An Explicit Formula for The Fibonacci Numbers[1]

written by Edison Hauptman, ERH129@pitt.edu

The Fibonacci sequence begins $1, 1, 2, 3, 5, 8, 13, 21, \ldots$, and is a **recursive** formula. Each number is the sum of the previous two numbers; that is, the Fibonacci sequence $\{f_i\}$ is defined by $f_n = f_{n-1} + f_{n-2}$ and $f_1 = f_2 = 1$. In this project, we will introduce a specific technique to calculate the Maclaurin Series of a function, and we will use this technique to derive an **explicit** formula for the Fibonacci sequence. An explicit formula makes it possible to write down the 100th number in the Fibonacci sequence without having to calculate the first 99 numbers.

1. Write down the Maclaurin series for the function $g(x) = \dfrac{1}{1-x}$ by relating it to the sum of a geometric series. You have most likely seen this series in class!

2. Let's verify that this equation is true in a different way. Write

$$g(x) = \frac{1}{1-x} = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \ldots = \sum_{n=0}^{\infty} c_n x^n.$$

   This is a generic representation for the Maclaurin series of a function. Now, multiply both sides by $1 - x$ to get

$$1 = (1-x)(c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \ldots).$$

   This equation must be true for every value of x. So, by expanding the right side and gathering like terms, show that every coefficient of the power series is 1.
   Hint: Solve for c_0 first, then use that value to get c_1, then use that value to get c_2, etc.

3. Use this same technique to find a Maclaurin series representation for

$$h(x) = \frac{1}{4+x^2}.$$

   Then, verify your answer by writing this function as the sum of a geometric series.
   Hint: Can you find a clear pattern to the coefficients? You may need to calculate several terms in order to see it.

4. Use this same technique to find a Maclaurin series representation for

$$f(x) = \frac{x}{1-x-x^2}.$$

[1] This project has been adapted from one of James Stewart's challenge problems for his textbook, Essential Calculus, Early Transcendentals, 2nd Edition. These can be found on his website.

Show that the coefficients $c_0, c_1, \ldots$ follow the Fibonacci sequence $\{f_i\}$, defined by $f_n = f_{n-1} + f_{n-2}$ and $f_1 = f_2 = 1$ (also, $f_0 = 0$).

5. Now, we will build the Maclaurin series for $f(x)$ in a different way. Find the (irrational) roots of $1 - x - x^2$ using the quadratic formula, and use these roots r_1 and r_2 to factor $1 - x - x^2$ into two linear terms, $-(r_1 - x)(r_2 - x)$. Then, based on this linear factorization, calculate the **partial fraction decomposition** of

$$f(x) = \frac{-x}{(r_1 - x)(r_2 - x)}.$$

6. For both terms in your partial fraction decomposition, build its Maclaurin series using the sum of a geometric series. Then, add the two series term-by-term to get one big Maclaurin series.

7. Our answer from question 6 gives us explicit values for each term in the series we calculated in question 4. Thus, we can write down the explicit formula for f_n. Use this to calculate the exact value of f_{100}, the 100th Fibonacci number in the sequence.