FOURIER SERIES OF SQUARE AND TRIANGLE WAVES

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Abstract. In this project, students practice integration techniques by computing the Fourier series of square and triangle waves. As an application, they derive a beautiful formula for π.

1. Introduction

In music production, sound engineering, and signal processing (a subfield of electrical engineering), we work with special real-valued functions called *signals*. These functions model light waves, sound waves, or other types of waveforms traveling through media like air or water.

Definition. A *signal* is a function $f : \mathbb{R} \to \mathbb{R}$ be satisfying the following conditions:

- f is periodic with period 2π. That is, $f(x) = f(x + 2\pi)$ for all real numbers $x \in \mathbb{R}$.
- f is piecewise continuous.
- f is bounded. That is, there exists a positive real number $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$.

When encoding signals into computers—for example, if we want to compress the sound waves of songs into MP3 files—we often need to minimize how much storage we use. This is where a theorem from a field of math called *harmonic analysis* comes in: We can write any signal (and, in fact, very many other functions) as an infinite sum of trigonometric functions. This sum is called a *Fourier series*, and it converges everywhere for signals.

The partial sums of the Fourier series approximate the original signal. To minimize storage usage on computers, we often encode signals as one of these partial sums. This loses information proportional to which partial sum we encode; this is why the compressed audio of an MP3 doesn't sound nearly as crisp as the audio of a live performance. Nevertheless, the partial sums are good enough approximations for most applications.

The following describes how to compute the Fourier series of signal functions that are odd.

Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$ be a signal. If f is* odd*—that is, if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$—then the Fourier series of f is*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx),$$

where the Fourier coefficients b_n are given by

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx.$$

An example of an odd signal is the *square wave* function; see Figure 1. Square waves play a fundamental role in signal processing, and they help model the sound profiles of certain musical instruments (including electric guitars with distortion, pipe organs, and clarinets). If you've ever been to a rave or played an old NES or Game Boy game, chances are you've heard a "pure" square wave before; they're used as leads (melodic "instruments") in EDM and chiptune.

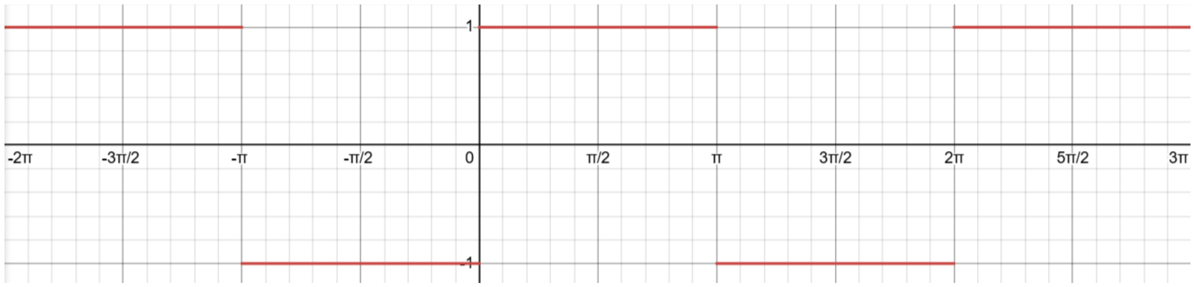

FIGURE 1. Plot of the square wave function f.

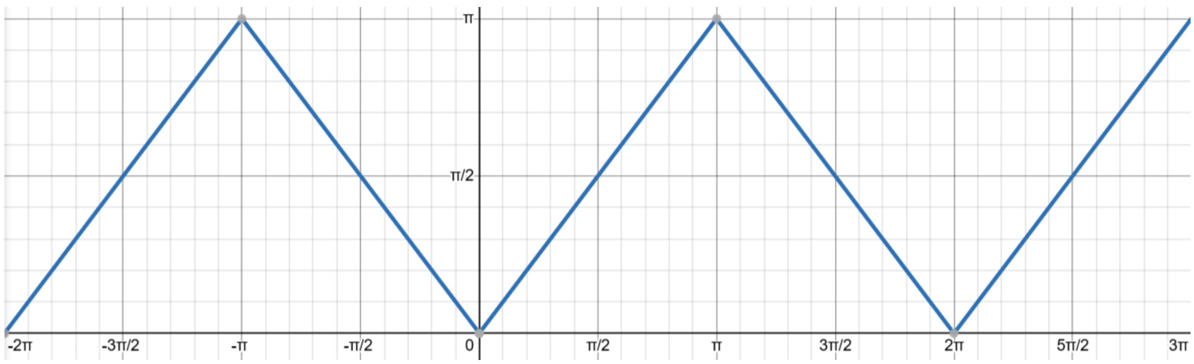

FIGURE 2. Plot of the triangle wave function g.

2. Exploration

Let f be the square wave function depicted in Figure 1.

Problem 1. Show that the Fourier series of f is

$$f(x) = \frac{4}{\pi} \sum_{n=1,\, n \text{ odd}}^{\infty} \frac{\sin(nx)}{n}.$$

Problem 2. Use a computer to plot the first few partial sums of the Fourier series of f. (Recall that the kth *partial sum* of a series $\sum_{n=1}^{\infty} a_n$ is defined to be the finite sum $\sum_{n=1}^{k} a_n$.

Problem 3. Compare your plots with Figure 1. How do the partial sums behave near the places where f has jump discontinuities? (This is called the *Gibbs phenomenon.*)

Problem 4. Here's a neat little application. By evaluating the Fourier series of f at $x = \pi/2$, derive a formula for π as the alternating series

$$\pi = 4\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right).$$

Now, let g be the *triangle wave* function depicted in Figure 2. (In contrast to square waves, triangle waves are used as bass "instruments" in electronic music, especially EDM and chiptune.)

Problem 5. Can you see why f is the derivative of g away from the "corners" of g? What happens to f at these corners?

Problem 6. Since $g' = f$ almost everywhere, we can obtain the Fourier series of g by integrating the Fourier series of f term-by-term.[1] Do this using $C := \pi/2$ as the constant of integration (why?).

Problem 7. Use a computer to plot the first few partial sums of the Fourier series of g. To verify your solution to Problem 6, compare your plots with Figure 2.

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[1] Usually, you'd need to justify integrating term-by-term, since that isn't always the same as integrating the whole series. But one of the miracles of harmonic analysis is that, for Fourier series of signals, those *are* the same!