Hopfield/Hopular Neural Networks in Clash

Rad Tudor-Andrei Radu Tudor-Eduard

Serie: CTIen | Grupa: 30432

1 Contents

2 Clash Syntax & Data Structures

Clash is a functional hardware description language based on Haskell. Key architectural constraints include:

- **Static Sizing:** All bit-widths and vector lengths must be known at compile time.
- **Time Modeling:** Time is modeled via the `Signal` keyword, representing values received at every clock cycle (translated as wires).
- **Explicit Clocks:** Clash does not assume a default clock; it must be explicitly defined.

2.1 Signal Type and Lifting

A `Signal dom a` represents a stream of values. Operations on signals must be **lifted**:

- **fmap ($<\$>$):** Applies a function at every clock cycle (combinational logic).
- **Applicative ($<*>$):** Lifting in the case of multiple signals.

Operation	Hardware Meaning
Lifted ($<\$>$, $<*>$)	Combinational Logic
`register`	Sequential Logic (Flip-Flops)

2.2 Standard Primitives

- **Mux:** `mux cond a b` (standard multiplexer).
- **Bundling:** `bundle` and `unbundle` for converting between tuples and signals.
- **Vectors:** `Vec n a` provides fixed-size, synthesizable arrays.
- **State Machines:** `mealy` and `moore` functions define stateful logic.

3 Clash Modus Operandi

The transformation from Haskell code to Hardware Description Language (HDL) follows these steps:

1. **Parsing:** GHC parses Haskell + Clash primitives.
2. **Normalization:** Functions are inlined, recursion is eliminated, and higher-order functions are resolved.
3. **Clash Core:** The final IR before hardware interpretation.
4. **Netlist Generation:** A graph of wires, registers, and combinational blocks is created.
5. **HDL Emission:** VHDL/Verilog code is generated.

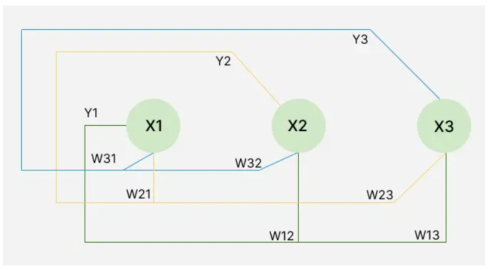
4 Hopfield Neural Network

A Hopfield network is a recurrent, fully connected neural network used for **Auto-Associative Memory**.

4.1 Architecture

The network consists of N neurons, connected to every other neuron with the following constraints:

- No self-connections: $w_{ii} = 0$
- Symmetric weights: $w_{ij} = w_{ji}$



For a specific neuron i, the local field h_i and activation x_i are defined as:

$$h_i = \sum w_{ij} x_j - \theta_i \tag{1}$$

$$x_i^{(t+1)} = \text{sign}(h_i) \tag{2}$$

4.2 Energy Function

The network minimizes a **Lyapunov function** to reach a stable state:

$$E = -\frac{1}{2}\sum w_{ij}s_is_j - \sum I_is_i \tag{3}$$

Here, s_i is the state of the neuron and I_i is the external input. Updates are performed asynchronously to ensure stability.

4.3 Training Steps

1. Initialize the weights w_{ij} to store the desired patterns.
2. For each input vector $\mathbf{x}$, initialize the activations of the network as:
$$y_i = x_i, \quad \text{for } i = 1, 2, \ldots, n.$$
3. Compute the total input to each neuron by combining the external input and the weighted feedback from all other neurons:
$$y_i^{\text{in}} = x_i + \sum_j y_j w_{ji}.$$
4. Apply the activation function to each output:
$$y_i = \begin{cases} 1, & \text{if } y_i^{\text{in}} > \theta_i, \\ y_i, & \text{if } y_i^{\text{in}} = \theta_i, \\ 0, & \text{if } y_i^{\text{in}} < \theta_i, \end{cases}$$
where the threshold θ_i is typically set to 0.
5. Feed back the updated outputs y_i to all neurons in the network and update the activation vector accordingly.
6. Repeat the computation of the total input and activation update until the network converges (the activation vector remains unchanged between successive iterations).

Due to its architecture, it has a maximum number of patterns to store:

$$P \approx 0.138N, \quad N \to \text{number of nodes}$$

4.4 Behavior example

In case of a small network with 3 binary neurons, $\{-1, +1\}$: Stored pattern:

$$\xi = \begin{bmatrix} +1 \\ -1 \\ +1 \end{bmatrix}$$

For each weight:

$$w_{ij} = \frac{1}{N}\xi_i\xi_j, i \neq j$$
$$w_{ii} = 0$$

Compute the weight matrix:

$$W = \frac{1}{3}\begin{bmatrix} 0 & -1 & +1 \\ -1 & 0 & -1 \\ +1 & -1 & 0 \end{bmatrix}$$

Give a noisy input, with one bit wrong:

$$y^{(0)} = \begin{bmatrix} +1 \\ +1 \\ +1 \end{bmatrix}$$

Apply (1) to update the neurons, with a θ_i of 0.

$$y_i = sign(\sum_j w_{ij} y_j)$$

For neuron #1:

$$h_1 = \frac{1}{3}[0 \cdot 1 + (-1) \cdot 1 + 1 \cdot 1] = 0$$

$$y_i = +1 \text{ } (h_i \text{ equal to } \theta_i)$$

For neuron #2:

$$h_2 = \frac{1}{3}[(-1) \cdot 1 + 0 \cdot 1 + (-1) \cdot 1] = -\frac{2}{3}$$

$$y_2 = -1$$

For neuron #3:

$$h_3 = \frac{1}{3}[1 \cdot 1 + (-1) \cdot (-1) + 0 \cdot 1] = \frac{2}{3}$$

$$y_3 = +1$$

If we update again, all neurons keep their values. Thus, the pattern is stable, and the network's energy converged.

5 Modern Hopfield Networks

Modern Hopfield Networks introduce non-linearity, either in the energy function or neurons' activation function, leading to super-linear (even exponential, depending on the chosen activation function) scaling of the memory capacity as a function of the number of neurons.

6 Hopular Networks

6.1 Introduction

Hopular is a modern **Hop**field network-based model for tab**ular** data, able to retain associative memory properties. It competes with other tabular ML models, such as Gradient-Boosted Decision Trees (e.g. XGBoost) and MLPs.

6.2 Performance Comparison

The model exceeds in small-medium ($\lesssim$ 10000 samples) sized datasets. This is due to the network being able to identify local neighborhoods in the data, which is useful for regression tasks when the underlying function is hard to determine without large amounts of data.

The drawback of this recurrent model is the *iterative refinement* step: re-accessing the entire training set at every layer. The performance of this network comes at the cost of refining the prediction in an iterative manner.

Table 1: Performance and computational comparison on tabular data

Model	**Median Rank**	**Accuracy**	**Training**	**Inference**
Hopular	≈ 1.5	Highest	High	Medium–High
CatBoost	≈ 2.0	Very High	Medium	Low
LightGBM	≈ 2.2	Very High	Medium	Low
XGBoost	≈ 4.5	High	Medium–High	Low
MLP	≈ 5.0	Medium	Low	Low

6.3 Hopfield vs. Hopular

Before commencing with the implementation in Haskell/Clash, it may be helpful to present a side-by-side comparison of the original Hopfield network and the advanced version, Hopular network:

Table 2: Comparison of architecture and behavior of the two networks

Feature	**Hopfield**	**Hopular**
States	Binary states: $\{-1, 1\}$	Continuous states (floats)
Update	Hard update - sign()	Smooth non-linearity (softmax/exp)
Energy function	Quadratic energy	Log-sum-exp energy*
Use case	Associative memory	Supervised learning for tabular data
Scalability	Small networks	Scales to modern datasets
Memory capacity	$O(N)$ patterns	Exponential in dimension**

* Hopfield networks use a log-sum-exp energy function:

$$E(x) = -\frac{1}{\beta} \log \sum_{i=1}^{P} e^{\beta x^\top M_i}, \quad \text{where } \beta \text{ controls sharpness, } x \in \mathbb{R}^d,\ M_i \in \mathbb{R}^d$$

** For d = dimensionality of the state vector, and N = number of neurons, there is:

$$P_{max} \approx 0.138N \text{ for Hopfield, and } P_{max} = O(e^d) \text{ for Hopular}$$

7 Implementation Overview

7.1 Data Preparation and Pattern Encoding

7.1.1 Dataset Description

The Hopular network was evaluated on a tabular dataset containing 13 input features and a binary target class. The features include both numerical attributes (e.g., CreditScore, Age, Balance) and categorical attributes (e.g., Geography, Gender).

Since the Clash hardware implementation requires statically sized, fixed-width vectors, the dataset was preprocessed in Python before being embedded into the hardware description.

7.1.2 Preprocessing Pipeline

The preprocessing steps were as follows:

1. **One-Hot Encoding:** Categorical features (Geography, Gender) were transformed into binary columns using one-hot encoding.
2. **Train/Test Split:** The dataset was split using stratified sampling to preserve class distribution.
3. **Standardization:** Continuous features were standardized using:
$$x' = \frac{x - \mu}{\sigma}$$
where μ and σ were computed only on the training set.
4. **Prototype Computation:** For each class $c \in \{0, 1\}$, a prototype vector was computed:
$$p_c = \frac{1}{N_c} \sum_{i \in \mathcal{D}_c} x_i$$
These vectors represent the class centroids in feature space.
5. **Normalization:** Each prototype was normalized to unit norm:
$$p_c \leftarrow \frac{p_c}{\|p_c\|}$$
6. **Fixed-Point Quantization:** Since Clash does not operate on floating-point values, the vectors were scaled by a factor of 256 and converted to 16-bit signed integers:
$$p^{(q)} = \mathrm{round}(256 \cdot p)$$

7.1.3 Rationale for Fixed-Point Representation

Floating-point arithmetic significantly increases hardware complexity and resource utilization.

Instead, a fixed-point representation was used:

- **State Width:** 16-bit signed integers
- **Intermediate Products:** 32-bit signed integers
- **Scaling Factor:** 256

This ensures:

- Sufficient dynamic range after z-score normalization
- Safe accumulation during dot products

7.1.4 Pattern Storage

The final stored patterns correspond to the two class prototypes:

$$\texttt{Patterns} = \{p_0^{(q)}, p_1^{(q)}\}$$

These vectors are embedded directly into the hardware as constant ROM-like structures.

7.1.5 Inference Input Encoding

Test samples were processed identically:

1. Standardized using training statistics
2. Scaled by 256
3. Converted to signed integers

The resulting vector becomes the initial state of the Hopular network:

$$x^{(0)} = x_{\text{test}}^{(q)}$$

The iterative refinement mechanism then projects this noisy vector toward the closest stored prototype.

7.2 Top Entity

The implementation of the Hopular Network starts off with the *TopEntity*, a component similar to the *main()* function of a computer program. The input will be a noisy row and the output either the closest pattern stored or a spurious state.

```
Project.hs
    topEntity ::
        Clock Dom50 ->
        Reset Dom50 ->
        Enable Dom50 ->
        Signal Dom50 State ->
        Signal Dom50 State
    topEntity =
        exposeClockResetEnable hopularCore
```

7.3 Hopular Functions

hopularStep performs attention-based update by computing dot-product similarities between current state and all stored patterns. It uses scaling and exponentiation to obtain softmax weights. It, then, computes the new neuron values using those weights in a weighted sum. Thus, it brings the current state to the most similar stored pattern.

```
    hopularStep :: Patterns -> State -> State
    hopularStep pats x =
        let scores = map (‘dotWide‘ x) pats
            scoresScaled = map (clampScore . (‘shiftR‘ softmaxShift)) scores
            exps = map expApprox scoresScaled
            sumExp = sum exps
            col i = map (\p -> p !! i) pats
            combine i =
                let ws = zipWith
                                (\e p -> (resize e :: AccWide) * (resize p :: AccWide))
```

```
                            exps
                            (col i)
                    wsum = sum ws
                    out = safeDiv wsum sumExp
                in resize out :: Acc
        in imap (\i _ -> combine i) x
```

In short,

$$x^{`} = \sum_{i} softmax(p_i \cdot x) \cdot p_i$$

hopularCore wraps hopularStep into a Mealy machine that updates the network's state on each clock cycle. When a new input arrives, the internal state and iteration counter are reset. The iteration happens sequentially until convergence or the maximum number of iterations is reached.

```
Hopular.hs
    hopularCore ::
        HiddenClockResetEnable Dom50 =>
        Signal Dom50 State ->
        Signal Dom50 State
    hopularCore x0 =
        mealy step initState x0
        where
            maxIter :: Iter
            maxIter = 8
            initState = (repeat 0, 0 :: Iter)
            step (st, it) x =
                if x /= st then
                    ((x, 0), x)
                else
                    let st' = hopularStep patterns st
                        done = st' == st || it == maxIter
                        it' = if done then it else it + 1
                    in ((st', it'), st')
```

7.4 Types

Since the data had thirteen columns, the dimension Dim has been set to 13. Although the initial data was quite large in values, after preprocessing (z-factored and scaled by 256), a width of 16 bits (-32,768 to 32,767) for the Acc was sufficient.

```
    type Dim = 13
    type Acc = Signed 16
    type AccWide = Signed 32
    type State = Vec Dim Acc
    type NumPats = 2
    type Patterns = Vec NumPats State
    type Iter = Unsigned 4
```

7.5 Utility functions

7.5.1 Dot product

The dot function was used for similarity between two state vectors using 16-bit & 32-bit arithmetic (dotWide, safer for larger intermediate sums). In the context of Hopfield update, the dot products represent attention scores between input and stored patterns.

```
dot :: State -> State -> Acc
dot a b = sum (zipWith (*) a b)
```

7.5.2 Exponential computation

For the softmax α_i value computation, a look-up table of $e^x, x \in [-8, 8]$ values was used:

$$\text{expLUT}[i] \approx 256 \cdot e^{(i-8)/2}$$

Instead of computing e^x, the implementation approximates $e^{x/2}$, reducing the growth rate.

```
expLUT :: Vec 17 AccWide
expLUT =
    5   :> 8   :> 13 :> 21 :> 35 :> 57 :> 94 :> 155 :>
    256 :> 422 :> 696 :> 1147 :> 1892 :> 3119 :> 5142 :> 8478 :>
    13977 :> Nil

expApprox :: AccWide -> AccWide
expApprox s =
    let idx = fromIntegral (s + 8) :: Unsigned 5
    in expLUT !! idx
```

7.5.3 Error metrics

The sqError function computes SSE between two states, corresponding to the squared Euclidean distance. Provides a magnitude-sensitive measure of the deviation.

```
sqError :: State -> State -> AccWide
sqError y t =
    sum (map (\d -> let w = resize d :: AccWide in w*w) (zipWith (-) y t))
```

The linfDist function, or the Chebyshev distance, L∞ metric, captures the worst-case deviation across all neurons.

```
linfDist :: State -> State -> Acc
linfDist y t =
    foldl max 0 (map abs (zipWith (-) y t))
```

8 Simulation Results

Example of pattern stored in the Hopular network:

```
ghci> p0
384 :> -128 :> 64 :> -256 :> 512 :> 128 :> -64 :> 0 :> 256 :> -192 :> 96 :> 320 :>
-32 :> Nil
```

Noisy vector derived from previous pattern example:

```
ghci> x0
350 :> -120 :> 80 :> -220 :> 500 :> 100 :> -40 :> 10 :> 240 :> -180 :> 120 :> 300
:> -20 :> Nil
```

Closest row given by the Hopular network:

```
ghci> let y1 = hopularStep patterns x0
ghci> y1
383 :> -128 :> 63 :> -256 :> 511 :> 127 :> -64 :> -1 :> 255 :> -192 :> 95 :> 319
:> -32 :> Nil
```

The small SSE and the maximum deviation of one unit indicate only minor errors.

```
ghci> sqError y1 p0
8
ghci> linfDist y1 p0
1
```