WHY $(1/2)!$ EQUALS $\sqrt{\pi}$: THE GAMMA FUNCTION

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ABSTRACT. In this project, students practice integration by parts and u-substitution by computing certain values of the gamma function Γ with an application to probability theory.

1. Introduction

One of the most famous open problems in mathematics, worth a one million-dollar prize to whoever solves it, is called the *Riemann hypothesis*. Considered part of an area of math called *analytic number theory*, the Riemann hypothesis concerns a certain complex-valued function Γ called the *gamma function*:

$$\Gamma(z) := \int_0^\infty x^{z-1} e^{-x}\, dx. \tag{1.1}$$

Outside of number theory, the gamma function appears in quantum physics, astrophysics, and fluid dynamics. Statisticians use it to define a probability distribution called the *gamma distribution* that models everything from earthquakes to rainfall to insurance claims.

Although we won't teach enough here to help you win that one million dollars, we will study a few interesting facts related to the gamma function. In particular, you've probably seen the *factorial* function, denoted by !, that sends a nonnegative integer $n \in \mathbb{N}$ to the positive integer

$$n! := \prod_{k=1}^{n} k = n(n-1)(n-2)\cdots 1.$$

It turns out that there's a way (called *analytic continuation*) to extend the factorial function into a differentiable function defined over almost all complex numbers $z \in \mathbb{C}$. Below, we'll see that the gamma function provides one way to do that.

2. Exploration

Let Γ denote the gamma function (1.1), and let $\mathbb{N}$ denote the set of nonnegative integers.

Problem 1 (Warm-up)**.** Show that $\Gamma(1) = 1$.

Problem 2. Show that

$$\Gamma(\varepsilon + 1) = \varepsilon\Gamma(\varepsilon)$$

for all positive real numbers $\varepsilon > 0$.

Problem 3. Deduce that

$$\Gamma(n+1) = n!$$

for all nonnegative integers $n \in \mathbb{N}$.

Remark 1. Problem 3 shows that Γ is an *analytic continuation* of the factorial function—that is, Γ allows us to compute "factorials" of almost any complex number $z \in \mathbb{C}$. (In other words, we've upgraded the factorial function, which is discrete, to Γ, which is differentiable!) Problem 6 does exactly this for a certain set of rational numbers.

Problem 4. Show that

$$\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^{\infty} e^{-u^2}\, du. \tag{2.1}$$

(Hint: The function e^{-u^2} is even.)

The improper integral (2.1) is called a *Gaussian integral*. It takes some Calc III concepts to evaluate, namely *double integrals* and integration with *polar coordinates*. (Find a proof online if you're curious!) For the purposes of this exploration, you can take for granted that this integral actually equals $\sqrt{\pi}$ (as you can verify using a computer).

Problem 5. Deduce that

$$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}, \qquad \Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}. \tag{2.2}$$

Problem 6. Can you generalize (2.2) to a general formula for

$$\Gamma\left(\frac{2n+1}{2}\right), \quad n \in \mathbb{N}?$$

Remark 2. Combining Problems 3 and 6, you've just computed the value of $\Gamma(k/2)$ for all positive integers $k \in \mathbb{Z}^+$.

One major application of the gamma function is to help us understand certain probability distributions in statistics.

Problem 7. Let X be a *random variable* following an *exponential distribution* with parameter $\lambda > 0$. (In the notation used by statisticians, $X \sim \operatorname{Exp}(\lambda)$.)

Let $n \in \mathbb{N}$. A theorem called the *law of the unconscious statistician*, or *LOTUS*, gives a formula for the *mean* or *expected value* of the random variable X^n:

$$\mathbb{E}(X^n) = \frac{1}{\lambda}\int_0^{\infty} x^n e^{-x/\lambda}\, dx.$$

Use this formula to show that

$$\mathbb{E}(X^n) = n!\lambda^n.$$

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