Extra Credit Vector Spaces Homework

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Subspaces

Problem 1 (Unique)

Statement: Your job is to construct "almost" subspaces that satisfy some, but not all, of the subspace conditions. Show that your construction verifies the given criteria.

a) Find a subset of $\mathbb{R}^3$ that contains the zero vector, is closed under multiplication, but is not closed under addition.

b) Find a subset of $\mathbb{R}^3$ that is closed under addition, but not under scalar multiplication and does not contain the zero vector.

Solution: a) Consider the subset H of $\mathbb{R}^3$ that is formed by the x axis and the y axis. That is:

$$H = \left\{ \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid (y = 0 \land z = 0) \lor (x = 0 \land z = 0) \right\}$$

Also let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ be two vectors in H and $c \in \mathbb{R}$

- H contains the zero vector since the zero vector has y and z coordinates zero (in fact its x coordinate is also zero!).
- Consider $c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ cv_3 \end{bmatrix}$. Since $\vec{v} \in H$ then either ($v_2 = 0$ and $v_3 = 0$) or ($v_1 = 0$ and $v_3 = 0$). Therefore either ($cv_2 = 0$ and $cv_3 = 0$) or ($cv_1 = 0$ and $cv_3 = 0$). This implies that $c\vec{v} \in H$ and so H is closed under scalar multiplication.

- Consider the case when $v_1 = 1, v_2 = 0, v_3 = 0$ and $v_1 = 0, v_2 = 1, v_3 = 0$. The sum is $\vec{v} + \vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Notice however that $\vec{v} + \vec{u} \notin H$ since $(1 = 0$ and $0 = 0)$ is false and $(1 = 0$ and $0 = 0)$ is false (The first one refers to $y = 0$ and $z = 0$ and the second one to $x = 0$ and $z = 0$). Therefore H is not closed under addition.

b) Consider the subset H of $\mathbb{R}^3$ that is formed by the first octant without including the axes, that is:

$$H = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 > 0 \wedge x_2 > 0 \wedge x_3 > 0 \right\}$$

Let also $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ be two vectors in H, and c a scalar. Then consider:

- The sum of $\vec{v}$ and $\vec{u}$ is $\vec{v} + \vec{u} = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ v_3 + u_3 \end{bmatrix}$ is in H since $\vec{v}, \vec{u} \in H$ and therefore $v_1, v_2, v_3, u_1, u_2, u_3$ are all greater than zero and the sum of two real numbers greater than zero is still greater than zero. H is closed under addition.

- Consider the case when $v_1 = 1, v_2 = 1, v_3 = 1$ and $c = -1$. Then $c\vec{v} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$ which is not in H since $-1 < 0$. H is not closed under scalar multiplication.

- The zero vector is not in H since $0 > 0$ is a false statement.

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General Vector Spaces

Problem 1 (Unique)

Statement: Consider the set of vectors $\{\sin^3(x), \sin(3x), \sin(x)\}$ in the vector space $C(-\infty, \infty)$. Use your trig identities to show that this is a linearly dependent set of vectors. Prove any facts you use beyond the Pythagorean identities and sum/difference of angle formulas.

Solution: We will prove that we can write one of the vectors in the set as a linear combination of the other vectors. Start from $\sin(3x)$:

$$\begin{aligned}
\sin(3x) &= \sin(2x + x) \\
&= \sin(2x)\cos(x) + \cos(2x)\sin(x) \\
&= (2\sin(x)\cos(x))\cos(x) + (\cos^2(x) - \sin^2(x))sin(x) \\
&= 2\sin(x)\cos^2(x) + (1 - \sin^2(x) - \sin^2(x))sin(x) \\
&= 2\sin(x)(1 - \sin^2(x)) + (1 - 2\sin^2(x))sin(x) \\
&= 2\sin(x) - 2\sin^3(x) + \sin(x) - 2\sin^3(x) \\
&= 3\sin(x) - 4\sin^3(x)
\end{aligned}$$

Therefore,

$$x_1 \sin^3(x) + x_2 \sin(3x) + x_3 \sin(x) = 0_\infty$$

Where 0_∞ is the function whose value is always zero in $(-\infty, \infty)$. Has a non-trivial solution $x_1 = 4, x_2 = 1, x_3 = -3$ since:

$$\begin{aligned}
(4)\sin^3(x) + (1)\sin(3x) + (-3)\sin(x) &= 4\sin^3(x) + (3\sin(x) - 4\sin^3(x)) - 3\sin(x) \\
&= 4\sin^3(x) + 3\sin(x) - 4\sin^3(x) - 3\sin(x) \\
&= (4\sin^3(x) - 4\sin^3(x)) + (3\sin(x) - 3\sin(x)) \\
&= 0_\infty + 0_\infty \\
&= 0_\infty
\end{aligned}$$

The properties we used (asociativity, commutativity, distributive) come from assumption since $C(-\infty, \infty)$ is a vector space and also from our knowledge of functions in calculus (trig identities).

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Transformations Between General Vector Spaces

Problem 1 (Unique)

Statement: Let $\mathrm{T} : \mathbb{P}_3 \longrightarrow \mathbb{R}^3$ be the transformation defined by $\mathrm{T}(\vec{p}) = (\vec{p}(-2), \vec{p}(0), \vec{p}(1))$. Each entry evaluates the polynomial $\vec{p}$ at the value $-2, 0$, and 1, respectively.

a) Show that T is linear.
b) Is T one-to-one? Why or Why not?
c) Is T onto? Why or Why not?
d) Describe the Kernel and the Range of this transformation.

Solution: a) I am gonna write vectors in $\mathbb{R}^3$ as column vectors for ease of writing. Let $\vec{v}, \vec{w} \in \mathbb{P}_3$. Then $\vec{v}(t) = v_3t^3 + v_2t^2 + v_1t + v_0$ and $\vec{w}(t) = w_3t^3 + w_2t^2 + w_1t + w_0$ for any chosen $v_0, \ldots, v_3, w_0, \ldots, w_3 \in \mathbb{R}$ and for all $t \in \mathbb{R}$. Let's show that T is a linear transformation:

$$\begin{aligned}
\mathrm{T}(\vec{v} + \vec{w}) &= \mathrm{T}((v_3t^3 + v_2t^2 + v_1t + v_0) + (w_3t^3 + w_2t^2 + w_1t + w_0)) \\
&= \mathrm{T}((v_3 + w_3)t^3 + (v_2 + w_2)t^2 + (v_1 + w_1)t + (v_0 + w_0)) \\
&= \begin{bmatrix} (v_3 + w_3)(-2)^3 + (v_2 + w_2)(-2)^2 + (v_1 + w_1)(-2) + (v_0 + w_0) \\ (v_3 + w_3)(0)^3 + (v_2 + w_2)(0)^2 + (v_1 + w_1)(0) + (v_0 + w_0) \\ (v_3 + w_3)(1)^3 + (v_2 + w_2)(1)^2 + (v_1 + w_1)(1) + (v_0 + w_0) \end{bmatrix} \\
&= \begin{bmatrix} (-8v_3 + 4v_2 - 2v_1 + v_0) + (-8w_3 + 4w_2 - 2w_1 + w_0) \\ v_0 + w_0 \\ (v_3 + v_2 + v_1 + v_0) + (w_3 + w_2 + w_1 + w_0) \end{bmatrix} \\
&= \begin{bmatrix} (-8v_3 + 4v_2 - 2v_1 + v_0) \\ v_0 \\ v_3 + v_2 + v_1 + v_0 \end{bmatrix} + \begin{bmatrix} (-8w_3 + 4w_2 - 2w_1 + w_0) \\ w_0 \\ w_3 + w_2 + w_1 + w_0 \end{bmatrix} \\
&= \begin{bmatrix} v_3(-2)^3 + v_2(-2)^2 + v_1(-2) + v_0 \\ v_3(0)^3 + v_2(0)^2 + v_1(0) + v_0 \\ v_3(1)^3 + v_2(1)^2 + v_1(1) + v_0 \end{bmatrix} + \begin{bmatrix} w_3(-2)^3 + w_2(-2)^2 + w_1(-2) + w_0 \\ w_3(0)^3 + w_2(0)^2 + w_1(0) + w_0 \\ w_3(1)^3 + w_2(1)^2 + w_1(1) + w_0 \end{bmatrix} \\
&= \begin{bmatrix} \vec{v}(-2) \\ \vec{v}(0) \\ \vec{v}(1) \end{bmatrix} + \begin{bmatrix} \vec{w}(-2) \\ \vec{w}(0) \\ \vec{w}(1) \end{bmatrix} \\
&= \mathrm{T}(\vec{v}) + \mathrm{T}(\vec{w})
\end{aligned}$$

And:

$$\begin{aligned}
\mathrm{T}(c\vec{v}) &= \mathrm{T}(c(v_3t^3 + v_2t^2 + v_1t + v_0)) \\
&= \mathrm{T}((cv_3)t^3 + (cv_2)t^2 + (cv_1)t + (cv_0)) \\
&= \begin{bmatrix} (cv_3)(-2)^3 + (cv_2)(-2)^2 + (cv_1)(-2) + (cv_0) \\ (cv_3)(0)^3 + (cv_2)(0)^2 + (cv_1)(0) + (cv_0) \\ (cv_3)(1)^3 + (cv_2)(1)^2 + (cv_1)(1) + (cv_0) \end{bmatrix} \\
&= \begin{bmatrix} c(v_3(-2)^3 + v_2(-2)^2 + v_1(-2) + v_0) \\ c(v_3(0)^3 + v_2(0)^2 + v_1(0) + v_0) \\ c(v_3(1)^3 + v_2(1)^2 + v_1(1) + v_0) \end{bmatrix} \\
&= c \begin{bmatrix} \vec{v}(-2) \\ \vec{v}(0) \\ \vec{v}(1) \end{bmatrix} \\
&= c\,\mathrm{T}(\vec{v})
\end{aligned}$$

Therefore T is linear.

b) Let $\vec{v}, \vec{w}$ be defined as in part (a). Let's see if T is one-to-one. Suppose $\mathrm{T}(\vec{v}) = \mathrm{T}(\vec{w})$. This is iff:

$$\mathrm{T}(\vec{v}) = \mathrm{T}(\vec{w})$$
$$\begin{bmatrix} v_3(-2)^3 + v_2(-2)^2 + v_1(-2) + v_0 \\ v_3(0)^3 + v_2(0)^2 + v_1(0) + v_0 \\ v_3(1)^3 + v_2(1)^2 + v_1(1) + v_0 \end{bmatrix} = \begin{bmatrix} w_3(-2)^3 + w_2(-2)^2 + w_1(-2) + w_0 \\ w_3(0)^3 + w_2(0)^2 + w_1(0) + w_0 \\ w_3(1)^3 + w_2(1)^2 + w_1(1) + w_0 \end{bmatrix}$$
$$\begin{bmatrix} -8v_3 + 4v_2 + -2v_1 + v_0 \\ v_0 \\ v_3 + v_2 + v_1 + v_0 \end{bmatrix} = \begin{bmatrix} -8w_3 + 4w_2 + -2w_1 + w_0 \\ w_0 \\ w_3 + w_2 + w_1 + w_0 \end{bmatrix}$$

Which implies:

$$\begin{cases} v_0 - 2v_1 + 4v_2 - 8v_3 - w_0 + 2w_1 - 4w_2 + 8w_3 = 0 \\ v_0 \qquad\qquad\qquad\qquad - w_0 \qquad\qquad\qquad\qquad = 0 \\ v_0 + \ v_1 + \ v_2 + \ v_3 - w_0 - \ w_1 - \ w_2 - \ w_3 = 0 \end{cases}$$

Which can be solved by Row reducing the augmented matrix associated to the homogeneous system of equations in the variables $v_0, v_1, v_2, v_3, w_0, w_1, w_2, w_3$:

$$\begin{bmatrix} 1 & -2 & 4 & -8 & -1 & 2 & -4 & 8 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 \end{bmatrix}$$

$$\begin{matrix} \xrightarrow{R_2 - R_1} \\ \xrightarrow{R_3 - R_1} \end{matrix} \begin{bmatrix} 1 & -2 & 4 & -8 & -1 & 2 & -4 & 8 & 0 \\ 0 & 2 & -4 & 8 & 0 & -2 & 4 & -8 & 0 \\ 0 & 3 & -3 & 9 & 0 & -3 & 3 & -9 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & -2 & 4 & -8 & -1 & 2 & -4 & 8 & 0 \\ 0 & 1 & -2 & 4 & 0 & -1 & 2 & -4 & 0 \\ 0 & 3 & -3 & 9 & 0 & -3 & 3 & -9 & 0 \end{bmatrix}$$

$$\begin{matrix} \xrightarrow{R_1 + 2R_2} \\ \\ \xrightarrow{R_3 - 3R_2} \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 & 0 & -1 & 2 & -4 & 0 \\ 0 & 0 & 3 & -3 & 0 & 0 & -3 & 3 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 & 0 & -1 & 2 & -4 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 + 2R_3} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & -1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

Which yields the system of equations:

$$\begin{cases} v_0 \qquad\qquad - w_0 \qquad\qquad\qquad = 0 \\ \quad v_1 \quad + 2v_3 \qquad - w_1 \qquad - 2w_3 = 0 \\ \qquad v_2 - \ v_3 \qquad\qquad - w_2 + \ w_3 = 0 \end{cases}$$

$$= \begin{cases} v_0 \qquad\qquad = w_0 \\ \quad v_1 \quad + 2v_3 = w_1 + 2w_3 \\ \qquad v_2 - \ v_3 = w_3 - w_3 \end{cases}$$

Which does not imply $\vec{v} = \vec{w}$. Choose for instance $v_0 = 0, v_1 = v_2 = v_3 = 1, w_1 = 3, w_0 = w_2 = w_3 = 0$ and the system above is satisfied but $\vec{v}(t) \neq \vec{w}(t)$ for all $t \in \mathbb{R}$ and therefore T is not one-to-one.

c) A vector $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$ is in the range of T if and only if there is some $\vec{v} \in \mathbb{P}_3$ such that if $\vec{v}(t) = v_3t^3 + v_2t^2 + v_1t + v_0$ then $\mathrm{T}(\vec{v}) = \vec{b}$:

$$\begin{aligned} \mathrm{T}(\vec{v}) &= \mathrm{T}(v_3t^3 + v_2t^2 + v_1t + v_0) \\ &= \begin{bmatrix} v_3(-2)^3 + v_2(-2)^2 + v_1(-2) + v_0 \\ v_3(0)^3 + v_2(0)^2 + v_1(0) + v_0 \\ v_3(1)^3 + v_2(1)^2 + v_1(1) + v_0 \end{bmatrix} \\ &= \begin{bmatrix} -8v_3 + 4v_2 - 2v_1 + v_0 \\ v_0 \\ v_3 + v_2 + v_1 + v_0 \end{bmatrix} \\ &= \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \end{aligned}$$

This yields the system of equations:

$$\begin{cases} v_0 - 2v_1 + 4v_2 - 8v_3 = b_1 \\ v_0 \qquad\qquad\qquad\qquad = b_2 \\ v_0 + \ v_1 + \ v_2 + \ v_3 = b_3 \end{cases}$$

Using theorem 4 in Chapter 1 we only have to look at the number of pivots in the coefficient matrix of the system.

$$\begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \underset{\xrightarrow{R_3 - R_1}}{\xrightarrow{R_2 - R_1}} \begin{bmatrix} 1 & -2 & 4 & -8 \\ 0 & 2 & -4 & 8 \\ 0 & 3 & -3 & 9 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & -2 & 4 & -8 \\ 0 & 1 & -2 & 4 \\ 0 & 3 & -3 & 9 \end{bmatrix} \underset{\xrightarrow{R_3 - 3R_2}}{\xrightarrow{R_1 + 2R_2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 3 & -3 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_2+2R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Which has a pivot in every row, therefore the system is always consistent and the transformation is onto.

d) Since the transformation is onto then the range of the transformation T is $\mathbb{R}^3$. Using the reduced echelon form of part (c) we can describe the kernel of the transformation by augmenting with the zero column and we obtain the equations:

$$\begin{cases} v_0 & & & = 0 \\ & v_1 & + 2v_3 & = 0 \ ; \ v_3 \text{ is free} \\ & & v_2 - \ \ v_3 & = 0 \end{cases}$$

or

$$\begin{cases} v_0 & = 0 \\ \quad v_1 & = -2s \\ \qquad v_2 & = s \\ \qquad\quad v_3 & = s \in \mathbb{R} \end{cases}$$

The Kernel is then given by the vectors $\vec{v}$ such that $\vec{v}(t) = st^3 + st^2 - 2st = s(t^3 + t^2 - 2t)$ and in set notation:

$$\text{Ker}(T) = \left\{ \vec{p}(t) \in \mathbb{P}_3 \mid \vec{p}(t) = s(t^3 + t^2 - 2t) \right\}$$

The fact that the dimension of the Kernel and the Range add up to four (the dimension of $\mathbb{P}_3$) adds confidence to our results, plus one can plug in vectors in the kernel obtained from the definition above and see that $\text{T}(\vec{v}) = \vec{0}$

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