Mastery Homework 6

Rafael Laya

Fall 2018

Section 3.3

Problem 25

Statement: Use the concept of volume to explain why the determinant of a 3 x 3 matrix A is zero if and only if A is not invertible.

Solution: Consider a matrix A of 3 x 3 with columns $\vec{a_1}, \vec{a_2}, \vec{a_3}$.

Suppose that $\det(A) = 0$, then by theorem 9 of this section the volume of the parallelepiped spanned by the columns of A, $\vec{a_1}, \vec{a_2}, \vec{a_3}$ is $\det(A)$ which is zero. That means that the vectors are colinear or coplanar or all zero, which implies one of the vectors $\vec{a_i}$ (with $i \in \{1, 2, 3\}$) is a linear combination of the other two and the set $S = \{\vec{a_1}, \vec{a_2}, \vec{a_3}\}$ is a linearly dependent set. By the Invertible Matrix Theorem, the matrix A is not invertible.

Now, suppose that A is not invertible. Using the Invertible Matrix Theorem, the columns of A must be linearly dependent. This means that the vectors $\vec{a_1}, \vec{a_2}, \vec{a_3}$ are either colinear or coplanar or all zero and the volume of the parallelepiped spanned by $\vec{a_1}.\vec{a_2}, \vec{a_3}$ is the degenerate parallelepiped of zero volume. By theorem 9 of this section, the determinant of A is zero.

Both paragraphs above prove what we wanted to show. ∎

Section 4.1

Problem 10

Statement: Let H be the set of all vectors of the form $\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix}$. Show that H is a subspace of $\mathbb{R}^3$.

Solution: Let's first re-write H in set notation:

$$H = \left\{ \vec{v} \in \mathbb{R}^3 \mid \vec{v} = t \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, t \in \mathbb{R} \right\}$$

Notice that any vector $\vec{x} \in H$ has the form $\vec{x} = t \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ and $H = \text{Span}\left(\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}\right)$, By theorem 1 of this section, H is a subspace of $\mathbb{R}^3$

∎

Problem 12

Statement: Let W be the set of all vectors of the form $\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix}$. Show that W is a subspace of $\mathbb{R}^4$,

Solution: Following the argument of problem 10, rewrite W in set notation:

$$W = \left\{ \vec{v} \in \mathbb{R}^4 \mid \vec{v} = t \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix} + s \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}, \quad s, t \in \mathbb{R} \right\}$$

Notice that any vector $\vec{x} \in W$ has the form $\vec{x} = t \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix} + s \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}$ and therefore $W = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -3 \\ 4 \end{bmatrix}\right)$ which by theorem 1 of this section, W is a subspace of $\mathbb{R}^4$ ∎

Problem 32

Statement: Let H and K be subspaces of a vector space V. The intersection of H and K, written as $H \cap K$, is the set of $\vec{v}$ in V that belong to both H and K. Show that $H \cap K$ is a subspace of V. Give an example in $\mathbb{R}^2$ to show that the union of two subspaces is not, in general, a subspace.

Solution: Rewriting the intersection between H and K in set notation and the logical binary operator $\wedge$ (and):

$$(H \cap K) = \{\vec{v} \in V \mid \vec{v} \in H \wedge \vec{v} \in K\}$$

Now, take any two vectors in the intersection between H and K, that is, let $\vec{v}, \vec{u} \in (H \cap K)$ and also let $\lambda \in \mathbb{R}$.

- The zero vector of V, $\vec{0}_V$ is in the intersection between H and K since H is a subspace of V and so is K (by assumption). Therefore, $\vec{0}_V \in K$ and $\vec{0}_V \in H$, which implies $\vec{0}_V \in (H \cap K)$

- Consider $\vec{v} + \vec{u}$. By definition, $\vec{v}, \vec{u} \in H$ and $\vec{v}, \vec{u} \in K$. By assumption H and K are subspaces, therefore both are closed under addition. Then $\vec{v} + \vec{u} \in H$ since $\vec{v}, \vec{u} \in H$ and $\vec{v} + \vec{u} \in K$ since $\vec{v}, \vec{u} \in K$. Finally, $\vec{v} + \vec{u} \in H$ and $\vec{v} + \vec{u} \in K$, which implies $\vec{v} + \vec{u} \in (H \cap K)$.

- Consider $\lambda\vec{v}$. We know by definition that $\vec{v} \in H$ and $\vec{v} \in K$. By assumption H and K are subspaces and therefore closed under scalar multiplication. Then $\lambda\vec{v} \in H$ since $\vec{v} \in H$ and $\lambda\vec{v} \in K$ since $\vec{v} \in K$. Finally, $\lambda\vec{v} \in K$ and $\lambda\vec{v} \in H$ which implies $\lambda\vec{v} \in (H \cap K)$

Considering the three items above, the intersection between H and K, $H \cap K$ is a subspace of the vector space V, which is what we wanted to show.

We are also asked to give a counter-example of the union between two subspaces in $\mathbb{R}^2$ being a subspace. Let H be the line with slope 1 through the origin, and K be the line with slope -1 through the origin. Then the union $H \cup K$ can be written as (Using the logical operator or, $\vee$):

$$H \cup K = \left\{\vec{v} \in \mathbb{R}^2 \mid \vec{v} = t\begin{bmatrix}1\\1\end{bmatrix} \vee \vec{v} = s\begin{bmatrix}1\\-1\end{bmatrix}\right\}$$

Now take a non-zero vector in each of the lines, $\vec{v} = \begin{bmatrix}1\\1\end{bmatrix} \in (H \cup K)$ and $\vec{w} = \begin{bmatrix}1\\-1\end{bmatrix} \in (H \cup K)$ and consider $\vec{v} + \vec{w} = \begin{bmatrix}2\\0\end{bmatrix}$. Notice that $t\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$ and $s\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$ neither have solution since the vectors $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}$ are not parallel to $\begin{bmatrix}2\\0\end{bmatrix}$ and therefore $\vec{v} + \vec{w} \notin (H \cup K)$ and $H \cup K$ is not a subspace of $\mathbb{R}^2$.

$\blacksquare$

Section 4.2

Problem 30

Statement: Let $\mathrm{T} : V \longrightarrow W$ be a linear transformation from a vector space V into a vector space W. Prove that the range of T is a subspace of W.

Solution: let $\mathrm{T}(\vec{v}), \mathrm{T}(\vec{w})$ be any two vectors in the range of T and let $c \in \mathbb{R}$. Now consider:

- The zero vector of W, $\vec{0}_W$ is in W since by assumption T is linear and therefore $\mathrm{T}(\vec{0_V}) = \vec{0}_W$
- $\mathrm{T}(\vec{v}+\vec{w}) = \mathrm{T}(\vec{v}) + \mathrm{T}(\vec{w})$ (since T is linear by assumption) and since V is a vector space we have that V is closed under addition and $\vec{v} + \vec{w} \in V$. This implies then that $\mathrm{T}(\vec{v}) + \mathrm{T}(\vec{u}) \in W$
- $\mathrm{T}(c\vec{v}) = c\,\mathrm{T}(\vec{v})$ (since T is linear by assumption) and since V is a vector space we have that $c\vec{v} \in V$ and this implies $c\,\mathrm{T}(\vec{v}) \in W$.

With the three bullet points above, we have shown that the range of T is a subspace of W ∎

Section 4.3

Problem 29

Statement: Let $S = \{\vec{v_1}, \ldots, \vec{v_k}\}$ be a set of k vectors in $\mathbb{R}^n$, with $k < n$. Use a theorem from section 1.4 to explain why S cannot be a basis for $\mathbb{R}^n$.

Solution: Because if $k < n$ then we have too few vectors to have enough information to span $\mathbb{R}^n$. With this in mind:

Consider the matrix A of order n x k with columns $\vec{v_1}, \ldots, \vec{v_k}$. That is, $A = \begin{bmatrix} \vec{v_1} & \ldots & \vec{v_k} \end{bmatrix}$. We know that A has n rows and k columns and by assumption $k < n$ (A has more rows than columns). Since A has k columns, A can have at most k pivot positions, and since A has n rows, there is atleast one row without a pivot. A cannot have a pivot in every row, and by theorem 4 in chapter 1, the columns of A, do not span $\mathbb{R}^n$, which is equivalent to the vectors in S do not span $\mathbb{R}^n$ and therefore by definition S cannot be a basis for $\mathbb{R}^n$. ∎

Problem 30

Statement: Let $S = \{\vec{v_1}, \ldots, \vec{v_k}\}$ be a set of k vectors in $\mathbb{R}^n$, with $k > n$. Use a theorem from Chapter 1 to explain why S cannot be a basis for $\mathbb{R}^n$.

Solution: Because we have too many vectors that implies we must have redundant information and the set of vectors cannot be linearly independent. With this argument in mind:

By direct usage of theorem 8 in chapter 1, we have that the set S is a linearly dependent set since $k > n$ (more vectors than entries in each vector). Therefore, S cannot be linearly independent set, and by definition of a basis S cannot be a basis of $\mathbb{R}^n$. ∎