Efficient Numerical Method for the Heston-Hull-White Model

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Presentation Overview

In recent years, the financial world has focused on accurately pricing exotic and hybrid products:

- contingent claims involving equity, cash, bonds or interest rate products;
- portfolio with different asset classes, like interest rates, stocks, foreign exchange, commodities, etc;
- long-dated contingent claims.

The hybrid Heston-Hull-White (HHW) model combines the Heston (1993) stochastic volatility and Hull-White (1990) short rate models.
Grzelak and Oosterlee [2011] have combined these two models to allow the correlations to be non-zero.

Zero Coupon Bond
Instantaneous Forward Rate

A basic interest rate product is the zero coupon bond, $P(t,T)$, which pays 1 currency unit at maturity time T. The t-price with payoff $H(T)$ is given by:

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(z)dz} H(T)|\mathcal{F}(t)\right] \quad H(T) = P(T,T) \equiv 1 \tag{1.1}$$

Assuming no arbitrage and market completeness the forward rate $R(t,S,T)$ is:

$$R(t,S,T) = -\frac{\log P(t,T) - \log P(t,S)}{T-S} \tag{1.2}$$

In the limit, $T - S \to 0$, we can define the instantaneous forward rate as:

$$f^r(t,T) \stackrel{\text{def}}{=} \lim_{S\to T} R(t,S,T) = -\frac{\partial}{\partial T}\log P(t,T) \tag{1.3}$$

The short rate is $r(t) \equiv f^r(t,t)$, and the money-savings account is:

$$M(t) = \exp\left(\int_0^t r(s)ds\right) = \exp\left(\int_0^t f^r(s,s)ds\right) \tag{1.4}$$

The Hull-White model [Hull and White, 1990] is a stochastic interest rate model in which the short rate is driven by a Generalized Ornstein-Uhlenbeck mean reverting process. Under the risk-neutral measure $\mathbb{Q}$ the dynamics $r(t)$ is given by:

$$\mathrm{d}r_t = \lambda(\theta_t - r_t)dt + \eta dW_t^r \tag{1.5}$$

where:

- θ_t is a time-dependent drift term:

$$\theta_t = \frac{1}{\lambda}\frac{\partial}{\partial t}f(0,t) + f(0,t) + \frac{\eta^2}{2\lambda^2}(1 - e^{-2\lambda t}) \tag{1.6}$$

- W_t^r is the Brownian motion;
- η is the volatility of the interest rate;
- λ is the reversion rate parameter.

The short rate is normally distributed, so it can be negative.

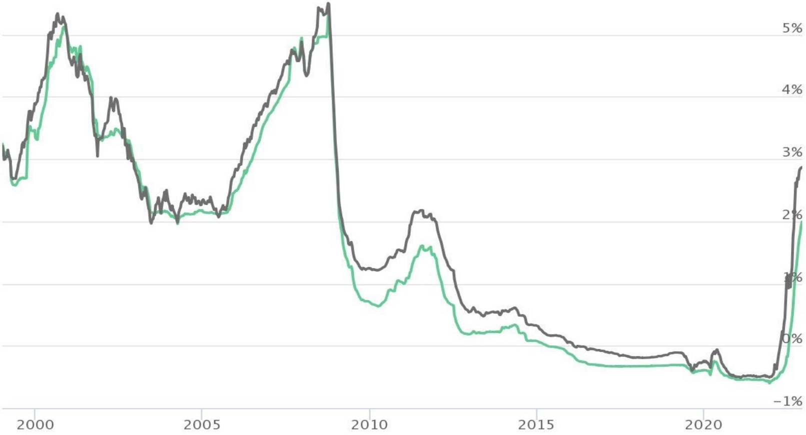


Figure: *Grey line - Euribor 12 months; Green line - Euribor 3 months; Source = Euribor*

It is important to properly choose the initial value for the process $r(t)$:

$$r(0) = f(0,0) \approx -\frac{\partial \log P(0,\epsilon)}{\partial \epsilon} \quad \text{for } \epsilon \to 0 \tag{1.7}$$

$$f(0,t) = -\frac{\log(P(t+dt)) - \log(P(t-dt))}{2dt} \tag{1.8}$$

$$\theta_t = \frac{1}{\lambda}\frac{f(0,t+dt) - f(0,t-dt)}{2dt} + f(0,t) + \frac{\eta^2}{2\lambda^2}(1 - e^{-2\lambda t}) \tag{1.9}$$

We can generate sample paths over the interval $[0,T]$ by discretizing this interval with $\Delta t_i = t_{i+1} - t_i$ and integrate over a time interval $[t_i, t_{i+1}]$:

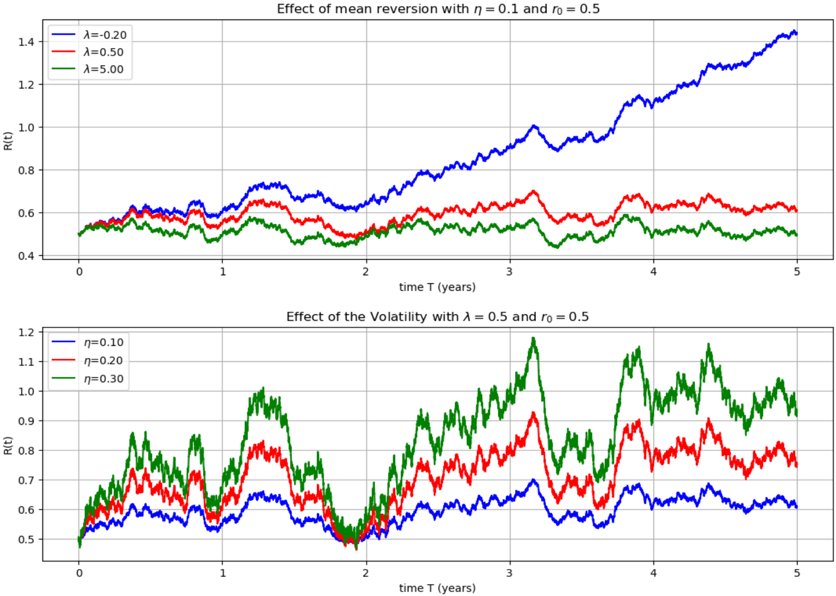
$$r_{i+1} = r_i + \int_{t_i}^{t_{i+1}} \lambda\left(\theta_s - r_s\right) ds + \eta \int_{t_i}^{t_{i+1}} dW_s^r \tag{1.10}$$

$\int_{t_i}^{t_{i+1}} dW_s^r = W_{i+1}^r - W_i^r$ and $\sqrt{dt}Z^r = \sqrt{\Delta t_i}Z^r$ are distributed in the same way, with $Z^r \sim \mathcal{N}(0,1)$ variables. Using the Euler discretization:

$$r_{i+1} = r_i + \lambda\theta_i\Delta t_i - \lambda r_i \Delta t_i + \eta\sqrt{\Delta t_i}Z^r \tag{1.11}$$

Simulation of the Hull-White SDE

Hull-White Model



The Heston model under the risk-neutral pricing measure $\mathbb{Q}$ is specified by the following system of SDEs:

$$\begin{cases} dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^s & S_0 > 0 \\ dv_t = \kappa(\bar{v} - v_t)dt + \gamma\sqrt{v_t} dW_t^v & v_0 > 0 \end{cases} \tag{1.12}$$

For the asset price process, we have:

- W_t^s is the Brownian motion;
- r is the risk-free interest rate (constant);

The volatility process v_t is a CIR-type process, where:

- W_t^v is the Brownian motion;
- κ is the speed of adjustment of the volatility towards its mean;
- $\bar{v}$ is the long-run mean;
- γ is the second-order volatility.

The correlation between the two Brownian motions is:

$$dW_t^s dW_t^v = \rho_{s,v} dt \quad \text{with} \quad |\rho_{s,v}| \leq 1.$$

The Heston-Hull-White model is formed by combining the Heston model and Hull-White model in a correlated manner. Under the $\mathbb{Q}$ measure this results in the following system of SDEs:

$$\begin{cases} dr_t = \lambda(\theta_t - r_t)dt + \eta dW_t^r & r_0 \in \mathbb{R} \\ dv_t = \kappa(\bar{v} - v_t)dt + \gamma\sqrt{v_t}dW_t^v & v_0 > 0 \\ dS_t = r_t S_t dt + \sqrt{v_t} S_t dW_t^s & S_0 > 0 \end{cases} \tag{2.1}$$

The correlations are given by:

$$dW_t^s dW_t^v = \rho_{s,v} dt, \quad dW_t^s dW_t^r = \rho_{s,r} dt, \quad dW_t^v dW_t^r = \rho_{v,r} dt \tag{2.2}$$

The price of any contingent claim under a risk-neutral measure can be obtained by risk-neutral valuation, i.e, by computing the expectation of the discounted payoff. With the payoff $H(S_T, v_T, r_T)$ at time T, the t-price of a claim is given by:

$$V(S_t, v_t, r_t) = \mathbb{E}^{\mathbb{Q}}\left[e^{\int_t^T r_s ds} H(S_T, v_T, r_T) | \mathcal{F}(t)\right] \tag{2.3}$$

However, due to the stochastic interest rate, the discount factor cannot be disentangled from the expectation.

With $x_t = \log S_t$ and with the application of Ito's formula, the model is described by:

$$\begin{cases} dr_t = \lambda\left(\theta_t - r_t\right)dt + \eta dW_t^r & r_0 \in \mathbb{R} \\ dv_t = \kappa\left(\bar{v} - v_t\right)dt + \gamma\sqrt{v_t}dW_t^v & v_0 > 0 \\ dx_t = \left(r_t - \frac{1}{2}v_t\right)dt + \sqrt{v_t}dW_t^x & x_0 = \log S_0 \end{cases} \tag{2.4}$$

With the Cholesky decomposition of the correlation matrix $\mathbf{C}$ we can substitute the correlated Brownian motion $\mathbf{W}_t$ with an independent n-dimensional Brownian motion $\mathbf{B}_t = [B_t^r, B_t^v, B_t^x]^\top$. Define $\mathbf{W}_t = [W_t^r, W_t^v, W_t^x]^\top$ and $\mathbf{C}$ as:

$$\mathbf{C} = \begin{bmatrix} 1 & \rho_{r,v} & \rho_{r,x} \\ \rho_{r,v} & 1 & \rho_{v,x} \\ \rho_{r,x} & \rho_{v,x} & 1 \end{bmatrix} \tag{2.5}$$

With $\mathbf{L}$ as the lower triangular matrix of $\mathbf{C} = \mathbf{L}\mathbf{L}^T$, we have: $\mathbf{W}_t = \mathbf{L}\mathbf{B}_t$

$$\begin{aligned} W_t^r &= B_t^r \\ W_t^v &= \rho_{r,v}B_t^r + \sqrt{1-\rho_{r,v}^2}B_t^v \\ W_t^x &= \rho_{r,x}B_t^r + \frac{\rho_{v,x} - \rho_{r,v}\rho_{r,x}}{\sqrt{1-\rho_{r,v}^2}}B_t^v + \sqrt{1-\rho_{r,x}^2 - \frac{(\rho_{v,x} - \rho_{r,v}\rho_{r,x})^2}{1-\rho_{r,v}^2}}B_t^x \end{aligned} \tag{2.6}$$

The log-dynamics can be expressed with $\mathbf{B}_t$ as:

$$d\mathbf{X}_t = \mu\left(\mathbf{X}_t\right)dt + \sigma\left(\mathbf{X}_t\right)d\mathbf{B}_t \tag{2.7}$$

with state vector $\mathbf{X}_t$, drift vector $\mu\left(\mathbf{X}_t\right)$ and volatility matrix $\sigma\left(\mathbf{X}_t\right)$ as:

$$\mathbf{X}_t = \begin{bmatrix} r_t \\ v_t \\ x_t \end{bmatrix} \quad \mu\left(\mathbf{X}_t\right) = \begin{bmatrix} \lambda\left(\theta - r_t\right) \\ \kappa\left(\bar{v} - v_t\right) \\ r_t - \frac{1}{2}v_t \end{bmatrix} \tag{2.8}$$

$$\sigma\left(\mathbf{X}_t\right) = \begin{bmatrix} \eta & 0 & 0 \\ \rho_{r,v}\gamma\sqrt{v_t} & \sqrt{1-\rho_{r,v}^2}\gamma\sqrt{v_t} & 0 \\ \rho_{r,x}\sqrt{v_t} & \frac{\left(\rho_{v,x}-\rho_{r,x}\rho_{r,v}\right)}{\sqrt{1-\rho_{r,v}^2}}\sqrt{v_t} & \sqrt{1-\rho_{r,x}^2-\left(\frac{\rho_{v,x}-\rho_{r,v}\rho_{r,x}}{\sqrt{1-\rho_{r,v}^2}}\right)^2}\sqrt{v_t} \end{bmatrix} \tag{2.9}$$

The matrix $\Sigma\left(\mathbf{X}_t\right) = \sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^{\mathrm{T}}$ is the symmetric instantaneous covariance matrix:

$$\Sigma\left(\mathbf{X}_t\right) = \begin{bmatrix} \eta^2 & \rho_{r,v}\gamma\eta\sqrt{v_t} & \rho_{r,x}\eta\sqrt{v_t} \\ \rho_{r,v}\gamma\eta\sqrt{v_t} & \gamma^2 v_t & \rho_{v,x}\gamma v_t \\ \rho_{r,x}\eta\sqrt{v_t} & \rho_{v,x}\gamma v_t & v_t \end{bmatrix} \tag{2.10}$$

The system is affine if $\rho_{r,v} = \rho_{r,x} = 0$. We will assume that the interest rate r_t and the variance v_t are not correlated, i.e., $\rho_{r,v} = 0$.

Deterministic approximation for the non-affine term in $\Sigma(\mathbf{X}_t)$:

$$\rho_{r,x}\eta\sqrt{v(t)} \approx \rho_{r,x}\eta\mathbb{E}[\sqrt{v(t)}] \tag{2.11}$$

And resulting covariance matrix:

$$\Sigma(\mathbf{X}_t) = \begin{bmatrix} \eta^2 & 0 & \rho_{r,x}\eta\mathbb{E}\left[\sqrt{v_t}\right] \\ 0 & \gamma^2 v_t & \rho_{v,x}\gamma v_t \\ \rho_{r,x}\eta\mathbb{E}\left[\sqrt{v_t}\right] & \rho_{v,x}\gamma v_t & v_t \end{bmatrix} \tag{2.12}$$

It was shown by [Cox et al., 1985], that, $v(t) \mid v(0)$ is distributed as:

$$v(t) \mid v(0) \sim \bar{c}(t,0)\chi^2(\delta, \bar{\kappa}(t,0)), \quad t > 0 \tag{2.13}$$

$$\bar{c}(t,0) = \frac{1}{4\kappa}\gamma^2\left(1 - \mathrm{e}^{-\kappa t}\right), \quad \delta = \frac{4\kappa\bar{v}}{\gamma^2}, \quad \bar{\kappa}(t,0) = \frac{4\kappa v(0)\mathrm{e}^{-\kappa t}}{\gamma^2\left(1 - \mathrm{e}^{-\kappa t}\right)} \tag{2.14}$$

By [Dufresne, 2001]:

$$\mathbb{E}[\sqrt{v(t)} \mid \mathcal{F}(0)] = \sqrt{2\bar{c}(t,0)}\frac{\Gamma\left(\frac{1+\delta}{2}\right)}{\Gamma\left(\frac{\delta}{2}\right)}\,{}_1F_1\left(-\frac{1}{2}, \frac{\delta}{2}, -\frac{\bar{\kappa}(t,0)}{2}\right) \tag{2.15}$$

with ${}_1F_1(a;b;z)$ the confluent hyper-geometric function.

Given a system of SDEs with a n-dimensional state vector $\mathbf{X}(t)$:

$$d\mathbf{X}(t) = \mu(\mathbf{X}(t))dt + \sigma(\mathbf{X}(t))d\mathbf{W}(t) \tag{2.16}$$

The system is said to be of the affine form if:

$$\begin{gathered}\mu(\mathbf{X}(t)) = a_0 + a_1\mathbf{X}(t), \\ \Sigma\left(\mathbf{X}_t\right) = (\sigma(\mathbf{X}(t))\sigma(\mathbf{X},(t))^T)_{i,j} = (c_0)_{i,j} + (c_1)_{i,j}^T\mathbf{X}(t), \\ r(\mathbf{X}(t)) = r_0 + r_1^T\mathbf{X}(t)\end{gathered} \tag{2.17}$$

$$(a_0, a_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}, \quad (c_0, c_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n}, \quad (r_0, r_1) \in \mathbb{R} \times \mathbb{R}^n \tag{2.18}$$

The conditions imply that for a model to be affine, each element of its drift and covariance matrices must be a linear function of the state variables.
The discounted **ChF** [Duffie et al., 2000] with $\mathbf{u} = [u, 0, \dots, 0]^T$ and $\tau = T - t$ is:

$$\phi(\mathbf{u}, \mathbf{X}(t), t, T) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(s)ds + i\mathbf{u}^T\mathbf{X}(T)} | \mathcal{F}(t)\right] = e^{A(\mathbf{u},\tau) + \mathbf{B}^T(\mathbf{u},\tau)\mathbf{X}(t)} \tag{2.19}$$

With $\mathbf{X}(t) = [X(t), r(t), v(t)]^T$, the discounted characteristic function is given by:

$$\phi(\mathbf{u}, \mathbf{X}(t), t, T) = \exp\left(A(u,\tau) + B(u,\tau)X(t) + C(u,\tau)r(t) + D(u,\tau)v(t)\right) \tag{2.20}$$

The function $A(\mathbf{u},\tau)$ and $\mathbf{B}^\top(\mathbf{u},\tau) = [B(u,\tau), C(u,\tau), D(u,\tau)]$ satisfy the following system of complex-valued ordinary differential equations:

$$\frac{d}{d\tau}\mathbf{B}(\mathbf{u},\tau) = -r_1 + a_1^\top \mathbf{B}(\mathbf{u},\tau) + \frac{1}{2}\mathbf{B}^\top(\mathbf{u},\tau) c_1 \mathbf{B}(\mathbf{u},\tau), \quad \mathbf{B}^\top(\mathbf{u},\tau) = i\mathbf{u} \tag{2.21}$$

$$\frac{d}{d\tau}A(\mathbf{u},\tau) = -r_0 + \mathbf{B}^\top(\mathbf{u},\tau) a_0 + \frac{1}{2}\mathbf{B}^\top(\mathbf{u},\tau) c_0 \mathbf{B}(\mathbf{u},\tau), \quad A(\mathbf{u},0) = 0 \tag{2.22}$$

The HHW affine decomposition is given by:

$$\begin{gathered} a_0 = \begin{bmatrix} 0 \\ \kappa\bar{v} \\ \lambda\theta \end{bmatrix}, \quad a_1 = \begin{bmatrix} 0 & -\frac{1}{2} & 1 \\ 0 & -\kappa & 0 \\ 0 & 0 & -\lambda \end{bmatrix}, \quad r_0 = 0, \quad r_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ c_0 = \begin{bmatrix} 0 & 0 & \rho_{r,x}\eta\mathbb{E}\left[\sqrt{v_t}\right] \\ 0 & 0 & 0 \\ \rho_{r,x}\eta\mathbb{E}\left[\sqrt{v_t}\right] & 0 & \eta^2 \end{bmatrix} \\ c_1 = \begin{bmatrix} (0,1,0) & (0,\rho_{v,x}\gamma,0) & (0,0,0) \\ (0,\rho_{v,x}\gamma,0) & (0,\gamma^2,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) \end{bmatrix} \end{gathered} \tag{2.23}$$

The functions A, B, C, D satisfy the following system of ODEs:

$$\begin{aligned}
\frac{\mathrm{d}B}{\mathrm{d}\tau} &= 0, \quad B(u,0) = iu \\
\frac{\mathrm{d}C}{\mathrm{d}\tau} &= -1 - \lambda C + B, \quad C(u,0) = 0 \\
\frac{\mathrm{d}D}{\mathrm{d}\tau} &= B(B-1)/2 + (\gamma\rho_{x,v}B - \kappa)\,D + \gamma^2 D^2/2, \quad D(u,0) = 0 \\
\frac{\mathrm{d}A}{\mathrm{d}\tau} &= \lambda\theta C + \kappa\bar{v}D + \eta^2 C^2/2 + \eta\rho_{x,r}\mathbb{E}[\sqrt{v(t)}]BC, \quad A(u,0) = 0
\end{aligned} \tag{2.24}$$

The solution of the system is given by:

$$\begin{aligned}
B(u,\tau) &= iu \\
C(u,\tau) &= (iu-1)\lambda^{-1}\left(1 - \mathrm{e}^{-\lambda\tau}\right) \\
D(u,\tau) &= \frac{1 - \mathrm{e}^{-D_1\tau}}{\gamma^2\left(1 - g\mathrm{e}^{-D_1\tau}\right)}\left(\kappa - \gamma\rho_{x,v}iu - D_1\right) \\
A(u,\tau) &= I_1(\tau) + \kappa\bar{v}I_2(\tau) + \frac{1}{2}\eta^2 I_3(\tau) + \eta\rho_{x,r}I_4(\tau)
\end{aligned} \tag{2.25}$$

with:

$$D_1 = \sqrt{(\gamma\rho_{x,v}iu - \kappa)^2 - \gamma^2 iu(iu-1)} \qquad g = \frac{\kappa - \gamma\rho_{x,v}iu - D_1}{\kappa - \gamma\rho_{x,v}iu + D_1}$$

The integrals $I_2(\tau)$ and $I_3(\tau)$ admit an analytic solution, and $I_1(\tau)$ and $I_4(\tau)$ admits a semi-analytic solution:

$$\begin{aligned}
I_1(\tau) &= (iu-1)\int_0^\tau (1-e^{-\lambda z})\theta_{(T-z)}\mathrm{d}z \\
I_2(\tau) &= \frac{\tau}{\gamma^2}\left(\kappa - \gamma\rho_{x,v}iu - D_1\right) - \frac{2}{\gamma^2}\log\left(\frac{1-g\mathrm{e}^{-D_1\tau}}{1-g}\right) \\
I_3(\tau) &= \frac{1}{2\lambda^3}(i+u)^2\left(3+\mathrm{e}^{-2\lambda\tau} - 4\mathrm{e}^{-\lambda\tau} - 2\lambda\tau\right) \\
I_4(\tau) &= iu\int_0^\tau \mathbb{E}[\sqrt{v_{(T-z)}}]C(u,z)\mathrm{d}z \\
&= -\frac{1}{\lambda}\left(iu+u^2\right)\int_0^\tau \mathbb{E}[\sqrt{v_{(T-z)}}]\left(1-\mathrm{e}^{-\lambda z}\right)\mathrm{d}z
\end{aligned} \tag{2.26}$$

So we have the Characteristic function.

The probability density function and its characteristic function, $f_X(y)$ and $\phi_X(u)$, form an example of a Fourier pair:

$$\phi_X(u) = \int_{\mathbb{R}} \mathrm{e}^{iyu} f_X(y)\mathrm{d}y \qquad f_X(y) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-iuy}\phi_X(u)\mathrm{d}u \tag{3.1}$$

The Fourier cosine expansion of a function $g(x)$ on an interval $[-\pi, \pi]$ is:

$$g(x) = \sum_{k=0}^{\infty} \bar{A}_k \cos(kx) \quad \text{with} \quad \bar{A}_k = \frac{2}{\pi}\int_0^{\pi} g(x)\cos(kx)\mathrm{d}x \tag{3.2}$$

For functions supported on any other finite interval, $[a, b] \in \mathbb{R}$, the Fourier cosine series expansion can be obtained via a change of variables:

$$x = \pi\frac{y-a}{b-a}, \quad y = \frac{b-a}{\pi}x + a \tag{3.3}$$

$$g(y) = {\sum_{k=0}^{\infty}}{}' \bar{A}_k \cdot \cos\left(k\pi\frac{y-a}{b-a}\right), \quad \bar{A}_k = \frac{2}{b-a}\int_a^b g(y)\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y \tag{3.4}$$

The integrands have to decay to zero at $\pm\infty$ so we can truncate the integration range:

$$\hat{\phi}_X(u) = \int_a^b \mathrm{e}^{iuy} f_X(y)\mathrm{d}y \approx \int_{\mathbb{R}} \mathrm{e}^{iuy} f_X(y)\mathrm{d}y = \phi_X(u) \tag{3.5}$$

For any random variable X, and constant $a \in \mathbb{R}$, the following equality holds:

$$\phi_X(u)e^{ia} = \mathbb{E}\left[e^{iuX+ia}\right] = \int_{-\infty}^{\infty} e^{i(uy+a)} f_X(y)dy \tag{3.6}$$

Substitute $u = \frac{k\pi}{b-a}$ and multiply by $\exp\left(-i\frac{ka\pi}{b-a}\right)$ and taking the real part:

$$\operatorname{Re}\left\{\hat{\phi}_X\left(\frac{k\pi}{b-a}\right) \cdot \exp\left(-i\frac{ka\pi}{b-a}\right)\right\} = \int_a^b \cos\left(k\pi\frac{y-a}{b-a}\right) f_X(y)dy \tag{3.7}$$

At the right-hand side we have the definition of $\bar{A}_k$ so:

$$\bar{A}_k = \frac{2}{b-a}\operatorname{Re}\left\{\hat{\phi}_X\left(\frac{k\pi}{b-a}\right) \cdot \exp\left(-i\frac{ka\pi}{b-a}\right)\right\} \tag{3.8}$$

$$\bar{F}_k = \frac{2}{b-a}\operatorname{Re}\left\{\phi_X\left(\frac{k\pi}{b-a}\right) \cdot \exp\left(-i\frac{ka\pi}{b-a}\right)\right\} \tag{3.9}$$

Replace $\bar{A}_k$ by $\bar{F}_k$ in the series expansion of $f_X(y)$ on $[a, b]$ and truncate the series:

$$\hat{f}_X(y) \approx \sum_{k=0}^{N-1}{}' \bar{F}_k \cos\left(k\pi\frac{y-a}{b-a}\right) \tag{3.10}$$

- Approximation for the standard normal density function.

The PDF and characteristic function are known in closed form:

$$f_{\mathcal{N}(0,1)}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad \phi_{\mathcal{N}(0,1)}(u) = e^{-\frac{1}{2}u^2} \tag{3.11}$$

- Approximation for the lognormal density function.

With Y a log-normal random variable and $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$ a normal random variable, we can write $Y = e^X$. The cumulative distribution function of Y in terms of the cumulative distribution function of X can be calculated as:

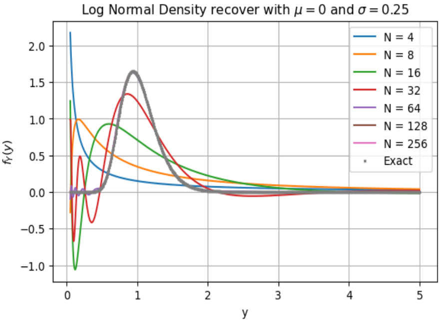
$$F_Y(y) \stackrel{\text{def}}{=} \mathbb{P}[Y \leq y] = \mathbb{P}\left[e^X \leq y\right] = \mathbb{P}[X \leq \log(y)] = F_X(\log(y)) \tag{3.12}$$

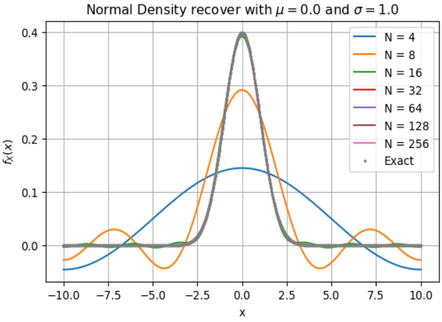
By differentiation, we get:

$$f_Y(y) \stackrel{\text{def}}{=} \frac{dF_Y(y)}{dy} = \frac{dF_X(\log(y))}{d\log y} \frac{d\log(y)}{dy} = \frac{1}{y} f_X(\log(y)) \tag{3.13}$$

Since X is normally distributed we have $\phi_X(u) = e^{i\mu u - \frac{1}{2}\sigma^2 u^2}$

Integration interval $[a, b] = [-10, 10]$ for both approximations.

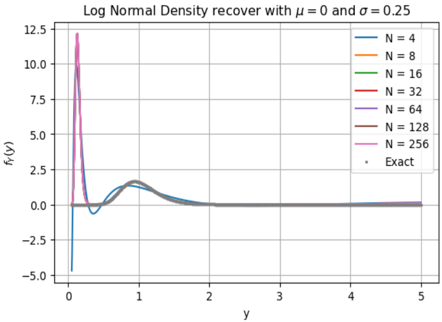




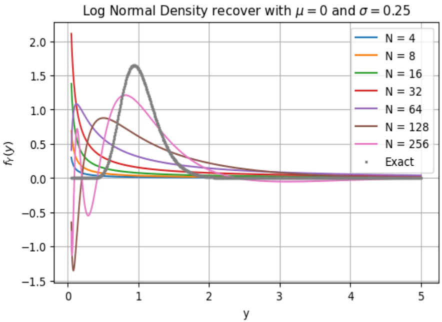
N	$Error$
4	$2.174111972e+00$
8	$1.2811026962e+00$
16	$1.242288484e+00$
32	$9.866610056e-01$
64	$9.404080678e-02$
128	$6.110817377e-06$
256	$1.075039460e-14$

N	$Error$
4	$2.537377837e-01$
8	$1.075173272e-01$
16	$7.172377614e-03$
32	$4.032340789e-07$
64	$2.486596021e-16$
128	$2.486596125e-16$
256	$2.4865961258e-16$

Integration interval $[a, b] = [-1, 1]$.



Integration interval $[a, b] = [-100, 100]$.



An interval which is chosen too small will lead to a significant integration-range truncation error, whereas an interval which is set very large would require a large value for N.

The COS formula for pricing European options is obtained by substituting the density function with its Fourier cosine series expansion in the risk-neutral valuation formula. With $X(t) = \log S(t)$ taking values $X(t_0) = x$, $X(T) = y$ and $\tau = T - t_0$ we have:

$$V\left(t_0, x\right) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_{t_0}^{T} r(z)\mathrm{d}z} V(T, y) \mid \mathcal{F}\left(t_0\right)\right] = \int_{\mathbb{R}} V(T, y) f_{\mathbf{X}}\left(T, y; t_0, x\right) \mathrm{d}y \tag{3.14}$$

Since $f_{\mathbf{X}}(y)$ rapidly decays to zero as $y \to \pm\infty$, we truncate the infinite integration range to $[a, b] \subset \mathbb{R}$ and without losing significant accuracy we obtain:

$$V\left(t_0, x\right) \approx V_{\mathrm{I}}\left(t_0, x\right) = \int_a^b V(T, y) f_X(y) dy \tag{3.15}$$

The density is approximated by its Fourier cosine expansion in y:

$$\hat{f}_{\mathbf{X}}(y) = \sum_{k=0}^{+\infty} \bar{A}_k(x) \cos\left(k\pi \frac{y-a}{b-a}\right), \quad \bar{A}_k(x) := \frac{2}{b-a} \int_a^b \hat{f}_{\mathbf{X}}(y) \cos\left(k\pi \frac{y-a}{b-a}\right) dy \tag{3.16}$$

So we have:

$$V_{\mathrm{II}}\left(t_0, x\right)=\int_a^b V(T, y) \sum_{k=0}^{+\infty} \bar{A}_k(x) \cos \left(k \pi \frac{y-a}{b-a}\right) \mathrm{d} y \tag{3.17}$$

After the interchange of the summation and the integration:

$$V_{\mathrm{II}}\left(t_0, x\right)=\frac{b-a}{2} \sum_{k=0}^{+\infty} \bar{A}_k(x) H_k, \qquad H_k=\frac{2}{b-a} \int_a^b V(T, y) \cos \left(k \pi \frac{y-a}{b-a}\right) \mathrm{d} y \tag{3.18}$$

The H_k are the cosine series coefficients of the payoff function, $V(T,y)$. We can truncate the series summation and approximate $\bar{A}_k(x)$ by $\bar{F}_k(x)$:

$$V\left(t_0, x\right) \approx V_{\mathrm{III}}\left(t_0, x\right)=\sum_{k=0}^{N-1} \operatorname{Re}\left\{\phi_X\left(\frac{k \pi}{b-a}\right) \exp \left(-i k \pi \frac{a}{b-a}\right)\right\} H_k \tag{3.19}$$

The cosine coefficients for e^x on $[c,d] \subset [a,b]$ are:

$$\begin{aligned}\chi_k(c,d) &= \int_c^d e^x \cos\left(k\pi \frac{x-a}{b-a}\right) dx \\ &= \frac{1}{1+\left(\frac{k\pi}{b-a}\right)^2} \left\{ \cos\left(k\pi\frac{d-a}{b-a}\right) e^d - \cos\left(k\pi\frac{c-a}{b-a}\right) e^c \right. \\ &\left. + \frac{k\pi}{b-a}\left[\sin\left(k\pi\frac{d-a}{b-a}\right) e^d - \sin\left(k\pi\frac{c-a}{b-a}\right) e^c\right]\right\}\end{aligned} \tag{3.20}$$

The cosine coefficients for 1 on $[c,d] \subset [a,b]$ are:

$$\begin{aligned}\psi_k(c,d) &= \int_c^d 1 \cos\left(k\pi\frac{x-a}{b-a}\right) dx \\ &= \begin{cases} \frac{b-a}{k\pi}\left[\sin\left(k\pi\frac{d-a}{b-a}\right) - \sin\left(k\pi\frac{c-a}{b-a}\right)\right] & k>0 \\ d-c & k=0 \end{cases}\end{aligned} \tag{3.21}$$

Cosine-coefficients for plain vanilla call options with integration bounds $[a, b]$ and $a < 0 < b$, in adjusted log-asset price, i.e., $y(T) = \log(\frac{S(T)}{K})$ and payoff $V(T, y)$ are:

$$V(T, y) = \max(S_T - K, 0) = \max(K(e^y - 1), 0) = K(e^y - 1)\mathbb{I}_{y>0}$$

$$\begin{aligned} H_k^{call} &= \frac{2}{b-a}\int_a^b V(T, y)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= \frac{2}{b-a}\int_a^b K(e^y - 1)\mathbb{I}_{y>0}\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= \frac{2}{b-a}\int_0^b K(e^y - 1)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \end{aligned} \tag{3.22}$$

$$H_k^{call} = \frac{2}{b-a}K(\chi_k(0, b) - \psi_k(0, b)) \tag{3.23}$$

Similarly, for a put option we have:

$$H_k^{put} = \frac{2}{b-a}K(-\chi_k(a, 0) + \psi_k(a, 0)) \tag{3.24}$$

When we deal with $a < b < 0$ we have $H_k^{call} = 0$, while for $0 < a < b$, the payoff coefficients H_k^{call} are defined by $c = a$ and $d = b$.

For a vector of strikes, $\mathbf{K}$, we consider the transformation:

$$\mathbf{x} = \log\left(\frac{S(t_0)}{\mathbf{K}}\right) \quad \text{and} \quad \mathbf{y} = \log\left(\frac{S(T)}{\mathbf{K}}\right)$$

The characteristic function can be represented by:

$$\phi_{\mathbf{X}}(u, \mathbf{X}(t), t_0, T) = \varphi_{X_{\mathcal{L}}}(u, T) \cdot \mathrm{e}^{iu\mathbf{x}} \tag{3.25}$$

The pricing formula can be simplified and we can write:

$$V(t_0, \mathbf{x}) \approx \sum_{k=0}^{N-1} \mathrm{Re}\left\{\varphi_{X_{\mathcal{L}}}\left(\frac{k\pi}{b-a}, T\right) \exp\left(ik\pi\frac{\mathbf{x}-a}{b-a}\right)\right\} \mathbf{H}_k \tag{3.26}$$

The H_k-coefficients for European options can be presented as $\mathbf{H}_k = U_k\mathbf{K}$, where:

$$U_k = \begin{cases} \frac{2}{b-a}\left(\chi_k(0,b) - \psi_k(0,b)\right) & \text{for a call} \\ \frac{2}{b-a}\left(-\chi_k(a,0) + \psi_k(a,0)\right) & \text{for a put} \end{cases} \tag{3.27}$$

As a result, the COS pricing formula is:

$$V(t_0, \mathbf{x}) \approx \mathbf{K} \cdot \mathrm{Re}\left\{\sum_{k=0}^{N-1} \varphi_{X_{\mathcal{L}}}\left(\frac{k\pi}{b-a}, T\right) U_k \cdot \exp\left(ik\pi\frac{\mathbf{x}-a}{b-a}\right)\right\} \tag{3.28}$$

The integration range given by [Fang and Oosterlee, 2008] centred at $x + \zeta_1$ is:

$$[a, b] := \left[(x+\zeta_1) - L\sqrt{\zeta_2 + \sqrt{\zeta_4}}, \quad (x+\zeta_1) + L\sqrt{\zeta_2 + \sqrt{\zeta_4}}\right] \quad L \in [6, 12] \tag{3.29}$$

with ζ_i cumulants of the underlying stochastic process $\log \frac{S(t)}{K}$. Cumulant ζ_4 is included because the density functions of many Lévy processes, for short maturity time T, will have sharp peaks and fat tails (represented by ζ_4). The HHW model cumulants are not available, and [Grzelak et al. 2012] suggest the approximation:

$$[a, b] := [0 - 8\sqrt{\tau}, 0 + 8\sqrt{\tau}] \tag{3.30}$$

This integration range is not tailored to the density function but it has the advantage that the integration range depends neither on the cumulants nor on the strike K. The price of a call option is sensitive to the size of the integration range because the payoff has an unbounded value in S_t (the put option payoff is bounded by K). Thus, one should price a put option and then use the put-call parity to find the value of a call option [Fang and Oosterlee 2008].

With $x_t = \log S_t$, we consider the Monte Carlo simulation of the system of SDEs:

$$\begin{cases} dr_t = \lambda\left(\theta_t - r_t\right)dt + \eta dW_t^r & r_0 \in \mathbb{R} \\ dv_t = \kappa\left(\bar{v} - v_t\right)dt + \gamma\sqrt{v_t}dW_t^v & v_0 > 0 \\ dx_t = \left(r_t - \frac{1}{2}v_t\right)dt + \sqrt{v_t}dW_t^x & x_0 = \log S_0 \end{cases} \tag{3.31}$$

$$dW_x(t)dW_v(t) = \rho_{x,v}dt, \quad dW_x(t)dW_r(t) = \rho_{x,r}dt, \quad dW_r(t)dW_v(t) = 0;$$

Based on the Cholesky decomposition of the correlation matrix, the system is:

$$\begin{aligned} dr_t &= \lambda(\theta_t - r_t)dt + \eta dW_t^r \\ dv_t &= \kappa(\bar{v} - v_t)dt + \gamma\sqrt{v_t}dW_t^v \\ dx_t &= \left(r_t - \frac{1}{2}v_t\right)dt + \rho_{x,r}\sqrt{v_t}dW_t^r + \rho_{x,v}\sqrt{v_t}dW_t^v + \sqrt{1-\rho_{x,r}^2-\rho_{x,v}^2}\sqrt{v_t}dW_t^x \end{aligned} \tag{3.32}$$

Define a time grid, $t_i = i\frac{T}{m}$, with $i = 0, \ldots, m$, $\Delta t_i = t_{i+1} - t_i$ and integrating over a time interval $[t_i, t_{i+1}]$, gives:

$$\begin{aligned} x_{i+1} = x_i &+ \int_{t_i}^{t_{i+1}} \left(r_s - \frac{1}{2}v_s\right)ds + \rho_{x,r}\int_{t_i}^{t_{i+1}} \sqrt{v_s}dW_s^r \\ &+ \rho_{x,v}\int_{t_i}^{t_{i+1}} \sqrt{v_s}dW_s^v + \sqrt{1-\rho_{x,r}^2-\rho_{x,v}^2}\int_{t_i}^{t_{i+1}} \sqrt{v_s}dW_s^x \end{aligned} \tag{3.33}$$

$$v_{i+1} = v_i + \int_{t_i}^{t_{i+1}} \kappa(\bar{v} - v_s)ds + \gamma \int_{t_i}^{t_{i+1}} \sqrt{v_s} dW_s^v \tag{3.34}$$

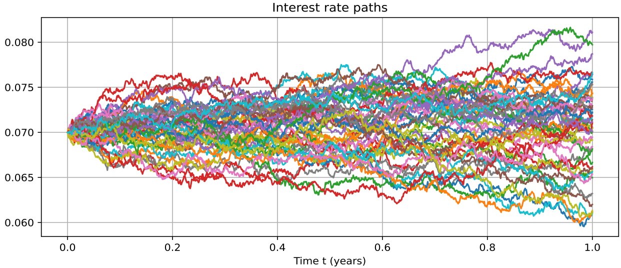
$$r_{i+1} = r_i + \int_{t_i}^{t_{i+1}} \lambda\left(\theta_s - r_s\right) ds + \eta \int_{t_i}^{t_{i+1}} dW_s^r \tag{3.35}$$

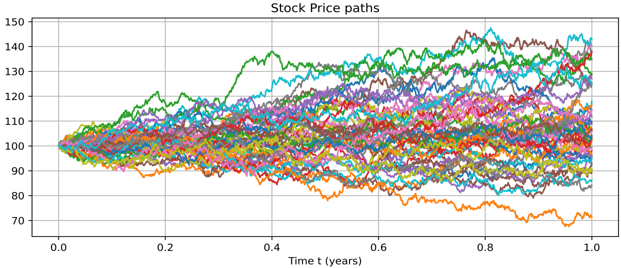
$\int_{t_i}^{t_{i+1}} dW^j = W_{i+1}^j - W_i^j$ and $\sqrt{dt}Z^j = \sqrt{\Delta t_i}Z^j$ are distributed in the same way:

$$W_{i+1}^j - W_i^j \overset{\mathrm{d}}{=} \sqrt{\Delta t_i} Z^j \tag{3.36}$$

with $Z^j \sim \mathcal{N}(0,1)$ variables and $j = r, v, x$.
Using Euler discretization and the full truncation scheme for the volatility process:

$$\begin{aligned}
r_{i+1} &= r_i + \lambda\theta_i\Delta t_i - \lambda r_i \Delta t_i + \eta\sqrt{\Delta t_i}Z^r \\
v_{i+1} &= v_i + \kappa(\bar{v} - v_i)\Delta t_i + \gamma\sqrt{v_i}\sqrt{\Delta t_i}Z^v \\
v_{i+1} &= \max\{v_{i+1}, 0\} \\
x_{i+1} &= x_i + (r_i - \frac{1}{2}v_i)\Delta t_i + \rho_{xr}\sqrt{v_i}\sqrt{\Delta t_i}Z^r + \\
&\quad \rho_{xv}\sqrt{v_i}\sqrt{\Delta t_i}Z^v + \sqrt{1 - \rho_{xr}^2 - \rho_{xv}^2}\sqrt{v_i}\sqrt{\Delta t_i}Z^x
\end{aligned} \tag{3.37}$$

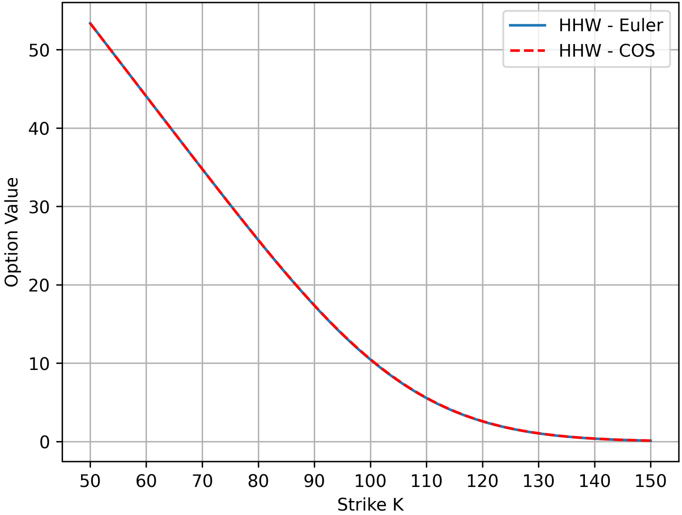




$S_0 = 100.0$
$T = 1$
$r_0 = 0.07$
$K \in [50, 150]$
$N_{Paths} = 25000$
$N_{Steps} = 1000$
$\lambda = 0.05$
$\eta = 0.005$
$\gamma = 0.0571$
$\bar{v} = 0.0398$
$v_0 = 0.0175$
$\kappa = 1.5768$
$\rho_{xr} = 0.2$
$\rho_{xv} = -0.5711$
$N = 500$
$L = 8$

Plain Vanilla European CALL Option under HHW model

Numerical Results



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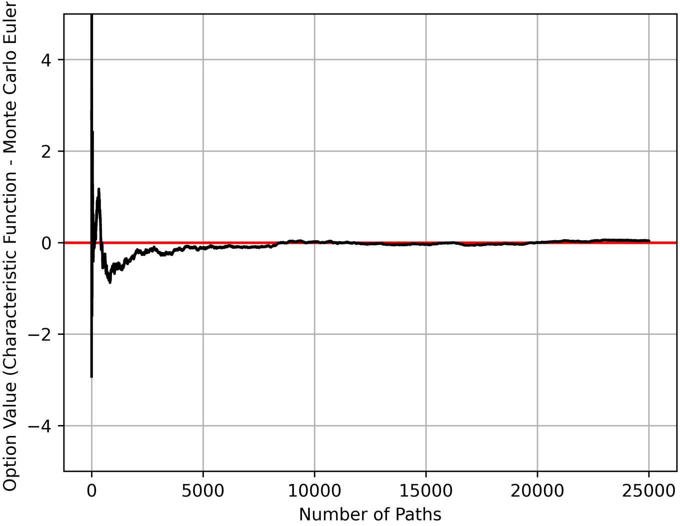

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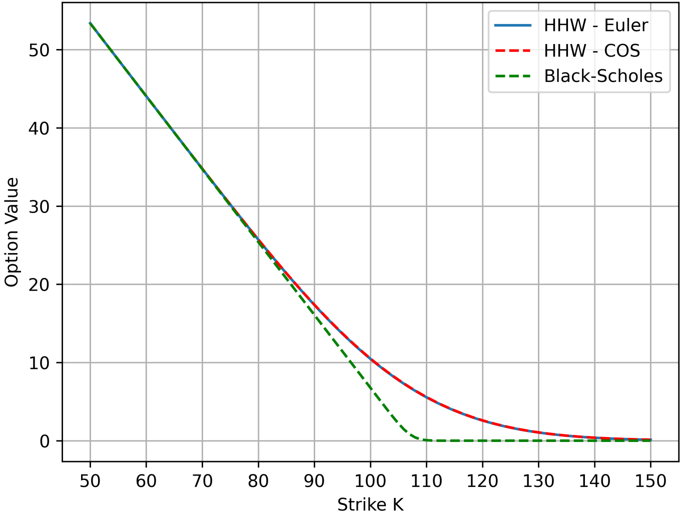
$L = 8$

K	Euler Value	COS Value	Diff. (Error)
50	53.38031	53.38040	0.00009
55	48.71869	48.71898	0.00029
60	44.05901	44.05967	0.00066
65	39.40608	39.40789	0.00181
70	34.77528	34.77759	0.00232
75	30.19377	30.19806	0.00429
80	25.70803	25.72022	0.01219
85	21.39669	21.41869	0.02200
90	17.35497	17.38597	0.03100
95	13.67948	13.71880	0.03932
100	10.46095	10.50016	0.03920
105	7.74729	7.78308	0.03580
110	5.55036	5.58167	0.03131
115	3.84677	3.87130	0.02454
120	2.57668	2.59698	0.02030
125	1.66961	1.68580	0.01619
130	1.04656	1.05978	0.01322
135	0.63683	0.64587	0.00904
140	0.37387	0.38205	0.00818
145	0.21232	0.21963	0.00731
150	0.11512	0.12288	0.00776

Time required for a simulation with 21 Stikes Price (1000 runs):

Euler = 22.47512 ± 0.06352 s

COS = 0.00649 ± 0.00002 s



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$r_0 = 0.07$
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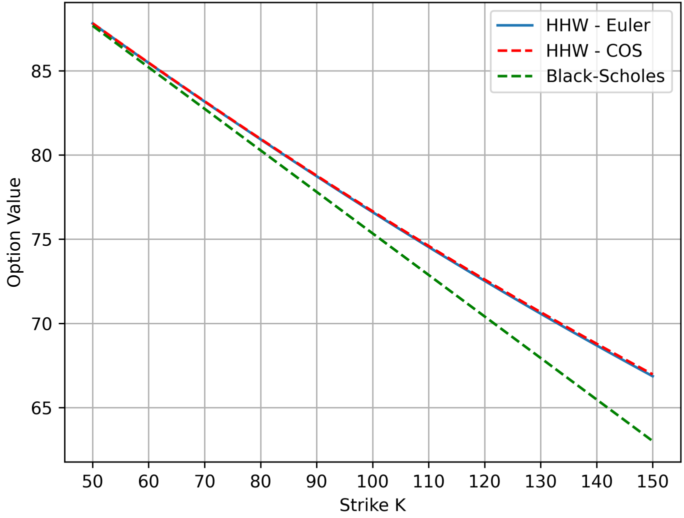
Black-Scholes vs Heston-Hull-White, $T = 20y$

Numerical Results



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Analytical-Numerical consideration

The above derivation for the H1-HW model is based on a zero correlation between the variance and interest rate processes.
A generalization of the H1-HW model to a full matrix of non-zero correlations between the processes can be made by similar approximations of the non-affine covariance matrix terms using their respective expectations.
A better theoretical understanding of the difference between the full-scale and the approximate H1-HW model may be based on the corresponding option pricing PDEs. For the Heston-Hull-White model, the option pricing PDE will be three-dimensional. By changing measures, from the spot measure to the T-forward measure, the pricing PDE reduces to a two-dimensional PDE to facilitate the analysis.

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