Cryptography Notes

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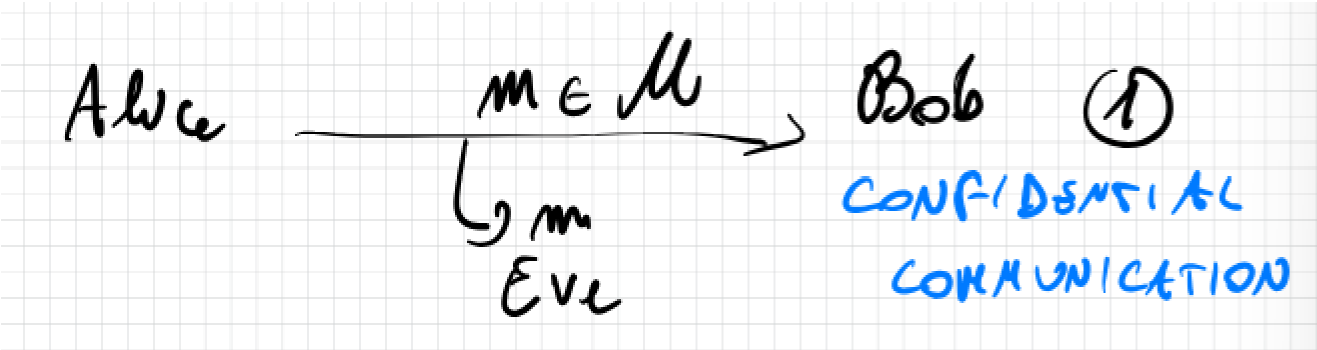
1 Intro to Cryptography

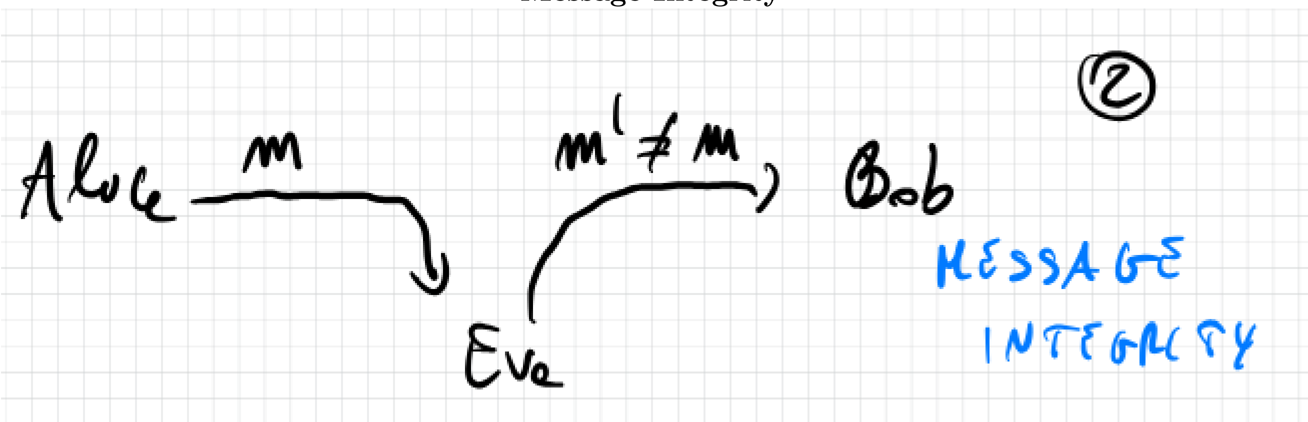
1.1 Secure Communication

We have multiple goals in cryptography, the most important ones being:

Confidential Integrity



Message Integrity



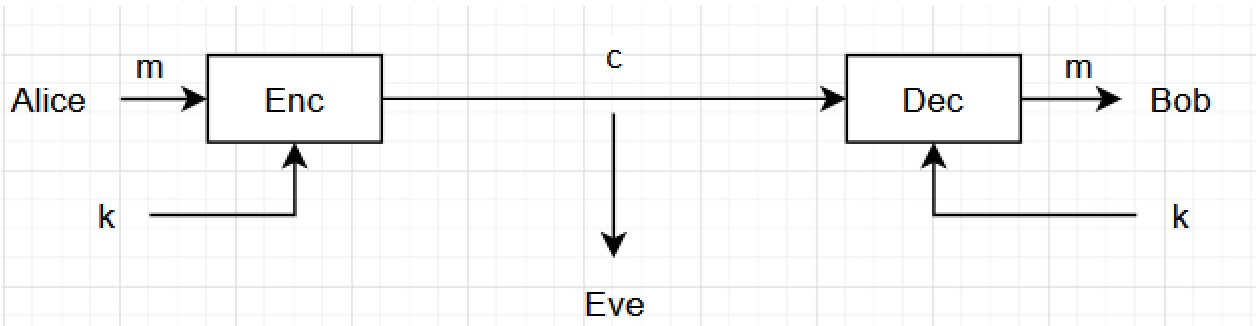
Basically we want our message to be both **confidential**, so no-one except the intended target sees it and we it to be unmodified, so that its **integrity** has not been compromised.

There are many different ways to do this, but in our case we only see two major ways:

- **Symmetric Cryptography**: Where Alice and Bob share a key $k \in \mathscr{K}$,the key is random and unknown to
- **Assymetric Cryptography**: Where Alice and Bob do not share a key, but they have each their own key pair (p_k, s_k) where p_k is the public key and s_k is the secret/private key

1.2 Unconditional Security

To achieve confidential communication, we use symmetric cryptography.



With $m \in \mathscr{M}, c \in \mathscr{C}, k \in \mathscr{K}$

In this case we have Alice sending a message m which is then encrypted utilizing a randomly generate key k to generate the cyphertext c, after that to get back to the initial message m, Bob will then need to decrypt it utilizing his own key k on cyphertext c.
In a more formal way we can define Symmetric encryption (SKE) as $\prod = (Enc, Dec)$ such that:

- Enc : $\mathscr{M} \times \mathscr{K} \rightarrow \mathscr{C}$
- Dec: $\mathscr{C} \times \mathscr{K} \rightarrow \mathscr{C}$
- k is uniform over $\mathscr{K}$ (k is chosen according to some distribution)

An encryption scheme must satisfy the correctness requirement:

Definition 1. $\forall k \in \mathscr{K}, \forall m \in \mathscr{M}$ *it holds that* $Dec(k, Enc(k, m)) = m$

Kerchoff's Principle:

Definition 2. *Security should not depend on the secrecy of the algorithm but on the secrecy of the key.*

1.3 Perfect Secrecy

Definition 3. *Let M be any distribution over* $\mathscr{M}$ *and K be uniform over* $\mathscr{K}$ *(Then observe C = Enc(K,M) in a distribution over C), we say that (Enc,Dec) =* $\prod$ *is* ***perfectly secret*** *if* $\forall M, \forall m \in \mathscr{M}, \forall c \in \mathscr{C} : Pr[M = m] = Pr(M = m|C = c)$ *(The probability that M is m is equal to the probability that M is m knowing that C is c, so by knowing the cyphertext, we dont gain additional information).*

Lemma 1. *The following are equivalent:*

- *Perfect Secrecy*
- *M and C are independant*
- $\forall m, m' \in \mathscr{M}, \forall c \in \mathscr{C} : Pr[Enc(k, m) = c] = Pr[Enc(k, m') = c]$ *with k being uniform over* $\mathscr{K}$

1.4 OTP

Let us see if OTP (*One Time Pad*) is perfectly secret
We know that the OTP uses $\oplus$ to generate and later decypher the cyphertext, we have that $K = M = C = \{0, 1\}^N$ with N being the length of the string, we know that:

- Enc (k,m) $= k \oplus m$
- Dec (k,c) $= c \oplus k = (k \oplus m) \oplus k = m$

To prove that it is perfectly secret let us utilize the third lemma:

$$Pr[C = c|M = m'] = Pr[\text{Enc}_k(m') = c] = Pr[m' \oplus K = c] = Pr[K = m' \oplus c] = 2^{-N}$$

and therefore:

$$Pr[Enc(k, m') = c] = 2^{-N}$$

There seem to be some limitations, the key can only be used once and it must as long as the message,lets assume we encrypt m" and m': $c_1 = k \oplus m_1$ $c_2 = k \oplus m_2$ therefore $c_1 \oplus c_2 = m_1 \oplus m_2$, so if I know a pair (m_1, c_1) then I could compute m_2, therefore we cannot encrypt two messages with the same key.

Theorem 1 (Shannon)**.** *Let* $\prod$ *be any perfectly secret SKE then we have* $|\mathscr{K}| \geq |\mathscr{M}|$.

Proof. Take $\prod$ to be uniform over $\mathscr{M}$. Take any c s.t. Pr[C=c] > 0.
Consider $\mathscr{M}' = \{Dec(k,c) : k \in \mathscr{K}\}$ and assume $|\mathscr{K}| < |\mathscr{M}|$ by contraddiction, then:

$$|\mathscr{M}|' \leq |\mathscr{K}| < |\mathscr{M}| \to |\mathscr{M}'| < |\mathscr{M}| \to \exists m \in \mathscr{M} \setminus \mathscr{M}'$$

Now:

$$Pr[M = m] = |\mathscr{M}|^{-1} \text{ but } Pr[M = m|C = c] = 0$$

□

1.5 Proof that the lemmas imply eachother

Let us prove that 1 $\implies$ 2 $\implies$ 3 $\implies$ 1

Let us start by proving that 1 $\implies$ 2:

Proof. We know that $Pr[M = m] = Pr[M = m|C = c] \to \frac{Pr[M=m \wedge C=c]}{Pr[C=c]} = Pr[M = m \wedge C = c]$
$= Pr[M = m] * Pr[C = c]$ and therefore we have proved their independence, so $I(M;C) = 0$ □

Let us prove that 2 $\implies$ 3

Proof. Let us fix an m from M and c from C:
$Pr[Enc(K,m) = c] = Pr[Enc(K,M) = c|M = m] \implies Pr[C = c|M = m] = Pr[C = c]$
Remember that Enc(...) is c!

We do the same thing for m' and we get: $Pr[C = c|M = m] = Pr[C = c]$ for both of them.
Therefore: $Pr[Enc(K,m') = c] = Pr[C = c]$ □

And now 3 $\implies$ 1: Take any c from C:
$Pr[C = c] = Pr[C = c|M = m]$ by 2 (we are claiming this)

If the claim is true then:
$Pr[M = m|C = c] * Pr[C = c] = Pr[M = m \wedge C = c] = Pr[C = c|M = m] * Pr[M = m] \implies$

$$\implies Pr[M = m] = \frac{Pr[M=m|C=c] * \cancel{Pr[C=c]}}{\cancel{Pr[C=c|M=m]}}$$

However we still need to prove the claim:

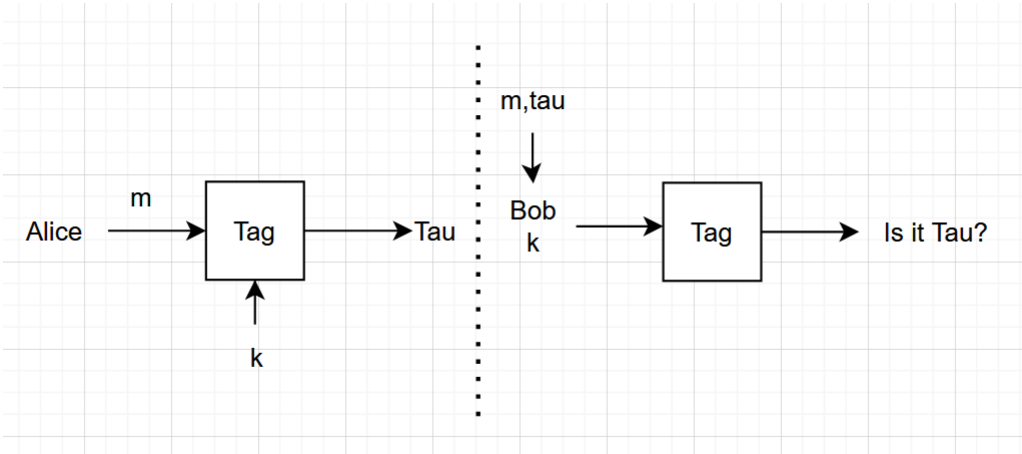
$$Pr[C = c] = \sum_{m'} Pr[C = c \wedge M = m'] = \sum_{m'} Pr[C = c|M = m'] * Pr[M = m'] =$$

$$\sum_{m'} Pr[Enc(K,m') = c|M = m'] * Pr[M = m'] = \sum_{m'} Pr[Enc(K,m') = c] * Pr[M = m']$$

$$\sum_{m'} Pr[Enc(k,m) = c] * Pr[M = m'] = Pr[Enc(k,m) = c] * \sum_{m'} Pr[M = m'] \impliedby 1$$

$$Pr[Enc(k,m) = c] = Pr[Enc(K,M) = c|M = m] \to Pr[C = c|M = m]$$

1.6 Message Authentication Codes



In case it is τ then we accept it, else no.
There is no need to prove correctness as τ is deterministic, so if we had the same k and m, we should get the same τ

Unforgeability

It should be hard to forge τ' such on msg m' and it should be hard to produce (m,τ) as long as m' $\neq$ m

Definition 4. ***Statistical secure MAC*** *We say that $\prod$ = Tag has ϵ-statistical security (unforgeability) if $\forall m, m' \in \mathscr{M}$ with $m \neq m'$ $\forall \tau, \tau' \in \mathscr{T}$:*

$$Pr[Tag(K, m') = \tau' \mid Tag(K, m) = \tau] \leq \epsilon$$

TLDR: Fix any m,m' with m' $\neq$ m take τ, τ' on the condition that τ is tag of m and given τ', it is always less than or equal to ϵ
Here ϵ is a parameter e.g. 2^{-80}
Exercise Let us prove that it is impossible to get $\epsilon = 0$
Because a random $\tau' \in \mathscr{T}$ has probability $\geq \frac{1}{|\mathscr{T}|}$ to be correct it is impossible.

Note that the definition is valid for One-Time!
We will show:

- The notion is Achievable
- It's inefficient, in fact:

Theorem 2. *Any t-time $2^{-\lambda}$ statistically secure Tag has a key of some (t+1)*λ*

We will now show that any form of hash function with a particular property satisfies the definition.

Definition 5. ***Pairwise independence*** *A family $\mathscr{H} = \{h_k : \mathscr{M} \to \mathscr{T}\}_{k \in \mathscr{K}}$ is pairwise independant if: $\forall m, m' \in \mathscr{M}$ s.t.m $\neq m'$ then: $(h(K, m), h(K, m'))$ is uniform over $\mathscr{T}^2 = \mathscr{T} \times \mathscr{T}$ for K uniform over $\mathscr{K}$*

Theorem 3. *Any family $\mathscr{H}$ of pairwise independent functions directly gives a $\epsilon = \frac{1}{|\mathscr{T}|}$ − statistically secure MAC.*

Proof. Fix any $m \in \mathscr{M}, \tau \in \mathscr{T}$:

$$Pr[Tag(K, m) = \tau] =$$

$$Pr_k[h(K,m)=\tau] = \frac{1}{|\mathscr{T}|} \text{ by pairwise independence}$$

Similiarly, for any m,m' s.t. m $\neq$ m', $\tau, \tau' \in \mathscr{T}$.

$$Pr_k[Tag(K,m)=\tau \wedge Tag(K,m')=\tau'] =$$

$$Pr_k[h(K,m)=\tau \wedge h(K,m')=\tau'] = \frac{1}{|\mathscr{T}|^2}$$

By Bayes:

$$Pr[Tag(K,m')=\tau' | Tag(K,m)=\tau] =$$

$$\frac{Pr[h(K,m;)=\tau' \wedge h(K,m)=\tau]}{Pr[h(K,m)=\tau]} =$$

$$\frac{\frac{1}{|\mathscr{T}|^2}}{\frac{1}{|\mathscr{T}|}} = \frac{1}{|\mathscr{T}|}$$

□

Now we need to instantiate it, here is a construction, Let p be a prime:

$$h_{a,b}(m) = am + b \mod p$$

$$k = (a,b) \in \mathbb{Z}_p^2 = \mathscr{K}$$

$$\mathbb{Z}_p = \mathscr{M} = \mathscr{T}$$

Lemma 2. *The above $\mathscr{H}$ is pairwise independant.*

Proof. For all $m, m' \in \mathbb{Z}_p, \tau, \tau' \in \mathbb{Z}_p$ with $m \neq m'$

$$\Pr_{(a,b)\in\mathbb{Z}_p^2}[h_{a,b}(m)=\tau \wedge h_{a,b}(m')=\tau'] =$$

$$\Pr_{(a,b)\in\mathbb{Z}_p^2}\left[\begin{pmatrix} m & 1 \\ m' & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{matrix} \tau \\ \tau' \end{matrix}\right] =$$

$$\Pr_{(a,b)\in\mathbb{Z}_p^2}\left[\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} m & 1 \\ m' & 1 \end{pmatrix} \begin{matrix} \tau \\ \tau' \end{matrix}\right] =$$

$$\frac{1}{p^2} = \frac{1}{|\mathbb{Z}_p|^2} = \frac{1}{|\mathscr{T}|^2}$$

□

1.7 Randomness Extraction

Alice and Bob need a **random** key, how can they generate it?
Randomness is crucial for crypto, and two components are necessary in any RNG (e.g. Fortuna, /dev/rand):

- Randomness extraction: By measuring physical quantities we can get an **unpredictable** sequence of bits (Not necessarily uniform or for cheap!)
  From this we extract a **ranom** Y which is short (e.g. 256 bits)
- Expand it to any amount (polynomial) using a psedorandom generator (PRG) - but this requires computational assumptions.

We want to understand how to extract from an unpredictable source X.
Example ***Von Neumann Extractor*** Assume $B \in 0,1$ s.t. $\Pr[B = 0] = p < \frac{1}{2}$.

- Sample $b_1 \in B, b_2 \in B$
- if $b_1 = b_2$ then Resample
- Else output 1 $\iff b_1 = 0, b_2 = 1,$ or 0 if $b_1 = 1, b_2 = 0$

Assuming it outputs something, this will be s.t.

$$\Pr[\text{Output } 0] = Pr[\text{Output } 1] = p * (1 - p)$$

$$\Pr[\text{No output after N tries}] = (1 - 2p(1 - p))^N \text{ which becomes small for large enough N}$$

We want to generalize this question, ideally we want to design a function Ext that takes a random variable X and outputs an uniform Ext(X), but this is impossible as the source must be unpredictable and Ext is deterministic

Definition 6 (Min-Entropy)**.** *The min-entropy of X is:* $H_\infty = -\log_2 \max \Pr[X = x]$

Example: Let $X \equiv U_m$ Uniform over $\{0,1\}^N$. $H_\infty = N$
If X is a costant we have $H_\infty(X) = 0$

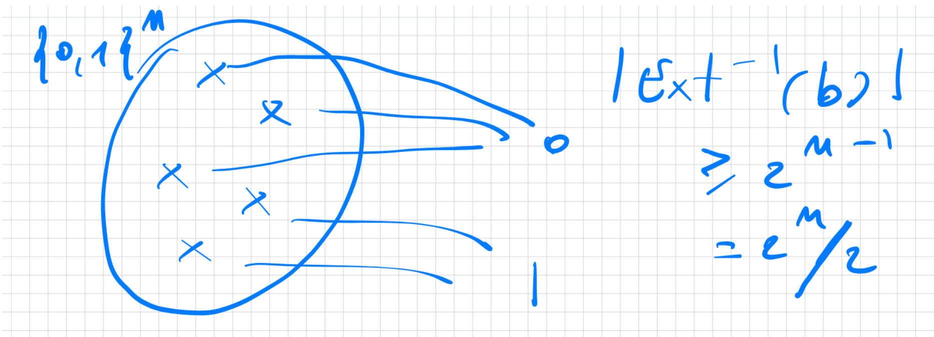
Here's the next best thing:
Design Ext that extracts UNIFORM $Y = Ext(X)$ for every X s.t. $H_\infty(X) \geq k$
But this is also impossible, even if

$$Ext(X) = b \in \{0,1\}$$

$$k = n - 1$$

$$x \in \{0,1\}^n$$

And here's why: fix any Ext:$\{0,1\}^n \to 0,1$ and let $b \in 0,1$ be the output of maximing $|Ext^{-1}(b)|$



The bad X: Define X to be Uniform over $Ext^{-1}(b)$. Since it is uniform: $H_\infty(X) \geq n-1$ but $Ext(X) = b$ so not uniform.

Solution: Swap the quantifiers.

Definition 7 (Seeded Extractor)**.** *A function* $Ext : \{0,1\}^n \times \{0,1\}^d \to \{0,1\}$ *is a (k,ϵ)-seeded extractor if for every X over s.t.* $H_\infty(X) \geq k$*:*

$$(S, Ext(S, X)) \approx_\epsilon (S, U_e)$$

for $S \equiv U_d$ *(uniform over* $\{0,1\}^d$*). (Note that* $(S, U_e) \equiv U_{d+e}$*).*

What does this mean? Thre is a standard way to measure distance between distributionsL

$$Z \equiv_\epsilon Z' \iff SD(Z, Z') \leq \epsilon$$

$$SD(Z, Z') = \frac{1}{2} \sum_z |Pr[Z = z] - Pr[Z' = z]|$$

This is equivalent: $\forall$ Unbounded adversary A:

$$|Pr[A(z) = 1 : z \in Z] - Pr[A(z) = 1 : z \in Z']| \leq \epsilon$$

Theorem 4 (Leftover Hash Lemma)**.** *Let $\mathscr{H} = \{h_s : \{0,1\}^n \to \{0,1\}^l\}_{s \in \{0,1\}^d}$ be a family of pairwise independent hash functions. Then Ext(x,s) = $h_s(x)$ is a (k,ϵ)-seeded extractor for $k \geq l + 2\log_2(\frac{1}{\epsilon})$-2.*

Lemma 3. *Let Y be a RV over $\mathscr{Y}$. Such that:*

$$Col(y) = \sum_{y \in \mathscr{Y}} Pr[Y = y]^2 \leq \frac{1}{|\mathscr{Y}|} * (1 + 4\epsilon^2)$$

Then, SD(Y,U) $\leq \epsilon$