Mastery Homework 8

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Section 6.1

Problem 26

Statement: Let $\vec{u} = \begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix}$, and let W be the set of all $\vec{x} \in \mathbb{R}^3$ such that $\vec{u} \cdot \vec{x} = 0$. What theorem in Chapter 4 can be used to show that W is a subspace of $\mathbb{R}^3$? Describe W in geometric language.

Solution: The geometric place of all vectors perpendicular to the vector $\vec{u}$ in $\mathbb{R}^3$ is the plane P that is perpendicular to $\vec{u}$ that goes through the origin. That is, the plane $5x - 6y + 7z = 0$. A plane in $\mathbb{R}^3$ is spanned by two vectors. The span of any set of vectors in a vector space V is a subspace of V by theorem 1 in chapter 4. Therefore W has to be subspace of $\mathbb{R}^3$. For instance, solve for z in the equation for the plane and find two vectors in the plane that are not parallel: $z = \frac{-5x+6y}{7}$. When $x = y = 1$ we have $z = \frac{-5+6}{7} = \frac{1}{7}$ and $x = 2y = 2$ we have $z = \frac{-10+6}{7} = -\frac{4}{7}$ And so $W = \text{Span}\left(\left\{\begin{bmatrix} 1 \\ 1 \\ \frac{1}{7} \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -\frac{4}{7} \end{bmatrix}\right\}\right)$. ∎

Problem 28

Statement: Suppose a vector $\vec{y}$ is orthogonal to vectors $\vec{u}$ and $\vec{v}$. Show that $\vec{y}$ is orthogonal to the vector $\vec{u} + \vec{v}$.

Solution: Suppose $\vec{y}$ is orthogonal to $\vec{u}$ and $\vec{v}$ in some vector space V with an inner product. Then $\vec{y} \cdot \vec{u} = 0$ and $\vec{y} \cdot \vec{v} = 0$. Now consider the following:

$$
\begin{aligned}
\vec{y} \cdot (\vec{u} + \vec{v}) &= \vec{y} \cdot \vec{u} + \vec{y} \cdot \vec{v} \\
&= 0 + 0 \\
&= 0
\end{aligned}
$$

Therefore $\vec{y}$ is orthogonal to the vector $\vec{u} + \vec{v}$.

∎

6.2

Problem 32

Statement: Let $\{\vec{v_1}, \vec{v_2}\}$ be an orthogonal set of nonzero vectors, and let c_1, c_2 be any nonzero scalars. Show that $\{c_1\vec{v_2}, c_2\vec{v_2}\}$ is also an orthogonal set. Since orthogonality of a set is defined in terms of pairs of vectors, this shows that if the vectors in an orthogonal set are normalized, the new set will still be orthogonal.

Solution: Suppose $\{\vec{v_1}, \vec{v_2}\}$ is an orthogonal set of nonzero vectors in a vector space V with inner product. Let c_1, c_2 be any nonzero scalars. Then $\vec{v_1} \cdot \vec{v_2} = 0$. Now consider the following:

$$
\begin{aligned}
(c_1\vec{v_1}) \cdot (c_2\vec{v_2}) &= (c_1)(c_2)(\vec{v_1} \cdot \vec{v_2}) \\
&= (c_1 c_2)(0) \\
&= 0
\end{aligned}
$$

Then $c_1\vec{v_1}$ is orthogonal to $c_2\vec{v_2}$ and the set $\{c_1\vec{v_1}, c_2\vec{v_2}\}$ is an orthogonal set. ∎

Problem 33

Statement: Given $\vec{u} \neq \vec{0}$ in $\mathbb{R}^n$, let $L = \text{Span}\,\{\vec{u}\}$. Show that the mapping $\vec{x} \mapsto \text{proj}_{\text{L}}(\vec{x})$ is a linear transformation.

Solution: Let $\text{T} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the transformation such that $\text{T}(\vec{x}) = \text{proj}_{\text{L}}(\vec{x})$ for some $\vec{u} \neq \vec{0}$ in $\mathbb{R}^n$. Where $L = Span(\vec{u})$. Since L contains one vector, we find an equation for T:

$$
\text{T}(\vec{x}) = \frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u}
$$

Now consider any two vectors $\vec{x_1}, \vec{x_2} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then:

$$\begin{aligned}
\mathrm{T}(\vec{x_1} + \vec{x_2}) &= \frac{(\vec{x_1} + \vec{x_2}) \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} \\
&= \frac{\vec{x_1} \cdot \vec{u} + \vec{x_2} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} \\
&= \left(\frac{\vec{x_1} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} + \frac{\vec{x_2} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\right)\vec{u} \\
&= \frac{\vec{x_1} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} + \frac{\vec{x_2} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} \\
&= \mathrm{T}(\vec{x_1}) + \mathrm{T}(x_2)
\end{aligned}$$

And,

$$\begin{aligned}
\mathrm{T}(c\vec{x_1}) &= \frac{(c\vec{x_1}) \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} \\
&= \frac{c(\vec{x_1} \cdot \vec{u})}{\vec{u} \cdot \vec{u}}\vec{u} \\
&= c\left(\frac{\vec{x_1} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u}\right) \\
&= c\,\mathrm{T}(\vec{x})
\end{aligned}$$

Therefore T is a linear transformation. $\blacksquare$