Mastery Homework 3

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Section 1.8

Problem 26.

Statement: Let $\vec{u}$ and $\vec{v}$ be linearly independent vectors in $\mathbb{R}^3$, and let P be the plane through $\vec{u}, \vec{v}$, and $\vec{0}$. The parametric equation of P is $\vec{x} = s\vec{u} + t\vec{v}$ (with $s, t \in \mathbb{R}$). Show that a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ maps P onto a plane through $\vec{0}$, or onto a line through $\vec{0}$, or onto just the origin in $\mathbb{R}^3$. What must be true about $\mathrm{T}(\vec{u})$ and $\mathrm{T}(\vec{v})$ in order for the image of the plane P to be a plane?

Solution: let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation. We want to apply T to the parametric equation of the plane P in order to apply linearity and see what corresponds to the parametric equation that we obtain. From there we will be able to deduce what has to be true about $\mathrm{T}(\vec{u})$ and $\mathrm{T}(\vec{v})$ in order for the image of the plane P to be a plane. Therefore using the linearity of T consider:

$$\begin{aligned}
\mathrm{T}(\vec{x}) &= \mathrm{T}(s\vec{u} + \mathrm{t}\tilde{\mathrm{v}}) \\
&= \mathrm{T}(s\vec{u}) + t\,\mathrm{T}(t\vec{v}) \\
&= s\,\mathrm{T}(\vec{u}) + t\,\mathrm{T}(\vec{v})
\end{aligned}$$

Every $\vec{x}$ gets mapped as a linear combination of $\mathrm{T}(\vec{u}), \mathrm{T}(\vec{v})$. Therefore the plane P gets mapped into $\mathrm{Span}(\{\mathrm{T}(\vec{u}), \mathrm{T}(\vec{v})\})$. Then:

- If $\mathrm{T}(\vec{u}), \mathrm{T}(\vec{v})$ are both the zero vector, then $\mathrm{Span}(\{\mathrm{T}(\vec{u}), \mathrm{T}(\vec{v})\}) = \{\vec{0}\}$ and the plane P gets mapped directly into the Origin, $\vec{0}$. See that if we let $\mathrm{T}(\vec{u}) = \mathrm{T}(\vec{v}) = \vec{0}$ then:

$$\mathrm{T}(\vec{x}) = s(\vec{0}) + t(\vec{0}) = \vec{0} + \vec{0} = \vec{0}$$

- If $\mathrm{T}(\vec{u}), \mathrm{T}(\vec{v})$ are linearly dependent and they are not both zero (say $\mathrm{T}(\vec{u}) \neq \vec{0}$, without loss of generality), then either one vector is zero and the Span is the line that goes through the origin and the other vector (say $\mathrm{T}(\vec{u})$), or one is a multiple $k \neq 0$ of the other and $\mathrm{Span}(\{\mathrm{T}(\vec{u}), \mathrm{T}(\vec{v})\}) = \mathrm{Span}(\{\mathrm{T}(\vec{u}), k\,\mathrm{T}(\vec{u})\}) = \mathrm{Span}(\{\mathrm{T}(\vec{u})\})$ or simply by theorem since $\{\mathrm{T}(\vec{u})\}$ is a linearly independent set and $\{\mathrm{T}(\vec{u}), \mathrm{T}(\vec{v})\}$ is linearly dependent (after we added only one vector to the set) they have the same Span, which is a line through the origin. See that if we let $\mathrm{T}(\vec{v}) = k\,\mathrm{T}(\vec{u})$

  $$\mathrm{T}(\vec{x}) = s\,\mathrm{T}(\vec{u}) + tk\,\mathrm{T}(\vec{v}) = (s + tk)\,\mathrm{T}(\vec{u})$$

  or if we let $\mathrm{T}(\vec{v}) = \vec{0}$ while $\mathrm{T}(\vec{u}) \neq \vec{0}$ then:

  $$\mathrm{T}(\vec{x}) = s\,\mathrm{T}(\vec{u}) + t(\vec{0}) = s\,\mathrm{T}(\vec{u}) + \vec{0} = s\,\mathrm{T}(\vec{u})$$

  In any case, this is the equation of a line through the origin.

- If $\mathrm{T}(\vec{u}), \mathrm{T}(\vec{v})$ are linearly independent, then $\mathrm{Span}(\{\mathrm{T}(\vec{u}), \mathrm{T}(\vec{v})\})$ is a Plane through the origin and through both vectors $\mathrm{T}(\vec{u}), \mathrm{T}(\vec{v})$ (the equation that led to this discussion is precisely the parametric equation of a plane).

Looking at what we have in our third bullet point we can also answer the last question: $\mathrm{T}(\vec{u}), \mathrm{T}(\vec{v})$ have to be linearly independent so that the image of the plane P is mapped into another plane under the transformation T.

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Problem 31

Statement: Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, and let $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ be a linearly dependent set in $\mathbb{R}^n$. Explain Why the Set $\{\mathrm{T}(\vec{v_1}), \mathrm{T}(\vec{v_2}), \mathrm{T}(\vec{v_3})\}$ is linearly dependent.

Solution: We will use the definition of linearly dependent set of vectors for $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ and then we will apply the transformation T to our resulting equation, by linearity we will be able to extract the coefficients of the linear combination and decide about the linearly dependency of the set $\{\mathrm{T}(\vec{v_1}), \mathrm{T}(\vec{v_2}), \mathrm{T}(\vec{v_3})\}$ (which will show atleast a non-trivial solution). Suppose then $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is a linearly dependent set in $\mathbb{R}^n$. That is,

$$x_1\vec{v_1} + x_2\vec{v_2} + x_3\vec{v_3} = \vec{0}$$

Has a non-trivial solution $(c_1, c_2, c_3) \neq (0, 0, 0)$. Apply T to the equation above when $(x_1, x_2, x_3) = (c_1, x_2, c_3)$

$$\mathrm{T}(c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}) = \mathrm{T}(\vec{0})$$

Using the linearity of T, and the fact that $\mathrm{T}(\vec{0}) = \mathrm{T}(\vec{0} + \vec{0}) = \mathrm{T}(\vec{0}) + \mathrm{T}(\vec{0})$ and therefore $\mathrm{T}(\vec{0}) = \vec{0}$ (keep in mind that the input is the vector zero in $\mathbb{R}^n$ and the output is the vector zero in $\mathbb{R}^m$),

$$\begin{aligned}\mathrm{T}(c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}) &= \mathrm{T}(c_1\vec{v_1}) + \mathrm{T}(c_2\vec{v_2}) + \mathrm{T}(c_3\vec{v_3}) \\ &= c_1\,\mathrm{T}(\vec{v_1}) + c_2\,\mathrm{T}(\vec{v_2}) + c_3\,\mathrm{T}(\vec{v_3}) = \vec{0}\end{aligned}$$

Therefore,

$$x_1\,\mathrm{T}(\vec{v_1}) + x_2\,\mathrm{T}(\vec{v_2}) + x_3\,\mathrm{T}(\vec{v_3}) = \vec{0}$$

Has a non-trivial solution $(x_1, x_2, x_3) = (c_1, c_2, c_3)$ and the set $\{\mathrm{T}(\vec{v_1}), \mathrm{T}(\vec{v_2}), \mathrm{T}(\vec{v_3})\}$ is a linearly dependent set.

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Problem 34

Statement: Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Show that if T maps two linearly independent vectors onto a linearly dependent set, then the equation $\mathrm{T}(\vec{x}) = \vec{0}$ has a nontrivial solution.

Solution: Let $\{\vec{v_1}, \vec{v_2}\}$ be a linearly independent set in $\mathbb{R}^n$ and suppose $\{\mathrm{T}(\vec{v_1}), \mathrm{T}(\vec{v_2})\}$ is a linearly dependent set. Then:

$$x_1\vec{v_1} + x_2\vec{v_2} = \vec{0}$$

only has trivial solution and,

$$x_1\,\mathrm{T}(\vec{v_1}) + x_2\,\mathrm{T}(\vec{v_2}) = \vec{0}$$

has a non-trivial solution $(x_1, x_2) = (c_1, c_2) \neq (0, 0)$. Then using linearity of T,

$$\begin{aligned}c_1\,\mathrm{T}(\vec{v_1}) + c_2\,\mathrm{T}(\vec{v_2}) &= \mathrm{T}(c_1\vec{v_1}) + \mathrm{T}(c_2\vec{v_2}) \\ &= \mathrm{T}(c_1\vec{v_1} + c_2\vec{v_2})\end{aligned}$$

And so,

$$\mathrm{T}(c_1\vec{v_1} + c_2\vec{v_2}) = \vec{0}$$

$\mathrm{T}(\vec{x}) = \vec{0}$ has a non-trivial solution since the vector $\vec{x} = c_1\vec{v_1} + c_2\vec{v_2}$ is a solution to $\mathrm{T}(\vec{x}) = \vec{0}$ and it is not the zero vector since by assumption $x_1\vec{v_1} + x_2\vec{v_2} = \vec{0}$ only has trivial solution but here $(c_1, c_2) \neq (0, 0)$.

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Problem 36

Statement: Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the transformation that projects each vector $\vec{x} = (x_1, x_2, x_3)$ onto the plane $x_2 = 0$, So $\mathrm{T}(\vec{x}) = (x_1, 0, x_3)$. Show that T is a linear transformation.

Solution: Let $\vec{v} = (v_1, v_2, v_3)$ and $\vec{u} = (u_1, u_2, u_3)$ be any two vectors in $\mathbb{R}^3$ and let $\lambda \in \mathbb{R}$. Now consider:

$$\begin{aligned}
\mathrm{T}(\vec{v}) + \mathrm{T}(\vec{u}) &= \mathrm{T}((v_1, v_2, v_3) + (u_1, u_2, u_3)) \\
&= \mathrm{T}(v_1 + u_1, v_2 + u_2, v_3 + u_3) \\
&= (v_1 + u_1, 0, v_3 + u_3) \\
&= (v_1, 0, v_3) + (u_1, 0, u_3) \\
&= \mathrm{T}(\vec{v}) + \mathrm{T}(\vec{u})
\end{aligned}$$

And,

$$\begin{aligned}
\mathrm{T}(\lambda\vec{v}) &= \mathrm{T}(\lambda(v_1, v_2, v_3)) \\
&= \mathrm{T}(\lambda v_1, \lambda v_2, \lambda v_3) \\
&= (\lambda v_1, 0, \lambda v_3) \\
&= \lambda(v_1, 0, v_3) \\
&= \lambda\,\mathrm{T}(\vec{v})
\end{aligned}$$

Since this is true for all $\vec{v}, \vec{u} \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$ Then T is a linear transformation by definition.

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Problem 22

Statement: Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a linear transformation such that $\mathrm{T}(x_1, x_2) = (x_1 - 2x_2, -x_1 + 3x_2, 3x_1 - 2x_2)$ find $\vec{x}$ such that $\mathrm{T}(\vec{x}) = (-1, 4, 9)$

Solution: Given the formula of the linear transformation we can find $\mathrm{T}(\vec{e_1}) = \mathrm{T}(1,0)$ and $\mathrm{T}(\vec{e_2}) = \mathrm{T}(0,1)$. By theorem 10 we can write the transformation matrix A and then we will solve the system $A\vec{x} = \vec{b}$ where $\vec{b} = \begin{bmatrix} -1 \\ 4 \\ 9 \end{bmatrix}$. Let's find the images of $\vec{e_1}, \vec{e_2}$ under T,

$$\mathrm{T}(\vec{e_1}) = \mathrm{T}(1,0) = (1 - 2(0), -1 + 3(0), 3(1) - 2(0)) = (1, -1, 3)$$
$$\mathrm{T}(\vec{e_2}) = \mathrm{T}(0,1) = (0 - 2(1), -0 + 3(1), 3(0) - 2(1)) = (-2, 3, -2)$$

Therefore the transformation matrix A is:

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & 2 \end{bmatrix}$$

We can rewrite $\mathrm{T}(\vec{x})$ as: $\mathrm{T}(\vec{x}) = A\vec{x}$. We want to find $\vec{x}$ such that $\mathrm{T}(\vec{x}) = \vec{b}$ which is solved by row reducing the augmented matrix $\begin{bmatrix} A & \vec{b} \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 4 \\ 3 & -2 & 9 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 + R_1} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 3 \\ 0 & 4 & 12 \end{bmatrix} \xrightarrow[R_3 - 4R_2]{R_1 + 2R_2} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution is

$$\begin{cases} x_1 \quad = 5 \\ \quad x_2 = 3 \end{cases}$$

In the desired form, $\vec{x} = (5, 3)$ is such that $\mathrm{T}(\vec{x}) = (-1, 4, 9)$

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