GEORGIA INSTITUTE OF TECHNOLOGY
School of Electrical and Computer Engineering

Pre-test

Date: January 12, 2017 Course: ECE 6254

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Last, First

- Open book/internet.
- No time limit.
- The test is worth 100 points. There are ten questions, and each one is worth 10 points. In multi-part questions, each part will be weighted equally.
- All work should be performed on the test itself. If more space is needed, use the backs of the pages.
- This test will be conducted under the rules and guidelines of the Georgia Tech Honor Code and no cheating will be tolerated (i.e., no discussing the test with other students).
- Make sure to look at the question titles, they will sometimes provide valuable hints.
- Please contact Prof. Davenport (in person or via email) directly with any questions if you are unclear what a question is asking.

Problem 1: Random variables. Suppose that three independent random variables X, Y, and Z are distributed according to

$$X \sim \text{Normal}(1,1) \quad Y \sim \text{Normal}(1,2) \quad Z \sim \text{Normal}(2,4)$$

What is the probability that $(X - Y)Z < 0$.

Let $D = (X-Y)$ ... $\mu_D = \mu_X - \mu_Y = 1 - 1 = 0$

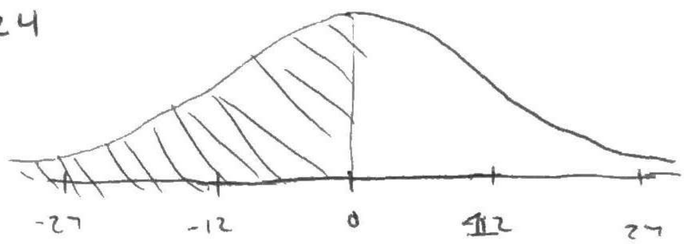
$D \sim N(0,3)$ $\quad \sigma_D^2 = \sigma_X^2 + \sigma_Y^2 = 1 + 2 = 3$

Let $A = DZ$ ... $\mu_A = \mu_D \mu_Z = 0(2) = 0$

$A \sim N(0,24)$ $\quad \sigma_A^2 = \sigma_D^2 \sigma_Z^2 + \sigma_Z^2 \mu_D^2 + \sigma_D^2 \mu_Z^2 = 3(4) + \cancel{4(0)} + 3(4)$

$\sigma_A^2 = 24$

$\boxed{P(A < 0) = 0.5}$



Problem 2: Independence: Suppose that the probability that A occurs is 0.75 and the probability that **both** A **and** B occur is 0.3. If A and B are independent events, what is the probability that **neither** A **nor** B occur?

$P(A) = 0.75$, $P(A \cap B) = 0.3$ $\xrightarrow{\text{indepence}}$ $P(B) = \dfrac{P(A \cap B)}{P(A)} = 0.4$

$P(A^c \cap B^c) = P(A^c)P(B^c) = (1 - 0.75)(1 - 0.4) = \boxed{0.15}$

Problem 3: Conditional probability density functions and derived distributions: Suppose that X and Y have joint pdf given by

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-x-2y} & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

(a) What are the marginal probability density functions for X and Y?

$$f_X(x) = \int_{-\infty}^{\infty} 2e^{(-x-2y)}dy = 2\left(-\tfrac{1}{2}e^{(-x-2y)}\right) = \boxed{\begin{cases} -e^{(-x-2y)} & , x \geq 0 \\ 0 & , x < 0 \end{cases}}$$

$$f_Y(y) = \int_{-\infty}^{\infty} 2e^{(-x-2y)}dx = 2\left(-e^{(-x-2y)}\right) = \boxed{\begin{cases} -2e^{(-x-2y)} & , y \geq 0 \\ 0 & , y < 0 \end{cases}}$$

(b) What is the probability distribution function for the random variable $R = \frac{X}{Y}$?

$$P_R(R) = \int_{-\infty}^{\infty} y f_{X,Y}(Ry, y)dy = \int_{-\infty}^{\infty} y2e^{(-Ry-2y)}dy$$

$$= 2e^{-(R+2)}\int_{-\infty}^{\infty} ye^{y}dy = \boxed{2e^{-(R+2)}e^{y}(y-1)}$$

Problem 4: The median and the cumulative distribution function: Let M be the number of miles your electric car can drive without running out of electricity, and suppose that M has probability density function given by

$$f_M(m) = \begin{cases} \frac{e^{-m/301}}{301} & m \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

What is the median range for your car? That is, how far can you drive before there is a 50% chance that your battery runs out?

$$\text{Median} \rightarrow \int_{-\infty}^{M} f_M(m) = \frac{1}{2}$$

$$\int^{M} \frac{1}{301} e^{-\frac{m}{301}} = \frac{1}{2}$$

$$-e^{-\frac{m}{301}} \Big|^{M} = \frac{1}{2}$$

$$-e^{-\frac{M}{301}} + e = \frac{1}{2}$$

$$e^{-\frac{M}{301}} = \frac{1}{2}$$

$$-\frac{m}{301} = \ln(1/2)$$

$$\boxed{m = -301 \ln\left(\tfrac{1}{2}\right) = 208.64 \text{ miles}}$$

Problem 5: Bayes rule and normal random variables: Suppose that you have a newborn baby at home. Let X be the amount of time it takes (in hours) until your baby wakes up when you put her down, which we will model as an exponential random variable with parameter λ. However, you have no idea what λ is for your baby. However, you read online that a good model for λ is the pdf

$$f_\Lambda(\lambda) = \begin{cases} \frac{1}{\ln(2)\lambda} & \frac{1}{2} \le \lambda \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that you put down your baby and she wakes up after $X = 2$ hours. Use this information to compute an updated pdf for Λ, i.e., $f_{\Lambda|X}(\lambda|2)$.

Exponential Distributions are memoryless

$$\boxed{\therefore\ f_{\Lambda|X}(\lambda|2) = f_\Lambda(\lambda) = \begin{cases} \frac{1}{\ln(2)\lambda}, & \frac{1}{2} \le \lambda \le 1 \\ 0, & \text{otherwise} \end{cases}}$$

Problem 6: Joint probability density functions: The *correlation coefficient* $\rho(X, Y)$ between a pair of random variables X and Y is given by

$$\rho(X, Y) = \frac{E\Big[(X - E[X])\ (Y - E[Y])\Big]}{\sigma_X \sigma_Y}$$

Let X and Y be independent random variables with $\mathrm{var}(X) = 3$ and $\mathrm{var}(Y) = 4$. We do not know $E[X]$ or $E[Y]$. Let $Z = 2X + Y$. What is the correlation coefficient $\rho(X, Z) = \mathrm{cov}(X, Z)/\sqrt{\mathrm{var}(X)\mathrm{var}(Z)}$?

$$\sigma_z^2 = 2\sigma_x^2 + \sigma_y^2 = 10$$

$$\rho(x,z) = \frac{E[(x - E[x]) \cdot (z - E[z])]}{\sigma_x \sigma_z}$$

$$\rho(x,z) = \frac{E[(x - E[x]) \cdot ((2x+y) - E(2x+y))]}{\sqrt{3}\sqrt{10}}$$

$$\rho(x,z) = \frac{E[x(2x+y) - E[x](2x+y) - xE[2x+y] + E[x]E[2x+y]]}{\sqrt{30}}$$

let $E[\,] = \mu$

$$\rho(x,z) = \frac{E[2x^2 + xY - 2x\mu_x - Y\mu_x - x\mu_{2x+y} + \mu_x\mu_{2x+y}]}{\sqrt{30}}$$

$$\boxed{\rho(x,z) = \frac{E[\mu_x(\mu_{2x+y} - 2x - y) + x(2x - \mu_{2x+y} + y)]}{\sqrt{30}}}$$

Problem 7: Pythagoras?

(a) Under what conditions on x and y is it true that

$$\|x + y\|_2^2 = \|x\|_2^2 + \|y\|_2^2 \ ?$$

$\boxed{x + y \text{ must be orthogonal}}$

$$\|x+y\|_2^2 = \langle x+y, x+y \rangle$$

$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$$

$$= \|x\|_2^2 + \langle x, y \rangle + \overline{\langle x, y \rangle} + \|y\|^2$$

$$\boxed{= \|x\|_2^2 + \|y\|_2^2 \quad \text{If: } \langle x, y \rangle + \overline{\langle x, y \rangle} = 0}$$

(b) Under what conditions on x and y is it true that

$$\|x + y\|_2 = \|x\|_2 + \|y\|_2 \ ?$$

Matrix Norm Identity that $\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$

To be equal, x + y must be orthogonal?

Problem 8: Singular value decomposition.: Let

$$A = \begin{bmatrix} -2 & 2 & 2 & -2 & 0 \\ -2 & 2 & 2 & -2 & 0 \\ 2 & -2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

(a) What is rank(A)?

$$\text{rank}(A) = 2 \quad b/c \quad \text{rref}(A) = \begin{bmatrix} 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Using Python or MATLAB (or whatever) find the singular value decomposition of A. That is, find matrices U, Σ, V such that

$$A = U\Sigma V^{\mathrm{T}} \qquad (5\times5) = (5\times5)\,($$

and $U^{\mathrm{T}}U = \mathbf{I}$, $V^{\mathrm{T}}V = \mathbf{I}$, and Σ has non-negative entries along its diagonal and is zero elsewhere.

$$A = \underbrace{\begin{bmatrix} -0.5774 & 0 & 0 & 0.5774 & -0.5774 \\ -0.5774 & 0 & 0 & 0.2113 & 0.7887 \\ 0.5774 & 0 & 0 & 0.7887 & 0.2113 \\ 0 & 0.7071 & -.7071 & 0 & 0 \\ 0 & 0.7071 & -0.7071 & 0 & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 6.93 & 0 & 0 & 0 & 0 \\ 0 & 2.83 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -0.5774 & -.2113 & -.7887 & 0 \\ 0 & -0.5774 & 0.7887 & .2113 & 0 \\ -0.866 & -.2887 & -.2887 & .2887 & 0 \end{bmatrix}}_{V^{T}}$$

(c) Describe, in words, the column space (or range) of A:

$$\text{Range}(A) = \{v \in \mathbb{R}^5 \ : \ v = Ax \text{ for some } x\}.$$

The column space (range) of A is spanned by the basis vectors $sp\left\{\begin{bmatrix} -2 \\ -2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 2 \end{bmatrix}\right\}$, and therefore equals the set of all linear combinations of those two vectors.

(d) Describe, in words, the row space of A (this is the column space of A^{T}):

$$\text{Range}(A^{\mathrm{T}}) = \{v \in \mathbb{R}^5 \ : \ v = A^{\mathrm{T}}x \text{ for some } x\}.$$

The row space of A is spanned by basis vectors $sp\left\{\begin{matrix}[-2 \ 2 \ 2 \ -2 \ 0] \\ [0 \ 0 \ 0 \ 0 \ 2]\end{matrix}\right\}$, and therefore is the set of vectors $(a, b, c, d, e) \in \mathbb{R}^5$ satisfying the equations $a = d$ and $b = c = e$.

Problem 9: Eigenvalues and eigenvectors. Suppose that $\boldsymbol{A}$ and $\boldsymbol{B}$ are square symmetric matrices.

(a) Show that if $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same eigenvectors, then they *commute*: $\boldsymbol{AB} = \boldsymbol{BA}$.

$\cancel{\Lambda_A \Lambda_B} \rightarrow A = V_A^{-1} \Lambda_A V_A$ and $B = V_B^{-1} \Lambda_B V_B$

$V_A = V_B$ and $V_A^{-1} = V_B^{-1}$

$$AB = (V_A^{-1} \Lambda_A V_A)(V_B^{-1} \Lambda_B V_B)$$

$$= V^{-1} \Lambda_A V V^{-1} \Lambda_B V$$

$$= V^{-1} \Lambda_A \Lambda_B V$$

$$= V^{-1} \Lambda_B \Lambda_A V$$

$$= V^{-1} \Lambda_B (V V^{-1}) \Lambda_A V$$

$$= (V^{-1} \Lambda_B V)(V^{-1} \Lambda_A V) = BA$$

(b) Show that if $\boldsymbol{AB} = \boldsymbol{BA}$ and $\boldsymbol{A}$ has no repeated eigenvalues, then $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same eigenvectors.

$$(V_A^{-1} \Lambda_A V_A)(V_B^{-1} \Lambda_B V_B) = (V_B^{-1} \Lambda_B V_B)(V_A^{-1} \Lambda_A V_A)$$

let $V_A = V_B = V$

$$V \cdot (V^{-1} \Lambda_A \Lambda_B V = V^{-1} \Lambda_B \Lambda_A V)$$

$$(\Lambda_A \Lambda_B V = \Lambda_B \Lambda_A V) \cdot V^{-1}$$

$$\Lambda_A \Lambda_B = \Lambda_B \Lambda_A$$

$$\Lambda_A \Lambda_B = \Lambda_A \Lambda_B$$

Problem 10: Orthogonal projections: Let

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \mathbf{p}_2 = \begin{bmatrix} 4 \\ -2 \\ -6 \\ -7 \end{bmatrix} \qquad \mathbf{p}_3 = \begin{bmatrix} 3 \\ 4 \\ -2 \\ 1 \end{bmatrix}$$

and

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 7 \end{bmatrix}.$$

Find a decomposition of $\mathbf{x}$ into $\mathbf{x} = \mathbf{x}^* + \mathbf{x}_e$ where $\mathbf{x}^*$ is in the span of $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, i.e., where $\mathbf{x}^* = \alpha_1\mathbf{p}_1 + \alpha_2\mathbf{p}_2 + \alpha_3\mathbf{p}_3$ for some suitable choice of $\alpha_1, \alpha_2, \alpha_3$. Make sure to give both $\mathbf{x}^*$ and $\mathbf{x}_e$, and show your work/describe your method, even if you use a computer to help with the calculations.

Let $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$

$$x^* = 1\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + 2\begin{bmatrix} 4 \\ -2 \\ -6 \\ -7 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 4 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ -5 \\ -2 \end{bmatrix}$$

for $x = x^* + x_e$, $x_e = x - x^* = \begin{bmatrix} -7 \\ -2 \\ 8 \\ 9 \end{bmatrix}$

so $x = x^* + x_e$

$$\boxed{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ -5 \\ -2 \end{bmatrix} + \begin{bmatrix} -7 \\ -2 \\ 8 \\ 9 \end{bmatrix}} \quad \boxed{\text{when } \alpha_1 = 1,\ \alpha_2 = 2,\ \alpha_3 = 3}$$