# Integrate Any Rational Function

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In this project, we will consider functions of the form $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions. Such functions are called **rational functions**. By writing an algorithm to compute integrals of this form, we will learn why every rational function has an elementary integral—that is, we can write the integral of any such function as a finite combination of functions/operations we're already familiar with.

1. Let's start with functions of the form $\frac{Dx + E}{Ax^2 + Bx + C}$, where $A, B, C, D, E$ are some constants.

   (a) Use partial fraction decomposition to compute the integral $\int \frac{2x+3}{x^2 - 7x - 9} dx$. Even when we don't know how to factor $Ax^2 + Bx + C$, we can sometimes use the quadratic formula to get two roots. This will tell us how to factor the denominator, so we can use a partial fraction decomposition!

   (b) Use partial fraction decomposition to compute the integral $\int \frac{-x-4}{9x^2 - 12 + 4} dx$.

   (c) Consider the integral $\int \frac{3x-5}{x^2 + 4x + 7} dx$. This integral cannot be solved using Partial Fraction Decomposition because the two roots of $Ax^2 + Bx + C$ aren't real numbers. Instead, we could complete the square for $Ax^2 + Bx + C$. Note that this will give a denominator which looks vaguely like the derivative of $\arctan(x)$ in the denominator. However, that anti-derivative won't work if there are $x$'s in the numerator, so you'll first need to split up the fraction into two pieces with the same denominator. One numerator will have $x$'s and one will not. Can you come up with a split that allows you to use a simple $u$-substitution on one piece and a trig substitution on the other piece?

   Note that the three integrals above involve handling the three separate cases for the quadratic function in the denominator. In the first one, the quadratic function has two real roots. In the second, the quadratic function has one real root. In the third, the quadratic function has no roots.

2. Now, let's think about larger denominators:

   (a) If the denominator of the rational function is a higher-degree polynomial, explain why it must have a real-number factorization into terms which are at most quadratic. *Hint: think about the roots of the polynomial.*

   (b) In general, it can be very hard to discover a real-number factorization of a polynomial into at-most-quadratic pieces. In fact, while there is a cubic formula and a quartic formula to write down the roots of any polynomial of degree 3 or 4, it has been proven that no formula for general higher-degree polynomials can exist. Try to factor the quartic polynomial $x^4 + 4$. If you don't solve it, verify that $x^4 + 4 = (x^2 - 2x + 2)(x^2 + 2x + 2)$ by multiplying the terms back out.

3. Write a program that takes in a rational function where the denominator is factored into at-most-quadratic pieces, and outputs the indefinite integral of the rational function. Consider the following specifications:

   - You may assume that the denominator is factored into no more than three at-most-quadratic pieces. Note that $x^2 - 2x + 1 = (x-1)^2$ has been factored into two pieces.
   - You may assume the numerator has degree at most 6.
   - Your initial input will look like

     $$\frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6}{(p_0 + p_1 x + p_2 x^2)(q_0 + q_1 x + q_2 x^2)(r_0 + r_1 x + r_2 x^2)},$$

     where all $a_i, p_i, q_i, r_i$ are arbitrary rational numbers.
   - Your algorithm should be able to use appropriate strategies to solve all three integrals in the first question. This means you may need to run some tests and employ different strategies depending on the outcome of each.
   - You may not assume that any quadratic pieces in the given denominator are fully factored.
   - An early step will be to perform a partial fraction decomposition on a large fraction with your factored denominator. You should create the system of equations to solve for the coefficients in your numerator, but you do not need to solve the resulting system from scratch. For example, in Python, you should freely use the command numpy.linalg.solve(). If you are not familiar with matrices, talk to your instructor!

4. Test your program on the following integrals. Use a computer algebra system (CAS) to verify your program's answer.

   - $\displaystyle\int \frac{4x^4 + 8x^2 + 7x - 10}{(x-2)(x^2+1)^2}$
   - $\displaystyle\int \frac{3x^3 + x^2 + x + 20}{(x^2 - 4x - 5)(2x^2 - 6x - 9)}$
   - $\displaystyle\int \frac{4x^6 - 28x^5 + 76x^4 - 278x^3 + 447x^2 - 676x + 891}{(x^2 - 6x + 9)(x^2+5)^2}$

5. Are you confident that every rational function can be integrated using an algorithm like yours?