Mastery Homework 9

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Section 6.3

Problem 24

Statement: Let W be a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\vec{w_1}, \ldots, \vec{w_p}\}$, and let $\{\vec{v_1}, \ldots, \vec{v_q}\}$ be an orthogonal basis for $W^\perp$.

a. Explain why $\{\vec{w_1}, \ldots, \vec{w_p}, \vec{v_1}, \ldots, \vec{v_q}\}$ is an orthogonal set.

b. Explain why the set in part (a) spans $\mathbb{R}^n$.

c. Show that $dim(W) + din(W^\perp) = n$.

Solution: a. Recall the definition of $W^\perp$:

$$W^\perp = \left\{\vec{v} \in \mathbb{R}^n \,|\, \vec{v} \cdot \vec{w} = 0, \forall \vec{w} \in W\right\}$$

We know $\vec{w_i} \cdot \vec{w_j} = 0$ for $i, j \in \{1, \ldots, p\}$ when $i \neq j$ and $\vec{v_i} \cdot \vec{v_j} = 0$ for $i, j \in \{1, \ldots, q\}$ when $i \neq j$ by hypothesis. We only have to show that $\vec{w_i} \cdot \vec{v_j}$ for $i = 1, \ldots, p$ and $j = 1, \ldots, q$. Now, from the definition of $W^\perp$ we know that for any $\vec{v} \in W$ and $\vec{w} \in W^\perp$ we have $\vec{v} \cdot \vec{w} = 0$. Notice then that $\vec{w_i} \in W$ and $\vec{v_j} \in W^\perp$ since they are part of a basis for W and $W^\perp$, respectively.

b. By the Orthogonal Decomposition Theorem, any vector $\vec{x}$ can be written as $\vec{x} = \vec{x}_w + \vec{x}_{w_\perp}$ where $\vec{x_w} \in W$ and $\vec{x}_{w_\perp} \in W^\perp$. Since $\{\vec{w_1}, \ldots, \vec{w_p}\}$ is a basis for W and $\{\vec{v_1}, \ldots, \vec{v_q}\}$ is a basis for $W^\perp$ there must exist scalars $\lambda_1, \ldots, \lambda_p, \beta_1, \ldots, \beta_q$ such that:

$$\lambda_1 \vec{w}_1 + \cdots + \lambda_p \vec{w_p} = \vec{x}_w$$

And,

$$\beta_1 \vec{v_1} + \cdots + \lambda_q \vec{v_q} = \vec{x}_{w_\perp}$$

Adding these two equations:

$$\lambda_1 \vec{w_1} + \cdots + \lambda_p \vec{w_p} + \beta_1 \vec{v_1} + \cdots + \beta_q \vec{v_q} = \vec{x}_w + \vec{x}_{w_\perp} = \vec{x}$$

Therefore any $\vec{x} \in \mathbb{R}^n$ is also in $\text{Span}(\lambda_1\vec{w_1}, \ldots, \lambda_p\vec{w_p}, \beta_1\vec{v_1}, \ldots, \beta_q\vec{v_q})$, or simply $\text{Span}(\lambda_1\vec{w_1}, \ldots, \lambda_p\vec{w_p}, \beta_1\vec{v_1}, \ldots, \beta_q\vec{v_q}) = \mathbb{R}^n$

c. Let $S = \{\lambda_1\vec{w_1}, \ldots, \lambda_p\vec{w_p}, \beta_1\vec{v_1}, \ldots, \beta_q\vec{v_q}\}$We know that S spans $\mathbb{R}^n$ and that S is an orthogonal set which by one of the first theorems of this chapter is a linearly independent set. This implies that S is a basis for $\mathbb{R}^n$ and therefore it must contain n vectors, $p + q = n$. Finally:

$$\dim(\mathbb{R}^n) = \dim(S) = \dim(W) + \dim(W^\perp) = p + q = n$$

Or simply:

$$\dim(W) + \dim(W^\perp) = n$$

Which is exactly what we wanted to show. ∎

Section 6.5

Problem 14

Statement: Let $A = \begin{bmatrix} 2 & 1 \\ -3 & -4 \\ 3 & 2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$. Compute $A\vec{u}$ and $A\vec{v}$, and compare them with $\vec{b}$. Is it possible that at least one of $\vec{u}$ or $\vec{v}$ could be a least-squares solution of $A\vec{x} = \vec{b}$?

Solution:

$$A\vec{u} = \begin{bmatrix} 2 & 1 \\ -3 & -4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} 2 & 1 \\ -3 & -4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 8 \end{bmatrix}$$

$$
\begin{aligned}
\left\|\vec{b} - A\vec{u}\right\| &= \left\| \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} \right\| \\
&= \sqrt{2^2 + 4^2 + 2^2} \\
&= \sqrt{24}
\end{aligned}
$$

$$
\begin{aligned}
\left\|\vec{b} - A\vec{v}\right\| &= \left\| \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 7 \\ 2 \\ 8 \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix} \right\| \\
&= \sqrt{2^2 + 2^2 + 4^2} \\
&= \sqrt{24}
\end{aligned}
$$

By the definition alone of least-squares we can have multiple least-squares solutions. However, notice that:

$$
\det(A^T A) = \det\left(\begin{bmatrix} 2 & -3 & 3 \\ 1 & -4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -3 & -4 \\ 3 & 2 \end{bmatrix} \right) = \det\left(\begin{bmatrix} 22 & 20 \\ 20 & 21 \end{bmatrix} \right) = 22(21) - 20^2 = 62 \neq 0
$$

Therefore $A^T A$ is invertible and the least squares solution is unique. Therefore $\vec{u}, \vec{v}$ cannot be a least squares solution to $A\vec{x} = \vec{b}$.

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Problem 25

Statement: Describe all least-squares solutions of the system

$$
\begin{cases} x + y = 2 \\ x + y = 4 \end{cases}
$$

Solution: The system is equivalent to the matrix equation $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Let's solve the normal system $A^T A = A^T \vec{b}$:

$$A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

∎

Let's reduce the augmented matrix associated to the system:

$$\begin{bmatrix} 2 & 2 & 6 \\ 2 & 2 & 6 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 2 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{R_1}{2}} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The least squares solutions is any vector in the line $x + y = 3$ which is simply a line in between $x + y = 2$ and $x + y = 4$.

And the image of any least-squares solution is ($x = 3 - y$ from $x + y = 3$):

$$A\vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 - y \\ y \end{bmatrix} = \begin{bmatrix} 3 - y + y \\ 3 - y + y \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

With error:

$$\left\| \vec{b} - A\vec{x} \right\| = \left\| \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\| = \sqrt{1^2 + 1^2} = \sqrt{2}$$