The Paradise of Georg Cantor

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"No one will drive us from the paradise which Cantor created for us."
David Hilbert

Abstract.

This article delves into the fundamental ideas of set theory, such as set definitions, functions, and cardinality. The authors examine many methods for proving theorems, such as proof by induction and proof by contradiction. Cantor's diagonal argument and theorem on the uncountability of real numbers, as well as its repercussions, such as the uncountability of the power set of natural numbers and the presence of an infinite hierarchy of infinities, are presented in this work. The proofs are provided in a straightforward and succinct way, allowing anyone with a basic familiarity of mathematical notation and vocabulary to understand them. The paper continues with a discussion of some of these conclusions' ramifications and their relevance in current mathematics.

Overall, this paper provides a solid introduction to the fundamental concepts of set theory and presents several important results and their proofs, making it a valuable resource for students and researchers alike.

1 Cantor's Life

In this section we will present the life story of Georg Cantor based on the article "*The Nature of Infinity*" by JÃ¸rgen Veisdal [1].

Cantor's family moved to Wiesbaden, Germany, when he was eleven years old. Cantor's interest in mathematics blossomed here. He attended Frankfurt High School and developed an early interest in mathematics, which he studied at the University of Berlin. Cantor was educated at the University of Berlin by some of the most notable mathematicians of the day, including Karl Weierstrass, Ernst Eduard Kummer, and Leopold Kronecker. He got his bachelor's degree in mathematics and physics from the University of Berlin in 1867, and two years later, he received his doctorate from the same institution for his dissertation on number theory.

Cantor's early work focused on algebraic number theory and the development of the concept of continuous fractions. He also contributed significantly to function theory and variational calculus. Cantor began his career as a lecturer at the University of Halle in 1872, where he remained for the rest of his life. He was recognised as an outstanding professor in 1872 and elevated to full professor in 1879.

Cantor began to build his most renowned work in the subject of set theory while studying at the University of Halle. Cantor's work in set theory began in the 1870s, with an emphasis on infinite sets. He created the idea of cardinality to compare the sizes of infinite sets. Cantor is credited with establishing that infinity has several levels, with some infinities being bigger than others. This concept was groundbreaking at the time, challenging conventional mathematical thought.

Cantor's work on set theory resulted in the creation of a new field of mathematics known as "axiomatic set theory." This new field of mathematics was founded on a series of axioms that outlined the characteristics of sets and their relationships. Cantor's work also contributed to the creation of transfinite number theory, which includes the invention of the idea of aleph numbers to characterise the size of infinite sets.

Cantor suffered substantial criticism and hostility from many of his contemporaries despite his pioneering contributions to mathematics. Because of his work on infinity, some accused him of being a mystic or even mad.

Cantor's theories were so revolutionary that even the famous mathematician David Hilbert reportedly said, "No one shall expel us from the paradise that Cantor has created." Cantor, on the other hand, persisted in developing his theories and was finally recognised for his contributions.

Cantor had a liking for music and was a skilled pianist in addition to his work in mathematics. He frequently utilised musical analogies to describe mathematical topics, and he thought the beauty of mathematics was akin to the beauty of music. Cantor's love in music prompted him to research the musician Johann Sebastian Bach's works, and he even produced a dissertation on the subject titled "Contributions to the Theory of the Transfinite in Music."

Cantor's latter years, on the other hand, were marred by personal and health issues, including depression and a decrease in his mental health. Cantor's influence has long inspired mathematicians and scientists across the world. His groundbreaking work on set theory and the idea of the infinite provided the groundwork for contemporary mathematics and has had an impact on many other disciplines of study, including physics, computer science, and engineering.

To summarise, Georg Cantor was a great mathematician who pioneered set theory and the idea of infinite. Cantor stayed devoted to his profession and continued to push the limits of mathematical reasoning despite severe opposition and criticism from his colleagues. His contributions to mathematics continue to have an impact on the discipline, and his legacy will no sure inspire future generations of mathematicians.

2 Cantor's Theorem

We start by defining a set.

Definition 2.1. A *set* is a collection of objects.

- The set of natural numbers $\mathbb{N} = 0, 1, 2, 3, \ldots$.
- The set of integers $\mathbb{Z} = \ldots, -2, -1, 0, 1, 2, \ldots$.
- The set of rational numbers $\mathbb{Q} = \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z} \setminus 0$.
- The set of real numbers $\mathbb{R}$, which includes all the rational numbers and all the irrational numbers (numbers that cannot be expressed as the ratio of two integers).
- The set of even numbers $E = 2n : n \in \mathbb{Z}$.
- The set of vowels $V = a, e, i, o, u$.

Definition 2.2. Let A and B be sets. We say that A is a *subset* of B, and write $A \subseteq B$ if every element of A is also an element of B. In other words, for all x, if $x \in A$, then $x \in B$.

- Let $A = 1, 2, 3$ and $B = 1, 2, 3, 4, 5$. Then $A \subseteq B$ because every element in A is also in B.
- Let C be the set of even integers and D be the set of integers. Then $C \subseteq D$ because every even integer is an integer.
- Let $E = a, b, c$ and $F = a, b, c, d, e$. Then $E \subseteq F$ because every element in E is also in F.

We next introduce the notion of a power set.

Definition 2.3. The *power set* $\mathcal{P}(S)$ of a set S is the set of all subsets of S.

Example 2.4. *Here are three examples of a set and its power set.*

1. *Let* $S = 1, 2$. *Then the power set of* S *is* $\mathcal{P}(S) = \emptyset, \{1\}, \{2\}, \{1, 2\}$.

2. *Let* $T = a, b, c$. *Then the power set of* T *is* $\mathcal{P}(T) = \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c$

3. *Let* $U =$ *apple, orange, banana. Then the power set of* U *is* $\mathcal{P}(U) = \emptyset, \{apple\}, \{orange\}, \{ba$

Let us now introduce the concept of a function and some of its properties.

Definition 2.5. Let A and B be sets.

- A *function* $f : A \to B$ from A to B is a rule that assigns to each element $a \in A$ a unique element $f(a) \in B$.

- A function $f : A \to B$ is called *injective* if for any $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

- A function $f : A \to B$ is called *surjective* if for any $b \in B$, there exists $a \in A$ such that $f(a) = b$.

- A function $f : A \to B$ is called *bijective* if it is both injective and surjective.

We next show that the composition of injective functions is also injective.

Theorem 2.6. *Suppose that* $f : A \to B$ *and* $g : B \to C$ *are injective functions. Then their composition* $g \circ f$ *is injective as well.*

Proof. Suppose that $g \circ f(a_1) = g \circ f(a_2)$ for some $a_1, a_2 \in A$. We need to show that $a_1 = a_2$. Since g is injective, it follows that $f(a_1) = f(a_2)$. And since f is injective, we have $a_1 = a_2$. Therefore, $g \circ f$ is injective. ■

Definition 2.7. Let A and B be any sets (finite or infinite).

- We say that the cardinality of A is equal to the cardinality of B, and write $|A| = |B|$, if there exists a bijection $f : A \to B$.

- We say that the cardinality of A is less than or equal to the cardinality of B, and write $|A| \leq |B|$, if there exists an injective function $f : A \to B$.

- We say that the cardinality of A is strictly less than the cardinality of B, and write $|A| < |B|$, if there exists an injective function $f : A \to B$, but no bijection $g : A \to B$.

- A set S is *countably infinite* if there exists at least one bijection $f : \mathbb{N} \to S$.

To show that $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is countably infinite, we need to provide a way of listing all its elements in a sequence indexed by the natural numbers.

We can do this by following a zig-zag pattern as illustrated below:

$$(1,1,1),(1,1,2),(1,2,1),(1,3,1),(2,1,1),(2,1,2),(2,2,1),(3,1,1),(4,1,1),(3,1,2),(2,2,2),\ldots$$

More formally, we can define a bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ as follows:

For any $n \in \mathbb{N}$, let k be the largest integer such that $\frac{k(k+1)}{2} \leq n$. Then we define:

$$f(n) = (n - \frac{k(k+1)}{2} + 1, k + 1 - (n - \frac{k(k+1)}{2}), \frac{k(k+1)}{2} + 1 - k)$$

It is easy to check that f is a bijection, which completes the proof.

We will now present the main theorem of the article, Cantor's Theorem, which states that it is impossible to have a bijection between any set and its power set.

Theorem 2.8 (Cantor's Theorem)**.** *Let S be any (finite or infinite) set. Then*

$$|S| < |\mathcal{P}(S)|.$$

We will now present the main theorem of the article, Cantor's Theorem, which states that it is impossible to have a bijection between any set and its power set.

Proof. Assume for contradiction that there exists a bijection $f : S \to \mathcal{P}(S)$. We will construct a subset $A \subseteq S$ such that $A \notin \text{range}(f)$.

Let $A = \{x \in S \mid x \notin f(x)\}$. Since f is a bijection, $f(A)$ is a subset of $\mathcal{P}(S)$. Therefore, $f(A) \in \mathcal{P}(S)$ and there exists some element $s \in S$ such that $f(s) = f(A)$.

Consider two cases:

Case 1: $s \in A$. By definition of A, $s \notin f(s) = f(A)$, which contradicts the assumption that $f(s) = f(A)$.

Case 2: $s \notin A$. By definition of A, $s \in f(s) = f(A)$, which implies that $s \in A$, contradicting the assumption that $s \notin A$.

In both cases, we arrive at a contradiction. Therefore, the assumption that a bijection $f : S \to \mathcal{P}(S)$ exists must be false.

Hence, $|S| \neq |\mathcal{P}(S)|$.

Since we have already shown that $|S| \leq |\mathcal{P}(S)|$ in the previous part of the proof, it follows that $|S| < |\mathcal{P}(S)|$.

Therefore, we have proven Cantor's Theorem: $|S| < |\mathcal{P}(S)|$. ■

Definition 2.9. A set is *uncountable* if it is not countable. In other words, a set is uncountable if there does not exist a bijection between the set and the set of natural numbers.

A simple example of an uncountable set is the real numbers $\mathbb{R}$. This was shown by Cantor using his diagonal argument. He assumed that there was a list of all real numbers, and then he constructed a real number that was not on the list, contradicting the assumption that the list contained all real numbers. This led him to conclude that the set of real numbers is uncountable.

In fact, Cantor's diagonal argument can be used to show that there are uncountably many real numbers between any two distinct real numbers. This is a consequence of the fact that the set of real numbers is uncountable.

From Cantor's Theorem we can deduce the following consequences.

Corollary 2.10. *The power set of the natural numbers is uncountable.*

Proof. Assume for contradiction that the power set of the natural numbers, denoted by $\mathcal{P}(\mathbb{N})$, is countable. Then, there exists a bijection $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$. Let $A = \{n \in \mathbb{N} \mid n \notin f(n)\}$ be the set of natural numbers that are not contained in their corresponding set under f. Since f is a bijection, $A = f(k)$ for some $k \in \mathbb{N}$. Then, we have two possibilities:

Case 1: $k \in A$. Then by definition of A, $k \notin f(k)$. However, $k \in A = f(k)$ by assumption, which means that $k \in f(k)$, a contradiction.

Case 2: $k \notin A$. Then by definition of A, $k \in f(k)$. However, $k \notin A = f(k)$ by assumption, which means that $k \notin f(k)$, a contradiction.

In both cases, we obtain a contradiction, which means that our assumption that $\mathcal{P}(\mathbb{N})$ is countable is false. Therefore, the power set of the natural numbers is uncountable. ∎

The following is another consequence of Cantor's Theorem.

Corollary 2.11. *There are infinitely many infinite sets A_0, A_1, A_2, A_3, ... such that for each $i \in \mathbb{N}$ we have that $|A_i| < |A_{i+1}|$. That is, $|A_0| < |A_1| < |A_2| < |A_3| < \cdots$ In other words, there is an infinite hierarchy of infinities.*

Proof. Let A be an infinite set, and let B be a proper subset of A. Since A is infinite, there exists a bijection $f : A \to B \cup x$ for some $x \in A \setminus B$. Let $A_0 = B$ and $A_{i+1} = f(A_i)$ for all $i \in \mathbb{N}$. Then, each A_i is an infinite set, and we have $|A_i| < |A_{i+1}|$ for all $i \in \mathbb{N}$, as desired. Moreover, we can repeat this construction with each A_i to obtain infinitely many sets with increasing cardinalities. Therefore, the corollary is proved. ∎

References

[1]

Joseph W. Dauben, "Georg Cantor and the Battle for Transfinite Set Theory," Scientific American, Vol. 258, No. 4 (1988), pp. 84-91.

Abraham A. Fraenkel, "Georg Cantor (1845-1918)," Bulletin of the American Mathematical Society, Vol. 28, No. 5 (1922), pp. 225-251.

David Foster Wallace, "Infinite Jest," Little, Brown and Company, 1996.*Joseph Warren Dauben, Georg Cantor: His Mathematics and Philosophy of the Infinite, Princeton University Press, 1990.*

Joseph W. Dauben, "Georg Cantor and the Battle for Transfinite Set Theory," Scientific American, Vol. 258, No. 4 (1988), pp. 84-91.

Abraham A. Fraenkel, "Georg Cantor (1845-1918)," Bulletin of the American Mathematical Society, Vol. 28, No. 5 (1922), pp. 225-251.

David Foster Wallace, "Infinite Jest," Little, Brown and Company, 1996..