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ECE 6254
29 Jan 17

HW # 1

① $P[k|n,p] = \binom{n}{k} p^k (1-p)^{n-k} = \left(\frac{n!}{(n-k)!\,k!}\right) p^k (1-p)^{n-k}$

$\hat{p}_i = \frac{\#\text{ heads}}{n}$, $\hat{p}_i = 0 = \frac{0}{n}$

a) $n = 10$, $p = 0.05$, what's probability at least 1 coin has $\hat{p}_i = 0$?

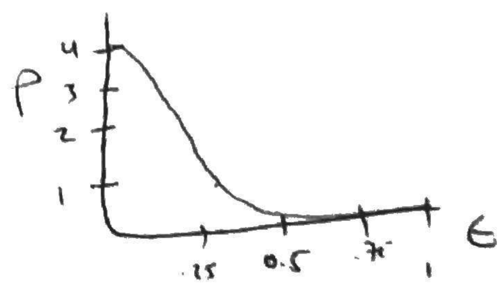
exactly 1 coin $= P[0|10, 0.05] = \frac{10!}{(10-0)!\,0!} (0.05)^0 (1-0.05)^{10-0} = 0.5987$

$P\left[\begin{smallmatrix}\text{at least 1 coin out}\\ \text{of } m \text{ has } k=0\end{smallmatrix}\right] = 1 - (1 - P[0/10, 0.05])^m$

	$m=1$	$m=1000$	$m=1,000,000$
$p=0.05$	0.5987	1	1
$p=0.75$	0	0.001	0.6147

1b) $\mathbb{P}\left[\max_i |\hat{p}_i - p_i| > \epsilon\right] \leq \sum_{j=1}^{\hat{m}} P[|\hat{R}_n(h_j) - R(h_j)| > \epsilon]$

$\leq 2me^{-2n\epsilon^2}$



② $X = N(0, \sigma^2)$, $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-t^2/2\sigma^2}$, find tail bound for X using Chernoff bounding method

a) $P[X \geq t] = e^{-\lambda t} E[e^{\lambda X}]$

$\hookrightarrow E[e^{\lambda x}] = M(\lambda) = e^{\mu\lambda} e^{\frac{\sigma^2\lambda^2}{2}} \overset{\mu=0}{=} e^{\frac{\sigma^2\lambda^2}{2}}$

$$\boxed{P[X \geq t] = e^{-\lambda t} e^{\frac{\sigma^2\lambda^2}{2}} = e^{\lambda\left(\frac{\sigma^2}{2}\lambda - t\right)}}$$

optimize over λ to minimize bound...

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② b) if $\{X_1, \dots, X_m\}$ are iid & $X_i \sim N(0, \sigma^2)$

$$\mathbb{P}\left[\max_{i=1,\dots,m} X_i > t\right] \leq \sum_{i=1}^{m} P[X_i > t]$$

$$\boxed{\mathbb{P}\left[\max_{i=1,\dots,m} X_i > t\right] \leq m e^{\lambda\left(\frac{\sigma^2}{2}\lambda - t\right)}}$$

c) for above X_i, with $\mathbb{P} \geq 0.9$ $\max_{i=1,\dots,m} X_i \leq$?

$$0.9 \leq m e^{\lambda\left(\frac{\sigma^2}{2}\lambda - t\right)}$$

$$\ln\left(\frac{0.9}{m}\right) \leq \lambda\left(\frac{\sigma^2}{2}\lambda - t\right)$$

$$\frac{\sigma^2}{2}\lambda - \frac{1}{\lambda}\ln\left(\frac{0.9}{m}\right) \geq t \Rightarrow \boxed{\max_{i=1,\dots,m} X_i \leq \frac{\sigma^2\lambda^2}{2} - \ln\left(\frac{.9}{m}\right)}$$

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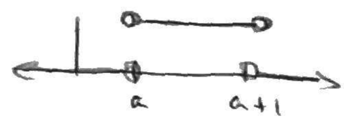
③ a) $f_{X|Y}(x|0) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

$f_{X|Y}(x|1) = \begin{cases} 1, & x \in [a, a+1] \\ 0, & \text{otherwise} \end{cases}$



b) $\text{Risk}(h_j) = \mathbb{P}[h_j(x) \neq Y]$

Empirical Risk $\hat{R}_n(h_j) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{i : h_j(x_j) \neq y_i\}}(i)$

minimization pick $\to$ $h^* = \arg\min_{h_j \in H} \hat{R}_n(h_j) = \arg\max_{k \in \{1,\dots,K\}} \eta_k(x)$

such that $R(h) \geq R(h^*)$

$$h^* = \arg\max_{k \in \{0,1\}} \frac{P[Y=k] f_{X|Y}(x|k)}{P[Y=0] f_{X|Y}(x|0) + P[Y=1] f_{X|Y}(x|1)}$$

$P[Y=k] = P[Y=0] = P[Y=1] = \frac{1}{2}$

$$\boxed{h^* = \frac{f_{X|Y}(x|k)}{e^{-x} + \mathbb{1}_{x \in [a,a+1]}(x)}}$$

(3c) Bayes Risk $= R^*(x) = 1 - \eta_1(x)$

$$\eta_1(x) = \frac{P[Y=1]f_{X|Y}(x|1)}{P[Y=0]f_{X|Y}(x|0) + P[Y=1]f_{X|Y}(x|1)}$$

$$\eta_1(x) = \frac{\mathbb{1}_{x\in[a,a+1]}}{e^{-x} + \mathbb{1}_{x\in[a,a+1]}}$$

$$R^*(x) = \frac{e^{-x} + \mathbb{1}_{x\in[a,a+1]} - \mathbb{1}_{x\in[a,a+1]}}{e^{-x} + \mathbb{1}_{x\in[a,a+1]}}$$

$$\boxed{R^*(x) = \frac{e^{-x}}{e^{-x} + \mathbb{1}_{x\in[a,a+1]}}}$$

(3d) Determine risk of classifier resulting (in part b) from $a \to \hat{a}$;

$$\boxed{h^* = \underset{k\in\{0,1\}}{\arg\max} \frac{f_{X|Y}(x|k)}{e^{-x} + \mathbb{1}_{x\in[|a-\hat{a}|,\, |a-\hat{a}|+1]}(x)}}$$

(3e) $$P[\hat{h}(x) \neq Y] = P[|a-\hat{a}| \geq \epsilon]P[\hat{h}(x) \neq Y \mid |a-\hat{a}| \geq \epsilon] + P[|a-\hat{a}| < \epsilon]P[\hat{h}(x) \neq Y \mid |a-\hat{a}| < \epsilon]$$

$$= \delta\, P[\hat{h}(x) \neq Y \mid |a-\hat{a}| \geq \epsilon] + (1-\delta)P[\hat{h}(x) \neq Y \mid |a-\hat{a}| < \epsilon]$$

$$\Rightarrow = \frac{P[\hat{h}(x) \neq Y \cap |a-\hat{a}| \geq \epsilon]}{\delta} = \frac{\frac{1}{2}(1)}{1} \quad \text{if } \delta_{max} = 1$$

$$= \frac{1}{2} + (1-1) = \boxed{0}$$