Econ 202A Macroeconomics: Section 1

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October 23, 25, 2024

Syllabus

Section Overview

- (1) Numerical solutions (2) using a finite difference method (3) for continuous time (4) heterogeneous agent models

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- (1) Numerical solutions (2) using a finite difference method (3) for continuous time (4) heterogeneous agent models
- Useful references:
    1. **Benjamin Moll's website**
    2. Online Appendix for Achdou et al. (2022)
    3. LeVeque, 2007, *"Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-dependent Problems."*
    4. Candler, 1999, *"Finite Difference Methods for Continuous Time Dynamic Programming."*

Schedule (Subject to Change)

1. **October 23, 25**: Discrete and Continuous-Time Dynamics & Introduction to Finite Difference Method
2. **October 30, November 1**: Neoclassical Growth Model
3. **November 6, 8**: Huggett Model (Partial Equilibrium)
4. **November 13, 15**: Huggett Model (Kolmogorov Forward Equation and General Equilibrium)
5. **November 20, 22**: Coding: Huggett Model
6. **November 27, 29**: **No Section: Thanksgiving Break**
7. **December 4, 6**: TBD

Communication and Office Hours

- Office hours
  — **Fridays from 11am to 1pm** in Evans 546
  — If this time doesn't work for you, email me to arrange an alternative meeting.
  — Please email your questions in advance (especially if they involve coding—please attach the relevant code) so that I can review them beforehand.
- Email
  — Include [ECON 202A] at the beginning of the subject line.
  — Please allow up to two business days for a response.

Problem Sets

- Submission
  - — Late assignments are not accepted.
  - — For coding problems:
    1. Copy the code into a text file (e.g., .txt, LaTeX, Word, etc.)
    2. Submit a PDF version.

- Programming Language
  - — All in-class demos will use MATLAB.
  - — Some assignments may require you to adapt pre-written MATLAB code, which will not be available in other languages.
  - — Berkeley offers free access to MATLAB. (`https://software.berkeley.edu/matlab`.)
  - — You are welcome to use other languages, but support will only be provided for MATLAB.

Section 1

Overview

1. Dynamics in Discrete and Continuous Time
   - Discrete Time: Difference Equations
   - Continuous Time: Differential Equations
2. Solving Ordinary Differential Equations (ODEs)
   - Analytical Solution: Integrating Factor Method
   - Numerical Solution: **Finite Difference Method**
3. Exercises: Solving and Comparing Solutions
   - Capital Accumulation
   - Consumption Euler Equation (Assignment)

Section 1-1: Dynamics in Discrete and Continuous Time

Dynamic Models

- In macroeconomics, many key questions revolve around how variables evolve over time.

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- Two broad categories of dynamic models:
    1. Discrete Time: $K_{t+1} = (1 - \delta)K_t + I_t$
    2. Continuous Time: $\dot{K}_t = I_t - \delta K_t$

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  1. Discrete Time: $K_{t+1} = (1-\delta)K_t + I_t$
  2. Continuous Time: $\dot{K}_t = I_t - \delta K_t$
- Models can be translated from discrete time to continuous time, and vice versa.

Difference Equations

The **first-order linear difference equation** is defined as:

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where $\{z_t\}$ is an exogenously given, bounded sequence.

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Exercise: Solve the First-Order Difference Equation

Solve the first-order linear autonomous difference equation:

$$x_{t+1} = bx_t + 1$$

Solving a First-Order Difference Equation

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Solve the first-order linear autonomous difference equation, **given an initial value $\mathbf{x_0}$**:

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Solving a First-Order Difference Equation

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Solve the first-order linear autonomous difference equation, **given an initial value $\mathbf{x_0}$**:

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General Solution Steps

1. Solve the homogeneous equation: $x_{t+1} = bx_t \;\Rightarrow\; x_t = Ab^t$
2. Find the stationary point or steady state: $x = \dfrac{c}{1-b}$ when $b \neq 1$
3. Solve for the general solution as the sum of the homogeneous solution and the particular solution: $x_t = Ab^t + x$
4. Determine A using the initial condition: $x_0 = Ab^0 + x \;\Rightarrow\; A = x_0 - x$
5. The general solution for the autonomous equation: $x_t = (x_0 - \dfrac{c}{1-b})b^t + \dfrac{c}{1-b}$

Difference Equations in Macroeconomics

Capital Accumulation Equation:

$$K_{t+1} = (1-\delta)K_t + I_t$$

- This is a first-order linear difference equation.
- It requires one boundary condition.
- This is a **forward equation** since it describes how the system evolves forward in time.
- Therefore, it requires an **initial condition** K_0.
- If $I_t = 0 \; \forall t$ and $0 < \delta < 1$, then $K_t \to 0$.
- If $I_t = c > 0 \; \forall t$, then K_t converges to $\frac{c}{\delta}$.

Difference Equations in Macroeconomics

Consumption Euler Equation:

$$\frac{1}{C_t} = \beta R_t \frac{1}{C_{t+1}}$$

- This is also a first-order linear difference equation.
- It requires one boundary condition.
- Unlike the capital accumulation equation, this is a **backward equation**, determining optimal consumption today based on future consumption.
- Therefore, it requires either a **terminal condition** or a transversality condition.
- In a finite-horizon model, the consumption level at a specific time $t = T$ must be given.
- In an infinite-horizon model, a transversality condition is commonly imposed to ensure that C_t converges to a specific value over time.

Differential Equations

The **first-order linear ordinary differential equation (ODE)** is defined as:

$$\dot{X(t)} = a(t)X(t) + b(t) \tag{2}$$

- If $b(t) = 0$, equation (2) is a homogeneous equation.
- If $a(t)$ and $b(t)$ are non-zero constants, we say equation (2) has constant coefficients.

From Discrete to Continuous Time

Derivation of Continuous-Time Capital Accumulation

Discrete time with unit time steps:

$$K_{t+1} = (1-\delta)K_t + \delta I_t$$

With an arbitrary Δ time step:

$$K_{t+\Delta} = (1-\Delta\delta)K_t + \Delta I_t$$

Rearranging:

$$\frac{K_{t+\Delta} - K_t}{\Delta} = I_t - \delta K_t$$

Taking the limit as $\Delta \to 0$:

$$\lim_{\Delta\to 0}\frac{K_{t+\Delta} - K_t}{\Delta} = \lim_{\Delta\to 0}(I_t - \delta K_t)$$

$$\therefore \dot{K}_t = I_t - \delta K_t$$

From Discrete to Continuous Time

Exercise: Discrete to continuous-Time Transformation

Transform the discrete-time consumption Euler equation into continuous-time counterpart:

$$u'(C_t) = \beta R_t u'(C_{t+1}), \quad \text{where} \quad u(C_t) = \frac{1}{C_t}$$

Denote:

$$\beta = 1 - \rho, \quad R_t = 1 + r_t$$

From Discrete to Continuous Time

Derivation of Continuous-Time Consumption Euler Equation

With an arbitrary Δ time step:

$$u'(C_t) = (1 - \Delta\rho)(1 + \Delta r_t)u'(C_{t+\Delta})$$

First approximation of $u'(C_{t+\Delta})$ around C_t:

$$u'(C_{t+\Delta}) = u'(C_t + (C_{t+\Delta} - C_t)) \simeq u'(C_t) + u''(C_t)(C_{t+\Delta} - C_t)$$

Plugging in and rearranging:

$$u'(C_t) = (1 - \Delta\rho)(1 + \Delta r_t)\left[u'(C_t) + u''(C_t)(C_{t+\Delta} - C_t)\right]$$

$$1 = (1 - \Delta\rho)(1 + \Delta r_t)\left[1 + \frac{u''(C_t)}{u'(C_t)}(C_{t+\Delta} - C_t)\right]$$

From Discrete to Continuous Time

Substitute:

$$\frac{u''(C_t)}{u'(C_t)} = -\frac{1}{C_t}$$

Rearranging:

$$1 = (1 - \Delta\rho + \Delta r_t - \Delta^2 \rho r_t)\left[1 - \frac{C_{t+\Delta} - C_t}{C_t}\right]$$

Dividing by Δ and taking the limit as $\Delta \to 0$:

$$\lim_{\Delta \to 0}\left[(1 - \Delta\rho + \Delta r_t - \Delta^2 \rho r_t)\frac{1}{C_t}\frac{C_{t+\Delta} - C_t}{\Delta}\right] = \lim_{\Delta \to 0}\left(\frac{-\Delta\rho + \Delta r_t - \Delta^2 \rho r_t}{\Delta}\right)$$

$$\therefore \frac{\dot{C}_t}{C_t} = r_t - \rho$$

Differential Equations in Macroeconomics

Capital Accumulation Equation:

$$\dot{K}_t = I_t - \delta K_t$$

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Differential Equations in Macroeconomics

Consumption Euler Equation:

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- Unlike the capital accumulation equation, this is a backward equation, determining optimal consumption today based on future consumption.
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- In a finite-horizon model, the consumption level at a specific time $t = T$ must be given.
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Section 1-2: Solving Ordinary Differential Equations (ODE)

Solving Ordinary Differential Equations (ODEs)

- Two approaches to solving ODEs:
  1. Analytical: Integrating Factor Method
  2. Numerical: Finite Difference Method

Solving Ordinary Differential Equations (ODEs)

- Two approaches to solving ODEs:
    1. Analytical: Integrating Factor Method
    2. Numerical: Finite Difference Method
- Continuous-time heterogeneous agent models are particularly analytically intractable, so they are typically solved numerically using the finite difference method.

Analytical Solution: Integrating Factor Method

1. **Write the ODE in Standard Form**

   The first step is to ensure the ODE is in its standard form:

   $$\dot{y}(t) + p(t)y(t) = q(t),$$

   which represents a first-order linear ODE. If the equation is not in this form, manipulate it so that it matches this structure.

2. **Find the Integrating Factor**

   The key idea is to find a function, known as the integrating factor, which simplifies the ODE. The integrating factor, denoted by $\mu(t)$, is chosen to make the left-hand side easier to integrate. It is defined as:

   $$\mu(t) = e^{\int p(t)\,dt}.$$

   This function will be used to multiply through the entire ODE.

Analytical Solution: Integrating Factor Method

3. **Multiply the ODE by the Integrating Factor**
   Multiply the entire ODE by $\mu(t)$:
   $$\mu(t)\dot{y}(t) + \mu(t)p(t)y(t) = \mu(t)q(t).$$
   Since $\mu(t) = e^{\int p(t)\,dt}$, this transforms the left-hand side into the derivative of a product:
   $$\frac{d}{dt}\left(\mu(t)y(t)\right) = \mu(t)q(t).$$
   This step is crucial, as it significantly simplifies the equation.

4. **Integrate Both Sides**
   Integrate both sides of the equation with respect to t:
   $$\int \frac{d}{dt}\left(\mu(t)y(t)\right)\,dt = \int \mu(t)q(t)\,dt.$$

Analytical Solution: Integrating Factor Method

This yields:

$$\mu(t)y(t) = \int \mu(t)q(t)\,dt + C,$$

where C is the constant of integration.

5. **Solve for $y(t)$**

   Finally, solve for $y(t)$ by dividing both sides by $\mu(t)$:

$$y(t) = \frac{1}{\mu(t)}\left(\int \mu(t)q(t)\,dt + C\right).$$

6. **Determine the Constant of Integration**

   Solve for C using given boundary conditions.

   This provides the general solution to the first-order linear ODE.

Solving Capital Accumulation

Exercise: Analytically Solve the Continuous-Time Capital Accumulation Equation

Solve the capital accumulation equation:

$$\dot{K}_t = I_t - \delta K_t$$

where I_t is the exogenously given investment and δ is the depreciation rate, using the initial condition K_0.

Solving Capital Accumulation

Analytical Solution of Continuous-Time Capital Accumultion

Step 1: Write the ODE in Standard Form

The first step is to rewrite the equation in its standard form:

$$\dot{K}_t + \delta K_t = I_t$$

which corresponds to the form $\dot{y}(t) + p(t)y(t) = q(t)$, where $p(t) = \delta$ and $q(t) = I_t$.

Step 2: Find the Integrating Factor

The integrating factor $\mu(t)$ is defined as:

$$\mu(t) = e^{\int \delta \, dt} = e^{\delta t}$$

Solving Capital Accumulation

Step 3: Multiply the ODE by the Integrating Factor

Multiply both sides of the equation by $\mu(t) = e^{\delta t}$:

$$e^{\delta t}\dot{K}_t + e^{\delta t}\delta K_t = e^{\delta t} I_t$$

The left-hand side simplifies to the derivative of a product:

$$\frac{d}{dt}\left(e^{\delta t} K_t\right) = e^{\delta t} I_t$$

Step 4: Integrate Both Sides

Integrate both sides with respect to t:

$$\int \frac{d}{dt}\left(e^{\delta t} K_t\right) dt = \int e^{\delta t} I_t \, dt$$

This gives:

$$e^{\delta t}K_t = \int e^{\delta t}I_t\,dt + C$$

where C is the constant of integration.

Step 5: Solve for K_t

Solve for K_t by dividing both sides by $e^{\delta t}$:

$$K_t = e^{-\delta t}\left(\int_0^t e^{\delta s}I_s\,ds + C\right)$$

Step 6: Determine the Constant of Integration

Use the initial condition K_0 to find C:

$$K_0 = e^{-\delta\cdot 0}\left(\int_0^0 e^{\delta s}I_s\,ds + C\right) = C$$

Solving Capital Accumulation

Analytical Solution

The analytical solution to the capital accumulation equation is:

$$K_t = e^{-\delta t} \left(\int_0^t e^{\delta s} I_s \, ds + K_0 \right)$$

If I_t is constant at I

The solution to the capital accumulation equation is:

$$K_t = \frac{I}{\delta}(1 - e^{-\delta t}) + e^{-\delta t} K_0$$

Numerical Solution: Finite Difference Method

- The finite difference method **approximates derivatives using finite differences**—calculating changes between function values at discrete time steps.

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- The key idea involves:
  1. Discretizing the domain (i.e., time) into a finite number of intervals.
  2. Approximating the first-order derivative by transitioning from continuous to discrete time, as shown below:
  $$\dot{X}_t = \frac{dX_t}{dt} \approx \frac{X_{t+\Delta} - X_t}{\Delta}$$
  where Δ is the small time step.

Numerical Solution: Finite Difference Method

- The finite difference method **approximates derivatives using finite differences**—calculating changes between function values at discrete time steps.

- The key idea involves:
  1. Discretizing the domain (i.e., time) into a finite number of intervals.
  2. Approximating the first-order derivative by transitioning from continuous to discrete time, as shown below:

     $$\dot{X}_t = \frac{dX_t}{dt} \approx \frac{X_{t+\Delta} - X_t}{\Delta}$$

     where Δ is the small time step.

- Finite difference methods transform differential equations, which may be nonlinear, into systems of linear equations that can be solved using matrix algebra techniques.

Finite Difference Approximation

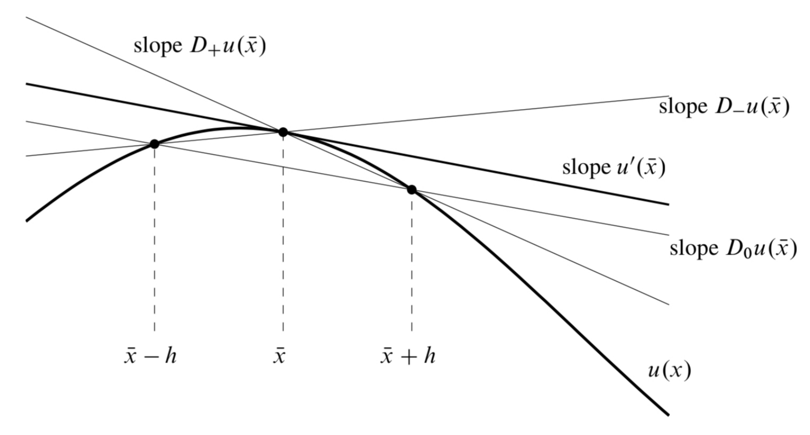

Figure 1: Various approximations to $u'(\overline{x})$ interpreted as the slope of secant lines.

Finite Difference Approximation

- One-sided approximations:
  - Forward difference:
$$D_+ u(\overline{x}) \equiv \frac{u(\overline{x}+h) - u(\overline{x})}{h}$$
  - Backward difference:
$$D_- u(\overline{x}) \equiv \frac{u(\overline{x}) - u(\overline{x}-h)}{h}$$
  - Both are first-order accurate approximations.
- Centered approximation (central difference):
$$D_0 u(\overline{x}) \equiv \frac{u(\overline{x}+h) - u(\overline{x}-h)}{2h} = \frac{1}{2}\left[D_+ u(\overline{x}) + D_- u(\overline{x})\right]$$
  - This is a second-order accurate approximation.

Section 1-3: Exercises: Solving and Comparing Solutions

Solving the Continuous-Time Capital Accumulation Equation

Exercise: Numerically Solve the Continuous-Time Capital Accumulation Equation

Solve the capital accumulation equation:

$$\dot{K}_t = I_t - \delta K_t$$

where $I_t = 5 \ \forall t$ is the exogenously given investment and $\delta = 0.05$ is the depreciation rate, with the initial condition $K_0 = 10$.

Solving the Continuous-Time Capital Accumulation Equation

Update Rule for Forward Difference Method

Since we are given an initial boundary condition, we will use the **forward difference method** to solve this equation. The update rule is:

$$K(t + \Delta t) = K(t) + \Delta t \left(I(t) - \delta K(t) \right)$$

where Δt is the time step size. You will iterate this equation forward in time, starting from $t = 0$ and progressing to $t = T$.

Solving the Continuous-Time Capital Accumulation Equation

Finite Difference Approximations

We **discretize the time domain** into uniformly spaced grid points, where $\Delta t = t(i+1) - t(i)$ represents the distance (time step) between consecutive grid points.

Using the shorthand notations $K_i = K(t_i)$, the **finite difference approximation** of the capital accumulation equation is expressed as:

$$K(i+1) = K(i) + \Delta t \left(I(i) - \delta K(i) \right)$$

Numerical (FDM) vs Analytical Solution

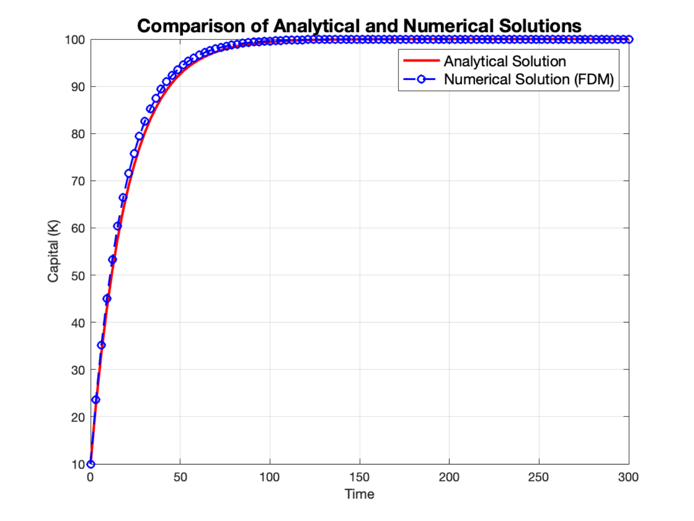

Figure 2: Numerical vs Analytical Solution for $I = 5$ with 100 Time Steps

Numerical (FDM) vs Analytical Solution

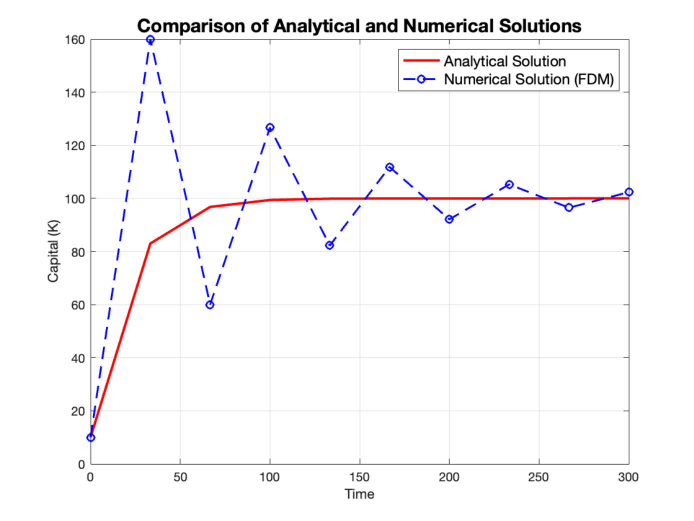

Figure 3: Numerical vs Analytical Solution for $I = 5$ with 10 Time Steps

Numerical (FDM) vs Analytical Solution

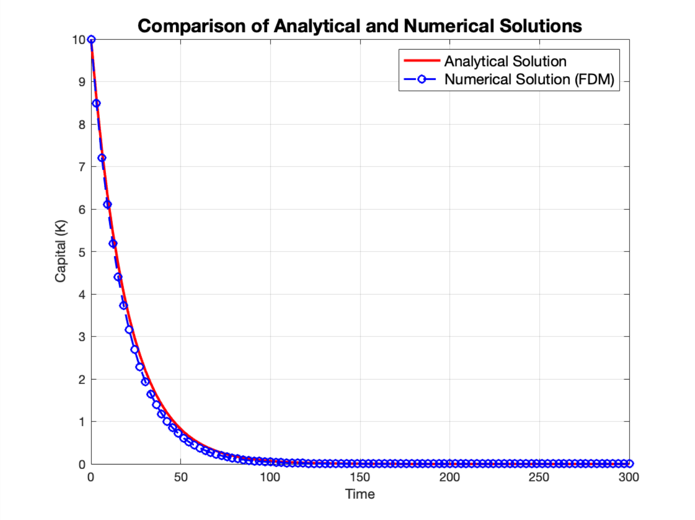

Figure 4: Numerical vs Analytical Solution for $I = 0$ with 100 Time Steps

Numerical (FD) vs Numerical (ODE45) Solution

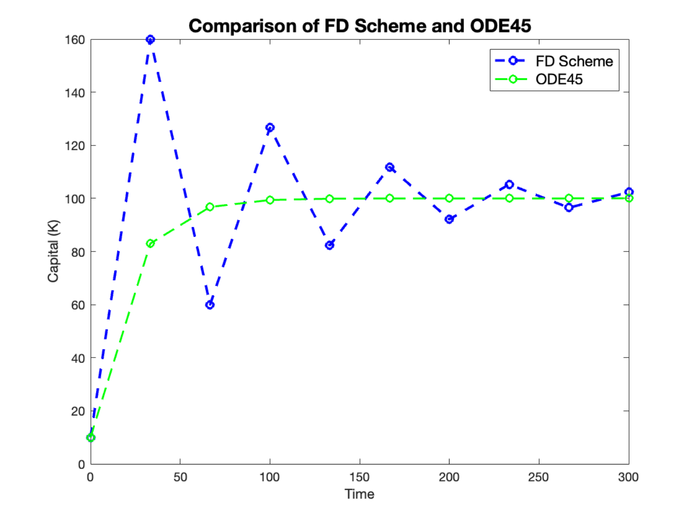

Figure 5: FD Scheme vs ODE45 Solution for I = 5 with 10 Time Steps

Numerical (FDM) vs Numerical (ODE45) Solution

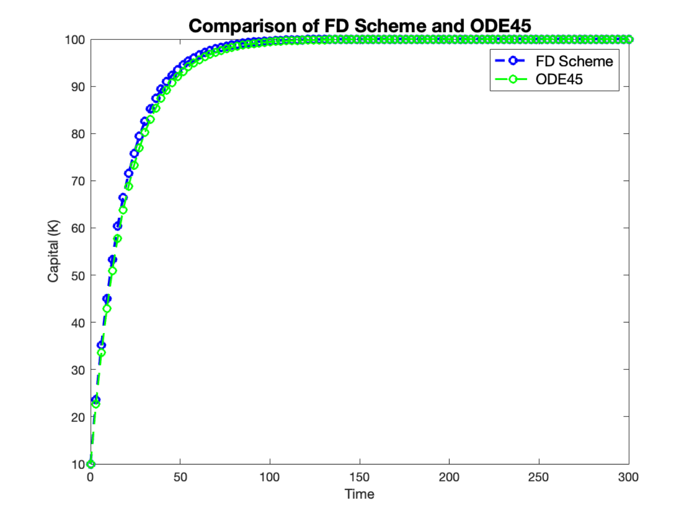

Figure 6: FD Scheme vs ODE45 Solution for I = 5 with 100 Time Steps

References

Achdou, Y., J. Han, J.-M. Lasry, P.-L. Lions, and B. Moll (2022). Income and wealth distribution in macroeconomics: A continuous-time approach. The review of economic studies 89(1), 45–86.

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