Hybrid Heston-Hull-White Model

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Presentation Overview

In recent years, the financial world has focused on the accurate pricing of exotic and hybrid products based on a combination of underlyings from different asset classes. From the Black-Scholes model a major step away in derivatives pricing was made by Hull and White [1990], Stein and Stein [1991] and Heston [1993] who defined volatility as a diffusion process. However, the development of new and more complex financial products now required the modelling of stochastic interest rate components.

Pricing hybrid contingent claims involving equity, cash, bonds or other products dependent on interest rates;

Pricing portfolio that consists of contracts belonging to different asset classes, like interest rates, stocks, foreign exchange, commodities, etc;

Pricing long-dated contingent claims accurately (movements in the interest rate may influence the behaviour of stock prices, especially in the long run).

The hybrid Heston-Hull-White (HHW) model combines the Heston (1993) stochastic volatility and Hull-White (1990) short rate models. Grzelak and Oosterlee [2011] have combined these two models to allow the correlations to be non-zero.

A basic interest rate product is the zero coupon bond, $P(t,T)$, which pays 1 currency unit at maturity time T.
The fundamental theorem of asset pricing states that the price at time t of any contingent claim with payoff $H(T)$, under the risk-neutral measure $\mathbb{Q}$, is given by:

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(z)dz} H(T) | \mathcal{F}(t)\right] \quad H(T) = P(T,T) \equiv 1 \tag{1.1}$$

Assuming no arbitrage and market completeness the forward rate $R(t,S,T)$ is:

$$R(t,S,T) = -\frac{\log P(t,T) - \log P(t,S)}{T-S} \tag{1.2}$$

In the limit, $T - S \to 0$, the instantaneous forward rate is defined as:

$$f^r(t,T) \stackrel{\text{def}}{=} \lim_{S \to T} R(t,S,T) = -\frac{\partial}{\partial T} \log P(t,T) \tag{1.3}$$

The short rate $r(t)$ is defined as the limit of the instantaneous forward rate $r(t) \equiv f^r(t,t)$. The money-savings account is defined as:

$$M(t) := \exp\left(\int_0^t r(s)ds\right) = \exp\left(\int_0^t f^r(s,s)ds\right) \tag{1.4}$$

Hull-White model

Short-Rate Model

The Hull-White model [Hull and White, 1990] is a stochastic interest rate no-arbitrage model in which the short rate is driven by a Generalized Ornstein-Uhlenbeck mean reverting process. Under the risk-neutral measure $\mathbb{Q}$ the dynamics $r(t)$ is given by:

$$\mathrm{d}r_t = \lambda(\theta_t - r_t)dt + \eta dW_t^r \tag{1.5}$$

where:

- θ_t is a deterministic time-dependent drift term chosen to fit the interest rate term structure observed in the market. For the Hull-White model is:

$$\theta_t = \frac{1}{\lambda}\frac{\partial}{\partial t}f(0,t) + f(0,t) + \frac{\eta^2}{2\lambda^2}(1 - e^{-2\lambda t}) \tag{1.6}$$

- W_t^r is the Brownian motion under measure $\mathbb{Q}$;
- η determines the volatility of the interest rate;
- λ is the reversion rate parameter (speed of reversion).

The short rate is normally distributed, so we have the possibility of a negative interest rate.

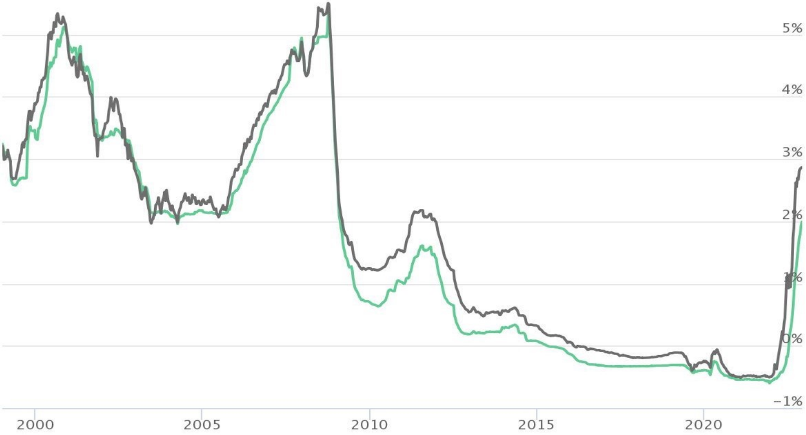


Figure: *Grey line - Euribor 12 months; Green line - Euribor 3 months; Source = Euribor*

It is important to properly choose the initial value for the process $r(t)$:

$$r(0) = f(0,0) \approx -\frac{\partial \log P(0,\epsilon)}{\partial \epsilon} \quad \text{for } \epsilon \to 0 \tag{1.7}$$

$$f(0,t) = -\frac{\log(P(t+dt)) - \log(P(t-dt))}{2dt} \tag{1.8}$$

$$\theta_t = \frac{1}{\lambda}\frac{f(0,t+dt) - f(0,t-dt)}{2dt} + f(0,t) + \frac{\eta^2}{2\lambda^2}(1-e^{-2\lambda t}) \tag{1.9}$$

We can generate sample paths over the interval $[0,T]$ by discretizing this interval:

$$0 = t_0 < t_1 < \ldots < t_i < \ldots < t_{n-1} < t_n = T$$

$$r_{i+1} = r_i + \lambda(\theta_{t_i} - r_i)dt + \eta(W^r_{i+1} - W^r_i) \tag{1.10}$$

with $i = 0,\ldots,n$ and $\Delta t_i = t_{i+1} - t_i$ we can integrate over a time interval $[t_i, t_{i+1}]$:

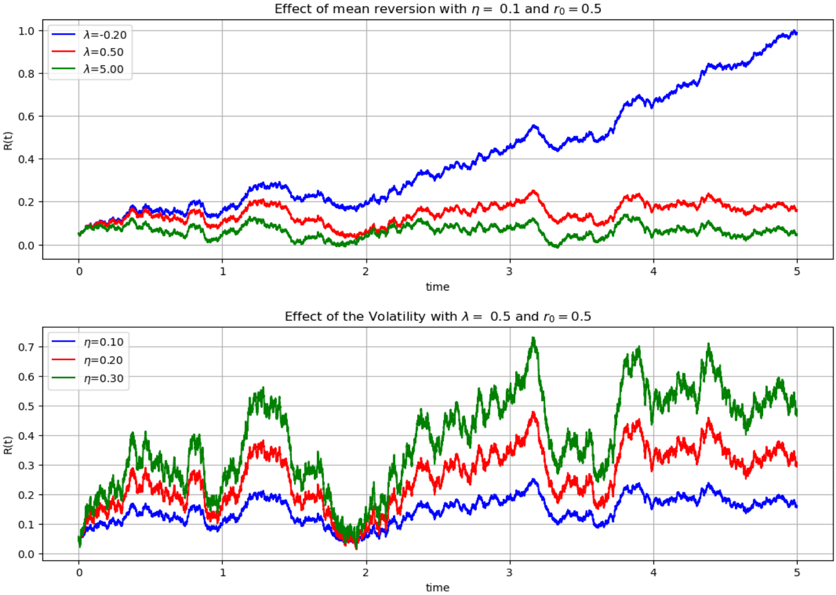
$$r_{i+1} = r_i + \int_{t_i}^{t_{i+1}} \lambda\,(\theta_s - r_s)\,ds + \eta\int_{t_i}^{t_{i+1}} dW^r_s \tag{1.11}$$

$\int_{t_i}^{t_{i+1}} dW^r_s = W^r_{i+1} - W^r_i$ and $\sqrt{dt}Z^r = \sqrt{\Delta t_i}Z^r$ are distributed in the same way, with $Z^r \sim \mathcal{N}(0,1)$ variables. Using the Euler discretization:

$$r_{i+1} = r_i + \lambda\theta_i\Delta t_i - \lambda r_i\Delta t_i + \eta\sqrt{\Delta t_i}Z^r \tag{1.12}$$

Simulation of the Hull-White SDE

Hull-White Model



Under the Heston stochastic volatility model, the asset price volatility is indirectly modelled through the variance process. The Heston model under the risk-neutral pricing measure $\mathbb{Q}$ is specified by the following system of SDEs:

$$\begin{cases} dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^x & S_0 > 0 \\ dv_t = \kappa(\bar{v} - v_t)dt + \gamma\sqrt{v_t} dW_t^v & v_0 > 0 \end{cases} \tag{1.13}$$

The asset price process S_t follows a process resembling geometric Brownian motion with non-constant instantaneous volatility, where:

- W_t^x is the Brownian motion under measure $\mathbb{Q}$;
- r is the risk-free interest rate (constant);

The volatility process v_t is a CIR-type process, where:

- W_t^v is the Brownian motion under measure $\mathbb{Q}$;
- κ is the speed of adjustment of the volatility towards its mean;
- $\bar{v}$ is the long-run mean;
- γ is the second-order volatility.

The correlation between the two Brownian motions is given by: $dW_t^s dW_t^v = \rho_{s,v} dt$ with $|\rho_{s,v}| \leq 1$.

The Heston-Hull-White (HHW) model is formed by combining the Heston model and Hull-White model in a correlated manner. Under the $\mathbb{Q}$ measure this results in the following system of SDEs:

$$\begin{cases} dr_t = \lambda(\theta_t - r_t)dt + \eta dW_t^r & r_0 \in \mathbb{R} \\ dv_t = \kappa(\bar{v} - v_t)dt + \gamma\sqrt{v_t}dW_t^v & v_0 > 0 \\ dS_t = r_t S_t dt + \sqrt{v_t} S_t dW_t^s & S_0 > 0 \end{cases} \tag{2.1}$$

For the HHW model, the correlations are given by:

$$dW_t^s dW_t^v = \rho_{s,v} dt, \quad dW_t^s dW_t^r = \rho_{s,r} dt, \quad dW_t^v dW_t^r = \rho_{v,r} dt \tag{2.2}$$

The price of any contingent claim under a risk-neutral measure can be obtained by risk-neutral valuation, i.e, by computing the expectation of the discounted payoff. With the payoff $H(S_T, v_T, r_T)$ at time T, the t-price of a claim is given by:

$$V(S_t, v_t, r_t) = \mathbb{E}^{\mathbb{Q}}\left[e^{\int_t^T r_s ds} H(S_T, v_T, r_T) | \mathcal{F}(t)\right] \tag{2.3}$$

However, due to the stochastic interest rate, the discount factor cannot be disentangled from the expectation.

Heston-Hull-White Model

Independent Brownian Motion

With $x_t = \log S_t$, the HHW log-dynamics are described by:

$$\begin{cases} dr_t = \lambda\left(\theta_t - r_t\right)dt + \eta dW_t^r & r_0 \in \mathbb{R} \\ dv_t = \kappa\left(\bar{v} - v_t\right)dt + \gamma\sqrt{v_t}dW_t^v & v_0 > 0 \\ dx_t = \left(r_t - \frac{1}{2}v_t\right)dt + \sqrt{v_t}dW_t^x & x_0 = \log S_0 \end{cases} \tag{2.4}$$

With the help of the Cholesky decomposition of the correlation matrix $\mathbf{C}$ we can substitute the correlated Brownian motion $\mathbf{W}_t$ with an independent n-dimensional Brownian motion $\mathbf{B}_t$. Define $\mathbf{W}_t = [W_t^r, W_t^v, W_t^x]^\top$ and $\mathbf{C}$ as:

$$\mathbf{C} = \begin{bmatrix} 1 & \rho_{r,v} & \rho_{r,x} \\ \rho_{r,v} & 1 & \rho_{v,x} \\ \rho_{r,x} & \rho_{v,x} & 1 \end{bmatrix} \tag{2.5}$$

With $\mathbf{L}$ as the lower triangular matrix of $\mathbf{C} = \mathbf{L}\mathbf{L}^T$, we have: $\mathbf{W}_t = \mathbf{L}\mathbf{B}_t$

$$\begin{aligned} W_t^r &= B_t^r \\ W_t^v &= \rho_{r,v}B_t^r + \sqrt{1-\rho_{r,v}^2}B_t^v \\ W_t^x &= \rho_{r,x}B_t^r + \frac{\rho_{v,x} - \rho_{r,v}\rho_{r,x}}{\sqrt{1-\rho_{r,v}^2}}B_t^v + \sqrt{1-\rho_{r,x}^2 - \frac{(\rho_{v,x} - \rho_{r,v}\rho_{r,x})^2}{1-\rho_{r,v}^2}}B_t^x \end{aligned} \tag{2.6}$$

The HHW log-dynamics can be expressed with $\mathbf{B}_t$ as:

$$d\mathbf{X}_t = \mu\left(\mathbf{X}_t\right)dt + \sigma\left(\mathbf{X}_t\right)d\mathbf{B}_t \tag{2.7}$$

with state vector $\mathbf{X}_t$, drift vector $\mu\left(\mathbf{X}_t\right)$ and volatility matrix $\sigma\left(\mathbf{X}_t\right)$ as:

$$\mathbf{X}_t = \begin{bmatrix} r_t \\ v_t \\ x_t \end{bmatrix} \quad \mu\left(\mathbf{X}_t\right) = \begin{bmatrix} \lambda\left(\theta - r_t\right) \\ \kappa\left(\bar{v} - v_t\right) \\ r_t - \frac{1}{2}v_t \end{bmatrix} \tag{2.8}$$

$$\sigma\left(\mathbf{X}_t\right) = \begin{bmatrix} \eta & 0 & 0 \\ \rho_{r,v}\gamma\sqrt{v_t} & \sqrt{1-\rho_{r,v}^2}\gamma\sqrt{v_t} & 0 \\ \rho_{r,x}\sqrt{v_t} & \frac{\left(\rho_{v,x}-\rho_{r,x}\rho_{r,v}\right)}{\sqrt{1-\rho_{r,v}^2}}\sqrt{v_t} & \sqrt{1-\rho_{r,x}^2-\left(\frac{\rho_{v,x}-\rho_{r,v}\rho_{r,x}}{\sqrt{1-\rho_{r,v}^2}}\right)^2}\sqrt{v_t} \end{bmatrix} \tag{2.9}$$

The matrix $\Sigma\left(\mathbf{X}_t\right) = \sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^{\mathrm{T}}$ is the symmetric instantaneous covariance matrix:

$$\Sigma\left(\mathbf{X}_t\right) = \begin{bmatrix} \eta^2 & \rho_{r,v}\gamma\eta\sqrt{v_t} & \rho_{r,x}\eta\sqrt{v_t} \\ \rho_{r,v}\gamma\eta\sqrt{v_t} & \gamma^2 v_t & \rho_{v,x}\gamma v_t \\ \rho_{r,x}\eta\sqrt{v_t} & \rho_{v,x}\gamma v_t & v_t \end{bmatrix} \tag{2.10}$$

The system is affine if $\rho_{r,v} = \rho_{r,x} = 0$. We will assume that the interest rate r_t and the variance v_t are not correlated, i.e., $\rho_{r,v} = 0$.

We will use a deterministic approximation for the non-affine term in $\Sigma\left(\mathbf{X}_t\right)$:

$$\rho_{r,x}\eta\sqrt{v(t)} \approx \rho_{r,x}\eta\mathbb{E}[\sqrt{v(t)}] \tag{2.11}$$

The resulting covariance matrix is given by:

$$\Sigma\left(\mathbf{X}_t\right) = \begin{bmatrix} \eta^2 & 0 & \rho_{r,x}\eta\mathbb{E}\left[\sqrt{v_t}\right] \\ 0 & \gamma^2 v_t & \rho_{v,x}\gamma v_t \\ \rho_{r,x}\eta\mathbb{E}\left[\sqrt{v_t}\right] & \rho_{v,x}\gamma v_t & v_t \end{bmatrix} \tag{2.12}$$

For a given time $t > 0$, the expectation and variance of $\sqrt{v(t)}$, where $v(t)$ is a CIR-type process [Cox et al., 1985] are given by:

$$\mathbb{E}[\sqrt{v_t} \mid \mathcal{F}(0)] = \sqrt{2\bar{c}(t,0)}\mathrm{e}^{-\bar{\kappa}(t,0)/2}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\bar{\kappa}(t,0)}{2}\right)^k\frac{\Gamma\left(\frac{1+\delta}{2}+k\right)}{\Gamma\left(\frac{\delta}{2}+k\right)}, \tag{2.13}$$

$$\mathrm{Var}[\sqrt{v_t} \mid \mathcal{F}(0)] = \bar{c}(t,0)(\delta+\bar{\kappa}(t,0)) - (\mathbb{E}[\sqrt{v_t}])^2 \tag{2.14}$$

where $\Gamma(k)$ is the Gamma function and:

$$\bar{c}(t,0) = \frac{1}{4\kappa}\gamma^2\left(1-\mathrm{e}^{-\kappa t}\right), \quad \delta = \frac{4\kappa\bar{v}}{\gamma^2}, \quad \bar{\kappa}(t,0) = \frac{4\kappa v(0)\mathrm{e}^{-\kappa t}}{\gamma^2\left(1-\mathrm{e}^{-\kappa t}\right)} \tag{2.15}$$

It was shown in [Cox et al., 1985; Broadie and Kaya, 2006], that, $v(t) \mid v(0)$ is distributed as $\bar{c}(t,0)$ times a noncentral chi-squared random variable, $\chi^2(\delta, \bar{\kappa}(t,0))$, with δ the degrees of freedom parameter and noncentrality parameter $\bar{\kappa}(t,0)$, i.e.,

$$v(t) \mid v(0) \sim \bar{c}(t,0)\chi^2(\delta, \bar{\kappa}(t,0)), \quad t > 0 \tag{2.16}$$

The density function for $v(t)$ can be expressed as:

$$f_{v(t)}(x) := \frac{\mathrm{d}}{\mathrm{d}x} F_{v(t)}(x) = \frac{1}{\bar{c}(t,0)} f_{\chi^2(\delta,\bar{\kappa}(t,0))}(x/\bar{c}(t,0)) \tag{2.17}$$

By [Dufresne, 2001], it follows that:

$$\begin{aligned}\mathbb{E}[\sqrt{v(t)} \mid \mathcal{F}(0)] &:= \int_0^\infty \frac{\sqrt{x}}{\bar{c}(t,0)} f_{\chi^2(\delta,\bar{\kappa}(t,0))} \left(\frac{x}{\bar{c}(t,0)}\right) \mathrm{d}x \\ &= \sqrt{2\bar{c}(t,0)} \frac{\Gamma\left(\frac{1+\delta}{2}\right)}{\Gamma\left(\frac{\delta}{2}\right)} {}_1F_1\left(-\frac{1}{2}, \frac{\delta}{2}, -\frac{\bar{\kappa}(t,0)}{2}\right)\end{aligned} \tag{2.18}$$

where ${}_1F_1(a;b;z)$ is a so-called confluent hyper-geometric function, which is also known as Kummer's function [Kummer, 1936] of the first kind.

Given a system of SDEs with a n-dimensional state vector $\mathbf{X}(t)$:

$$d\mathbf{X}(t) = \mu(\mathbf{X}(t))dt + \sigma(\mathbf{X}(t))d\mathbf{W}(t) \tag{2.19}$$

The system is said to be of the affine form if:

$$\begin{aligned} \mu(\mathbf{X}(t)) &= a_0 + a_1\mathbf{X}(t), \\ \Sigma\left(\mathbf{X}_t\right) = (\sigma(\mathbf{X}(t))\sigma(\mathbf{X},(t))^T)_{i,j} &= (c_0)_{i,j} + (c_1)_{i,j}^T\mathbf{X}(t), \\ r(\mathbf{X}(t)) &= r_0 + r_1^T\mathbf{X}(t) \end{aligned} \tag{2.20}$$

$$(a_0, a_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}, \quad (c_0, c_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n}, \quad (r_0, r_1) \in \mathbb{R} \times \mathbb{R}^n \tag{2.21}$$

The conditions imply that for a model to be affine, each element of its drift and covariance matrices must be a linear function of the state variables.
The discounted **ChF** [Duffie et al., 2000] with $\mathbf{u}$ a complex n-element row vector $\mathbf{u} = [u, 0, \ldots, 0]$ and $\tau = T - t$ the time-lag between today and the maturity time, is:

$$\phi(\mathbf{u}, \mathbf{X}(t), t, T) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(s)ds + i\mathbf{u}^T\mathbf{X}(T)}|\mathcal{F}(t)\right] = e^{A(\mathbf{u},\tau)+\mathbf{B}^T(\mathbf{u},\tau)\mathbf{X}(t)} \tag{2.22}$$

With $\mathbf{X}(t) = [X(t), r(t), v(t)]$, the discounted characteristic function is given by:

$$\phi(\mathbf{u}, \mathbf{X}(t), t, T) = \exp\left(A(u,\tau) + B(u,\tau)x(t) + C(u,\tau)r(t) + D(u,\tau)v(t)\right) \tag{2.23}$$

The function $A(\mathbf{u},\tau)$ and $\mathbf{B}^\top(\mathbf{u},\tau) = [B(u,\tau), C(u,\tau), D(u,\tau)]$ satisfy the following system of complex-valued ordinary differential equations:

$$\frac{d}{d\tau}\mathbf{B}(\mathbf{u},\tau) = -r_1 + a_1^\top \mathbf{B}(\mathbf{u},\tau) + \frac{1}{2}\mathbf{B}^\top(\mathbf{u},\tau) c_1 \mathbf{B}(\mathbf{u},\tau), \quad \mathbf{B}^\top(\mathbf{u},\tau) = i\mathbf{u} \tag{2.24}$$

$$\frac{d}{d\tau}A(\mathbf{u},\tau) = -r_0 + \mathbf{B}^\top(\mathbf{u},\tau) a_0 + \frac{1}{2}\mathbf{B}^\top(\mathbf{u},\tau) c_0 \mathbf{B}(\mathbf{u},\tau), \quad A(\mathbf{u},0) = 0 \tag{2.25}$$

The HHW affine decomposition is given by:

$$a_0 = \begin{bmatrix} 0 \\ \kappa\bar{v} \\ \lambda\theta \end{bmatrix}, \quad a_1 = \begin{bmatrix} 0 & -\frac{1}{2} & 1 \\ 0 & -\kappa & 0 \\ 0 & 0 & -\lambda \end{bmatrix}, \quad r_0 = 0, \quad r_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$c_0 = \begin{bmatrix} 0 & 0 & \rho_{r,x}\eta\mathbb{E}\left[\sqrt{v_t}\right] \\ 0 & 0 & 0 \\ \rho_{r,x}\eta\mathbb{E}\left[\sqrt{v_t}\right] & 0 & \eta^2 \end{bmatrix} c_1 = \begin{bmatrix} (0,1,0) & (0,\rho_{v,x}\gamma,0) & (0,0,0) \\ (0,\rho_{v,x}\gamma,0) & (0,\gamma^2,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) \end{bmatrix}$$

With $\mathbf{X}(t) = [X(t), r(t), v(t)]$, the discounted characteristic function is given by:

$$\phi := \phi(\mathbf{u}, \mathbf{X}(t), t, T) = \exp\left(A(u,\tau) + B(u,\tau)X(t) + C(u,\tau)r(t) + D(u,\tau)v(t)\right) \tag{2.26}$$

The functions $A(u,\tau)$, $B(u,\tau)$, $C(u,\tau)$, $D(u,\tau)$ with the parameters of the HHW model and for $u \in \mathbb{C}$ and $\tau = T - t \geq 0$, satisfy the following system of ODEs:

$$
\begin{aligned}
\frac{\mathrm{d}B}{\mathrm{d}\tau} &= 0, \quad B(u,0) = iu \\
\frac{\mathrm{d}C}{\mathrm{d}\tau} &= -1 - \lambda C + B, \quad C(u,0) = 0 \\
\frac{\mathrm{d}D}{\mathrm{d}\tau} &= B(B-1)/2 + (\gamma\rho_{x,v}B - \kappa)\, D + \gamma^2 D^2/2, \quad D(u,0) = 0 \\
\frac{\mathrm{d}A}{\mathrm{d}\tau} &= \lambda\theta C + \kappa\bar{v}D + \eta^2 C^2/2 + \eta\rho_{x,r}\mathbb{E}[\sqrt{v(t)}]BC, \quad A(u,0) = 0
\end{aligned}
\tag{2.27}
$$

The solution of the ODEs system is given by:

$$
\begin{aligned}
B(u,\tau) &= iu \\
C(u,\tau) &= (iu-1)\lambda^{-1}\left(1 - \mathrm{e}^{-\lambda\tau}\right) \\
D(u,\tau) &= \frac{1 - \mathrm{e}^{-D_1\tau}}{\gamma^2\left(1 - g\mathrm{e}^{-D_1\tau}\right)}\left(\kappa - \gamma\rho_{x,v}iu - D_1\right) \\
A(u,\tau) &= I_1(\tau) + \kappa\bar{v}I_2(\tau) + \frac{1}{2}\eta^2 I_3(\tau) + \eta\rho_{x,r}I_4(\tau)
\end{aligned}
\tag{2.28}
$$

with:

$$D_1 = \sqrt{(\gamma \rho_{x,v} iu - \kappa)^2 - \gamma^2 iu(iu-1)} \qquad g = \frac{\kappa - \gamma \rho_{x,v} iu - D_1}{\kappa - \gamma \rho_{x,v} iu + D_1}$$

The integrals $I_2(\tau)$ and $I_3(\tau)$ admit an analytic solution, and $I_1(\tau)$ and $I_4(\tau)$ admits a semi-analytic solution:

$$\begin{aligned}
I_1(\tau) &= (iu-1)\int_0^\tau (1-e^{-\lambda z})\theta_{(T-z)}\mathrm{d}z \\
I_2(\tau) &= \frac{\tau}{\gamma^2}(\kappa - \gamma\rho_{x,v} iu - D_1) - \frac{2}{\gamma^2}\log\left(\frac{1-g\mathrm{e}^{-D_1\tau}}{1-g}\right) \\
I_3(\tau) &= \frac{1}{2\lambda^3}(i+u)^2\left(3+\mathrm{e}^{-2\lambda\tau} - 4\mathrm{e}^{-\lambda\tau} - 2\lambda\tau\right) \\
I_4(\tau) &= iu\int_0^\tau \mathbb{E}[\sqrt{v_{(T-z)}}]C(u,z)\mathrm{d}z \\
&= -\frac{1}{\lambda}(iu+u^2)\int_0^\tau \mathbb{E}[\sqrt{v_{(T-z)}}]\left(1-\mathrm{e}^{-\lambda z}\right)\mathrm{d}z
\end{aligned} \tag{2.29}$$

The probability density function and its characteristic function, $f_X(y)$ and $\phi_X(u)$, form an example of a Fourier pair:

$$\phi_X(u) = \int_{\mathbb{R}} \mathrm{e}^{iyu} f_X(y)\mathrm{d}y \qquad f_X(y) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-iuy}\phi_X(u)\mathrm{d}u \tag{3.1}$$

The definition of the Fourier expansion of a function $g(x)$ on an interval $[-1,1]$ is:

$$g(\theta) = \sum_{k=0}^{\infty}{}' \bar{A}_k \cos(k\pi\theta) + \sum_{k=1}^{\infty} \bar{B}_k \sin(k\pi\theta) \tag{3.2}$$

where the prime at the sum, $\sum'$, indicates that the first term in the summation is weighted by one-half, and the coefficients are given by:

$$\bar{A}_k = \int_{-1}^{1} g(\theta)\cos(k\pi\theta)d\theta, \quad \bar{B}_k = \int_{-1}^{1} g(\theta)\sin(k\pi\theta)d\theta \tag{3.3}$$

By setting $\bar{B}_k = 0$, we obtain the classical Fourier cosine expansion, by which we can represent even functions around $\theta = 0$ exactly.

For functions supported on any other finite interval, $[a, b] \in \mathbb{R}$, the Fourier cosine series expansion can be obtained via a change of variables:

$$\theta := \frac{y-a}{b-a}\pi, \quad y = \frac{b-a}{\pi}\theta + a \tag{3.4}$$

$$g(y) = \sum_{k=0}^{\infty}{}' \bar{A}_k \cdot \cos\left(k\pi \frac{y-a}{b-a}\right), \quad \bar{A}_k = \frac{2}{b-a}\int_a^b g(y)\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y \tag{3.5}$$

The integrands have to decay to zero at $\pm\infty$ so we can truncate the integration range without losing significant accuracy. Suppose $[a, b] \in \mathbb{R}$ is chosen such that the truncated integral approximates the infinite counterpart very well, i.e.:

$$\hat{\phi}_X(u) := \int_a^b \mathrm{e}^{iuy} f_X(y)\mathrm{d}y \approx \int_{\mathbb{R}} \mathrm{e}^{iuy} f_X(y)\mathrm{d}y = \phi_X(u). \tag{3.6}$$

For any random variable X, and constant $a \in \mathbb{R}$, the following equality holds:

$$\phi_X(u)\mathrm{e}^{ia} = \mathbb{E}\left[\mathrm{e}^{iuX+ia}\right] = \int_{-\infty}^{\infty} \mathrm{e}^{i(uy+a)} f_X(y)dy \tag{3.7}$$

Substitute $u = \frac{k\pi}{b-a}$ and multiply by $\exp\left(-i\frac{ka\pi}{b-a}\right)$ and taking the real part:

$$\mathrm{Re}\left\{\hat{\phi}_X\left(\frac{k\pi}{b-a}\right)\cdot\exp\left(-i\frac{ka\pi}{b-a}\right)\right\} = \int_a^b \cos\left(k\pi\frac{y-a}{b-a}\right) f_X(y)dy \tag{3.8}$$

At the right-hand side we have the definition of $\bar{A}_k$ so:

$$\bar{A}_k \equiv \frac{2}{b-a}\mathrm{Re}\left\{\hat{\phi}_X\left(\frac{k\pi}{b-a}\right)\cdot\exp\left(-i\frac{ka\pi}{b-a}\right)\right\} \tag{3.9}$$

$$\bar{F}_k := \frac{2}{b-a}\mathrm{Re}\left\{\phi_X\left(\frac{k\pi}{b-a}\right)\cdot\exp\left(-i\frac{ka\pi}{b-a}\right)\right\} \tag{3.10}$$

Than follows that $\bar{A}_k \approx \bar{F}_k$ and we can replace $\bar{A}_k$ by $\bar{F}_k$ in the series expansion of $f_X(y)$ on $[a,b]$ and truncate the series summation, so that:

$$\hat{f}_X(y) \approx {\sum_{k=0}^{N-1}}' \bar{F}_k \cos\left(k\pi\frac{y-a}{b-a}\right) \tag{3.11}$$

The cosine series expansions of functions without singularities and discontinuities in any of its derivatives anywhere in the complex plane, except at ∞ exhibit an exponential convergence [Boyd, 1989], we can expect to give highly accurate approximations, with a small value for N, to density functions that have no singularities on $[a,b]$.

- Approximation for the standard normal density function.

The PDF and characteristic function are known in closed-form, and read, respectively:

$$f_{\mathcal{N}(0,1)}(y) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}y^2}, \quad \phi_{\mathcal{N}(0,1)}(u) = \mathrm{e}^{-\frac{1}{2}u^2} \tag{3.12}$$

- Approximation for the lognormal density function.

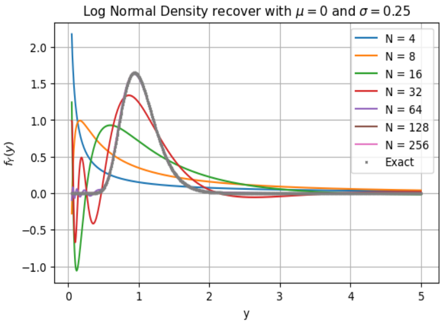
The characteristic function of lognormal random variables is not known in closed form, and therefore we cannot simply apply the COS method to a log-normal characteristic function. With $Y = \mathrm{e}^X$ a log-normal random variable and $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$. The CDF of Y in terms of the CDF of X can be calculated as:

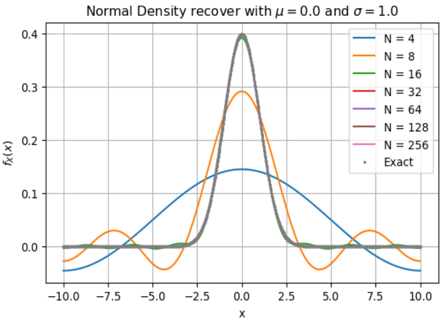
$$F_Y(y) \stackrel{\text{def}}{=} \mathbb{P}[Y \leq y] = \mathbb{P}\left[\mathrm{e}^X \leq y\right] = \mathbb{P}[X \leq \log(y)] = F_X(\log(y)) \tag{3.13}$$

By differentiation, we get:

$$f_Y(y) \stackrel{\text{def}}{=} \frac{\mathrm{d}F_Y(y)}{\mathrm{d}y} = \frac{\mathrm{d}F_X(\log(y))}{\mathrm{d}\log y} \frac{\mathrm{d}\log(y)}{\mathrm{d}y} = \frac{1}{y} f_X(\log(y)) \tag{3.14}$$

Since X is normally distributed, and $\phi_X(u) = \mathrm{e}^{i\mu u - \frac{1}{2}\sigma^2 u^2}$, by the COS method, we may recover the PDF of a lognormal random variable.





N	$Error$
4	$2.174111972e+00$
8	$1.2811026962e+00$
16	$1.242288484e+00$
32	$9.866610056e-01$
64	$9.404080678e-02$
128	$6.110817377e-06$
256	$1.075039460e-14$

N	$Error$
4	$2.537377837e-01$
8	$1.075173272e-01$
16	$7.172377614e-03$
32	$4.032340789e-07$
64	$2.486596021e-16$
128	$2.486596125e-16$
256	$2.4865961258e-16$

The COS formula for pricing European options is obtained by substituting the density function with its Fourier cosine series expansion. The risk-neutral valuation formula is the point of departure for pricing European options.
With $X(t) := \log S(t)$ and with $X(t)$ taking values $X(t_0) = x$ and $X(T) = y$, the value at time t of a plain vanilla European option with $\tau = T - t_0$ is:

$$V(t_0, x) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_{t_0}^{T} r(z)\mathrm{d}z} V(T, y) \mid \mathcal{F}(t_0)\right] = \int_{\mathbb{R}} V(T, y) f_{\mathbf{X}}(T, y; t_0, x)\, \mathrm{d}y \tag{3.15}$$

$f_{\mathbf{X}}(y) \equiv f_{\mathbf{X}}(T, y; t_0, x) = \int_{\mathbb{R}} f_{X,z}(T, y; t_0, x)\mathrm{d}z$ is the transition probability density of $X(T)$ with $z(t) = -\int_{t_0}^{T} r(z)\mathrm{d}z$.
Since $f_X(y)$ rapidly decays to zero as $y \to \pm\infty$, we truncate the infinite integration range to $[a, b] \subset \mathbb{R}$, without losing significant accuracy we obtain:

$$V(t_0, x) \approx V_{\mathrm{I}}(t_0, x) = \int_a^b V(T, y) f_X(y) dy \tag{3.16}$$

Since $f_X(y)$ is usually not known whereas the characteristic function is, the density is approximated by its Fourier cosine expansion in y:

$$\hat{f}_X(y) = \sum_{k=0}^{+\infty} \bar{A}_k(x) \cos\left(k\pi \frac{y-a}{b-a}\right), \quad \bar{A}_k(x) := \frac{2}{b-a} \int_a^b \hat{f}_X(y) \cos\left(k\pi \frac{y-a}{b-a}\right) dy \tag{3.17}$$

so that:

$$V_{\mathrm{I}}\left(t_0, x\right)=\int_a^b V(T, y) \sum_{k=0}^{+\infty} \bar{A}_k(x) \cos \left(k \pi \frac{y-a}{b-a}\right) \mathrm{d} y \tag{3.18}$$

After the interchange of the summation and the integration:

$$V_{\mathrm{I}}\left(t_0, x\right)=\frac{b-a}{2} \cdot \sum_{k=0}^{+\infty} \bar{A}_k(x) \cdot H_k \tag{3.19}$$

with:

$$H_k:=\frac{2}{b-a} \int_a^b V(T, y) \cos \left(k \pi \frac{y-a}{b-a}\right) \mathrm{d} y \tag{3.20}$$

The H_k are the cosine series coefficients of the payoff function, $V(T, y)$.
Thus, we have transformed an integral of the product of two real functions, $f_X(y)$ and $V(T, y)$, into a product of their Fourier cosine series coefficients.
Due to the rapid decay rate of these coefficients, we can truncate the series summation approximates $\bar{A}_k(x)$ by $\bar{F}_k(x)$:

$$V\left(t_0, x\right) \approx V_{\mathrm{III}}\left(t_0, x\right)=\sum_{k=0}^{N-1} \operatorname{Re}\left\{\phi_X\left(\frac{k \pi}{b-a}\right) \exp \left(-i k \pi \frac{a}{b-a}\right)\right\} H_k \tag{3.21}$$

The payoff for European options in an adjusted log-asset price, i.e., $y(T) = \log \frac{S(T)}{K}$ is:

$$V(T, y) := \max\left[\bar{\alpha} \cdot K\left(\mathrm{e}^{y} - 1\right), 0\right] \quad \text{with} \quad \bar{\alpha} = \begin{cases} 1 & \text{for a call} \\ -1 & \text{for a put} \end{cases} \tag{3.22}$$

Cosine-coefficients for vanilla call options with integration bounds $[a, b]$ and payoff:

$$V(T, y) = \max\left(S_T - K, 0\right) = \max\left(K\left(e^{y} - 1\right), 0\right) = K\left(e^{y} - 1\right)\mathbb{I}_{y>0}$$

$$\begin{aligned} V_k^{call} &= \frac{2}{b-a}\int_a^b v(y, T)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= \frac{2}{b-a}\int_a^b K\left(e^{y} - 1\right)\mathbb{I}_{y>0}\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= \frac{2}{b-a}\int_0^b K\left(e^{y} - 1\right)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \end{aligned} \tag{3.23}$$

$$V_k^{call} = \frac{2}{b-a}K\left(\chi_k(0, b) - \psi_k(0, b)\right) \tag{3.24}$$

Similarly, for a put option we have:

$$V_k^{put} = \frac{2}{b-a} K \left(-\chi_k(a,0) + \psi_k(a,0) \right) \tag{3.25}$$

The cosine coefficients for e^x on $[c,d] \subset [a,b]$ are:

$$\begin{aligned}\chi_k(c,d) &= \int_c^d e^x \cos\left(k\pi \frac{x-a}{b-a}\right) dx \\ &= \frac{1}{1+\left(\frac{k\pi}{b-a}\right)^2} \left\{ \cos\left(k\pi\frac{d-a}{b-a}\right) e^d - \cos\left(k\pi\frac{c-a}{b-a}\right) e^c \right. \\ &\quad \left. + \frac{k\pi}{b-a}\left[\sin\left(k\pi\frac{d-a}{b-a}\right) e^d - \sin\left(k\pi\frac{c-a}{b-a}\right) e^c\right] \right\}\end{aligned} \tag{3.26}$$

The cosine coefficients for 1 on $[c,d] \subset [a,b]$ are:

$$\begin{aligned}\psi_k(c,d) &= \int_c^d 1 \cos\left(k\pi \frac{x-a}{b-a}\right) dx \\ &= \begin{cases} \frac{b-a}{k\pi}\left[\sin\left(k\pi\frac{d-a}{b-a}\right) - \sin\left(k\pi\frac{c-a}{b-a}\right)\right] & k>0 \\ d-c & k=0 \end{cases}\end{aligned} \tag{3.27}$$

The COS pricing formula can be greatly simplified for exponential Lévy processes, as multiple options for different strike prices can be computed simultaneously.
For a given column vector of strikes, $\mathbf{K}$, we consider the following transformation:

$$\mathbf{x} = \log\left(\frac{S(t_0)}{\mathbf{K}}\right) \quad \text{and} \quad \mathbf{y} = \log\left(\frac{S(T)}{\mathbf{K}}\right)$$

The characteristic function of $X(t) = \log(S(t)/K)$ is:

$$\phi_X(u; t_0, t) := \mathrm{e}^{iuX(t_0)} \varphi_X(u, t) = \mathrm{e}^{iux} \mathbb{E}\left[\mathrm{e}^{iuX(t)}\right] \tag{3.28}$$

and is denoted by $\phi_X(u) := \phi_X(u; t_0, t)$.
For a Lévy process, the characteristic function can be represented by:

$$\phi_{\mathbf{X}}(u; t_0, T) = \varphi_{X_{\mathcal{L}}}(u, T) \cdot \mathrm{e}^{iu\mathbf{x}} \tag{3.29}$$

In this case, the pricing formula simplifies and we can write:

$$V(t_0, \mathbf{x}) \approx \sum_{k=0}^{N-1} \mathrm{Re}\left\{\varphi_{X_{\mathcal{L}}}\left(\frac{k\pi}{b-a}, T\right) \exp\left(ik\pi \frac{\mathbf{x}-a}{b-a}\right)\right\} \mathbf{H}_k \tag{3.30}$$

Recalling the H_k-formulas for vanilla European options they can be presented as a vector multiplied by a scalar, i.e.: $\mathbf{H}_k = U_k \mathbf{K}$ where:

$$U_k = \begin{cases} \frac{2}{b-a}\left(\chi_k(0,b) - \psi_k(0,b)\right) & \text{for a call} \\ \frac{2}{b-a}\left(-\chi_k(a,0) + \psi_k(a,0)\right) & \text{for a put} \end{cases} \tag{3.31}$$

As a result, the COS pricing formula is:

$$V\left(t_0, \mathbf{x}\right) \approx \mathbf{K} \cdot \operatorname{Re}\left\{\sum_{k=0}^{N-1} \varphi_{X_{\mathcal{L}}}\left(\frac{k\pi}{b-a}, T\right) U_k \cdot \exp\left(ik\pi \frac{\mathbf{x}-a}{b-a}\right)\right\} \tag{3.32}$$

The summation can be written as a matrix-vector product and we can obtain the option prices for different strikes if $\mathbf{K}$ and $\mathbf{x}$ are vectors. The integration range given by [Fang and Oosterlee, 2008] with the domain centred at $x + \zeta_1$ is:

$$[a,b] := \left[(x+\zeta_1) - L\sqrt{\zeta_2 + \sqrt{\zeta_4}}, \quad (x+\zeta_1) + L\sqrt{\zeta_2 + \sqrt{\zeta_4}}\right] \tag{3.33}$$

with $L \in [6, 12]$ and ζ_i cumulants of the underlying stochastic process $\log(S(T)/K)$. Cumulant ζ_4 is included because the density functions of many Lévy processes, for short maturity time T, will have sharp peaks and fat tails (represented by ζ_4).

The HHW model cumulants are not available, and [Grzelak et al. 2012] suggest the following approximation with τ the time to maturity:

$$[a, b] := [0 - 8\sqrt{\tau}, 0 + 8\sqrt{\tau}] \tag{3.34}$$

An interval which is chosen too small will lead to a significant integration-range truncation error, whereas an interval which is set very large would require a large value for N. This integration range is not tailored to the density function for $\log(S(t)/K)$, and it should be considered with some care, but it has the advantage that the integration range depends neither on the cumulants nor on the strike K. This is useful when we consider option values for multiple strike prices simultaneously. When pricing call options with the COS method, the results are sensitive to the size of the integration range; i.e. the choice of L. This is due to a call option payoff having an unbounded value in S_t and is not observed with put option payoffs, whose values are bounded by K. Thus, one should price a put option and then use the put-call parity to find the value of a call option [Fang and Oosterlee 2008].

With $x_t = \log S_t$, we consider the Monte Carlo simulation of the system of SDEs:

$$\begin{cases} dr_t = \lambda\left(\theta_t - r_t\right)dt + \eta dW_t^r & r_0 \in \mathbb{R} \\ dv_t = \kappa\left(\bar{v} - v_t\right)dt + \gamma\sqrt{v_t}dW_t^v & v_0 > 0 \\ dx_t = \left(r_t - \frac{1}{2}v_t\right)dt + \sqrt{v_t}dW_t^x & x_0 = \log S_0 \end{cases} \tag{3.35}$$

$$dW_x(t)dW_v(t) = \rho_{x,v}dt, \quad dW_x(t)dW_r(t) = \rho_{x,r}dt, \quad dW_r(t)dW_v(t) = 0;$$

Based on the Cholesky decomposition of the correlation matrix, the system is:

$$\begin{gathered} dr_t = \lambda(\theta_t - r_t)dt + \eta dW_t^r \\ dv_t = \kappa(\bar{v} - v_t)dt + \gamma\sqrt{v_t}dW_t^v \\ dx_t = \left(r_t - \frac{1}{2}v_t\right)dt + \rho_{x,r}\sqrt{v_t}dW_t^r + \rho_{x,v}\sqrt{v_t}dW_t^v + \sqrt{1 - \rho_{x,r}^2 - \rho_{x,v}^2}\sqrt{v_t}dW_t^x \end{gathered} \tag{3.36}$$

Define a time grid, $t_i = i\frac{T}{m}$, with $i = 0, \ldots, m$, $\Delta t_i = t_{i+1} - t_i$ and integrating over a time interval $[t_i, t_{i+1}]$, gives:

$$\begin{aligned} x_{i+1} = x_i + \int_{t_i}^{t_{i+1}} \left(r_s - \frac{1}{2}v_s\right)ds + \rho_{x,r}\int_{t_i}^{t_{i+1}} \sqrt{v_s}dW_s^r \\ + \rho_{x,v}\int_{t_i}^{t_{i+1}} \sqrt{v_s}dW_s^v + \sqrt{1 - \rho_{x,r}^2 - \rho_{x,v}^2}\int_{t_i}^{t_{i+1}} \sqrt{v_s}dW_s^x \end{aligned} \tag{3.37}$$

$$v_{i+1} = v_i + \int_{t_i}^{t_{i+1}} \kappa(\bar{v} - v_s)ds + \gamma \int_{t_i}^{t_{i+1}} \sqrt{v_s} dW_s^v \tag{3.38}$$

$$r_{i+1} = r_i + \int_{t_i}^{t_{i+1}} \lambda\left(\theta_s - r_s\right) ds + \eta \int_{t_i}^{t_{i+1}} dW_s^r \tag{3.39}$$

$\int_{t_i}^{t_{i+1}} dW^j = W_{i+1}^j - W_i^j$ and $\sqrt{dt}Z^j = \sqrt{\Delta t_i}Z^j$ are distributed in the same way:

$$W_{i+1}^j - W_i^j \overset{\mathrm{d}}{=} \sqrt{\Delta t_i} Z^j \tag{3.40}$$

with $Z^j \sim \mathcal{N}(0,1)$ variables and $j = r, v, x$.
Using Euler discretization and the full truncation scheme for the volatility process:

$$\begin{aligned}
r_{i+1} &= r_i + \lambda\theta_i\Delta t_i - \lambda r_i \Delta t_i + \eta\sqrt{\Delta t_i} Z^r \\
v_{i+1} &= v_i + \kappa(\bar{v} - v_i)\Delta t_i + \gamma\sqrt{v_i}\sqrt{\Delta t_i} Z^v \\
v_{i+1} &= \max\{v_{i+1}, 0\} \\
x_{i+1} &= x_i + (r_i - \frac{1}{2}v_i)\Delta t_i + \rho_{xr}\sqrt{v_i}\sqrt{\Delta t_i}Z^r + \\
&\quad \rho_{xv}\sqrt{v_i}\sqrt{\Delta t_i}Z^v + \sqrt{1 - \rho_{xr}^2 - \rho_{xv}^2}\sqrt{v_i}\sqrt{\Delta t_i}Z^x
\end{aligned} \tag{3.41}$$

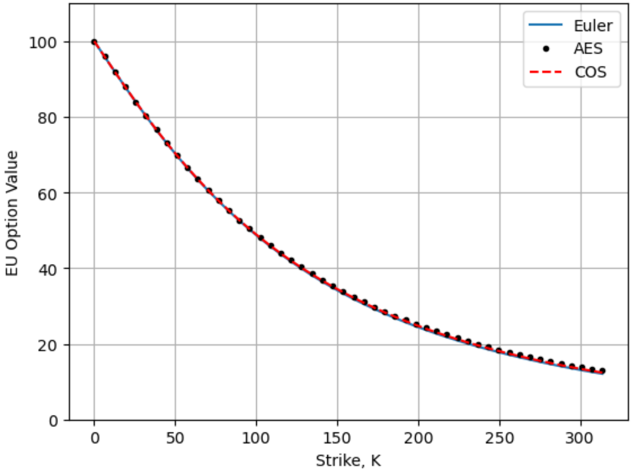


Figure: *Comparison for MC, AES and COS method.*

Heston vs Black-Scholes vs Heston-Hull-White

Conclusion

Analytical-Numerical consideration

The above derivation for the H1-HW model is based on a zero correlation between the variance and interest rate processes.
A generalization of the H1-HW model to a full matrix of non-zero correlations between the processes can be made by similar approximations of the non-affine covariance matrix terms using their respective expectations.
A better theoretical understanding of the difference between the full-scale and the approximate H1-HW model may be based on the corresponding option pricing PDEs.
For the Heston-Hull-White model, the option pricing PDE will be three-dimensional.
By changing measures, from the spot measure to the T-forward measure, the pricing PDE reduces to a two-dimensional PDE to facilitate the analysis.

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