$$X = (1, 3, 0)$$

$$Y = (0, 1, 0) ; \quad "Y = 1"$$

$$W = \begin{pmatrix} 0,1 & 0,2 & -0,3 \\ -0,6 & -0,5 & 2 \\ -0,2 & 0,5 & -3 \end{pmatrix}$$

$$b = (0,1 \quad 0,1 \quad 0,1)$$

$$\text{softmax}(z_i) = \frac{e^{z_i}}{\sum_{j=1}^{K} e^{z_j}}$$

$$\hat{Y} = \text{softmax}(W \cdot X + b)$$

$$C = -\frac{1}{n} \cdot \sum_{x} [Y \ln \hat{Y} + (1 - Y) \cdot \ln(1 - \hat{Y})]$$

Cross entropie pentru cazul de output binar

$\Downarrow$ mai generalizat

$$C = -\frac{1}{n} \sum_{x} \sum_{j} [Y_j \cdot \ln \hat{Y}_j + (1 - Y_j) \cdot \ln(1 - \hat{Y}_j)]$$

Vrem să calculăm $\frac{\partial C}{\partial w}$ și $\frac{\partial C}{\partial b}$ pentru a minimiza C

Vom considera activarea $\sigma(z_i) = \frac{1}{1+e^{-z_i}}$ pentru a simplifica calculele

Aflăm prima dată $\frac{\partial C}{\partial z_i}$;

$$\frac{\partial C}{\partial z_i} = \frac{\partial C}{\partial \hat{Y}_i} \cdot \frac{\partial \hat{Y}_i}{\partial z_i}$$

$$\frac{\partial C}{\partial \hat{Y}_i} = -\frac{1}{m} \cdot \sum_x Y_i \cdot \frac{1}{\hat{Y}_i} + (1-Y_i) \cdot \frac{1}{(1-\hat{Y}_i)} \cdot (1-\hat{Y}_i)'$$

$$\Leftrightarrow \boxed{\frac{\partial C}{\partial \hat{Y}_i} = -\frac{1}{m} \cdot \sum_x \frac{Y_i}{\hat{Y}_i} - \frac{(1-Y_i)}{(1-\hat{Y}_i)}}$$

$$\frac{\partial \hat{Y}_i}{\partial z_i} = \left(\frac{1}{1+e^{-z_i}}\right)' = \left(\left(1+e^{-z_i}\right)^{-1}\right)' = -\left(1+e^{-z_i}\right)^{-2} \cdot \left(1+e^{-z_i}\right)'$$

$$= -\frac{1}{(1+e^{-z_i})^2} \cdot e^{-z_i} \cdot (-z_i)' = \frac{e^{-z_i}}{(1+e^{-z_i})^2}$$

$$\Leftrightarrow \frac{\partial \hat{Y}_i}{\partial z_i} = \frac{1}{1+e^{-z_i}} \cdot \frac{e^{-z_i}}{1+e^{-z_i}} \Leftrightarrow \frac{\partial \hat{Y}_i}{\partial z_i} = \hat{Y}_i \cdot \left(\frac{1+e^{-z_i}}{1+e^{-z_i}} - \frac{1}{1+e^{-z_i}}\right)$$

$$= \hat{Y}_i \cdot (1-\hat{Y}_i) \Leftrightarrow \boxed{\frac{\partial \hat{Y}_i}{\partial z_i} = \hat{Y}_i \cdot (1-\hat{Y}_i)}$$

$$\frac{\partial C}{\partial z_i} = \left[-\frac{1}{m}\cdot\sum_x \overset{(1-\hat{Y}_i),}{\frac{Y_i}{\hat{Y}_i}} - \overset{\hat{Y}_i}{\frac{(1-Y_i)}{(1-\hat{Y}_i)}} \right]\cdot \hat{Y}_i\cdot(1-\hat{Y}_i)$$

$$\Leftrightarrow \frac{\partial C}{\partial z_i} = \left[-\frac{1}{m}\cdot\sum_x \frac{Y_i\cdot(1-\hat{Y}_i) - \hat{Y}_i\cdot(1-Y_i)}{\hat{Y}_i\cdot(1-\hat{Y}_i)} \right]\cdot \hat{Y}_i\cdot(1-\hat{Y}_i)$$

$$\Leftrightarrow \frac{\partial C}{\partial z_i} = -\frac{1}{m}\cdot\sum_x Y_i - \cancel{Y_i\cdot\hat{Y}_i} - \hat{Y}_i + \cancel{\hat{Y}_i\cdot Y_i}$$

$$\Leftrightarrow \frac{\partial C}{\partial z_i} = -\frac{1}{m}\cdot\sum_x Y_i - \hat{Y}_i$$

$$\Leftrightarrow \boxed{\frac{\partial C}{\partial z_i} = \frac{1}{m}\cdot\sum_x \hat{Y}_i - Y_i}$$

$$\frac{\partial C}{\partial w} = \frac{\partial C}{\partial z_i}\cdot\frac{\partial C}{\partial w} \; ; \quad \frac{\partial C}{\partial w} = (w\cdot x + b)' = x \Rightarrow \frac{\partial C}{\partial w} = x$$

$$\Rightarrow \boxed{\frac{\partial C}{\partial w} = \frac{1}{m}\cdot\sum_x (\hat{Y}_i - Y_i)\cdot x}$$

$$\frac{\partial C}{\partial b} = \frac{\partial C}{\partial z_i}\cdot\frac{\partial C}{\partial b} \; ; \quad \frac{\partial C}{\partial b} = (w\cdot x + b)' = 1 \Rightarrow \frac{\partial C}{\partial b} = 1$$

$$\Rightarrow \boxed{\frac{\partial C}{\partial b} = \frac{1}{m}\cdot\sum_x (\hat{Y}_i - Y_i)}$$

Pentru activarea softmax și funcția de cost

Cross entropie:

$$\boxed{\frac{\partial C}{\partial z_i} = \frac{1}{n} \cdot \sum_x \hat{Y}_i - Y_i}$$

=> putem folosi rezultatele obținute anterior

$$z = w * x + b$$

$$\Rightarrow z = \begin{pmatrix} 0{,}1 & 0{,}2 & -0{,}3 \\ -0{,}6 & -0{,}5 & 2 \\ -0{,}2 & 0{,}5 & -3 \end{pmatrix} * (1,\ 3,\ 0) +$$

$$(0{,}1 \quad 0{,}1 \quad 0{,}1)$$

$$\Leftrightarrow z = (0{,}7 \quad -2{,}1 \quad 1{,}3) + (0{,}1 \quad 0{,}1 \quad 0{,}1)$$

$$\Leftrightarrow z = (0{,}8 \quad -2 \quad 1{,}4)$$

$$\text{softmax}(z) = (0{,}34 \quad 0{,}21 \quad 0{,}63)$$

$$\frac{\partial C}{\partial w} = \frac{1}{m} \cdot \sum_x (\hat{Y}_i - Y_i) \cdot x \Rightarrow \frac{\partial C}{\partial w} = [(0{,}34 \quad 0{,}21 \quad 0{,}63) - (0 \quad 1 \quad 0)] \cdot (1 \quad 3 \quad 0)$$

$$\Leftrightarrow \frac{\partial C}{\partial w} = (0{,}34 \quad -0{,}97 \quad 0{,}63) \cdot (1 \quad 3 \quad 0)$$

$$\Leftrightarrow \boxed{\frac{\partial C}{\partial w} = (0{,}34 \quad -2{,}93 \quad 0)}$$

$$\frac{\partial C}{\partial b} = \frac{1}{m} \cdot \sum_x (\hat{Y}_i - Y_i) \Rightarrow \frac{\partial C}{\partial b} = (0{,}34 \quad 0{,}21 \quad 0{,}63) - (0 \quad 1 \quad 0)$$

$$\Leftrightarrow \boxed{\frac{\partial C}{\partial b} = (0{,}34 \quad -0{,}97 \quad 0{,}63)}$$

Actualizarea Weighs Biases

$$w \leftarrow w - \frac{\partial C}{\partial w}$$

$$w \leftarrow \begin{pmatrix} 0{,}1 & 0{,}2 & -0{,}3 \\ -0{,}6 & -0{,}5 & 2 \\ -0{,}2 & 0{,}5 & -3 \end{pmatrix} - \begin{pmatrix} 0{,}34 & -2{,}93 & 0 \\ 0{,}34 & -2{,}93 & 0 \\ 0{,}34 & -2{,}93 & 0 \end{pmatrix}$$

$$\Rightarrow \boxed{w \leftarrow \begin{pmatrix} -0{,}24 & 3{,}1 & -0{,}3 \\ -0{,}94 & 2{,}4 & 2 \\ -0{,}54 & 3{,}4 & -3 \end{pmatrix}}$$

$$b \leftarrow b - \frac{\partial C}{\partial b}$$

$$b \leftarrow (0{,}1 \quad 0{,}1 \quad 0{,}1) - (0{,}34 \quad -0{,}97 \quad 0{,}63)$$

$$\Rightarrow \boxed{b \leftarrow (-0{,}24;\ 1{,}07;\ -0{,}53)}$$

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