

Physics-Informed Neural Networks: A Deep Learning Framework for Solving Forward and Inverse Problems Involving Nonlinear Partial Differential Equations

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Abstract

We introduce *physics-informed neural networks* – neural networks that are trained to solve supervised learning tasks while respecting any given laws of physics described by general nonlinear partial differential equations. In this work, we present our developments in the context of solving two main classes of problems: data-driven solution and data-driven discovery of partial differential equations. Depending on the nature and arrangement of the available data, we devise two distinct types of algorithms, namely continuous time and discrete time models. The first type of models forms a new family of data-efficient spatio-temporal function approximators, while the latter type allows the use of arbitrarily accurate implicit Runge-Kutta time stepping schemes with unlimited number of stages. The effectiveness of the proposed framework is demonstrated through a collection of classical problems in fluids, quantum mechanics, reaction-diffusion systems, and the propagation of nonlinear shallow-water waves.

Keywords: Data-driven scientific computing, Machine learning, Predictive modeling, Runge-Kutta methods, Nonlinear dynamics

¹ 1. Introduction

² With the explosive growth of available data and computing resources,
³ recent advances in machine learning and data analytics have yielded trans-

4 formative results across diverse scientific disciplines, including image recognition [1], cognitive science [2], and genomics [3]. However, more often than
5 not, in the course of analyzing complex physical, biological or engineering
6 systems, the cost of data acquisition is prohibitive, and we are inevitably
7 faced with the challenge of drawing conclusions and making decisions under
8 partial information. In this *small data* regime, the vast majority of state-
9 of-the-art machine learning techniques (e.g., deep/convolutional/recurrent
10 neural networks) are lacking robustness and fail to provide any guarantees
11 of convergence.

13

14 At first sight, the task of training a deep learning algorithm to accurately
15 identify a nonlinear map from a few – potentially very high-dimensional –
16 input and output data pairs seems at best naive. Coming to our rescue, for
17 many cases pertaining to the modeling of physical and biological systems,
18 there exists a vast amount of prior knowledge that is currently not being uti-
19 lized in modern machine learning practice. Let it be the principled physical
20 laws that govern the time-dependent dynamics of a system, or some empiri-
21 cally validated rules or other domain expertise, this prior information can
22 act as a regularization agent that constrains the space of admissible solutions
23 to a manageable size (e.g., in incompressible fluid dynamics problems by dis-
24 carding any non realistic flow solutions that violate the conservation of mass
25 principle). In return, encoding such structured information into a learning
26 algorithm results in amplifying the information content of the data that the
27 algorithm sees, enabling it to quickly steer itself towards the right solution
28 and generalize well even when only a few training examples are available.

29

30 The first glimpses of promise for exploiting structured prior information to
31 construct data-efficient and physics-informed learning machines have already
32 been showcased in the recent studies of [4–6]. There, the authors employed
33 Gaussian process regression [7] to devise functional representations that are
34 tailored to a given linear operator, and were able to accurately infer solu-
35 tions and provide uncertainty estimates for several prototype problems in
36 mathematical physics. Extensions to nonlinear problems were proposed in
37 subsequent studies by Raissi *et. al.* [8, 9] in the context of both inference and
38 systems identification. Despite the flexibility and mathematical elegance of
39 Gaussian processes in encoding prior information, the treatment of nonlinear
40 problems introduces two important limitations. First, in [8, 9] the authors
41 had to locally linearize any nonlinear terms in time, thus limiting the applica-

space in between single step region can be non-linear

42 ability of the proposed methods to discrete-time domains and compromising
43 the accuracy of their predictions in strongly nonlinear regimes. Secondly,
44 the Bayesian nature of Gaussian process regression requires certain prior as-
45 sumptions that may limit the representation capacity of the model and give
46 rise to robustness/brittleness issues, especially for nonlinear problems [10].

47 **2. Problem setup**

48 In this work we take a different approach by employing deep neural net-
49 works and leverage their well known capability as universal function ap-
50 proximators [11]. In this setting, we can directly tackle nonlinear problems
51 without the need for committing to any prior assumptions, linearization, or
52 local time-stepping. We exploit recent developments in automatic differenti-
53 ation [12] – one of the most useful but perhaps under-utilized techniques in
54 scientific computing – to differentiate neural networks with respect to their
55 input coordinates and model parameters to obtain *physics-informed neural*
56 *networks*. Such neural networks are constrained to respect any symmetries,
57 invariances, or conservation principles originating from the physical laws that
58 govern the observed data, as modeled by general time-dependent and non-
59 linear partial differential equations. This simple yet powerful construction
60 allows us to tackle a wide range of problems in computational science and in-
61 troduces a potentially transformative technology leading to the development
62 of new data-efficient and physics-informed learning machines, new classes of
63 numerical solvers for partial differential equations, as well as new data-driven
64 approaches for model inversion and systems identification.

65
66 The general aim of this work is to set the foundations for a new paradigm
67 in modeling and computation that enriches deep learning with the longstand-
68 ing developments in mathematical physics. To this end, our manuscript is
69 divided into two parts that aim to present our developments in the con-
70 text of two major classes of problems: data-driven solution and data-driven
71 discovery of partial differential equations. All code and data-sets accom-
72 panying this manuscript are available on GitHub at <https://github.com/maziarraissi/PINNs>. Throughout this work we have been using relatively
73 simple deep feed-forward neural networks architectures with hyperbolic tan-
74 gent activation functions and no additional regularization (e.g., L1/L2 penal-
75 ties, dropout, etc.). Each numerical example in the manuscript is accompa-
76 nied with a detailed discussion about the neural network architecture we
77

78 employed as well as details about its training process (e.g. optimizer, learning
 79 rates, etc.). Finally, a comprehensive series of systematic studies that
 80 aims to demonstrate the performance of the proposed methods is provided
 81 in Appendix A and Appendix B.

82

83 In this work, we consider parametrized and nonlinear partial differential
 84 equations of the general form

$$u_t + \mathcal{N}[u; \lambda] = 0, \quad x \in \Omega, \quad t \in [0, T], \quad (1)$$

85 where $u(t, x)$ denotes the latent (hidden) solution, $\mathcal{N}[\cdot; \lambda]$ is a nonlinear op-
 86 erator parametrized by λ , and Ω is a subset of \mathbb{R}^D . This setup encapsulates a
 87 wide range of problems in mathematical physics including conservation laws,
 88 diffusion processes, advection-diffusion-reaction systems, and kinetic equa-
 89 tions. As a motivating example, the one dimensional Burgers equation [13]
 90 corresponds to the case where $\mathcal{N}[u; \lambda] = \lambda_1 u u_x - \lambda_2 u_{xx}$ and $\lambda = (\lambda_1, \lambda_2)$.
 91 Here, the subscripts denote partial differentiation in either time or space.
 92 Given noisy measurements of the system, we are interested in the solution
 93 of two distinct problems. The first problem is that of inference, filtering and
 94 smoothing, or data-driven solutions of partial differential equations [4, 8]
 95 which states: *given fixed model parameters λ what can be said about the*
 96 *unknown hidden state $u(t, x)$ of the system?* The second problem is that of
 97 learning, system identification, or data-driven discovery of partial differential
 98 equations [5, 9, 14] stating: *what are the parameters λ that best describe the*
 99 *observed data?*

100 3. Data-driven solutions of partial differential equations

101 Let us start by concentrating on the problem of computing data-driven so-
 102 lutions to partial differential equations (i.e., the first problem outlined above)
 103 of the general form

$$u_t + \mathcal{N}[u] = 0, \quad x \in \Omega, \quad t \in [0, T], \quad (2)$$

104 where $u(t, x)$ denotes the latent (hidden) solution, $\mathcal{N}[\cdot]$ is a nonlinear differ-
 105 ential operator, and Ω is a subset of \mathbb{R}^D . In sections 3.1 and 3.2, we put
 106 forth two distinct types of algorithms, namely continuous and discrete time
 107 models, and highlight their properties and performance through the lens of
 108 different benchmark problems. In the second part of our study (see section
 109 4), we shift our attention to the problem of data-driven discovery of partial
 110 differential equations [5, 9, 14].

111 3.1. *Continuous Time Models*

112 We define $f(t, x)$ to be given by the left-hand-side of equation (2); i.e.,

$$f := u_t + \mathcal{N}[u], \quad (3)$$

113 and proceed by approximating $u(t, x)$ by a deep neural network. This as-
 114 sumption along with equation (3) result in a *physics-informed neural net-*
 115 *work* $f(t, x)$. This network can be derived by applying the chain rule for
 116 differentiating compositions of functions using automatic differentiation [12],
 117 and has the same parameters as the network representing $u(t, x)$, albeit with
 118 different activation functions due to the action of the differential operator
 119 \mathcal{N} . The shared parameters between the neural networks $u(t, x)$ and $f(t, x)$
 120 can be learned by minimizing the mean squared error loss

$$MSE = MSE_u + MSE_f, \quad (4)$$

121 where

$$MSE_u = \frac{1}{N_u} \sum_{i=1}^{N_u} |u(t_u^i, x_u^i) - u^i|^2,$$

122 and

$$MSE_f = \frac{1}{N_f} \sum_{i=1}^{N_f} |f(t_f^i, x_f^i)|^2.$$

123 Here, $\{t_u^i, x_u^i, u^i\}_{i=1}^{N_u}$ denote the initial and boundary training data on $u(t, x)$
 124 and $\{t_f^i, x_f^i\}_{i=1}^{N_f}$ specify the collocations points for $f(t, x)$. The loss MSE_u
 125 corresponds to the initial and boundary data while MSE_f enforces the struc-
 126 ture imposed by equation (2) at a finite set of collocation points. Although
 127 similar ideas for constraining neural networks using physical laws have been
 128 explored in previous studies [15, 16], here we revisit them using modern
 129 computational tools, and apply them to more challenging dynamic problems
 130 described by time-dependent nonlinear partial differential equations.

131

132 Here, we should underline an important distinction between this line of
 133 work and existing approaches in the literature that elaborate on the use of
 134 machine learning in computational physics. The term *physics-informed ma-*
 135 *chine learning* has been also recently used by Wang *et. al.* [17] in the context
 136 of turbulence modeling. Other examples of machine learning approaches for
 137 predictive modeling of physical systems include [18–29]. All these approaches

138 employ machine learning algorithms like support vector machines, random
139 forests, Gaussian processes, and feed-forward/convolutional/recurrent neural
140 networks merely as *black-box* tools. As described above, the proposed work
141 aims to go one step further by revisiting the construction of “custom” activa-
142 tion and loss functions that are tailored to the underlying differential opera-
143 tor. This allows us to open the black-box by understanding and appreciating
144 the key role played by automatic differentiation within the deep learning field.
145 Automatic differentiation in general, and the back-propagation algorithm in
146 particular, is currently the dominant approach for training deep models by
147 taking their derivatives with respect to the parameters (e.g., weights and
148 biases) of the models. Here, we use the exact same automatic differentiation
149 techniques, employed by the deep learning community, to physics-inform
150 neural networks by taking their derivatives with respect to their input co-
151 ordinates (i.e., space and time) where the physics is described by partial
152 differential equations. We have empirically observed that this structured ap-
153 proach introduces a regularization mechanism that allows us to use relatively
154 simple feed-forward neural network architectures and train them with small
155 amounts of data. The effectiveness of this simple idea may be related to
156 the remarks put forth by Lin, Tegmark and Rolnick [30] and raises many
157 interesting questions to be quantitatively addressed in future research. To
158 this end, the proposed work draws inspiration from the early contributions of
159 Psichogios and Ungar [16], Lagaris *et. al.* [15], as well as the contemporary
160 works of Kondor [31, 32], Hirn [33], and Mallat [34].

161

162 In all cases pertaining to data-driven solution of partial differential equa-
163 tions, the total number of training data N_u is relatively small (a few hundred
164 up to a few thousand points), and we chose to optimize all loss functions using
165 L-BFGS, a quasi-Newton, full-batch gradient-based optimization algorithm
166 [35]. For larger data-sets, such as the data-driven model discovery examples
167 discussed in section 4, a more computationally efficient mini-batch setting can
168 be readily employed using stochastic gradient descent and its modern vari-
169 ants [36, 37]. Despite the fact that there is no theoretical guarantee that this
170 procedure converges to a global minimum, our empirical evidence indicates
171 that, if the given partial differential equation is well-posed and its solution is
172 unique, our method is capable of achieving good prediction accuracy given
173 a sufficiently expressive neural network architecture and a sufficient num-
174 ber of collocation points N_f . This general observation deeply relates to the
175 resulting optimization landscape induced by the mean square error loss of

176 equation 4, and defines an open question for research that is in sync with
 177 recent theoretical developments in deep learning [38, 39]. To this end, we
 178 will test the robustness of the proposed methodology using a series of sys-
 179 tematic sensitivity studies that are provided in Appendix A and Appendix B.

180

181 3.1.1. Example (*Schrödinger Equation*)

182 This example aims to highlight the ability of our method to handle pe-
 183 riodic boundary conditions, complex-valued solutions, as well as different
 184 types of nonlinearities in the governing partial differential equations. The
 185 **one-dimensional nonlinear Schrödinger equation** is a classical field equation
 186 that is used to study quantum mechanical systems, including nonlinear wave
 187 propagation in optical fibers and/or waveguides, Bose-Einstein condensates,
 188 and plasma waves. In optics, the nonlinear term arises from the intensity
 189 dependent index of refraction of a given material. Similarly, the nonlinear
 190 term for Bose-Einstein condensates is a result of the mean-field interactions
 191 of an interacting, N-body system. The **nonlinear Schrödinger equation along**
 192 **with periodic boundary conditions is given by**

divide h into real and complex part: do the same for the whole eqn

$$\begin{aligned} ih_t + 0.5h_{xx} + |h|^2h &= 0, \quad x \in [-5, 5], \quad t \in [0, \pi/2], \\ h(0, x) &= 2 \operatorname{sech}(x), \\ h(t, -5) &= h(t, 5), \\ h_x(t, -5) &= h_x(t, 5), \end{aligned} \tag{5}$$

193 where $h(t, x)$ is the complex-valued solution. Let us define $f(t, x)$ to be given
 194 by

$$f := ih_t + 0.5h_{xx} + |h|^2h,$$

195 and proceed by placing a complex-valued neural network prior on $h(t, x)$.
 196 In fact, if u denotes the real part of h and v is the imaginary part, we
 197 are placing a multi-out neural network prior on $h(t, x) = [u(t, x) \ v(t, x)]$.
 198 This will result in the **complex-valued (multi-output) physic-informed neural**
 199 **network $f(t, x)$** . The shared parameters of the neural networks $h(t, x)$ and
 200 $f(t, x)$ can be learned by minimizing the mean squared error loss

$$MSE = MSE_0 + MSE_b + MSE_f, \tag{6}$$

201 where

$$MSE_0 = \frac{1}{N_0} \sum_{i=1}^{N_0} |h(0, x_0^i) - h_0^i|^2,$$

this condition will be true for both u and v portion of h

202

$$MSE_b = \frac{1}{N_b} \sum_{i=1}^{N_b} (|h^i(t_b^i, -5) - h^i(t_b^i, 5)|^2 + |h_x^i(t_b^i, -5) - h_x^i(t_b^i, 5)|^2),$$

203 and

$$MSE_f = \frac{1}{N_f} \sum_{i=1}^{N_f} |f(t_f^i, x_f^i)|^2.$$

204 Here, $\{x_0^i, h_0^i\}_{i=1}^{N_0}$ denotes the initial data, $\{t_b^i\}_{i=1}^{N_b}$ corresponds to the collocation points on the boundary, and $\{t_f^i, x_f^i\}_{i=1}^{N_f}$ represents the collocation points on $f(t, x)$. Consequently, MSE_0 corresponds to the loss on the initial data, 207 MSE_b enforces the periodic boundary conditions, and MSE_f penalizes the Schrödinger equation not being satisfied on the collocation points.

209

210 In order to assess the accuracy of our method, we have simulated equation
 211 (5) using conventional spectral methods to create a high-resolution data set.
 212 Specifically, starting from an initial state $h(0, x) = 2 \operatorname{sech}(x)$ and assuming
 213 periodic boundary conditions $h(t, -5) = h(t, 5)$ and $h_x(t, -5) = h_x(t, 5)$, we
 214 have integrated equation (5) up to a final time $t = \pi/2$ using the Chebfun
 215 package [40] with a spectral Fourier discretization with 256 modes and a
 216 fourth-order explicit Runge-Kutta temporal integrator with time-step $\Delta t =$
 217 $\pi/2 \cdot 10^{-6}$. Under our data-driven setting, all we observe are measurements
 218 $\{x_0^i, h_0^i\}_{i=1}^{N_0}$ of the latent function $h(t, x)$ at time $t = 0$. In particular, the training
 219 set consists of a total of $N_0 = 50$ data points on $h(0, x)$ randomly parsed
 220 from the full high-resolution data-set, as well as $N_b = 50$ randomly sampled
 221 collocation points $\{t_b^i\}_{i=1}^{N_b}$ for enforcing the periodic boundaries. Moreover,
 222 we have assumed $N_f = 20,000$ randomly sampled collocation points used
 223 to enforce equation (5) inside the solution domain. All randomly sampled
 224 point locations were generated using a space filling Latin Hypercube Sampling
 225 strategy [41].

226  <https://www.youtube.com/watch?v=H0RZ1uezuuw>

227 Here our goal is to infer the entire spatio-temporal solution $h(t, x)$ of the
 228 Schrödinger equation (5). We chose to jointly represent the latent function
 229 $h(t, x) = [u(t, x) \ v(t, x)]$ using a 5-layer deep neural network with 100 neurons per layer and a hyperbolic tangent activation function. In general, the
 231 neural network should be given sufficient approximation capacity in order to
 232 accommodate the anticipated complexity of $u(t, x)$. Although more systematic
 233 procedures such as Bayesian optimization [42] can be employed in order

234 to fine-tune the design of the neural network, in the absence of theoretical
235 error/convergence estimates, the interplay between the neural architecture/-
236 training procedure and the complexity of the underlying differential equation
237 is still poorly understood. One viable path towards assessing the accuracy
238 of the predicted solution could come by adopting a Bayesian approach and
239 monitoring the variance of the predictive posterior distribution, but this goes
240 beyond the scope of the present work and will be investigated in future stud-
ies.
241

242 In this example, our setup aims to highlight the robustness of the pro-
243 posed method with respect to the well known issue of over-fitting. Specifi-
244 cally, the term in MSE_f in equation (6) acts as a regularization mechanism
245 that penalizes solutions that do not satisfy equation (5). Therefore, a key
246 property of *physics-informed neural networks* is that they can be effectively
247 trained using small data sets; a setting often encountered in the study of
248 physical systems for which the cost of data acquisition may be prohibitive.
249 Figure 1 summarizes the results of our experiment. Specifically, the top
250 panel of figure 1 shows the magnitude of the predicted spatio-temporal solu-
251 tion $|h(t, x)| = \sqrt{u^2(t, x) + v^2(t, x)}$, along with the locations of the initial and
252 boundary training data. The resulting prediction error is validated against
253 the test data for this problem, and is measured at $1.97 \cdot 10^{-3}$ in the rela-
254 tive \mathbb{L}_2 -norm. A more detailed assessment of the predicted solution is pre-
255 sented in the bottom panel of Figure 1. In particular, we present a compar-
256 ison between the exact and the predicted solutions at different time instants
257 $t = 0.59, 0.79, 0.98$. Using only a handful of initial data, the *physics-informed*
258 *neural network* can accurately capture the intricate nonlinear behavior of the
259 Schrödinger equation.
260

261 One potential limitation of the continuous time neural network models
262 considered so far stems from the need to use a large number of colloca-
263 tion points N_f in order to enforce physics-informed constraints in the en-
264 tire spatio-temporal domain. Although this poses no significant issues for
265 problems in one or two spatial dimensions, it may introduce a severe bot-
266 tleneck in higher dimensional problems, as the total number of collocation
267 points needed to globally enforce a physics-informed constrain (i.e., in our
268 case a partial differential equation) will increase exponentially. Although
269 this limitation could be addressed to some extend using sparse grid or quasi
270 Monte-Carlo sampling schemes [43, 44], in the next section, we put forth a
271 different approach that circumvents the need for collocation points by in-

solution:
discrete time
modeling

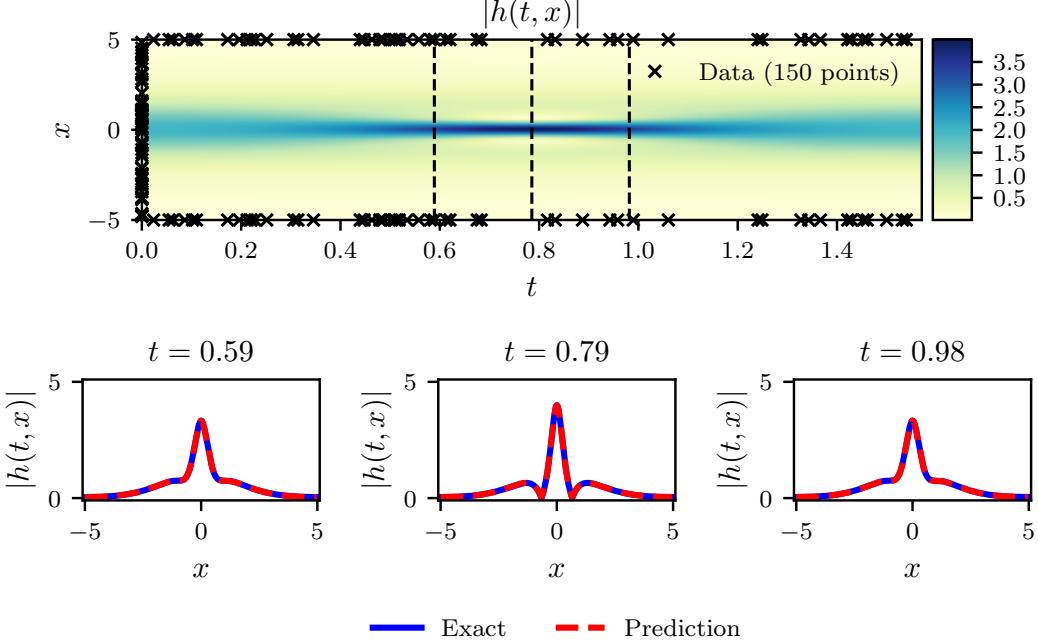


Figure 1: *Shrödinger equation:* Top: Predicted solution $|h(t, x)|$ along with the initial and boundary training data. In addition we are using 20,000 collocation points generated using a Latin Hypercube Sampling strategy. Bottom: Comparison of the predicted and exact solutions corresponding to the three temporal snapshots depicted by the dashed vertical lines in the top panel. The relative \mathbb{L}_2 error for this case is $1.97 \cdot 10^{-3}$.

272 introducing a more structured neural network representation leveraging the
 273 classical Runge-Kutta time-stepping schemes [45].

274 3.2. Discrete Time Models

275 Let us apply the general form of Runge-Kutta methods with q stages [45]
 276 to equation (2) and obtain

$$u^{n+c_i} = u^n - \Delta t \sum_{j=1}^q a_{ij} \mathcal{N}[u^{n+c_j}], \quad i = 1, \dots, q, \quad (7)$$

$$u^{n+1} = u^n - \Delta t \sum_{j=1}^q b_j \mathcal{N}[u^{n+c_j}].$$

C_j should be less than 1, delt time step

277 Here, $u^{n+c_j}(x) = u(t^n + c_j \Delta t, x)$ for $j = 1, \dots, q$. This general form en-
 278 capsulates both implicit and explicit time-stepping schemes, depending on
 279 the choice of the parameters $\{a_{ij}, b_j, c_j\}$. Equations (7) can be equivalently

280 expressed as

$$\begin{aligned} u^n &= u_i^n, \quad i = 1, \dots, q, \\ u^n &= u_{q+1}^n, \end{aligned} \tag{8}$$

281 where

in time

$$\begin{aligned} u_i^n &:= u^{n+c_i} + \Delta t \sum_{j=1}^q a_{ij} \mathcal{N}[u^{n+c_j}], \quad i = 1, \dots, q, \\ u_{q+1}^n &:= u^{n+1} + \Delta t \sum_{j=1}^q b_j \mathcal{N}[u^{n+c_j}]. \end{aligned} \tag{9}$$

282 We proceed by placing a multi-output neural network prior on

$$[u^{n+c_1}(x), \dots, u^{n+c_q}(x), u^{n+1}(x)]. \tag{10}$$

283 This prior assumption along with equations (9) result in a *physics-informed*
 284 *neural network* that takes x as an input and outputs

$$[u_1^n(x), \dots, u_q^n(x), u_{q+1}^n(x)]. \tag{11}$$

285 *3.2.1. Example (Allen-Cahn Equation)*

286 This example aims to highlight the ability of the proposed discrete time
 287 models to handle different types of nonlinearity in the governing partial dif-
 288 ferential equation. To this end, let us consider the *Allen-Cahn equation* along
 289 with periodic boundary conditions

$$\begin{aligned} u_t - 0.0001u_{xx} + 5u^3 - 5u &= 0, \quad x \in [-1, 1], \quad t \in [0, 1], \\ u(0, x) &= x^2 \cos(\pi x), \\ u(t, -1) &= u(t, 1), \\ u_x(t, -1) &= u_x(t, 1). \end{aligned} \tag{12}$$

290 The *Allen-Cahn equation* is a well-known equation from the area of reaction-
 291 diffusion systems. It describes the process of phase separation in multi-
 292 component alloy systems, including order-disorder transitions. For the Allen-
 293 Cahn equation, the nonlinear operator in equation (9) is given by

$$\mathcal{N}[u^{n+c_j}] = -0.0001u_{xx}^{n+c_j} + 5(u^{n+c_j})^3 - 5u^{n+c_j},$$

294 and the shared parameters of the neural networks (10) and (11) can be
 295 learned by minimizing the sum of squared errors

$$SSE = SSE_n + SSE_b, \tag{13}$$

296 where

applying implicit Runge-Kutta method constraint through loss

$$SSE_n = \sum_{j=1}^{q+1} \sum_{i=1}^{N_n} |u_j^n(x^{n,i}) - u^{n,i}|^2,$$

for all stages, for a unique input, the true output is the same

297 and

using neural network multi-outputs directly

$$\begin{aligned} SSE_b = & \sum_{i=1}^q |u^{n+c_i}(-1) - u^{n+c_i}(1)|^2 + |u^{n+1}(-1) - u^{n+1}(1)|^2 \\ & + \sum_{i=1}^q |u_x^{n+c_i}(-1) - u_x^{n+c_i}(1)|^2 + |u_x^{n+1}(-1) - u_x^{n+1}(1)|^2. \end{aligned}$$

298 Here, $\{x^{n,i}, u^{n,i}\}_{i=1}^{N_n}$ corresponds to the data at time-step t^n . In classical nu-
299 matical analysis, these time-steps are usually confined to be small due to sta-
300 bility constraints for explicit schemes or computational complexity constrains
301 for implicit formulations [45]. These constraints become more severe as the
302 total number of Runge-Kutta stages q is increased, and, for most problems
303 of practical interest, one needs to take thousands to millions of such steps
304 until the solution is resolved up to a desired final time. In sharp contrast to
305 classical methods, here we can employ implicit Runge-Kutta schemes with
306 an arbitrarily large number of stages at effectively very little extra cost.¹
307 This enables us to take very large time steps while retaining stability and
308 high predictive accuracy, therefore allowing us to resolve the entire spatio-
309 temporal solution in a single step.

310

311 In this example, we have generated a training and test data-set set by
312 simulating the Allen-Cahn equation (12) using conventional spectral meth-
313 ods. Specifically, starting from an initial condition $u(0, x) = x^2 \cos(\pi x)$ and
314 assuming periodic boundary conditions $u(t, -1) = u(t, 1)$ and $u_x(t, -1) =$
315 $u_x(t, 1)$, we have integrated equation (12) up to a final time $t = 1.0$ using the
316 Chebfun package [40] with a spectral Fourier discretization with 512 modes
317 and a fourth-order explicit Runge-Kutta temporal integrator with time-step
318 $\Delta t = 10^{-5}$.

319

320 Our training data-set consists of $N_n = 200$ initial data points that are
321 randomly sub-sampled from the exact solution at time $t = 0.1$, and our goal

¹To be precise, it is only the number of parameters in the last layer of the neural network that increases linearly with the total number of stages.

322 is to predict the solution at time $t = 0.9$ using a single time-step with size
 323 $\Delta t = 0.8$. To this end, we employ a discrete time *physics-informed neural*
 324 *network* with 4 hidden layers and 200 neurons per layer, while the output
 325 layer predicts 101 quantities of interest corresponding to the $q = 100$ Runge-
 326 Kutta stages $u^{n+c_i}(x)$, $i = 1, \dots, q$, and the solution at final time $u^{n+1}(x)$.
 327 The theoretical error estimates for this scheme predict a temporal error ac-
 328 cumulation of $\mathcal{O}(\Delta t^{2q})$ [45], which in our case translates into an error way
 329 below machine precision, i.e., $\Delta t^{2q} = 0.8^{200} \approx 10^{-20}$. To our knowledge, this
 330 is the first time that an implicit Runge-Kutta scheme of that high-order has
 331 ever been used. Remarkably, starting from smooth initial data at $t = 0.1$ we
 332 can predict the nearly discontinuous solution at $t = 0.9$ in a single time-step
 333 with a relative \mathbb{L}_2 error of $6.99 \cdot 10^{-3}$, as illustrated in Figure 2. This error is
 334 entirely attributed to the neural network's capacity to approximate $u(t, x)$,
 335 as well as to the degree that the sum of squared errors loss allows interpola-
 336 tion of the training data.

337

338 The key parameters controlling the performance of our discrete time al-
 339 gorithm are the total number of Runge-Kutta stages q and the time-step
 340 size Δt . As we demonstrate in the systematic studies provided in Appendix
 341 A and Appendix B, low-order methods, such as the case $q = 1$ correspond-
 342 ing to the classical trapezoidal rule, and the case $q = 2$ corresponding to the
 343 4th-order Gauss-Legendre method, cannot retain their predictive accuracy for
 344 large time-steps, thus mandating a solution strategy with multiple time-steps
 345 of small size. On the other hand, the ability to push the number of Runge-
 346 Kutta stages to 32 and even higher allows us to take very large time steps,
 347 and effectively resolve the solution in a single step without sacrificing the
 348 accuracy of our predictions. Moreover, numerical stability is not sacrificed
 349 either as implicit Gauss-Legendre is the only family of time-stepping schemes
 350 that remain A-stable regardless of their order, thus making them ideal for
 351 stiff problems [45]. These properties are unprecedented for an algorithm of
 352 such implementation simplicity, and illustrate one of the key highlights of
 353 our discrete time approach.

Inverse Problems

354 4. Data-driven discovery of partial differential equations

355 In the current part of our study, we shift our attention to the problem of
 356 data-driven discovery of partial differential equations [5, 9, 14]. In sections 4.1
 357 and 4.2, we put forth two distinct types of algorithms, namely continuous

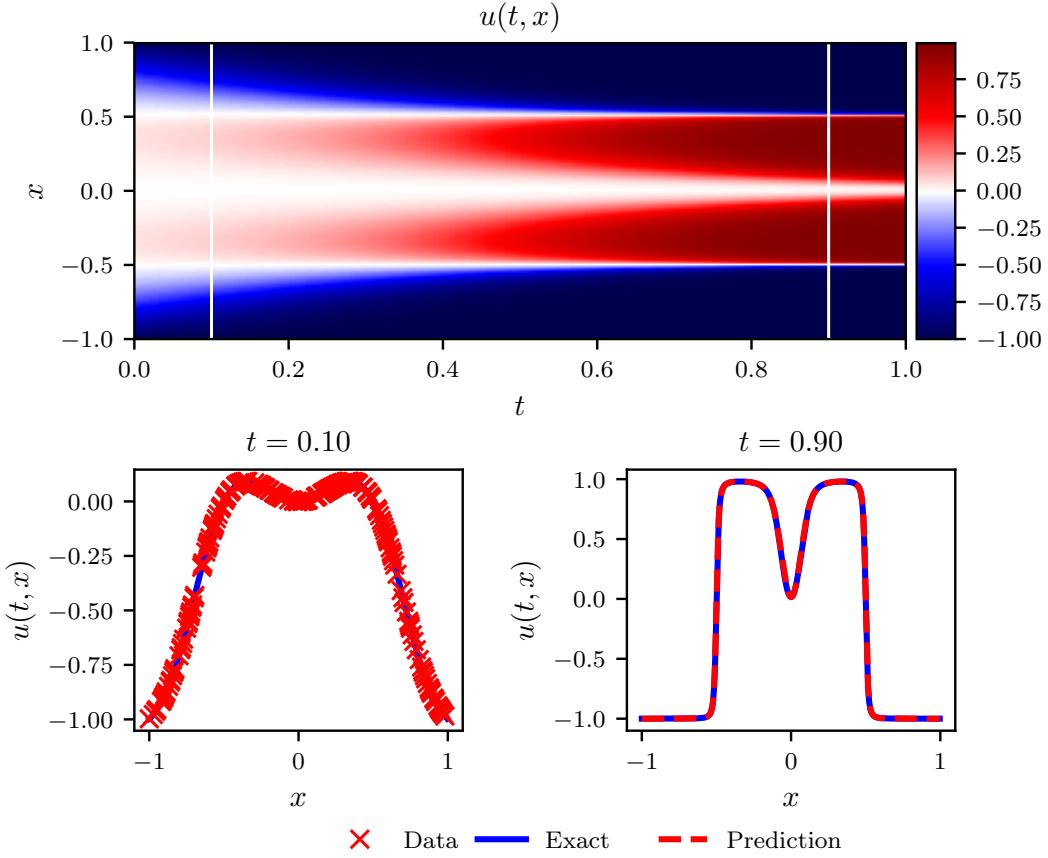


Figure 2: *Allen-Cahn equation:* Top: Solution $u(t, x)$ along with the location of the initial training snapshot at $t = 0.1$ and the final prediction snapshot at $t = 0.9$. Bottom: Initial training data and final prediction at the snapshots depicted by the white vertical lines in the top panel. The relative \mathbb{L}_2 error for this case is $6.99 \cdot 10^{-3}$.

and discrete time models, and highlight their properties and performance through the lens of various canonical problems.

4.1. Continuous Time Models

Let us recall equation (1) and similar to section 3.1 define $f(t, x)$ to be given by the left-hand-side of equation (1); i.e.,

$$f := u_t + \mathcal{N}[u; \lambda]. \quad (14)$$

We proceed by approximating $u(t, x)$ by a deep neural network. This assumption along with equation (14) result in a *physics-informed neural net*

Question:

In forward problem, we only need boundary points and unsupervised collocation points. But here, we see the use of a lot of supervised points. What would happen if we use a similar setting as the forward problem to estimate the eqn??

How come this method works without the need of any boundary conditions??

365 work $f(t, x)$. This network can be derived by applying the chain rule for
 366 differentiating compositions of functions using automatic differentiation [12].
 367 It is worth highlighting that the parameters of the differential operator λ
 368 turn into parameters of the *physics-informed neural network* $f(t, x)$.

369 4.1.1. Example (Navier-Stokes Equation)

370 Our next example involves a realistic scenario of incompressible fluid flow
 371 described by the ubiquitous Navier-Stokes equations. Navier-Stokes equa-
 372 tions describe the physics of many phenomena of scientific and engineering
 373 interest. They may be used to model the weather, ocean currents, water flow
 374 in a pipe and air flow around a wing. The Navier-Stokes equations in their
 375 full and simplified forms help with the design of aircrafts and cars, the study
 376 of blood flow, the design of power stations, the analysis of the dispersion of
 377 pollutants, and many other applications. Let us consider the Navier-Stokes
 378 equations in two dimensions² (2D) given explicitly by

$$\begin{aligned} u_t + \lambda_1(uu_x + vu_y) &= -p_x + \lambda_2(u_{xx} + u_{yy}), \\ v_t + \lambda_1(uv_x + vv_y) &= -p_y + \lambda_2(v_{xx} + v_{yy}), \end{aligned} \quad (15)$$

379 where $u(t, x, y)$ denotes the x -component of the velocity field, $v(t, x, y)$ the
 380 y -component, and $p(t, x, y)$ the pressure. Here, $\lambda = (\lambda_1, \lambda_2)$ are the unknown
 381 parameters. Solutions to the Navier-Stokes equations are searched in the set
 382 of divergence-free functions; i.e.,

$$u_x + v_y = 0. \quad (16)$$

383 This extra equation is the continuity equation for incompressible fluids that
 384 describes the conservation of mass of the fluid. We make the assumption
 385 that

$$u = \psi_y, \quad v = -\psi_x, \quad (17)$$

386 for some latent function $\psi(t, x, y)$.³ Under this assumption, the continuity
 387 equation (16) will be automatically satisfied. Given noisy measurements

$$\{t^i, x^i, y^i, u^i, v^i\}_{i=1}^N$$

²It is straightforward to generalize the proposed framework to the Navier-Stokes equa-
 tions in three dimensions (3D).

³This construction can be generalized to three dimensional problems by employing the
 notion of vector potentials.

388 of the velocity field, we are interested in learning the parameters λ as well as
 389 the pressure $p(t, x, y)$. We define $f(t, x, y)$ and $g(t, x, y)$ to be given by

$$\begin{aligned} f &:= u_t + \lambda_1(uu_x + vu_y) + p_x - \lambda_2(u_{xx} + u_{yy}), \\ g &:= v_t + \lambda_1(uv_x + vv_y) + p_y - \lambda_2(v_{xx} + v_{yy}), \end{aligned} \quad (18)$$

390 and proceed by jointly approximating $[\psi(t, x, y) \ p(t, x, y)]$ using a single
 391 neural network with two outputs. This prior assumption along with equations
 392 (17) and (18) results into a *physics-informed neural network* $[f(t, x, y) \ g(t, x, y)]$.
 393 The parameters λ of the Navier-Stokes operator as well as the parameters of
 394 the neural networks $[\psi(t, x, y) \ p(t, x, y)]$ and $[f(t, x, y) \ g(t, x, y)]$ can be
 395 trained by minimizing the mean squared error loss

$$\begin{aligned} MSE &:= \frac{1}{N} \sum_{i=1}^N \left(|u(t^i, x^i, y^i) - u^i|^2 + |v(t^i, x^i, y^i) - v^i|^2 \right) && \text{this first term corresponds to given measurements} \\ &+ \frac{1}{N} \sum_{i=1}^N \left(|f(t^i, x^i, y^i)|^2 + |g(t^i, x^i, y^i)|^2 \right). && \text{the given measurements also have to satisfy the PDE constraints, this term is for that} \end{aligned} \quad (19)$$

396 Here we consider the prototype problem of incompressible flow past a circular
 397 cylinder; a problem known to exhibit rich dynamic behavior and transitions
 398 for different regimes of the Reynolds number $Re = u_\infty D / \nu$. Assuming a
 399 non-dimensional free stream velocity $u_\infty = 1$, cylinder diameter $D = 1$, and
 400 kinematic viscosity $\nu = 0.01$, the system exhibits a periodic steady state
 401 behavior characterized by a asymmetrical vortex shedding pattern in the
 402 cylinder wake, known as the Kármán vortex street [46].

403

404 To generate a high-resolution data set for this problem we have employed
 405 the spectral/ hp -element solver NekTar [47]. Specifically, the solution domain
 406 is discretized in space by a tessellation consisting of 412 triangular elements,
 407 and within each element the solution is approximated as a linear combination
 408 of a tenth-order hierarchical, semi-orthogonal Jacobi polynomial expansion
 409 [47]. We have assumed a uniform free stream velocity profile imposed at the
 410 left boundary, a zero pressure outflow condition imposed at the right bound-
 411 ary located 25 diameters downstream of the cylinder, and periodicity for the
 412 top and bottom boundaries of the $[-15, 25] \times [-8, 8]$ domain. We integrate
 413 equation (15) using a third-order stiffly stable scheme [47] until the system
 414 reaches a periodic steady state, as depicted in figure 3(a). In what follows,
 415 a small portion of the resulting data-set corresponding to this steady state

416 solution will be used for model training, while the remaining data will be
417 used to validate our predictions. For simplicity, we have chosen to confine
418 our sampling in a rectangular region downstream of cylinder as shown in
419 figure 3(a).

420

421 Given scattered and potentially noisy data on the stream-wise $u(t, x, y)$
422 and transverse $v(t, x, y)$ velocity components, our goal is to identify the un-
423 known parameters λ_1 and λ_2 , as well as to obtain a qualitatively accurate
424 reconstruction of the entire pressure field $p(t, x, y)$ in the cylinder wake, which
425 by definition can only be identified up to a constant. To this end, we have
426 created a training data-set by randomly sub-sampling the full high-resolution
427 data-set. To highlight the ability of our method to learn from scattered and
428 scarce training data, we have chosen $N = 5,000$, corresponding to a mere
429 1% of the total available data as illustrated in figure 3(b). Also plotted are
430 representative snapshots of the predicted velocity components $u(t, x, y)$ and
431 $v(t, x, y)$ after the model was trained. The neural network architecture used
432 here consists of 9 layers with 20 neurons in each layer.

433

434 A summary of our results for this example is presented in figure 4. We
435 observe that the *physics-informed neural network* is able to correctly identify
436 the unknown parameters λ_1 and λ_2 with very high accuracy even when the
437 training data was corrupted with noise. Specifically, for the case of noise-
438 free training data, the error in estimating λ_1 and λ_2 is 0.078%, and 4.67%,
439 respectively. The predictions remain robust even when the training data are
440 corrupted with 1% uncorrelated Gaussian noise, returning an error of 0.17%,
441 and 5.70%, for λ_1 and λ_2 , respectively.

442

443 A more intriguing result stems from the network's ability to provide a
444 qualitatively accurate prediction of the entire pressure field $p(t, x, y)$ in the
445 absence of any training data on the pressure itself. A visual comparison
446 against the exact pressure solution is presented in figure 4 for a represen-
447 tative pressure snapshot. Notice that the difference in magnitude between
448 the exact and the predicted pressure is justified by the very nature of the
449 *incompressible* Navier-Stokes system, as the pressure field is only identifiable
450 up to a constant. This result of inferring a continuous quantity of interest
451 from auxiliary measurements by leveraging the underlying physics is a great
452 example of the enhanced capabilities that *physics-informed neural networks*
453 have to offer, and highlights their potential in solving high-dimensional in-

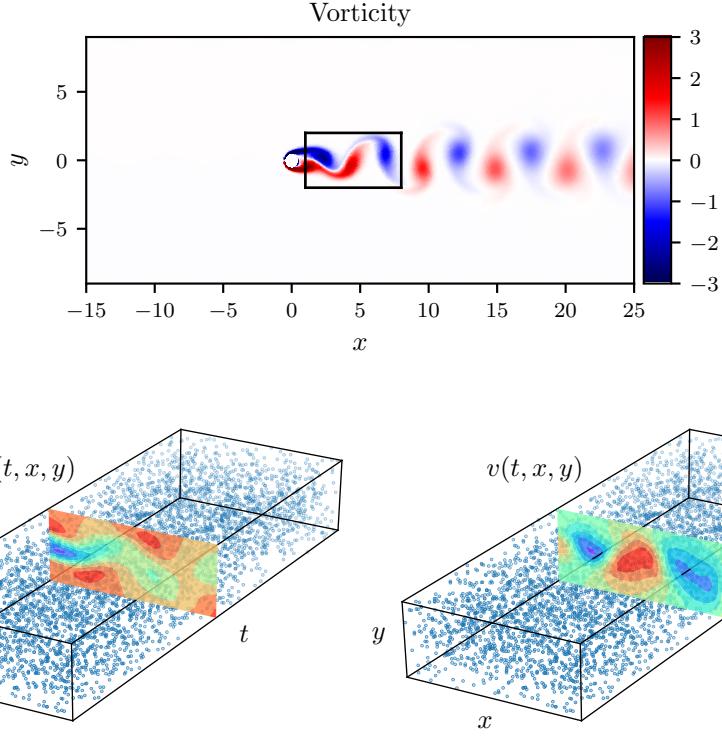


Figure 3: *Navier-Stokes equation:* *Top:* Incompressible flow and dynamic vortex shedding past a circular cylinder at $Re = 100$. The spatio-temporal training data correspond to the depicted rectangular region in the cylinder wake. *Bottom:* Locations of training data-points for the stream-wise and transverse velocity components, $u(t, x, y)$ and $v(t, x, t)$, respectively.

454 verse problems.

455

456 Our approach so far assumes availability of scattered data throughout the
 457 entire spatio-temporal domain. However, in many cases of practical interest,
 458 one may only be able to observe the system at distinct time instants. In the
 459 next section, we introduce a different approach that tackles the data-driven
 460 discovery problem using only two data snapshots. We will see how, by lever-
 461 aging the classical Runge-Kutta time-stepping schemes, one can construct
 462 discrete time *physics-informed neural networks* that can retain high predic-
 463 tive accuracy even when the temporal gap between the data snapshots is
 464 very large.

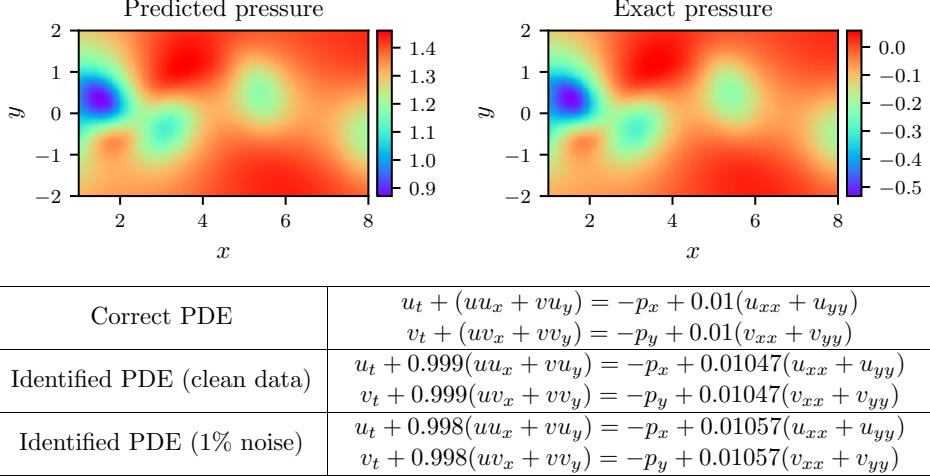


Figure 4: *Navier-Stokes equation*: Top: Predicted versus exact instantaneous pressure field $p(t, x, y)$ at a representative time instant. By definition, the pressure can be recovered up to a constant, hence justifying the different magnitude between the two plots. This remarkable qualitative agreement highlights the ability of *physics-informed neural networks* to identify the entire pressure field, despite the fact that no data on the pressure are used during model training. Bottom: Correct partial differential equation along with the identified one obtained by learning λ_1, λ_2 and $p(t, x, y)$.

limitation: can only approximate the unk parameters (lambdas), but can't estimate latent variables (such as p in the prev example)

465 4.2. Discrete Time Models

466 We begin by applying the general form of Runge-Kutta methods [45] with
467 q stages to equation (1) and obtain

$$\begin{aligned} u^{n+c_i} &= u^n - \Delta t \sum_{j=1}^q a_{ij} \mathcal{N}[u^{n+c_j}; \lambda], \quad i = 1, \dots, q, \\ u^{n+1} &= u^n - \Delta t \sum_{j=1}^q b_j \mathcal{N}[u^{n+c_j}; \lambda]. \end{aligned} \quad (20)$$

468 Here, $u^{n+c_j}(x) = u(t^n + c_j \Delta t, x)$ is the hidden state of the system at time
469 $t^n + c_j \Delta t$ for $j = 1, \dots, q$. This general form encapsulates both implicit and
470 explicit time-stepping schemes, depending on the choice of the parameters
471 $\{a_{ij}, b_j, c_j\}$. Equations (20) can be equivalently expressed as

$$\begin{aligned} u^n &= u_i^n, \quad i = 1, \dots, q, \\ u^{n+1} &= u_i^{n+1}, \quad i = 1, \dots, q. \end{aligned} \quad (21)$$

⁴⁷² where

$$u_i^n := u^{n+c_i} + \Delta t \sum_{j=1}^q a_{ij} \mathcal{N}[u^{n+c_j}; \lambda], \quad i = 1, \dots, q, \quad (22)$$

^{eqn 2 - 1 of (20) ->} $u_i^{n+1} := u^{n+c_i} + \Delta t \sum_{j=1}^q (a_{ij} - b_j) \mathcal{N}[u^{n+c_j}; \lambda], \quad i = 1, \dots, q.$

⁴⁷³ We proceed by placing a multi-output neural network prior on

$$[u^{n+c_1}(x), \dots, u^{n+c_q}(x)]. \quad (23)$$

⁴⁷⁴ This prior assumption along with equations (22) result in two *physics-informed*
⁴⁷⁵ *neural networks*

$$[u_1^n(x), \dots, u_q^n(x), u_{q+1}^n(x)], \quad (24)$$

⁴⁷⁶ and

$$[u_1^{n+1}(x), \dots, u_q^{n+1}(x), u_{q+1}^{n+1}(x)]. \quad (25)$$

⁴⁷⁷ Given noisy measurements at two distinct temporal snapshots $\{\mathbf{x}^n, \mathbf{u}^n\}$ and
⁴⁷⁸ $\{\mathbf{x}^{n+1}, \mathbf{u}^{n+1}\}$ of the system at times t^n and t^{n+1} , respectively, the shared
⁴⁷⁹ parameters of the neural networks (23), (24), and (25) along with the ⁴⁸⁰ pa-
⁴⁸¹ rameters λ of the differential operator can be trained by minimizing the sum of squared errors

$$SSE = SSE_n + SSE_{n+1}, \quad (26)$$

⁴⁸² where

time step n and (n+1) which has delta t difference

$$SSE_n := \sum_{j=1}^q \sum_{i=1}^{N_n} |u_j^n(x^{n,i}) - u^{n,i}|^2,$$

Question:

Why are we considering all u's to have the same ground truth output for same n irrespective of difference in stage no.??

⁴⁸³ and

j: stage, i: training sample

$$SSE_{n+1} := \sum_{j=1}^q \sum_{i=1}^{N_{n+1}} |u_j^{n+1}(x^{n+1,i}) - u^{n+1,i}|^2.$$

⁴⁸⁴ Here, $\mathbf{x}^n = \{x^{n,i}\}_{i=1}^{N_n}$, $\mathbf{u}^n = \{u^{n,i}\}_{i=1}^{N_n}$, $\mathbf{x}^{n+1} = \{x^{n+1,i}\}_{i=1}^{N_{n+1}}$, and $\mathbf{u}^{n+1} = \{u^{n+1,i}\}_{i=1}^{N_{n+1}}$.

⁴⁸⁶ 4.2.1. Example (Kortewegde Vries Equation)

⁴⁸⁷ Our final example aims to highlight the ability of the proposed frame-
⁴⁸⁸ work to handle governing partial differential equations involving higher or-
⁴⁸⁹ der derivatives. Here, we consider a mathematical model of waves on shallow
⁴⁹⁰ water surfaces; the Korteweg-de Vries (KdV) equation. This equation can
⁴⁹¹ also be viewed as Burgers equation with an added dispersive term. The KdV

492 equation has several connections to physical problems. It describes the evolution
 493 of long one-dimensional waves in many physical settings. Such physical
 494 settings include shallow-water waves with weakly non-linear restoring forces,
 495 long internal waves in a density-stratified ocean, ion acoustic waves in a
 496 plasma, and acoustic waves on a crystal lattice. Moreover, the KdV equa-
 497 tion is the governing equation of the string in the Fermi-Pasta-Ulam problem
 498 [48] in the continuum limit. The KdV equation reads as

$$u_t + \lambda_1 u u_x + \lambda_2 u_{xxx} = 0, \quad (27)$$

499 with (λ_1, λ_2) being the unknown parameters. For the KdV equation, the
 500 nonlinear operator in equations (22) is given by

$$\mathcal{N}[u^{n+c_j}] = \lambda_1 u^{n+c_j} u_x^{n+c_j} - \lambda_2 u_{xxx}^{n+c_j}$$

501 and the shared parameters of the neural networks (23), (24), and (25) along
 502 with the parameters $\lambda = (\lambda_1, \lambda_2)$ of the KdV equation can be learned by
 503 minimizing the sum of squared errors (26).

504 To obtain a set of training and test data we simulated (27) using con-
 505 ventional spectral methods. Specifically, starting from an initial condition
 506 $u(0, x) = \cos(\pi x)$ and assuming periodic boundary conditions, we have inte-
 507 grated equation (27) up to a final time $t = 1.0$ using the Chebfun package
 508 [40] with a spectral Fourier discretization with 512 modes and a fourth-order
 509 explicit Runge-Kutta temporal integrator with time-step $\Delta t = 10^{-6}$. Using
 510 this data-set, we then extract two solution snapshots at time $t^n = 0.2$ and
 511 $t^{n+1} = 0.8$, and randomly sub-sample them using $N_n = 199$ and $N_{n+1} = 201$
 512 to generate a training data-set. We then use these data to train a discrete
 513 time *physics-informed neural network* by minimizing the sum of squared error
 514 loss of equation (26) using L-BFGS [35]. The network architecture used here
 515 comprises of 4 hidden layers, 50 neurons per layer, and an output layer pre-
 516 dicting the solution at the q Runge-Kutta stages, i.e., $u^{n+c_j}(x)$, $j = 1, \dots, q$,
 517 where q is empirically chosen to yield a temporal error accumulation of the
 518 order of machine precision ϵ by setting⁴

$$q = 0.5 \log \epsilon / \log(\Delta t), \quad (28)$$

⁴This is motivated by the theoretical error estimates for implicit Runge-Kutta schemes suggesting a truncation error of $\mathcal{O}(\Delta t^{2q})$ [45].

520 where the time-step for this example is $\Delta t = 0.6$.

521

522 The results of this experiment are summarized in figure 5. In the top
523 panel, we present the exact solution $u(t, x)$, along with the locations of the
524 two data snapshots used for training. A more detailed overview of the exact
525 solution and the training data is given in the middle panel. It is worth noticing
526 how the complex nonlinear dynamics of equation (27) causes dramatic
527 differences in the form of the solution between the two reported snapshots.
528 Despite these differences, and the large temporal gap between the two training
529 snapshots, our method is able to correctly identify the unknown parameters
530 regardless of whether the training data is corrupted with noise or not.
531 Specifically, for the case of noise-free training data, the error in estimating
532 λ_1 and λ_2 is 0.023%, and 0.006%, respectively, while the case with 1% noise
533 in the training data returns errors of 0.057%, and 0.017%, respectively.

534 **5. Conclusions**

535 We have introduced *physics-informed neural networks*, a new class of
536 universal function approximators that is capable of encoding any underlying
537 physical laws that govern a given data-set, and can be described by partial
538 differential equations. In this work, we design data-driven algorithms for
539 inferring solutions to general nonlinear partial differential equations, and con-
540 structing computationally efficient physics-informed surrogate models. The
541 resulting methods showcase a series of promising results for a diverse collec-
542 tion of problems in computational science, and open the path for endowing
543 deep learning with the powerful capacity of mathematical physics to model
544 the world around us. As deep learning technology is continuing to grow
545 rapidly both in terms of methodological and algorithmic developments, we
546 believe that this is a timely contribution that can benefit practitioners across
547 a wide range of scientific domains. Specific applications that can readily en-
548 joy these benefits include, but are not limited to, data-driven forecasting of
549 physical processes, model predictive control, multi-physics/multi-scale mod-
550 eling and simulation.

551

552 We must note however that the proposed methods should not be viewed
553 as replacements of classical numerical methods for solving partial differential
554 equations (e.g., finite elements, spectral methods, etc.). Such methods have
555 matured over the last 50 years and, in many cases, meet the robustness and

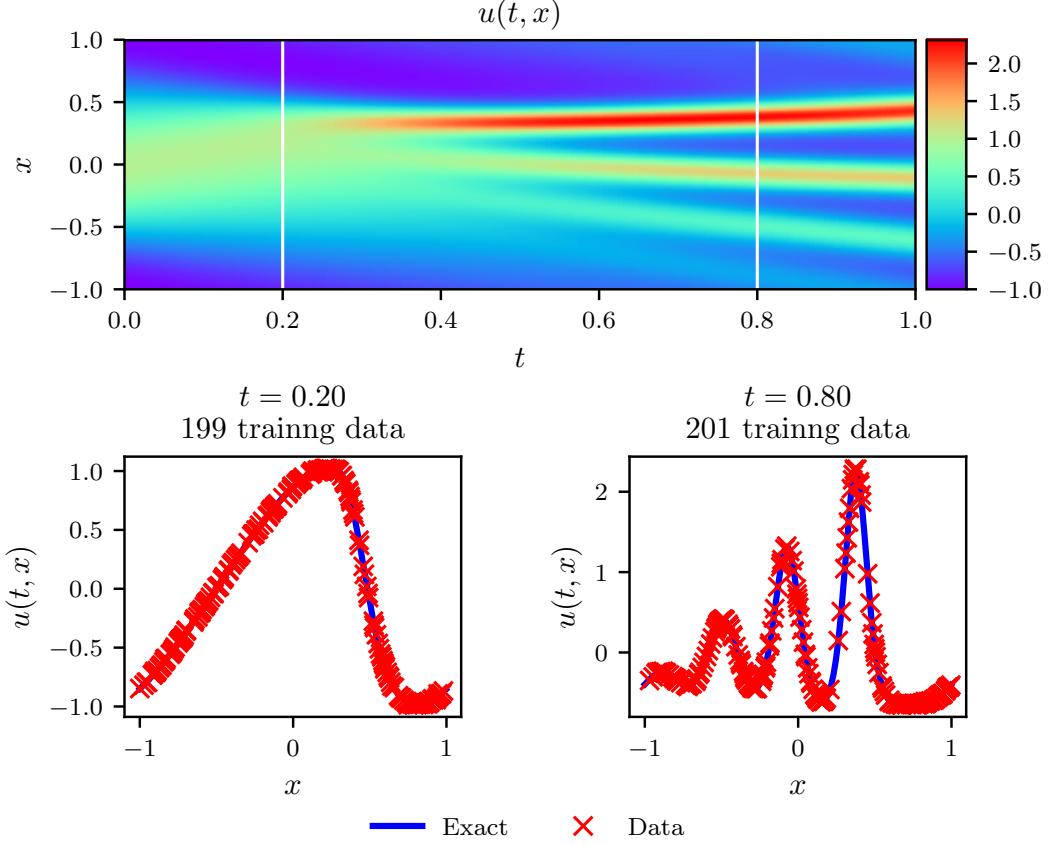


Figure 5: *KdV equation:* *Top:* Solution $u(t, x)$ along with the temporal locations of the two training snapshots. *Middle:* Training data and exact solution corresponding to the two temporal snapshots depicted by the dashed vertical lines in the top panel. *Bottom:* Correct partial differential equation along with the identified one obtained by learning λ_1, λ_2 .

computational efficiency standards required in practice. Our message here, as advocated in Section 3.2, is that classical methods such as the Runge-Kutta time-stepping schemes can coexist in harmony with deep neural networks, and offer invaluable intuition in constructing structured predictive

560 algorithms. Moreover, the implementation simplicity of the latter greatly
561 favors rapid development and testing of new ideas, potentially opening the
562 path for a new era in data-driven scientific computing.

563

564 Although a series of promising results was presented, the reader may per-
565 haps agree this work creates more questions than it answers. How deep/wide
566 should the neural network be? How much data is really needed? Why does
567 the algorithm converge to unique values for the parameters of the differen-
568 tial operators, i.e., why is the algorithm not suffering from local optima for
569 the parameters of the differential operator? Does the network suffer from
570 vanishing gradients for deeper architectures and higher order differential op-
571 erators? Could this be mitigated by using different activation functions?
572 Can we improve on initializing the network weights or normalizing the data?
573 Are the mean square error and the sum of squared errors the appropriate
574 loss functions? Why are these methods seemingly so robust to noise in the
575 data? How can we quantify the uncertainty associated with our predictions?
576 Throughout this work, we have attempted to answer some of these questions,
577 but we have observed that specific settings that yielded impressive results for
578 one equation could fail for another. Admittedly, more work is needed collec-
579 tively to set the foundations in this field.

580

581 In a broader context, and along the way of seeking answers to those
582 questions, we believe that this work advocates a fruitful synergy between
583 machine learning and classical computational physics that has the potential
584 to enrich both fields and lead to high-impact developments.

585 **Acknowledgements**

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588 **Appendix A. Data-driven solution of partial differential equations**

589 This Appendix accompanies the main manuscript, and contains a series
590 of systematic studies that aim to demonstrate the performance of the pro-
591 posed algorithms for problems pertaining to *data-driven solution of partial*
592 *differential equations*. Throughout this document, we will use the Burgers'
593 equation as a canonical example.

594 *Appendix A.1. Continuous Time Models*

595 In one space dimension, the Burger's equation along with Dirichlet bound-
 596 ary conditions reads as

$$\begin{aligned} u_t + uu_x - (0.01/\pi)u_{xx} &= 0, \quad x \in [-1, 1], \quad t \in [0, 1], \\ u(0, x) &= -\sin(\pi x), \\ u(t, -1) &= u(t, 1) = 0. \end{aligned} \quad (\text{A.1})$$

597 Let us define $f(t, x)$ to be given by

$$f := u_t + uu_x - (0.01/\pi)u_{xx},$$

598 and proceed by approximating $u(t, x)$ by a deep neural network. To highlight
 599 the simplicity in implementing this idea we have included a Python code
 600 snippet using Tensorflow [49]; currently one of the most popular and well
 601 documented open source libraries for machine learning computations. To
 602 this end, $u(t, x)$ can be simply defined as

```
603 def u(t, x):  
604     u = neural_net(tf.concat([t, x], 1), weights, biases)  
605     return u
```

606 Correspondingly, the *physics-informed neural network* $f(t, x)$ takes the form

```
607 def f(t, x):  
608     u = u(t, x)  
609     u_t = tf.gradients(u, t)[0]  
610     u_x = tf.gradients(u, x)[0]  
611     u_xx = tf.gradients(u_x, x)[0]  
612     f = u_t + u*u_x - (0.01/tf.pi)*u_xx  
613     return f
```

614 The shared parameters between the neural networks $u(t, x)$ and $f(t, x)$ can
 615 be learned by minimizing the mean squared error loss

$$MSE = MSE_u + MSE_f, \quad (\text{A.2})$$

616 where

$$MSE_u = \frac{1}{N_u} \sum_{i=1}^{N_u} |u(t_u^i, x_u^i) - u^i|^2,$$

617 and

$$MSE_f = \frac{1}{N_f} \sum_{i=1}^{N_f} |f(t_f^i, x_f^i)|^2.$$

618 Here, $\{t_u^i, x_u^i, u^i\}_{i=1}^{N_u}$ denote the initial and boundary training data on $u(t, x)$
619 and $\{t_f^i, x_f^i\}_{i=1}^{N_f}$ specify the collocations points for $f(t, x)$. The loss MSE_u
620 corresponds to the initial and boundary data while MSE_f enforces the struc-
621 ture imposed by equation (A.1) at a finite set of collocation points. Although
622 similar ideas for constraining neural networks using physical laws have been
623 explored in previous studies [15, 16], here we revisit them using modern
624 computational tools, and apply them to more challenging dynamic problems
625 described by time-dependent nonlinear partial differential equations.

626
627 The Burgers equation is often considered as a prototype example of a
628 hyperbolic conservation law (as $\nu \rightarrow 0$). Notice that if we want to “fabri-
629 cate” an “exact” solution to this equation we would select a solution $u(t, x)$
630 (e.g., $e^{-t} \sin(\pi x)$) and obtain the corresponding right hand side $f(t, x)$ by
631 differentiation. The resulting $u(t, x)$ and $f(t, x)$ are “guaranteed” to satisfy
632 the Burgers equation and conserve all associated invariances by construction.
633 In our work, we replace $u(t, x)$ by a neural network $u(t, x; W, b)$ and obtain
634 a physics-informed neural network $f(t, x; W, b)$ by automatic differentiation.
635 Consequently, the resulting pair $u(t, x; W, b)$ and $f(t, x; W, b)$ must satisfy
636 the Burgers equation regardless of the choice of the weights W and bias b
637 parameters. Hence, at this “prior” level, i.e. before we train the networks
638 on a given set of data, our model should exactly preserves the continuity
639 and momentum equations by construction. During training, given a data-set
640 t_i, x_i, u_i and t_j, x_j, f_j , we then try to find the “correct” parameters W^* and b^*
641 such that we get as good a fit as possible to both the observed data and the
642 differential equation residual. During this process the residual, albeit small,
643 will not be exactly zero, and therefore our approximation will conserve mass
644 and momentum within the accuracy of the residual loss. Similar behavior is
645 observed in classical Galerkin finite element methods, while the only numer-
646 ical methods that are known to have exact conservation properties in this
647 setting are discontinuous Galerkin and finite volumes.

648
649 In all benchmarks considered in this work, the total number of training
650 data N_u is relatively small (a few hundred up to a few thousand points), and
651 we chose to optimize all loss functions using L-BFGS a quasi-Newton, full-

batch gradient-based optimization algorithm [35]. For larger data-sets a more computationally efficient mini-batch setting can be readily employed using stochastic gradient descent and its modern variants [36, 37]. Despite the fact that there is no theoretical guarantee that this procedure converges to a global minimum, our empirical evidence indicates that, if the given partial differential equation is well-posed and its solution is unique, our method is capable of achieving good prediction accuracy given a sufficiently expressive neural network architecture and a sufficient number of collocation points N_f . This general observation deeply relates to the resulting optimization landscape induced by the mean square error loss of equation 4, and defines an open question for research that is in sync with recent theoretical developments in deep learning [38, 39]. Here, we will test the robustness of the proposed methodology using a series of systematic sensitivity studies that accompany the numerical results presented in the following.

Figure A.6 summarizes our results for the data-driven solution of the Burgers equation. Specifically, given a set of $N_u = 100$ randomly distributed initial and boundary data, we learn the latent solution $u(t, x)$ by training all 3021 parameters of a 9-layer deep neural network using the mean squared error loss of (A.2). Each hidden layer contained 20 neurons and a hyperbolic tangent activation function. The top panel of Figure A.6 shows the predicted spatio-temporal solution $u(t, x)$, along with the locations of the initial and boundary training data. We must underline that, unlike any classical numerical method for solving partial differential equations, this prediction is obtained without any sort of discretization of the spatio-temporal domain. The exact solution for this problem is analytically available [13], and the resulting prediction error is measured at $6.7 \cdot 10^{-4}$ in the relative \mathbb{L}_2 -norm. Note that this error is about two orders of magnitude lower than the one reported in our previous work on data-driven solution of partial differential equation using Gaussian processes [8]. A more detailed assessment of the predicted solution is presented in the bottom panel of figure A.6. In particular, we present a comparison between the exact and the predicted solutions at different time instants $t = 0.25, 0.50, 0.75$. Using only a handful of initial and boundary data, the *physics-informed neural network* can accurately capture the intricate nonlinear behavior of the Burgers' equation that leads to the development of a sharp internal layer around $t = 0.4$. The latter is notoriously hard to accurately resolve with classical numerical methods and requires a laborious spatio-temporal discretization of equation (A.1).

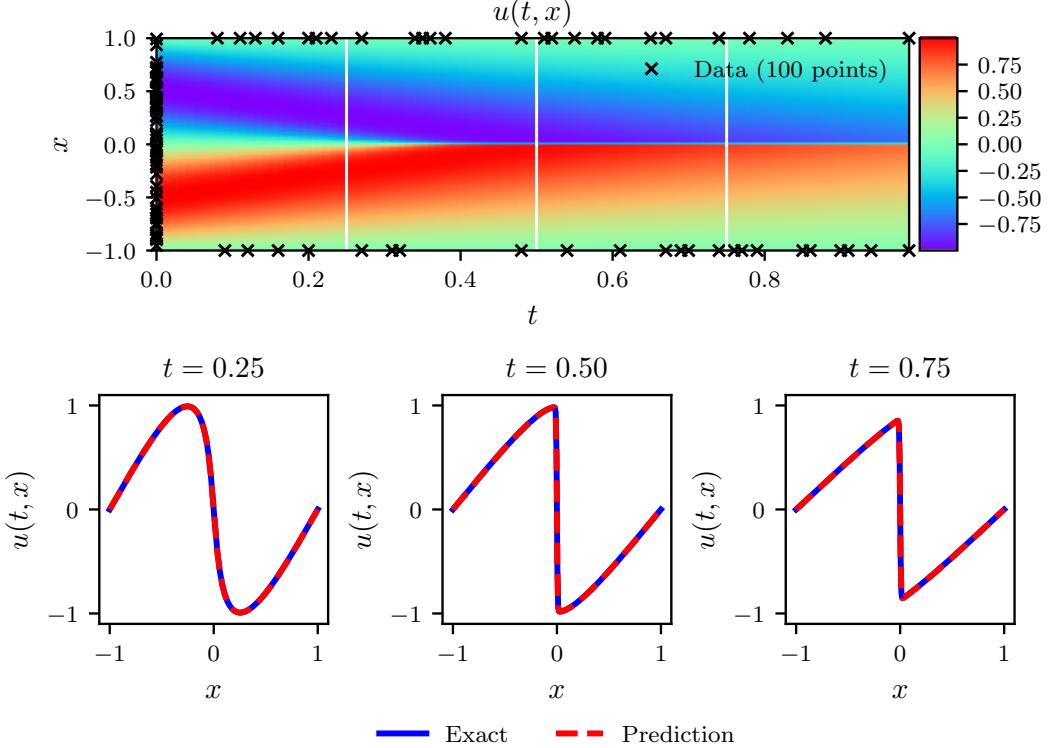


Figure A.6: *Burgers' equation:* Top: Predicted solution $u(t, x)$ along with the initial and boundary training data. In addition we are using 10,000 collocation points generated using a Latin Hypercube Sampling strategy. Bottom: Comparison of the predicted and exact solutions corresponding to the three temporal snapshots depicted by the white vertical lines in the top panel. The relative \mathbb{L}_2 error for this case is $6.7 \cdot 10^{-4}$. Model training took approximately 60 seconds on a single NVIDIA Titan X GPU card.

690

691 To further analyze the performance of our method, we have performed
 692 the following systematic studies to quantify its predictive accuracy for differ-
 693 ent number of training and collocation points, as well as for different neural
 694 network architectures. In table A.1 we report the resulting relative \mathbb{L}_2 error
 695 for different number of initial and boundary training data N_u and different
 696 number of collocation points N_f , while keeping the 9-layer network archi-
 697 tecture fixed. The general trend shows increased prediction accuracy as the
 698 total number of training data N_u is increased, given a sufficient number of
 699 collocation points N_f . This observation highlights a key strength of *physics-
 700 informed neural networks*: by encoding the structure of the underlying phys-

There needs to sufficient number of boundary points

$N_u \backslash N_f$	2000	4000	6000	7000	8000	10000
20	2.9e-01	4.4e-01	8.9e-01	1.2e+00	9.9e-02	4.2e-02
40	6.5e-02	1.1e-02	5.0e-01	9.6e-03	4.6e-01	7.5e-02
60	3.6e-01	1.2e-02	1.7e-01	5.9e-03	1.9e-03	8.2e-03
80	5.5e-03	1.0e-03	3.2e-03	7.8e-03	4.9e-02	4.5e-03
100	6.6e-02	2.7e-01	7.2e-03	6.8e-04	2.2e-03	6.7e-04
200	1.5e-01	2.3e-03	8.2e-04	8.9e-04	6.1e-04	4.9e-04

Table A.1: *Burgers' equation*: Relative \mathbb{L}_2 error between the predicted and the exact solution $u(t, x)$ for different number of initial and boundary training data N_u , and different number of collocation points N_f . Here, the network architecture is fixed to 9 layers with 20 neurons per hidden layer.

Layers \ Neurons	10	20	40
2	7.4e-02	5.3e-02	1.0e-01
4	3.0e-03	9.4e-04	6.4e-04
6	9.6e-03	1.3e-03	6.1e-04
8	2.5e-03	9.6e-04	5.6e-04

Table A.2: *Burgers' equation*: Relative \mathbb{L}_2 error between the predicted and the exact solution $u(t, x)$ for different number of hidden layers and different number of neurons per layer. Here, the total number of training and collocation points is fixed to $N_u = 100$ and $N_f = 10,000$, respectively.

ical law through the collocation points N_f , one can obtain a more accurate and data-efficient learning algorithm.⁵ Finally, table A.2 shows the resulting relative \mathbb{L}_2 for different number of hidden layers, and different number of neurons per layer, while the total number of training and collocation points is kept fixed to $N_u = 100$ and $N_f = 10,000$, respectively. As expected, we observe that as the number of layers and neurons is increased (hence the capacity of the neural network to approximate more complex functions), the predictive accuracy is increased.

⁵Note that the case $N_f = 0$ corresponds to a standard neural network model, i.e., a neural network that does not take into account the underlying governing equation.

709 *Appendix A.2. Discrete Time Models*

710 Let us apply the general form of Runge-Kutta methods with q stages [45]
 711 to a general equation of the form

$$u_t + \mathcal{N}[u] = 0, \quad x \in \Omega, \quad t \in [0, T], \quad (\text{A.3})$$

712 and obtain

$$\begin{aligned} u^{n+c_i} &= u^n - \Delta t \sum_{j=1}^q a_{ij} \mathcal{N}[u^{n+c_j}], \quad i = 1, \dots, q, \\ u^{n+1} &= u^n - \Delta t \sum_{j=1}^q b_j \mathcal{N}[u^{n+c_j}]. \end{aligned} \quad (\text{A.4})$$

713 Here, $u^{n+c_j}(x) = u(t^n + c_j \Delta t, x)$ for $j = 1, \dots, q$. This general form en-
 714 capsulates both implicit and explicit time-stepping schemes, depending on
 715 the choice of the parameters $\{a_{ij}, b_j, c_j\}$. Equations (7) can be equivalently
 716 expressed as

$$\begin{aligned} u^n &= u_i^n, \quad i = 1, \dots, q, \\ u^n &= u_{q+1}^n, \end{aligned} \quad (\text{A.5})$$

717 where

$$\begin{aligned} u_i^n &:= u^{n+c_i} + \Delta t \sum_{j=1}^q a_{ij} \mathcal{N}[u^{n+c_j}], \quad i = 1, \dots, q, \\ u_{q+1}^n &:= u^{n+1} + \Delta t \sum_{j=1}^q b_j \mathcal{N}[u^{n+c_j}]. \end{aligned} \quad (\text{A.6})$$

718 We proceed by placing a multi-output neural network prior on

$$[u^{n+c_1}(x), \dots, u^{n+c_q}(x), u^{n+1}(x)]. \quad (\text{A.7})$$

719 This prior assumption along with equations (A.6) result in a *physics-informed*
 720 *neural network* that takes x as an input and outputs

$$[u_1^n(x), \dots, u_q^n(x), u_{q+1}^n(x)]. \quad (\text{A.8})$$

721 To highlight the key features of the discrete time representation we revisit
 722 the problem of data-driven solution of the Burgers' equation. For this case,
 723 the nonlinear operator in equation (A.6) is given by

$$\mathcal{N}[u^{n+c_j}] = u^{n+c_j} u_x^{n+c_j} - (0.01/\pi) u_{xx}^{n+c_j},$$

724 and the shared parameters of the neural networks (A.7) and (A.8) can be
 725 learned by minimizing the sum of squared errors

$$SSE = SSE_n + SSE_b, \quad (\text{A.9})$$

726 where

$$SSE_n = \sum_{j=1}^{q+1} \sum_{i=1}^{N_n} |u_j^n(x^{n,i}) - u^{n,i}|^2,$$

727 and

$$SSE_b = \sum_{i=1}^q (|u^{n+c_i}(-1)|^2 + |u^{n+c_i}(1)|^2) + |u^{n+1}(-1)|^2 + |u^{n+1}(1)|^2.$$

728 Here, $\{x^{n,i}, u^{n,i}\}_{i=1}^{N_n}$ corresponds to the data at time t^n . The Runge-Kutta
729 scheme now allows us to infer the latent solution $u(t, x)$ in a sequential fashion.
730 Starting from initial data $\{x^{n,i}, u^{n,i}\}_{i=1}^{N_n}$ at time t^n and data at the
731 domain boundaries $x = -1$ and $x = 1$, we can use the aforementioned loss
732 function (A.9) to train the networks of (A.7), (A.8), and predict the solution
733 at time t^{n+1} . A Runge-Kutta time-stepping scheme would then use this
734 prediction as initial data for the next step and proceed to train again and
735 predict $u(t^{n+2}, x)$, $u(t^{n+3}, x)$, etc., one step at a time.

736

737 The result of applying this process to the Burgers' equation is presented
738 in figure A.7. For illustration purposes, we start with a set of $N_n = 250$ initial
739 data at $t = 0.1$, and employ a *physics-informed neural network* induced by an
740 implicit Runge-Kutta scheme with 500 stages to predict the solution at time
741 $t = 0.9$ in a single step. The theoretical error estimates for this scheme predict
742 a temporal error accumulation of $\mathcal{O}(\Delta t^{2q})$ [45], which in our case translates
743 into an error way below machine precision, i.e., $\Delta t^{2q} = 0.8^{1000} \approx 10^{-97}$. To
744 our knowledge, this is the first time that an implicit Runge-Kutta scheme
745 of that high-order has ever been used. Remarkably, starting from smooth
746 initial data at $t = 0.1$ we can predict the nearly discontinuous solution at
747 $t = 0.9$ in a single time-step with a relative \mathbb{L}_2 error of $8.2 \cdot 10^{-4}$. This error is
748 two orders of magnitude lower than the one reported in [8], and it is entirely
749 attributed to the neural network's capacity to approximate $u(t, x)$, as well as
750 to the degree that the sum of squared errors loss allows interpolation of the
751 training data. The network architecture used here consists of 4 layers with
752 50 neurons in each hidden layer.

753

754 A detailed systematic study to quantify the effect of different network
755 architectures is presented in table A.5. By keeping the number of Runge-
756 Kutta stages fixed to $q = 500$ and the time-step size to $\Delta t = 0.8$, we have

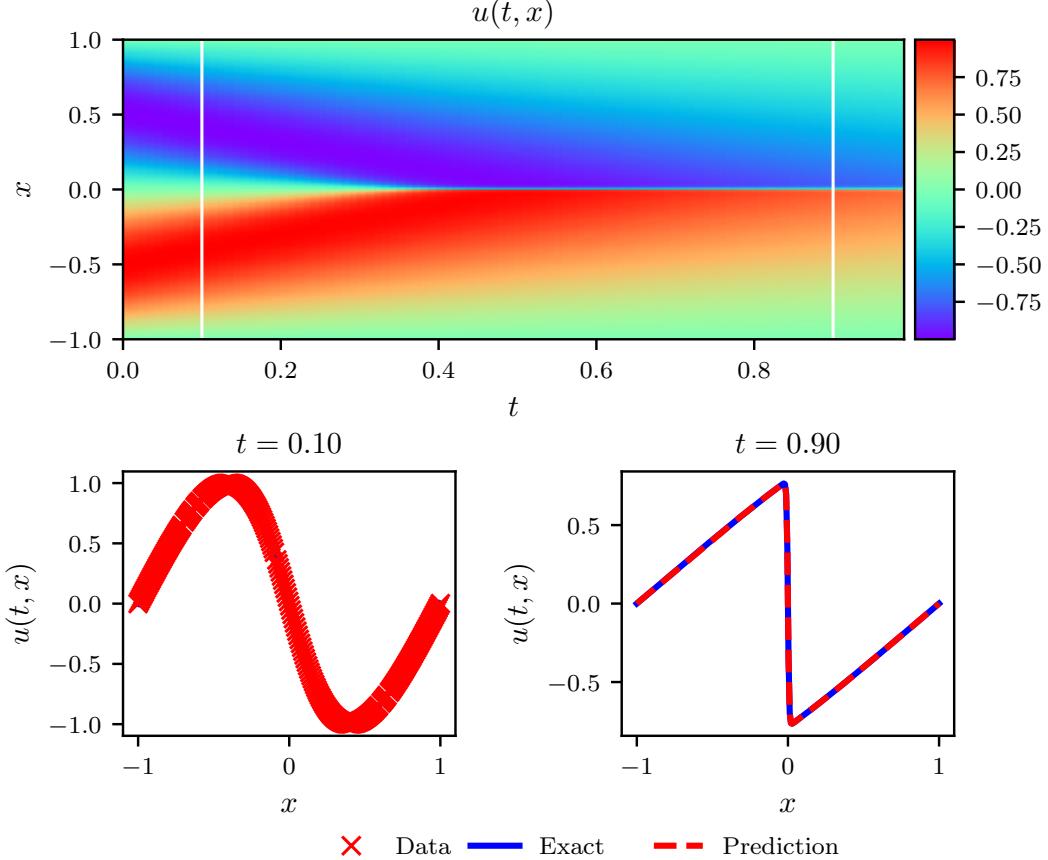


Figure A.7: *Burgers equation:* Top: Solution $u(t, x)$ along with the location of the initial training snapshot at $t = 0.1$ and the final prediction snapshot at $t = 0.9$. Bottom: Initial training data and final prediction at the snapshots depicted by the white vertical lines in the top panel. The relative \mathbb{L}_2 error for this case is $8.2 \cdot 10^{-4}$.

varied the number of hidden layers and the number of neurons per layer, and monitored the resulting relative \mathbb{L}_2 error for the predicted solution at time $t = 0.9$. Evidently, as the neural network capacity is increased the predictive accuracy is enhanced.

The key parameters controlling the performance of our discrete time algorithm are the total number of Runge-Kutta stages q and the time-step size Δt . In table A.4 we summarize the results of an extensive systematic study where we fix the network architecture to 4 hidden layers with 50 neurons

Layers \ Neurons	10	25	50
1	4.1e-02	4.1e-02	1.5e-01
2	2.7e-03	5.0e-03	2.4e-03
3	3.6e-03	1.9e-03	9.5e-04

Table A.3: *Burgers' equation*: Relative final prediction error measure in the \mathbb{L}_2 norm for different number of hidden layers and neurons in each layer. Here, the number of Runge-Kutta stages is fixed to 500 and the time-step size to $\Delta t = 0.8$.

766 per layer, and vary the number of Runge-Kutta stages q and the time-step
 767 size Δt . Specifically, we see how cases with low numbers of stages fail to
 768 yield accurate results when the time-step size is large. For instance, the case
 769 $q = 1$ corresponding to the classical trapezoidal rule, and the case $q = 2$
 770 corresponding to the 4th-order Gauss-Legendre method, cannot retain their
 771 predictive accuracy for time-steps larger than 0.2, thus mandating a solu-
 772 tion strategy with multiple time-steps of small size. On the other hand, the
 773 ability to push the number of Runge-Kutta stages to 32 and even higher
 774 allows us to take very large time steps, and effectively resolve the solution
 775 in a single step without sacrificing the accuracy of our predictions. More-
 776 over, numerical stability is not sacrificed either as implicit Gauss-Legendre is
 777 the only family of time-stepping schemes that remain A-stable regardless of
 778 their order, thus making them ideal for stiff problems [45]. These properties
 779 are unprecedented for an algorithm of such implementation simplicity, and
 780 illustrate one of the key highlights of our discrete time approach.

781 Finally, in table A.5 we provide a systematic study to quantify the accu-
 782 racy of the predicted solution as we vary the spatial resolution of the input
 783 data. As expected, increasing the total number of training data results in
 784 enhanced prediction accuracy.

785 Appendix B. Data-driven discovery of partial differential equations

786 This Appendix accompanies the main manuscript, and contains a series
 787 of systematic studies that aim to demonstrate the performance of the pro-
 788 posed algorithms for problems pertaining to *data-driven discovery of partial*
 789 *differential equations*. Throughout this document, we will use the Burgers'
 790 equation as a canonical example.

$q \backslash \Delta t$	0.2	0.4	0.6	0.8
q	1	2	4	8
1	3.5e-02	1.1e-01	2.3e-01	3.8e-01
2	5.4e-03	5.1e-02	9.3e-02	2.2e-01
4	1.2e-03	1.5e-02	3.6e-02	5.4e-02
8	6.7e-04	1.8e-03	8.7e-03	5.8e-02
16	5.1e-04	7.6e-02	8.4e-04	1.1e-03
32	7.4e-04	5.2e-04	4.2e-04	7.0e-04
64	4.5e-04	4.8e-04	1.2e-03	7.8e-04
100	5.1e-04	5.7e-04	1.8e-02	1.2e-03
500	4.1e-04	3.8e-04	4.2e-04	8.2e-04

Table A.4: *Burgers' equation*: Relative final prediction error measured in the \mathbb{L}_2 norm for different number of Runge-Kutta stages q and time-step sizes Δt . Here, the network architecture is fixed to 4 hidden layers with 50 neurons in each layer.

N	250	200	150	100	50	10
Error	4.02e-4	2.93e-3	9.39e-3	5.54e-2	1.77e-2	7.58e-1

Table A.5: *Burgers equation*: Relative \mathcal{L}_2 error between the predicted and the exact solution $u(t, x)$ for different number of training data N_n . Here, we have fixed $q = 500$, and used a neural network architecture with 3 hidden layers and 50 neurons per hidden layer.

791 Appendix B.1. Continuous Time Models

792 As a first example, let us consider the Burgers' equation. This equation
 793 arises in various areas of applied mathematics, including fluid mechanics,
 794 nonlinear acoustics, gas dynamics, and traffic flow [13]. It is a fundamen-
 795 tal partial differential equation and can be derived from the Navier-Stokes
 796 equations for the velocity field by dropping the pressure gradient term. For
 797 small values of the viscosity parameters, Burgers' equation can lead to shock
 798 formation that is notoriously hard to resolve by classical numerical methods.
 799 In one space dimension the equation reads as

$$u_t + \lambda_1 u u_x - \lambda_2 u_{xx} = 0. \quad (\text{B.1})$$

800 Let us define $f(t, x)$ to be given by

$$f := u_t + \lambda_1 u u_x - \lambda_2 u_{xx}, \quad (\text{B.2})$$

801 and proceed by approximating $u(t, x)$ by a deep neural network. This will
 802 result in the *physics-informed neural network* $f(t, x)$. The shared parameters

803 of the neural networks $u(t, x)$ and $f(t, x)$ along with the parameters $\lambda =$
 804 (λ_1, λ_2) of the differential operator can be learned by minimizing the mean
 805 squared error loss

$$MSE = MSE_u + MSE_f, \quad (B.3)$$

806 where

$$MSE_u = \frac{1}{N} \sum_{i=1}^N |u(t_u^i, x_u^i) - u^i|^2,$$

807 and

$$MSE_f = \frac{1}{N} \sum_{i=1}^N |f(t_u^i, x_u^i)|^2.$$

808 Here, $\{t_u^i, x_u^i, u^i\}_{i=1}^N$ denote the training data on $u(t, x)$. The loss MSE_u cor-
 809 responds to the training data on $u(t, x)$ while MSE_f enforces the structure
 810 imposed by equation (B.1) at a finite set of collocation points, whose number
 811 and location is taken to be the same as the training data.

812 To illustrate the effectiveness of our approach, we have created a train-
 813 ing data-set by randomly generating $N = 2,000$ points across the entire
 814 spatio-temporal domain from the exact solution corresponding to $\lambda_1 = 1.0$
 815 and $\lambda_2 = 0.01/\pi$. The locations of the training points are illustrated in the
 816 top panel of figure B.8. This data-set is then used to train a 9-layer deep
 817 neural network with 20 neurons per hidden layer by minimizing the mean
 818 square error loss of (B.3) using the L-BFGS optimizer [35]. Upon training,
 819 the network is calibrated to predict the entire solution $u(t, x)$, as well as the
 820 unknown parameters $\lambda = (\lambda_1, \lambda_2)$ that define the underlying dynamics. A
 821 visual assessment of the predictive accuracy of the *physics-informed neural*
 822 *network* is given in the middle and bottom panels of figure B.8. The network
 823 is able to identify the underlying partial differential equation with remark-
 824 able accuracy, even in the case where the scattered training data is corrupted
 825 with 1% uncorrelated noise.

827 To further scrutinize the performance of our algorithm, we have per-
 828 formed a systematic study with respect to the total number of training data,
 829 the noise corruption levels, and the neural network architecture. The results
 830 are summarized in tables B.6 and B.7. The key observation here is that the
 831 proposed methodology appears to be very robust with respect to noise levels
 832 in the data, and yields a reasonable identification accuracy even for noise

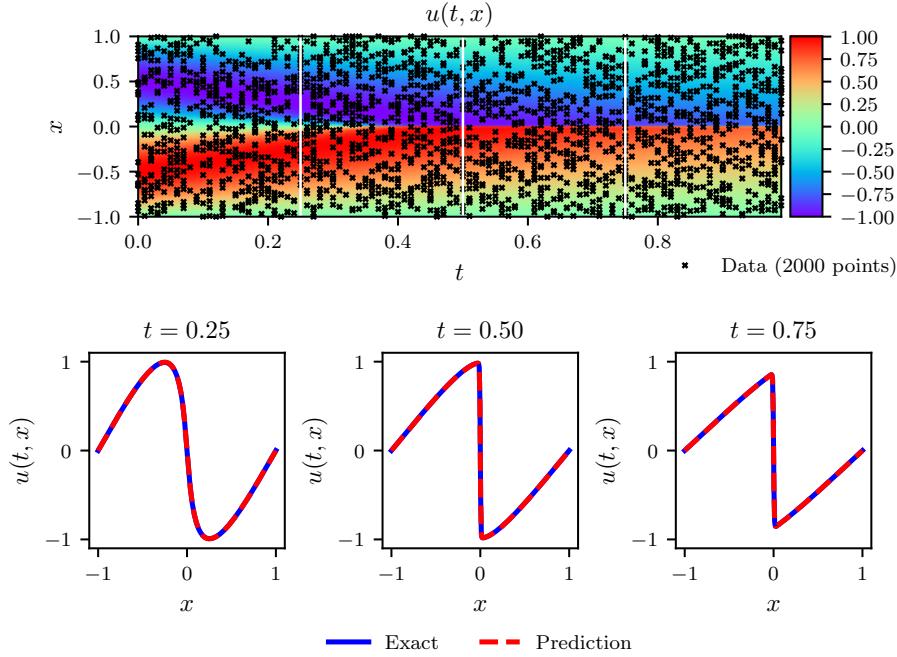


Figure B.8: *Burgers equation:* *Top:* Predicted solution $u(t, x)$ along with the training data. *Middle:* Comparison of the predicted and exact solutions corresponding to the three temporal snapshots depicted by the dashed vertical lines in the top panel. *Bottom:* Correct partial differential equation along with the identified one obtained by learning λ_1 and λ_2 .

corruption up to 10%. This enhanced robustness seems to greatly outperform competing approaches using Gaussian process regression as previously reported in [9] as well as approaches relying on sparse regression that require relatively clean data for accurately computing numerical gradients [50]. We also observe some variability and non monotonic trends in tables B.6 and B.7 as the network architecture and total number of training points are changed. This variability could be potentially attributed to different factors pertaining to the equation itself as well as the particular neural network setup, and gives rise to a series of questions that require further investigation, as discussed in the concluding remarks section of this paper.

very little consistency of error % with increasing training points

noise N_u	% error in λ_1				% error in λ_2			
	0%	1%	5%	10%	0%	1%	5%	10%
500	0.131	0.518	0.118	1.319	13.885	0.483	1.708	4.058
1000	0.186	0.533	0.157	1.869	3.719	8.262	3.481	14.544
1500	0.432	0.033	0.706	0.725	3.093	1.423	0.502	3.156
2000	0.096	0.039	0.190	0.101	0.469	0.008	6.216	6.391

Table B.6: *Burgers' equation*: Percentage error in the identified parameters λ_1 and λ_2 for different number of training data N corrupted by different noise levels. Here, the neural network architecture is kept fixed to 9 layers and 20 neurons per layer.

Neurons Layers	% error in λ_1			% error in λ_2		
	10	20	40	10	20	40
2	11.696	2.837	1.679	103.919	67.055	49.186
4	0.332	0.109	0.428	4.721	1.234	6.170
6	0.668	0.629	0.118	3.144	3.123	1.158
8	0.414	0.141	0.266	8.459	1.902	1.552

Table B.7: *Burgers' equation*: Percentage error in the identified parameters λ_1 and λ_2 for different number of hidden layers and neurons per layer. Here, the training data is considered to be noise-free and fixed to $N = 2,000$.

844 Appendix B.2. Discrete Time Models

845 Our starting point here is similar to as described in section 3.2. Now
846 equations (7) can be equivalently expressed as

$$\begin{aligned} u^n &= u_i^n, \quad i = 1, \dots, q, \\ u^{n+1} &= u_i^{n+1}, \quad i = 1, \dots, q. \end{aligned} \tag{B.4}$$

847 where

$$\begin{aligned} u_i^n &:= u^{n+c_i} + \Delta t \sum_{j=1}^q a_{ij} \mathcal{N}[u^{n+c_j}; \lambda], \quad i = 1, \dots, q, \\ u_i^{n+1} &:= u^{n+c_i} + \Delta t \sum_{j=1}^q (a_{ij} - b_j) \mathcal{N}[u^{n+c_j}; \lambda], \quad i = 1, \dots, q. \end{aligned} \tag{B.5}$$

848 We proceed by placing a multi-output neural network prior on

$$[u^{n+c_1}(x), \dots, u^{n+c_q}(x)]. \tag{B.6}$$

849 This prior assumption along with equations (22) result in two *physics-informed*
 850 *neural networks*

$$[u_1^n(x), \dots, u_q^n(x), u_{q+1}^n(x)], \quad (\text{B.7})$$

851 and

$$[u_1^{n+1}(x), \dots, u_q^{n+1}(x), u_{q+1}^{n+1}(x)]. \quad (\text{B.8})$$

852 Given noisy measurements at two distinct temporal snapshots $\{\mathbf{x}^n, \mathbf{u}^n\}$ and
 853 $\{\mathbf{x}^{n+1}, \mathbf{u}^{n+1}\}$ of the system at times t^n and t^{n+1} , respectively, the shared
 854 parameters of the neural networks (B.6), (B.7), and (B.8) along with the
 855 parameters λ of the differential operator can be trained by minimizing the
 856 sum of squared errors

$$SSE = SSE_n + SSE_{n+1}, \quad (\text{B.9})$$

857 where

$$SSE_n := \sum_{j=1}^q \sum_{i=1}^{N_n} |u_j^n(x^{n,i}) - u^{n,i}|^2,$$

858 and

$$SSE_{n+1} := \sum_{j=1}^q \sum_{i=1}^{N_{n+1}} |u_j^{n+1}(x^{n+1,i}) - u^{n+1,i}|^2.$$

859 Here, $\mathbf{x}^n = \{x^{n,i}\}_{i=1}^{N_n}$, $\mathbf{u}^n = \{u^{n,i}\}_{i=1}^{N_n}$, $\mathbf{x}^{n+1} = \{x^{n+1,i}\}_{i=1}^{N_{n+1}}$, and $\mathbf{u}^{n+1} =$
 860 $\{u^{n+1,i}\}_{i=1}^{N_{n+1}}$.

861 Appendix B.3. Example (Burgers' Equation)

862 Let us illustrate the key features of this method through the lens of the
 863 Burgers' equation. Recall the equation's form

$$u_t + \lambda_1 uu_x - \lambda_2 u_{xx} = 0, \quad (\text{B.10})$$

864 and notice that the nonlinear spatial operator in equation (B.5) is given by

$$\mathcal{N}[u^{n+c_j}] = \lambda_1 u^{n+c_j} u_x^{n+c_j} - \lambda_2 u_{xx}^{n+c_j}.$$

865 Given merely two training data snapshots, the shared parameters of the neu-
 866 ral networks (B.6), (B.7), and (B.8) along with the parameters $\lambda = (\lambda_1, \lambda_2)$
 867 of the Burgers' equation can be learned by minimizing the sum of squared
 868 errors in equation (B.9). Here, we have created a training data-set compris-
 869 ing of $N_n = 199$ and $N_{n+1} = 201$ spatial points by randomly sampling the

870 exact solution at time instants $t^n = 0.1$ and $t^{n+1} = 0.9$, respectively. The
 871 training data are shown in the top and middle panel of figure B.9. The neural
 872 network architecture used here consists of 4 hidden layers with 50 neurons
 873 each, while the number of Runge-Kutta stages is empirically chosen to yield
 874 a temporal error accumulation of the order of machine precision ϵ by setting⁶

$$\checkmark q = 0.5 \log \epsilon / \log(\Delta t), \quad (\text{B.11})$$

875 where the time-step for this example is $\Delta t = 0.8$. The bottom panel of figure B.9 summarizes the identified parameters $\lambda = (\lambda_1, \lambda_2)$ for the cases of
 876 noise-free data, as well as noisy data with 1% of Gaussian uncorrelated noise
 877 corruption. For both cases, the proposed algorithm is able to learn the cor-
 878 rect parameter values $\lambda_1 = 1.0$ and $\lambda_2 = 0.01/\pi$ with remarkable accuracy,
 879 despite the fact that the two data snapshots used for training are very far
 880 apart, and potentially describe different regimes of the underlying dynamics.
 881

882 A sensitivity analysis is performed to quantify the accuracy of our predic-
 883 tions with respect to the gap between the training snapshots Δt , the noise
 884 levels in the training data, and the *physics-informed neural network* architec-
 885 ture. As shown in table B.8, the proposed algorithm is quite robust to both
 886 Δt and the noise corruption levels, and it returns reasonable estimates for
 887 the unknown parameters. This robustness is mainly attributed to the flexi-
 888 bility of the underlying implicit Runge-Kutta scheme to admit an arbitrarily
 889 high number of stages, allowing the data snapshots to be very far apart in
 890 time, while not compromising the accuracy with which the nonlinear dynam-
 891 ics of equation (B.10) are resolved. This is the key highlight of our discrete
 892 time formulation for identification problems, setting it apart from competing
 893 approaches [9, 50]. Lastly, table B.9 presents the percentage error in the
 894 identified parameters, demonstrating the robustness of our estimates with
 895 respect to the underlying neural network architecture. Despite the overall
 896 positive results, the variability observed in tables B.8 and B.9 is still largely
 897 unexplained and naturally motivates a series of questions provided in the
 898 concluding remarks section of this paper.

⁶This is motivated by the theoretical error estimates for implicit Runge-Kutta schemes suggesting a truncation error of $\mathcal{O}(\Delta t^{2q})$ [45].

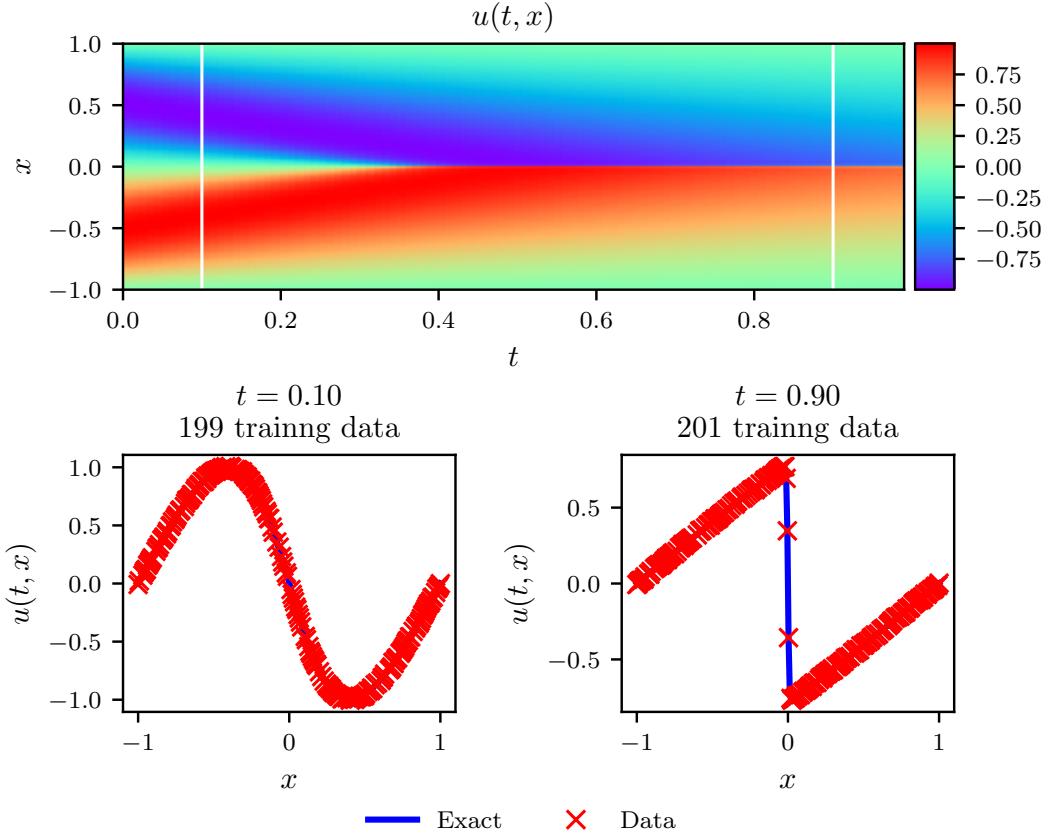


Figure B.9: *Burgers equation:* *Top:* Solution $u(t, x)$ along with the temporal locations of the two training snapshots. *Middle:* Training data and exact solution corresponding to the two temporal snapshots depicted by the dashed vertical lines in the top panel. *Bottom:* Correct partial differential equation along with the identified one obtained by learning λ_1, λ_2 .

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Δt	noise	% error in λ_1				% error in λ_2			
		0%	1%	5%	10%	0%	1%	5%	10%
0.2		0.002	0.435	6.073	3.273	0.151	4.982	59.314	83.969
0.4		0.001	0.119	1.679	2.985	0.088	2.816	8.396	8.377
0.6		0.002	0.064	2.096	1.383	0.090	0.068	3.493	24.321
0.8		0.010	0.221	0.097	1.233	1.918	3.215	13.479	1.621

Table B.8: *Burgers' equation*: Percentage error in the identified parameters λ_1 and λ_2 for different gap size Δt between two different snapshots and for different noise levels.

Layers	Neurons	% error in λ_1			% error in λ_2		
		10	25	50	10	25	50
1		1.868	4.868	1.960	180.373	237.463	123.539
2		0.443	0.037	0.015	29.474	2.676	1.561
3		0.123	0.012	0.004	7.991	1.906	0.586
4		0.012	0.020	0.011	1.125	4.448	2.014

Table B.9: *Burgers' equation*: Percentage error in the identified parameters λ_1 and λ_2 for different number of hidden layers and neurons in each layer.

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