Mastery Homework 4

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Section 1.9

Problem 26

Statement: Determine if the linear transformation T is a) One to One b) onto. With $\mathrm{T} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $\mathrm{T}(\vec{e_1}) = (1,3), \mathrm{T}(\vec{e_2}) = (4,-7)$, and $\mathrm{T}(\vec{e_3}) = (-5,4)$ and where $\vec{e_1}, \vec{e_2}, \vec{e_3}$ are the columns of the identity matrix.

Solution: a) The linear transformation T is One to One if and only if the equation $\mathrm{T}(\vec{x}) = \vec{0}$ only has trivial solution. Given the information in the statement we know the standard matrix A of the linear transformation:

$$A = \begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix}$$

Then, $\mathrm{T}(\vec{x}) = A\vec{x} = \vec{0}$ can be solved by row reducing the augmented matrix $\begin{bmatrix} A & \vec{0} \end{bmatrix}$

$$\begin{bmatrix} 1 & 4 & -5 & 0 \\ 3 & -7 & 4 & 0 \end{bmatrix}$$

Notice, however, that A has two rows and three columns, therefore it can have at most two pivots (one pivot per row). If A has at most two pivot columns then the system is either inconsistent (which is false since $A\vec{x} = \vec{0}$ always has trivial solution) or there is a free variable in the linear system with augmented matrix $\begin{bmatrix} A & \vec{0} \end{bmatrix}$ and therefore there are infinitely many solutions. If there are infinitely many solutions then we have non-trivial solutions to $\mathrm{T}(\vec{x}) = \vec{0}$ and so T is not One to One.

b) The linear transformation T is Onto if and only if the columns of the matrix A span R^2. Let $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$ and then we only have to row reduce $\begin{bmatrix} A & \vec{b} \end{bmatrix}$ which is equivalent to solving the equation $x_1 a_1 + x_2 a_2 + x_3 a_3 = \vec{b}$

$$\begin{bmatrix} 1 & 4 & -5 & b_1 \\ 3 & -7 & 4 & b_2 \end{bmatrix} \xrightarrow{R_2-3R_1} \begin{bmatrix} 1 & 4 & -5 & b_1 \\ 0 & -19 & 19 & -3b_1+b_2 \end{bmatrix} \xrightarrow{-\frac{1}{19}R_2} \begin{bmatrix} 1 & 4 & -5 & b_1 \\ 0 & 1 & -1 & \frac{3}{19}b_1 - \frac{1}{19}b_2 \end{bmatrix}$$

We do not have to reach the row reduced echelon form since at this stage we know we have one pivot in every row of A and therefore the right-most column of the augmented matrix cannot be a pivot column and the system is always consistent for any $\vec{b}$. The linear transformation T is onto.

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Problem 28

Statement: Determine if the linear transformation T is a) One to One b) onto.

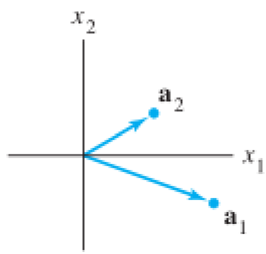

Figure 1: The columns of the Standard Matrix A of the linear transformation T

Solution: a) Since the image shows that $\vec{a_1}, \vec{a_2}$ are not multiples of each others then the set $\{\vec{a_1}, \vec{a_2}\}$ is a linearly independent set. The linear transformation T is (by theorem) one to one if and only if the columns of A are linearly independent, which is the case here. T is a one to one linear transformation.

b) Since $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ the matrix A has two columns and two rows. Since the columns of A are linearly independent then A has two pivots, therefore each row of A has a pivot and by theorem the columns of A span $\mathbb{R}^2$. Because the columns of A span $\mathbb{R}^2$, again by theorem, T is onto.

∎

Problem 34

Statement: Why is the question "Is the linear transformation T onto? an existence question?

Solution: Let $\mathrm{T} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. The question whether T is onto is an existence question because it is equivalent (by definition) to asking that if for every $\vec{b} \in \mathbb{R}^m$ there EXISTS some $\vec{x} \in \mathbb{R}^n$ such that $\mathrm{T}(\vec{x}) = \vec{b}$ ∎

Section 2.1

Problem 22

Statement: Show that if the columns of B are linearly dependent, then so are the columns of AB.

Solution: Let A be a matrix of order mxn and B of order nxp. Let $\mathrm{col_i}(M)$ be the notation for the ith column of a matrix M. Suppose the columns of B are linearly dependent, that is, there is non-trivial solution $(x_1, \ldots, x_p) = (c_1, \ldots, c_p)$ to the equation:

$$x_1 \,\mathrm{col_1}(B) + \cdots + x_p \,\mathrm{col_p}(B) = \vec{0}$$

Which is equivalent to having a non-trivial solution $\vec{x} = \vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$ to: $B\vec{x} = \vec{0}$. Now, multiply by A on the left and apply properties of matrix multiplication:

$$\begin{aligned} B\vec{c} &= \vec{0} \\ A(B\vec{c}) &= A\vec{0} \\ (AB)\vec{c} &= \vec{0} \end{aligned}$$

Therefore there is a non-trivial solution $\vec{c}$ to the equation $(AB)\vec{x} = \vec{0}$ which is equivalent to a non-trivial solution $(x_1, \ldots, x_p) = (c_1, \ldots, c_p)$ to:

$$x_1 \,\mathrm{col_1}(AB) + \cdots + x_p \,\mathrm{col_p}(AB) = \vec{0}$$

Therefore the columns of AB are linearly dependent. ∎

Section 2.2

Problem 14

Statement: Suppose $(B - C)D = 0$ where B and C are mxn matrices and D is invertible. Show that $B = C$.

Solution: Take the original equation and multiply by the inverse of D, D^{-1} on the right (in order to get, say, "rid" of D).

$$
\begin{aligned}
(B-C)D &= 0 \\
((B-C)D)D^{-1} &= 0D^{-1} \\
(B-C)(DD^{-1}) &= 0 \\
(B-C)I_n &= 0 \\
B-C &= 0
\end{aligned}
$$

Add C and then:

$$
\begin{aligned}
(B-C)+C &= 0+C \\
B+(-C+C) &= C \\
B+0 &= C \\
B &= C
\end{aligned}
$$

Which is exactly what we wanted to show.

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Problem 15

Statement: Suppose A, B, C are invertible matrices of nxn. Show that ABC is also invertible by producing a matrix D such that $(ABC)D = I$ and $D(ABC) = I$

Solution: Recall that: $(AB)^{-1} = B^{-1}A^{-1}$. We can apply this property multiple times to figure out D:

$$
\begin{aligned}
(ABC)^{-1} &= C^{-1}(AB)^{-1} \\
&= C^{-1}B^{-1}A^{-1}
\end{aligned}
$$

So let $D = C^{-1}B^{-1}A^{-1}$ and then:

$$
\begin{aligned}
(ABC)D &= (ABC)(C^{-1}B^{-1}A^{-1}) \\
&= (AB)(CC^{-1})(B^{-1}A^{-1}) \\
&= (AB)(I)(B^{-1}A^{-1} \\
&= (AB)(B^{-1}A^{-1} \\
&= A(BB^{-1})A^{-1} \\
&= (AI)A^{-1} \\
&= AA^{-1} \\
&= I
\end{aligned}
$$

And,

$$\begin{aligned}D(ABC) &= (C^{-1}B^{-1}A^{-1})(ABC)\\ &= (C^{-1}B^{-1})(A^{-1}A)(BC)\\ &= (C^{-1}B^{-1})(I)(BC)\\ &= (C^{-1}B^{-1})(BC)\\ &= C^{-1}(B^{-1}B)C\\ &= C^{-1}(I)C\\ &= C^{-1}C\\ &= I\end{aligned}$$

Therefore ABC is invertible and $D = C^{-1}B^{-1}A^{-1} = (ABC)^{-1}$

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Problem 20

Statement: Suppose A, B and X are nxn matrices with A, X and $A - AX$ invertible, and suppose $(A - AX)^{-1} = X^{-1}B$. a) Explain why B is invertible. b) Solve for X. If you need to invert a matrix explain why it is invertible.

Solution: a) By assumption A, X and $A - AX$ are invertible. We can multiply by X on the left (since $(X^{-1})^{-1} = X$) and this will tell us B in terms of $(A - AX)^{-1}$ and X, all of which exist:

$$\begin{aligned}(A - AX)^{-1} &= X^{-1}B\\ X(A - AX)^{-1} &= X(X^{-1}B)\\ X(A - AX)^{-1} &= (XX^{-1})B\\ X(A - AX)^{-1} &= IB\\ X(A - AX)^{-1} &= B\end{aligned}$$

Since X is invertible and so is $(A - AX)^{-1}$ (the inverse of the inverse is the original) then by theorem, the product of two invertible matrices, B, is also invertible. In fact its inverse is the product of the inverses in opposite order ($B^{-1} = (A - AX)X^{-1}$)

b) In order to solve for X first take inverse on both sides (which is possible since the product of two invertible matrices is invertible by theorem). B is invertible by part (a), and $(A - AX)^{-1}, X^{-1}$ are invertible since $A - AX, X$ are invertible by assumption and the inverse of the inverse is the original,

$$\begin{aligned}(A - AX)^{-1} &= X^{-1}B\\ ((A - AX)^{-1})^{-1} &= B^{-1}(X^{-1})^{-1}\\ A - AX &= B^{-1}X\end{aligned}$$

Add AX to both sides,

$$\begin{aligned} A - AX + AX &= B^{-1}X + AX \\ A &= (B^{-1} + A)X \end{aligned}$$

Since A is invertible then $(B^{-1} + A)X$ must be invertible (they are equal). We know by assumption X is invertible and then the other factor must be invertible (because we can multiply by X^{-1} on the right and AX^{-1} is the product of two invertible matrices and therefore $(B^{-1} + A) = AX^{-1}$ is invertible too). Multiply by $(B^{-1} + A)^{-1}$ on the left,

$$\begin{aligned} &(A + B^{-1})^{-1}A = (A + B^{-1})^{-1}(A + B^{-1})X \\ &(A + B^{-1})^{-1}A = IX \\ &(A + B^{-1})^{-1}A = X \end{aligned}$$

Which is what we wanted (to solve for X). $\blacksquare$