Gabriel's Horn

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In this project, we will investigate the shape formed by rotating the region "bounded" by $y = \frac{1}{x}$, $y = 0$, $x = 1$, and $x = \infty$ about the x-axis. This 3-dimensional solid, known as "Gabriel's Horn", has infinite surface area but only finite volume!

1. Use rotational discs/washers to compute the volume of Gabriel's Horn. Note that this is an improper integral; you should verify that the volume is a finite number.

2. The formula to compute the the surface area of a solid of revolution rotated about the x-axis is
$$\int_a^b 2\pi f(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$
Deriving this formula involves techniques similar to the ones we used to derive formulas for the volumes of solids of revolution. Using this formula, set up an integral for the surface area of Gabriel's Horn.

3. From the formula you've set up in question 2, compute the indefinite integral. Verify that the indefinite integral is
$$\ln\left(\frac{\sqrt{1+x^{-4}}+1}{\sqrt{1+x^{-4}}-1}\right) - 2\sqrt{1+x^{-4}} + C.$$

4. Use the result from the previous question to show that Gabriel's Horn has infinite volume.

5. Surface area integrals like the one you're asked to compute in question 3 can be very difficult. However, we can also verify that the integral in question 2 diverges without having to compute the hard integral directly. Instead, we compare our difficult integral to a smaller, easier-to-compute integral which also diverges. In other words,

   If $g(x) < f(x)$ on an interval (a,b) and $\int_a^b g(x)$ diverges to infinity, then $\int_a^b f(x)$ also diverges.

   Come up with a smaller function that is easier to integrate, and show that it also diverges when you integrate it on the same bounds.