Phenomenological Introduction to Neutrino Oscillations

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I. INTRODUCTION

II. EXPERIMENTAL EVIDENCE OF NEUTRINO OSCILLATION PHENOMENON

III. THEORETICAL FRAMEWORK

Neutrinos interacts with other particles via weak interactions which are described by the charge current (CC) and neutral current (NC) interaction Lagrangians:

$$\mathcal{L}_{CC} = -\frac{g}{2\sqrt{2}}\mathcal{J}_{\rho}^{CC}W^{\rho} + h.c, \tag{1}$$

$$\mathcal{L}_{NC} = -\frac{g}{2\cos(\theta_W)}\mathcal{J}_{\rho}^{NC}Z^{\rho}, \tag{2}$$

where g represents the $SU(2)_L$ gauge coupling constant and θ_W the weak angle. Furthermore, the charged and neutral current that are respectively denoted by $\mathcal{J}_{\rho}^{CC}$ and $\mathcal{J}_{\rho}^{NC}$ are defined as:

$$\mathcal{J}_{\rho}^{CC} = 2\sum_{l}\bar{\nu}_{lL}\gamma_{\rho}l_L, \tag{3}$$

$$\mathcal{J}_{\rho}^{NC} = \sum_{l}\bar{\nu}_{lL}\gamma_{\rho}\nu_{lL}, \tag{4}$$

where the charged leptonic field l can be one of the neutrino flavors (e, μ, τ).

If the neutrinos have zero-masses, then the left-handed neutrino field $\nu_{\alpha L}$ with flavor α can be written as a superposition of the left-handed components ν_{iL} of the neutrino field with mass m_i. In an ultra-relativistic scenario, we have:

$$\nu_{\alpha L} = U_{\alpha i}\nu_{iL} \tag{5}$$

where repeated indices are summed over. Henceforth, we use Greek letters to refer to the neutrino masses and Latin letters to refer to the flavours. The neutrino masses i runs from 1 to N where N denotes the number of massive neutrinos. In this project, we are only going to focus in the case $N = 3$.

In Eq. (5), U is a unitary matrix. This implies that a flavor eigenstate $|\nu_{\alpha}\rangle$ can be written as a superposition of different mass eigenstates $|\nu_i\rangle$ in the following way:

$$|\nu_{\alpha L}\rangle = U_{\alpha i}^{\star}|\nu_i\rangle, \tag{6}$$

$$|\bar{\nu}_{\alpha L}\rangle = U_{\alpha i}|\bar{\nu}_i\rangle. \tag{7}$$

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Assuming that we have three (03) massive neutrinos and that the neutrinos are Dirac particles, the mixing unitary matrix U can be written as:

$$U = \cdots, \tag{8}$$

where $c_{ij} = \cos(\theta_{ij})$ and $s_{ij} = \sin(\theta_{ij})$. Here, θ_{ij} denote the mixing angles while δ denotes the Dirac-type CP-phase. Defining $\Delta_{ij} = m_i^2 - m_j^2$ and ordering the masses such that $\Delta_{21}^2 > 0$ and $\Delta_{21}^2 < \Delta_{31}^2$, we have the following constraints:

$$0 \leq \theta_{ij} \leq \frac{\pi}{2} \quad (i \neq j), \quad 0 \leq \delta \leq 2\pi. \tag{9}$$

III.1. Neutrino evolution equation in vacuum and in matter

The evolution equation of a generic neutrino state $|\nu(t)\rangle$ is described by a Shrödinger-like equation:

$$\mathrm{i}\partial_t|\nu(t)\rangle = H|\nu(t)\rangle, \tag{10}$$

where H represents the Hamiltonian operator. Expressed in the flavor eigenstate basis $|\nu_{\alpha}\rangle$, the above equation translates into

$$\mathrm{i}\partial_t v^{(f)}(t) = H^{(f)}\nu^{(f)}(t), \tag{11}$$

where $v^{(f)}(t)$ denotes the vector describing the flavor content of the neutrino state $|\nu(t)\rangle$. Elements of the Hamiltonian matrix $H^{(f)}$ are given by

$$H_{\alpha\beta}^{(f)} = \langle\nu_{\alpha}|H|\nu_{\beta}\rangle. \tag{12}$$

In the mass eigenstate basis, the vacuum Hamiltonian $H^{(m)}$ (where m indicates the mass eigenstate representation) is determined in terms of the neutrino masses

$$\begin{aligned} H_{vac}^{(m)} &= \mathrm{diag}\left(\sqrt{\vec{p}^2 + m_1^2}, \sqrt{\vec{p}^2 + m_2^2}, \sqrt{\vec{p}^2 + m_3^2}\right), \\ &\approx |\vec{p}| + \frac{1}{2|\vec{p}|}\mathrm{diag}\left(m_1^2, m_2^2, m_3^2\right). \end{aligned} \tag{13}$$

In the first equality we assumed that the neutrino state $|\nu(t)\rangle$ can be described as a superposition of states with fixed momentum $\vec{p}$. In the last line, we used the ultra-relativistic approximation $\sqrt{\vec{p}^2 + m_i^2} \sim |\vec{p}| + m_i^2/2|\vec{p}|$. The new Hamiltonian in the flavor eigenstates therefore reads

$$H_{\alpha\beta,vac}^{(f)} = U_{\alpha i}H_{ij,vac}^{(m)}U_{i\beta}^{\dagger} \tag{14}$$

In the presence of matter, we have to add the vacuum Hamiltonian an effective potential V in the evolution equation

$$i\partial_t|\nu(t)\rangle = (H_{vac} + V)\,|\nu(t)\rangle, \tag{15}$$

where in the context of SM, the effective potential is a matrix that is diagonal in the flavor basis $V^{(f)} = \mathrm{diag}\,(V_e, V_\nu, V_\tau)$.

III.2. Survival probability

Assuming that a neutrino is created a a time $t_0 = 0$ at a point $x_0 = 0$, the flavor state $|\nu(t)\rangle$ in the flavor eigenstate basis can be written as

$$\nu^{(f)}(0) = \left(\langle\nu_e|\nu(0)\rangle, \langle\nu_\mu|\nu(0)\rangle, \langle\nu_\tau|\nu(0)\rangle\right)^{\mathrm{T}}. \tag{16}$$

After some interval of time t at a given point x, its flavor has evolved according to

$$\nu^{(f)}(x) = S^{(f)}(x)\nu^{(f)}(0), \tag{17}$$

where the evolution operator S is expressed as

$$S^{(f)} = T\left[\exp\left(-\mathrm{i}\int_0^x \mathrm{d}\tilde{x}\, H^{(f)}(\tilde{x})\right)\right]. \tag{18}$$

In the above equation, T represents the time ordering operator. Therefore, the probability to detect a neutrino of flavor ν_β at a distance L from its initial position (where its initial flavor is known) is given by

$$P(\nu_\beta \longleftarrow \nu_\alpha) = |S^{(f)}_{\beta\alpha(L)}|^2. \tag{19}$$

This implies that the survival probability $P(\nu_\alpha \longleftarrow \nu_\alpha)$ is given by the following

$$P(\nu_\alpha \longleftarrow \nu_\alpha) = 1 - P(\nu_\beta \longleftarrow \nu_\alpha). \tag{20}$$

The above expression is a consequence of the unitarity of the mixing matrix $\sum_\beta P(\nu_\beta \longleftarrow \nu_\alpha) = 1$.

III.3. Vacuum neutrino oscillations

As it was presented in the previous section, in vacuum the neutrino Hamiltonian H is constant. Hence, the evolution operator can be written as

$$S^{(f)} = US^{(m)}U^\dagger. \tag{21}$$

The evolution operator in the mass eigenstate (denoted by the (m)) is a diagonal matrix that is function of $\phi_i = -m_i^2 x/2|\vec{p}|$

$$S^{(m)} = \mathrm{diag}\left(\exp(i\phi_1), \exp(i\phi_2), \exp(i\phi_3)\right). \tag{22}$$

The probability of observing a neutrino of flavor ν_α to change into a neutrino of flavor ν_β is given by

$$P(\nu_\beta \longleftarrow \nu_\alpha) = \left(U_{\beta i}U^\star_{\alpha i}U^\star_{\beta j}U_{\alpha j}\right)\exp(i\phi_{ij}), \tag{23}$$

where in the ultra-relativistic limit $|\vec{p}| \sim E$ and hence $\phi_{ij} = (\Delta_{ij}L)/(2E)$. From this result, we are now equipped with the tools needed to compute the oscillation probabilities. For the case of two neutrino flavor mixing, we only consider one non-vanishing mixing angle θ_{ij} in the evolution operator U described by Eq. (8). The oscillation probability in Eq. (23) then leads to the following well known result

$$P(\nu_\beta \longleftarrow \nu_\alpha) = \sin^2(2\theta_{ij})\sin^2\left(\frac{\Delta^2_{ij}L}{4E}\right). \tag{24}$$

Notice that in the above expression, α has to be different from β ($\alpha \neq \beta$) and depending on the non-vanishing mixing angle, we end up with different flavor changes, i.e.:

$$\theta_{12} \neq 0 \Longleftrightarrow P(\nu_\mu \longleftarrow \nu_e), \tag{25}$$
$$\theta_{23} \neq 0 \Longleftrightarrow P(\nu_\tau \longleftarrow \nu_\mu), \tag{26}$$
$$\theta_{13} \neq 0 \Longleftrightarrow P(\nu_\tau \longleftarrow \nu_e). \tag{27}$$

In the three-neutrino case, by performing the same steps and using further constraint on the masses as introduced previously ($\Delta^2_{21} << |\Delta^2_{31}|$), we have for instance, for $\nu_e \longrightarrow \nu_\mu$, the following probability

$$\begin{aligned} P(\nu_\mu \longleftarrow \nu_e) =& s^2_{23}S_{23}\sin^2(2\theta_{13}) \\ &+ c^2_{23}S_{12}\sin^2(2\theta_{12}) - 8JS_{12}S_{13}, \end{aligned} \tag{28}$$

where $S_{ij} = \sin^2\left(\Delta^2_{ij}L/(4E)\right)$ and

$$J = \cdots. \tag{29}$$

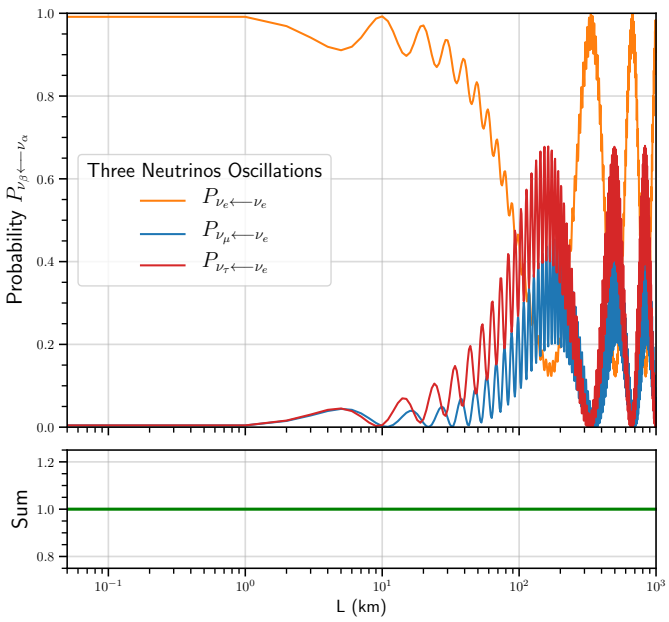
IV. PHENOMENOLOGY OF NEUTRINO OSCILLATION PROBABILITIES

In this section,

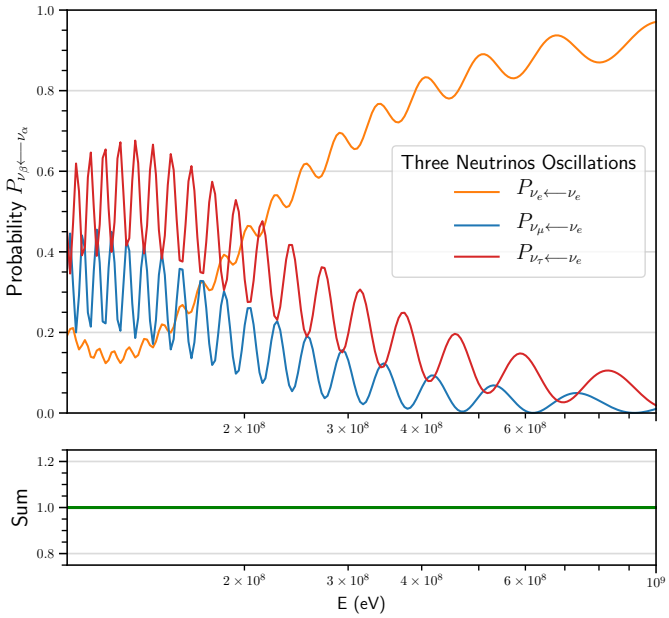
IV.1. Numerical insights

From the analytical point of view, computing probabilities of flavor transition involve diagonalizing the Hamiltonian operator. This procedure, however, can be complicated especially when one studies neutrino oscillations in matter. As is often done, careful studies of perturbative methods that lead to some approximations are used in order to computed these probabilities. As described in [], numerical methods can bring new insights into understanding how these probabilities are computed. The resulting methods provide strategies to explore non-standard oscillations where the Hamiltonians do not have generic analytical solutions.

In this project, we follow closely the method described in Ref. [] first introduced by Ohlsson and Snellman in Ref.[]. It consists on expanding the Hamiltonian operator that enters in the expression of the evolution operator in terms of $SU(2)$ and $SU(3)$ matrices. The steps for such a calculation can be broken down into the following steps:



(a) Varying distance (baseline).



(b) Varying energy.

FIG. 1: Three-neutrino oscillation probabilities in vacuum with varying (a) baseline and (b) varying energy. The top panels show the actual probabilities while the bottom panels show the sum of each probability.

- First, the Hamiltonian is expanded in terms of the Pauli matrices in the case of two-neutrino oscillation, and in terms of the Gell-Mann matrices in the case of three-neutrino flavors.

- Compute the coefficients of the expansions in terms of the components of the Hamiltonian.

- The exponent $\exp(\mathrm{i}Ht)$ is then expanded using the Cayley-Hamilton theorem which states that any analytic function of an $n \times n$ matrix can be written as a polynomial of degree $(n-1)$ in that matrix.

- Compute the evolution operator in terms of the coefficients in the Hamiltonian series and derive the corresponding probability.

In this report, we only illustrate the case for two-neutrino oscillations. For the three-neutrino case, refer to Ref. []. Let us denote the two-neutrino Hamiltonian by H_2. Its expansion in terms of the Pauli matrices σ^i is given by

$$\mathrm{H}_2 = h_0 + h_i\sigma^i \qquad (i = 1, 2, 3). \tag{30}$$

The coefficients h_k are fully determined by the components of the Hamiltonian matrix and can be easily computed using the explicit expression of the Pauli matrices.

$$h_0 = \tag{31}$$

The evolution operator U_2 for the twp-neutrino oscillations case can therefore be written as

$$\mathrm{U}_2 = \exp\left(-\mathrm{i}(h_0 + h_i\sigma^i)L\right). \tag{32}$$

The first term in the exponent does not affect the probability and therefore can be ignored. Hence, the evolution operator just becomes $\mathrm{U}_2 = \exp(-\mathrm{i}h_i\sigma^i L)$. Using Euler's formula, the above expression yields

$$\mathrm{U}_2 = \cos(|h|L) - \frac{\mathrm{i}}{|h|}\sin(|h|L)h_i\sigma^i, \tag{33}$$

where $|h| = \sqrt{|h_1|^2 + |h_2|^2 + |h_3|^2}$. We can now compute the survival probability

$$P(\nu_\alpha \longleftarrow \nu_\alpha) = |\nu_\alpha^\dagger \mathrm{U}_\alpha \nu_\alpha|^2. \tag{34}$$

Considering $\nu_\alpha = (1, 0)^{\mathrm{T}}$, we can compute the terms that enter in the expression of the probability in Eq. (34)

$$\nu_\alpha^\dagger \mathrm{U}_\alpha \nu_\alpha = \cos(|h|L) - \mathrm{i}\frac{h_3}{|h|}\sin(|h|L). \tag{35}$$

Putting this expression back into Eq. (34), and with some algebras we can derive the final expression of the survival probability of a neutrino of flavor α,

$$P(\nu_\alpha \longleftarrow \nu_\alpha) = \cos^2(|h|L) + \frac{|h_3|^2}{|h|^2}\sin^2(|h|L). \tag{36}$$

Therefore, the expression of the two-neutrino oscillation probabilities that a neutrino of flavor α is detected with a flavor β is given by the following simple relation:

$$P(\nu_\beta \longleftarrow \nu_\alpha) = 1 - P(\nu_\alpha \longleftarrow \nu_\alpha). \tag{37}$$

Let us now show that using this approach, we re-derive the two-neutrino oscillation probability given by Eq. (24). For two-neutrino flavor oscillation, the vacuum Hamiltonian operator is defined as

$$\mathrm{H}_2^{vac} = \frac{1}{2E}\mathrm{R}_{2,\theta}\mathrm{H}_2^{(m)}\mathrm{R}_{2,\theta}^\dagger, \tag{38}$$

where $R_{2,\theta}$ represents an Euler rotation with angle θ_{ij} and $H_2^{(m)}$ is given by

$$H_2^{(m)} = \text{diag}\left(\frac{\Delta_{ij}^2}{2}, -\frac{\Delta_{ij}^2}{2}\right). \tag{39}$$

Using the coefficients h_i to compute $|h|$, putting the explicit expression of the coefficients into Eq. (34), and doing some simplification, we arrive at the following expression

$$P(\nu_\beta \longleftarrow \nu_\alpha) = \sin^2(2\theta_{ij}) \sin^2\left(\frac{\Delta_{ij}^2 L}{4E}\right), \tag{40}$$

which is exactly the same as in Eq. (34). The difference being that this approach can be easily implemented numerically to compute oscillations in matter. Going into full details of using this method to compute the three-oscillation probabilities or to study neutrino oscillations in matter is beyond the scope of this project and will be left aside. For complete details and computations, refer to Ref. []. The next section, however, will present results beyond the two-neutrino oscillations both in vacuum and in matter.

IV.2. Phenomenological results

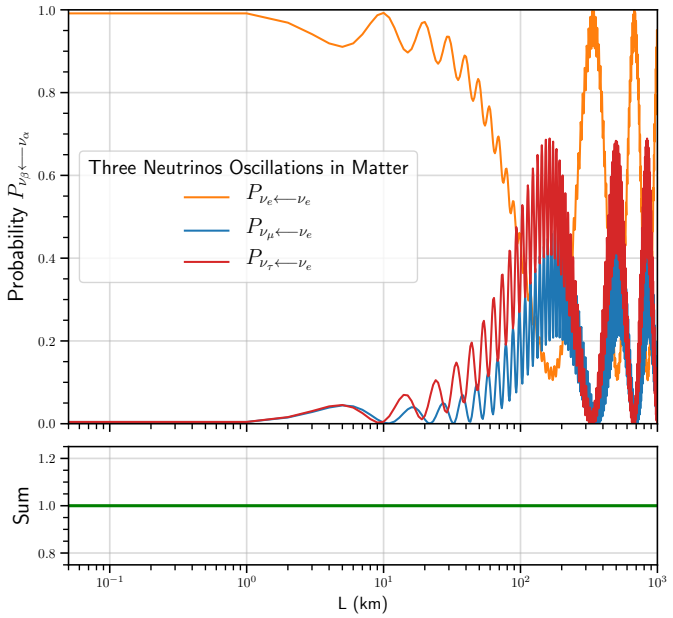
This section is dedicated to the phenomenological study of the three-neutrino oscillations both in vacuum and in presence of matter with constant density. To compute the oscillation probabilities, the only input parameters that we vary are the energy E and the distance traveled by the neutrino L. The other parameters such as the masses and the mixing angles that enter in the expression of the PMNS mixing matrix are extracted from the NuFit code that fix the parameters to experimental data in order to find the best-fit values. These parameters are summarized in Table I.

Parameters	Numerical Values
δ_{CP}	217°
$\sin^2\theta_{12}$	0.310
$\sin^2\theta_{23}$	0.582
$\sin^2\theta_{13}$	0.022
Δ_{21}	$7.391 \cdot 10^{-5}$
Δ_{31}	$2.525 \cdot 10^{-3}$

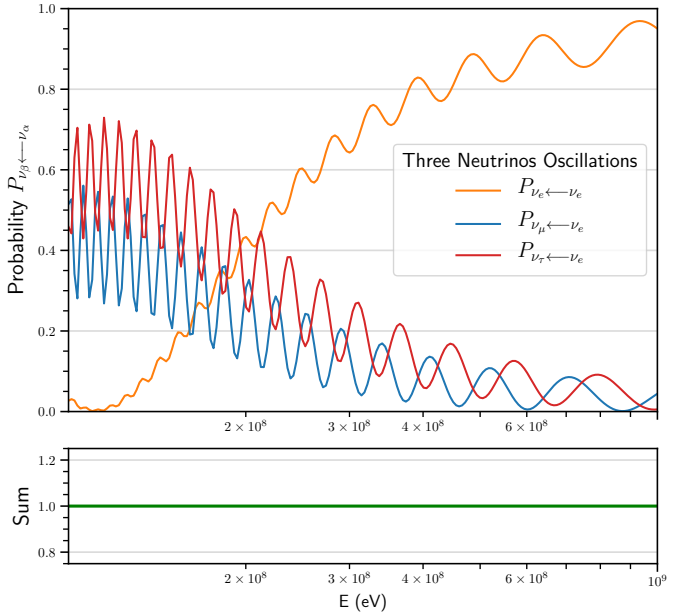
TABLE I: Numerical values of the mass differences Δ_{ij} and mixing angles θ_{ij} extracted from a global fit to oscillation data [**?** **?**].

The plots shown in this project were produced using python codes [] which in turn rely on an external python library called `NuOsc` [].

In Fig. 1b, we plot the three-neutrino oscillation probabilities as a function of the baseline L and energy E. In particular, we plot the probability that a neutrino with



(a) Varying distance (baseline).



(b) Varying energy.

FIG. 2: Three-neutrino oscillation probabilities in matter with varying (a) baseline and (b) varying energy.

an initial flavor ν_e oscillates between flavors ν_β (with $\beta = e, \mu, \tau$). We can clearly see in the bottom panels that the sum of all probabilities exactly gives one. For small values of L, we see that the probability that the electron-neutrino does not oscillate is high. As the neutrino traverses more distances, this probability fluctuates, and in some regions smaller than the probability of the neutrino to have a flavor ν_μ or ν_τ.