

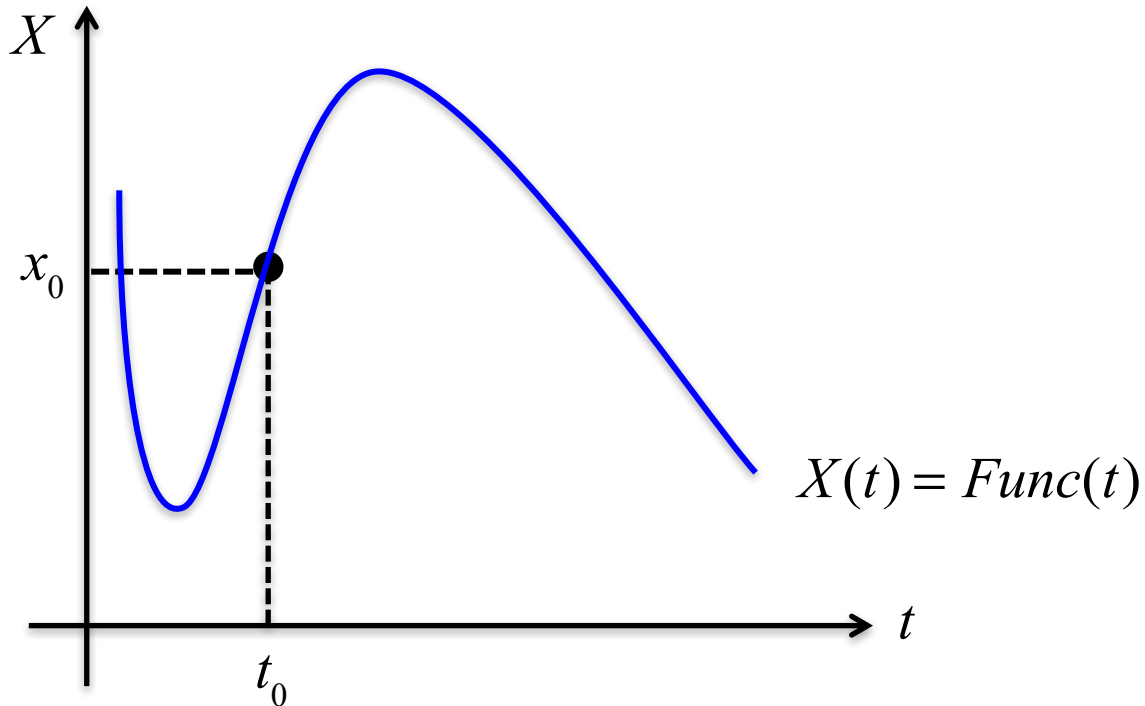
Brownian motion 1: basic theories

Basic knowledge of stochastic process

Stochastic process

A deterministic process:

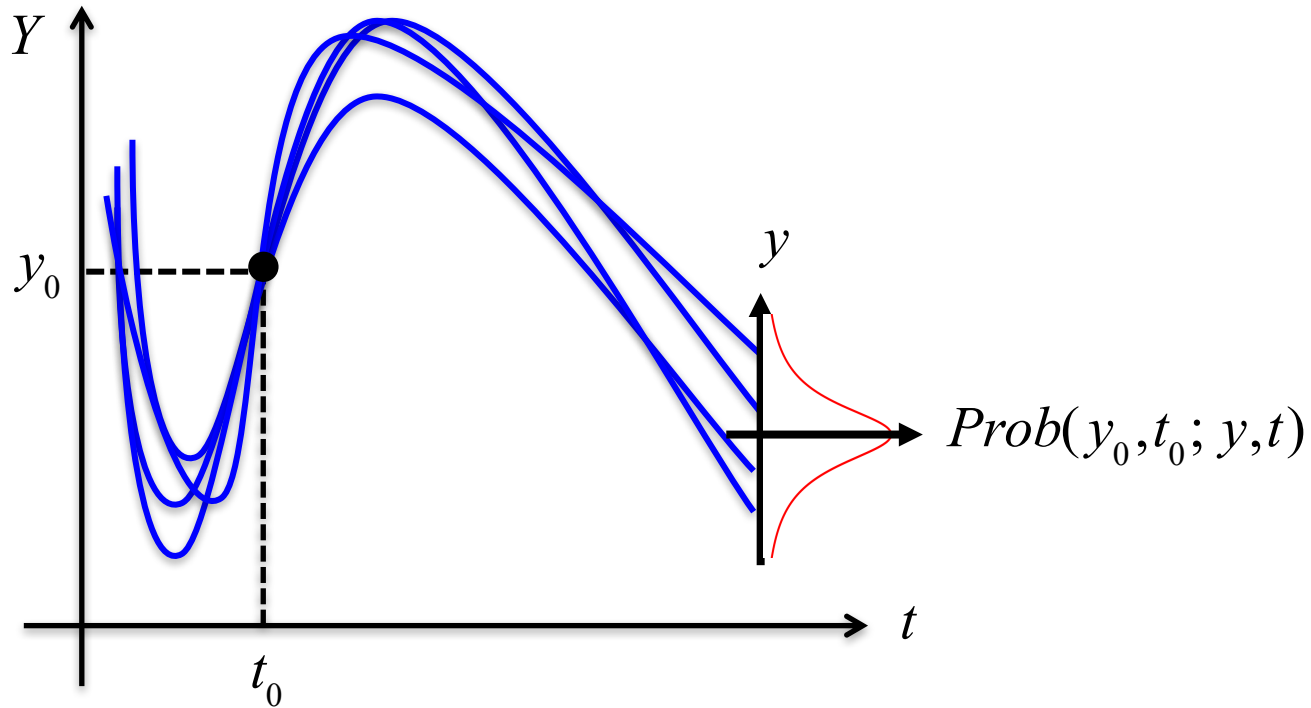
$$X(t) = \text{Func}(t)$$



Stochastic process

A stochastic process:

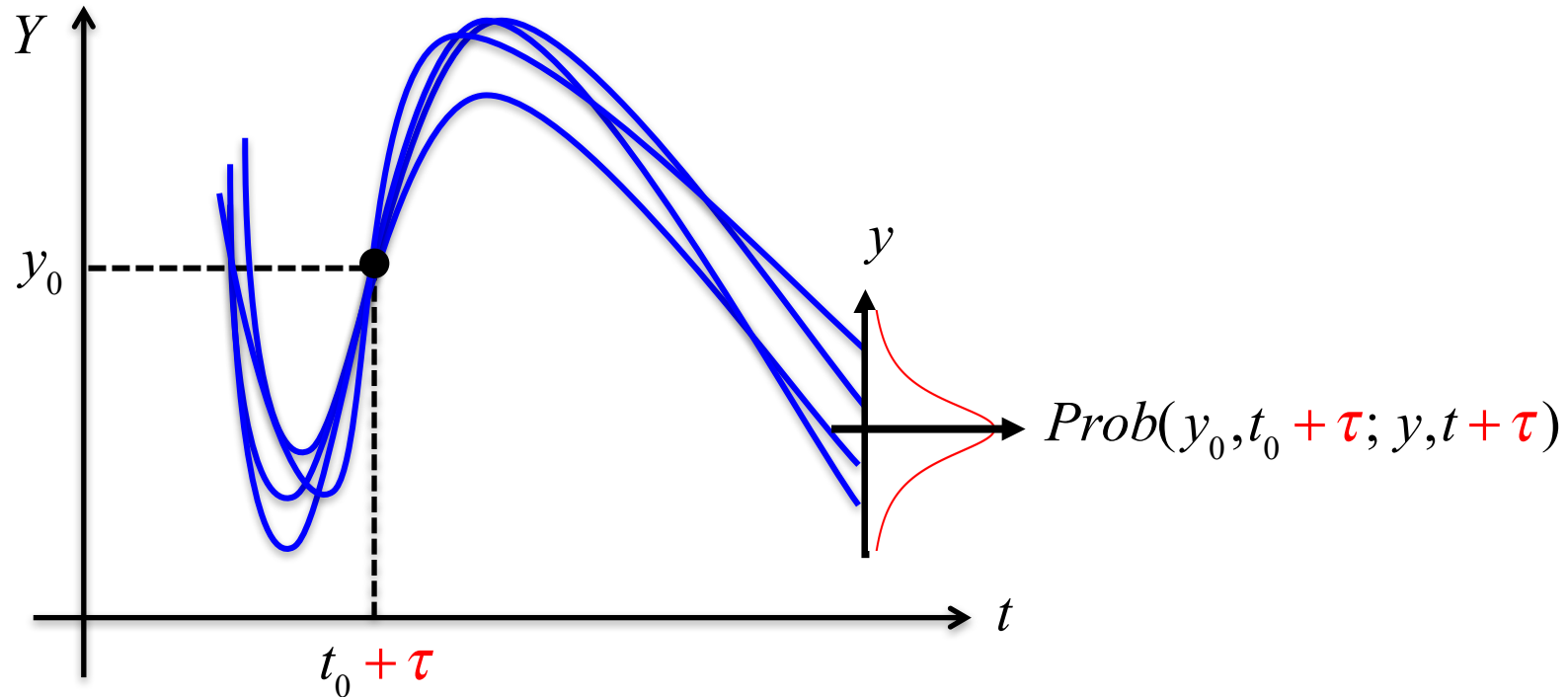
$$Y(t) \neq \text{Func}(t) \rightarrow \text{Prob}(y_0, t_0; y, t)$$



Stochastic process

A stochastic process:

$$Y(t) \neq \text{Func}(t) \rightarrow \text{Prob}(y_0, t_0; y, t)$$



A steady stochastic process:

$$\text{Prob}(y_0, t_0 + \tau; y, t + \tau) = \text{Prob}(y_0, t_0; y, t)$$

Stochastic process

Consider a steady stochastic process $Y(t)$ with its mean $\langle Y(t) \rangle = 0$ and define Fourier transformation

$$\tilde{Y}_T(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} Y_T(t) \quad (1)$$

and inverse Fourier transformation

$$Y_T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \tilde{Y}_T(\omega) \quad (2)$$

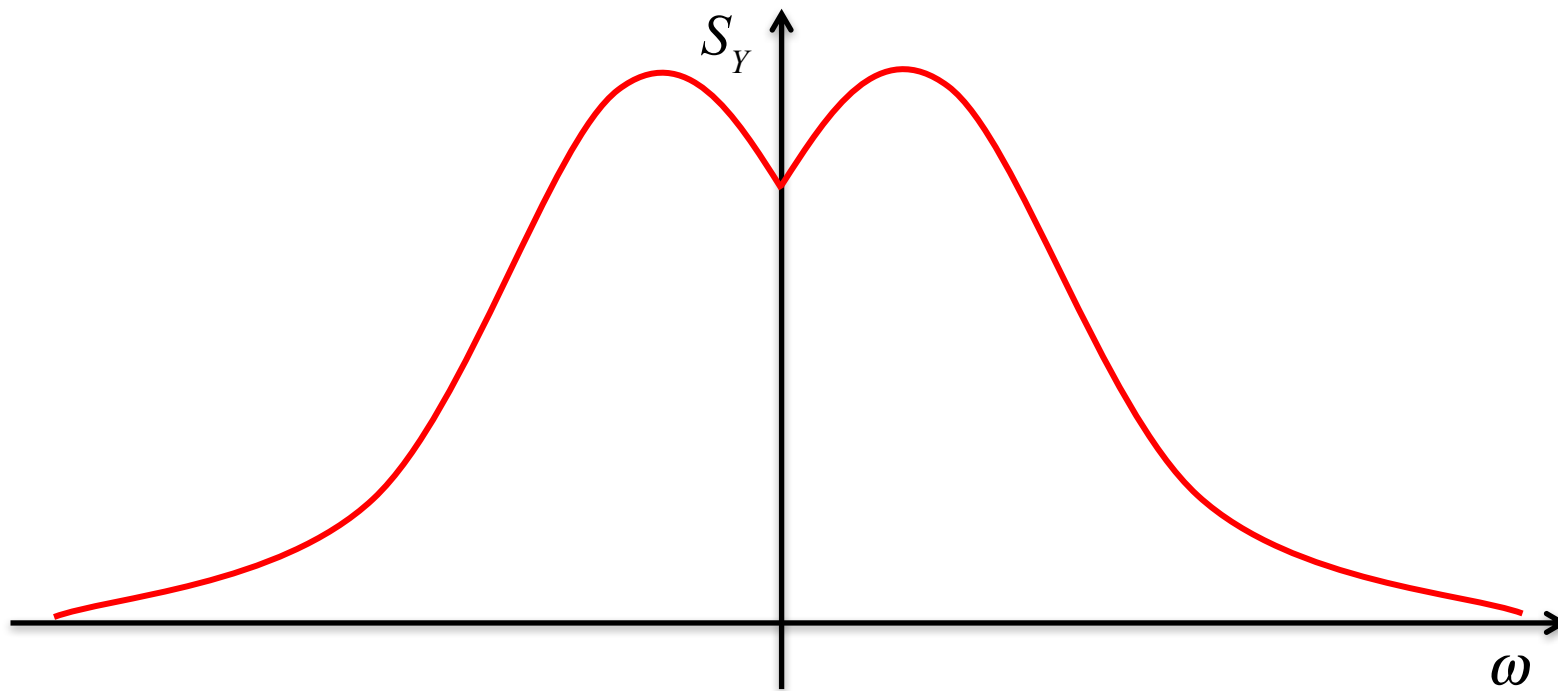
using

$$\begin{aligned} Y_T(t) &= Y(t) & (|t| \leq T/2) \\ Y_T(t) &= 0 & (|t| > T/2) \end{aligned} \quad (3)$$

Stochastic process

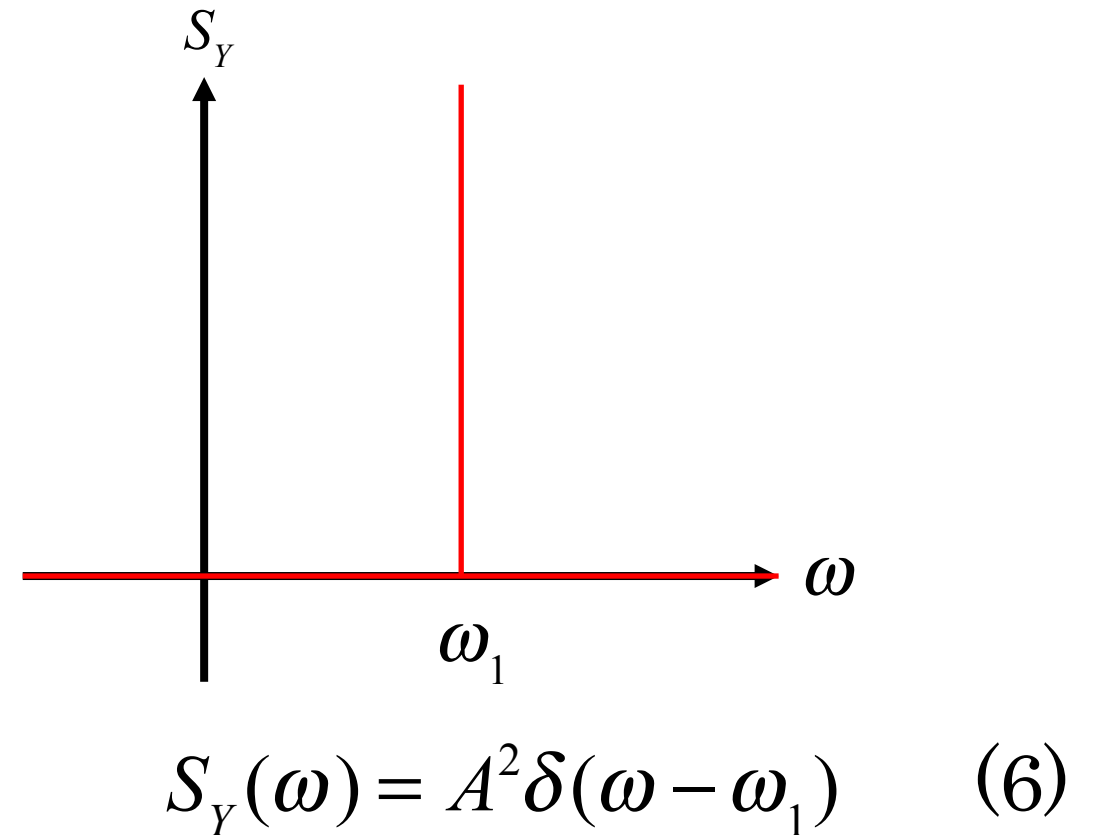
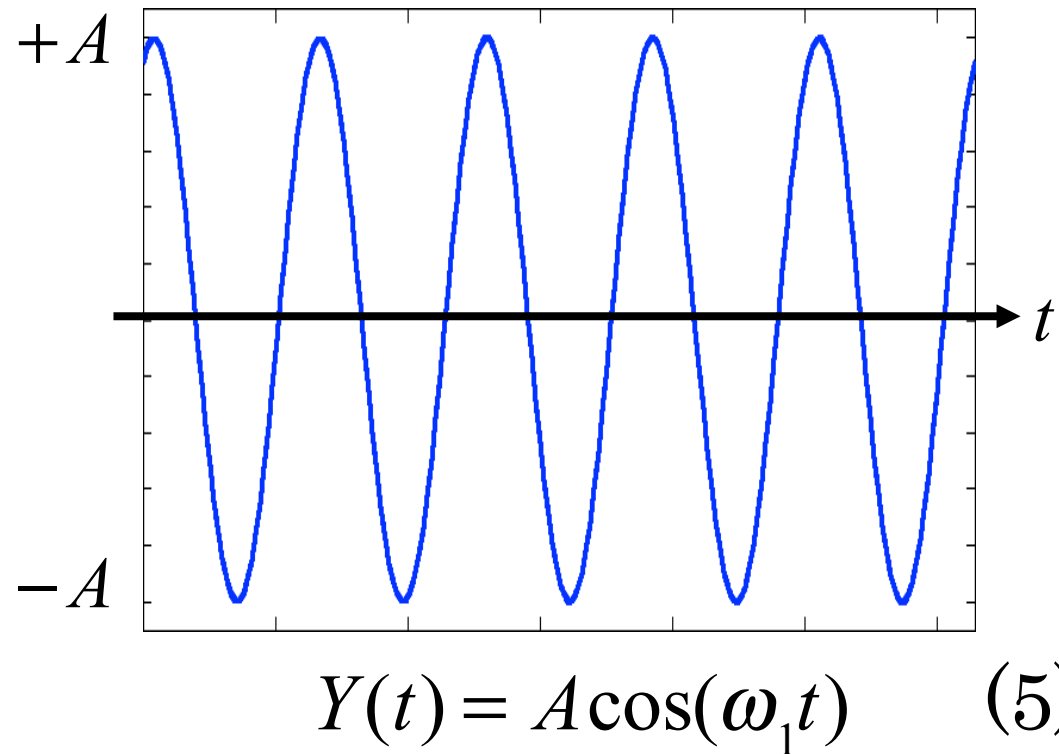
Spectral density / Power spectrum

$$S_Y(\omega) \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \left| \tilde{Y}_T(\omega) \right|^2 \quad (4)$$



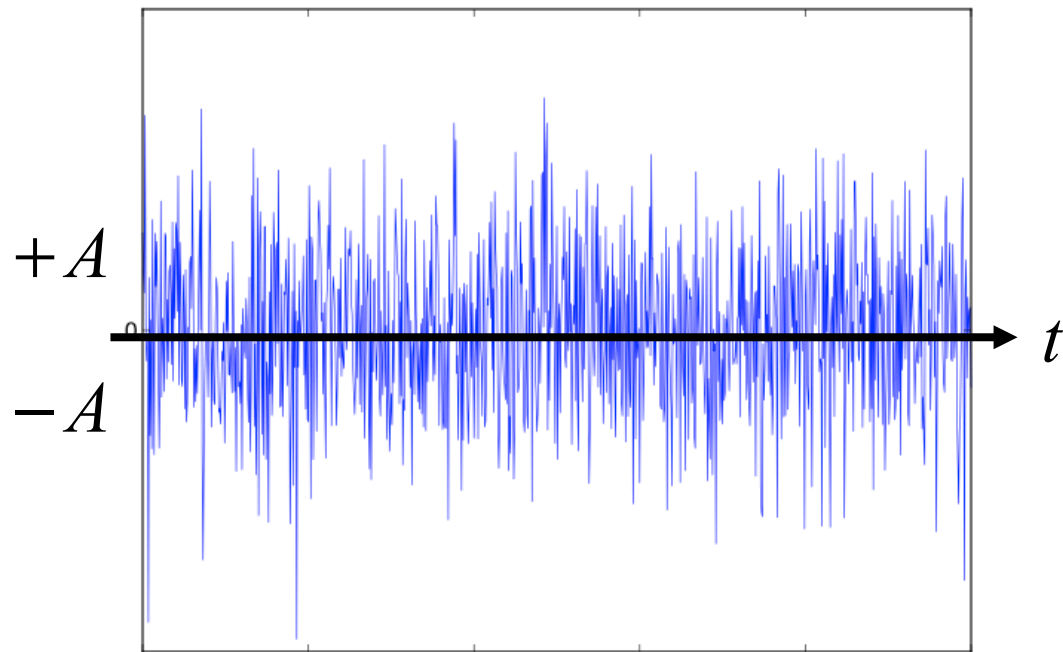
Stochastic process

Case 1: Single cosine wave

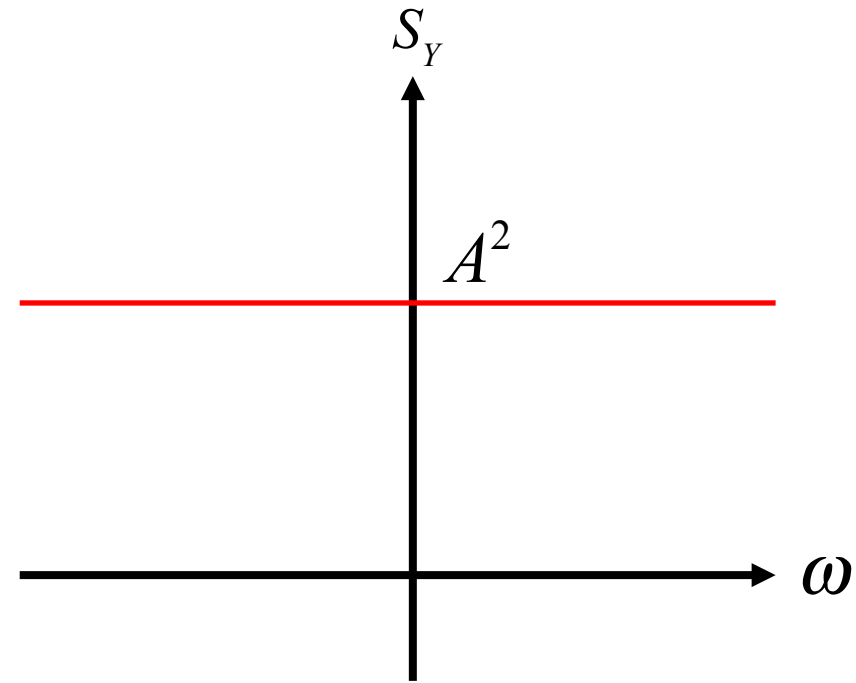


Stochastic process

Case 2: White noise



$$Y(t) = A\xi(t) \quad (7)$$

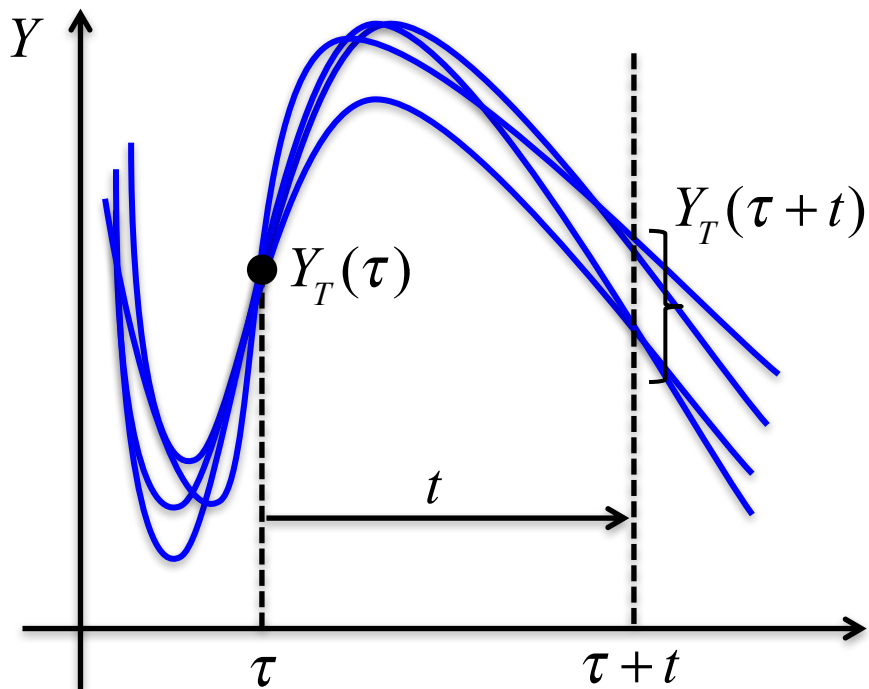


$$S_Y(\omega) = A^2 \quad (8)$$

Stochastic process

Auto-correlation function

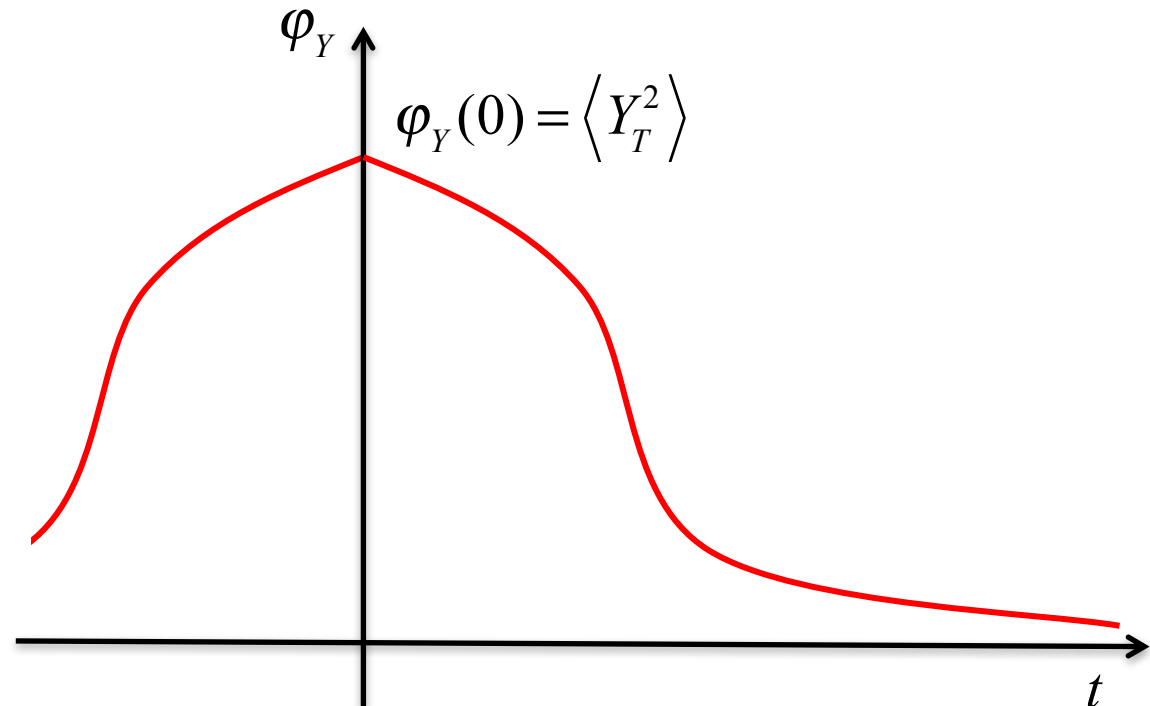
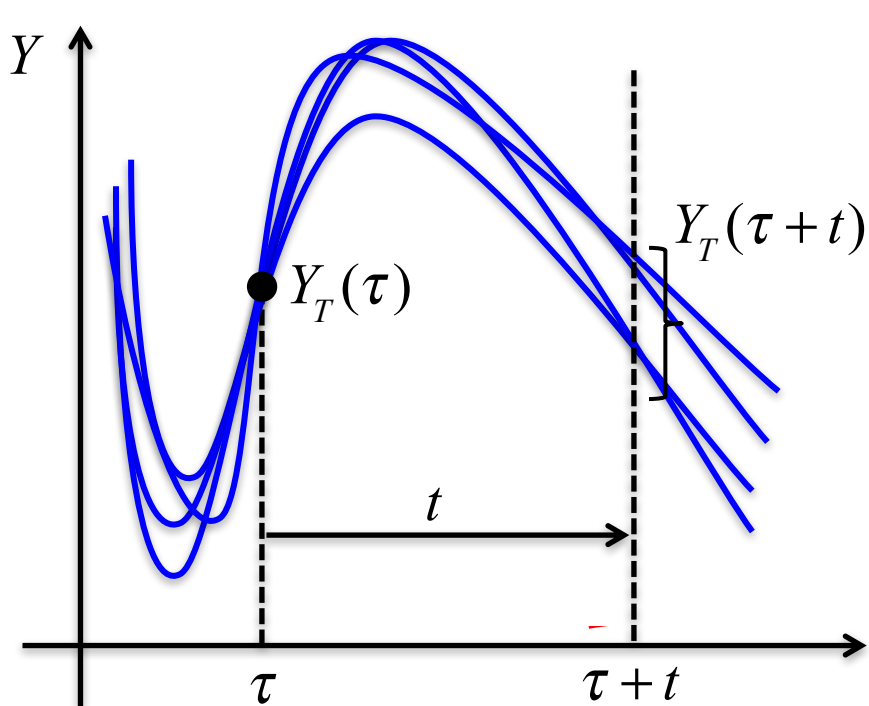
$$\varphi_Y(t) \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{\infty} d\tau Y_T(\tau) Y_T(\tau + t) \equiv \langle Y(\tau) Y(\tau + t) \rangle_{\tau} \quad (9)$$



Stochastic process

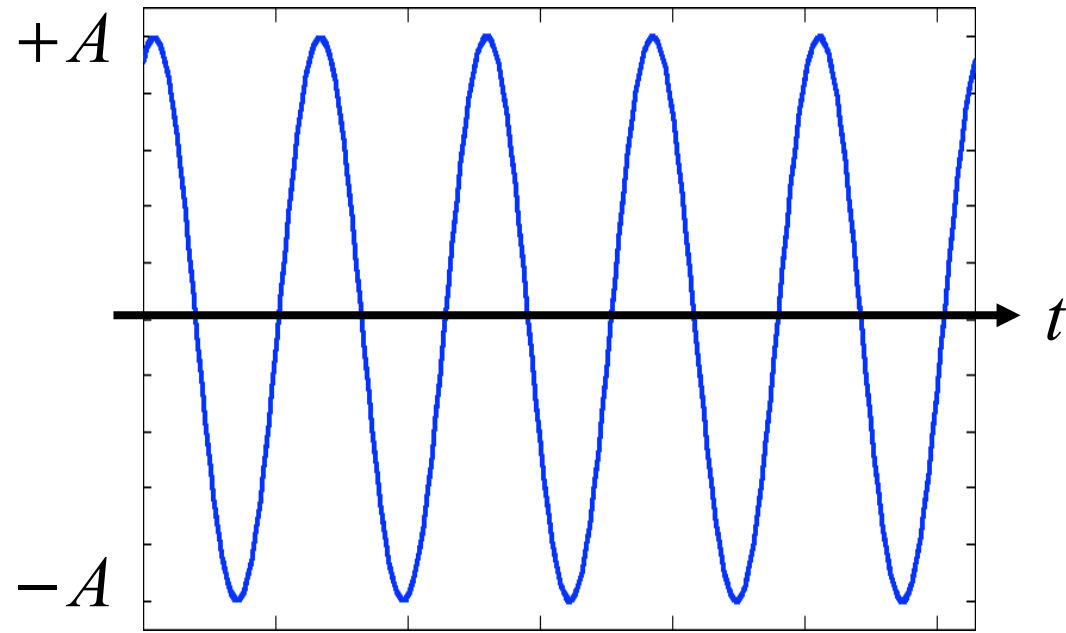
Auto-correlation function

$$\varphi_Y(t) \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{\infty} d\tau Y_T(\tau) Y_T(\tau + t) \equiv \langle Y(\tau) Y(\tau + t) \rangle_{\tau} \quad (9)$$

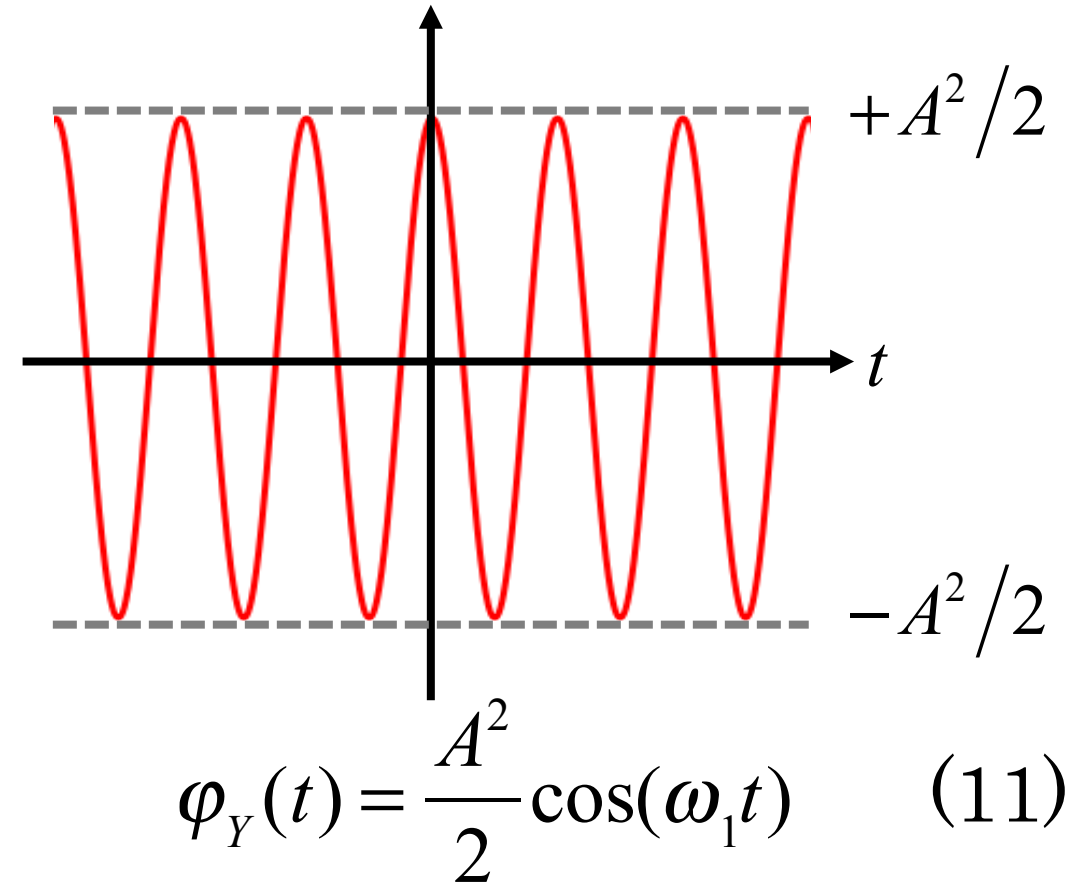


Stochastic process

Case 1: Single cosine wave



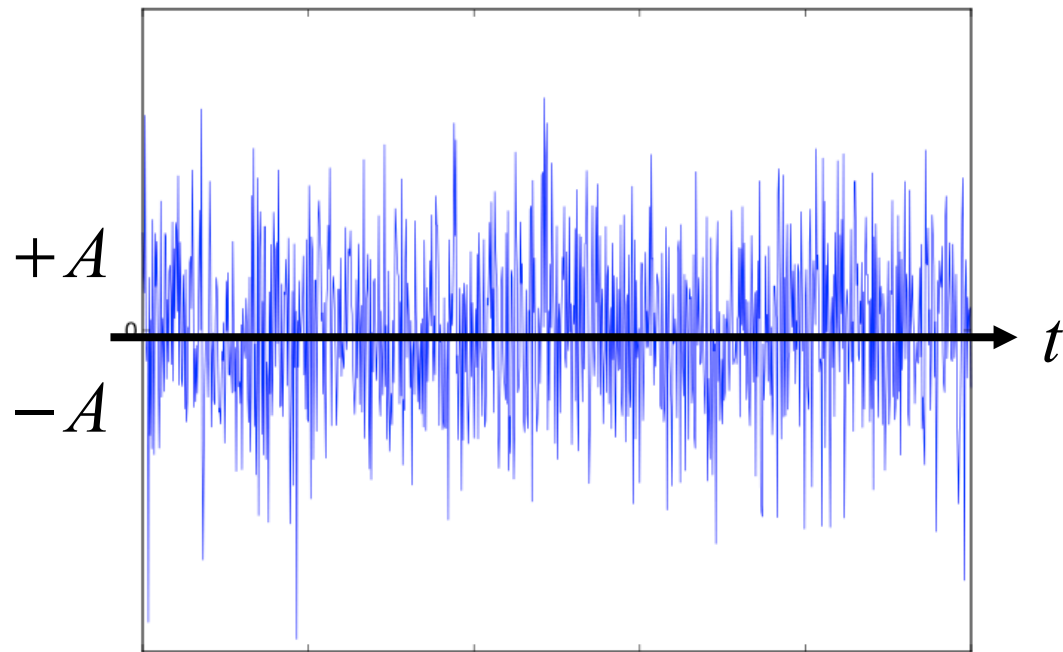
$$Y(t) = A \cos(\omega_1 t) \quad (10)$$



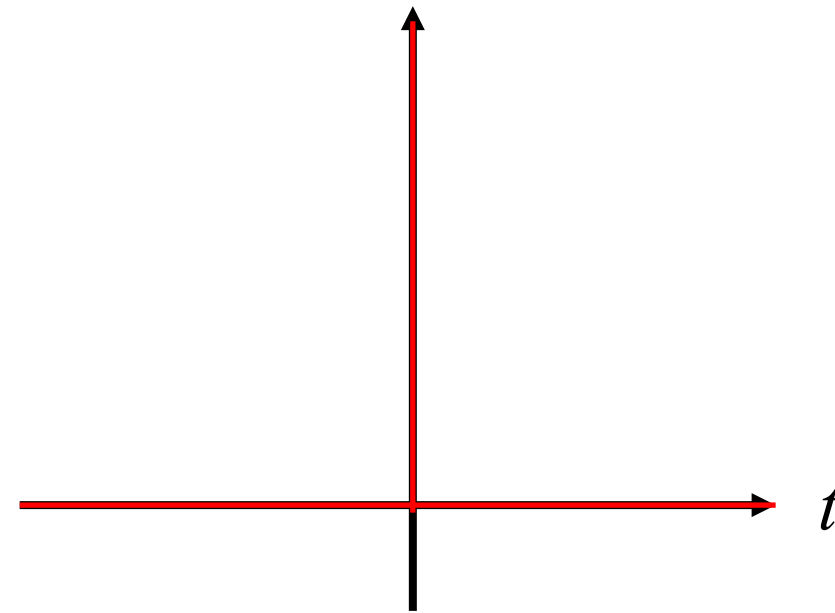
$$\varphi_Y(t) = \frac{A^2}{2} \cos(\omega_1 t) \quad (11)$$

Stochastic process

Case 2: White noise



$$Y(t) = A\xi(t) \quad (12)$$



$$\varphi_Y(\omega) = A^2\delta(t) \quad (13)$$

Stochastic process

From Eq.(9),

$$\begin{aligned}\varphi_Y(t) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{\infty} d\tau \left[Y_T(\tau) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(\tau+t)} \tilde{Y}_T(\omega) \right] \right] \\ &= \lim_{T \rightarrow \infty} \frac{1}{2\pi T} \int_{-\infty}^{\infty} d\omega \left[e^{-i\omega t} \tilde{Y}_T(\omega) \int_{-\infty}^{\infty} d\tau \left[e^{-i\omega\tau} Y_T(\tau) \right] \right] \\ &= \lim_{T \rightarrow \infty} \frac{1}{2\pi T} \int_{-\infty}^{\infty} d\omega \left[e^{-i\omega t} \tilde{Y}_T(\omega) \tilde{Y}_T^*(\omega) \right] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \left[\lim_{T \rightarrow \infty} \frac{1}{T} |\tilde{Y}_T(\omega)|^2 \right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} S_Y(\omega) \quad (14)\end{aligned}$$

Stochastic process

And also,

$$S_Y(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \varphi_Y(t) \quad (15)$$

Wiener-Khintchine theorem:

$$\boxed{\varphi_Y(t) \begin{array}{c} \xrightarrow{\text{inverse Foulter Eq.(14)} \\ \xleftarrow{\text{Foulter Eq.(15)}} \end{array} S_Y(\omega)}$$

Sum rules:

$$\varphi_Y(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega S_Y(\omega) \quad (16)$$

$$S_Y(0) = \int_{-\infty}^{\infty} dt \varphi_Y(t) \quad (17)$$

Stochastic process

Supplemental note for the derivation of Eq. (11)

$$\begin{aligned}\varphi_Y(t) &= \lim_{T \rightarrow \infty} \frac{A^2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \cos(\omega_1 \tau) \cos(\omega_1(\tau + t)) \\&= \lim_{T \rightarrow \infty} \frac{A^2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \sin\left(\omega_1 \tau + \frac{\pi}{2}\right) \sin\left(\omega_1(\tau + t) + \frac{\pi}{2}\right) \\&= \lim_{T \rightarrow \infty} \frac{A^2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \sin\left(\omega_1 \tau + \frac{\pi}{2}\right) \left[\sin\left(\omega_1 \tau + \frac{\pi}{2}\right) \cos(\omega_1 t) + \cos\left(\omega_1 \tau + \frac{\pi}{2}\right) \sin(\omega_1 t) \right] \\&= \lim_{T \rightarrow \infty} \frac{A^2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \left[\sin^2\left(\omega_1 \tau + \frac{\pi}{2}\right) \cos(\omega_1 t) + \sin\left(\omega_1 \tau + \frac{\pi}{2}\right) \cos\left(\omega_1 \tau + \frac{\pi}{2}\right) \sin(\omega_1 t) \right] \\&= \cos(\omega_1 t) \lim_{T \rightarrow \infty} \frac{A^2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \sin^2\left(\omega_1 \tau + \frac{\pi}{2}\right) + \sin(\omega_1 t) \lim_{T \rightarrow \infty} \frac{A^2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \sin\left(\omega_1 \tau + \frac{\pi}{2}\right) \cos\left(\omega_1 \tau + \frac{\pi}{2}\right) \\&= \frac{A^2}{2} \cos(\omega_1 t) + 0 \quad \dots \quad \text{Eq.(11)}\end{aligned}$$