Optimal incentives for collective intelligence

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Collective intelligence is the ability of a group to perform more effectively than any individual alone. Diversity among group members is a key condition for the emergence of collective intelligence, but maintaining diversity is challenging in the face of social pressure to imitate one's peers. Through an evolutionary game-theoretic model of collective prediction, we investigate the role that incentives may play in maintaining useful diversity. We show that market-based incentive systems produce herding effects, reduce information available to the group, and restrain collective intelligence. Therefore, we propose an incentive scheme that rewards accurate minority predictions and show that this produces optimal diversity and collective predictive accuracy. We conclude that real world systems should reward those who have shown accuracy when the majority opinion has been in error.

collective intelligence | game theory | democracy | diversity | markets

The financial crisis and its aftermath have reopened long-standing debates about the collective wisdom of our societal organizations (1–3). Financial and prediction markets seem unable to foresee major economic and political upheavals, such as the credit crunch or Brexit. This lack of collective foresight could be the result of insufficient diversity among decision-making individuals (4). Diversity has been identified as a key ingredient of successful groups across many facets of collective behavior (5–7). It is a crucial condition for collective intelligence (6–10) that can be more important than the intelligence of individuals within a group (11). Because collective behavior ultimately results from individual actions, incentives play a major role for diversity and collective performance (12, 13). Although most previous research has focused on explaining how collective intelligence emerges (14), less is known about how to optimize the wisdom of crowds in a quantitative sense.

Harnessing collective wisdom is important. Global systems of communication, governance, trade, and transport grow rapidly in complexity every year. Many of these real world problems have a large number of contributing factors. For example, predicting future economic fluctuations requires integrating knowledge about credit markets and supply chains across the world as well as the ramifications of political developments in different countries and the shifting sentiments of individual investors and consumers. Political developments are themselves the result of many factors: both direct (e.g., political parties' strategies) and indirect (e.g., technological change). Scientific questions are also increasingly complex. For instance, building a complete model of an ecosystem requires bringing together expertise on many scales from individual animal behavior to complex networks of predation and codependency (15). In each case, knowledge about the diverse contributing factors is dispersed. For these high-dimensional problems, it is becoming impossible for any single individual or agency to gather and process enough data to understand the entire system (16). In many cases, we do not even have full knowledge of what the potential causal factors are, let alone a full understanding of them.

Attention is, therefore, shifting toward distributed systems as a means of bringing together the local knowledge and private expertise of many individuals (12, 17). In machine learning, researchers have found that a pluralistic modeling approach maximizes prediction accuracy (18). In politics, the forecasts of prediction markets (19, 20) are now commonly reported alongside opinion polls during elections. Scientists are also turning to crowdsourcing collective wisdom as a validation tool (21–23). However, as highlighted by the failure of financial and prediction markets to foresee the results of recent elections in the United Kingdom and the United States, collective wisdom is not a guaranteed property of a distributed system (2), partly because of herding effects (24, 25). In science as well, the incentive structure undervalues diversity: low-risk projects with assured outcomes are more likely to be funded than highly novel or interdisciplinary work (26, 27). Rewards for conformity with institutional cultures can severely limit useful diversity (28). Previous work (29) has investigated mechanisms to elicit truthful minority views to counter herding effects in expressed opinion. This work raises the question: how can minority viewpoints be fostered in the first place to enhance diversity and its potential benefits for collective intelligence?

Here, we analyze an evolutionary game-theoretic model of collective intelligence among unrelated agents motivated by individual rewards. We show that previously proposed incentive structures (13) are suboptimal from the standpoint of collective intelligence and in particular, produce too little diversity between individuals. We propose an incentive system that we term "minority rewards," wherein agents are rewarded for expressing accurate minority opinions, and show that this produces stable, near-optimal collective intelligence at equilibrium. Our results show that common real world reward structures are unlikely to produce optimal collectively intelligent behavior, and we present a superior alternative that can inform the design of reward systems.

Results

To investigate the effect of incentives on collective intelligence, we use an abstract model of collective information gathering and

aggregation (13). Complex outcomes are modeled as a result of n independent causal factors. A large population of individual agents gathers information in a decentralized fashion, each being able to pay attention to just one of these factors at any given time. Collective prediction is achieved by aggregation of individual predictions via simple voting. Agents are motivated by an incentive scheme that offers rewards for making accurate predictions. It is assumed that the accuracy of an individual's prediction can be judged after the event. We exclude cases where either the ground truth is never discoverable or no such ground truth exists (for instance, in questions regarding taste or voter preferences). Instead, we consider questions such as the prediction of future events (which are known after they occur) or scientific questions (which may be resolved at some later point in time). For example, one might consider whether national gross domestic product (GDP) will rise above trend in the coming year, whether a certain party will win an election, or whether global temperatures will change by more than 1 °C in the next decade. The proportion of agents attending to different sources of information evolves depending on the rewards that they receive, where less successful agents tend to imitate their more successful peers.

Consider a binary outcome, Y, which is the result of many factors, $x_1, x_2, \ldots, x_n$. We model this outcome as the sign of a weighted sum of the contributing factors:

$$Y = \text{sign}\left(\sum_{i=1}^{n} \beta_i x_i\right). \qquad [1]$$

For simplicity, we assume that each contributing factor takes binary values, such that $Y, x_i \in \{-1, 1\}$, and that the values of these factors are uncorrelated (*SI Appendix* discusses instances with correlated factors). Without loss of generality, $\beta_i > 0$ for all factors.

An individual attending to factor i observes the value of x_i. Having observed the value of x_i, this individual then votes in line with that observation. Thus, if the proportion of individuals attending to factor i is ρ_i, the collective prediction $\hat{Y}$ is given by

$$\hat{Y} = \text{sign}\left(\sum_{i=1}^{n} \rho_i x_i\right). \qquad [2]$$

Collective accuracy, C, is the probability that the collective vote agrees with the ground truth given the distribution, $\{\rho\}$, of agents attending to each factor:

$$C = P\left(\hat{Y} = Y \mid \{\rho\}\right). \qquad [3]$$

The reward given to an agent for an accurate vote depends on the proportion of other correct votes in any given collective decision. Let z_i be the proportion of agents who will vote identically to those attending to factor i (i.e., the proportion of agents attending to factors with values that match x_i): $z_i = \sum_{j=1}^{n} \rho_j \delta_{x_i, x_j}$, where δ is the Kronecker delta. Then, the reward is determined by a function, $f(z_i)$, such that an agent receives a reward proportional to $f(z_i)$ if and only if his/her prediction is accurate. We will investigate three potential reward systems for deciding how each agent is rewarded for his/her accurate votes, the first two of which are taken from previous work by Hong et al. (13). The first of these is "binary rewards": agents receive a fixed reward if they make an accurate prediction, corresponding to the reward function $f(z_i) = 1$. The second is "market rewards": a fixed total reward is shared equally among all agents who vote accurately, corresponding to the reward function $f(z_i) = 1/z_i$. This reward scheme adds an incentive to be accurate when others are not and closely mimics the reward system of actual prediction markets. Finally, we introduce minority rewards: agents are rewarded for an accurate prediction when fewer than one-half of the other agents also vote accurately, corresponding to the reward function $f(z_i) = 1 - H(z_i - 1/2)$, where $H(\cdot)$ is the Heaviside step

function. This system explicitly rewards agents who hold accurate minority opinions and incentivizes agents to be accurate on questions where the majority prediction is wrong.

The expected reward that a player receives by attending to factor i is determined by the expected value of $f(z_i)$, conditioned on voting accurately (Eq. **8**). Players adapt their behavior in response to the rewards that they and others receive. In alignment with previous evolutionary game theory work, we model changes in individual attention to factors as being the result of imitation; agents who are observed to be gaining greater rewards are imitated by those gaining fewer. This model leads to the classic replicator equation (30), describing the evolution of the proportion of agents, ρ_i, who pay attention to factor i (Eq. **6**).

We studied the behavior of the model under three incentive schemes described above. We initialized the model by assigning uniform proportions of agents to each factor, with values of β randomly drawn from a uniform distribution (the absolute scale of β does not affect the model). We followed the evolutionary dynamics described by the replicator equation until the population converged to equilibrium. This calculation was repeated over a range of problem dimensionalities from $n = 3$ to $n = 10,000$. Expected rewards were calculated by either exhaustive search over all possible values of $x_1, \ldots, x_n$ (for $n < 10$) or using appropriate normal distribution limits for large numbers of factors (*Materials and Methods*).

Fig. 1 shows how collective accuracy and diversity evolve toward equilibrium for three rewards systems of binary, market, and minority rewards in simulations with $n = 100$, $n = 1,000$, and $n = 10,000$ independent factors. Note the logarithmic scale on the x axis to better illustrate the early evolution. For each reward system, two initial allocations of agents' attention are used: (*i*) a uniform allocation to each factor and (*ii*) an allocation where one-half of all agents attend to the single most important factor, with others allocated uniformly across the other factors. This figure shows that the equilibrium distribution of attention is the same no matter whether agents initially attend to arbitrary factors or initially favor the most obvious ones. The convergence time to equilibrium depends on the magnitude of rewards; in our simulations, we normalize rewards, such that the mean reward per agent is one at each time step.

Fig. 2 shows how the resulting collective accuracy varies across problem dimensionalities from $n = 3$ to $n = 10,000$ for three different reward systems and a uniform allocation of agents to factors. For simple problems ($n < 10$), all reward schemes produce high collective accuracy (over 90%). In these cases, the strong predictive power of only one or two meaningful independent factors means that individual accuracy is high, and collective aggregation only leads to relatively small increases in collective accuracy. However, even for these "small n" problems, we observe that minority rewards outperform other schemes. The differences in collective accuracy become more substantial as n increases. As Fig. 1 shows, these differences become apparent after only a few iterations, well before equilibrium is reached. Consistent with ref. 13, we find that market rewards increase diversity and collective accuracy relative to binary rewards. However, collective accuracy under market rewards declines rapidly with increasing n, falling to $\sim 65\%$ for $n = 10,000$. For comparison, we also show the accuracy achieved under a uniform allocation of agents, which reaches a stable value of $\sim 80\%$ for large n. Market rewards, therefore, produce lower accuracy than a uniform allocation for all but the lowest values of n. In contrast, minority rewards lead to a far higher accuracy than any of the investigated alternative reward systems, regardless of system complexity, and achieve close to 100% accuracy up to $n = 10,000$. Our mathematical analysis shows that minority rewards will continue to produce near-perfect accuracy for any problem size if the population of agents is large enough (*SI Appendix*). Our analysis of finite group sizes shows that minority
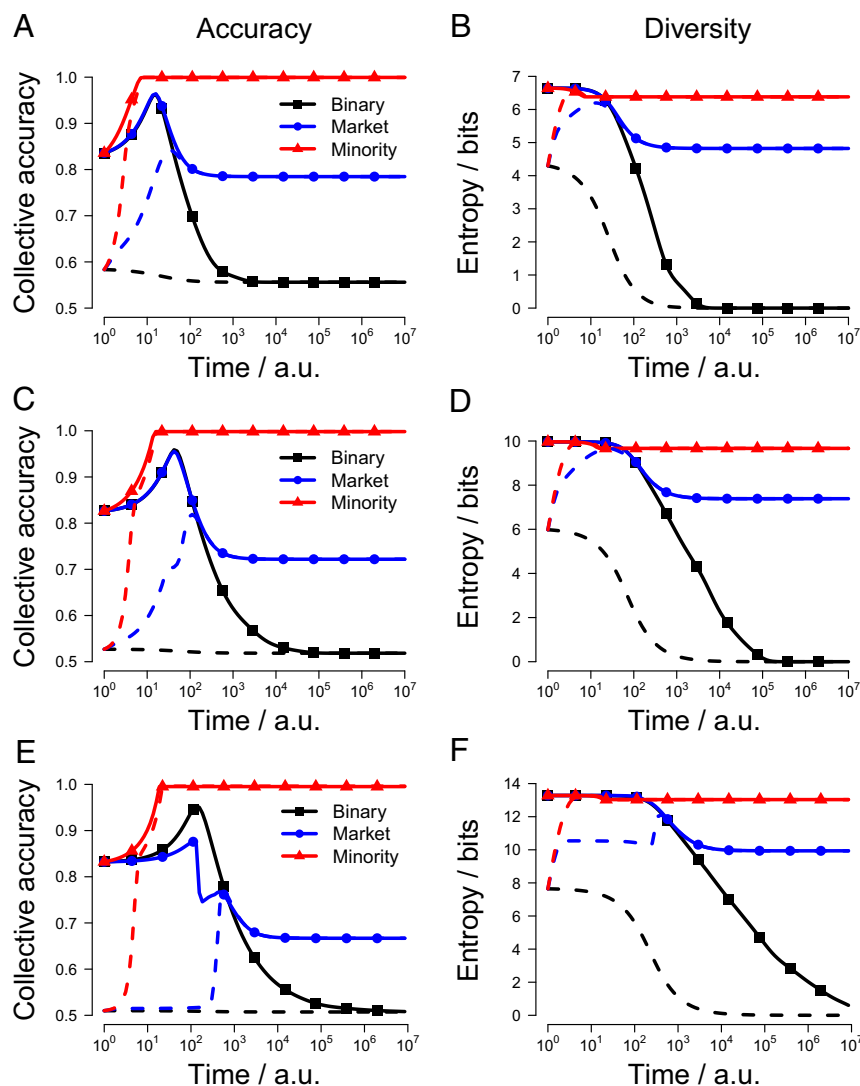
Fig. 1. Evolution of (*A*, *C*, and *E*) collective accuracy and (*B*, *D*, and *F*) diversity for binary rewards (black lines and squares), market rewards (blue lines and circles), and minority rewards (red lines and triangles) in simulations with (*A* and *B*) $n = 100$, (*C* and *D*) $n = 1{,}000$, and (*E* and *F*) $n = 10{,}000$ independent factors. Lines indicate results from a uniform initial allocation of agents over factors, whereas dashed lines indicate an initial allocation of 50% of agents to the single most important factor, with the remainder allocated uniformly over the remaining factors. Note that the number of time steps is plotted on a logarithmic scale.

rewards outperform other reward schemes for problem dimensions up to 10 times bigger than the population size, assuming best response dynamics (*SI Appendix*, Fig. S1).

The different levels of collective accuracy across reward systems are a reflection of the differing equilibrium distributions of the proportion of agents attending to each factor. Minority rewards outperform both market rewards and unweighted approaches, because attention is automatically redirected if the collective prediction would otherwise be wrong; only those outcomes where the majority opinion is wrong contribute to agents' rewards. Under minority rewards, the system converges toward a state where the number of agents paying attention to any factor is proportional to factor importance. This optimal distribution is both a stationary and a stable state of the minority rewards system (our mathematical analysis is in *SI Appendix*). Additional analysis (*SI Appendix*, Fig. S2) shows that varying the cutoff value for minority rewards (for example, by rewarding those voting with less than 40% of the group or 60%) invariably reduces collective accuracy. In Fig. 3, we plot the equilibrium dis-

tribution for each reward system for a high-dimensional problem ($n = 10{,}000$). Using binary rewards, almost all agents attend to the single most important factor. Under market rewards, agents distribute themselves in proportion to the predictive value of the factors but only among the top 10% of factors; 90% of factors receive essentially no attention at all (this proportion decreases as n increases and is, therefore, larger for smaller values of n). By comparison, under minority rewards, the proportion of agents paying attention to a factor is also proportional to its importance, but agents cover the full range of factors down to the least important ones, thereby providing more information to the group and improving predictions. The evolution of this distribution toward equilibrium is shown in detail in *SI Appendix*, Fig. S3.

Discussion

We proposed a reward system, minority rewards, that incentivizes individual agents in their choice of which informational factors to pay attention to when operating as part of a group. This system rewards agents for both making accurate predictions
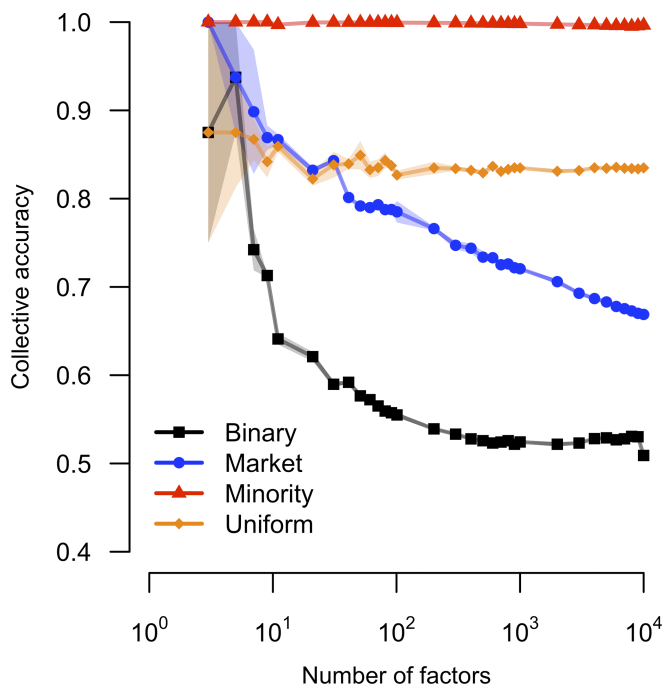
Fig. 2. Collective accuracy at equilibrium as a function of the number of independent factors across different reward systems. Lines and shaded regions show the mean and SD of 10 independent simulations with different randomly generated values for the factor coefficients. Points on each curve show the precise values of n for which simulations were carried out equally spaced within each multiple of 10.

and being in the minority of their peers or conspecifics. As such, it encourages a balance between seeking useful information that has substantive predictive value for the ground truth and seeking information that is currently underutilized by the group. Conversely, where the collective opinion is already correct, no rewards are offered, and therefore, no agent is motivated to change their strategy. Over time, therefore, agents are motivated to change their behavior only in ways that benefit collective accuracy.

The poor performance of market rewards relative to a uniform unweighted allocation for $n > 10$ shows that a market reward system incentivizes herding behavior and suppresses useful diversity as illustrated by the equilibrium distribution in Fig. 3B. This result suggests that stock markets and prediction markets tend to systematically underweight a large pool of informational factors that are of limited predictive power individually but that can contribute powerfully to aggregate predictions if agents can be persuaded to pay attention to them. This finding sheds doubt on the accuracy of existing markets as a tool for aggregating dispersed knowledge to predict future profits or events and motivates additional work on how to design collectively more accurate market mechanisms. The relatively high performance of uniform allocations of attention supports work showing that models with equally weighted predictors can match or even improve on more closely fitted prediction models (31, 32). The inclusion of all relevant predictors is often more important than determining their appropriate weights in making predictions; too much diversity is less harmful than too little, especially for complex problems.

Incentives are a fundamental part of any effort to harness the potential of collective intelligence. In this paper, we have presented evidence that rewarding accurate minority opinions can induce near-optimal collective accuracy within a model of collective prediction. Therefore, to maximize the collective wisdom

of a group, we suggest that individuals should not be rewarded simply for having made successful predictions or findings and also that a total reward should not be equally distributed among those who have been successful or accurate. Instead, rewards should be primarily directed toward those who have made successful predictions in the face of majority opposition from their peers. This proposal can be intuitively understood as rewarding those who contribute information that has the potential to change collective opinion, because it contradicts the current mainstream view. In our model, groups rapidly converge to an equilibrium with very high collective accuracy, after which the rewards for each agents become less frequent. We anticipate that, after this occurs, agents would move on to new unsolved problems. This movement would produce a dynamic system in which agents are incentivized to not only solve problems collectively but also, address issues where collective wisdom is currently weakest. Future work should investigate how our proposed reward system can be best implemented in practice from scientific career schemes to funding and reputation systems (33) to prediction markets and democratic procedures (34). We suggest experiments to determine how humans respond to minority rewards and additional theoretical work to determine the effects of stochastic rewards, agent learning, and finite group dynamics. In conclusion, how best to foster collective intelligence is an important problem that we need to solve collectively.

Materials and Methods

Terminology. Throughout this paper, we use the following conventions for describing probability distributions.

$\mathbb{E}(x)$ denotes the expectation of x.
$\mathcal{N}(x; \mu, \sigma^2)$ denotes the normal probability density function with mean μ and variance σ^2 evaluated at x.
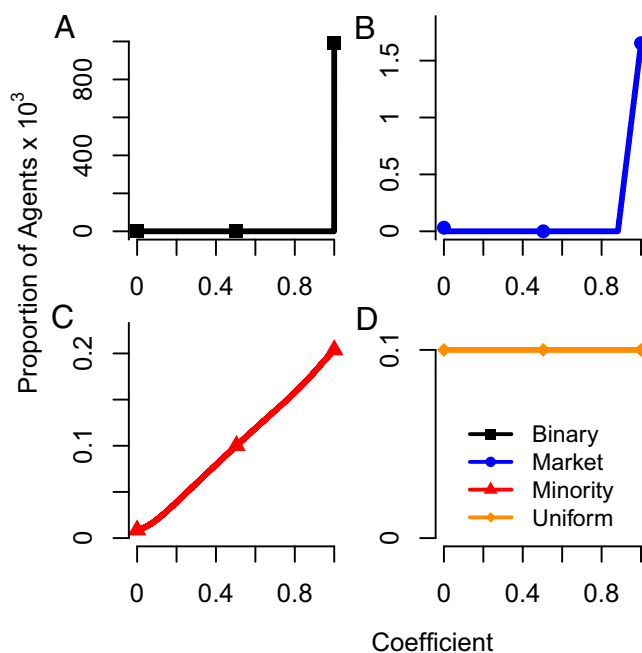

Fig. 3. Equilibrium proportions of agents paying attention to each factor as a function of the coefficient associated with that factor. Results are shown for simulations with $n = 10{,}000$ factors and three reward systems of (A) binary rewards, (B) market rewards, and (C) minority rewards as well as (D) the uniform allocation. Binary rewards drive almost all agents to the single most important factor (the greatest coefficient). Market rewards create a distribution proportional to coefficient size across the most important 10% of factors, whereas minority rewards distribute agents almost perfectly in proportion to the magnitude of the coefficient.

$\mathcal{N}(x; \mu, \Sigma)$ for vector-valued x and μ and matrix Σ denotes the multivariate normal probability density function with mean μ and covariance matrix Σ evaluated at x.

$\Phi(x)$ denotes the standard normal cumulative probability distribution function with mean = 0 and SD = 1.

Ground Truth and Voting. We consider a binary outcome, Y, that is the result of many independent factors, $x_1, x_2, \ldots, x_n$ (correlated factors are in *SI Appendix*). We model this outcome as being determined by the sign of ψ: a weighted sum of the contributing factors

$$Y = \text{sign}(\psi), \ \psi = \sum_{i=1}^{n} \beta_i x_i. \quad [4]$$

In computational implementation of this model, we sample values of $\{\beta\}$ independently from a uniform distribution (the scale of which is arbitrary and does not influence the analysis). We assume without loss of generality that factors are ordered such that $\beta_i \geq \beta_{i+1}$, and furthermore, we normalize the values of the coefficients such that $\sum_{i=1}^{n} \beta_i = 1$ without affecting the value of Y. Our analytical results (*SI Appendix*) do not depend on the exact distribution of $\{\beta\}$. Any sampling distribution for $\{\beta\}$ that has a finite moment of order m, $m > 2$ will obey the Ljapunov and Lindeberg conditions (35), guaranteeing convergence in distribution of ψ to a normal distribution, from which our results are obtained.

Each individual attends to one factor at a given time; an individual attending to factor i, therefore, observes the value of x_i. Having observed the value of x_i, this individual then votes in line with that observation. The collective prediction, $\hat{Y}$, is given by the sign of the collective vote V, which is a sum over the contributing factors weighted by the proportion of individuals attending to each factor:

$$\hat{Y} = \text{sign}(V), \ V = \sum_{i=1}^{n} \rho_i x_i. \quad [5]$$

Evolutionary Dynamics. We model changes in individual attention to factors as being motivated by imitation; agents who are observed to be gaining greater rewards are imitated by those gaining fewer (30), leading to the classic replicator equation (36–38) describing the evolution of p_i, the proportion of agents attending to factor i:

$$\dot{\rho}_i = \rho_i \left(\mathbb{E}(R_i) - \sum_{j=1}^{n} \rho_j \mathbb{E}(R_j) \right), \quad [6]$$

where $\sum_{i=1}^{n} \rho_i = 1$ by definition. The expected reward $[\mathbb{E}(R_i) = 1]$ is the mean reward that an agent attending factor i will receive averaging over all possible values of both x_i and the other factors x_j. It is, thus, determined by both the proportion of times that the agent will vote correctly (when $x_i = Y$) and the magnitude of the reward received on those occasions (determined by the reward system). To calculate this expectation, we either exhaustively enumerate all possibilities (for $n < 10$) or numerically evaluate an approximation considering the normally distributed limiting behavior (see below). When solving these n equations (one for each factor) numerically, we normalize the rewards given to all agents, such that $\sum_{i=1}^{n} \rho_i \mathbb{E}(R_i) = 1$. This normalization is equivalent to adaptive variation of the time step and does not change the relative rewards between options or the final steady state, but it ensures smoother convergence to that state. This normalization also mimics a real constraint on any practical reward system where the total reward available may be fixed. In our model, we assume that agents reliably receive the expected reward for the factor to which they attend. Similar models with stochastic rewards (13) may show slower convergence to equilibrium. In our simulation of the collective dynamics of the system, we used the Runge–Kutta order 2(3) algorithm as implemented in R by Soetaert et al. (39).

The Three Reward Schemes. We present three possible systems for rewarding agents for making accurate predictions. Each reward scheme corresponds to a choice of reward function, $f(z)$, which determines the magnitude of the reward when an agent makes an accurate prediction as a function of the proportion, z, of other agents who also do so. These reward schemes are

i) binary rewards: $f(z) = 1$,
ii) market rewards: $f(z) = 1/z$, and
iii) minority rewards: $f(z) = 1 - H(z - 1/2)$, where H is the Heaviside step function.

The expected reward that an agent receives for attending to factor i is, therefore, the expected value of $f(z_i)$ conditional on his/her vote being accurate:

$$\mathbb{E}(R_i) = \int_{\epsilon}^{1} f(z_i) P(Y = x_i \mid z_i) p(z_i) dz, \quad [7]$$

where z_i is the proportion of agents voting identically to those attending to factor i: $z_i = \sum_{j=1}^{n} \rho_j \delta_{x_i, x_j}$, where δ is the Kronecker delta. The lower limit of the integral above is $\epsilon > 0$ to account for the limiting case of a single individual attending to the factor. As the population size N tends to infinity, ϵ tends to zero. For our implementation, we take $\epsilon = 10^{-6}$.

Normal Approximation for Expected Rewards. For $n \geq 10$, an exhaustive search over all 2^n combinations of $x_1, \ldots x_n$ is computationally infeasible. Instead, we use the Central Limit Theorem to approximate the expected reward received for attending to any given factor. Focusing on a single individual who attends factor i, we can calculate the expected reward received by the individual as follows. We assume without loss of generality by symmetry that the focal individual observes $x_i = 1$. The expected reward, $\mathbb{E}(R_i)$, is then

$$\mathbb{E}(R_i) = \int_{\epsilon}^{1} f(z_i) P(\psi > 0 \mid x_i = 1, z_i) p(z_i) dz. \quad [8]$$

Given the independence of the individual values of x_i, the mean and variance of ψ can be determined by the linearity of expectations and the sum rule for variances of independent variables:

$$\mathbb{E}(\psi \mid x_i = 1) = \beta_i \sum_{j \neq i}^{n} \beta_j \mathbb{E}(x_j) = \beta_i,$$

$$\text{VAR}(\psi \mid x_i = 1) = \sum_{j \neq i}^{n} \beta_j^2 \mathbb{E}(x_j^2) = \sum_{j \neg i}^{n} \beta_j^2$$

$$\Rightarrow p(\psi \mid x_i = 1) \simeq \mathcal{N}\left(\psi; \beta_i, \sum_{j \neq i} \beta_j^2 \right). \quad [9]$$

In the case of binary rewards, where $f(z) = 1$, the value of z_i does not impact on the reward for attending to any factor. In this case, the expected reward is calculated directly from the distribution of ψ:

$$\mathbb{E}_{\text{binary}}(R_i) = P(\psi > 0 \mid x_i = 1)$$
$$= \Phi\left(\frac{\beta_i}{\sum_{j \neq i} \beta_j^2} \right). \quad [10]$$

For other reward schemes where the value of z_i affects the reward, we also require an approximation for $p(z_i)$. Again, we calculate the mean and variance of z_i:

$$\mathbb{E}(z_i \mid x_i = 1) = \rho_i + \sum_{j \neq i} \frac{\rho_j(\mathbb{E}(x_j) + 1)}{2} = \frac{(1 + \rho_i)}{2},$$

$$\text{VAR}(z_i \mid x_i = 1) = \sum_{j \neq i}^{n} \left(\frac{\rho_j}{2} \right)^2 \mathbb{E}(x_j^2) = \frac{1}{4} \sum_{j \neq i} \rho_j^2$$

$$\Rightarrow p(z_i \mid x_i = 1) \simeq \mathcal{N}\left(z; \frac{1 + \rho_i}{2}, \frac{1}{4} \sum_{j \neq i} \rho_j^2 \right). \quad [11]$$

The convergence of z_i in distribution to a normal distribution depends on the values of $\{\rho\}$ meeting the Lindeberg condition (35). In practice, this condition means that all elements of $\{\rho\}$ should tend to zero as the number of dimensions, n, tends to infinity (i.e., the distribution should not be dominated by a small subset of elements). As illustrated in Fig. 1, when the system is initialized in a state conforming to these requirements, it will remain so for market and minority reward systems but will not remain so for the binary reward system. Because the binary reward system does not depend on the value of z_i, the failure of this approximation in this case does not have any repercussions for our results.

Both ψ and z_i are correlated because of the shared dependence on the values of $x_1, \ldots, x_n$ with a covariance of

$$\text{cov}(z_i, \psi \mid x_i = 1) = \frac{1}{2} \sum_{j \neq i} \sum_{k \neq i} \beta_j \rho_k \mathbb{E}(x_j x_k) = \frac{1}{2} \sum_{j \neq i} \beta_j \rho_j. \quad [12]$$

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In the normal distribution limit, the joint distribution may be approximated as

$$p(\psi, z_i \mid x_i = 1) = \mathcal{N}\left(\begin{bmatrix} \psi \\ z_i \end{bmatrix}; \begin{bmatrix} \mu_\psi \\ \mu_z \end{bmatrix}, \begin{bmatrix} K_{\psi, \psi} & K_{\psi, z} \\ K_{\psi, z} & K_{z, z} \end{bmatrix} \right) \qquad [13]$$

with

$$\mu_\psi = \mathbb{E}(\psi \mid x_i = 1)$$
$$\mu_z = \mathbb{E}(z_i \mid x_i = 1)$$
$$K_{\psi, \psi} = \mathrm{VAR}(\psi \mid x_i = 1)$$
$$K_{z, z} = \mathrm{VAR}(z_i \mid x_i = 1)$$
$$K_{\psi, z} = \mathrm{COV}(\psi, z_i \mid x_i = 1).$$

Using standard relations for conditional normal distributions, we, therefore, have

$$p(\psi \mid x_i = 1, z_i) = \mathcal{N}\left(\psi; \mu_\psi + (z_i - \mu_z)\frac{K_{\psi, x}}{K_{z, z}}, K_{\psi, \psi} - \frac{K_{\psi, x}^2}{K_{z, z}} \right)$$

$$\Rightarrow P(\psi > 0 \mid x_i = 1, z) = \Phi\left(\frac{\mu_\psi + (z_i - \mu_z)\frac{K_{\psi, x}}{K_{z, z}}}{K_{\psi, \psi} - \frac{K_{\psi, x}^2}{K_{z, z}}} \right). \qquad [14]$$

Combining the above expressions gives the complete equation for the expected reward of attending to factor i conditioned on the values of β, the current distribution of attention ρ, and the reward function $f(z)$:

$$\mathbb{E}(R_i) = \int_\epsilon^1 f(z_i) \mathcal{N}(z_i; \mu_z, K_{z, z}) \Phi\left(\frac{\mu_\psi + (z_i - \mu_z)\frac{K_{\psi, x}}{K_{z, z}}}{K_{\psi, \psi} - \frac{K_{\psi, x}^2}{K_{z, z}}} \right) dz_i. \qquad [15]$$

This integral may be evaluated numerically to give the expected reward for any general reward modulation function $f(z)$.

Calculating Collective Accuracy. The collective accuracy, C, is the probability that the collective vote will correctly predict the ground truth conditioned on the current distribution of attention to different factors. For small numbers of factors (we use $n < 10$), this probability can be determined exactly by exhaustive search over all 2^n possible combinations of the values of

$x_1, \ldots x_n$. For larger values of n, we use the following normal approximation (similarly defined as above) for the joint distribution of the latent ground truth function ψ and the collective vote V:

$$p(\psi, V) \simeq \mathcal{N}\left(\begin{bmatrix} \psi \\ V \end{bmatrix}; \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} S_{\psi, \psi} & S_{\psi, v} \\ S_{\psi, v} & S_{V, V} \end{bmatrix} \right), \qquad [16]$$

where

$$S_{\psi, \psi} = \sum_{i=1}^n \beta_i^2, \quad S_{V, V} = \frac{1}{4}\sum_{i=1}^n \rho_i^2, \quad S_{\psi, v} = \frac{1}{2}\sum_{i=1}^n \beta_i \rho_i, \qquad [17]$$

implying the following conditional probability distribution for V given ψ:

$$p(V \mid \psi) \simeq \mathcal{N}\left(V; \psi\frac{S_{\psi, v}}{S_{\psi, \psi}}, S_{V, V} - \frac{S_{\psi, v}^2}{S_{\psi, \psi}} \right). \qquad [18]$$

Considering without loss of generality the case where $Y = 1$,

$$C = P(\hat{Y} = 1 \mid Y = 1)$$
$$= P(V > 0 \mid \psi > 0)$$
$$= 2\int_0^\infty \int_0^\infty \mathcal{N}\left(V; \psi\frac{S_{\psi, v}}{S_{\psi, \psi}}, S_{V, V} - \frac{S_{\psi, v}^2}{S_{\psi, \psi}} \right) dV \mathcal{N}(\psi; 0, S_{\psi, \psi}) d\psi$$
$$= 2\int_0^\infty \Phi\left(\frac{\psi\frac{S_{\psi, v}}{S_{\psi, \psi}}}{S_{V, V} - \frac{S_{\psi, v}^2}{S_{\psi, \psi}}} \right) dV \mathcal{N}(\psi; 0, S_{\psi, \psi}) d\psi, \qquad [19]$$

which can be evaluated numerically. The normal approximation limit becomes invalid when the distribution of $\{\rho\}$ is concentrated on very few elements; in these cases (which we identify as 99% of the distribution mass being concentrated on fewer than 10 elements), we use exhaustive search over the values of $\{x\}$ corresponding to the remaining factors with nonnegligible values of ρ.

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