Mastery Homework 5

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Section 2.3

Problem 20

Statement: If nxn matrices E and F have the property that $EF = I$, then E and F commute. Explain why.

Solution: Because if $EF = I$ then we know by the Invertible Matrix Theorem and by the result or problem 27 solved in fundamental homework that they are inverses of each other and therefore E and F multiply and result on the identity whether it is EF or FE. Let's show this:

Let E and F be two matrices of order nxn. Suppose that $EF = I$ where I is the identity matrix of order nxn. Applying the Invertible Matrix Theorem we know that F is invertible. Consider:

$$EF = I$$

Multiply both sides of the equality above by F^{-1} on the right:

$$\begin{aligned}(EF)F^{-1} &= IF^{-1}\\ E(FF^{-1}) &= F^{-1}\\ E(I) &= F^{-1}\\ E &= F^{-1}\end{aligned}$$

Multiply both sides of the equality above by F on the left:

$$\begin{aligned}E &= F^{-1}\\ FE &= FF^{-1}\\ FE &= I\end{aligned}$$

Therefore,

$$EF = I = FE$$

Which shows that E and F commute. ∎

Problem 32

Statement: Suppose A is a nxn matrix with the property that the equation $A\vec{x} = \vec{0}$ has only trivial solution. Without using the Invertible Matrix Theorem, explain directly why the equation $A\vec{x} = \vec{b}$ must have a solution for each $\vec{b}$ in $\mathbb{R}^n$.

Solution: Simply because if $A\vec{x} = \vec{0}$ has only trivial solution we know that the row reduced echelon form of A must not have a row of zero-es and therefore the system will be consistent for any vector $\vec{b}$ that we use in the augmented column of $\begin{bmatrix} A & \vec{b} \end{bmatrix}$ with this argument in mind:

Let A be a nxn matrix with the property that $A\vec{x} = \vec{0}$ has only trivial solution. Then, due to theorem 2 in chapter 1 we know that the matrix A must not have any free variables since the system is consistent and it has unique solution. Because A has n columns it must have n pivot positions that correspond to n basic variables. A must also have a pivot in every row since it has n pivots and n rows, using theorem 4 from chapter 1 we know that there is a solution for each $\vec{b} \in \mathbb{R}^n$. ∎

Problem 38

Statement: Suppose a linear transformation $\mathrm{T} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ has the property that $\mathrm{T}(\vec{u}) = \mathrm{T}(\vec{v})$ for some pair of distinct vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$. Can T map $\mathbb{R}^n$ onto $\mathbb{R}^n$? Why or Why not?

Solution: Let $\mathrm{T} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation. Assume $\mathrm{T}(\vec{u}) = \mathrm{T}(\vec{v})$ for some $\vec{u}, \vec{v} \in \mathbb{R}^n$ with $\vec{u} \neq \vec{v}$. By theorem 10 in chapter 1 we know that there exists a matrix A such that:

$$\mathrm{T}(\vec{x}) = A\vec{x}$$

For all $\vec{x} \in \mathbb{R}^n$. By assumption the transformation T is not one-to-one, since $\mathrm{T}(\vec{u}) = \mathrm{T}(\vec{v})$ with $\vec{u} \neq \vec{v}$. Using the Invertible Matrix Theorem, since T is not one-to-one the linear transformation T does not map $\mathbb{R}^n$ onto $\mathbb{R}^n$. ∎

Section 3.1

Problem 41

Statement: Let $\vec{u} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Compute the area of the parallelogram determined by $\vec{u}, \vec{v}, \vec{u} + \vec{v}$, and $\vec{0}$, and compute the determinant of $\begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix}$. How do they compare? Replace the first entry of $\vec{v}$ by an arbitrary number x, and repeat the problem. Draw a picture and explain what you find.

Solution: First draw a picture of the parallelogram determined by $\vec{u}, \vec{v}, \vec{u} + \vec{v}, \vec{0}$

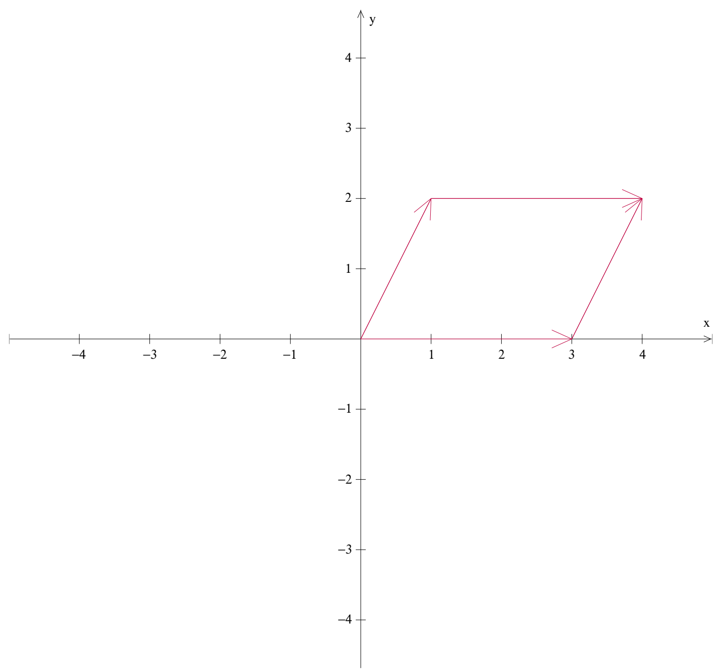

Figure 1: Parallelogram determined by $\vec{u}, \vec{v}, \vec{u} + \vec{v}, \vec{0}$

Since the base of the parallelogram is 3 and height 2 its area is $3 \cdot 2 = 6$. Let's calculate the determinant of $\begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix}$

$$\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} = 3(2) - 1(0) = 6$$

They compare by being exactly the same. That is, in this case: $\begin{vmatrix} \vec{u} & \vec{v} \end{vmatrix} = 6 =$ Area of the parallelogram spanned by $\vec{v}, \vec{u}, \vec{v} + \vec{u}, \vec{0}$

Replacing the first entry of $\vec{v}$ by a number x we have:

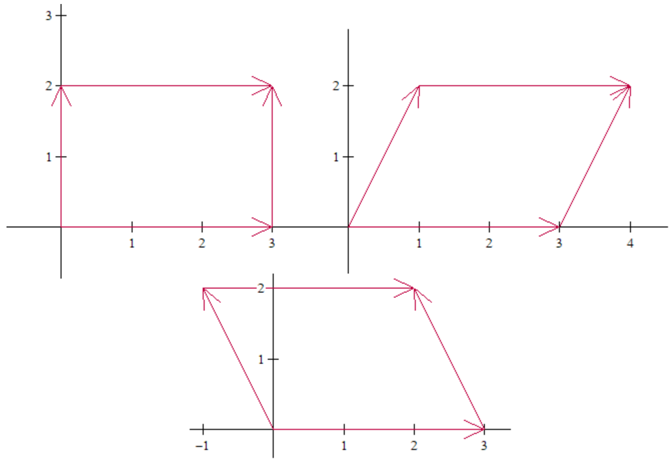

Figure 2: The parallelogram with a few different values of x

Notice that no matter the value of x, the parallelogram is always of base three and height two, therefore its area is always 6. Also:

$$\begin{vmatrix} 3 & x \\ 0 & 2 \end{vmatrix} = 3(2) - x(0) = 6$$

$\blacksquare$

This gives us a hint that the determinant $\begin{vmatrix} \vec{u} & \vec{v} \end{vmatrix}$ is closely related to the area of the parallelogram spanned by the vectors $\vec{u}, \vec{v}$. Namely, $\begin{vmatrix} \vec{u} & \vec{v} \end{vmatrix}$ is exactly the area of the parallelogram spanned by $\vec{u}, \vec{v}$ (In reality we have to add absolute value to the determinant because it might turn out negative, using knowledge from Calculus 3).

Section 3.2

Problem 34

Statement: Let A and P be square matrices, with P invertible. Show that $\det(PAP^{-1}) = \det(A)$.

Solution: Suppose A and P are matrices of nxn with P invertible. That is, there exists P^{-1} such that:

$$PP^{-1} = I$$

Where I is the identity matrix of order nxn. We also know that:

$$\det(PP^{-1}) = \frac{1}{\det(P)}$$

Now, using the multiplicative property of determinants:

$$\begin{aligned}
\det(PAP^{-1}) &= det(PA)\dot{\det}(P^{-1}) \\
&= \det(P)\det(A)\frac{1}{\det(P^{-1})} \\
&= \det(A)\left(\det(P)\frac{1}{\det(P^{-1})}\right) \\
&= \det(A)(1) \\
&= \det(A)
\end{aligned}$$

Which is precisely what we wanted to show. ∎

Problem 36

Statement: Find a formula for $\det(rA)$ when A is a nxn matrix.

Solution: Let $r \in \mathbb{R}$ and A be a matrix of nxn. Then, define R as the diagonal matrix of nxn where the entries in the diagonal are r. Because R is diagonal matrix, by theorem its determinant is the product along the diagonal (a diagonal matrix is also triangular matrix). Because R is of nxn, it has n entries on its diagonal. Therefore $\det(R) = r^n$. Let's denote a_{ij} as the entry in A at row i, column j. Notice, however that $rA = RA$,

$$\begin{bmatrix} r & 0 & \dots & 0 \\ 0 & r & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r \end{bmatrix} \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} ra_{11} & \dots & ra_{1n} \\ \vdots & & \vdots \\ ra_{n1} & \dots & ra_{nn} \end{bmatrix} = r \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

Because R is the result of multiplying every row of the identity matrix of order nxn by r. Now, using the multiplicative propery of determinants:

$$\begin{aligned}
\det(rA) &= \det(RA) \\
&= \det(R)\det(A) \\
&= r^n \det(A)
\end{aligned}$$

Which is the formula we wanted. ∎