5. Niech $X \sim \chi^2(k)$. Wyznaczyć $\mathrm{E}(X)$ oraz $\mathrm{V}(X)$.

$$f(x) = \frac{x^{\frac{k}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} \quad dla\ x>0$$

$$E(X) = \int_0^\infty x f(x)\,dx = \frac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} \int_0^\infty x^{\frac{k}{2}} e^{-\frac{x}{2}}\,dx = \left| \begin{matrix} t = \frac{x}{2} \\ dt = \frac{1}{2}dx \\ dx = 2dt \\ x = 2t \end{matrix} \right| =$$

$$= \frac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} \int_0^\infty (\cancel{2}t)^{\frac{k}{2}} e^{-t}\, 2dt = \frac{2}{\Gamma\left(\frac{k}{2}\right)} \underbrace{\int_0^\infty t^{\frac{k}{2}} e^{-t}\,dt}_{\Gamma\left(\frac{k}{2}+1\right)} =$$

$$\frac{2\Gamma\left(\frac{k}{2}+1\right)}{\Gamma\left(\frac{k}{2}\right)} = 2\cdot\frac{k}{2} = k$$

$$\Gamma(z+1) = z\Gamma(z)$$

$$E(X^2) = \int_0^\infty x^2 f(x)\,dx = \frac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} \int_0^\infty x^2 \cdot x^{\frac{k}{2}-1} e^{-\frac{x}{2}}\,dx = \left| \begin{matrix} t = \frac{x}{2} \\ dx = 2dt \\ x = 2t \end{matrix} \right| =$$

$$= \frac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} \int_0^\infty (\cancel{2}t)^{\frac{k}{2}+1} e^{-t}\,\cancel{2}dt = \frac{2\cdot 2^{\frac{k}{2}+1}}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} \int_0^\infty t^{\frac{k}{2}+1} e^{-t}\,dt =$$

$$= \frac{2^2\Gamma\left(\frac{k}{2}+2\right)}{\Gamma\left(\frac{k}{2}\right)} = \frac{4\Gamma\left(\frac{k}{2}+1\right)\left(\frac{k}{2}+1\right)}{\Gamma\left(\frac{k}{2}\right)} = \frac{4\cancel{\Gamma\left(\frac{k}{2}\right)}\left(\frac{k}{2}\right)\left(\frac{k}{2}+1\right)}{\cancel{\Gamma\left(\frac{k}{2}\right)}} =$$

$$= k^2 + 2k$$

$$VX = E(X^2) - E(X)^2 = k^2 + 2k - (k)^2 = 2k$$

6. Niech $X \sim \chi^2(k)$, $\alpha > 2$. Znaleźć oszacowania dla $P(X \geqslant k\alpha)$ (Markov, Chebyshev).

Markov $P(X \geq a) \leq \frac{EX}{a}$

$$P(X \geq k\alpha) \leq \frac{EX}{k\alpha} = \frac{k}{k\alpha} = \frac{1}{\alpha} = \frac{1}{2}$$

Chebyshev $P(|X - EX| \geq a) \leq \frac{VX}{a^2}$

$$P(X \geq k\alpha) = P(X - k \geq k(\alpha - 1)) \leq$$

$$\leq P(|X - k| \geq k(\alpha - 1)) \leq \frac{2k}{k^2(\alpha-1)^2} = \frac{2}{k(\alpha-1)^2}$$

7. Niech $X_1 \sim N(2,4)$, $X_2 \sim N(3,9)$, $\mathrm{Cov}(X_1, X_2) = 1$. Niech dalej $Y_1 = X_1 + X_2$, $Y_2 = 2X_1 - X_2$. Obliczyć wartości oczekiwane, wariancje i kowariancję zmiennych Y_1, Y_2.

$EX_1 = 2 \quad VX_1 = 4$

$\mathrm{Cov}(X_1, X_2) = 1$

$EX_2 = 3 \quad VX_2 = 9$

$EY_1 = E(X_1 + X_2) = EX_1 + EX_2 = 5$

$VY_1 = V(X_1 + X_2) = VX_1 + VX_2 + 2\mathrm{Cov}(X_1, X_2) = 15$

$EY_2 = E(2X_1 - X_2) = 2EX_1 - EX_2 = 1$

$VY_2 = V(2X_1 - X_2) = 4VX_1 + VX_2 - 4\mathrm{Cov}(X_1, X_2) = 16 + 9 - 4 = 21$

$\mathrm{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2)$

$E(Y_1 Y_2) = E(2X_1^2 - X_1X_2 + 2X_1X_2 - X_2^2) =$

$= 2E(X_1^2) + E(X_1X_2) - E(X_2^2)$

$E(X_1^2) = VX_1 + (EX_1)^2 = 8$

z wzoru na V

$E(X_1 X_2) = \mathrm{Cov}(X_1, X_2) + E(X_1) \cdot E(X_2) = 7$

$E(X_2^2) = VX_2 + E(X_2)^2 = 9 + 3^2 = 18$

$E(Y_1 Y_2) = 27$

$\mathrm{Cov}(Y_1, Y_2) = 22$

8. Niech $X \sim \mathrm{N}(\mu, \sigma^2)$. Znaleźć oszacowanie Chernoffa dla $\mathrm{P}(X \geqslant \mu + 3\sigma)$.

Chernhoff $P(X \geq a) \leq e^{-at} M_X(t) \quad \forall t > 0$

$$P(X \geq \mu + 3\sigma) \leq e^{-(\mu+3\sigma)t} e^{\mu t} e^{\frac{\sigma^2 t^2}{2}} =$$

$$= e^{-(\mu+3\sigma)t + \mu t + \frac{\sigma^2 t^2}{2}} = f(t)$$

$$f'(t) = \underbrace{e^{-(\mu+3\sigma)t + \mu t + \frac{\sigma^2 t^2}{2}}}_{>0} \cdot (-\mu - 3\sigma + \mu + \sigma^2 t) = 0$$

$$t\sigma^2 - 3\sigma = 0$$

$$t = \frac{3}{\sigma} \qquad f''(t) = e^{\cdots} \cdot (\ldots)^2 > 0$$

$$f\left(\frac{3}{\sigma}\right) = e^{\frac{3}{\sigma}\left(-3\sigma + \frac{\sigma 3}{2}\right)} = e^{-9 + \frac{9}{2}} = e^{-\frac{9}{2}}$$

9. Niech $X \sim \mathrm{N}(\mu, \sigma^2)$. Znaleźć oszacowanie Chernoffa dla $\mathrm{P}(X \leqslant \mu - 3\sigma)$.
Wsk: Dla $t > 0$, $P(X \leqslant a) = P(\exp(-tX) \geqslant \exp(-ta)) = \ldots$

$$P(X \leq \mu - 3\sigma) = P(e^{-tX} \geq e^{-t(\mu-3\sigma)}) \leq e^{-t(\mu-3\sigma)} M(t)$$

$$= e^{-t(\mu-3\sigma)} e^{\mu t} e^{\frac{\sigma^2 t^2}{2}} = f(t)$$

$$f'(t) = e^{-t(\mu-3\sigma)+\mu t+\frac{\sigma^2 t^2}{2}} (-(\mu-3\sigma)+\mu+\sigma^2 t) =$$

$$\sigma^2 t + 3\sigma = 0 \Rightarrow t = -\frac{3}{\sigma}$$

$$f(-\tfrac{3}{\sigma}) = e^{-\frac{3}{\sigma}(3\sigma - \frac{\sigma^3}{2})} = e^{-9+\frac{9}{2}} = e^{-\frac{9}{2}}$$

10. Z nierówności Czebysheva oszacować $P(|X-\mu| \geqslant 3\sigma)$. Porównać z zadaniami 8,9.

$$P(|X-\mu| \geq 3\sigma) \leq \frac{VX}{(3\sigma)^2} = \frac{\sigma^2}{9\sigma^2} = \frac{1}{9} > e^{-\frac{9}{2}}$$