Learning and Decision Making

Laboratory 1: Markov chains

In the end of the lab, you should submit all code/answers written in the tasks marked as "Activity n. XXX", together with the corresponding outputs and any replies to specific questions posed to the e-mail adi.tecnico@gmail.com. Make sure that the subject is of the form [<group n.>] LAB <lab n.>.

1. Modeling

Consider once again the train modeling problem described in the Homework and for which you wrote a Markov chain model:

Recall that your chain should describe the motion of the single train traveling the network, where:

- Stations A and B are just like regular stops;
- The travel time between any two consecutive stops is exactly 10 minutes. The train stops exactly 2 minutes in each location.
- At the intersection marked with a bold $\times$, the train follows the branch 1-3 with probability 0.5, the branch 4 with probability 0.15, and the branch 5-6 with probability 0.35.

Activity 1.

Implement your Markov chain model in Python. In particular,

- Create a list with all the states;
- Define a `numpy` array with the corresponding transition probabilities.

The order for the states used in the transition probability matrix should match that in the list of states.

Note 1: Don't forget to import `numpy`. If you need additional matrix operations (such as matrix powers or eigenvalues and eigenvectors), you may also import the library `numpy.linalg`.

Note 2: Make sure to print the result in the end.

In [1]:
```python
import numpy as np

X = ("Station A", "Stop 1", "Stop 2", "Stop 3", "Stop 4", "Stop 5",
     "Stop 6", "Station B")
print("States: {}".format(X))

P = np.zeros((8, 8))
P[0, 1] = 0.5
P[0, 4] = 0.15
P[0, 5] = 0.35
P[1, 2] = 1
P[2, 3] = 1
P[3, 7] = 1
P[4, 7] = 1
P[5, 6] = 1
P[6, 7] = 1
P[7, 0] = 1
print("Transition probability matrix:")
print(P)
```

```
States: ('Station A', 'Stop 1', 'Stop 2', 'Stop 3', 'Stop 4', 'Stop 5', 'Stop
6', 'Station B')
Transition probability matrix:
[[0.   0.5  0.   0.   0.15 0.35 0.   0.  ]
 [0.   0.   1.   0.   0.   0.   0.   0.  ]
 [0.   0.   0.   1.   0.   0.   0.   0.  ]
 [0.   0.   0.   0.   0.   0.   0.   1.  ]
 [0.   0.   0.   0.   0.   0.   0.   1.  ]
 [0.   0.   0.   0.   0.   0.   1.   0.  ]
 [0.   0.   0.   0.   0.   0.   0.   1.  ]
 [1.   0.   0.   0.   0.   0.   0.   0.  ]]
```

Activity 2.

Compute, using the proper transition matrix manipulations, the probability of the following trajectories:

- 4 - B - A - 4
- A - 2 - 3 - B - A
- 5 - 6 - B - A - 4

Note: Make sure to print the result in the end.

In [2]:
```python
def calculateProb(path):
    prob = 1
    for step in range(len(path) - 1):
        prob *= P[path[step], path[step + 1]]
    return prob

path1 = [4, 7, 0, 4]
path2 = [0, 2, 3, 7, 0]
path3 = [5, 6, 7, 0, 4]

print("Probability of path '{}': \t{}".format(path1, calculateProb(path1)))
print("Probability of path '{}': \t{}".format(path2, calculateProb(path2)))
print("Probability of path '{}': \t{}".format(path3, calculateProb(path3)))
```

```
Probability of path '[4, 7, 0, 4]':     0.15
Probability of path '[0, 2, 3, 7, 0]':  0.0
Probability of path '[5, 6, 7, 0, 4]':  0.15
```

2. Stability

Activity 3

Compute the stationary distribution for the chain. Confirm, computationally, that it is indeed the stationary distribution.

Note: The stationary distribution is a *left* eigenvector of the transition probability matrix associated to the eigenvalue 1. As such, you may find useful the numpy function `numpy.linalg.eig`. Also, recall that the stationary distribution is *a distribution*.

In [3]:
```python
# (@ is the "dot" operator)
# u @ P = u
# <=> (u @ P).T = u.T
# <=> P.T @ u.T = u.T
# => u.T is an eigenvector of P.T with eigenvalue 1
# (The eigenvector v with eigenvalue lambda of a matrix A satisfies A @ v = lamb
da @ v)

# Gets all the eigenvalues and vectors for P transposed
eigenValues, eigenVectors = np.linalg.eig(P.T)

# Chooses the eigenvector corresponding to the eigenvalue equal to 1, which is t
he first one,
# and normalizes it
# We want the column and not the line because we're working with the eigenvector
s of P transposed
u = eigenVectors.T[0].real / np.sum(eigenVectors[:, 0].real)

# Verifies if u = uP
test = u @ P
if np.allclose(u, test):
    print("Stationary distribution for the chain:\n{}".format(u))
```

```
Stationary distribution for the chain:
[0.22988506 0.11494253 0.11494253 0.11494253 0.03448276 0.08045977
 0.08045977 0.22988506]
```

Activity 4.

Empirically show that the chain is ergodic.

Note: Recall that a chain is ergodic if, given any initial distribution, it converges to the stationary distribution.

In [4]:
```python
import random

# Multiplying a random (normalized) initial distribution by the matrix P to the
 power of 100000000,
# we conclude the chain is ergodic as the result of the multiplication is simila
r to the stationary distribution.
# That is done for 1000 randomly generated initial distributions.

def act4():
    for i in range(1000):
        newDist = np.random.rand(8)
        newDist /= np.sum(newDist)
        uInf = newDist @ np.linalg.matrix_power(P, 100000000)
        if not np.allclose(uInf, u):
            print("Different from stationary distribution!")
            return
    print("The chain is ergodic, as the chain converges to the stationary distri
bution calculated in Activity 3" \
          + " with several different initial distributions.")

act4()
```

```
The chain is ergodic, as the chain converges to the stationary distribution cal
culated in Activity 3 with several different initial distributions.
```

3. Simulation

You are now going to *simulate* the Markov chain that you defined in Question #1.

Activity 5

Generate a 10,000-step long trajectory of the chain defined in Activity #1.

In [5]:
```python
def checkTrajPossible(traj):
    size = len(traj)
    for step in range(size - 1):
        if not P[traj[step], traj[step + 1]] != 0:
            raise Exception("The generated trajectory is not possible.")
    print("The generated trajectory is possible.")

trajectory = []
stop = 0
for i in range(0, 10000):
    trajectory.append(stop)
    stop = np.random.choice((np.arange(8)), p = P[stop])

print("Trajectory length:", len(trajectory))
print("First 10 stops:", trajectory[0:10])
checkTrajPossible(trajectory)
```

```
Trajectory length: 10000
First 10 stops: [0, 5, 6, 7, 0, 5, 6, 7, 0, 1]
The generated trajectory is possible.
```

Activity 6

Draw a histogram of the trajectory generated in Activity #5. Make sure that the histogram has one bin for each state. Compare the relative frequencies with the result of Activity #3.

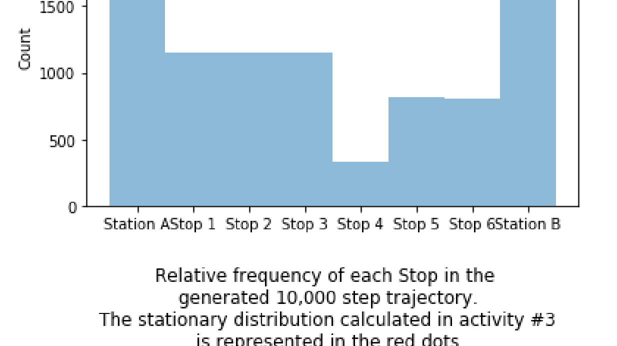
Note: Don't forget to load `matplotlib`.

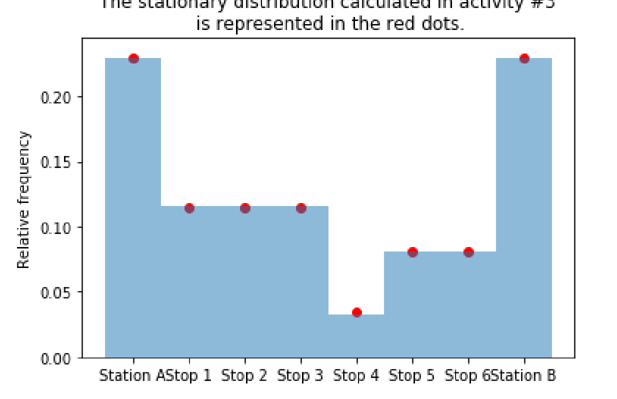
In [7]:
```python
import matplotlib.pyplot as plt

x = np.arange(8)
y = np.bincount(trajectory)

plt.bar(x, y, align='center', width=1, alpha=0.5)
plt.xticks(x, X)
plt.ylabel('Count')
plt.title('Count of each Stop in the generated 10,000 step trajectory.')
plt.show()

plt.figure()
plt.bar(x, y / np.sum(y), align='center', width=1, alpha=0.5)
plt.scatter(x, u, zorder=1, color="red")
plt.xticks(x, X)
plt.ylabel('Relative frequency')
plt.title('Relative frequency of each Stop in the \ngenerated 10,000 step trajec
tory.\n' \
    + 'The stationary distribution calculated in activity #3\n is represented in
 the red dots.')
plt.show()
```





As we can see in the last histogram, the relative frequency of the stops in the generated trajectory aproximates the stationary distribtution represented as the red dots.