True/False Questions: Disc. Board.

Rafael Laya

Fall 2018

Problem 1.

Proposition: If the columns of A are linearly independent, then the equation $A\vec{x} = \vec{0}$ has a unique solution.

Proof: Let A be a matrix of order mxn whose columns are represented by the vectors $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n} \in \mathbb{R}^m$. Assume that the columns of A are linearly independent. Then, by definition the vector equation:

$$x_1\vec{a_1} + x_2\vec{a_2} + \cdots + x_n\vec{a_n} = \vec{0}$$

has unique solution where $x_i = 0$ for $i = 1, \ldots, n$ The system above is equivalent to the matrix equation:

$$A\vec{x} = \vec{0}$$

And therefore the solution set is the same. That is, the solution to the equation $A\vec{x} = \vec{0}$ has a unique solution (which is $\vec{x} = \vec{0}$).

∎

Problem 2.

Proposition: If $A\vec{x} = \vec{0}$ has a unique solution, then the columns of A are linearly independent.

Proof: Let A be a matrix of order mxn whose columns are represented by the vectors $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n} \in \mathbb{R}^m$. Assume that $A\vec{x} = \vec{0}$ has unique solution (the trivial solution). Then the equivalent vector equation with $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$ ∎

$$x_1\vec{a_1} + \cdots + x_n\vec{a_n} = \vec{a_n}$$

has the same solution set as the matrix equation, a unique solution where $x_i = 0$ for $i = 1, \ldots, n$. By definition the set of the columns of A, $\{\vec{a_1}, \ldots, \vec{a_n}\}$ is linearly independent.

Problem 3.

Proposition: If the columns of A are linearly independent, then the equation $A\vec{x} = \vec{b}$ has a unique solution for any vector

This is false since $\vec{v_1} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ are linearly independent but

$$\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

has no solution since the equivalent system of linear equations implies $1 = 0$

Problem 4.

Proposition: If the columns of A are linearly dependent, the equation $A\vec{x} = \vec{b}$ has infinitely many solutions for any vector $\vec{b}$.

This is false since $\vec{v_1} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix}1\\0\end{bmatrix}$ are linearly dependent but

$$\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

has no solution since the equivalent system of linear equations implies $0 = 1$

Problem 5.

Proposition: If the columns of A are linearly dependent and the equation $A\vec{x} = \vec{b}$ has at least one solution, then it has infinitely many solutions.

Proof: Let A be a matrix of order mxn whose columns are represented by the vectors $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n} \in \mathbb{R}^m$. Suppose that the columns of A are linearly dependent. By definition there is a non-trivial solution for:

$$x_1\vec{a_1} + \cdots + x_n\vec{n} = \vec{0}$$

Let then $\vec{\lambda} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$ be any non-trivial solution to the equation above. Assume that $A\vec{x} = \vec{b}$ has at least a solution and let $\vec{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$ be one solution. Now consider $k \in \mathbb{R}$ and:

$$\begin{aligned} A(k\vec{\lambda} + \vec{p}) &= A(k\vec{\lambda}) + A\vec{p} \\ &= k(A\vec{\lambda}) + \vec{b} \\ &= k\vec{0} + \vec{b} \\ &= \vec{0} + \vec{b} \\ &= \vec{b} \end{aligned}$$

We know the solution $k\vec{\lambda} + \vec{p} \neq \vec{p}$ since $\vec{\lambda} \neq \vec{0}$ and given that we can choose any value for k then there must be infinite solutions to the equation $A\vec{x} = \vec{b}$

∎

Problem 6.

Proposition: If the equation $A\vec{x} = \vec{b}$ has a unique solution for some vector $\vec{b}$, the columns of A are linearly independent.

Proof: Let A be a matrix of order mxn whose columns are represented by the vectors $\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n} \in \mathbb{R}^m$. Suppose that $A\vec{x} = \vec{b}$ has a unique solution for some vector b, then:

$$x_1\vec{a_1} + \cdots + x_n\vec{a_n} = \vec{b}$$

has unique solution. Suppose now that the columns of A are linearly dependent. That is, there exists $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that:

$$\lambda_1\vec{a_1} + \cdots + \lambda_n\vec{a_n} = \vec{0}$$

Now consider $\vec{\lambda} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$ and so:

$$\begin{aligned} A(\vec{x} + \vec{\lambda}) &= A\vec{x} + A\vec{\lambda} \\ &= \vec{b} + \vec{0} \\ &= \vec{b} \end{aligned}$$

This solution $\vec{x} + \vec{\lambda} \neq \vec{x}$ since $\vec{\lambda} \neq \vec{0}$ Therefore there is at least a second solution to $A\vec{x} = \vec{b}$ which is a contradiction to our hypothesis. By contradiction the columns of A must be linearly independent.

∎

Problem 7.

Proposition: If the equation $A\vec{x} = \vec{b}$ has a unique solution for some vector $\vec{b}$, then the equation $A\vec{x} = \vec{0}$ has a unique solution.

Proof: Suppose the equation $A\vec{x} = \vec{b}$ has a unique solution $\vec{x_1}$ for some vector $\vec{b}$, and suppose $A\vec{x} = \vec{0}$ has a solution other than the trivial solution $\vec{x_2}$. Then:

$$\begin{aligned} A(\vec{x_1} + \vec{x_2}) &= A\vec{x_1} + A\vec{x_2} \\ &= \vec{b} + \vec{0} \\ &= \vec{b} \end{aligned}$$

Then $A\vec{x} = \vec{b}$ has two distinct solutions (since $\vec{x_2} \neq \vec{0}$) and by contradiction we are left with $A\vec{x} = \vec{0}$ must have an unique solution. ∎

Problem 8.

Proposition: if the columns of A are linearly dependent, then the columns of A do not span $\mathbb{R}^m$.

This is false. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The first two columns Span $\mathbb{R}^2$ and the third column is the sum of the first two thus the set of all three columns of A is linearly dependent.

Problem 9.

Proposition: If A is an nxn matrix and the columns of A are linearly dependent, then the equation $A\vec{x} = \vec{b}$ always has a unique solution.

This is false, consider:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This is an inconsistent system but the columns of A contain the zero vector and thus are always linearly dependent.

Problem 10.

Proposition: If A is an nxn matrix and the columns of A are linearly independent, then the columns of A span $\mathbb{R}^n$.

Proof: Suppose A is a nxn matrix and that its columns $\vec{a_1}, \ldots, \vec{a_n}$ are linearly independent. Then:

$$x_1\vec{a_1} + \cdots + x_n\vec{a_n} = \vec{0}$$

only has trivial solution $x_i = 0$ for $i = 1, \ldots, n$

Therefore the reduced echelon form of A must have a pivot in each of its n columns, otherwise there would be free variables in $A\vec{x} = \vec{0}$ and infinitely many solutions (since $A\vec{x} = \vec{0}$ is always consistent with the trivial solution). Now, choose any vector $\vec{b} \in \mathbb{R}^n$ and the augmented matrix $\begin{bmatrix} A & \vec{b} \end{bmatrix}$ has already n pivots in the first n columns, it can only have n pivots since it has n rows and the rightmost column cannot be a pivot column therefore the system is consister (theorem 2). Because each variable is a basic variable (given the n correspond to n variables) there are no free variables and the solution is unique. ∎