Network Elements
Part 02

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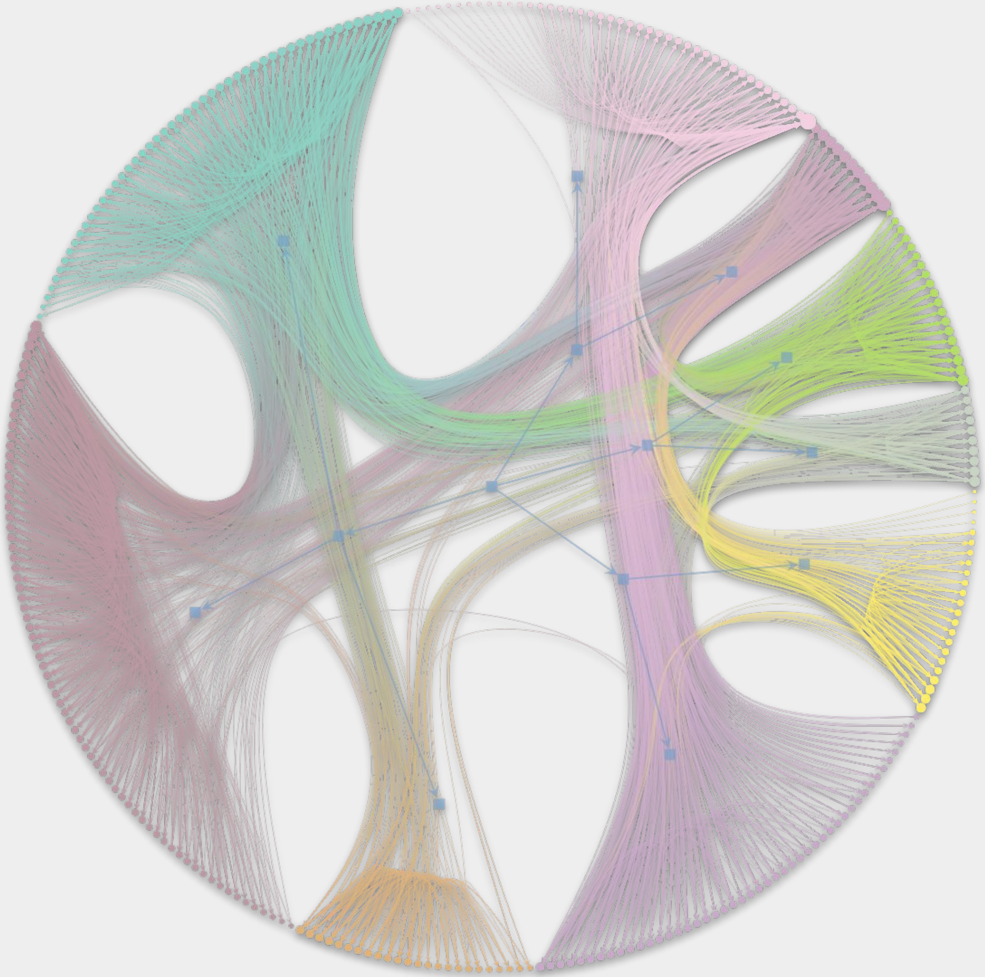
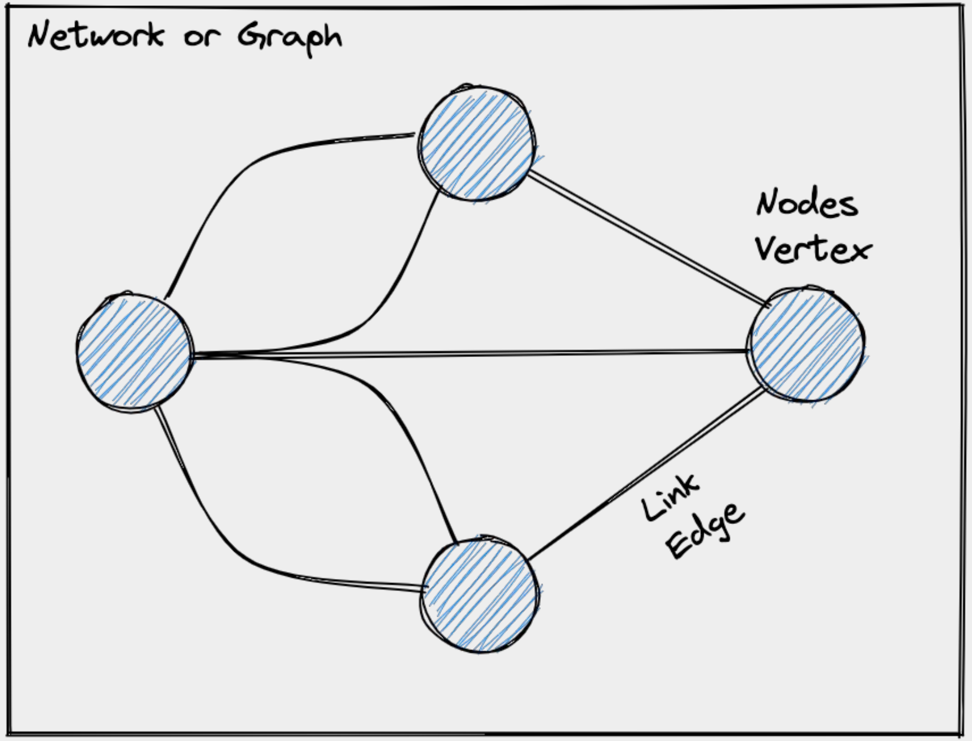


Basic Definition



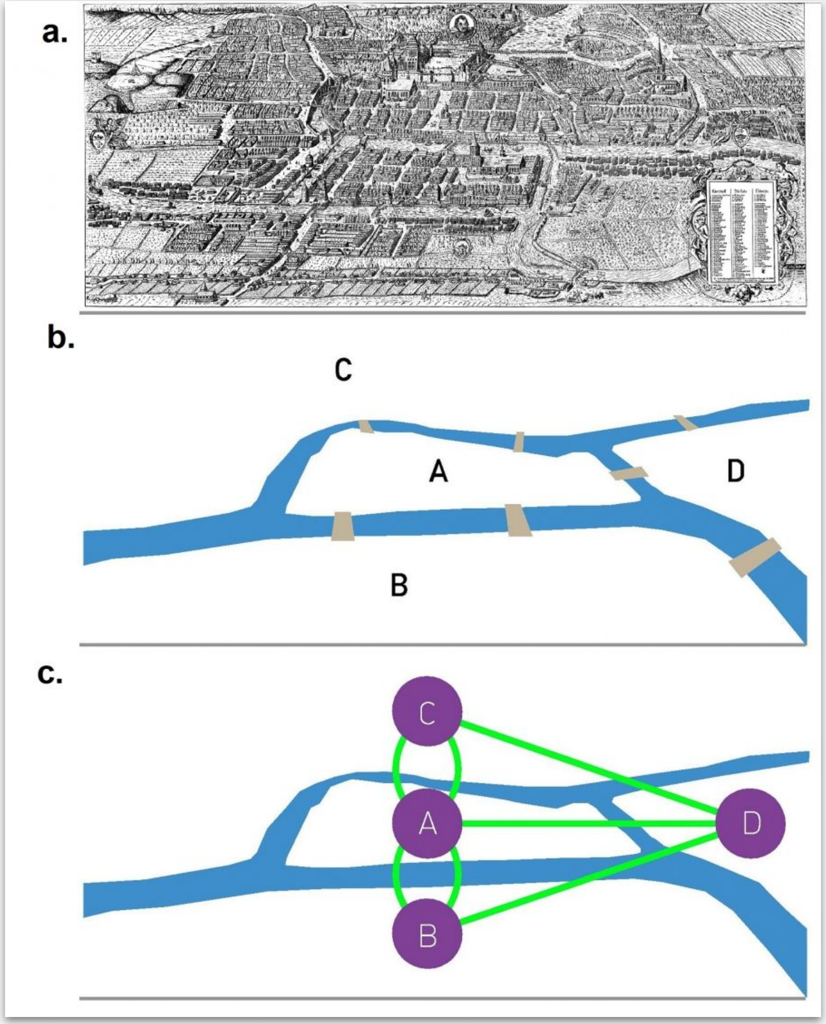
In very general terms a network, or graph, is a set of elements, which we call **nodes**, along with a set of connections between pairs of nodes, which we call **links**. The links represent the presence of a relation among the elements represented by the nodes.


Leonhard Euler, 1735

Seven Bridges of Königsberg

Can one walk across all seven bridges and never cross the same one twice?

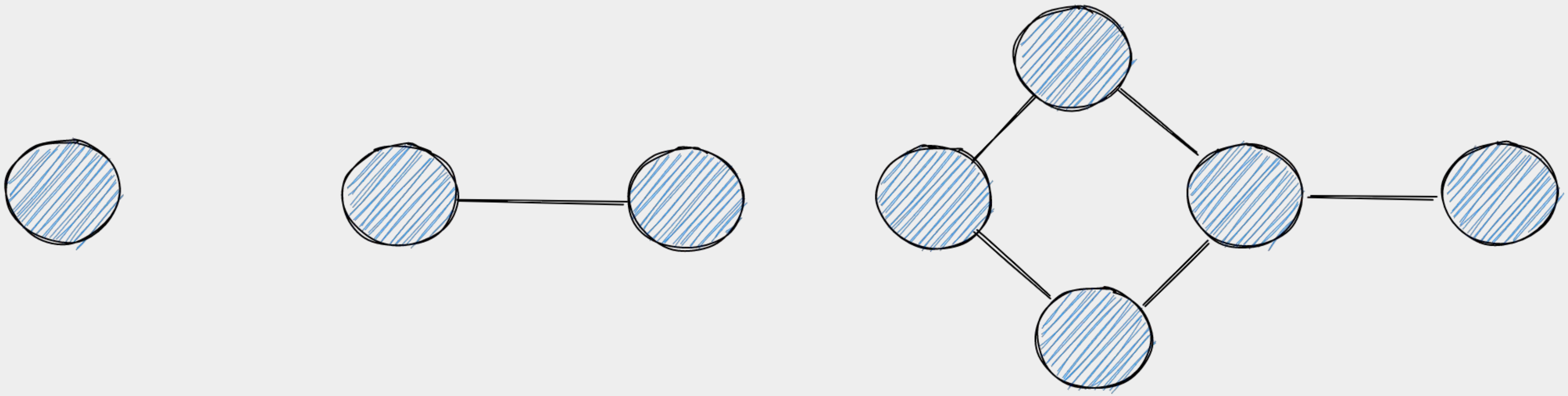
The rigorous language for the description of networks is found in graph theory, a field of mathematics that can be traced back to the pioneering work of Leonhard Euler in the eighteenth century.



Despite many attempts, no one could find such path. The problem remained unsolved until 1735, when Leonhard Euler, offered a rigorous mathematical proof that such path does not exist.

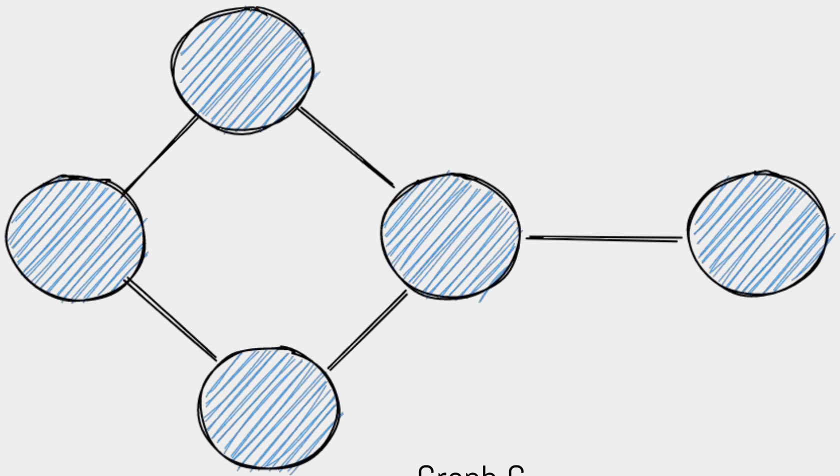
A more rigorous definition of a network

$$G = (V, E), with \;\; E \subseteq V \times V$$

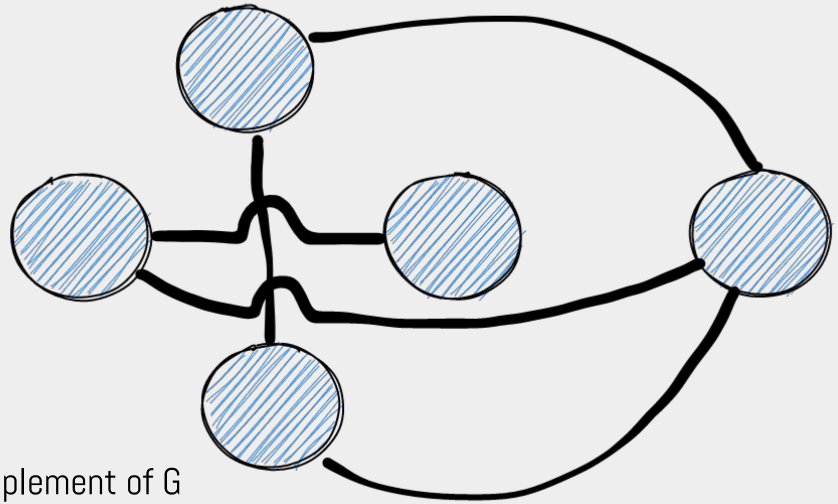


We can derive a series of special graphs

For instance, we can derive the complement of G. This is equivalent to remove all of the original edges of G, and then connect all the unconnected pairs of nodes in G.



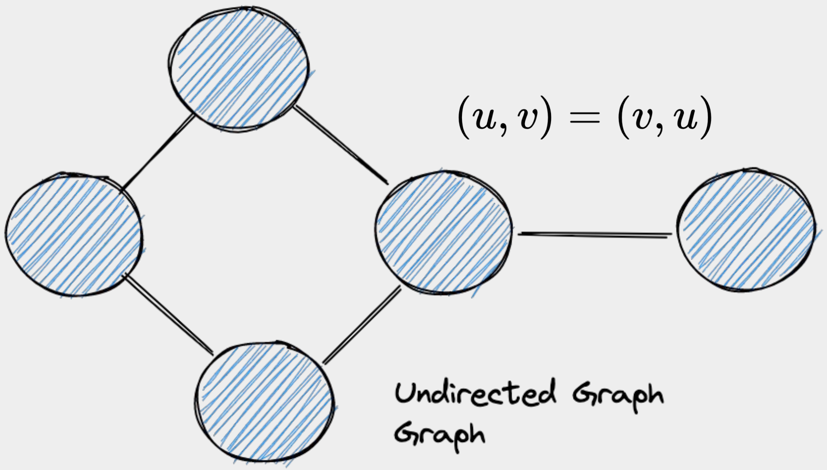
Graph G

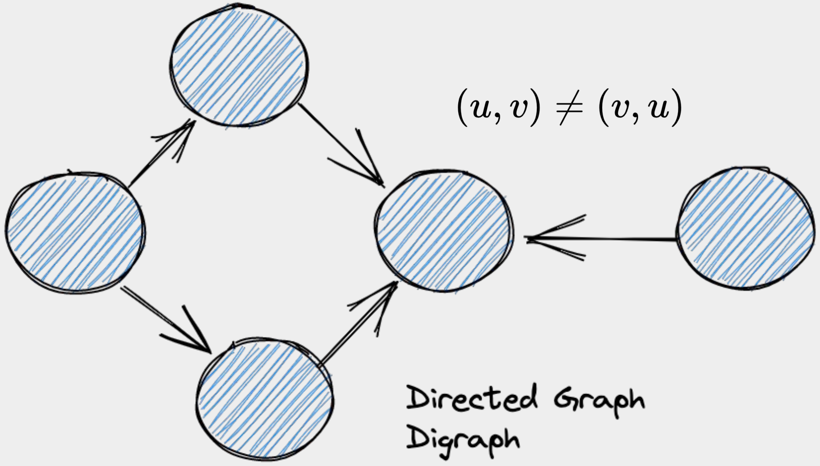


Complement of G

Sometimes you really need to complicate stuff

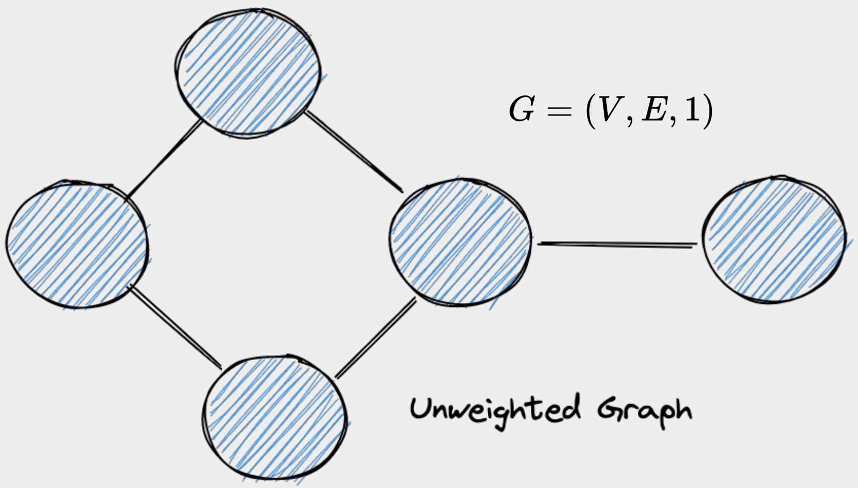
A first thing we will do is realizing that **not all relations are reciprocal**. We can introduce this asymmetry in the graph model.

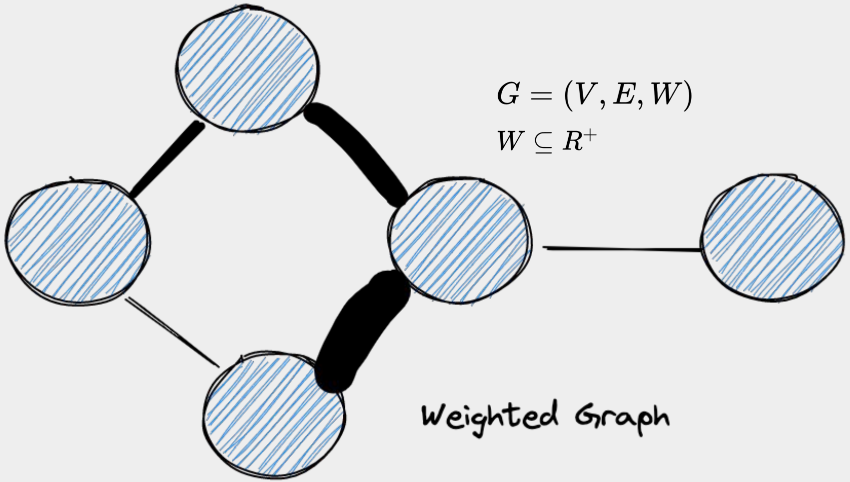


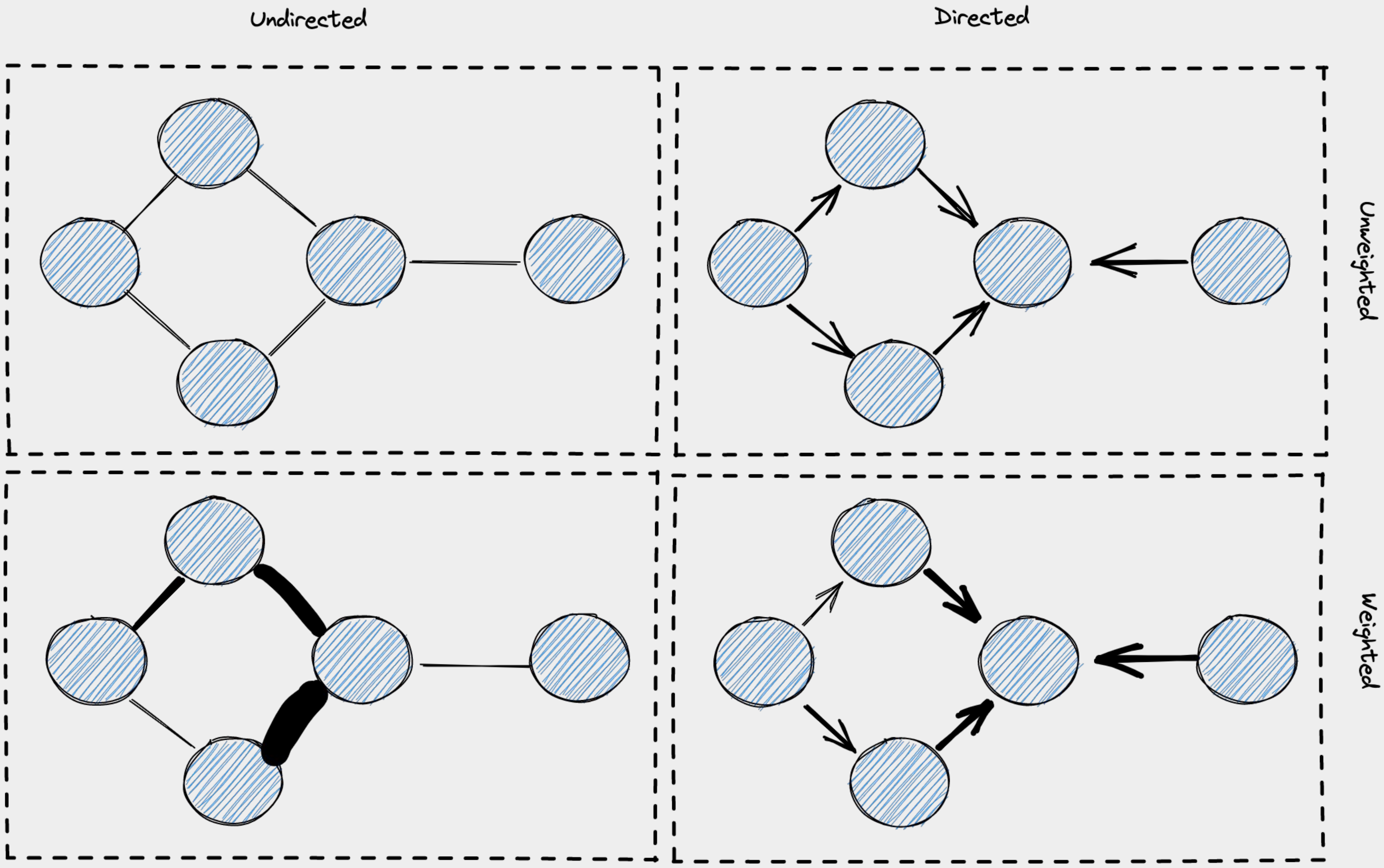


Sometimes you really need to complicate stuff

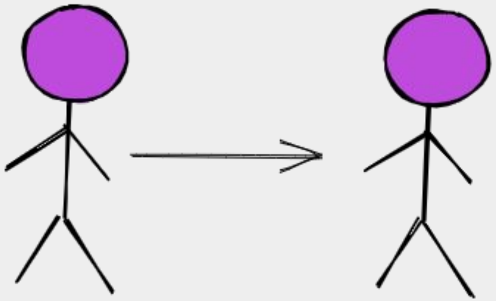
Another way to make edges more interesting is realizing that **two connections are not necessarily equally important** in the network.



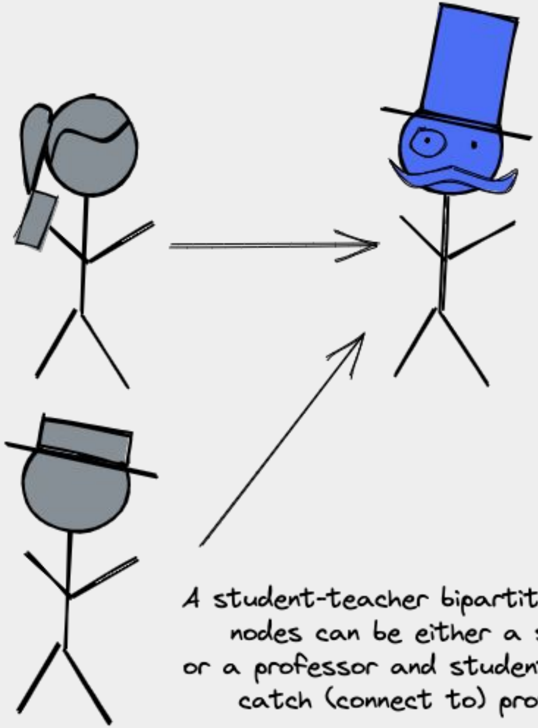



Undirected
Directed
Unweighted
Weighted

Extended Graphs (bipartite network)

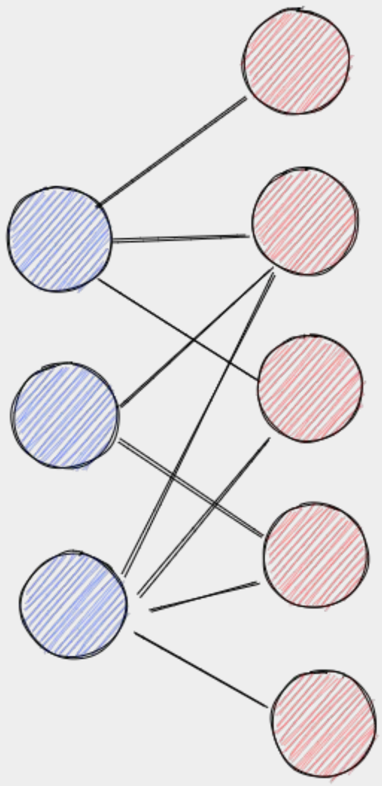


A simple graph representing
a social network with
no additional constraints.



A student-teacher bipartite network:
nodes can be either a student
or a professor and students can only
catch (connect to) professor.

Extended Graphs (bipartite network)



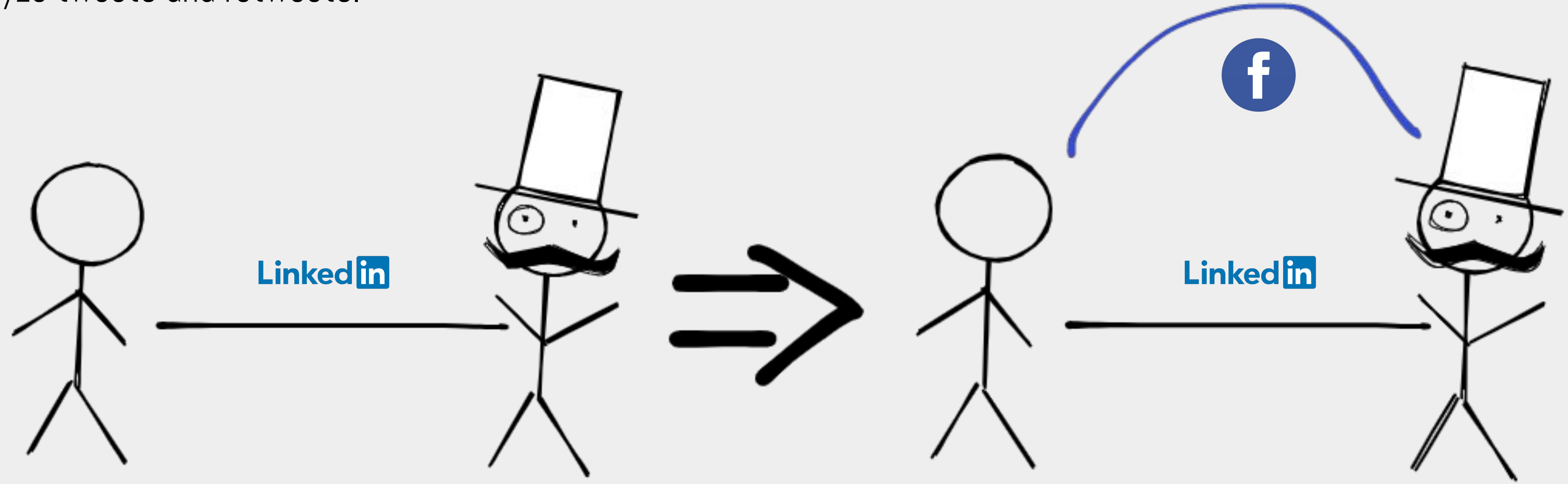
Bipartite networks are used for countless things, connecting:

1. countries to the products they export
2. hosts to guest in symbiotic relationships
3. users to the social media items they tag
4. bank-firm relationships in financial networks
5. players-bands in jazz
6. listener-band in music consumption
7. plant-pollinators in ecosystems

Extended Graphs (multilayer graph)

Two people might have started working in the same company and thus first connected on Linkedin, and then became friends and connected on Facebook. Such scenario could not be captured by simply looking at one of the two networks.

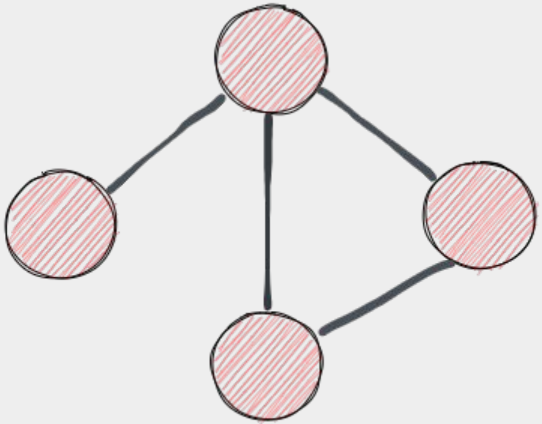
Traditionally, network scientists try to focus on one thing at a time. For instance, they will download a sample of the Linkedin graph. Or they will analyze tweets and retweets.



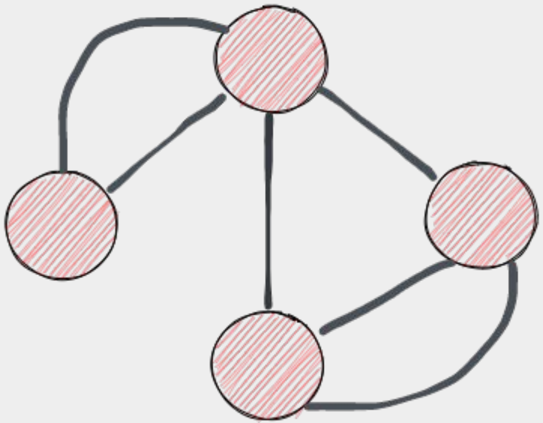
Extended Graphs (multilayer graph)

$$G = (V, E, L)$$

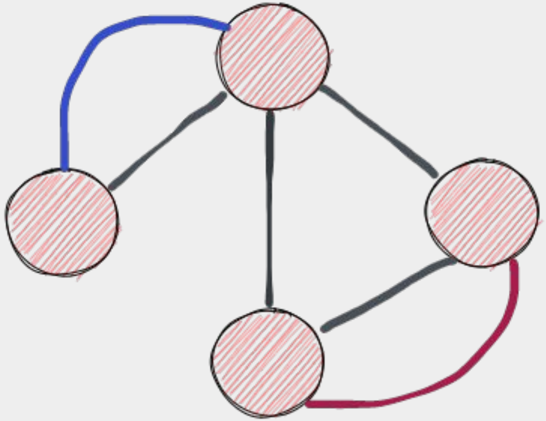
$$(u, v, l) \in E,\ with\, u, v \in V and\, l \in L$$



A simple graph



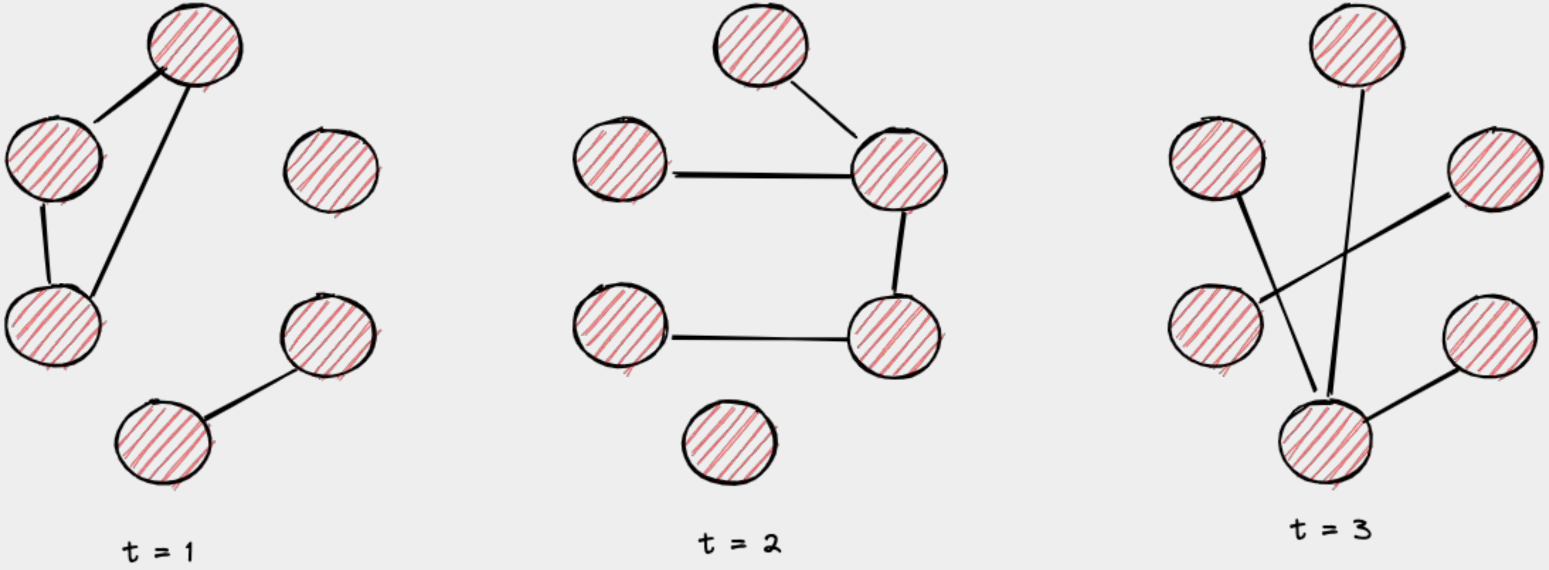
A multigraph, with multiple edges between the same node pairs



A multilayer network, where each edge has a type (represented by its color)

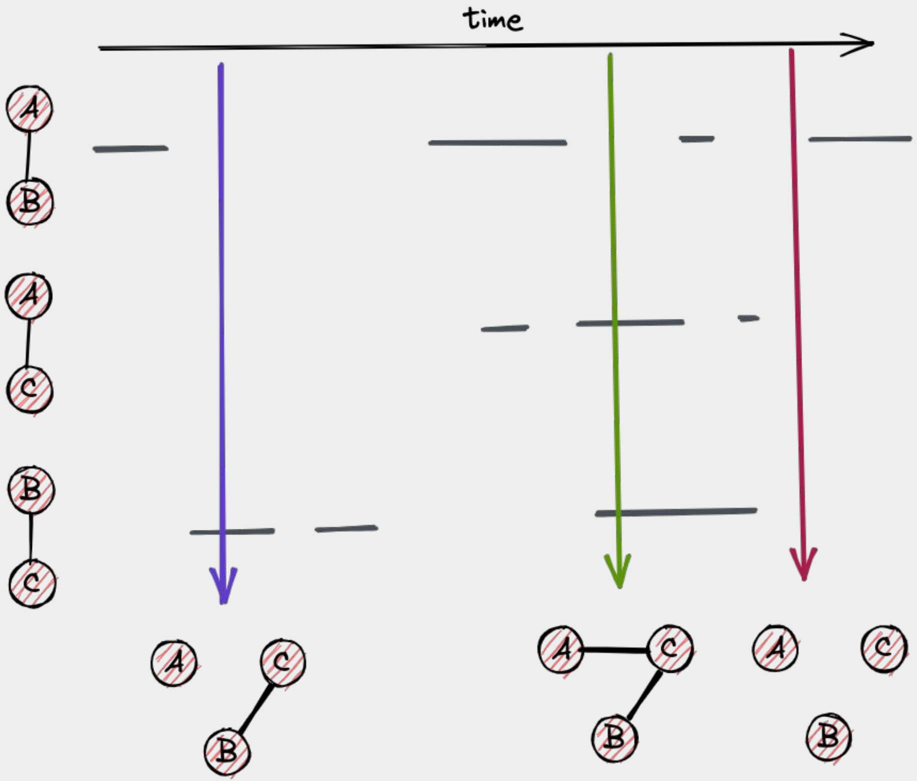
Extended Graphs (dynamic graph)

Imagine that your network represents a road graph. Nodes are intersections, and edges are stretches of the street connecting them. Roadworks might cut off a segment for a few days. If your network model cannot take this into account, you would end up telling drivers to use a road that is blocked, creating traffic jams and a lot of discomfort

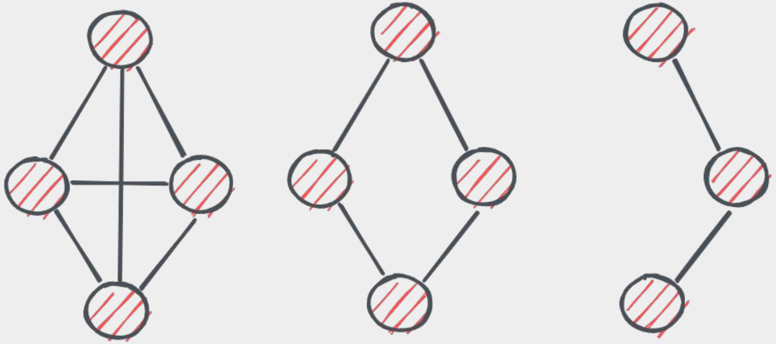


Extended Graphs (dynamic graph)

$$G = (G_1, G_2, \ldots, G_n)$$
$$G_i = (V_i, E_i)$$

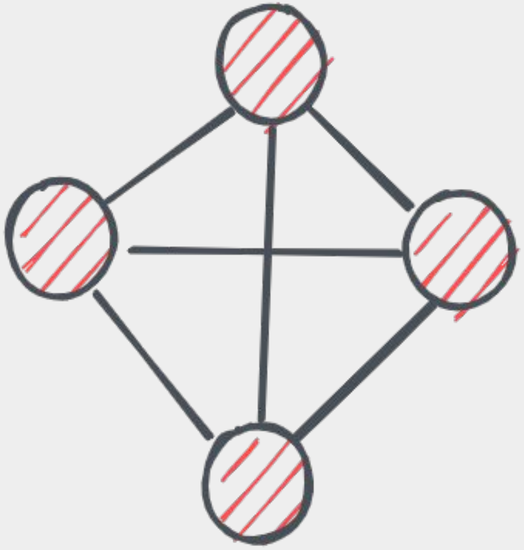


Density and Sparsity, Subnetworks



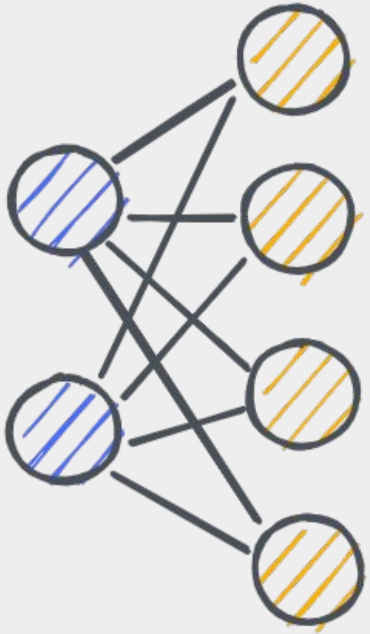
Density and Sparsity

The maximum number of links in a network is bounded by the possible number of distinct connections among the nodes of the system. The maximum number of links is therefore given by the number of pairs of nodes. A network with the maximum number of links, in which all possible pairs of nodes are connected by links, is called a **complete network.**



$$L_{max} = \binom{N}{2} = \frac{N(N-1)}{2}$$

$$L_{max}^{directed} = N(N-1)$$



$$L_{max} = N_1 \times N_2$$

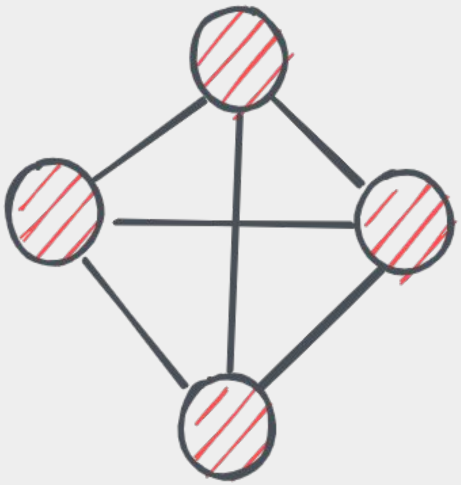
Density and Sparsity

$$d = \frac{L}{L_{max}} = \frac{2L}{N(N-1)}$$

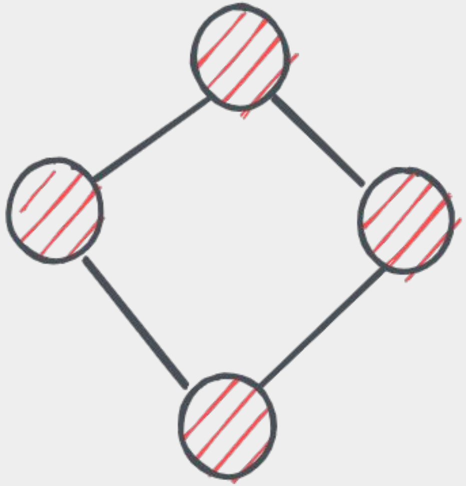
$$d = \frac{L}{L_{max}^{directed}} = \frac{L}{N(N-1)}$$

The fraction of possible links that actually exist, which is the same as the fraction of pairs of nodes that are actually connected, is called the **density of the network.**

$$d = \frac{L}{L_{max}}$$



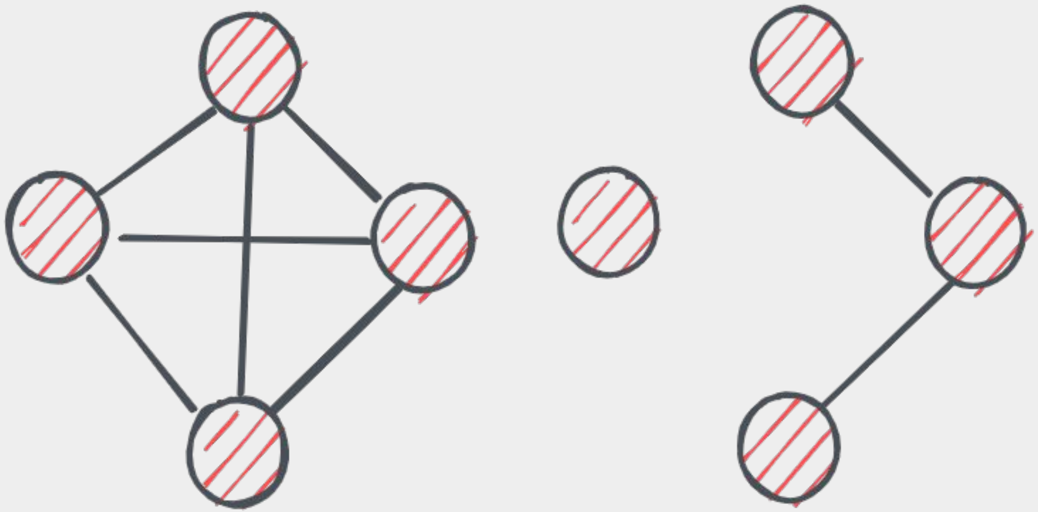
$d = 1$



$d < 1$

Density and Sparsity

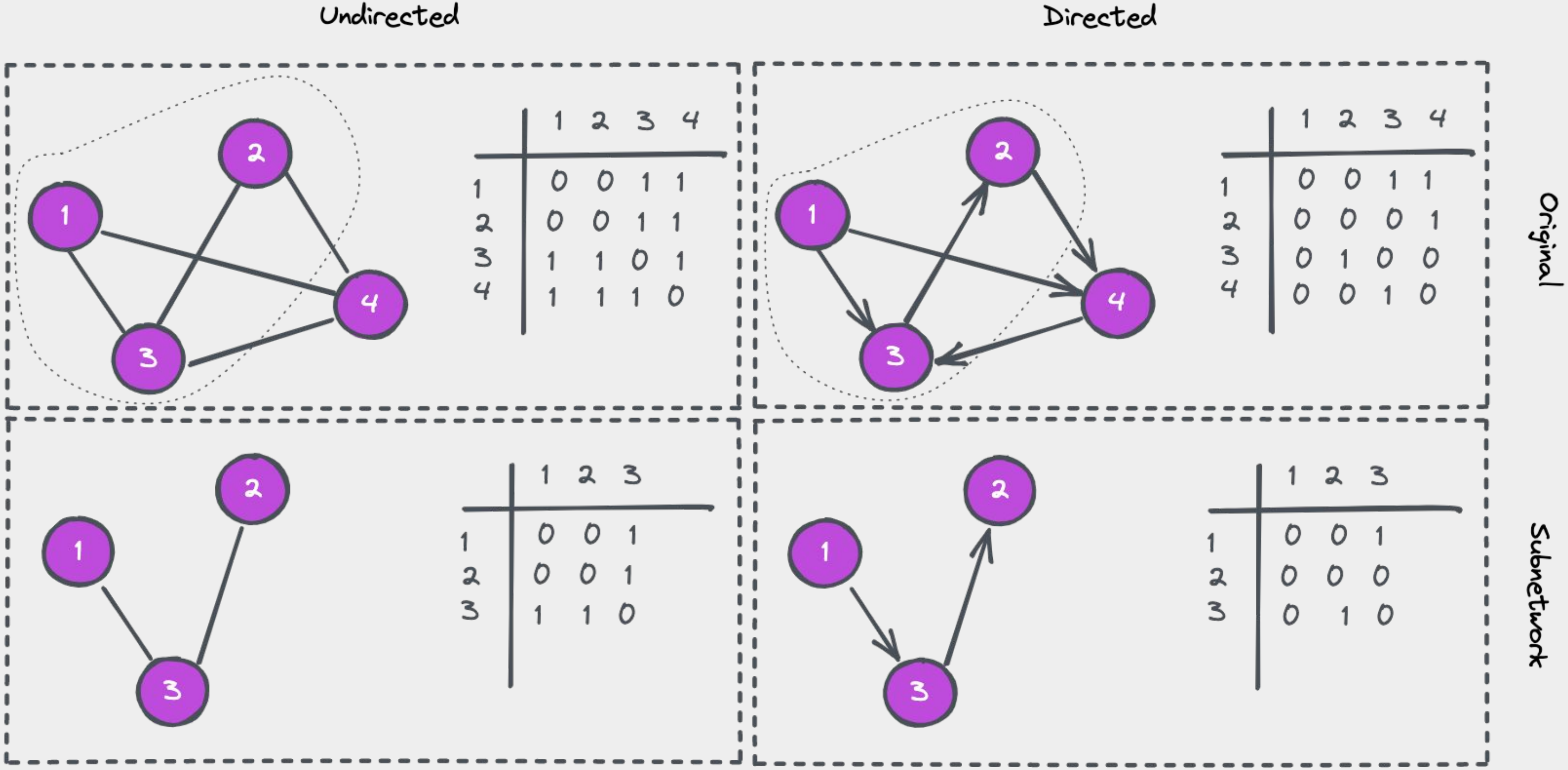
We say that the **network is sparse** if the number of links grows proportionally to the number of nodes (L ~ N), or even slower. If instead the number of links grows faster, e.g. quadratically with network size (L ~ N^2), then we say that the **network is dense.**



$$L \ll L_{max} \rightarrow d \ll 1$$

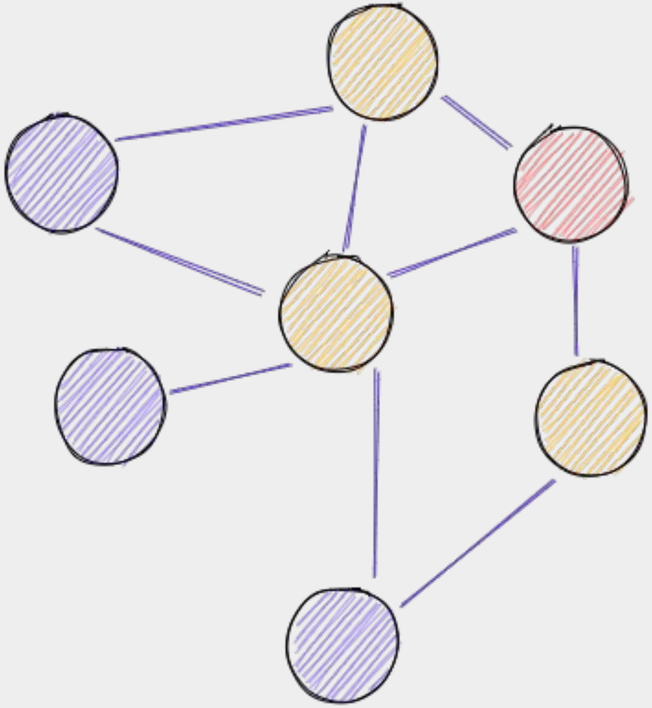
Network	Type	Nodes (N)	Links (L)	Density (d)	Average degree ($\langle k \rangle$)
Facebook Northwestern Univ.		10,567	488,337	0.009	92.4
IMDB movies and stars		563,443	921,160	0.000006	3.3
IMDB co-stars	W	252,999	1,015,187	0.00003	8.0
Twitter US politics	DW	18,470	48,365	0.0001	2.6
Enron email	DW	87,273	321,918	0.00004	3.7
Wikipedia math	D	15,220	194,103	0.0008	12.8
Internet routers		190,914	607,610	0.00003	6.4
US air transportation		546	2,781	0.02	10.2
World air transportation		3,179	18,617	0.004	11.7
Yeast protein interactions		1,870	2,277	0.001	2.4
C. elegans brain	DW	297	2,345	0.03	7.9
Everglades ecological food web	DW	69	916	0.2	13.3

Subnetwork

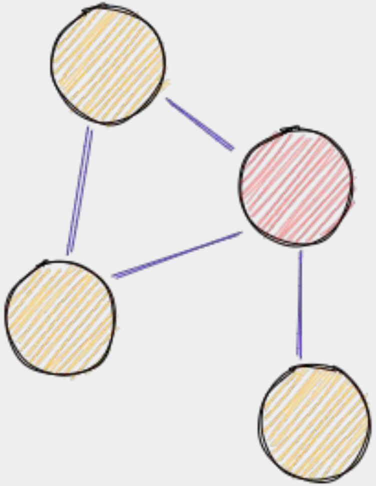

Undirected
Directed
Original
Subnetwork
1 2 3 4
1 0 0 1 1
2 0 0 1 1
3 1 1 0 1
4 1 1 1 0
1 2 3 4
1 0 0 1 1
2 0 0 0 1
3 0 1 0 0
4 0 0 1 0
1 2 3
1 0 0 1
2 0 0 1
3 1 1 0
1 2 3
1 0 0 1
2 0 0 0
3 0 1 0

Subnetwork

A special type of subnetwork is the ego network of a node, which is the subnetwork consisting of the chosen node — called the ego — and its neighbors. Ego networks are often studied in social network analysis.

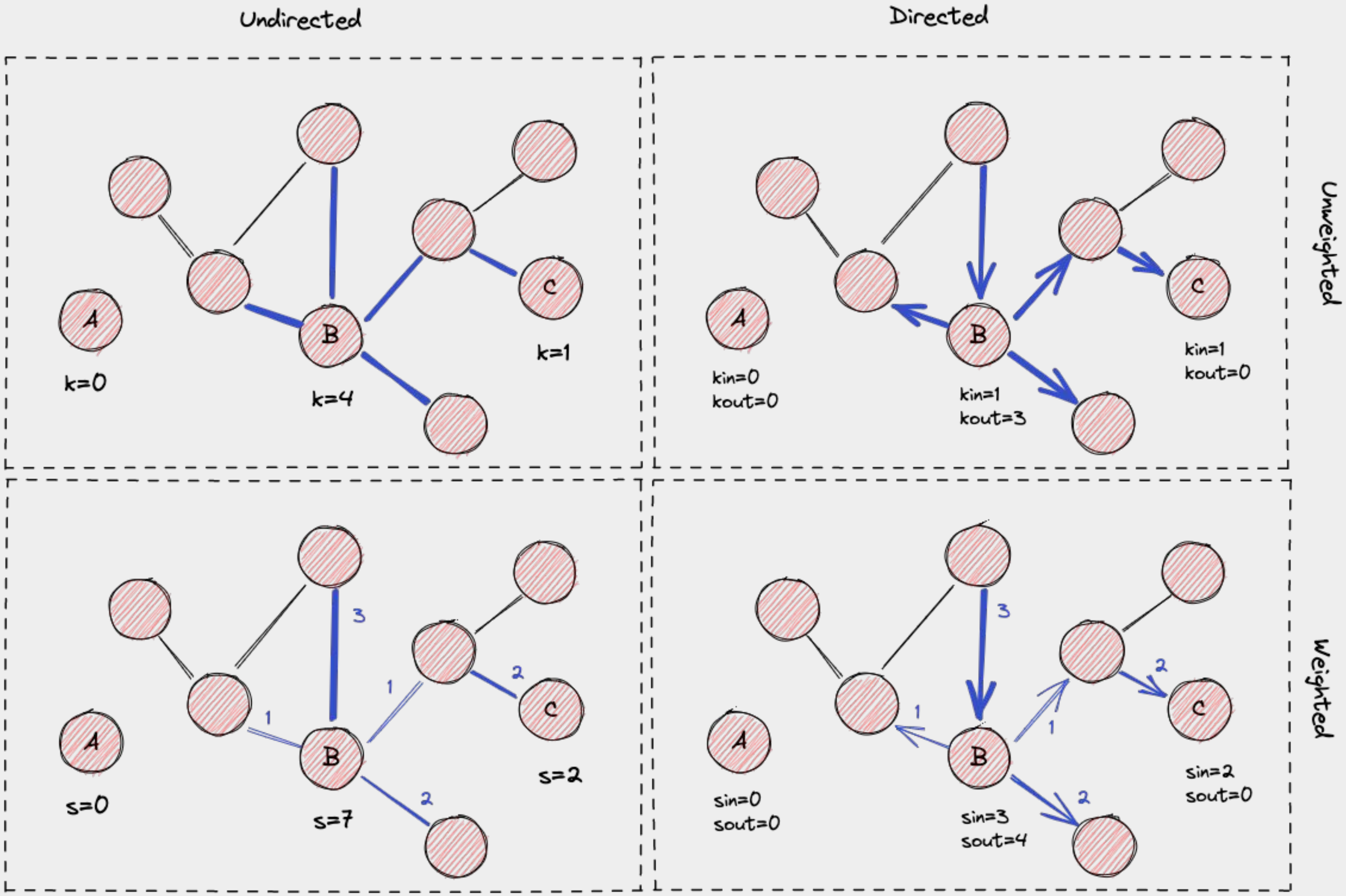


Original Network

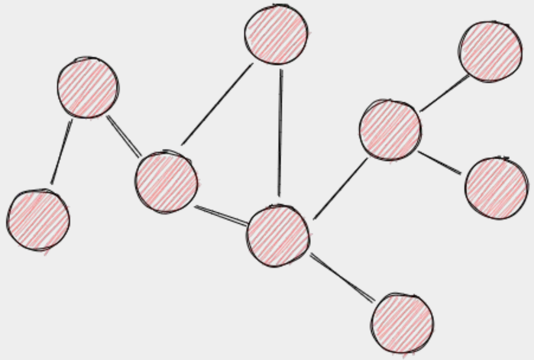


Ego Network

Degree and Network Representation


Undirected
Directed
Unweighted
Weighted
A
B
C
k=0
k=4
k=1
kin=0
kout=0
kin=1
kout=3
kin=1
kout=0
3
1
1
2
2
s=0
s=7
s=2
3
1
1
2
2
sin=0
sout=0
sin=3
sout=4
sin=2
sout=0

The average degree of a network is defined as:



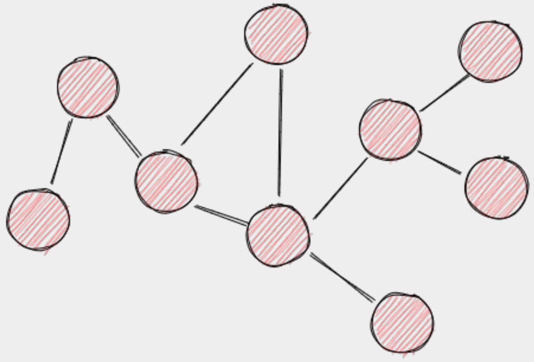
$$\langle \boldsymbol{k} \rangle = \frac{\sum_i \boldsymbol{k}_i}{N}$$

Since each link contributes to the degree of two nodes in an undirected network:

$$\langle k \rangle = \frac{2L}{N} = d(N-1)$$

$$d = \frac{2L}{N(N-1)}$$

The average degree of a network is defined as:

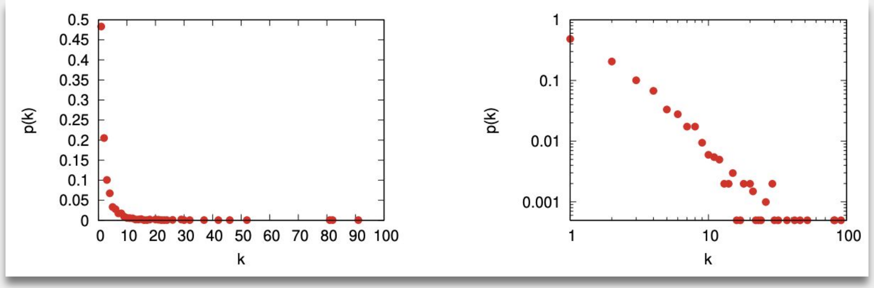


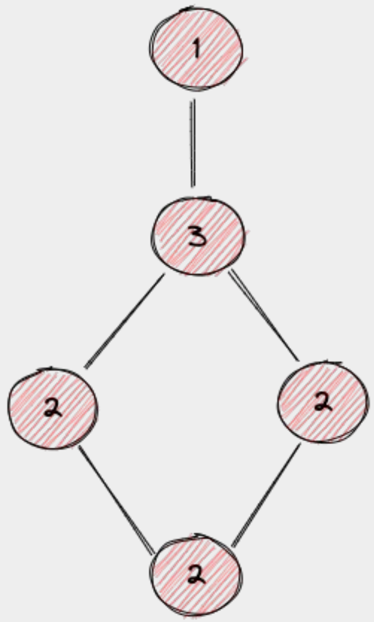
$$\langle k \rangle = \frac{2L}{N} = d(N-1)$$

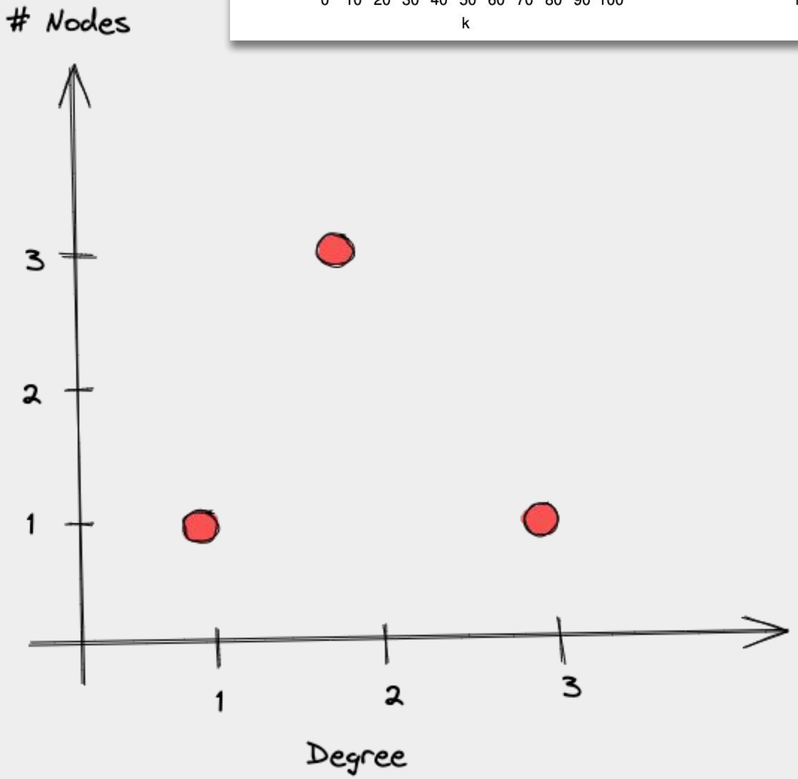
$$d = \frac{\langle k \rangle}{N-1}$$

This makes sense because the **maximum possible degree** of a node is k_{max} = *N - 1*, obtained when the node is connected to every other node. Intuitively, the **density is the ratio between the average and maximum degree**.

Degree Distribution

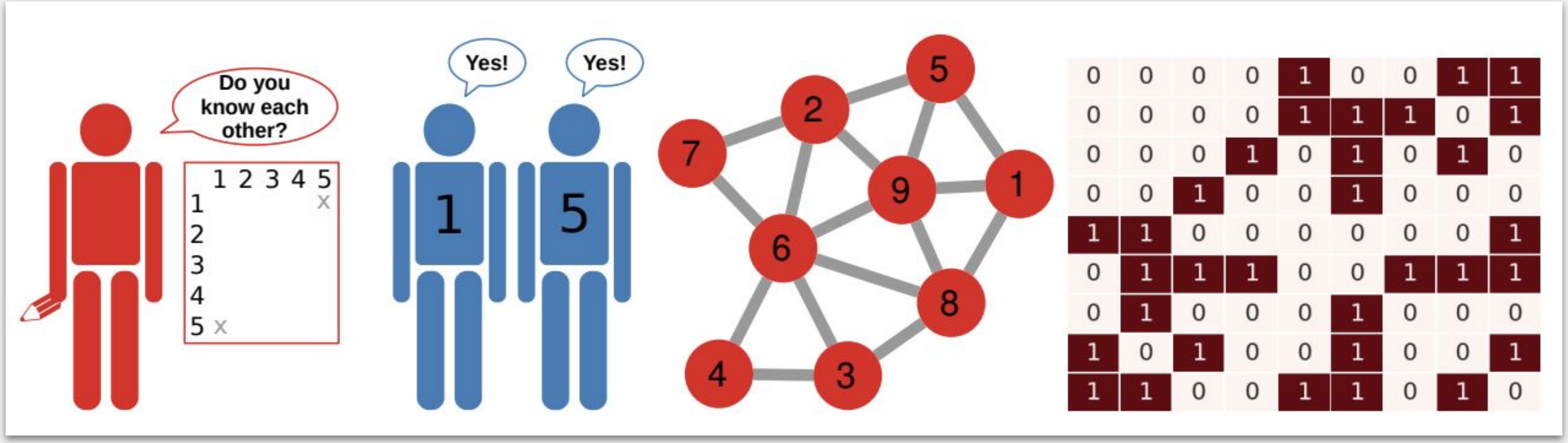






Network Representation

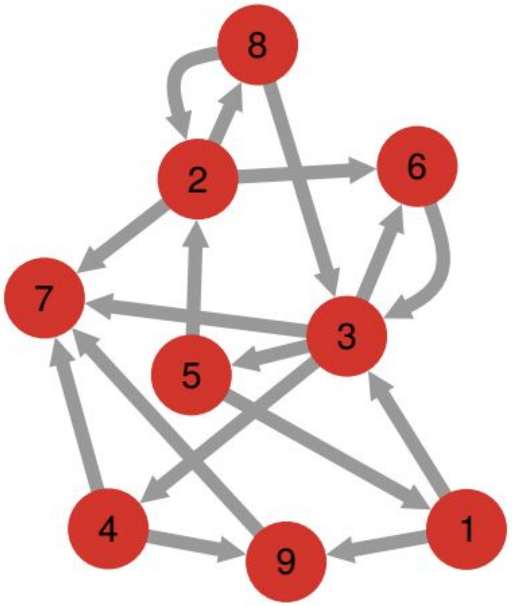
The **adjacency matrix** is the basic representation of a graph as a **matrix**. Each row/column corresponds to a node. Each cell represents an edge, set to one if the edge exists, and zero otherwise.



Network Representation

A non-symmetric adjacency matrix corresponding to a directed graph

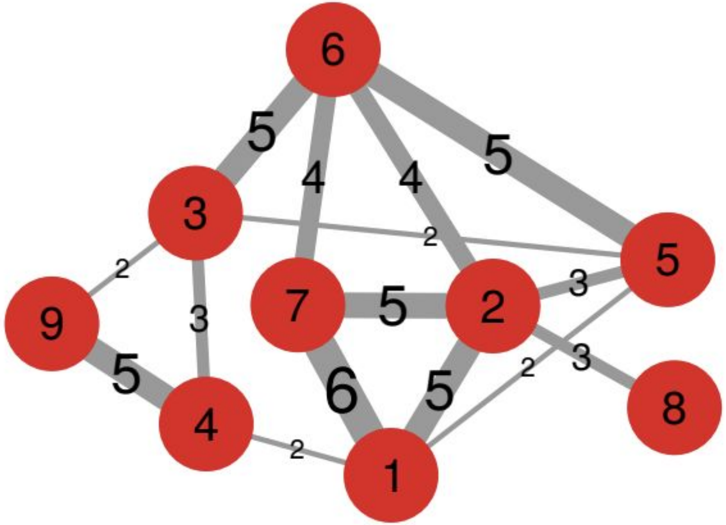
0	0	1	0	0	0	0	0	1
0	0	0	0	0	1	1	1	0
0	0	0	1	1	1	1	0	0
0	0	0	0	0	0	1	0	1
1	1	0	0	0	0	0	0	1
0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	1	1	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0



Network Representation

A non-binary adjacency matrix corresponding a weighted graph

0	5	0	2	2	0	6	0	0
5	0	0	0	3	4	5	3	0
0	0	0	3	2	5	0	0	2
2	0	3	0	0	0	0	0	5
2	3	2	0	0	5	0	0	0
0	4	5	0	5	0	4	0	0
6	5	0	0	0	4	0	0	0
0	3	0	0	0	0	0	0	0
0	0	2	5	0	0	0	0	0



Network Representation

$$\begin{array}{c|cc} & |V_1| & |V_2| \\ \hline |V_1| & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$}} \\ |V_2| & & \end{array}$$

A non-square adjacency matrix corresponding a bipartite graph

0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	1	1	1
0	1	0	0	1	1	1	1	0	0
1	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0	0	0
0	0	0	0	0	1	0	0	1	1