Calculus 2 Project:

Power Reduction Formulas and High Dimensional Balls

In this project, you will use Integration by Parts to solve some trigonometric integrals, then apply this formula to compute the volumes of unit balls in more than 3 dimensions.

1. Consider integrals of the form $\int \sin^n(x)dx$:

   (a) Use Integration by Parts to compute $\int \sin^2(x)dx$. Choose $u = \sin(x)$ and $dv = \sin(x)dx$, and make use of the trig identity $\cos^2(x) = 1 - \sin^2(x)$ along the way.

   (b) Use multiple iterations of Integration by Parts to solve $\int \sin^4(x)dx$. For the first iteration, choose $u = \sin^3(x)$ and $dv = \sin(x)dx$.

   (c) Write a formula for $\int \sin^n(x)dx$ with one iteration of Integration by Parts, using a similar strategy to part (b). Your resulting integral will also be a power of $\sin(x)$, but the exponent will be smaller. This is called a **Power Reduction Formula**.

   (d) Use your Power Reduction Formula from part (c) to solve the integrals $\int \sin^8(x)dx$ and $\int \sin^9(x)dx$. How do these answers compare?

2. Consider integrals of the form $\int \sec^n(x)dx$:

   (a) Note that $\int \sec(x)dx = \ln|\sec(x)+\tan(x)|+C$. Prove this using differentiation or a u-substitution.

   (b) Use Integration by Parts to compute $\int \sec^3(x)dx$. Choose $u = \sec(x)$ and $dv = \sec^2(x)dx$, and make use of the trig identity $\tan^2(x) = \sec^2(x) - 1$ along the way.

   (c) Write a formula for $\int \sec^n(x)dx$ with one iteration of Integration by Parts, using a similar strategy to part (b). Your resulting integral will also be a power of $\sec(x)$, but the exponent will be smaller.

   (d) Use your Power Reduction Formula from part (c) to solve the integrals $\int \sec^7(x)dx$ and $\int \sec^8(x)dx$. How do these answers compare?

3. You have learned that the area of a circle (2-dimensional ball) with radius r is $V_2(r) = \pi r^2$, and the volume of a sphere (3-dimensional ball) with radius r is $V_3(r) = \frac{4}{3}\pi r^3$. There is a recursive formula to compute the volume of a unit ball in n dimensions:

$$V_n(1) = \int_{-1}^{1} V_{n-1}\left(\sqrt{1-x^2}\right) dx.$$

   (a) Verify that this formula works for $V_3(1)$. Take the top half of a unit circle, and consider the solid obtained by rotating it about the x-axis—this is the unit sphere. Use the method of rotational discs/washers to set up an integral for the volume of the sphere, and rewrite the integral to verify that you obtain

$$V_3(1) = \int_{-1}^{1} V_2\left(\sqrt{1-x^2}\right) dx.$$

(b) **Bonus:** Explain how this argument can generalize to n dimensions.

Hint: for the sphere, you added up many circles. For the n-ball, add up many $(n-1)$-balls.

(c) We use the fact that balls of radii r and s have volumes in the proportion $\dfrac{V_{n-1}(r)}{V_{n-1}(s)} = \dfrac{r^{n-1}}{s^{n-1}}$ to write a nicer formula:

$$V_n(1) = V_{n-1}(1) \int_{-1}^{1} \left(\sqrt{1-x^2}\right)^{n-1} dx.$$

Then, using the trig subtitution $x = \sin(\theta)$, we obtain

$$V_n(1) = V_{n-1}(1) \int_{-\pi/2}^{\pi/2} \cos^n(x) dx.$$

Use this recursive formula to compute the volumes of the unit balls with dimension 4, 5, and 6.