phi_h)\\
&= (q\nabla v_h,\ \nabla\phi_h) + ((q_h - q)\nabla v_h,\ \nabla\phi) - (f, \phi_h)\\
&\quad +((q_h - q)\nabla v_h,\ \nabla(\phi_h - \phi)).
\end{aligned} \tag{3.8}$$

Then using the assumed convergence on v_h, we know that $\{\|\nabla v_h\|_{L^2(\Omega)}\}$ is bounded; combining this with the Lebesgue dominant convergence theorem we derive

$$\Big|((q_h - q)\nabla v_h,\ \nabla\phi)\Big| \le \|\nabla v_h\|_{L^2(\Omega)}\Big\{\int_\Omega |q_h - q|^2|\nabla\phi|^2dx\Big\}^{1/2} \to 0 \quad \text{as} \quad h \to 0.$$

Similarly, we can show that all other terms in (3.8) converge; we then take the limit in (3.8) and use the definition of $e(\cdot,\cdot)$ to yield

$$\lim_{h\to 0}(\nabla e_h(q_h, v_h),\ \nabla\phi) = (q\nabla v,\ \nabla\phi) - (f, \phi) = (\nabla e(q, v),\ \nabla\phi) \quad \forall\, \phi \in V.$$

Thus we have proved Lemma 3.5. ∎

Now we are ready to prove Theorem 3.2.

Proof of Theorem 3.2. Let $(q_h^*, v_h^*, \lambda_h^*) \in K_h\times \overset{\circ}{V}_h \times \overset{\circ}{V}_h$ be the saddle-point of L_r, that is,

$$L_r(q_h^*, v_h^*; \mu_h) \le L_r(q_h^*, v_h^*; \lambda_h^*) \le L_r(q_h, v_h; \lambda_h^*) \quad \forall\, (q_h, v_h, \mu_h) \in K_h\times \overset{\circ}{V}_h \times \overset{\circ}{V}_h\ .$$

The first inequality implies immediately that $e_h(q_h^*, v_h^*) = 0$, and the second inequality gives us

$$J_h(q_h^*, v_h^*) \le J_h(q_h, v_h) + (\nabla\lambda_h^*,\ \nabla e_h(q_h, v_h)) + \frac{r}{2}\|\nabla e_h(q_h, v_h)\|^2_{L^2(\Omega)} \tag{3.9}$$
$$\forall\, (q_h, v_h) \in K_h\times \overset{\circ}{V}_h\ .$$

By letting $q_h = \alpha_1$, a constant, and $v_h \in\overset{\circ}{V}_h$ be the unique solution of the equation

$$(\nabla v_h,\ \nabla\phi) = \left(\frac{1}{\alpha_1}f,\ \phi\right) \quad \forall\, \phi \in\overset{\circ}{V}_h$$

and hence $e_h(q_h, v_h) = 0$, we deduce from (3.9) that $\|q_h^*\|_{BV(\Omega)} + \|v_h^*\|_{H^1(\Omega)} \le C$. But taking $q_h = q_h^*$ in (3.9) and using (3.2) and the definition of $e_h(\cdot,\cdot)$ we get for any $v_h \in\overset{\circ}{V}_h$ that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega q_h^*|\nabla(v_h^* - z)|^2dx\\
&\le \frac{1}{2}\int_\Omega q_h^*\,|\nabla(v_h - z)|^2dx + (q_h^*\nabla v_h, \nabla\lambda_h^*) - (f, \lambda_h^*) + \frac{r}{2}\|\nabla e_h(q_h^*, v_h)\|^2_{L^2(\Omega)}\\
&\le (q_h^*\nabla v_h, \nabla\lambda_h^*) + \eta\|\nabla\lambda_h^*\|^2_{L^2(\Omega)} + \frac{C}{\eta}\|f\|^2_{H^{-1}(\Omega)} + C\{\|\nabla v_h\|^2_{L^2(\Omega)} + \|\nabla z\|^2_{L^2(\Omega)}\}
\end{aligned}$$

for any $\eta > 0$. Now we take $v_h = -\varepsilon\lambda_h^*$ for some constant $\varepsilon > 0$ and $\eta = \frac{1}{2}\alpha_1\varepsilon$ and we derive

$$\frac{1}{2}\alpha_1\varepsilon\|\nabla\lambda_h^*\|^2_{L^2(\Omega)} \le C\Big\{\varepsilon^2\|\nabla\lambda_h^*\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\|f\|^2_{H^{-1}(\Omega)} + \|\nabla z\|^2_{L^2(\Omega)}\Big\}.$$

Then choosing $\varepsilon = \alpha_1/(4C)$ above gives $\|\nabla\lambda_h^*\|_{L^2(\Omega)} \le C$. Hence each subsequence of $\{(q_h^*, v_h^*, \lambda_h^*)\}_{h>0}$ has a subsequence, still denoted by $\{(q_h^*, v_h^*, \lambda_h^*)\}$, satisfying

(3.10)
$$q_h^* \to q^* \text{ in } L^1(\Omega), \qquad v_h^* \to v^* \text{ weakly in } H_0^1(\Omega), \qquad \lambda_h^* \to \lambda^* \text{ weakly in } H_0^1(\Omega)$$

or

$$q_h^* \to q^* \text{ in } L^1(\Omega), \qquad v_h^* \to v^* \text{ in } L^2(\Omega), \qquad \lambda_h^* \to \lambda^* \text{ in } L^2(\Omega)$$

for some $(q^*, v^*, \lambda^*) \in K \times V \times V$. By Lemma 3.5 we have $e_h(q_h^*, v_h^*) \to e(q^*, v^*)$ weakly in $H_0^1(\Omega)$. Thus $e_h(q_h^*, v_h^*) = 0$ also implies that $e(q^*, v^*) = 0$, and the following holds:

$$\mathcal{L}_r(q^*, v^*; \mu) \le \mathcal{L}_r(q^*, v^*; \lambda^*) \quad \forall\, \mu \in V. \tag{3.11}$$

On the other hand, for any $(q, v) \in K \times V$ and any $\varepsilon > 0$, by Lemma 3.3 we can find a function $q_\varepsilon \in C^\infty(\bar{\Omega})$ satisfying

$$\int_\Omega |q_\varepsilon - q| dx < \varepsilon, \qquad \Big| \int_\Omega |\nabla q_\varepsilon| dx - \int_\Omega |Dq| \Big| < \varepsilon. \tag{3.12}$$

Now we define

$$\tilde{q}_\varepsilon = \begin{cases} \alpha_1 & \text{if} \quad q_\varepsilon < \alpha_1, \\ q_\varepsilon & \text{if} \quad \alpha_1 \le q_\varepsilon \le \alpha_2, \\ \alpha_2 & \text{if} \quad q_\varepsilon > \alpha_2. \end{cases} \tag{3.13}$$

Then $\tilde{q}_\varepsilon \in K \cap W^{1,\infty}(\Omega)$ since

$$\nabla\tilde{q}_\varepsilon = \begin{cases} \nabla q_\varepsilon & \text{on} \quad A_\varepsilon = \{x \in \Omega : \alpha_1 \le q_\varepsilon \le \alpha_2\}, \\ 0 & \text{on} \quad \Omega \setminus A_\varepsilon. \end{cases} \tag{3.14}$$

Now we take $(q_h, v_h) = (I_h\tilde{q}_\varepsilon, R_h v) \in K_h \times \overset{\circ}{V}_h$ in (3.9) and get

(3.15)
$$J_h(q_h^*, v_h^*) \le J_h(I_h\tilde{q}_\varepsilon, R_h v) + (\nabla\lambda_h^*, \nabla e_h(I_h\tilde{q}_\varepsilon, R_h v)) + \frac{r}{2}\|\nabla e_h(I_h\tilde{q}_\varepsilon, R_h v)\|_{L^2(\Omega)}^2.$$

Then by the lower semicontinuity of the BV-norm (cf. [9]) we derive

$$\begin{aligned} &\liminf_{h\to 0} J_h(q_h^*, v_h^*) \\ &\ge \liminf_{h\to 0} \Big\{ \frac{1}{2}\int_\Omega q_h^* |\nabla(v_h^* - z)|^2 dx + \beta\int_\Omega |Dq_h^*| \Big\} \\ &\ge \liminf_{h\to 0} \frac{1}{2}\int_\Omega q_h^* |\nabla(v_h^* - z)|^2 dx + \liminf_{h\to 0} \beta\int_\Omega |Dq_h^*| \\ &\ge \frac{1}{2}\int_\Omega q^* \,|\nabla(v^* - z)|^2 dx + \beta\int_\Omega |Dq^*| = J(q^*, v^*), \end{aligned} \tag{3.16}$$

where we have used the following result:

$$\lim_{h\to 0} \frac{1}{2}\int_\Omega q_h^* |\nabla(v_h^* - z)|^2 dx = \frac{1}{2}\int_\Omega q^* \,|\nabla(v^* - z)|^2 dx,$$

which can be proved in exactly the same way as for (2.5).

Now by (3.6) and (3.7) we know that

$$\lim_{h\to 0} I_h\tilde{q}_\varepsilon = \tilde{q}_\varepsilon \text{ in } W^{1,1}(\Omega), \qquad \lim_{h\to 0} R_h v = v \text{ in } H_0^1(\Omega);$$

combining this with Lemma 3.4 gives

$$\lim_{h\to 0} e_h(I_h\tilde{q}_\varepsilon, R_h v) = e(\tilde{q}_\varepsilon, v) \text{ in } H_0^1\Omega).$$

Then letting $h \to 0$ in (3.15) and using (3.16) we obtain

$$J(q^*, v^*) \le J(\tilde{q}_\varepsilon, v) + (\nabla\lambda^*, \nabla e(\tilde{q}_\varepsilon, v)) + \frac{r}{2}\|\nabla e(\tilde{q}_\varepsilon, v)\|^2_{L^2(\Omega)}. \tag{3.17}$$

Since $q \in K$, we have from (3.12) and (3.13) that

$$\|\tilde{q}_\varepsilon - q\|_{L^1(\Omega)} \le \|q_\varepsilon - q\|_{L^1(\Omega)} < \varepsilon.$$

Thus $\lim_{\varepsilon\to 0} \tilde{q}_\varepsilon = q$ in $L^1(\Omega)$, which implies that $\lim_{\varepsilon\to 0} e(\tilde{q}_\varepsilon, v) = e(q, v)$ in $H_0^1(\Omega)$. Hence as $\varepsilon \to 0$, we derive

$$(\nabla\lambda^*, \nabla e(\tilde{q}_\varepsilon, v)) + \frac{r}{2}\|\nabla e(\tilde{q}_\varepsilon, v)\|^2_{L^2(\Omega)} \to (\nabla\lambda^*, \nabla e(q, v)) + \frac{r}{2}\|\nabla e(q, v)\|^2_{L^2(\Omega)}. \tag{3.18}$$

But by (3.14) and (3.12) we obtain

$$\int_\Omega |\nabla\tilde{q}_\varepsilon| dx = \int_{A_\varepsilon} |\nabla q_\varepsilon| dx \le \int_\Omega |\nabla q_\varepsilon| dx \le \int_\Omega |Dq| + \varepsilon;$$

therefore

$$\liminf_{\varepsilon\to 0} J(\tilde{q}_\varepsilon, v) \le \frac{1}{2}\int_\Omega q|\nabla(v-z)|^2 dx + \beta\int_\Omega |Dq| = J(q, v). \tag{3.19}$$

By substituting (3.18), (3.19) into (3.17) and passing to the limit $\varepsilon \to 0$ we finally get

$$\mathcal{L}_r(q^*, v^*; \lambda^*) = J(q^*, v^*) \le \mathcal{L}_r(q, v; \lambda^*) \quad \forall\, (q, v) \in K \times V.$$

This, together with (3.11), indicates that $(q^*, v^*; \lambda^*)$ is a saddle-point of $\mathcal{L}_r$. ∎

4. An Uzawa algorithm. In this section, we study an algorithm of the Uzawa type to find the saddle-points of the discrete augmented Lagrangian $L_r:\ K_h\times \overset{\circ}{V}_h \times \overset{\circ}{V}_h \to R$ defined in (3.3). We consider the following algorithm.

Uzawa Algorithm 1. We are given $\lambda^0 \in \overset{\circ}{V}_h$. Then for $n \ge 0$, with λ^n known, determine the pair $\{p^n, u^n\} \in K_h\times \overset{\circ}{V}_h$ such that

$$L_r(p^n, u^n; \lambda^n) \le L_r(q, v; \lambda^n) \quad \forall\, (q, v) \in K_h\times \overset{\circ}{V}_h; \tag{4.1}$$

then compute λ^{n+1} by

$$\lambda^{n+1} = \lambda^n + \rho_n e_h(p^n, u^n). \tag{4.2}$$

THEOREM 4.1. *Assume that* $0 < \beta_0 \le \rho_n \le \beta_1 < r$ *for any* $n = 1, 2, \ldots$. *Then any subsequence of* $\{p^n, u^n; \lambda^n\}$ *computed in the Uzawa algorithm* (4.1), (4.2) *has a subsequence (still denoted by)* $\{p^n, u^n; \lambda^n\}$ *such that*

$$p^n \to p \quad in \quad L^1(\Omega), \qquad u^n \to u \quad in \quad L^2(\Omega), \qquad \lambda^n \to \lambda \quad in \quad L^2(\Omega),$$

and $J_h(p^n, u^n) \to J_h(p, u)$ *as* $n \to \infty$. *Furthermore,* $\{p, u; \lambda\} \in K_h \times \overset{\circ}{V}_h \times \overset{\circ}{V}_h$ *is a saddle-point of* $L_r : K_h \times \overset{\circ}{V}_h \times \overset{\circ}{V}_h \to R$.

Proof. First, by Theorem 3.1 there exists a saddle-point $(q_h^*, v_h^*; \lambda_h^*)$ of $L_0 : K_h \times \overset{\circ}{V}_h \times \overset{\circ}{V}_h$, namely,

$$L_0(q_h^*, v_h^*; \mu_h) \le L_0(q_h^*, v_h^*; \lambda^*) \le L_0(q_h, v_h; \lambda^*) \quad \forall\, (q_h, v_h, \mu_h) \in K_h \times \overset{\circ}{V}_h \times \overset{\circ}{V}_h\ .$$

The first inequality immediately gives $e_h(q_h^*, v_h^*) = 0$, and the second implies that

$$J_h(q_h^*, v_h^*) \le J_h(q_h, v_h) + (\nabla\lambda_h^*,\ \nabla e_h(q_h, v_h)), \qquad \forall\, (q_h, v_h) \in K_h \times \overset{\circ}{V}_h\ . \tag{4.3}$$

Then taking $(q, v) = (q_h^*, v_h^*)$ in (4.1) and using (4.3) we obtain

$$\begin{aligned} &J_h(p^n, u^n) + (\nabla\lambda^n,\ \nabla e_h(p^n, u^n)) + \frac{r}{2}\|\nabla e_h(p^n, u^n)\|^2_{L^2(\Omega)} \\ &\le L_r(q_h^*, v_h^*; \lambda^n) = J_h(q_h^*, v_h^*) \\ &\le J_h(p^n, u^n) + (\nabla\lambda_h^*,\ \nabla e_h(p^n, u^n)). \end{aligned}$$

Hence

$$(\nabla(\lambda^n - \lambda_h^*),\ \nabla e_h(p^n, u^n)) + \frac{r}{2}\|\nabla e_h(p^n, u^n)\|^2_{L^2(\Omega)} \le 0. \tag{4.4}$$

Now let $\bar\lambda^n = \lambda^n - \lambda_h^*$; then we have

$$\bar\lambda^{n+1} = \bar\lambda^n + \rho_n e_h(p^n, u^n)$$

and thus

$$\begin{aligned} (\nabla\bar\lambda^n,\ \nabla e_h(p^n, u^n)) &= \frac{1}{\rho_n}(\nabla\bar\lambda_h,\ \nabla(\bar\lambda^{n+1} - \bar\lambda^n)) \\ &= \frac{1}{2\rho_n}\left\{\|\nabla\bar\lambda^{n+1}\|^2_{L^2(\Omega)} - \|\nabla\bar\lambda^n\|^2_{L^2(\Omega)} - \|\nabla(\bar\lambda^{n+1} - \bar\lambda^n)\|^2_{L^2(\Omega)}\right\} \\ &= \frac{1}{2\rho_n}\left\{\|\nabla\bar\lambda^{n+1}\|^2_{L^2(\Omega)} - \|\nabla\bar\lambda^n\|^2_{L^2(\Omega)} - \rho_n^2\|\nabla e_h(p^n, u^n)\|^2_{L^2(\Omega)}\right\}. \end{aligned}$$

Substituting this into (4.4), we get

$$\frac{1}{2\rho_n}\left\{\|\nabla\bar\lambda^{n+1}\|^2_{L^2(\Omega)} - \|\nabla\bar\lambda^n\|^2_{L^2(\Omega)}\right\} + \frac{1}{2}(r - \rho_n)\|\nabla e_h(p^n, u^n)\|^2_{L^2(\Omega)} \le 0.$$

Thus if $0 < \rho_n < r$, the sequence $\{\|\nabla\bar\lambda^n\|^2_{L^2(\Omega)}\}$ is monotonically decreasing and $\|\nabla e_h(p^n, u^n)\|_{L^2(\Omega)} \to 0$ as $n \to \infty$. Now letting $(q, v) = (q_h^*, v_h^*)$ in (4.1) we derive

$$J_h(p^n, u^n) \le J_h(q_h^*, v_h^*) - (\nabla\lambda^n,\ \nabla e_h(p^n, u^n)) \le C$$

with constant C independent of n. Therefore

$$\|p^n\|_{BV(\Omega)} + \|\nabla u^n\|_{L^2(\Omega)} \le C,$$

which implies that each subsequence of $\{p^n, u^n, \lambda^n\}$ has a subsequence (still denoted by) $\{p^n, u^n, \lambda^n\}$ such that

$$(p^n, u^n, \lambda^n) \to (p, u, \lambda) \text{ in } L^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \quad \text{as} \quad n \to \infty$$

for some $(p, u, \lambda) \in K_h \times \overset{\circ}{V}_h \times \overset{\circ}{V}_h$. Note that in a finite-dimensional space all the convergences are equivalent. Thus $e_h(p, u) = 0$ by means of

$$\|\nabla e_h(p^n, u^n)\|_{L^2(\Omega)} \to 0 \quad \text{and} \quad e_h(p^n, u^n) \to e_h(p, u) \quad \text{as} \quad n \to \infty.$$

Now letting $n \to \infty$ in (4.1) we easily obtain

$$L_r(p, u; \lambda) \leq L_r(q, v; \lambda) \quad \forall\, (q, v) \in K_h \times \overset{\circ}{V}_h \,.$$

Therefore $(p, u, \lambda) \in K_h \times \overset{\circ}{V}_h \times \overset{\circ}{V}_h$ is a saddle-point of L_r. □

Remark. To reduce the size of the minimization problem in (4.1), one may further divide the problem into two minimization subproblems with each seeking only one of the first two variables of the discrete augmented Lagrangian $L_r(\cdot, \cdot; \cdot)$. See Uzawa Algorithm 2 in the next section and [10, 3] for more algorithms of the same kind.

5. Numerical experiments. We now show some numerical experiments on the proposed method for parameter identification. We first describe how to implement the optimization step in (4.1). In order to solve the system (4.1) for the pair $\{p^n, u^n\}$, we use the following alternative iteration.

UZAWA ALGORITHM 2. We are given $\lambda^0 \in \overset{\circ}{V}_h$ and $q^0 \in K_h$. Set $n = 1$.

1. Set $k = 1$ and $q^{n,0} = q^{n-1}$.

2. Compute $u^{n,k}$ by solving

$$L_r(q^{n,k-1}, u^{n,k}; \lambda^{n-1}) = \min_{v_h \in V_h^0} L_r(q^{n,k-1}, v_h; \lambda^{n-1}), \tag{5.1}$$

and then compute $q^{n,k}$ by solving

$$L_r(q^{n,k}, u^{n,k}; \lambda^{n-1}) = \min_{p_h \in V_h} L_r(p_h, u^{n,k}; \lambda^{n-1}). \tag{5.2}$$

Compute $q^{n,k} = \max\{\alpha_1, \min\{q^{n,k}, \alpha_2\}\}$.
If $\|q^{n,k} - q^{n,k-1}\| \leq$ tolerance, set $u^n = u^{n,k}$ and $q^n = q^{n,k}$, GOTO **3**;
Otherwise set $k = k + 1$, GOTO **2.**

3. Compute λ^n by

$$\lambda^n = \lambda^{n-1} + \frac{3}{4}\, r\, e_h(p^n, u^n). \tag{5.3}$$

Set $n = n + 1$, GOTO **1.**

We use the Armijo algorithm (cf. Keung and Zou [15]) to solve problem (5.2). As the problem corresponds to a nonlinear algebraic system of equations, one may also use some other more efficient iterative methods. Problem (5.1), combining with the equation for $e_h(q^{n,k-1}, u^{n,k})$, corresponds to two linear algebraic systems of equations (both are positive definite), which are solved here by the conjugate gradient method.

We apply Uzawa Algorithm 2 to identify the discontinuous coefficients in the following test problem:

$$-\frac{d}{dx}\Big(q(x)\frac{d}{dx}u(x)\Big) = f(x), \qquad x \in (0, 1) \quad \text{with} \quad u(0) = u(1) = 0. \tag{5.4}$$

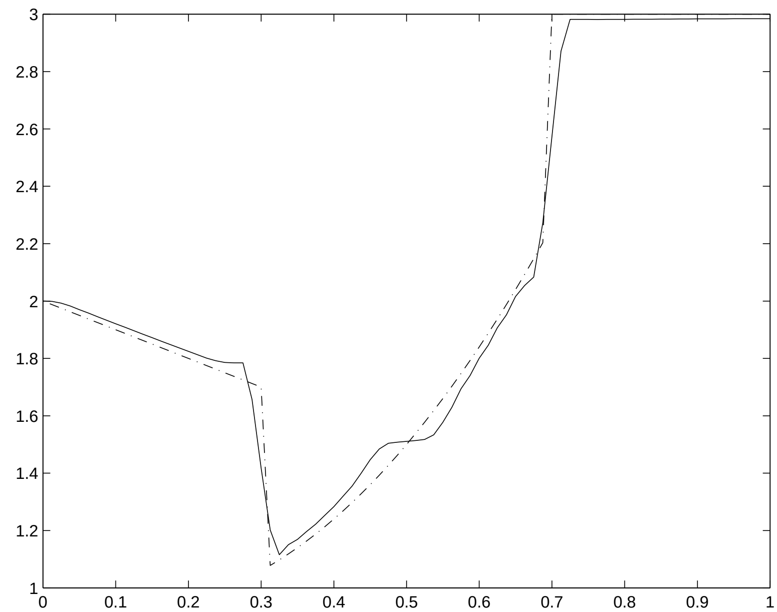

FIG. 5.1. $q_h^0 = 5.0$, $\beta = 10^{-3}$, $error = 0.032$, $iter = 5$.

Most parameters used in the algorithm are given below each figure. The error shown is the relative L^2-norm error between the exact parameter $q(x)$ to be identified and the computed parameter q_h. The regularization and smoothing parameters β and $\delta(h)$ (see (3.3)) are chosen to be 10^{-3} and 0.01. The augmented Lagrangian coefficient r is taken to be 1, and the finite element mesh size h to be 1/80. The lower and upper bounds α_1 and α_2 in the constrained set K are taken to be 0.5 and 20.0, respectively.

Example 1. We take the following discontinuous coefficient:

$$q(x) = \begin{cases} 2 - x, & x \in [0,\ 0.3], \\ 1 - x + 4x^2, & x \in (0.3,\ 0.7), \\ 3, & x \in [0.7,\ 1], \end{cases}$$

and compare it with the numerically identified solution q_h obtained by using Uzawa Algorithm 2. The exact observation data z is taken as $z(x) = u(q)(x) = \sin(\pi x)$, and the function $f(x)$ is then computed by (5.4) using $u(x)$ and $q(x)$. Figure 5.1 shows the exact solution $q(x)$ (the dotted line) and the numerically identified solution $q_h(x)$ (the solid line). The initial guesses λ^0 and q_h^0 are taken to be the constants 0 and 5.0, respectively. $q_h^0 = 5.0$ is not a good initial guess at all, but the numerical method converges very stably and fast; Figure 5.1 gives the result of the 5th iteration ($n = 5$).

We now add some random noise to the gradient of the true solution u. (Recall that we used the energy-norm in the output least squares formulation. If the L^2-norm is used, one should consider the noised observation data z of the true solution u directly, instead of the gradient.) Namely, we replace the gradient ∇z in the cost functional L_r with the noised data

$$\nabla z^\delta(x) = \nabla z(x) + \delta\,\mathrm{rand}\,(x),$$

where rand (x) is a uniformly distributed random function in $[-1, 1]$ and δ is the noise level parameter. The numerical result of the 5th iteration is shown in Figure 5.2 with

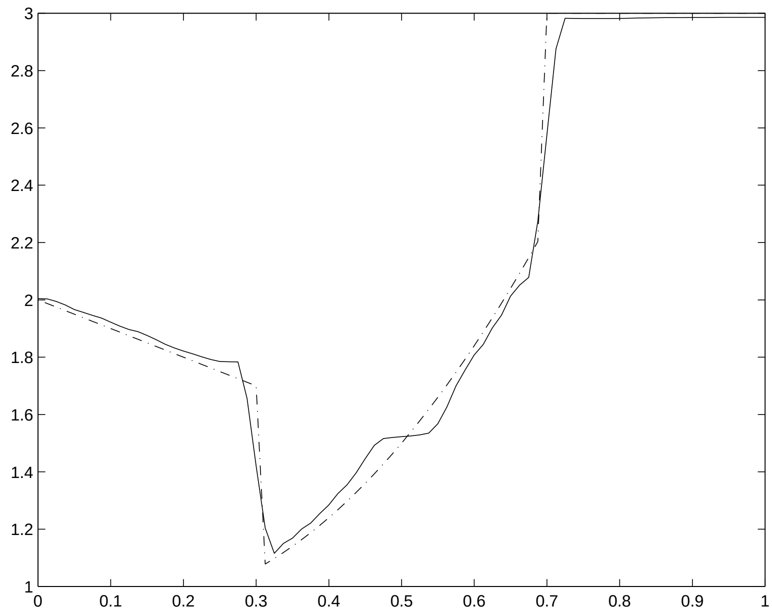

FIG. 5.2. $q_h^0 = 5.0$, $\beta = 10^{-3}$, *noise level* $\delta = 1\%$, *error* $= 0.033$, *iter* $= 5$.

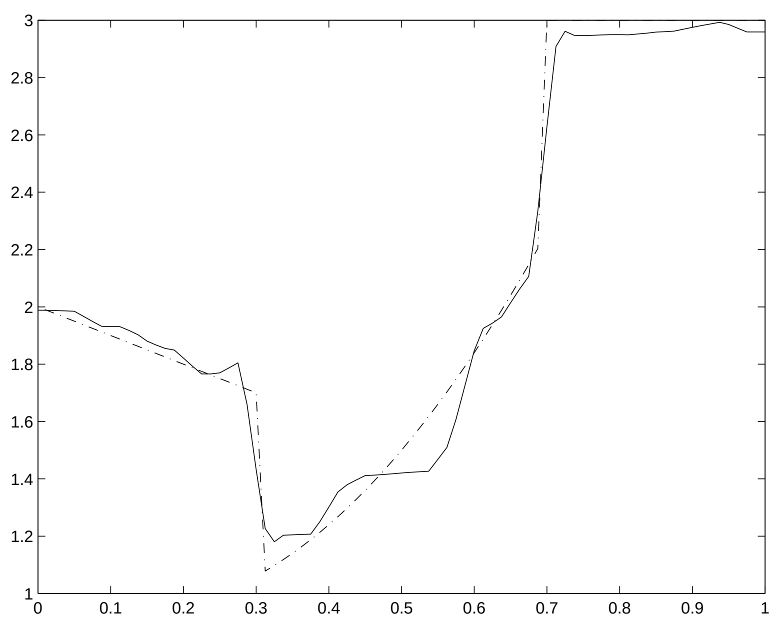

FIG. 5.3. $q_h^0 = 5.0$, $\beta = 10^{-3}$, *noise level* $\delta = 10\%$, *error* $= 0.038$, *iter* $= 5$.

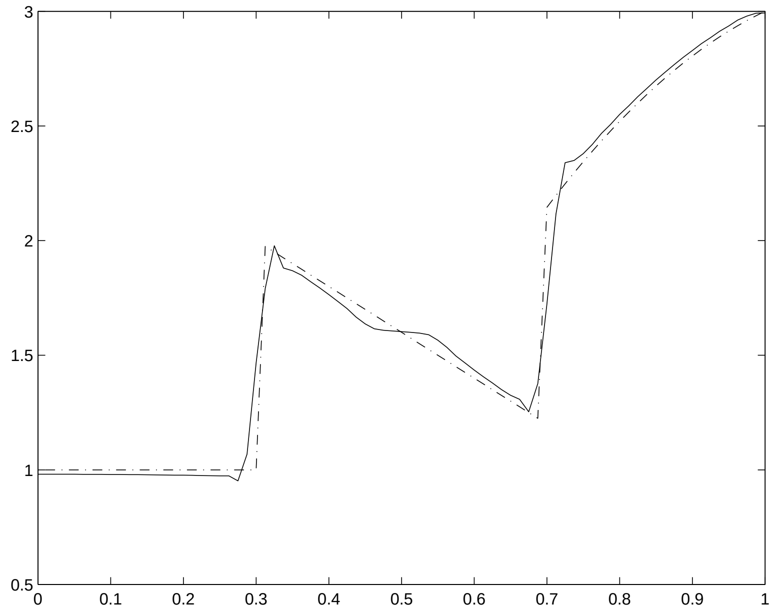

FIG. 5.4. $q_h^0 = 5.0$, $\beta = 10^{-3}$, *error* $= 0.043$, *iter* $= 5$.

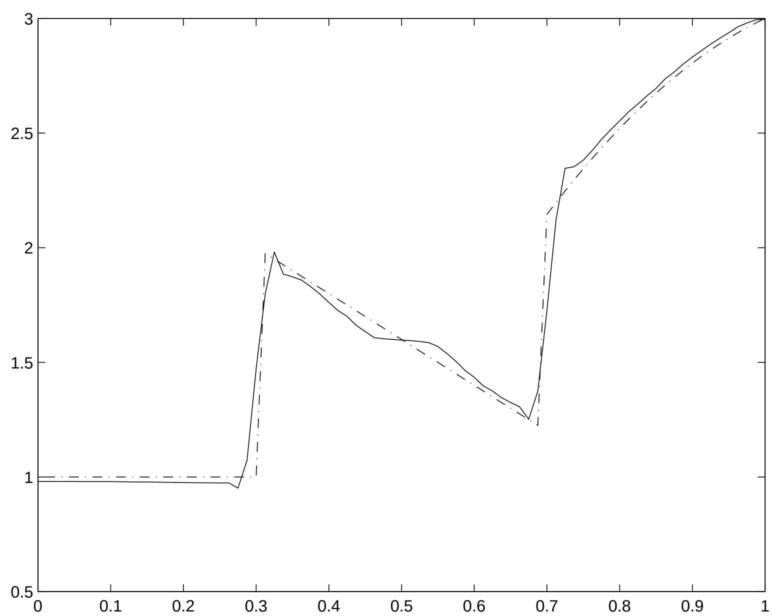

FIG. 5.5. $q_h^0 = 5.0$, $\beta = 10^{-3}$, *noise level* $\delta = 1\%$, *error* $= 0.045$, *iter* $= 5$.

the noise level parameter $\delta = 1\,\%$. We do not see much difference compared with the noise-free case (Figure 5.1). When the noise increases to 10%, the numerical identified solution is still very satisfactory; see Figure 5.3. This indicates that the numerical

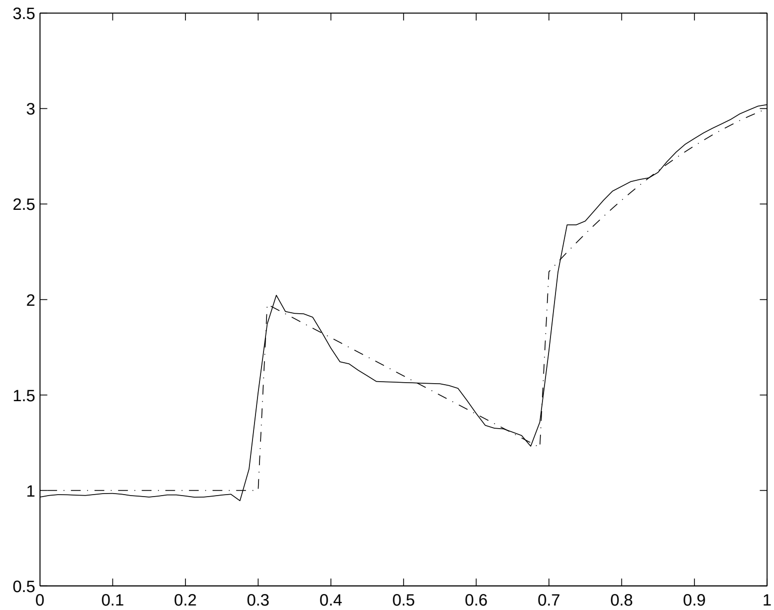

FIG. 5.6. $q_h^0 = 5.0$, $\beta = 10^{-3}$, *noise level* $\delta = 10\%$, *error* $= 0.051$, *iter* $= 5$.

method is not very sensitive to the noise.

Example 2. We take the discontinuous coefficient:

$$q(x) = \begin{cases} 1, & x \in [0,\ 0.3], \\ 2.6 - 2x, & x \in (0.3,\ 0.7), \\ -9\,x^2/2 + 21\,x/2 - 3, & x \in [0.7,\ 1], \end{cases}$$

and compare it again with the numerical solution q_h recovered by Uzawa Algorithm 2. Figure 5.4 shows the exact solution $q(x)$ (the dotted line) and the numerically identified solution $q_h(x)$ (the solid line), where we have taken the initial guesses $\lambda^0 = 0$ and $q_h^0 = 5.0$. We see again that the numerical method converges very stably and fast. Figure 5.4 is the result of the 5th iteration ($n = 5$).

Again, we add some random noise to the gradient of the true solution u; namely, we assume that the available data are the following noised data:

$$\nabla z^\delta(x) = \nabla z(x) + \delta\,\mathrm{rand}\,(x).$$

Figure 5.5 gives the numerical result of the 5th iteration with the noise level parameter $\delta = 1\,\%$. We can see that noise of this level has very little effect on the accuracy and stability of the numerical method. When the noise increases to 10%, the numerical identified solution is still very satisfactory; see Figure 5.6.

Our numerical experiences show that the numerical method proposed in the paper converges very fast (5 iterations for the considered examples) and globally, which is consistent with our theory. In fact one can take much worse initial guesses than the preceding ones ($q_h^0 = 5.0$). More importantly, the method seems to be not very sensitive to the noise.

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