Power Reduction Formulas and High Dimensional Balls

written by Edison Hauptman, ERH129@pitt.edu

In this project, you will use Integration by Parts to solve some trigonometric integrals, then apply this formula to compute the volumes of unit balls in more than 3 dimensions.

1. Consider integrals of the form $\int \sin^n(x)dx$:

   (a) Use Integration by Parts to compute $\int \sin^2(x)dx$. Choose $u = \sin(x)$ and $dv = \sin(x)dx$, and make use of the trig identity $\cos^2(x) = 1 - \sin^2(x)$ along the way.

   (b) Use multiple iterations of Integration by Parts to solve $\int \sin^4(x)dx$. For the first iteration, choose $u = \sin^3(x)$ and $dv = \sin(x)dx$.

   (c) Write a formula for $\int \sin^n(x)dx$ with one iteration of Integration by Parts, using a similar strategy to part (b). Your resulting integral will also be a power of $\sin(x)$, but the exponent will be smaller. This is called a **Power Reduction Formula**.

   (d) Use your Power Reduction Formula from part (c) to solve the integrals $\int \sin^8(x)dx$ and $\int \sin^9(x)dx$. How do these answers compare?

2. Consider integrals of the form $\int \sec^n(x)dx$:

   (a) Note that $\int \sec(x)dx = \ln|\sec(x)+\tan(x)|+C$. Prove this using differentiation or a u-substitution.

   (b) Use Integration by Parts to compute $\int \sec^3(x)dx$. Choose $u = \sec(x)$ and $dv = \sec^2(x)dx$, and make use of the trig identity $\tan^2(x) = \sec^2(x) - 1$ along the way.

   (c) Write a formula for $\int \sec^n(x)dx$ with one iteration of Integration by Parts, using a similar strategy to part (b). Your resulting integral will also be a power of $\sec(x)$, but the exponent will be smaller.

   (d) Use your Power Reduction Formula from part (c) to solve the integrals $\int \sec^7(x)dx$ and $\int \sec^8(x)dx$. How do these answers compare?

3. You have learned that the area of a circle (2-dimensional ball) with radius r is $V_2(r) = \pi r^2$, and the volume of a sphere (3-dimensional ball) with radius r is $V_3(r) = \frac{4}{3}\pi r^3$. There is a recursive formula to compute the volume of a unit ball in n dimensions:

$$V_n(1) = \int_{-1}^{1} V_{n-1}\left(\sqrt{1-x^2}\right) dx.$$

   (a) Verify that this formula works for $V_3(1)$. Take the top half of a unit circle, and consider the solid obtained by rotating it about the x-axis—this is the unit sphere. Use the method of rotational discs/washers to set up an integral for the volume of the sphere, and rewrite the integral to verify that you obtain

$$V_3(1) = \int_{-1}^{1} V_2\left(\sqrt{1-x^2}\right) dx.$$

(b) **Bonus:** Explain how this argument can generalize to n dimensions.

Hint: for the sphere, you added up many circles. For the n-ball, add up many $(n-1)$-balls.

(c) We use the fact that two n-dimensional balls of radii r and s have volumes in the proportion $\dfrac{V_{n-1}(r)}{V_{n-1}(s)} = \dfrac{r^{n-1}}{s^{n-1}}$ to derive a nicer formula:

$$V_n(1) = V_{n-1}(1) \int_{-1}^{1} \left(\sqrt{1-x^2}\right)^{n-1} dx.$$

Then, using the trig subtitution $x = \sin(\theta)$, we obtain

$$V_n(1) = V_{n-1}(1) \int_{-\pi/2}^{\pi/2} \cos^n(x) dx.$$

Use this recursive formula to compute the volumes of the unit balls with dimension 4, 5, and 6.