Cairo University
Faculty of Engineering
Computer Engineering Department

2018

Game theory and Lights-Out Game

Submitted to:

Dr. Samah Elshafiey Eltantawy

Minders

Authors:

Radwa Samy Khattab

Ayman Shawky Azzam

Ahmed Essam Eldeen

Abdelrahman Mohamed Badr

Abdullah Ezzat Abdelsataar

Ahmed Mohamed Zakaria

Abstract

This research focuses on Game theory and its applications in various fields. Our main concern is a game called Lights-Out game, that can be solved using Game theory, or in other words, using a mathematical model having matrices. Our goal in this research is to reach a better mathematical model than the one already used, one with less cost—faster output. This was achieved after lots of researching through books, papers and articles, and then followed by some experimenting on our software and coding, to make sure our algorithm works just fine, getting the least number of moves to solve the game, with least cost.

Table of Contents

List of Figures

I. Introduction

In our daily life, hanging with our friends, or doing whatever, we don't ever think about the math behind our decisions. Actually, we don't even realize that decisions have some mathematics and science behind them. That science is what is called Game theory.

As described in *Game theory: Analysis of Conflict*, "*Game theory is the study of mathematical models of strategic interaction between rational decision-makers*."

In this study, games aren't what we think it is; the entertainment games. No, a game is any interaction between people, where each player's outcome depends on the decisions made by other players. Which means, this concept can be applied in many situations, serious ones as well as those for the sake of entertainment. So, you can practically apply it in any decisions you are making through your daily-life, that is if, of course, you knew exactly how to model the situation into on of the models in Game theory.

Game theory was first introduced by the beautiful mind, John Nash, in 1950s. And he also defined what is known as Nash equilibria. Soon afterwards, the study started to develop more rapidly. It was used in many fields, including war and military, that it was used in World War II.

Game theory has basic two types; cooperative, and non-cooperative.

In cooperative games, players (called coalition in cooperative games) want to reach one common goal, and there is no losers and winners among them. This may vary in situations from group of friends splitting restaurant bill among them fairly, to a coalition of nations trying to figure out how to reduce climate change. In such games, Game theory tells you how much each player should contribute to the coalition, and hence how much they should benefit from it. That is to reach what is known as Shapley value; a method of dividing up gains or costs among players according to the value of their individual contributions.

In non-cooperative games, players are divided into winners and losers, and hence there is a competitive social interaction among them. In such games, Game theory basically tells you how to analyze your data to choose a course of actions that will benefit you the most, no matter what everyone else chooses to do. Nash equilibria was found most beneficial in non-cooperative games.

At the beginning, Game theory's main concern was zero-sum games, where only one player can win and others must lose in such case. However, today Game theory is much wider than that. It considers mixed-strategies, as well.

After some development in Game theory, now we look at variety of types of games and situations that can be studied using Game theory and Evolutionary Game theory. They can be represented in various methods. There is Extensive form, Normal form and Characteristic form.

From the many games studied using Game theory and solved through mathematical models, one game we considered in details in this research is Lights-Out game. In a nutshell, this game consists of 5 x 5 grid of lights, all lit at first, and we want to turn them off. There are some restricted rules on what happens when you turn-on or turn-off a lid, which makes the game much harder than it seems. However, with the right mathematical model, nothing is really that hard.

II. History

The first appearance of Game theory and its analysis was way before the recognition of the mathematical models and the modern explanation of it. That actually happened when Charles Waldegrave was trying to provide a minimax mixed strategy to solve a two-person card game, known as le Her, in the 17 hundred. This game is still known up till now as Waldegrave problem.

In the 1830s, the first game-theoretic analysis was made by James Madison when studying how states would behave under different systems of taxation.

Game theory started to develop more after that, especially with the beginning of the 20th century.

A. Emile Borel (the father of Game theory)

Emile Borel published several papers on Game theory in 1921. In one of them, he used poker as an example for more illustration and addressed the concept of bluffing. In another paper, he used to guess the opponent in a game of imperfect information. Borel is the first professor to put the Game theory in economic and military applications.

B. John Von Neumann

John Von Neumann was born in 1903. He published several papers in his lifetime. His first paper was on Game theory, called Theory of Parlor Games. It was published in 1928, and he was only eighteen by then. By his mid-twenties, John Neumann was known as a young mathematical genius, that his fame had spread worldwide in the academic community. John Neumann loved games and toys, which probably caused his great contribution to Game theory development.

1. John von Neumann in the 1940s

For John Neumann, the inspiration for Game theory was poker game, which he played occasionally and pretty well. John Neumann realized that poker was not guided by probability theory only, as any unfortunate player would think, but is guided by decision-making, which is what concerns Game theory.

In his 1928 paper, Theory of Parlor Games, John Neumann first approached the discussion of Game theory, and proved the famous minimax mixed strategy. From the outset, John Neumann knew that Game theory would prove itself invaluable to economists. He collaborates with Oskar Morgenstern, an Austrian economist at Princeton, to develop his theory. Their book, *Theory of Games and Economic Behavior*, revolutionized the field of economic science. Although the work itself was dedicated only to economists, its applications to psychology, sociology, politics, warfare, recreational games, and many other fields soon became apparent.

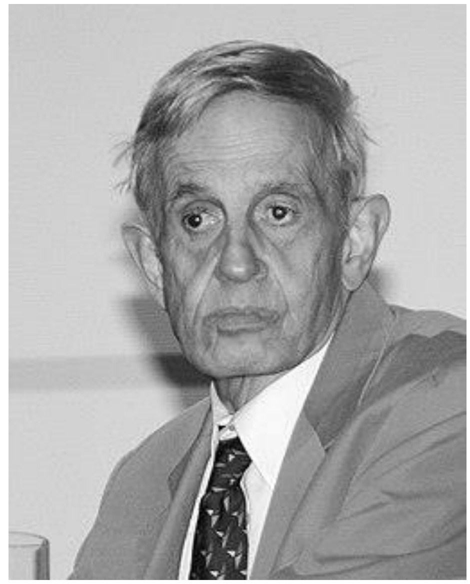
Although John Von Neumann appreciated Game theory's applications to economics, he was most interested in applying his methods to politics and warfare. He used his methods to model the Cold War interaction between the U.S.A. and the USSR, viewing them as two players in a zero-sum game. From the very beginning of World War II, John Neumann was confident of the Allies' victory. He built a mathematical model of the conflict from which he deduced that the Allies would win, by applying some of the methods of Game theory analysis to his predictions.

Even afterwards, John Neumann used his theories and mathematical models in more ways than anyone could have thought game theory might be helpful in. Unfortunately, he used his brilliance in warfare more often.

John Neumann also came up with ideas for a better computer, using his mathematical models to improve the computer's logic design. After the war, he worked on a machine, which he claimed would be able to accurately predict weather patterns and capable of 2000 operations a second. Even though the computer did not predict weather very well, it became quite useful doing a set of calculations necessary for the design of the hydrogen bomb.

John Neumann took credit for the idea of basing computer calculations to binary numbers, having programs stored in computer's memory in coded form instead of punch cards, and several other important developments, that articles wrote, "*From here the idea of computer started*".

C. John Forbes Nash Jr. (The Beautiful Mind)



2. *John Nash in November 2006 at a game theory conference in Cologne, Germany*

While John Neumann was publishing his first paper in the Game theory, another beautiful mind, John Nash, was born in 1928. A movie telling his life story was made, called “A Beautiful Mind”.

In his last days, John Nash had said “Game theory is useful in that way you can study a traffic network”. John Nash studied in MIT, got a Nobel Prize in Economic sciences in 1994.

Even though he suffered from schizophrenia disease, he managed to put his fingerprint in history by defining *Nash Equilibrium*. He died in an accident in 2015.

D. Nash Equilibrium

If in any game, all players have chosen a decision, and no player can benefit by changing their decisions for as long as the other players keep their decisions unchanged, then at such case, the available choices and their payoff hold a Nash equilibrium.

For further illustration, say Azzam and Akwah are in Nash equilibrium. If Azzam is making the best decision he can, taking into account Akwah's decision while Akwah's decision remains unchanged, and Akwah is making the best decision he can, taking into account Azzam's decision while Azzam's decision remains unchanged, then the two players are in Nash equilibrium state. Likewise, a group of players are in Nash equilibrium if each one is making the best decision possible, taking into account the decisions of the others in the game while the other players' decisions still as is.

Nash equilibrium provides a way of predicting what will happen if several people or several institutions are making decisions at the same time, and if the outcome for each of them depends on the decisions of the others. The simple insight underlying John Nash's idea is that one can't predict the result of the choices of multiple decisions of players if one analyzes those decisions in isolation. Each player must ask what other players would do, taking into account the decision-making of the others.

Nash equilibrium has been used to study what extents people with different preferences can cooperate, and whether they will take risks to achieve a cooperative outcome (as in stag hunt). It has also been used to study the adoption of technical standards, and also the occurrence of bank runs and currency crises (as in coordination game). It has also been used to analyze hostile situations like war and arms races (as in prisoner's dilemma), and also in how conflict may be mitigated by repeated interaction (as in tit-for-tat). Other applications include traffic flow (as in Wardrop's principle), how to organize auctions (as in auction theory), the outcome of efforts exerted by multiple parties in the education process, regulatory legislation such as environmental regulations (as in tragedy of the Commons), natural resource management, analyzing strategies in marketing, and even penalty kicks in football (as in matching pennies).

E. Prisoner’s Dilemma (Application on Nash Equilibrium)

It is a situation in which two players each have two options whose outcome depend on the simultaneous choice made by the other. The origin of these concept was formulated from an experiment by that name.

In such experiment, there is two persons who committed a crime and where caught by the police. In the investigation, the D.A. told each of them, separately, that they may serve none, two, five or ten years in jail, depending on their decisions. If one of them confessed and the other didn’t, they shall get immunity and serve none, while their partner would serve 10 years in jail. If they both ratted out each other, they both would serve five years in jail. And if none of them confessed anything, they both would serve two years only in jail.

Even though it seems like the two years for both is the best option, that is not the case from Game theory point of view. That is because none of the two criminals can trust the other fully that they won't rat them out. So, they would see that the 5 years in jail is the best option they can have, considering that the other person will confess. This is exactly like Nash equilibria. The below figure (fig.3) can explain the prisoner's dilemma experiment in a normal form.

Player 1 / Player 2	Don't Confess	Confess
Don't Confess	2 / 2	10 / 0
Confess	0 / 10	5 / 5

3. Prisoner's Dilemma illustration model

F. Modern days

Nowadays, Game theory is used most in several fields of computer science and in algorithms. It has many applications and serve important problems, way beyond the scope it started with in economics and warfare.

III. Representation of Games

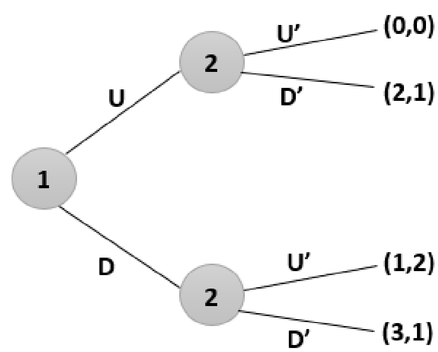
The Game theory studies games that contain mathematical objects. Every game has some elements, and we need to be aware of these elements, like the players of the game, information and strategies available to each player at each point of separation (decision point), and the payoffs of each outcome for each player. These four essential elements were referred to the acronym "PAPI" by Eric Rasmussen. The game that has all of these elements is a fully-defined game.

These four elements are usually used by a game theorist, as well as the solution concept of the game, to reach a set of strategies for each player, such that no players can profit by deviating from these strategies once employed.

We have three forms that we can use to present games; the characteristic function, and the extensive and the normal forms. The type of the game is the main factor in choosing the suitable model for its solution. Usually, cooperative games are presented in the characteristic function form, and non-cooperative games are presented in the extensive or the normal forms.

A. Extensive Form

This form is a description of a game in Game theory that allow (as the name suggests) us to represent a number of key aspects, like the possible sequential moves for each player, their choices at every point of separation (decision point), the information that each player has about the other player's moves when they make a decision and their payoffs for all possible game outcomes regardless this information is perfect or imperfect. Extensive form games also allow for modeled as "moves by nature", and a move by nature is a decision or move in an extensive form game made by a player who hasn’t strategic interests in the outcome. The effect is to add a player, who is practical role is to act like a random number generator.



4. Extensive form representation

For example, as in the following diagram (fig.4), the initial node belongs to first player, indicating that first player moves first. Play according to the tree is as follows: first player chooses between U and D; second player observes first player's choice and then chooses between U' and D’. The payoffs are as specified in the tree. The four outcomes are represented by the four terminal nodes of the tree: (U, U'), (U, D'), (D, U') and (D, D'). Say the payoffs associated with each outcome are as follows (0,0), (2,1), (1,2) and (3,1).

So, if player A chose to play D, he will get one of two options, 3 or 1. However, when it’s player B’s turn, he will definitely choose U’ to maximize his profit and get 2 instead of 1. And if player A chose U, he will get one of two options, 0 or 2. But assuming player B will act rationally, he will choose D’ to maximize his profit for 1 instead of 0.

Game theory’s solution, assuming both acts rationally, would be that player A chooses U and player B chooses D’. The payoff of such strategy is (2,1) (represented in extensive form).

B. Normal Form

This form is also a method of description of a game in Game theory, but unlike extensive form, normal form representations aren’t graphical. It represents the game using matrices. Even though this form can be used in identifying strictly dominated strategies and Nash equilibria, however, some information is lost in the representation, unlike in extensive form representations. This form includes all perceptible and conceivable strategies, and their corresponding payoffs, for each player.

In static games of complete/perfect information, this form is a description of players' strategy spaces and payoff functions. The set of all strategies available to the player is a strategy space for that player, while the strategy is a complete plan of action for every stage in the game, without taking into account that stage actually arises in play.

A payoff function for a player is the mapping from the cross-product of the space of players’ strategy to the set of payoffs of the player (normally the set of real numbers, where the number represents a cardinal or ordinal utility often cardinal in the normal-form representation), i.e. the payoff function of a player takes an input a strategy profile (that is a specification of strategies for every player) and return a representation of payoff as its output.

The matrix below is a normal form representation of a game in which players move simultaneously or non-simultaneously but at least don't observe the move of other player before making their move, and receive the payoffs as specified for the combinations of actions played.

For example, as in figure 5, if first player plays top and second player plays left, first player receives 4 and second player receives 3. In each cell, the first number represents the payoff to the row player (in this case first player), and the second number represents the payoff to the column player (in this case second player).

Player 1 \ Player 2	Left	Right
Top	4 , 3	-1 , -1
Bottom	0 , 0	3 , 4

5. Normal form representation

IV. Uses and applications

Although it may not occur to many of us, we all use Game theory in our daily life. That is because Game theory is applied to decision-making. That is actually rather interesting, because if we really considered the math behind our decisions, we shall be able to take the most optimum decisions, that is of course, assuming we think 100% rationally, even though that is not the case.

By now, I think it is clear that a game is not necessarily how we usually think about them, like board games or video games, where there is usually a winner and a loser. Actually, an unknown source defined a game as "*Any interaction between multiple people in which each person's payoff is affected by the decisions made by others*". Which means, that a game is basically any situation where there is an interaction between people and decisions are made.

Considering this, Game theory was applied to many fields in various applications and problems. Game theory's analysis was first used in economics, specifically by Antoine Augustin Cournot in 1838 while solving Cournot duopoly.

Cournot duopoly was an economic model that shows the structure of industry where there is a competition between companies on the output they produce, however, decisions are made independently of one another, meaning that each company makes the best decision from its point of view without knowing what other companies will decide to do. This case actually similar to a case known in Game theory called Nash equilibria.

Nash equilibria is a situation where a player is taking a decision that leaves them in best case scenario, no matter what other players decide to do. In Evolutionary Game theory book, Nash equilibria is defined as, "*a strategy profile* $x \in \vartheta$ *is a Nash equilibrium if it is a best reply to itself, namely if it is a fixed point of the mixed-strategy best-reply correspondence* β*:* $x \in \vartheta\, is\, a\, Nash\, equibria\, if\, x \in \beta(x).\ [eq. 1]$

Nash equilibria is used to analyze many situations in various fields up till now, and is most famous in analyzing prisoner's dilemma.

After using Game theory in economics only for quite a few, the use of Game theory expanded to many other fields, like politics, sociology and others. It was found of high importance in some fields more than in others, like in biology, where Game theory was found of high importance, rather more Evolutionary Game theory. John Maynard Smith talked about this evolution in Biology and its relation with Game theory in his book, Evolution and the Theory of Games, stating that *"paradoxically, it has turned out that Game theory is more readily applied to biology than to the field of economic behavior for which it was originally designed".*

Here we are to mention some of the fields that used Game theory in few details.

A. Description and modeling

This can be called the primary use of Game theory, in which mathematical analysis is put to use in modeling and describing how humans behave naturally. But there is a slight difference here that in this field, many more aspects are considered in modeling. In other words, although basic Game theory considers that players act rationally, that is not the case in description and modeling.

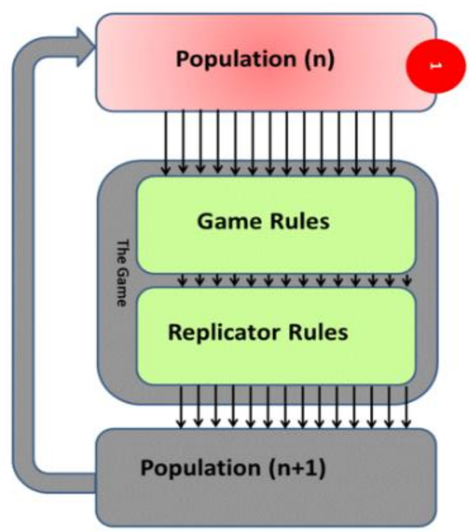
To really predict how a human would behave, we need to consider that humans don't always act rationally, and hence Game theory's solutions are violated when applied on real-life situations, and people don't even reach Nash equilibria situations.

To reach more real-life solutions, scientists have turned to Evolutionary Game theory for more realistic situations. In Evolutionary Game theory, models consider no rationality or rather limited rationality from players. It even considers biological aspects or cultural evolution that may affect such decisions. It focuses more on the dynamics of strategy change and how often there is a competing strategy in population.

From these models, is a system model based on Darwinian mechanism, having four phases, as shown in the diagram.

B. Prescriptive or normative analysis

In this analysis, Game theory is considered as a tool for creating suggestions for actions, rather than predicting actions. Which makes more sense in some ways, considering that typical Game theory models aren't always the case in real-life situations. So that even Nash equilibria here is considered as the most optimum solution and not the actual solution. The following figure (fig.6) shows an example for a model used for prescriptive analysis on population.



6. Evolutionary Game Theory analysis of Darwanian mechanism model

C. Economics and business

In mathematical economics and business perspective, Game theory comes in handy in predicting how competitors might behave in similar situations to which you are in. It is used in modeling behavior of interacting agents, and focuses more on specific strategies as “equilibria” or “solution concepts”.

Usually in economics and business, models of cooperative games are considered more than non-cooperative games. In non-cooperative games, there is always a winner and losers, for a player to win, others must lose. However, in cooperative games, it is not a necessity. A player can win and make most optimum decisions without having others to lose.

Application for these studies are various, from auctions, bargaining, social network formations to voting systems, industrial organization, and political economy.

D. Political science

In political sciences applications, players of games are voters, states, or politicians. One of the first uses of Game theory in political science was provided by Anthony Downs in his book “An Economic Theory of Democracy”, where he applied a model known as “Hoteling firm location” on a political process. In this model, Downs showed how political candidates behave depending on their voters, and how voters will act if they were fully informed. However, voters are not totally rational, and this is actually the cause of candidate divergence.

Game theory is also used in predicting nation’s response to each other when there is a new stated rule or law applied, or any big event that will cause some action from nations’ rulers.

In a research on reducing climate change, and how nations can contribute in doing so, Peter John Wood thought that climate change can actually be reduced if nations agreed on reducing the usage of greenhouse gas, due to its emissions. However, in the end of his paper, he stated that such case won’t be accomplished, due to prisoner’s dilemma situation which will surely happen among nations.

E. Biology

In biological applications, Game theory is rather more evolutionary. Typical Game theory models are not taken into account, rather studies drifted to Evolutionary Game theory for models that can be applied on real-life situations and more biological models.

The first use of Game theory analysis in biology was a study made by Fisher in 1930, to study and explain evolution and stability of sex ratios depending on the data of people living at the time. However, soon enough they noticed that equilibria models of Game theory are not of much use in biology and are not precise. And hence they started shifting towards more evolutionary models.

In 1973, Smith and Price introduced a state of biological equilibria known as evolutionary stable state (EES). And it is thought that every EES is a Nash equilibrium.

After so, biologists studied many biological behaviors using EES and evolutionary Game theory, as in communication between animals, or signaling. From those biological phenomena, is what is known as biological altruism. Which differs from traditional altruism.

In biological altruism, the organism (or the player) don't act rationally, but rather takes decisions to protect others even if that puts the player in a dangerous position. This, of course, contradicts with the traditional definition of Game theory where players act rationally, thinking of themselves before any.

But considering species as ants of bees, we know that bees protect their queen more than themselves, so in some cases, we need to put biological aspects into consideration in modeling. Which is exactly made by evolutionary Game theory.

F. Computer science and logic

Game theory play and an important role in such field, and this role has been increasing for the past decades. Many logical theories used game semantics in their basis. Computer scientists used games to model interactive computations.

Moreover, multi-agent systems field had several theoretical bases based on Game theory.

Game theory is most beneficial in this field in algorithms, such that it provides solutions for complicated problems, especially in online algorithms. One of those algorithms is K-server problem. It is beneficial too in calculating computational complexities of random algorithms.

Ever since the introduction of the internet, and algorithms for different purposes related to the internet network has been developed more and more. Such algorithms may be for finding equilibria in games, markets, computational auctions, peer-to-peer systems, security and information markets. All of those use the algorithmic Game theory.

Algorithmic Game theory is used in several fields of computer science like Artificial Intelligence (AI), Cloud Computing, Network Security, Social Networks and Problem Solving. In problem solving it's used to tell you if you can win in the game or not and tell you the steps you should take to win, that is if you can win. It's used in several games like Tic-Tac-Toe, Nim Games, piles game and other games. And we got the Nim theory from it. It's a theory solved several games in problem solving.

In cloud computing, the Game theory is used for modeling complex interactions between cloud providers (whose aim is to minimize cost while maximizing resource utilization) and a number of service providers (often with contradictory objectives of maximizing Quality of Service at minimal cost). In such a situation, a game is set up based on a utility function that will eventually steer game play towards an equilibrium state (Nash Equilibrium), where no players could change their strategy, and objectives of all players are balanced. There are many cases in cloud resource management involving spot pricing of cloud resource addressed by auction/bidding games, solved as well with algorithm Game theory.

In Network Security, Game theory is used, specifically Nash equilibrium case. In Network Security's literals, it is like a game of two players; attackers and defenders. The attacker's goal is to hack the computers while the defender's goal is to prevent these attacks. This theory is used in several scenarios with different number of attackers and different number of defenders.

V. Types of Games

Game theory has lots of types depending on various factors. It may be number of players, their goal or even just the symmetry in game rules. In this section, we will define some of these types and their differences, and sometimes, some applications for further illustration.

A. Cooperative and Non-cooperative games

In cooperative games, players are called to in a coalition with each other, where each player participates in this coalition to some extent and hence shall have a specific share in this coalition, as according to the output or payoff of the game. And this coalition of players are working towards one common goal. So there is no winners and losers.

Game theory here exactly tells us the contribution of each player to the coalition, and how much they shall benefit from it.

So, a cooperative game theoretical analysis addresses two important questions.

1) What coalitions will form when all individuals have decided to cooperate?
2) How would they share their gains through a mutual cooperation?

Cooperative Game theory gives us a high-level approach as it describes the game's structure, strategies, and payoffs of such coalition.

An example for cooperative game situation, is assuming two persons, say Rou and Joey, are making cookies, which are sold by one dollar each. Separately, Rou can make 20 cookies an hour, while Joey can make only 10 cookies an hour. However, when they teamed up and baked together (formed a coalition) they made 40 cookies an hour.

So, how much should each of Joey and Rou gain at such case? If Rou considered that she makes 20 an hour, then she shall gain 20 dollars an hour, and hence leaving 20$ for Joey. However, if she considered that Joey only makes 10 cookies an hour and hence get only 10 dollars an hour, then that shall leave her with 30 dollars an hour. Game theory here solves this situation, telling you that you should take the average between the two cases, and this is what is called the *Shapely Value*. So, in other words, Rou should take 25$ an hour, while Joey should take 15$ an hour. That is the coalition's best outcome, for both players.

In non-cooperative games, players are in a competitive social interaction where there will be some losers and winners among the players. And hence, there is no coalition, and no common goal. Rather they work separately—which is an important situation in non-cooperative game—to reach their goal through their best course of actions, taking into account the decisions of other players, while they remain unchanged.

Since non-cooperative Game theory is more general, cooperative games can be analyzed through the non-cooperative Game theory approach, provided that assumptions are made to consider all possible strategies available for the players due to the possibility of external enforcement of cooperation.

B. Zero-sum and Non-zero-sum games

In this kind of games, a player's gain is equal to other player's losses. A zero-sum game may have two players or any number of players. Games like chess and tennis are zero-sum games as there is one winner and one loser, Poker is also a specific example of zero-sum games since the sum of the amounts won by one player equals to the combination of losses of the others.

For further illustration, here is an example of a zero-sum game.

The game of matching pennies , it involves two players, A and B, simultaneously placing a penny on the table. The payoff depends on whether the pennies match or not. If

A \ B	Heads	Tails
Heads	+1, -1	-1, +1
Tails	-1, +1	+1, -1

7. *Zero-sum game payoffs*

both pennies are tails or both are heads, player A wins and keeps player B's penny. However, if they do not match, Player B wins and keeps Player A's penny.

This is a zero-sum game because one player's gain is the other's loss. The payoffs for Players A and B are shown in the table front. As can be seen in the above figure (fig.7), the sum of payoff of both A and B, in all four cases is, of course, zero.

Non-zero-sum games are completely different from the zero-sum ones as there is always an optimal solution, this is closer to the real-world life situations. Problems in the real world do not usually have straightforward results like zero sum games.

Non-zero-sum games differ from zero-sum games in a couple of things. One is that in non-zero-sum games, the sum of the payoff of all players don't add up to zero, rather be more or less than zero. Second is, in most of these types of games, players may be working towards the same goal, as in cooperative games, unlike in zero-sum games, where the usual is that one player's win depend on the loss of other players, as in non-cooperative games. But even though, at some cases, players of non-zero-sum games may have different interests, rather complementary.

In other words, in non-zero-sum games, there is no optimal strategy that all players work toward, and moreover, there is no predictable outcome. One example of such type is gambling.

For further illustration, here is an example.

Say a man and his wife want to go out in one evening. They decide to go either to a ballet show, or to a boxing match. They both prefer going together, and not alone. Even though the man prefers to go to the boxing match, he would prefer to go with his wife to the ballet show than going to the boxing match alone. Similarly, the wife would prefer to go to the ballet, but she too would rather go to the boxing match with her husband than going to the ballet alone.

The following table (fig.8) sums up the situation and its payoffs.

The wife's payoff matrix is represented by the first element of the ordered pair while the husband's payoff matrix is represented by the second of the ordered pair.

Husband / Wife	Boxing Match	Ballet
Boxing Match	2, 3	1, 1
Ballet	1, 1	3, 2

8. Non-zero-sum game payoffs

From the matrix above, it can be seen that the situation represents a non-zero-sum game, and not a competitive conflict. The common interest between the husband and wife is that they would both prefer to be together than going to the events alone. However, the opposing interests is that the wife prefers to go to the ballet while her husband prefers to go to the boxing match.

C. Perfect information and Imperfect information games

A game is considered from perfect information type, if all players know the moves previously made by all other players. So that if it’s one’s turn to play (make a move or take a decision), they always know what each of the other players has made up till now. Chess and tic-tac-toe games are good example on perfect information games.

A game is one of imperfect information if a player doesn’t know exactly what action other player took up to that point. Like in poker.

In other words, if it’s one’s turn to play, they don’t know what each of the other players have made up till to that point, keeping in mind Bayes rule (rules concerning probability), they can make such decision according to the Game theory model at hand.

There is also what is called complete information games, where each player knows all other player’s previous moves, as well as all the possible solutions or strategies they may consider ahead, and their payoffs. This type of games is commonly mistaken with perfect information games.

D. Combinatorial games

A combinational game is called on the type of games where to find an optimal strategy, you have to multiply all possible moves of the players. And the game's difficulty depends upon which.

In the preface of *Combinational Games: Tic Tac Toe Theory* book, Jozsef Beck said, "*Traditional Game theory focuses on games of incomplete information*", and then defined Combinational Games as, "*Combinatorial Game means a 2-player zero-sum game of skill (no chance moves) with complete information, and the payoff function has 3 values only: win, draw, and loss*".

What is interesting about combinational games is, although it seems like they have infinite ways of solution, it keeps the game alive. Some say that such solution can be reached at any time by backtracking algorithms. But this algorithm goes through all possible combinations. So, if we have a $5 \times 5 \times 5 \ = \ 5^3$ Tic-Tac-Toe is unsolved, if considered, you would find that you have around 3^125 position. Which means that you may need a 3^{125} case of backtracking! That will probably take weeks on the fastest computers on Earth.

Jozsef wrote his book on ways to solve combinational games, trying to avoid combinational chaos, to win a hopeless war. And in such, he did not consider the games of chance.

E. Symmetric Games

A game is called symmetric if a player's strategy doesn't depend on the other players nor their identity, just the other strategies provided, regardless who plays which. In other words, if during a game, players can be changed while strategies remain unchanged, then this game is symmetric.

For example, as in following table (fig.9), in rock-paper-scissor game, if first player A goes for rock then scissor and player B goes for paper then rock then player A's payoff is -2 and player B's is +2.

Then if we exchange their positions, meaning that player A goes for paper then rock and player B goes for rock then scissor, player A's payoff is 2 and player B's is -2.

Which means that a symmetric games payoff depends only on the strategy of the player not the players' identities.

	Rock	Paper	Scissor
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissor	-1, 1	1, -1	0, 0

9. Symmetric game representation

F. Simultaneous and Sequential Games

A game is called simultaneous if all players take action at the exact same time, without previous knowledge of what other players would decide, or play. Just like in rock-paper-scissors game.

A game is called sequential, or dynamic, if players take turns during the game, such that each player has knowledge of previous decisions or moves made by other players, even if the knowledge isn't complete. As in chess game.

In chess, the two players take turns playing and when it's a player's turn to play he has knowledge about the whole chess board and the other player's exact last action.

Here's a short comparison between the two types:

	Simultaneous Games	Sequential Games
Denoted By	Payoff matrices	Decision Trees
Prior Knowledge about the other player's action	No	Yes
Has time axis	No	Yes

10. Simultaneous VS. Sequential games

G. Infinitely long games

Infinitely long games are games that has infinitely many moves and winner of it can't be known until all those moves are done.

Those games usually don't concentrate on what the best way to play it is but whether the player has a winning strategy or not. Meaning that even with the perfect knowledge, each player may not have a winning strategy and the game would go on till one of them has one.

H. Discrete and Continuous games

While most of Game theory is concerned with finite discrete games which have finite players, moves, events and outcomes, continuous games extend those limits allowing the players to choose from a finite set of pure strategies.

Continuous games concepts include more general sets of pure strategies that can be uncountable.

I. Evolutionary Game theory

Evolutionary Game theory is concerned with players whose strategies change depending on rules. Those rules aren't always rational or farsighted and may feature imitation, optimization or survival of the strongest.

This model has uses in other fields such as biology and social sciences. It can represent biological evolution, as the parent who have more successful strategies, have a greater number of offspring.

Social sciences use this model to represent adjustments in players' strategies through their lifetime as the play a game more and more, whether that adjustment was done consciously or unconsciously.

J. Stochastic outcomes

Stochastic outcomes can be represented by adding a player whose actions are random to a game. This player isn't considered a third player in a two-player game but is used to get a random outcome when it's needed.

This model can be used to represent Individual decision problems which is used in some areas of artificial intelligence (AI), AI planning and multi-agent system. These fields use the same mathematics as the Game theory of stochastic outcomes.

In some problems, there can be different solutions to them due to the different models of stochastic outcomes.

As an example of different models, some problems consider the worst-case always rather than reasoning the expectation using a fixed probability distribution. This approach can be better where stochastic models are not available but still is overestimating costly unlikely to happen events.

VI. Lights-Out Game

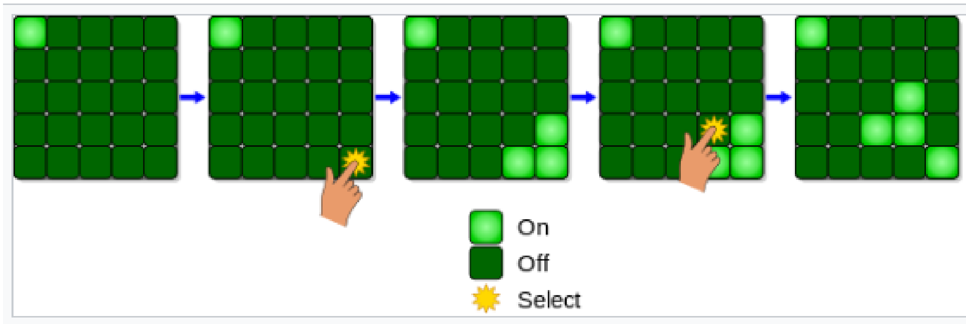
Lights-out game is a one-player game, analyzed using game theory, such that it considers the player to act rationally, trying to reach a specific payoff—a goal, in the most optimum way. In order to do so, the game is modelled using normal form to a set of matrices. After using mathematical equations and rules, we will get a vector that tells us the steps to follow to solve the game in the minimum number of moves.

It is a non-zero-sum, perfect-information, combinatorial game. It is indeed one-player game, however, in game theory, it can be considered as two player game between the player and the table. Hence, it can be said that it is non-cooperative game.

A. Game Description

We have a grid (like a matrix) with dimensions 5x5, each 1x1 block having a LED (so we have 5x5 = 25 lids), and some of them is turned on, initially. If you push the button which refers to a certain lid, its state would toggle, in addition to the lids upwards of it, downwards of it, to its right and to its left. Your goal is to turn off all lids, in the minimum number of moves, which is exactly what may concern game theorists.

For further explanation, this figure (fig.11) might help.



11. Steps of Lights-Out game

B. Solution of the Game: Algorithm 1

When we searched about the problem, we found that it was well-studied by mathematicians and they think that it can be easily solved and modeled by using an application of linear algebra and finite fields.

We can notice that we have only two states; Off-lid or On-lid, in a 5x5 grid. So, it can be like a 5x5 matrix with only two value; ones and zeros. 1 represent an On-lid while 0 represent an Off-lid.

Considering the above observation, if someone pushes a button it will toggle itself and those to its up, down, right and left too. So, let's assume we have state 1 and we pushed the button (3,3), this results in state 2. $[eq. 2]$

$$\underset{\text{(state 1)}}{\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}} \longrightarrow \underset{\text{(state 2)}}{\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}}$$

Since we only replace 0s with 1s and vice-versa, it is equivalent to add 1 modulo 2 to the postion which would toggle. For example we add the pushing effect below. $[eq. 3]$

$$\begin{matrix} 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&1&0&1&0 \\ 0&0&1&1&0 \\ 0&0&0&0&0 \end{matrix} + \begin{matrix} 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&1&1&1&0 \\ 0&0&1&0&0 \\ 0&0&0&0&0 \end{matrix} = \begin{matrix} 0&0&0&0&0 \\ 0&0&2&0&0 \\ 0&2&1&2&0 \\ 0&0&2&1&0 \\ 0&0&0&0&0 \end{matrix}$$

(state 1) (pushing effect) (result)

Then we take modulo 2 of each element. Hence we get the correct result of operation. $[eq. 4]$

$$\begin{matrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \end{matrix}$$

(state 2)

The previous example shows a very interesting thing about pushing a button. Every button has a unique matrix which identifies which elements would toggle. For example, the pushing-effect Matrix of button (3,3) is

$$\begin{matrix} 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&1&1&1&0 \\ 0&0&1&0&0 \\ 0&0&0&0&0 \end{matrix}$$

(pushing effect)

The reason that it is an interesting thing is the clear and easy way to describe the problem with a mathematical model then get a solution with just a few steps.

Suppose that we have Matrix A (n x n) which represent our grid. It has an initial state S. Every lid in this grad has a unique toggle Matrix T (n x n). From that we can put an expression to solve the problem and reach our goal turning off all lids. $[eq. 5]$

$$S+\sum_{k=1}^{n}\sum_{j=1}^{n}\mathrm{T}(a_k,b_j)=(0,0,0,0,\dots,0)$$

Where a,b is row and column of a button and $a,b\in[1,n]$.

This expression means we want set of toggle matrices which make the initial state go to 0 state (all light off state).

We have a question here; IS the correct answer to solve the problem is to push all buttons?

Of course not. there are buttons to push and buttons not, in the general case. So, let's suppose we have coefficients to detect which buttons would be pushed assuming the button is pushed only one time to have the minimum number of moves. Coefficients would be 1 or 0.

1 refers to which button would be pushed and 0 refers to which button wouldn't be pushed. Then the experession would be: $[eq.6]$

$$S+\sum_{k=1}^{n}\sum_{j=1}^{n}x*\mathrm{T}\left(a_k,b_j\right)=(0,0,0,0,\dots,0)$$

Where x is the coefficient of button (a,b).

It's more complex to deal with integers while having only two values; 0 and 1, and addition and multiplication are followed by modulo 2. In addition, we put more constraints on coefficients which have the same two values. So we would rather deal with the field the Galois field of order 2—GF(2)—(for more details look at appendix).

The definition of operations in GF(2):

$$0 + 0 = 0 \qquad 0 * 0 = 0$$
$$0 + 1 = 1 \qquad 0 * 1 = 0$$
$$1 + 0 = 1 \qquad 1 * 0 = 0$$
$$1 + 1 = 0 \qquad 1 * 1 = 1$$

$$-0 = 0 \qquad -1 = 1$$
$$\frac{1}{1} = 1$$

From the table above, we can notice that if the values are considered to be binary, then the truth table of addition is the same as the XOR between the two bits, and the truth table of Multiplication is the same as the AND between the two bits. The last three rules show that the inverse of the number is the same as the number, which will come in handy soon enough.

Another important property of these definitions of arithmetic is that multiplication distributes over addition, that is, $A(x + y) = Ax + Ay$ and we care about this because it means that all of the common properties of arithmetic of real numbers still apply to the field GF(2).

So, if we have a linear equation, we can add or multiply a number to both sides, which we will do now to simplify the linear equation above.

First, we shall add $-S$ to both sides of the equation to get $[eq. 7]$

$$\sum_{k=1}^{n} \sum_{j=1}^{n} x * \mathrm{T}(a_k, b_j) = -S$$

And since $-S = S$, then: $[eq. 8]$

$$\sum_{k=1}^{n} \sum_{j=1}^{n} x * \mathrm{T}(a_k, b_j) = S$$

Or in a more simplified form, such that X is the matrix representing all coefficients, of in other words, buttons to be pressed to solve the game, the equation is: $[eq. 9]$

$$\mathrm{T} * \mathrm{X} = \mathrm{S}$$

Now it is obvious that we want to know what X will lead to. Since $T(a, b)$ is a matrix which can be repersent as a vector, where the values are in row-major order and represent the X's and $S's$ as column vectors then the simplified equation makes sense and now form a set linear equation.

To know the values of X, we will solve the equation with a standard solving method for a set of linear equations, like Gaussian-Jordan elimination method. Why Gaussian-Jordan elimination method? Because I don't know if T has an inverse or not, if there is a solution for the initial value of S or not.

C. Result of Algorithm 1

After solving with Gaussian-Jordan elimination, it is founded that, T matrix has 23 independent columns, while the last 2 are dependent column which means we have many solutions (not infinite, though!) or no solution at all, depending on values of S.

If there are solutions, we have a relation between the last 2 columns, and all other columns. And hence, by assuming values for dependent coefficients, then make backward substitution on others, we get a solution for X, which means a solution for the game.

D. Cost of Algorithm 1

First, we compute the toggle matrices for the different buttons. There are $n\ x\ n = N$ button. Each of them needs $O(N)$ to compute the values of each pushing-effect matrix, or as we last considered, a vector, so the cost of this is $O(N^2)$ where N is the number of buttons.

Secondly, we should apply Gaussian-Jordan elimination on the NxN matrix, which takes $O(N^3)$.

If there is a solution then we have to make backward substitution, which takes $O(N^2)$.

Hence, the total cost of the algorithm is $O(N^3)$.

E. Solution of the Game: Our Updated Algorithm

Since the left-hand side of the equation $T * \mathrm{X} = S$ include matrix T which is constant for all states, considering the same dimensions for the game, then we can use LU -factorization concept but with some slight differences. The L wouldn't be exactly a lower-triangular matrix and the U matrix wouldn't be exactly upper-triangular matrix, due to the last two dependent rows which will cause last two lines in U to be zeros, so we will settle with the reduced-row-echelon form from matrix U, say named U'. And its lower-triangular triangle is L'. Hence our equation will be $[eq. 10]$

$$PL'U'X = S$$

Where P is the permutation matrix. We can get L' matrix—the inverse of elimination matrices applied—then multiply the two sides by $L'^{-1}P^{-1}$, to get this result $[eq. 11]$

$$UX = L'^{-1}P^{-1}S$$

The U', L', P matrices are constant for any initial S.

Now, we can multiply $L'^{-1} \times P^{-1}$ to have a CONSTANT matrix C for all states. Which will get us our main equation: $[eq. 12]$

$$U'X = CS$$

When we have an initial state values, we multiply C by S, then check if there are solutions or not.

If there are solutions, we have a relation between the last 2 columns of U' and all other columns.

We will assume values for the dependent coefficients, then use backward substitution to get a solution for X.

F. Cost of Our Algorithm

In our new solution, we can get matrix C ONCE before playing the game, which would cost $O(N^3)$ by Gaussian-Jordan elimination OR, we can get them from a source solved before, so it won't cost us anything. Once we get C and save it for any initial case in game afterwards, we can multiply C by S for any state, which will be matrix-vector multiplication, and will cost only $O(N^2)$.

Checking if there are solutions, then the backward-substitution costs $O(N^2)$.

Then, the total average of the updated solution's cost will be $O(N^2)$.

VII. Conclusion

Game theory is a classic theory which is applicable on lots of fields; economics, industry, biology, wars...etc. Games aren't as simple as some people think. Some Games rely on complex mathematical theories and calculations. There are a lot of mathematical models and algorithms discussed by Game theory which help in solving many problems with the best solution.

Our problem here called Lights-Out game, of type combinatorial games which has many possible moves and can have number of solutions.

The solution which used widely costs $O(N^3)$. We noticed it can be decreased if we keep in mind there is a constant-elements matrix. So, we used factorization concept and got a new constant matrix, C, before playing. For any initial state we multiply the new matrix by the initial state.

Because the initial state, S, is a vector then the multiplication of C by S costs $O(N^2)$. If there are solutions, then we make backward-substitution getting one of the answers, and this will cost $O(N^2)$. Hence, we can solve this game, using Game theory analysis and mathematical modeling in matrices and vectors, to solve the game in minimum number of moves, costing less, and solving faster.

Finally, we wrote C++ code for our algorithm, testing our theory and trying it to solve different states of the game, which we also made in C# language as a small desktop application.

VIII. Appendix

A. Galois field of Order 2

Galois field of order 2 is a mathematical structure named a field that provides a formal definition of arithmetic modulo 2.

In particular, it defines arithmetic over 0 and 1 by defining

- How to do addition of numbers modulo two?
- How to do multiplication of numbers modulo two?
- How to get the arithmetic inverse of a number modulo two?
- How to get the multiplicative inverse of a nonzero number modulo two?

These definitions are as follows:

$$0+0=0 \qquad 0*0=0$$
$$0+1=1 \qquad 0*1=0$$
$$1+0=1 \qquad 1*0=0$$
$$1+1=0 \qquad 1*1=1$$

$$-0=0 \qquad -1=1$$
$$\frac{1}{1}=1$$

The rules of addition and multiplication of numbers modulo two probably makes sense, but the rules for inverse is a bit strange. It should be pretty clear why $-0 = 0$.

The reason is that it is defined in a way that we want to have $1 + (-1) = (-1) + 1 + 0$. Looking at the above table of arithmetic, we see that this is only possible if $-1 = 1$.

B. Code of Algorithm 2

We have written our own C++ code that works on the algorithm we have described above; the updated one. The output of the code is instructions on which lids to press on. The code project is attached with the report. Along with our C# application that runs the game at initial state where all lids are turned on.

IX. Resources

- R. B. Myerson, *Game theory: Analysis of Conflict*, 1st edition, Harvard University Press 1991
- J. v. Neumann and O. Morgenstern, *The Theory of Games and Economics Behavior*, 16th edition, Princeton University Press 1994
- J. M. Smith, *Evolution and the Theory of Game*s, 1st edition, Cambridge University Press 1982
- J. W. Weibull, *Evolutionary Game theory*, 1st edition, The MIT Press 1997
- T. Vincent and J. S. Brown, *Evolutionary Game theory, Natural Selection, and Darwinian Dynamics*, 1st edition, Cambridge press 2005
- A. Downs, *An Evolution Theory of Democracy*, 1st edition, Harper Press 1957
- P. J. Wood, *Climate Change and Game theory* paper, [Abstract], PMID: 21332497, 2011 accessed from https://www.ncbi.nlm.nih.gov/pubmed/21332497
- N. Nisam, T. Roughgarden, E. Tardos, and V. V. Vazirani, *Algorithmic Game theory*, 1st edition, Cambridge Press 2007
- A. Rapoport, *Two-Person Game theory*, 1st edition, Dover Publications inc. 2013
- K. G. Binmore, *Game theory and the Social Contract: Just Playing*, 2nd vol., The MIT Press 1994
- T. Roughgarder, *Twenty Lectures on Algorithmic Game theory*, 1st edition, Cambridge University Press 2016
- H. I. Meinhardt, *Coopeative Decision Making in Common Pool Situations*, 1st edition, Springer 2002
- A. Washburn, *Two-Person Zero-Sum Games*, 4th edition, Springer 2013
- John Nash, *Two Person Cooperative Games* paper, [Abstract], The Econometric Society publish, vol. 21, no. 1, (Jan. 1953), pp. 128-140, accessed at https://www.jstor.org/stable/1906951
- J, Beck, *Combinatiorial Games: Tic-Tac-Toe Theory*, Cambridge University Press 2008
- J. Nash, *Non-cooperative Games*, Princeton University Press 1950

- C. Shannon, *Programming a Computer for Playing Chess* paper, [Abstract], pp 2-13, 1950
- E. Rasmusen, *Games and Information:An Introduction to Game Theory*, 4th edition, Wiley 2007
- D. M. Kreps, *Game theory and Economic Modelling*, Clarendon Press, Oxford 1990
- R. Aumann and S. Hart, *Handbook of Game theory with Economic Applications*, vol.2, North Holland 2002
- D. Robinson and D. Goforth, *The Topology of the 2x2 Games: A New Periodic Table*, Routledge 2005
- J. N. Webb, *Game theory: Decisions, Interactions and Evolution*, Springer 2007
- M. Anderson and T. Feil, *Turning Lights Out with Linear Algebra*, Mathematics Magazine 1998
- R. Eisele, *LightsOut Soltuion using Linear Algebra* article, [Online document], 2018, available at https://www.xarg.org/2018/07/lightsout-solution-using-linear-algebra/