AMORO Lab: Kinematics and Dynamics of a Biglide

Raffaele Pumpo, Emanuele Buzzurro

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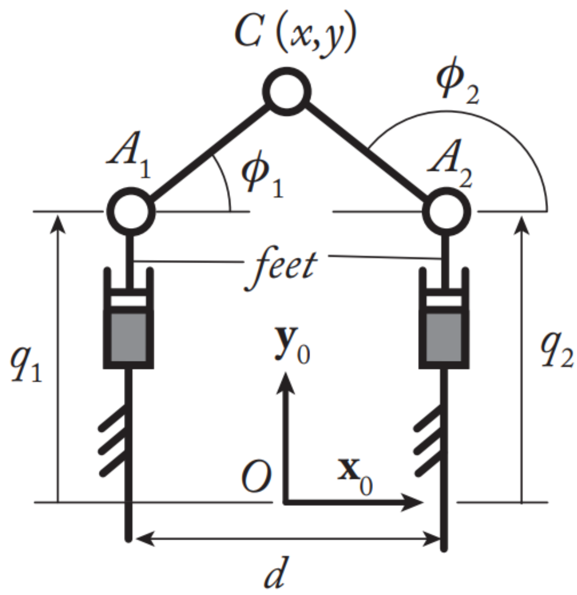

Figure 1: Kinematic model of the Biglide

1 Introduction

The main objective of the present lab is to compute the geometric, kinematic and dynamic models of a Biglide mechanism and to compare them with the results obtained with GAZEBO.
In this report we will only explain, in detail, the computation of the biglide mechanism models and how each step has been done.
In particular, we solved it in two different ways, and they will both be showed below, side by side.

2 Model

The kinematic architecture of the five-bar mechanism is shown in Fig.1. For the GAZEBO model, the geometric parameters are:

- $d = 0.4$ m
- $l_{A1C} = 0.3606$ m
- $l_{A2C} = 0.3606$ m

The two prismatic joints are actuated.

The base dynamic parameters are:

- $m_p = 3$ kg the mass of the end-effector
- $m_f = 1$ kg the mass of each foot

All other dynamic parameters are neglected.

3 Geometric models

3.1 Direct geometric model

The direct geometric model gives the position of the end-effector (x, y) as a function of the active joints coordinates (q1, q2) and the assembly mode.

The final objective of this methodology will be about obtaining the vector $\vec{OC}$ as a function of those variables. The plan is to build up the vector as a sum of others which depend on the active joints coordinates.
To do that, we followed the right leg path, in which has been used an ausiliary point: H, which represents the mid-point between the A1 and A2. The geometric model obtained by the following scheme:

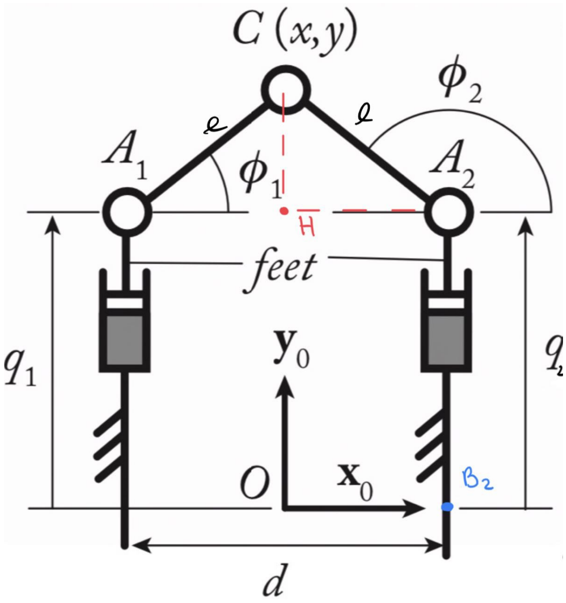

Figure 2: Biglide Model with segments

can be summarized as the following vectorial sum:

$\vec{OC} = \vec{OB_2} + \vec{B_2A_2} + \vec{A_2H} + \vec{HC}$
Where:

- $\vec{OB_2} = \begin{bmatrix} d/2 \\ 0 \end{bmatrix}$
- $\vec{B_2A_2} = \begin{bmatrix} 0 \\ q_2 \end{bmatrix}$
- $\vec{A_2H} = \frac{1}{2} * \vec{A_2A_1}$ with $\vec{A_2A_1} = -\vec{OA_1} + \vec{OA_2}$
- $\vec{OA_1} = \begin{bmatrix} -d/2 \\ q_1 \end{bmatrix}$
- $\vec{OA_2} = \begin{bmatrix} d/2 \\ q_2 \end{bmatrix}$
- $\vec{HC} = \gamma * \sqrt{l^2 + a^2}/a * U * \vec{A_2H}$ where:

  a $=|| \vec{A_2H} ||$

  U $= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

3.2 Passive joints geometric model

In this segment of the lab, we shift our focus towards the evaluation of the passive joint variable:

$q_d = [\phi_1, \phi_2]$

To do that, we start from the following equation, founded following the right leg path:

$\vec{OC} = \vec{OB_2} + \vec{B_2A_2} + l_{A2C} * \begin{bmatrix} \cos\phi_2 \\ \sin\phi_2 \end{bmatrix}$

And we divide the y term for the x term, obtaining:

$\frac{y}{x} = -2 * q_1/d + \tan(\phi_2)$

From which we can explicit the desired value of ϕ_2 as function of x, y:

$\phi_2 = \arctan\left(\frac{2q_2}{d} - \frac{y}{x}\right)$

In a similar way we can derive, for the left leg:

$\phi_1 = \arctan\left(\frac{2q_1}{d} + \frac{y}{x}\right)$

3.3 Inverse geometric model

The inverse geometric model in robotics refers to the mathematical description or algorithm that determines the active joint configuration of a robotic manipulator given a desired end-effector position and orientation. In other words, it involves calculating the joint variables necessary to achieve a specific pose (position and orientation) of the robot's end effector.
In our case we want to find:

$q_a = [q_1, q_2]$

As function of x and y .
Starting always from the following equation, founded following the right leg path:

$\vec{OC} = \vec{OB_2} + \vec{B_2A_2} + l_{A2C} * \begin{bmatrix} \cos\phi_2 \\ \sin\phi_2 \end{bmatrix}$

In order to obtain ,explicitly, the two active variable only as function of (x, y), we decided to sum the squared value of the elements along $\vec{x}$ and along $\vec{y}$, remaining with the following equation:

$(x - d/2)^2 + (y - q_2)^2 = (l_{A2C})^2$

As well as for the left leg:

$(x + d/2)^2 + (y - q_1)^2 = (l_{A1C})^2$

From where we derive the desired outcome, by isolating the active joint term in each equation:

$q_1 = \text{y} \pm\sqrt{(}(\text{l}_{A1C})^2 \text{ - } (x - d/2)^2)$
$q_2 = \text{y} \pm\sqrt{(}(\text{l}_{A2C})^2 \text{ - } (x + d/2)^2)$

In the end, we are left with 4 different possible solution.

4 First order kinematic models

The first-order Kinematic Model provides a mathematical relationship between the joint velocities and the resulting end-effector velocity of a robotic manipulator. In simpler terms, it describes how changes in joint positions affect the velocity of the robot's end effector.

4.1 Forward and Inverse Kinematic model

In this section, we want to find a relation between the active joint velocities and the end effector velocity.

4.1.1 Methodology 1

To do that we can start from the usual equation for the right leg:
$\vec{OC} = \vec{OB_2} + \vec{B_2A_2} + l_{A2C} * \begin{bmatrix} \cos\phi_2 \\ \sin\phi_2 \end{bmatrix}$

And rewrite it considering that:

- $\vec{OC} = \xi$
- $\vec{OB_2} = \frac{d}{2}\ \vec{x}$
- $\vec{B_2A_2} = q_1\ \vec{y}$
- $u_2 = \begin{bmatrix} \cos\phi_2 \\ \sin\phi_2 \end{bmatrix}$

So that we obtain the following equation:

$\xi = q_2\ \vec{y} + l_{A2C}\ \cdot \mathrm{u}_2 + \frac{d}{2}\ \vec{x}$

In a similar way we obtain the equation for the left leg:

$\xi = q_1\ \vec{y} + l_{A1C}\ \cdot \mathrm{u}_1$ - $\frac{d}{2}\ \vec{x}$

Since we want to find relation between velocities, we have to derive this equation with respect to time, considering that:

$\frac{d}{dt}\ \cdot \mathrm{u}_1 = \dot{\phi}_1 \cdot \begin{bmatrix} \text{- } \sin\phi_1 \\ \cos\phi_1 \end{bmatrix} = \mathrm{v}_1$

$\frac{d}{dt}\ \cdot \mathrm{u}_2 = \dot{\phi}_2 \cdot \begin{bmatrix} \text{-}\sin\phi_2 \\ \cos\phi_2 \end{bmatrix} = \mathrm{v}_2$

We obtain:

$\dot{\xi} = \dot{q_2}\ \vec{y} + l_{A2C} \cdot v_2 \cdot \dot{\phi_2}$

$\dot{\xi} = \dot{q_1}\ \vec{y} + l_{A1C} \cdot v_1 \cdot \dot{\phi_1}$

Now, to obtain explicitly the active joint velocities variable as function of the end effector one, we can multiply each terms of the first equation for u_2^T, and each term of the second equation for u_2^T, obtaining:

$u_2^T \cdot \dot{\xi} = u_2^T \cdot \dot{q_2}\ \vec{y}$

$u_1^T\ \cdot \dot{\xi} = u_1^T \cdot \dot{q_1}\ \vec{y}$

Which can be written as a matrix form, as following:

$$\begin{bmatrix} \mathrm{u}_1^T \\ \mathrm{u}_2^T \end{bmatrix} \cdot \dot{\xi} = \begin{bmatrix} \mathrm{u}_1^T \cdot \vec{y} & 0 \\ 0 & \mathrm{u}_2^T \cdot \vec{y} \end{bmatrix} \cdot \dot{q_a}$$

Where:

- $\mathrm{A} = \begin{bmatrix} \mathrm{u}_1^T \\ \mathrm{u}_2^T \end{bmatrix}$
- $\mathrm{B} = \begin{bmatrix} \mathrm{u}_1^T \cdot \vec{y} & 0 \\ 0 & \mathrm{u}_2^T \cdot \vec{y} \end{bmatrix}$

Which leads to the much more simpler form:

$\mathrm{A} \cdot \dot{\xi} = \mathrm{B} \cdot \dot{q_a}$

From this equation it's easy to analyse the singularities of the mechanism in which we coukd run into:

- $\det(A) = 0$, called "type-2 singularities". It happens when u_1 is parallel to u_2 , the two legs are aligned, and the robot could gain an uncontrollable motion along y coordinates, of the end-effector, even if the actuator are fixed.

- $\det(B) = 0$, "serial singularity", it happens when u_1 or u_2 are orthogonal to y, in this case the robot lose a DOF of the end effector: the motion along the x direction.

The forward and inverse kinematic model are obtained by inverting either A or B.

4.1.2 Methodology 2

In this alternative methodology, we always start from the partial loop closure equations:

h_1: $(x + d/2)^2 + (y - q_1)^2$ -$(l_{A1C})^2$

h_2: $(x - d/2)^2 + (y - q_2)^2$ -$(l_{A2C})^2$

And we can directly find the desired matrices, as following:

$$A = \frac{\partial \vec{H}}{\partial \vec{\xi}} = \begin{bmatrix} 2 \cdot (x + d/2) & 2 \cdot (y - q_1) \\ 2 \cdot (x - d/2) & 2 \cdot (y - q_2) \end{bmatrix}$$

$$B = \frac{\partial \vec{H}}{\partial \vec{q_a}} = \begin{bmatrix} \text{-}2 \cdot (y - q_1)\ 0 & 0 \\ 0 & \text{-}2 \cdot (y - q_2) \end{bmatrix}$$

In order to correctly explicit the following, desired, relation:

$$\mathrm{A} \cdot \dot{\xi} = \mathrm{B} \cdot \dot{q_a}$$

from which , by rearranging the terms, we are able to find the inverse/direct kinematic model, remembering that: $J = \text{-}A^{-1}B$

4.2 Passive joints kinematic model

In this section we are interested in finding a relation between the velocities of: end-effector, passive joints, active joints.

4.2.1 Methodology 1

To obtain explicitly the relation between passive joints and end-effector velocities, we can start by the following equation, previosuly evaluated:

$$\dot{\xi} = \dot{q_2}\ \vec{y} + l_{A2C} \cdot v_2 \cdot \dot{\phi}_2$$

$$\dot{\xi} = \dot{q_1}\ \vec{y} + l_{A1C} \cdot v_1 \cdot \dot{\phi}_1$$

And we can multiply each terms of the first equation for v_2^T, and each term of the second equation for v_1^T , obtaining:

$$v_2^T \dot{\xi} = v_2^T\ \dot{q_2}\ \vec{y} + l_{A2C} \cdot \dot{\phi}_2$$

$$v_1^T\ \dot{\xi} = v_1^T\ \dot{q_1}\ \vec{y} + l_{A1C} \cdot \dot{\phi}_1$$

Which can be written under the matrix form, as following:

$$\begin{bmatrix} \mathrm{v}_1^T \\ \mathrm{v}_2^T \end{bmatrix} \cdot \dot{\xi} = \begin{bmatrix} \mathrm{l}_{A1C} & 0 \\ 0 & \mathrm{l}_{A2C} \end{bmatrix} \cdot \dot{\phi} + \begin{bmatrix} \mathrm{v}_1^T \vec{y} & 0 \\ 0 & \mathrm{v}_2^T \vec{y} \end{bmatrix} \cdot \dot{q_a}$$

Where:

- $A_{p1} = \begin{bmatrix} \mathrm{v}_1^T \\ \mathrm{v}_2^T \end{bmatrix}$
- $d_p = \begin{bmatrix} \mathrm{l}_{A1C} & 0 \\ 0 & \mathrm{l}_{A2C} \end{bmatrix}$
- $B_{p1} = \begin{bmatrix} \mathrm{v}_1^T \vec{y} \\ \mathrm{v}_2^T \vec{y} \end{bmatrix}$

We can now write the final desired equation:

$A_p\dot{\xi} = d_p\dot{\phi} + B_{p1} \cdot \dot{q_a}$
Inverting the matrices we can easily obtain the value of $\dot{\phi}$ as a function of the active joint (DKM of the passive joint).

4.2.2 Methodology 2

To do that, we can find a new relation from the following equation, given by considering the loop equation of the parallel robot:

Φ_1: $l\cos\phi_1 - l\cos\phi_2$ -d

Φ_2: $q_1 + l\sin\phi_1 - l\sin\phi_2 - q_2$

From where, we can derive the two matrices, needed for the desired relation, as following:

$A_p = \frac{\partial\vec{\Phi}}{\partial\vec{q_p}} = \begin{bmatrix} \text{-l } \sin\phi_1 & \text{l } \sin\phi_2 \\ \text{l } \cos\phi_1 & \text{-l } \cos\phi_2 \end{bmatrix}$
$B_p = \frac{\partial\vec{\Phi}}{\partial\vec{q_a}} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$
Now we are able to compute the following relation:

$A_p\dot{q_d} = -B_p\dot{q_a}$

Where: J_p = -$A_p^{-1}B_p$

5 Second order kinematic models

The second-order kinematic model provides a comprehensive description of how joint accelerations influence the motion of a robotic system. In contrast to the first-order kinematic model, which primarily addresses the relationship between joint velocities and end-effector velocities, the second-order model delves into the impact of joint accelerations on the overall dynamics of the robot.

This model is crucial for understanding the dynamic behavior of robotic manipulators during motion planning, trajectory generation, and control.

5.1 Forward and inverse kinematic model

5.1.1 Methodology 1

We can easily compute the Forward and inverse kinematic model for the second order kinematic models, from the following equation, founded in the analog section of the first order kinematic:

$\dot{\xi} = \dot{q}_2 \; \vec{y} + l_{A2C} \cdot v_2 \cdot \dot{\phi}_2$

$\dot{\xi} = \dot{q}_1 \; \vec{y} + l_{A1C} \cdot v_1 \cdot \dot{\phi}_1$

Operating a derivation step with respect to time, we obtain:

$\ddot{\xi} = \ddot{q}_2 \; \vec{y} + l_{A2C} \cdot v_2 \cdot \ddot{\phi}_2 \text{ - } l_{A2C} \cdot u_2 \cdot \dot{\phi}_2^2$

$\ddot{\xi} = \ddot{q}_1 \; \vec{y} + l_{A1C} \cdot v_1 \cdot \ddot{\phi}_1 \text{ - } l_{A1C} \cdot u_1 \cdot \dot{\phi}_1^2$

This result has been obtained, considering that:

$\frac{d}{dt} \cdot \mathrm{v}_1 = \dot{\phi}_1 \cdot \begin{bmatrix} \text{- } \cos\phi_1 \\ \text{- } \sin\phi_1 \end{bmatrix} = -\mathrm{u}_1$

$\frac{d}{dt} \cdot \mathrm{v}_2 = \dot{\phi}_2 \cdot \begin{bmatrix} \text{-}\cos\phi_2 \\ \text{- } \sin\phi_2 \end{bmatrix} = -\mathrm{u}_2$

Now, to obtain the desired relation between end-effector and active joint acceleration, we have to multiply both sides, respectively for u_2^T, u_1^T, remaining with:

$u_2^T \cdot \ddot{\xi} = u_2^T \cdot \ddot{q}_2 \; \vec{y} \text{ - } l_{A2C} \cdot \dot{\phi}_2^2$

$u_1^T \cdot \ddot{\xi} = u_1^T \cdot \ddot{q}_1 \; \vec{y} \text{ - } l_{A1C} \cdot \dot{\phi}_1^2$

Which can be rewritten under the matrix form:

$$\begin{bmatrix} \mathrm{u}_1^T \\ \mathrm{u}_2^T \end{bmatrix} \cdot \ddot{\xi} = \begin{bmatrix} \mathrm{u}_1^T \cdot \vec{y} & 0 \\ 0 & \mathrm{u}_2^T \cdot \vec{y} \end{bmatrix} \cdot \ddot{q}_a \text{ - } \begin{bmatrix} \mathrm{l}_{A1c} \cdot \dot{\phi}_1^2 \\ \mathrm{l}_{A2c} \cdot \dot{\phi}_2^2 \end{bmatrix}$$

Where:

- $A = \begin{bmatrix} \mathrm{u}_1^T \\ \mathrm{u}_2^T \end{bmatrix}$
- $B = \begin{bmatrix} \mathrm{u}_1^T \cdot \vec{y} & 0 \\ 0 & \mathrm{u}_2^T \cdot \vec{y} \end{bmatrix}$
- $d = \text{- } \begin{bmatrix} \mathrm{l}_{A1c} \cdot \dot{\phi}_1^2 \\ \mathrm{l}_{A2c} \cdot \dot{\phi}_2^2 \end{bmatrix}$

Which leads to the much more simpler form:

$A \cdot \ddot{\xi} = B \cdot \ddot{q_a} + d$

That can be rewritten as:

$\ddot{\xi} = -A^{-1} \cdot [B \cdot \ddot{q_a} + d\,]$

5.1.2 Methodology 2

An alternative methodology approach, can be done deriving wit respect to time, the already known matrices A, B:

$$\dot{A} = \begin{bmatrix} 2 \cdot \dot{x} & 2 \cdot (\dot{y} - \dot{q_1}) \\ 2 \cdot \dot{x} & 2 \cdot (\dot{y} - \dot{q_2}) \end{bmatrix}$$
$$\dot{B} = \begin{bmatrix} -2 \cdot (\dot{y} - \dot{q_1}) & 0 \\ 0 & -2 \cdot (\dot{y} - \dot{q_2}) \end{bmatrix}$$

And now it is possible to explicit the desired relation as:

$A \cdot \ddot{\xi} + \dot{A} \cdot \dot{\xi} = B \cdot \ddot{q_a} + \dot{B} \cdot \dot{q_a}$

That can be rewritten as:

$$\ddot{\xi} = -A^{-1} \cdot [B \cdot \ddot{q_a} + \dot{B} \cdot \dot{q_a} + \dot{A} \cdot \dot{\xi}]$$

5.2 Passive Joint second order kinematic model

5.2.1 Methodology 1

To obtain explicitly the relation between passive joint and end-effector accelerations, we can start by the following equation, previosuly evaluated:

$\ddot{\xi} = \ddot{q_2}\,\vec{y} + l_{A2C} \cdot v_2 \cdot \ddot{\phi}_2 - l_{A2C} \cdot u_2 \cdot \dot{\phi}_2^2$

$\ddot{\xi} = \ddot{q_1}\,\vec{y} + l_{A1C} \cdot v_2 \cdot \ddot{\phi}_1 - l_{A1C} \cdot u_1 \cdot \dot{\phi}_1^2$

And we can multiply each terms of the first equation for v_2^T, and each term of the second equation for v_1^T , obtaining:

$v_2^T \ddot{\xi} = v_2^T\ \ddot{q_2}\,\vec{y} + l_{A2C} \cdot \ddot{\phi}_2$

$v_1^T \ddot{\xi} = v_1^T\ \ddot{q_1}\,\vec{y} + l_{A1C} \cdot \ddot{\phi}_1$

Which can be written under the matrix form, as following:

$$\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} \cdot \ddot{\xi} = \begin{bmatrix} l_{A1C} & 0 \\ 0 & l_{A2C} \end{bmatrix} \cdot \ddot{\phi} + \begin{bmatrix} v_1^T \vec{y} & 0 \\ 0 & v_2^T \vec{y} \end{bmatrix} \cdot \dot{q_a}$$

Where:

- $A_{p1} = \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}$
- $d_p = \begin{bmatrix} l_{A1C} & 0 \\ 0 & l_{A2C} \end{bmatrix}$
- $B_{p1} = \begin{bmatrix} v_1^T \vec{y} & 0 \\ 0 & v_2^T \vec{y} \end{bmatrix}$

Finally, we obtained our desired equation:

$A_{p1}\ddot{\xi} = d_p\ddot{\phi} + B_{p1} \cdot \dot{q_a}$

5.2.2 Methodology 2

Here, we just have to derive, with respect to time, the matrices A_p, B_p previously founded, obtaining:

$$\dot{A}_p = \begin{bmatrix} \text{-l} \cos\phi_1\dot{\phi}_1 & \text{l} \cos\phi_2\dot{\phi}_2 \\ \text{-l} \sin\phi_1\dot{\phi}_1 & \text{l} \sin\phi_2\dot{\phi}_2 \end{bmatrix}$$

$$\dot{B}_p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

That allowed us to find the desired relation:

$$A_p\ddot{q}_d + \dot{A}_p\dot{q}_d + B_p\ddot{q}_a + \dot{B}_p\dot{q}_a = 0$$

6 Dynamic model

The dynamic model of a robot is a mathematical representation that characterizes the forces and torques involved in the robot's motion. Unlike the kinematic model, which focuses on the relationship between joint positions, velocities, and end-effector motion, the dynamic model considers the forces and torques acting on the robot due to both its own motion and external influences.
This model is used to describe how the robot responds to applied forces, accelerates, and interacts with its environment. The dynamic model is essential for understanding the complex dynamics of robotic systems, aiding in tasks such as control, trajectory planning, and stability analysis.

The dynamic model of the Bi-glide is based only on the following dynamic parameters:

- mp = 3 kg the mass of the end-eector
- mf = 1 kg the mass of each foot

Since no other information is given regarding the dynamic parameters of the structure, we can assume that the mass of the diagonal links is null. Moreover, their kinetic energy will not be considered in the calculation of the energy of the entire structure.
Due to the closed loop architecture of the biglide mechanism, we can virtually cut it at the end end-effector joint and deal with a virtual tree structure and virtually free-moving platform. For the computation of the dynamic model of the structure, we started by finding the Lagrangian equation.
Afterwards, the Lagrangian's derivatives will be calculated to retrieve the components of the vector of generalized forces and moments τ . The Lagrangian equation is defined as follows:

L = E - U

Where:

- E: Kinetic energy of the system
- U: Potential energy of the system

Since it is a planar mechanism, the gravity effect can be neglected and The potential energy of our system will then equate to 0.
The Lagrange's equation will then be considered as equal to the Kinetic energy of the system. Meaning:

L = E

The total kinetic energy of the system, will be expressed as the sum of all energies of the bodies components. In our case the bodies with have a kinetic energy term, are the two legs and the end effector. Therefore, The total energy will be the following:

$$E_{tot} = E_{legR} + E_{legL} + E_{ee}$$

The dynamic model of the Biglide mechanism is obtained by computing, first, the dynamic model of the tree structure without the end end-effector. The formulation of the kinetic energy for a generic rigid-body is given by the following equation:

$E_i = \frac{1}{2}(m_i v_i^T v_i + w_i^T I_{oi} w_i + 2^i m s_i (v_i \times w_i))$

In this robot, the legs action is only translational and not rotational, and the equation for both bodies simplifies to:

$E_i = \frac{1}{2}(m_i v_i^T v_i)$ for i = 1,2

From which we derive the Lagrangian of the tree structure, which is:

$L_{legs} = E_{legs} = \frac{1}{2}(m_1 v_1^T v_1 + m_2 v_2^T v_2)$

While for the computation of the Lagrangian of the end end-effector, it can be easily computed, considering that its velocity is given by:

$v_{ee} = (\dot{x}\vec{x} + \dot{y}\vec{y})$

The Lagrangian turns out to be:

$L_{ee} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) m_{ee}$

Finally, by summing up both terms we obtain:

$L = L_{ee} + L_{legs} = \frac{1}{2}(m_1 v_1^T v_1 + m_2 v_2^T v_2 + (\dot{x}^2 + \dot{y}^2) m_{ee})$

Then using the Lagrange formalism, we can calculate τ_a (virtual input torques of the actuated robot's joints), τ_d (virtual input torques of the passive robot's joints),w_p (force and moments applied by the end end-effector), respectively:

- $\tau_a = \frac{d}{dt}(\frac{\partial L}{\partial \dot{\vec{q_a}}})^T - (\frac{\partial L}{\partial \vec{q_a}})^T = \begin{bmatrix} \mathrm{m}_1 & 0 \\ 0 & \mathrm{m}_2 \end{bmatrix} \cdot \ddot{\vec{q_a}}$
- $\tau_d = \frac{d}{dt}(\frac{\partial L}{\partial \dot{\vec{q_d}}})^T - (\frac{\partial L}{\partial \vec{q_d}})^T = 0$
- $w_p = \frac{d}{dt}(\frac{\partial L}{\partial \dot{\vec{x}}})^T - (\frac{\partial L}{\partial \vec{x}})^T = m_{ee} \cdot \ddot{\vec{x}}$

Thanks to the Lagrangian multipliers, we close the virtual tree structure and we consider as active joint the real one since until now we are considering as all active. Then the total vector of generalized forces, is given by the following relation:

$\tau = \tau_a + J_p^T \tau_d + J^T w_p$

Where J_p, J are the matrices evaluated in the previous steps.

It is also possible to obtain the expression of the dynamic model as a function of the active joint acceleration, under the form:

$\tau = M \ddot{q_a} + c$

Where:

- M is the inertia matrix and it is positive definite.
- c is the vector of Coriolis and centrifugal effects.

And they can be founded, starting from the following equation:

$w_p = m_{ee} \ddot{\vec{\xi}}$

$\ddot{\vec{\xi}} = -A^{-1} \cdot [B \cdot \ddot{q_a} + \dot{B} \cdot \dot{q_a} + \dot{A} \cdot \dot{\xi}]$

By substituting the second equation, in the first one, we obtain:

$w_p = m_{ee}(J\ddot{q_a} - A^{-1}(\dot{A}\dot{\xi} + \dot{B}\dot{q_a})) = m_{ee} J \ddot{q_a} + m_{ee} \cdot b$

Where: b $= -A^{-1}(\dot{A}\dot{\xi} + \dot{B}\dot{q_a})$.
Then we can write:

$\tau = \tau_a + J_p^T \tau_d + J^T w_p = \begin{bmatrix} \mathrm{m}_1 & 0 \\ 0 & \mathrm{m}_2 \end{bmatrix} \cdot \ddot{\vec{q_a}} + J^T m_{ee} \ (J \cdot \ddot{q_a} + b)$

Regrouping the terms that multiply $\ddot{q_a}$:

$$\tau = (\begin{bmatrix} \mathrm{m}_1 & 0 \\ 0 & \mathrm{m}_2 \end{bmatrix} J^T m_p \ J) \cdot \vec{\ddot{q_a}} + J^T m_{ee} \ b$$

We can finally write:

- $\mathrm{M} = \begin{bmatrix} \mathrm{m}_1 & 0 \\ 0 & \mathrm{m}_2 \end{bmatrix} + m_{ee} J^T J$
- $\mathrm{c} = m_{ee} J^T b$