Complex impedance of L, C and R

As a start we derive the formulas for the complex impedance for a capacitor, an inductor and a resistor. Subsequently, this is applied in simple passive analogue filters using these devices.

We consider an (harmonic) electric current $I(t) = I_0 \cos(\omega t)$ with I_0 the current amplitude and $\omega = 2\pi f$ the radial frequency (and f the frequency in Hz). In complex notation the current reads

$$I(t) = I_0 e^{j\omega t} \tag{1}$$

For an inductor, see figure 1, the relation between the voltage V across the inductor and the current through it is given by

$$V(t) = L\frac{dI(t)}{dt} \tag{2}$$

with L the coefficient of self-induction of the inductor.

Figure 1: Inductor symbol.

Substitution of the equation for the current, equation (1), into equation (2) yields

$$V(t) = L\, j\omega I_0 e^{j\omega t} \quad .$$

Using $j = e^{\frac{1}{2}\pi j}$ this can be written as

$$V(t) = \omega L I_0 e^{j\left(\omega t + \frac{1}{2}\pi\right)} = V_0 e^{j\left(\omega t + \frac{1}{2}\pi\right)} \tag{3}$$

where we have introduced the voltage amplitude $V_0 = \omega L I_0$.

Going back to non-complex notation, this would read $V(t) = V_0 \cos\left(\omega t + \frac{1}{2}\pi\right)$, i.e. there is a phase shift of 90° between the current and the voltage across an inductor.

Now, the complex impedance of an inductor is defined as $Z_L = \frac{V}{I} = \frac{j\omega L\, I_0 e^{j\omega t}}{I_0 e^{j\omega t}}$, i.e.

$$Z_L = j\omega L \tag{4}$$

The inductive reactance is given as $X_L = |Z_L| = \omega L$ and has units Ohm.

Note: The same result can be obtained by going to the frequency domain using the Fourier transform, which is defined for signal $x(t)$ as

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{5a}$$

The inverse Fourier transform reads

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{-j\omega t} d\omega \tag{5b}$$

We rewrite equation (2) as

$$v(t) = L \frac{di(t)}{dt} \tag{6}$$

now using lowercase symbols to indicate that we are in the time domain. We use the property that if X is the Fourier transform of x then $j\omega X$ is the Fourier transform of the derivative $x' = \frac{dx}{dt}$. Hence, Fourier transforming equation (6) yields $V(\omega) = j\omega L \, I(\omega)$ with V and I the Fourier transform of v and i, respectively. Hence

$$Z_L = \frac{V(\omega)}{I(\omega)} = j\omega L \tag{7}$$

For a capacitor, see figure 2, the relation between the voltage V across the capacitor and the accumulated charge Q is given by

$$V = \frac{Q}{C} \tag{8}$$

with C the capacitance of the capacitor. We have $I(t) = \frac{dQ(t)}{dt} = I_0 \cos(\omega t)$ or in complex notation $I(t) = I_0 \, e^{j\omega t}$.

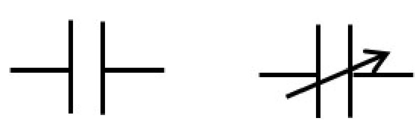

Figure 2: Capacitor symbols (left: fixed, right: variable).

Now

$$V = \frac{Q}{C} = \frac{1}{C}\int I(t)\,dt = \frac{1}{C}\int I_0\, e^{j\omega t} dt = \frac{I_0}{j\omega C} e^{j\omega t} = \frac{I}{j\omega C}.$$

Hence, the complex impedance of a capacitor, being the ratio of V and I, becomes

$$\boxed{Z_C = \frac{1}{j\omega C}} \tag{9}$$

The capacitive reactance is given as $X_C = |Z_C| = \frac{1}{\omega C}$ and has units Ohm.

Using $\frac{1}{j} = e^{-\frac{1}{2}\pi j}$ we see that $V(t) = V_0 \cos\left(\omega t - \frac{1}{2}\pi\right)$ with voltage amplitude $V_0 = \frac{I_0}{\omega C}$.

Finally, as for a resistor $V = IR$ (Ohm's law), the complex impedance of a resistor is simply

$$\boxed{Z_R = R} \tag{10}$$

with R the resistance of the resistor (which is of course real-valued).

We will now consider the *LRC* series circuit depicted in figure 3. The current at all points in the circuit is the same and assumed to be given as $I(t) = I_0 \cos(\omega t)$ (or $I(t) = I_0\, e^{j\omega t}$ in complex notation). With the expressions for the complex impedances derived above, the expression for the (AC) voltage V is easily found. The total impedance of the circuit is given by

$$\boxed{Z = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right)} \tag{11}$$

Hence, the voltage reads

$$V = ZI = \left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right] I_0\, e^{j\omega t}$$

which can be written as

$$V = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\; e^{j\phi}\, I_0\, e^{j\omega t} = V_0\, e^{j(\omega t + \phi)}$$

with voltage amplitude $V_0 = I_0 \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$ and phase angle $\phi = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$.

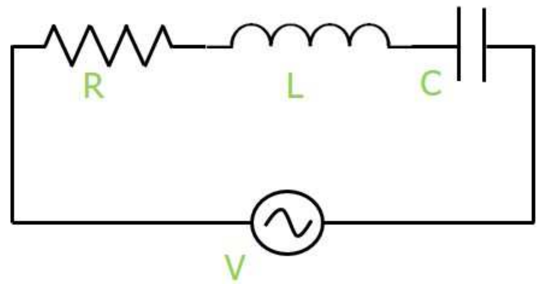

Figure 3: *RLC* series circuit.

The peak current in the *LRC* series circuit is given by

$$I_0 = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \tag{12}$$

which is maximum when $\omega L - \frac{1}{\omega C} = 0$ or $\omega_0 = \frac{1}{\sqrt{LC}}$ at which frequency the circuit is in resonance, see figure 4 where we plotted $\frac{I_0}{V_0}$ versus frequency for two values of R.

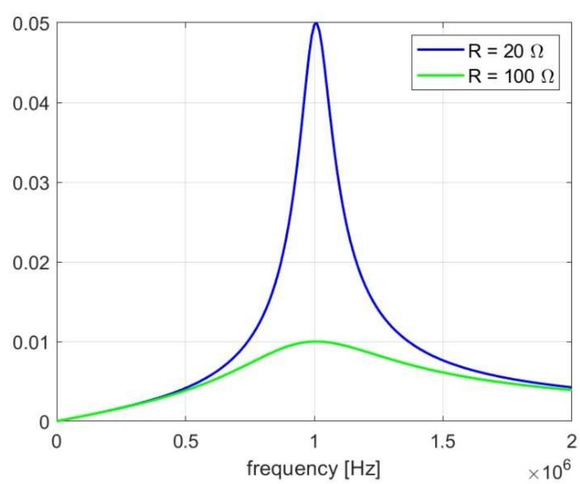

Figure 4: Resonance in an LRC series circuit for *L* = 25 µH and *C* = 1 nF (f_0 = 1 MHz)

As a second example we consider the *LRC* parallel circuit of figure 5. Given the voltage $V(t) = V_0 \cos(\omega t)$ (or $V(t) = V_0\, e^{j\omega t}$ in complex notation), what is the expression for total current leaving the source?

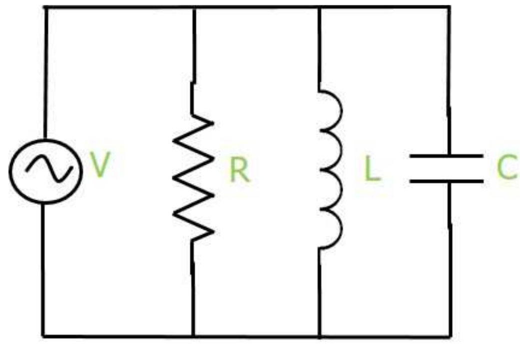

Figure 5: *RLC* parallel circuit.

Now, the total impedance Z of the circuit is determined by

$$\frac{1}{Z}=\frac{1}{Z_R}+\frac{1}{Z_L}+\frac{1}{Z_C}=\frac{1}{R}+\frac{1}{j\omega L}+j\omega C \tag{13}$$

Hence, the current is

$$I=\frac{V}{Z}=V\left(\frac{1}{R}+\frac{1}{j\omega L}+j\omega C\right)=V\left[\frac{1}{R}+j\left(\omega C-\frac{1}{\omega L}\right)\right]$$

which can be written as

$$I=V_0\,e^{j\omega t}\sqrt{\frac{1}{R^2}+\left(\omega C-\frac{1}{\omega L}\right)^2}\;e^{j\phi}=I_0\,e^{j(\omega t+\phi)}$$

with current amplitude $I_0=V_0\sqrt{\frac{1}{R^2}+\left(\omega C-\frac{1}{\omega L}\right)^2}$ and phase angle $\phi=\tan^{-1}\left[R\left(\omega C-\frac{1}{\omega L}\right)\right]$.

Passive analog filters

Passive low-pass, high-pass, band-pass and band-stop filters consisting of combinations of resistors, capacitors and inductors can be assessed by the complex impedance method developed in the previous section. In addition, the formula for a voltage divider using two resistors, see figure 6, is needed and given by

$$\frac{v_{out}}{v_{in}}=\frac{R_2}{R_1+R_2}\,. \tag{14a}$$

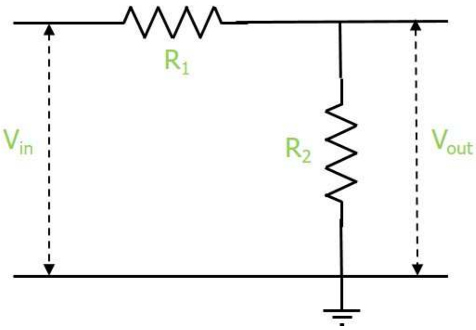

Figure 6: A simple voltage divider using two resistors *R₁* and *R₂* (which may be replaced by complex impedances *Z₁* and *Z₂*, respectively).

When the resistors R_1 and R_2 are replaced by the complex impedances Z_1 and Z_2, respectively, then the relation between output voltage v_{out} and input voltage v_{in} is still given by

$$\frac{v_{out}}{v_{in}} = \frac{Z_2}{Z_1 + Z_2}. \qquad (14b)$$

The ratio $\frac{v_{out}}{v_{in}}$ is called the transmission function $H(j\omega)$ of the filter.

As a first example we consider the filter depicted in figure 7. The transmission function or voltage amplification of this simple low-pass filter is given as

$$H(j\omega) = \frac{v_{out}}{v_{in}} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}. \qquad (15)$$

A so-called Bode diagram of $H(j\omega)$ is given in figure 8 for R = 850 Ω and C = 1 µF. A Bode diagram shows plots of the absolute value (in dB) and phase (in degrees) of $H(j\omega)$ as a function of frequency (using a logarithmic frequency axis). The roll-off in the stop band of the filter is 6 dB/octave (1 octave being a factor 2 in frequency) or 20 dB/decade. In this case the cut-off frequency is $f_c = \frac{1}{2\pi RC}$ = 187 Hz at which $|H(j\omega)| = \frac{1}{\sqrt{2}}$ (= - 3 dB).

<u>Note</u>: A filter is fully characterized by its transmission function, which gives its behavior in the frequency domain. A filter is also fully determined by its impulse response $h(t)$ being the response of the filter to a Dirac delta function, i.e. $v_{in}(t) = \delta(t)$. $h(t)$ is the inverse Fourier transform of $H(j\omega)$.

In this case $h(t) = \frac{1}{RC} e^{-\frac{t}{RC}}, \quad t \geq 0$.

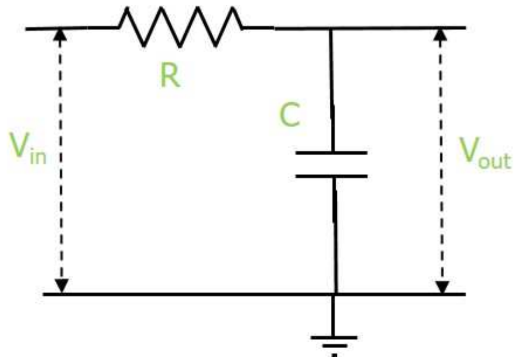

Figure 7: The simplest low-pass filter.

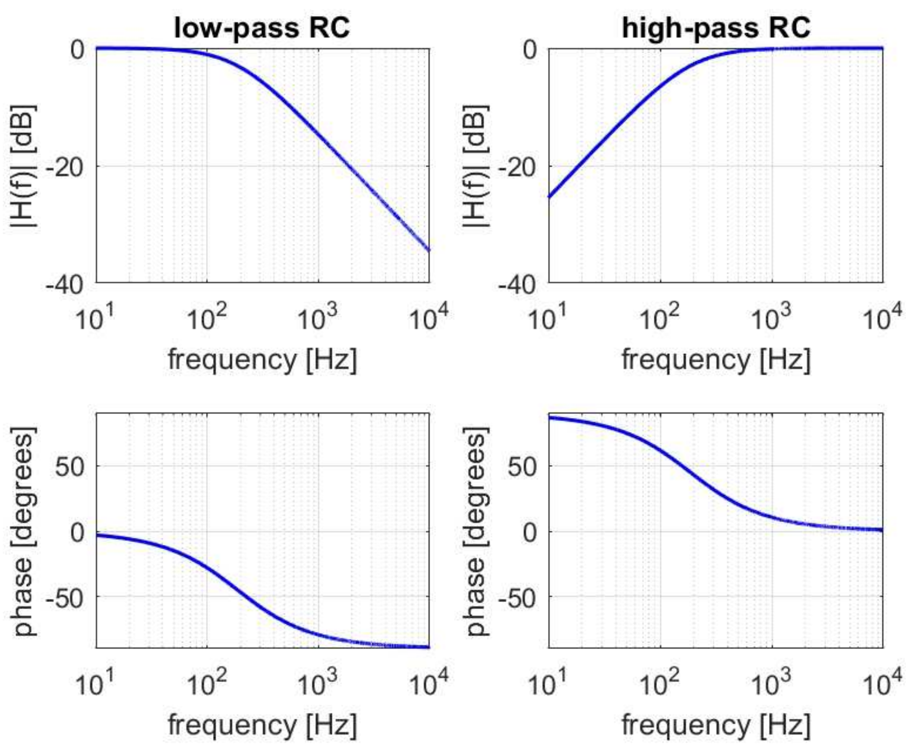

Figure 8: Bode plots of the low-pass RC filter (left) and high-pass RC filter (right).

Figure 9 shows the simplest high-pass filter, the transmission function of which is

$$H(j\omega) = \frac{v_{out}}{v_{in}} = \frac{R}{\dfrac{1}{j\omega C} + R} = \frac{j\omega RC}{1 + j\omega RC}. \tag{16}$$

The corresponding Bode plot is shown in figure 8, again for R = 850 Ω and C = 1 μF.

Note: The transmission function of this filter can be rewritten as $H(j\omega)=1-\frac{1}{1+j\omega RC}$. The inverse Fourier transform of this, i.e. the impulse response, is $h(t)=\delta(t)-\frac{1}{RC}e^{-\frac{t}{RC}}, \quad t\geq 0$.

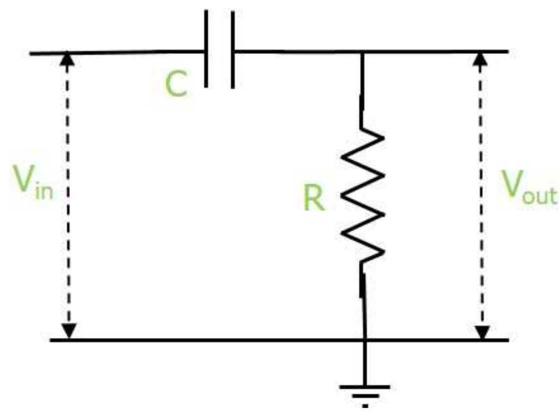

Figure 9: the simplest high-pass filter.

A simple band-pass filter can be made by adding an inductor parallel to the capacitor in the low-pass filter of figure 7, see figure 10. The transmission function of this filter is derived as follows. The impedances Z_1 and Z_2 (of equation 14b) are given by $Z_1 = R$ and $\frac{1}{Z_2}=\frac{1}{j\omega L}+j\omega C = j\left(\omega C-\frac{1}{\omega L}\right)$, i.e. the transmission function $H(j\omega)$ is thus $\frac{Z_2}{Z_1+Z_2}=\frac{\frac{1}{j\left(\omega C-\frac{1}{\omega L}\right)}}{R+j\left(\omega C-\frac{1}{\omega L}\right)}$. Hence,

$$H(j\omega)=\frac{1}{1+jR\left(\omega C-\frac{1}{\omega L}\right)}. \tag{17}$$

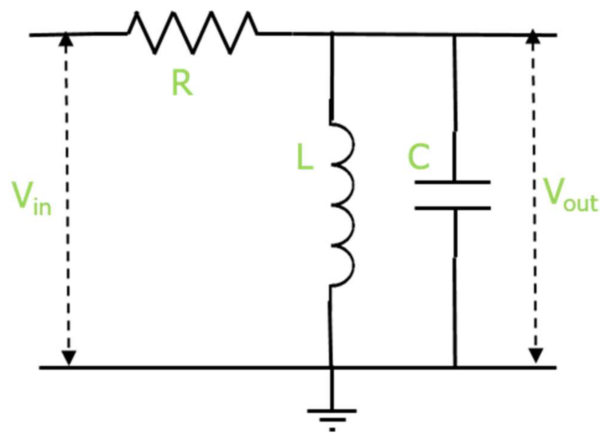

Figure 10: The simplest band-pass LRC filter.

The simplest band-stop filter using an inductor is shown in figure 11, the transmission function of which is

$$H(j\omega)=\frac{j\left(\omega L-\frac{1}{\omega C}\right)}{R+j\left(\omega L-\frac{1}{\omega C}\right)} \tag{18}$$

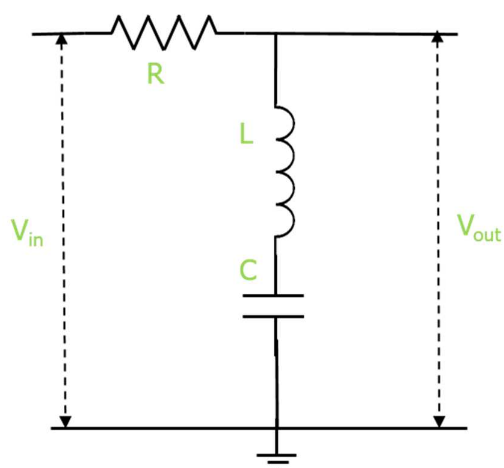

Figure 11: The simplest band-stop LRC filter.

Taking C = 230 pF and L = 110 µH, i.e. resonance frequency $f_0=\frac{1}{2\pi\sqrt{LC}}$ = 1 MHz, we have plotted $|H(j\omega)|$ and the phase of $H(j\omega)$ as function of frequency (on a linear frequency axis) in figure 12. The resistance R for the band-pass filter and the band-stop filter is chosen as 10 kΩ and 100 Ω, respectively.

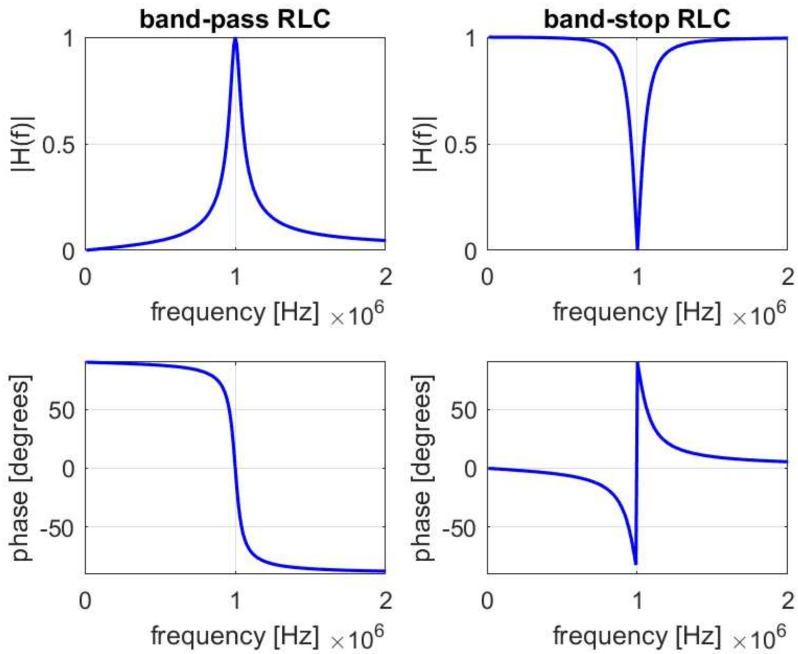

Figure 12: Plots of $H(j\omega)$ (absolute value and phase) for the RLC band-pass (left) and the RLC band-stop filter (right). The same values for L and C are taken (i.e. same resonance frequency being 1 MHz). For the band-pass filter R = 10 kΩ and for the band-stop filter R = 100 Ω.