scratch (/github/ryo0921/scratch/tree/master) / 01 (/github/ryo0921/scratch/tree/master/01)

Stochastic Processes: Data Analysis and Computer Simulation

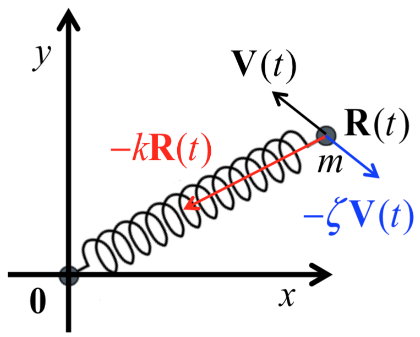
Python programming for beginners

4. Simulating a damped harmonic oscillator

4.1. A damped harmonic oscillator

Model system

- Spring constant: k
- Particle mass: m
- Friction constant: ζ
- Particle position: $\mathbf{R}(t)$
- Particle velocity: $\mathbf{V}(t)$
- Friction force: $-\zeta\mathbf{V}(t)$
- Spring force: $-k\mathbf{R}(t)$



Time evolution equations

$$\frac{d\mathbf{R}(t)}{dt} = \mathbf{V}(t) \quad \text{(B1)}$$

$$m\frac{d\mathbf{V}(t)}{dt} = -\zeta\mathbf{V}(t) - k\mathbf{R}(t) \quad \text{(B2)}$$

4.2. Computer simulation

Euler method

- Use Eq.(A8) in the previous lessen

$$\mathbf{R}_{i+1} = \mathbf{R}_i + \int_{t_i}^{t_{i+1}} dt\mathbf{V}(t) \simeq \mathbf{R}_i + \mathbf{V}_i\Delta t \tag{B3}$$

$$\begin{aligned}\mathbf{V}_{i+1} &= \mathbf{V}_i - \frac{\zeta}{m}\int_{t_i}^{t_{i+1}} dt\mathbf{V}(t) - \frac{k}{m}\int_{t_i}^{t_{i+1}} dt\mathbf{R}(t) \\ &\simeq \left(1 - \frac{\zeta}{m}\Delta t\right)\mathbf{V}_i - \frac{k}{m}\mathbf{R}_i\Delta t \end{aligned} \tag{B4}$$

Import libraries

In [1]:
```
% matplotlib nbagg
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.animation as animation
plt.style.use('ggplot')
```

Define variables

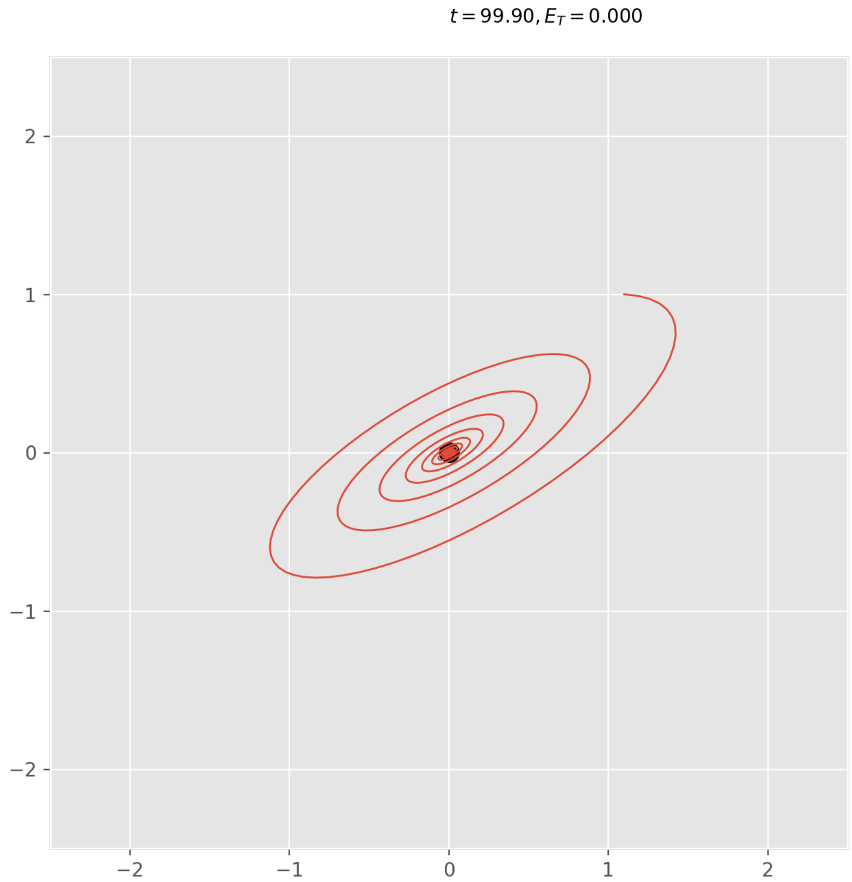
In [2]:
```
dim  = 2    # system dimension (x,y)
nums = 1000 # number of steps
R    = np.zeros(dim) # particle position
V    = np.zeros(dim) # particle velocity
Rs   = np.zeros([dim,nums]) # particle position (at all steps)
Vs   = np.zeros([dim,nums]) # particle velocity (at all steps)
Et   = np.zeros(nums) # total enegy of the system (at all steps)
time = np.zeros(nums) # time (at all steps)
```

Define functions

In [3]:
```
def init(): # initialize animation
    particles.set_data([], [])
    line.set_data([], [])
    title.set_text(r'')
    return particles,line,title
def animate(i): # define amination using Euler
    global R,V,F,Rs,Vs,time,Et
    R, V = R+V*dt, V*(1-zeta/m*dt)-k/m*dt*R # Euler method Eqs.(B3)&(B4)
    Rs[0:dim,i]=R
    Vs[0:dim,i]=V
    time[i]=i*dt
    Et[i]=0.5*m*np.linalg.norm(V)**2+0.5*k*np.linalg.norm(R)**2
    particles.set_data(R[0], R[1])      # current position
    line.set_data(Rs[0,0:i], Rs[1,0:i]) # add latest position Rs
    title.set_text(r"$t = {0:.2f},E_T = {1:.3f}$".format(i*dt,Et[i]))
    return particles,line,title
```

Perform the simulation

In [4]:
```
# System parameters
# particle mass, spring & friction constants
m, k, zeta = 1.0, 1.0, 0.25
# Initial condition
R[0], R[1] = 1., 1. # Rx(0), Ry(0)
V[0], V[1] = 1., 0. # Vx(0), Vy(0)
dt   = 0.1*np.sqrt(k/m) # set \Delta t
box  = 5 # set size of draw area
# set up the figure, axis, and plot element for animatation
fig, ax = plt.subplots(figsize=(7.5,7.5)) # setup plot
ax = plt.axes(xlim=(-box/2,box/2),ylim=(-box/2,box/2)) # draw range
particles, = ax.plot([],[],'ko', ms=10) # setup plot for particle
line,=ax.plot([],[],lw=1) # setup plot for trajectry
title=ax.text(0.5,1.05,r'',transform=ax.transAxes,va='center') # title
anim=animation.FuncAnimation(fig,animate,init_func=init,
     frames=nums,interval=5,blit=True,repeat=False) # draw animation
# anim.save('movie.mp4',fps=20,dpi=400)
```



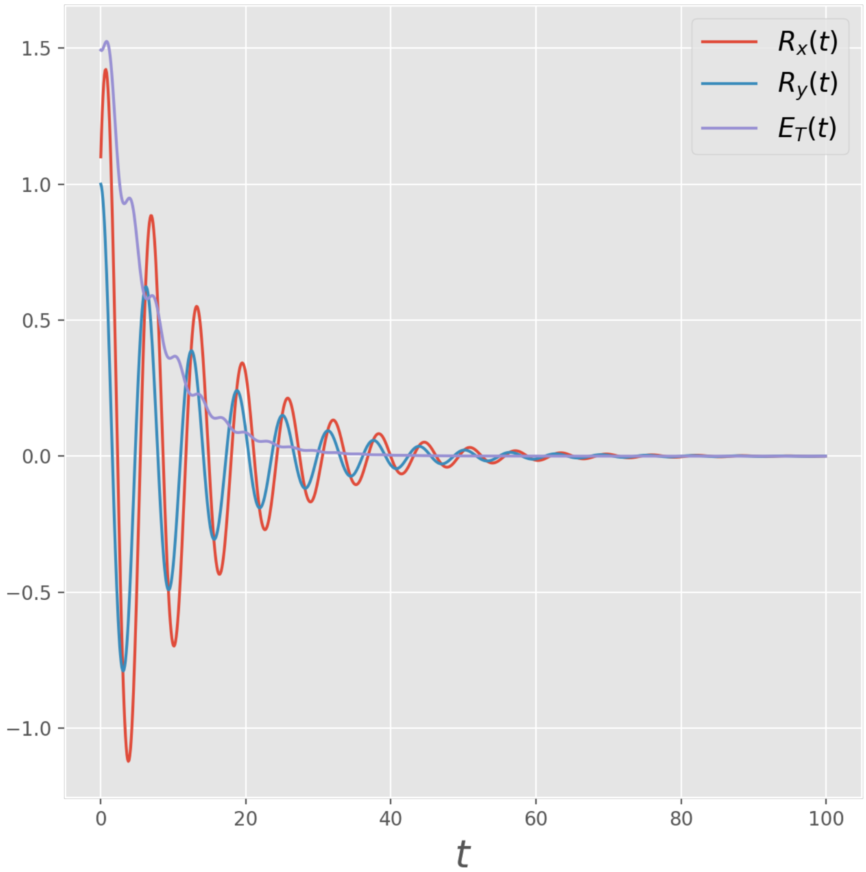
Analyze the simulation results

Temporal values of $R_x(t)$, $R_y(t)$, $E_T(t)$

- Total energy of the harmonic oscillator

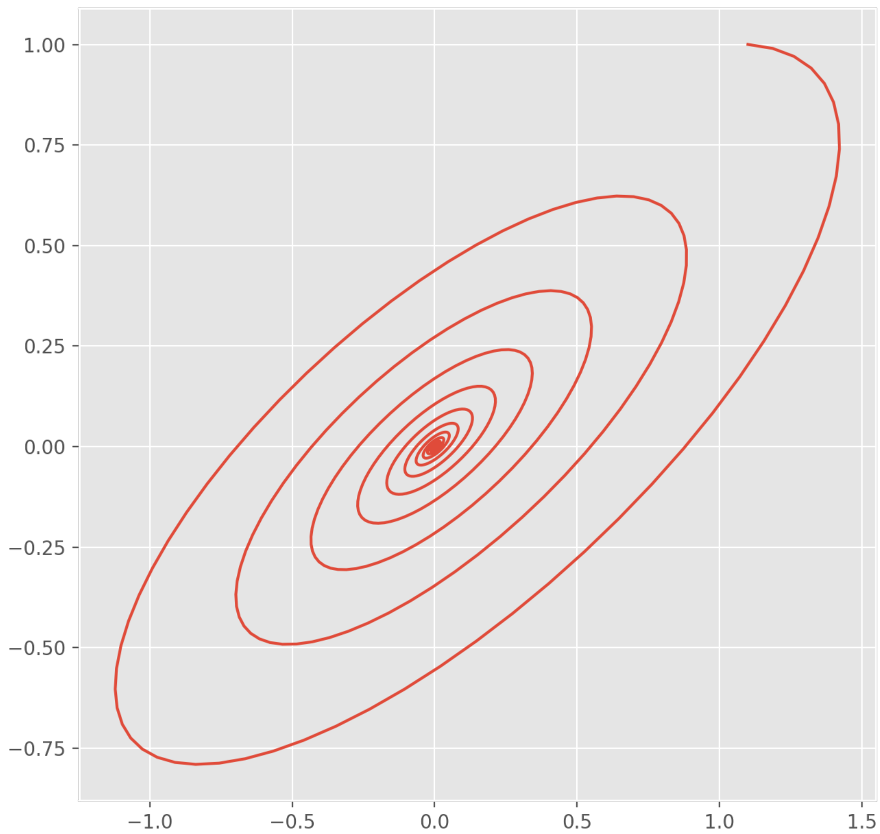
$$E_T(t) = E_{kinetic}(t) + E_{potential}(t) = \frac{1}{2}m\mathbf{V}^2(t) + \frac{1}{2}k\mathbf{R}^2(t)$$

In [5]:
```
fig, ax = plt.subplots(figsize=(7.5,7.5))
ax.set_xlabel(r"$t$", fontsize=20)
ax.plot(time,Rs[0]) # plot R_x(t)
ax.plot(time,Rs[1]) # plot R_y(t)
ax.plot(time,Et) # plot E(t) (ideally constant if \deta=0)
ax.legend([r'$R_x(t)$',r'$R_y(t)$',r'$E_T(t)$'], fontsize=14)
plt.show()
```



Trajectory plot

In [6]:
```
fig, ax = plt.subplots(figsize=(7.5,7.5))
ax.plot(Rs[0,0:nums],Rs[1,0:nums]) # parameteric plot Rx(t) vs. Ry(t)
plt.show()
```



Supplemental Note: Simulation schemes with other methods

Runge-Kutta 2nd order method

- Use Eqs.(A15) and (A16) in the previous lessen

$$\mathbf{R}'_{i+\frac{1}{2}} = \mathbf{R}_i + \frac{\Delta t}{2}\mathbf{V}_i \tag{B5}$$

$$\mathbf{V}'_{i+\frac{1}{2}} = \mathbf{V}_i - \frac{\zeta}{m}\frac{\Delta t}{2}\mathbf{V}_i - \frac{k}{m}\frac{\Delta t}{2}\mathbf{R}_i \tag{B6}$$

$$\mathbf{R}_{i+1} = \mathbf{R}_i + \Delta t\mathbf{V}'_{i+\frac{1}{2}} \tag{B7}$$

$$\mathbf{V}_{i+1} = \mathbf{V}_i - \frac{\zeta}{m}\Delta t\mathbf{V}'_{i+\frac{1}{2}} - \frac{k}{m}\Delta t\mathbf{R}'_{i+\frac{1}{2}} \tag{B8}$$

Runge-Kutta 4th order method

- Use Eqs.(A18) - (A21) in the previous lessen

$$\begin{aligned}
\mathbf{R}'_{i+\frac{1}{2}} &= \mathbf{R}_i + \frac{\Delta t}{2}\mathbf{V}_i && \text{(B9)}\\
\mathbf{V}'_{i+\frac{1}{2}} &= \mathbf{V}_i - \frac{\zeta}{m}\frac{\Delta t}{2}\mathbf{V}_i - \frac{k}{m}\frac{\Delta t}{2}\mathbf{R}_i && \text{(B10)}\\
\mathbf{R}''_{i+\frac{1}{2}} &= \mathbf{R}_i + \frac{\Delta t}{2}\mathbf{V}'_i && \text{(B11)}\\
\mathbf{V}''_{i+\frac{1}{2}} &= \mathbf{V}_i - \frac{\zeta}{m}\frac{\Delta t}{2}\mathbf{V}'_i - \frac{k}{m}\frac{\Delta t}{2}\mathbf{R}'_i && \text{(B12)}\\
\mathbf{R}'''_{i+1} &= \mathbf{R}_i + \Delta t\mathbf{V}''_i && \text{(B13)}\\
\mathbf{V}'''_{i+1} &= \mathbf{V}_i - \frac{\zeta}{m}\Delta t\mathbf{V}''_i - \frac{k}{m}\Delta t\mathbf{R}''_i && \text{(B14)}\\
\mathbf{R}_{i+1} &= \mathbf{R}_i + \frac{\Delta t}{6}\left(\mathbf{V}_i + 2\mathbf{V}'_{i+\frac{1}{2}} + 2\mathbf{V}''_{i+\frac{1}{2}} + \mathbf{V}'''_{i+1}\right) && \text{(B15)}\\
\mathbf{V}_{i+1} &= \mathbf{V}_i - \frac{\zeta}{m}\frac{\Delta t}{6}\left(\mathbf{V}_i + 2\mathbf{V}'_{i+\frac{1}{2}} + 2\mathbf{V}''_{i+\frac{1}{2}} + \mathbf{V}'''_{i+1}\right)\\
&\quad -\frac{k}{m}\frac{\Delta}{6}t\left(\mathbf{R}_i + 2\mathbf{R}'_{i+\frac{1}{2}} + 2\mathbf{R}''_{i+\frac{1}{2}} + \mathbf{R}'''_{i+1}\right) && \text{(B16)}
\end{aligned}$$