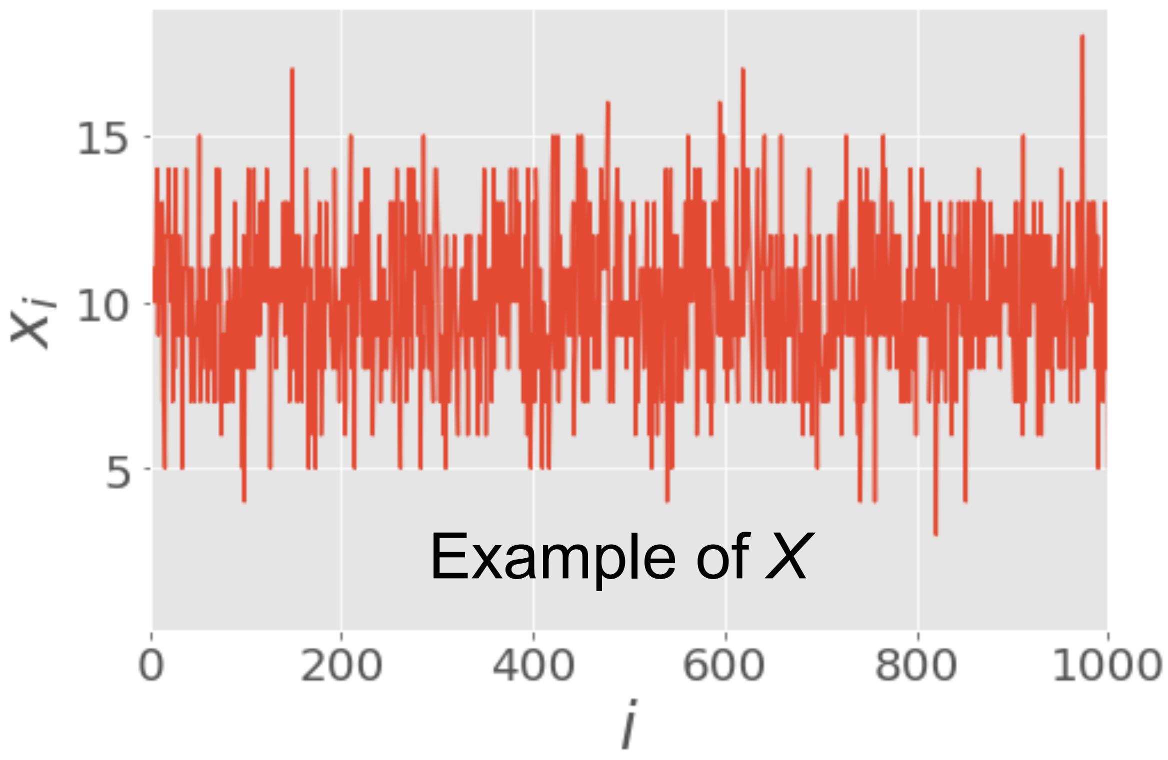
Distribution functions & random numbers

Stochastic variables and distribution functions

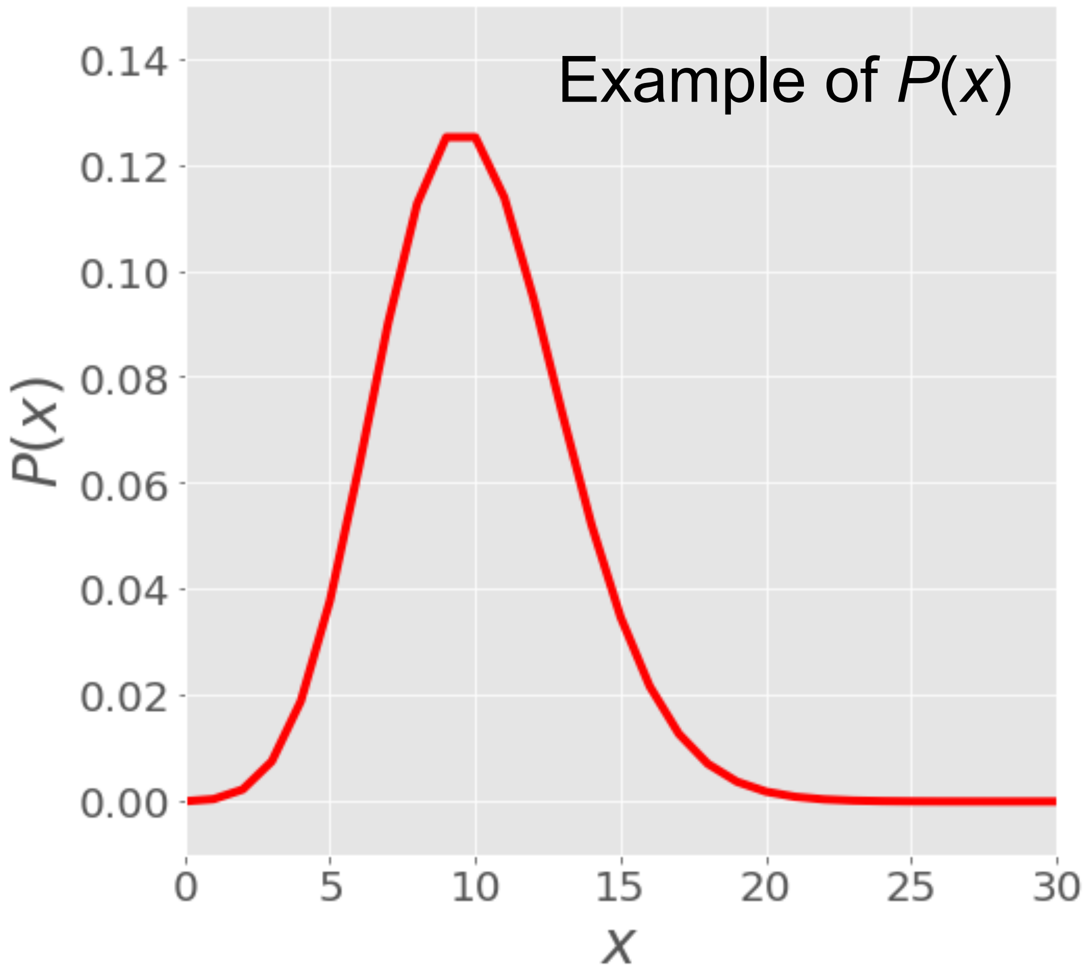
Stochastic variable and distribution functions

Stochastic variable:

$$X(=x_1, x_2, \cdots)$$



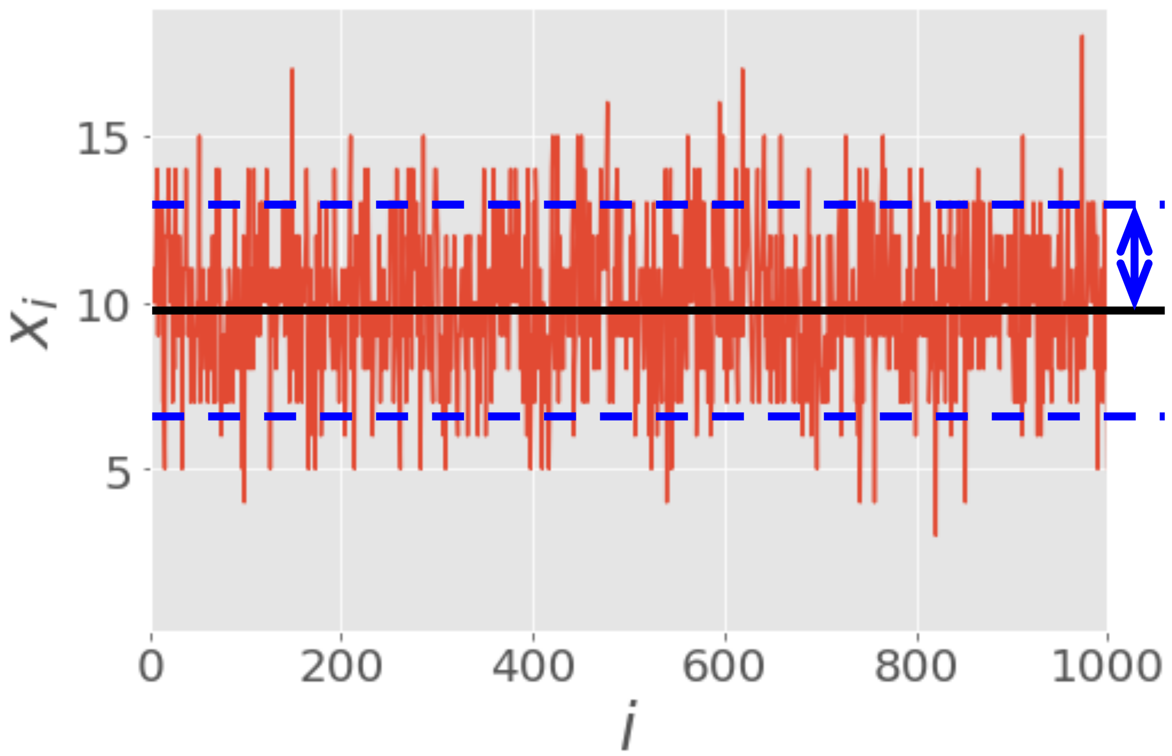
Distribution function: $P(x)$

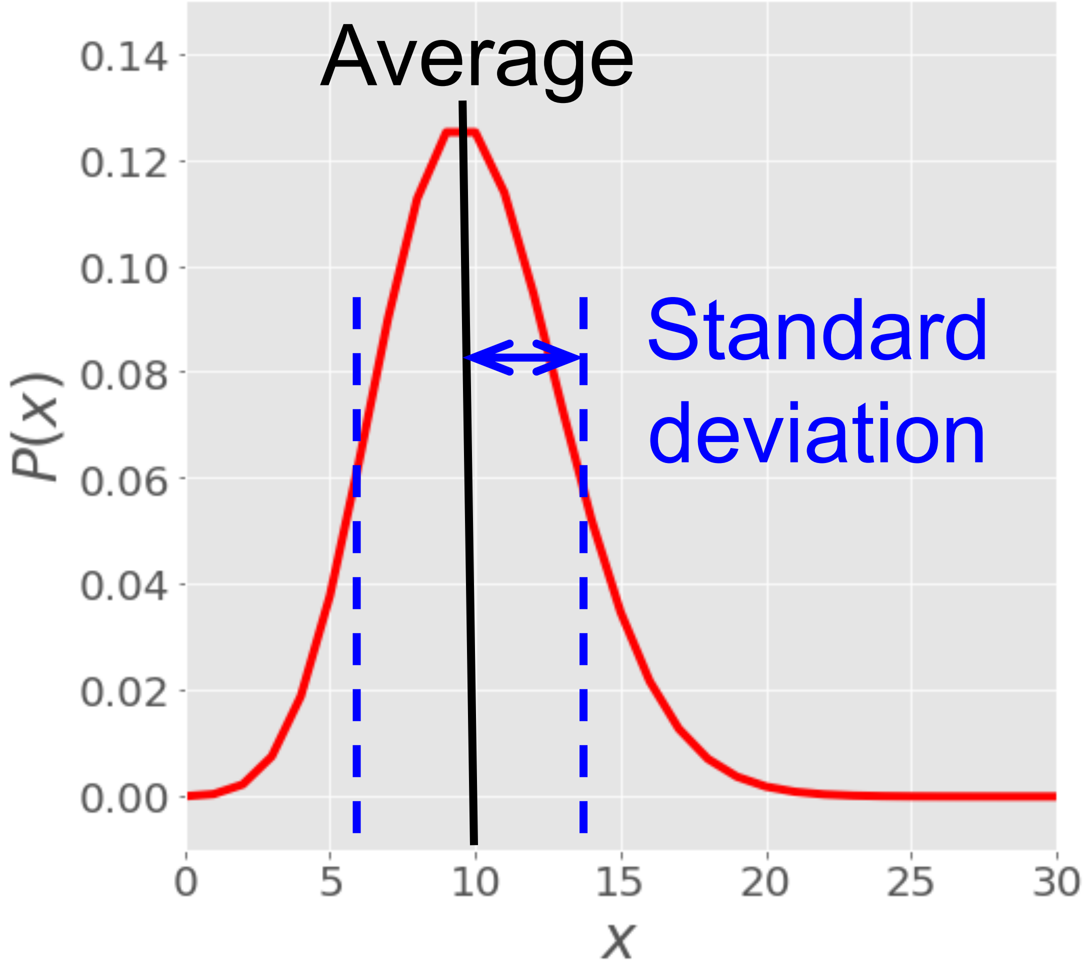


Stochastic variable and distribution functions

Stochastic variable: $X(=x_1, x_2, \cdots)$

Distribution function: $P(x)$





Stochastic variable and distribution functions

Basic things to know about $P(x)$ for x = real & continuum

1. Positive : $P(x) \geq 0$

2. Normalization : $\int_{-\infty}^{\infty} P(x)\,dx = 1$

3. Moment (m-th) : $\mu_m \equiv \langle X^m \rangle = \int_{-\infty}^{\infty} x^m P(x)\,dx$

4. Average : $\langle f(X) \rangle = \int_{-\infty}^{\infty} f(x) P(x)\,dx$

$$\langle X \rangle = \int_{-\infty}^{\infty} x P(x)\,dx = \mu_1$$

Stochastic variable and distribution functions

Basic things to know about $P(x)$ for x = real & continuum

5. Variance : $\sigma^2 \equiv \left\langle \left(X - \langle X \rangle \right)^2 \right\rangle = \langle X^2 \rangle - \langle X \rangle^2 = \mu_2 - \mu_1^2$

(σ is the Standard deviation)

6. Generating function : $G(k) \equiv \left\langle e^{-ikx} \right\rangle = \sum_{n=0}^{\infty} (ik)^n \frac{\mu_n}{n!}$

$$G'(0) = \mu_1, \quad G''(0) = \mu_2/2!, \quad G'''(0) = \mu_3/3!, \quad \cdots$$

$$\left(\because e^{-ikx} = 1 - ikx + \frac{1}{2!}(-ik)^2 x^2 + \frac{1}{3!}(-ik)^3 x^3 + \cdots \right)$$

Stochastic variable and distribution functions

Basic things to know about $P(n)$ for n = non-negative integer

1. Positive : $0 \le P(n) \le 1$
2. Normalization : $\sum_{n=0}^{\infty} P(n) = 1$
3. Moment (m-th) : $\mu_m \equiv \langle N^m \rangle = \sum_{n=0}^{\infty} n^m P(n)$
4. Average : $\langle f(N) \rangle = \sum_{n=0}^{\infty} f(n) P(n)$

$$\langle N \rangle = \sum_{n=0}^{\infty} n P(n) = \mu_1$$

Stochastic variable and distribution functions

Basic things to know about $P(n)$ for n = non-negative integer

5. Variance : $\sigma^2 \equiv \left\langle \left(N - \langle N \rangle\right)^2 \right\rangle = \langle N^2 \rangle - \langle N \rangle^2 = \mu_2 - \mu_1^2$

6. Generating function :

$$G(k) \equiv \langle e^{-ikn} \rangle = \sum_{n=0}^{\infty} e^{-ikn} P(n)$$

$$G(z \equiv e^{-ik}) = \sum_{n=0}^{\infty} z^n P(n)$$

$$G(z)\big|_{z=1} = 1, \quad G'(z)\big|_{z=1} = \mu_1$$

$$G''(z)\big|_{z=1} = \langle N(N-1) \rangle, \quad \cdots$$

Stochastic variable and distribution functions

Normal distribution = Gaussian distribution

$$P(x) = \frac{1}{\sqrt{2\pi s^2}} \exp\left[-\frac{(x - x_0)^2}{2s^2} \right] \qquad \text{(C1)}$$

$$\mu_1 \equiv \langle X \rangle = x_0 \qquad \text{(C2)}$$

$$\sigma^2 \equiv \langle X^2 \rangle - \langle X \rangle^2 = s^2 \qquad \text{(C3)}$$

Stochastic variable and distribution functions

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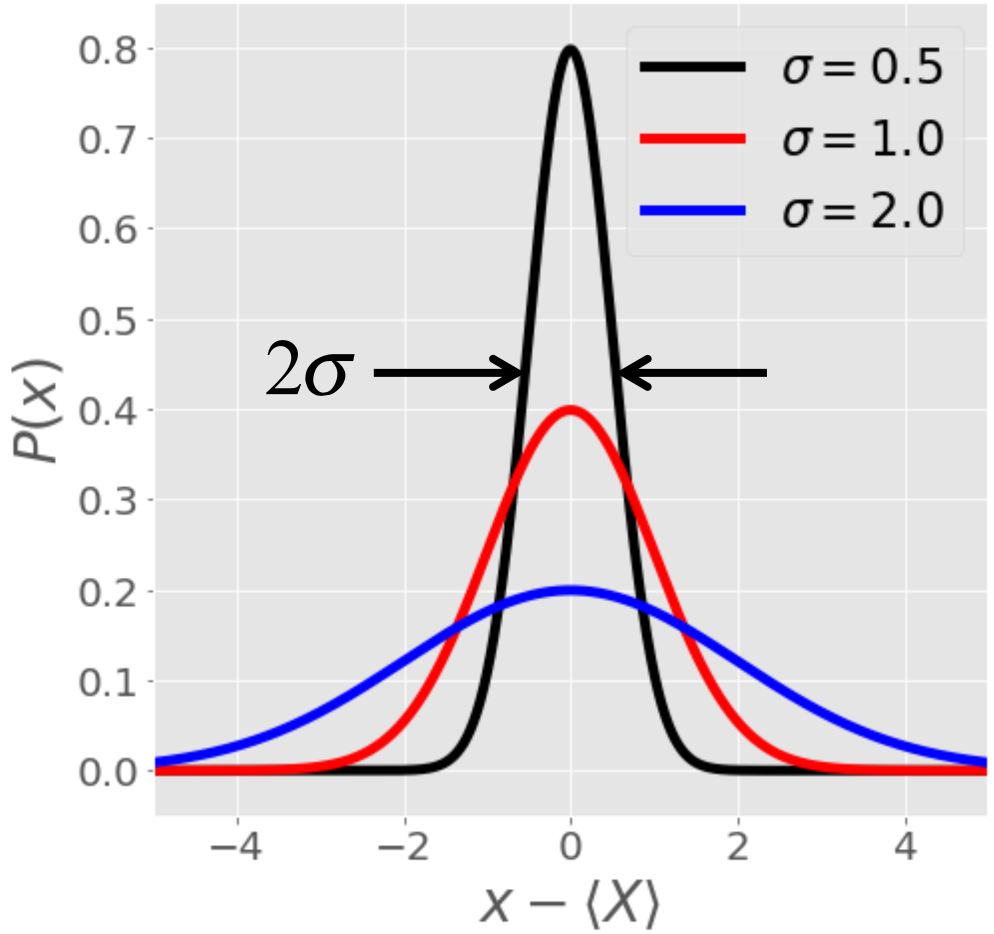

Stochastic variable and distribution functions

Ex. Maxwell-Boltzmann distribution

- Velocity of molecules : $V_\alpha \quad (\alpha = x, y, z)$
- Mass of molecule : m
- Temperature : T
- Boltzmann constant : k_B

$$P(v_\alpha) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{{v_\alpha}^2}{2\sigma^2} \right] \qquad \text{(C4)}$$

$$\langle V_\alpha \rangle = 0, \quad \langle V_\alpha^2 \rangle = \sigma^2 = \frac{k_B T}{2m} \qquad \text{(C5)}$$

Stochastic variable and distribution functions

Binomial distribution

$$P(n) = \frac{M!}{n!(M-n)!} p^n (1-p)^{M-n} \qquad \text{(C6)}$$

$$\mu_1 = \sum_{n=0}^{M} nP(n) = Mp \qquad \text{(C7)}$$

$$\sigma^2 = Mp(1-p) \qquad \text{(C8)}$$

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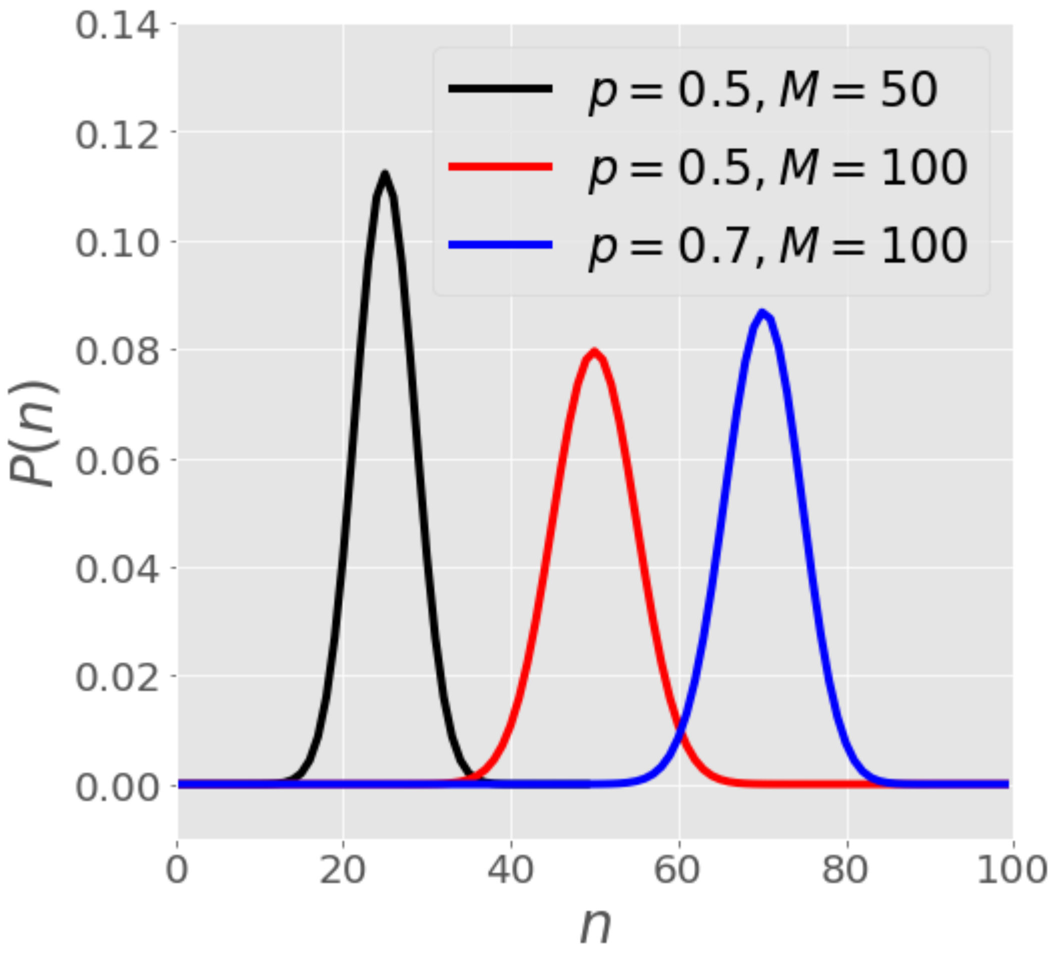

Stochastic variable and distribution functions

Poisson distribution

$$P(n) = \frac{a^n e^{-a}}{n!} \qquad \text{(C9)}$$

$$G(z) = e^{a(z-1)} \qquad \text{(C10)}$$

$$\mu_1 = G'(z)\Big|_{z=0} = a \qquad \text{(C11)}$$

$$\sigma^2 = \left\langle N^2 \right\rangle - \left\langle N \right\rangle^2 = a \qquad \text{(C12)}$$

$$\left(\because \left\langle N(N-1) \right\rangle = G''(z)\Big|_{z=0} = a^2 \right) \qquad \text{(C13)}$$

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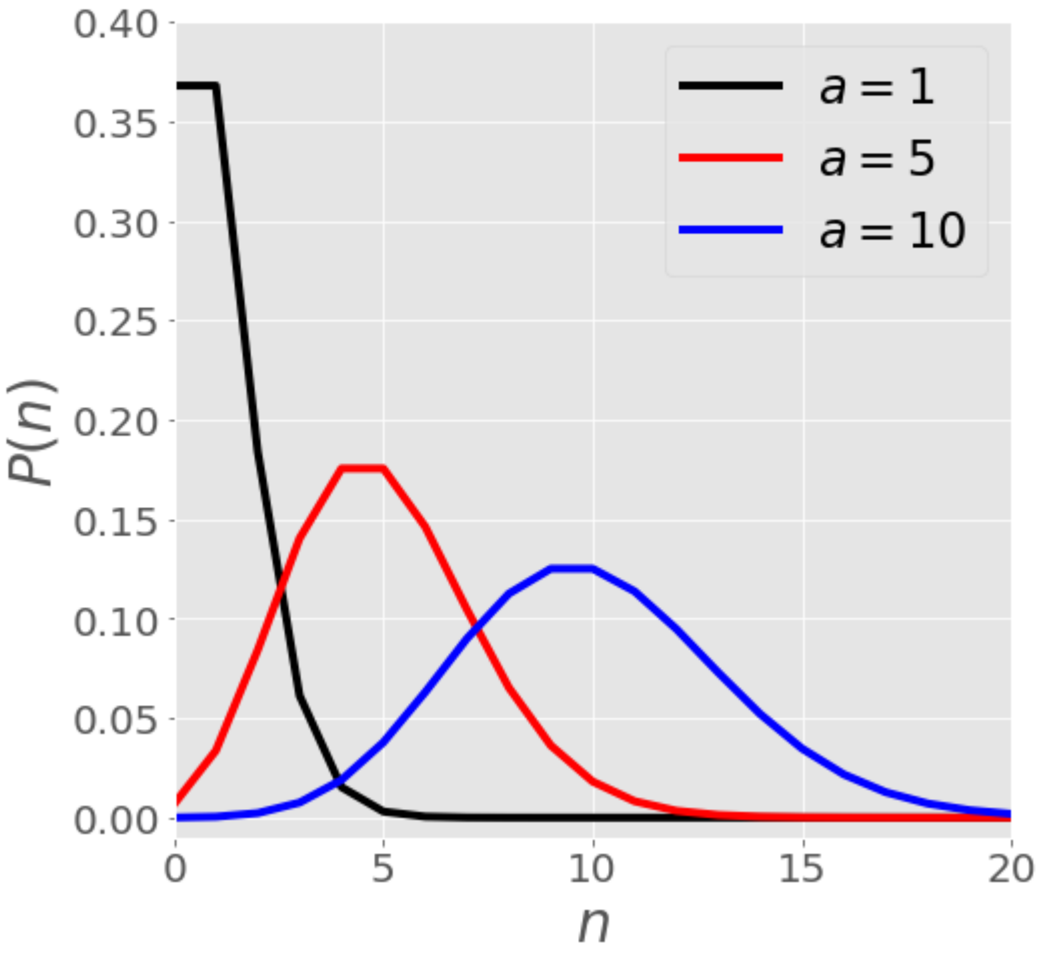

Stochastic variable and distribution functions

Binomial distribution $\rightarrow$ Normal distribution

See supplemental note for derivation.

$$P(n) = \frac{M!}{n!(M-n)!} p^n (1-p)^{M-n} \quad \text{(C6) Binomial}$$

$$\xrightarrow[n \to \textbf{cont.}]{n, M \gg 1} P(n) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(n-\mu_1)^2}{2\sigma^2} \right] \quad \text{(C1) Normal}$$

$$\mu_1 \equiv \langle N \rangle = Mp \quad \text{(C7)}$$

$$\sigma^2 \equiv \langle N^2 \rangle - \langle N \rangle^2 = Mp(1-p) \quad \text{(C8)}$$

Stochastic variable and distribution functions

Binomial distribution $\rightarrow$ Poisson distribution

See supplemental note for derivation.

$$P(n) = \frac{M!}{n!(M-n)!} p^n (1-p)^{M-n} \quad \text{(C6) Binomial}$$

$$\xrightarrow[Mp = a = \mathbf{const.}]{M \gg 1} \quad P(n) = \frac{a^n e^{-a}}{n!} \quad \text{(C9) Poisson}$$

$$\mu_1 \equiv a = Mp \quad \text{(C11)}$$

$$\sigma^2 \equiv a = Mp(1-p) \quad \text{(C12)}$$

$$\simeq Mp \quad \text{(C14)}$$