Homework 1. Markov chains

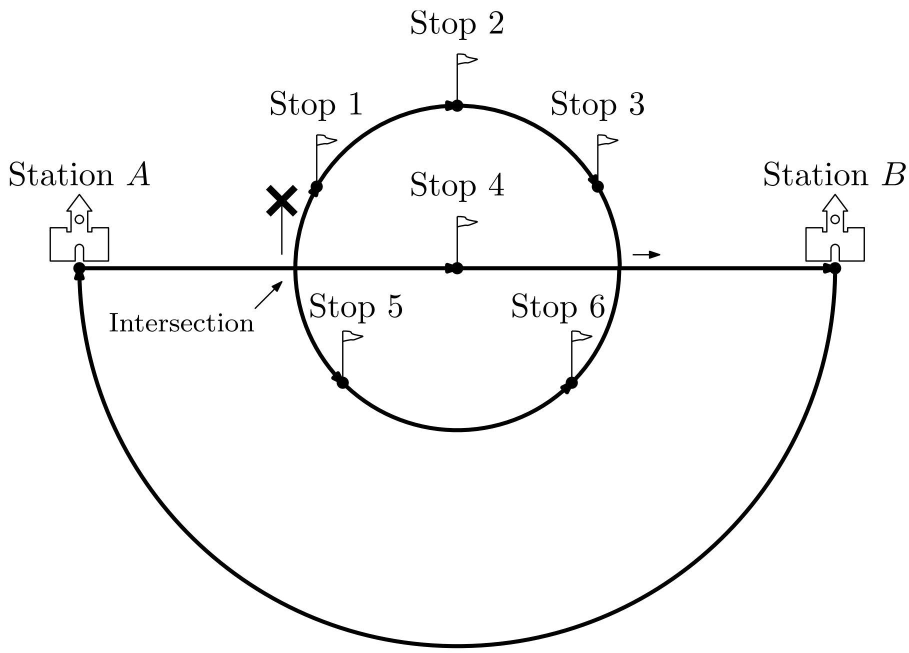

Figure 1: Train network including 2 major stations and 6 additional stops.

Consider the train network depicted in Fig. 1. In this homework, you will describe the motion of a train in this network using a Markov chain. To that purpose, consider the following information:

- For all purposes, Stations A and B are just like regular stops;
- There is a single train traveling in the network;
- The travel time between any two consecutive stops is exactly 10 minutes. For example, it takes 10 minutes to travel from Station A to Stop 1, or between Stop 5 and Stop 6.

- At the intersection marked with a bold $\times$, the train follows the branch with most people waiting. This corresponds to the branch with Stops 1-3 with probability 0.5, to the branch with Stop 4 with probability 0.15, and to the branch with Stops 5-6 with probability 0.35.

Exercise 1.

(a) Write down the Markov chain model (state space and transition probabilities) representing the motion of the train.

(b) Suppose that the train departs from Station A at time step $t = 0$. Compute the probability of the train being in each stop at time step $t = 3$.

(c) Consider a passenger arriving at Stop 4 just as the train leaves the station. What is the expected waiting time for such a passenger? Assume that the train stops for exactly 2 minutes in each station.

Solution:

(a) The Markov chain is specified as a pair $(\mathcal{X}, \boldsymbol{P})$, where the states correspond to train positions. We have that

$$\mathcal{X} = \{A, B, 1, 2, 3, 4, 5, 6\},$$

where the states are labeled after the corresponding stops. The transition probabilities can be extracted from the provided description of the motion.

$$\boldsymbol{P} = \begin{bmatrix} 0 & 0 & 0.5 & 0 & 0 & 0.15 & 0.35 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) The matrix $\boldsymbol{P}^3$ comes

$$\boldsymbol{P}^3 = \begin{bmatrix} 0.15 & 0.35 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0.15 & 0 & 0.5 & 0 & 0 & 0 & 0.35 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.15 & 0.35 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.15 & 0.35 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.15 & 0.35 & 0 \end{bmatrix},$$

leading to the distribution

$$\boldsymbol{\mu}_0 \boldsymbol{P}^3 = \begin{bmatrix} 0.15 & 0.35 & 0 & 0 & 0.5 & 0 & 0 & 0 \end{bmatrix}.$$

(c) The expected time taken by the passenger can be computed as

$$\mathbb{E}\left[\text{Waiting time}(4)\right] = \mathbb{E}\left[\text{Time}(4 \to A)\right] + \mathbb{E}\left[\text{Time}(A \to 4)\right].$$

The first time (from Stop 4 to Station A is constant:

$$\mathbb{E}\left[\text{Time}(4 \to A)\right] = 10 + 2 + 10 + 2 = 24 \text{ min}.$$

To determine the expected time from Station A to Stop 4, we note the following. With probability 0.15, the train will take the branch with Stop 4 and arrive within 10 minutes. However, with a probability 0.85 it will take one of the other branched and be back in Station A after an expected time of 55 minutes. However, after this roundtrip, we are back at Station A.

Therefore, we have:

$$\mathbb{E}\left[\text{Time}(A \to 4)\right] = 0.15 \times 10 + 0.85 \cdot \big(55 + \mathbb{E}\left[\text{Time}(A \to 4)\right]\big).$$

Solving for $\mathbb{E}\left[\text{Time}(A \to 4)\right]$ yields

$$\mathbb{E}\left[\text{Time}(A \to 4)\right] = 322 \text{ min.}$$

Finally, we get that

$$\mathbb{E}\left[\text{Waiting time}(4)\right] = 24 + 322 = 346 \text{ min.}.$$

The expected waiting time is, therefore, 5 hours and 46 minutes!