1 Pump my Sodium qubit

In this exercise we are going to make an optical atomic qubit for neutral Sodium (assume its center of mass trapped by an optical tweezer), and we will use no external magnetic fields.

When the external electron occupies the state $|3P_{3/2}, j_z = +\frac{3}{2}\rangle$ it has a spontaneous emission probability rate of $6.14 \cdot 10^7 s^{-1}$.

The excitation energy of the shell $3P_{3/2}$ over the ground state (that is, $3S_{1/2}$) is exactly 2.104429202(6) eV, a warm saffron yellow.

1. From this data, reconstruct the vector of electric dipole transitions

$$\langle 3S_{1/2}, j_z = +\tfrac{1}{2}|d_u|3P_{3/2}, j_z = +\tfrac{3}{2}\rangle$$

   in Cartesian coordinates $u = x, y, z$.

2. Consequently, calculate the dipole transition vector

$$\langle 3S_{1/2}, j_z|d_u|3P_{3/2}, j'_z\rangle$$

   for all possible (allowed) values of j_z and j'_z. Arrange all the 24 matrix elements in a table.

3. Then, calculate all spontaneous emission probability rates from any state $|3P_{3/2}, j'_z\rangle$ to any state $|3S_{1/2}, j_z\rangle$. Arrange the 8 probability rates in a table.

4. Write a Born-Markov model (a Master Equation) that captures the dynamics of spontaneous emission from the $|3P_{3/2}, j'_z\rangle$ levels to the $|3S_{1/2}, j_z\rangle$. Do you manage using only six Lindblad operators?

5. Now we also add laser light, propagating along z and circularly polarized (say anticlockwise) in the xy plane. The light is resonant to the $3S_{1/2} \leftrightarrow 3P_{3/2}$ transition (ignore the AC stark shift and ignore the $3P_{1/2}$ levels). Write the Hamiltonian in the lab frame, then in a rotating frame where it does not depend on time.

6. What amplitude E_0 of the laser electric field produces a Rabi frequency comparable to the spontaneous emission rate?

7. Using this E_0, now start from an initial mixed state ρ_0 that is the equal mixture of $|3S_{1/2}, j_z = +\frac{1}{2}\rangle$ and $|3S_{1/2}, j_z = -\frac{1}{2}\rangle$. Integrate the Lindblad master equation (with Laser on), analytically or numerically. Prove that the population of one of the initial two states dies out over time.

2 Measuring Ionic Phonons

Measuring the vibrational center-of-mass motion (i.e. phonons) for a trapped ion is tough. The internal atomic qubit state is easy: one can measure σ_z with Fluorescence measurement and the other Paulis, σ_x and σ_y, by rotating the atomic qubit before Fluorescence. But there is no direct way to measure phonons. So let us see what can be done with standard tools.

Consider a single two-level ion optical qubit, of frequency ω_a around the visible spectrum, trapped in a uni-axial harmonic trap ω_p in the MHz. Assume we are working in Lamb-Dicke regime $\eta \simeq 5\%$.

1. I turn on bichromatic laser light. The blue-detuned tone, with laser frequency ω_b and Rabi frequency Ω_b, is perfectly resonant with the blue sideband. The red-detuned tone, with laser frequency ω_r and Rabi frequency Ω_r, is perfectly resonant with the red sideband. The relative phase ϕ between the two beams is locked to zero. Write the conditions of being 'perfectly resonant' as large inequalities.

2. Then, write the conditions for the Carrier transition to be off-resonant, as a large inequality.

3. Write the total Hamiltonian (Atoms, Phonon, and Laser coupling) of the system in the Lab frame. Then transform it into a rotating frame where the Hamiltonian is completely time independent, once RWA is applied.

4. Can you formally integrate this Hamiltonian for the case $\Omega_b = \Omega_r$? You should be able to. Write the explicit time evolution operator $U(t)$ in the rotating frame.

5. Now we start with the qubit prepared in the state $|0\rangle$, and the phonon state unknown ρ_p. We evolve the system with $U(t)$ for a time t, then measure σ^y on the qubit. Show that we actually measured a property of the initial phonon state.

6. Verify numerically that if we start with the phonon in a coherent state $|\alpha\rangle$, the result of our measurement (for various t values), gives us the information we expected.

3 Cavity blockade: a way to make photons interact

In this exercise you will explore a regime in which a cavity, once filled with a single photon, blocks the possibility to insert another one. This can be effectively interpreted as a photon-photon interaction mediated by the atom in the cavity, a possibility that photons in vacuum do not have (the photon-photon cross section predicted by QED in vacuum is extremely small). The first observation of this phenomenon was done in Birnbaum et al. Nature 436, 87 (2005).

A driven cavity is described by a Jaynes-Cummings model

$$\hat{H} = \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar\omega_a |e\rangle\langle e| + \hbar\tilde{g}(\hat{\sigma}^+ \hat{a} + \hat{\sigma}^- \hat{a}^\dagger) + \hat{H}_d \,,$$

where $\hat{H}_d = \mathcal{E} e^{i\omega t} \hat{a} + \text{h.c.}$ is a (pumping) driving field.

1. Explain the physical meaning of $\hat{H}_d$. How would you realize such a term?
2. Rewrite the hamiltonian in the frame of the drive and use the RWA.

Consider $\omega = \omega_a = \omega_c$ and no coupling between cavity and atom, $\tilde{g} = 0$. Given an initially empty cavity, $|g, 0\rangle$, you will numerically solve the corresponding master equation at resonance (you can use the QuTiP library). Assume a small decay rate of the atom in the photon modes outside the cavity, Γ, and assume also that the cavity is not perfect and can lose photons with a small decay rate κ.

3. Integrate the master equation for enough large times to find the steady state. What is the mean photon number $\langle \hat{a}^\dagger \hat{a} \rangle$? What is the steady-state occupation of the n-photon states? Plot the histogram of occupations and compare with a coherent state distribution. [Suggestion: work in a regime of parameters that allow you to keep the photon Hilbert space sufficiently small (e.g. max $10 - 50$ photon states) and the average photon number of the steady state small, $\langle \hat{a}^\dagger \hat{a} \rangle \leq 10$. Explain how you made your choice.]
4. In the presence of a coupling $\tilde{g}$, the previous resonance condition is not valid anymore. Set ω to resonance with the lowest (one-photon) dressed state of the Jaynes-Cummings model and find again the steady state in the strong-coupling condition $\tilde{g} \gg \Gamma, \kappa$ and weak drive $\mathcal{E}$. How important is the value of the pumping strength $\mathcal{E}$? (Be careful that increasing $\mathcal{E}$ too much could lead to instability/bistability)
5. Can you have occupation of two photon states? If not, why? What happens if you increase the values of the decay rates? And if you change $\tilde{g}$?
6. A way to understand the suppression of two-photon states is to compute the two-photon correlation function $g^{(2)} \equiv \langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle / |\langle \hat{a}^\dagger \hat{a} \rangle|^2$, which compares the probability to find two simultaneous photons with the one of having two independent photons at random. In the blockade regime, you should find $g^{(2)} < 1$.
7. Scan the values of ω around the resonance and graphically represent your findings as in Fig. 2a of the paper by Birnbaum et al.
8. (Bonus) Can you make analytical predictions for the steady state in a convenient regime?

4 Two-particle dynamics in an optical lattice

We consider two indistinguishable photons or bosonic atoms on a one-dimensional lattice with $L = 4$ sites and periodic boundary conditions described by the Bose-Hubbard model:

$$\hat{H} = -J \sum_{i=1}^{3} \left(\hat{a}_i^\dagger \hat{a}_{i+1} + \text{h.c.} \right) + \frac{U}{2} \sum_{i=1}^{4} \hat{n}_i (\hat{n}_i - 1) \, . \tag{1}$$

The aim of this problem is to show that there are bound states in this model, known as doublon states, with both particles occupying the same site: $|d_i\rangle \equiv \hat{a}_i^\dagger \hat{a}_i^\dagger |0\rangle / \sqrt{2} = |2_i\rangle$, $i = 1, 2, 3, 4$, and to identify signatures of these that would be observed in quantum simulators. We focus on the parameter regime $U = 10J$.

1. List all d states of the Hilbert space. How many are there? How many states would there be for a generic chain of length L?

2. Show that doublons are not eigenstates of $\hat{H}$. Which part of the Hamiltonian do they diagonalize?

3. Implement the Hamiltonian matrix, diagonalize it numerically and plot the energy spectrum. Which states could correspond to $|d_i\rangle$?

4. The problem has several symmetries, for example translation or reflection. Pick one, write the basis in the new symmetry sector and then the Hamiltonian in blocks to reduce its complexity. Verify that the eigenvalues coincide with those obtained in the previous case.

5. We consider the hopping term (J) of the Hamiltonian as a perturbation and the doublons $|d_i\rangle$ as degenerate states. Derive the doublon energies up to second order using degenerate perturbation theory and verify with the previous numerical result. [Note: from now on work again in the initial basis, not the symmetric one].

6. A doublon can be pictured as a (composite) free particle moving with hopping $J' = -2J^2/U$. Can you see that from the previous result?

7. At time $t = 0$, a doublon $|d_i\rangle$ is prepared in the site $i = 2$. Numerically evolve the state with the Bose-Hubbard Hamiltonian up to t_{max} and check that the dynamics corresponds to the one of a single particle with hopping J' (for example by computing $\langle \hat{n}_i \rangle$, for all the sites). [Note: check various values of t_{max} to identify the best range].

8. Verify whether the results for the quantum walk derived in class hold. If you find disagreement, where does it come from?