

# Neuron Firing Models

For many biological processes, it is important to be able to understand when neurons are firing. Knowing the firing rate of a neuron and how it reacts to different stimuli can be useful for analyzing mechanisms such as muscle contraction and decision making. Neuron firing models can also help us understand the conditions that lead to seizures and consequently can be useful in developing treatments. In computational neuroscience, one useful family of models for periodically firing neurons are integrate-and-fire models. In these models, we have an activation variable  $x$  described by a differential equation of the form

$$\frac{dx}{dt} = f(x) + I$$

where  $f(x)$  is a given function and  $I$  is a constant input current. In addition to the differential equation for the activation variable, we also have a fixed threshold  $x_{th}$  and a fixed reset point  $x_r$ . Typically, we take  $x_r < x_{th}$ . In order to model neuronal firing, we add the condition that if  $\lim_{t \rightarrow T^-} x(t) = x_{th}$ , then we set  $x(T) = x_r$  and say that the neuron has fired (see fig. 1 and 2).

1. We begin by considering a linear integrate-and-fire (LIF) model:

$$\frac{dx}{dt} = x + I.$$

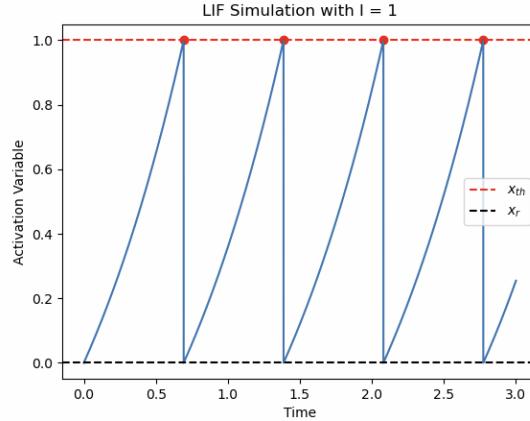


Figure 1: Example trajectory for the LIF model with  $I = 1$ ,  $x_{th} = 1$ , and  $x_r = 0$ . The threshold value is shown as a red dashed line and the reset value is shown as a black dashed line. The red dots indicate that the activation reached the threshold for firing and so we reset  $x$  to  $x_r$ .

- (a) Take the threshold to be  $x_{th} = 1$  and the reset to be  $x_r = 0$ . We can determine an explicit formula for the action variable by solving

$$t = \int_0^{x(t)} \frac{dx}{x + I}.$$

Compute the integral and show that  $x(t) = I(e^t - 1)$ . If  $I \leq 0$ , will the neuron ever fire?

- (b) Assuming  $I > 0$ , what is the period of the neuron  $P_l(I)$  as a function of  $I$ ?

Hint: The neuron fires when  $x(t) = x_{th} = 1$ , so the period  $T = \int_0^1 \frac{dx}{x + I}$ .

2. Another neuronal model we can consider is the quadratic integrate-and-fire (QIF) model:

$$\frac{dx}{dt} = x^2 + I.$$

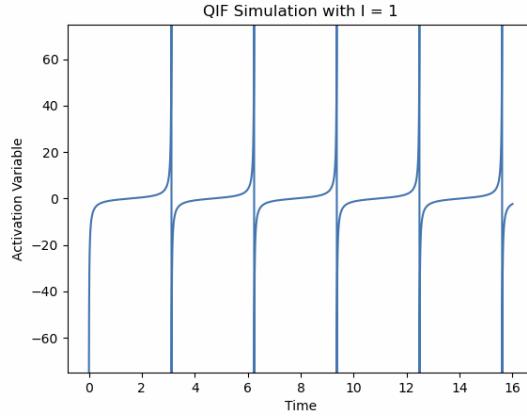


Figure 2: Example trajectory for the LIF model with  $I = 1$ ,  $x_{th} = \infty$ , and  $x_r = -\infty$ . The neuron fires when the activation reaches infinity which occurs at the vertical asymptotes.

- (a) Take the threshold to be  $x_{th} = \infty$  and the reset to be  $x_r = -\infty$ . We can again get an explicit formula for the action variable by solving an integral:

$$t = \int_{-\infty}^{x(t)} \frac{dx}{x^2 + I}.$$

Compute the integral and show that  $x(t) = \sqrt{I} \tan\left(\sqrt{I}t - \frac{\pi}{2}\right)$ . If  $I \leq 0$ , does the neuron ever fire?

- (b) Assuming  $I > 0$ , what is the period of the neuron  $P_q(I)$  as a function of  $I$ ?

3. One thing we can compare the LIF and QIF models is how the period changes with asymptotically large input current.

- (a) Show that for large positive  $I$  the period of the LIF model is inversely proportional to  $I$  by computing that  $\lim_{I \rightarrow \infty} IP_l(I)$  converges to a finite non-zero value.

Hint: Consider writing the limit as  $\lim_{I \rightarrow \infty} \frac{P_l(I)}{\left(\frac{1}{I}\right)}$  and use L'Hopital's rule.

- (b) Show that for large positive  $I$  the period of the QIF model is inversely proportional to  $\sqrt{I}$  by computing that  $\lim_{I \rightarrow \infty} \sqrt{I}P_q(I)$  converges to a finite non-zero value.