

Background on Wasserstein Spaces

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1 Introduction

This manuscript's intention is define Wasserstein spaces and show their properties.

We assume that the reader is already familiar with optimal transport, which in turn assumes a solid knowledge of measure theory and basic functional analysis. An intuitive understanding of Riemannian geometry could help as well, but is by no means required.

1.1 Informal introduction to metric geometry

Metric geometry is the field of mathematics that studies abstraction of key ideas from differential geometry, relying only on the intrinsic notion of distance.

Let us fix a metric space (\mathcal{S}, d) . Suppose we have enough regularity to be able to define paths $\omega : I \subseteq \mathbb{R} \rightarrow \mathcal{S}$, and their lengths $L(\omega)$ in the usual way.

As in the familiar Euclidean settings, two paths $\omega_1 : I_1 \rightarrow \mathcal{S}$ and $\omega_2 : I_2 \rightarrow \mathcal{S}$ are *equivalent* if there exists a continuous, non-decreasing and surjective function $\phi : I_1 \rightarrow I_2$ such that $\omega_1 = \omega_2 \circ \phi$. In this case, ω_2 is called a *reparametrization* of ω_1 (and vice-versa by symmetry), and it is trivial to check that $L(\omega_1) = L(\omega_2)$.

We say that a path $\omega : [a, b] \rightarrow \mathcal{S}$ has *constant speed* if for all $a \leq s \leq t \leq b$,

$$L(\omega_{[s,t]}) = \frac{t-s}{b-a} L(\omega),$$

and we have that any rectifiable path $\omega : [a, b] \rightarrow \mathcal{S}$ has a constant-speed reparametrization $\bar{\omega} : [0, 1] \rightarrow \mathcal{S}$ by usual multivariate calculus.

For fixed $x_0, x_1 \in \mathcal{S}$, a *geodesic* is defined as a path $\omega : [0, 1] \rightarrow \mathcal{S}$ such that

$$d(x_0, x_1) = L(\omega).$$

A metric space (\mathcal{S}, d) is said to be a *geodesic space* if for any $x_0, x_1 \in \mathcal{S}$ we can find a (constant speed) geodesic.

In a geodesic space, given $x_0, x_1 \in \mathcal{S}$, we can consider the constant speed geodesic $\omega : [0, 1] \rightarrow \mathcal{S}$ such that $\omega(0) = x_0, \omega(1) = x_1$ and define the *mid-point* between x_0 and x_1 to be $x_{\frac{1}{2}} := \omega(0.5)$.

Let us denote with λ the Lebesgue measure on \mathbb{R}^d .

Notably, if we take $(\mathcal{S}, d) = (\mathcal{P}_2^{ac}(\lambda), L_2(\lambda))$ the space of absolutely continuous measures with finite second moment with the metric induced by the Lebesgue L_2 norm, we see that geodesics represent *teleportation*, i.e. the vertical interpolation, as shown below.

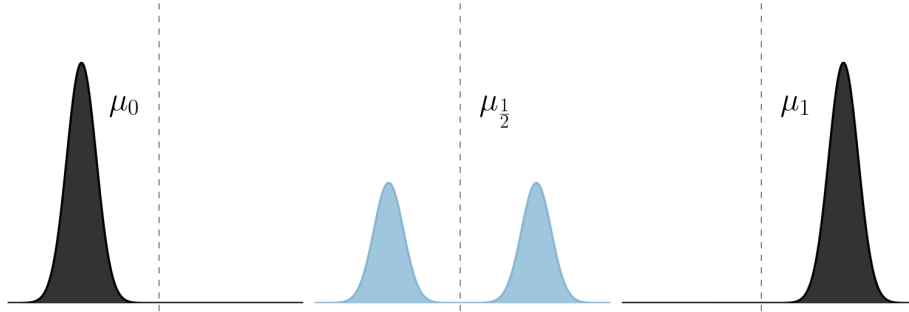


Figure 1: Geodesic in $(\mathcal{P}_2^{ac}(\lambda), L_2(\lambda))$.

The natural horizontal interpolation will be achieved in the next sections by $(\mathcal{S}, d) = (\mathcal{P}_2^{ac}(\lambda), \mathcal{W}_2)$, as shown below.

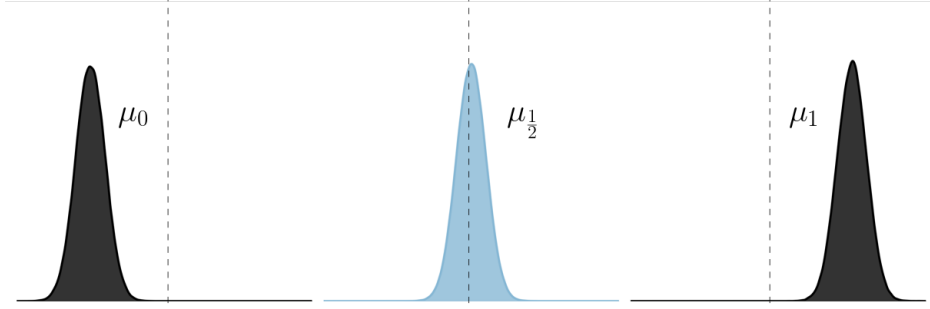


Figure 2: Geodesic in $(\mathcal{P}_2^{ac}(\lambda), \mathcal{W}_2)$.

1.2 Optimal transport

We are going to denote with \mathcal{P} the set of probability measures on \mathbb{R}^d and with \mathcal{P}_2 the subset of probability measures with finite second moment.

Definition 1.1. Given $\mu_0 \in \mathcal{P}$ and $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$, we define the *push forward measure*

$$T\#\mu_0 := \mu_0(T^{-1}(A)), \text{ for any } A \subseteq \mathbb{R}^d.$$

Remarkably, $\mu_1 = T\#\mu_0$ if and only if $\int_{\mathbb{R}^d} \phi d\mu_1 = \int_{\mathbb{R}^d} \phi \circ T d\mu_0$ for any $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ measurable and bounded, as shown in [3].

We have the following theorem which is useful in computations.

Theorem 1.2. Let $\mu_0, \mu_1 \in \mathcal{P}$, with $\mu_0, \mu_1 \ll \lambda$, where λ is the Lebesgue measure on \mathbb{R}^d . If $f = \frac{d\mu_0}{d\lambda}$ and $g = \frac{d\mu_1}{d\lambda}$ are the Radon-Nykodim derivatives, and $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a diffeomorphism such that $\mu_1 = T\#\mu_0$, we have that

$$f(x) = |\det J_{T(x)}| g(T(x)).$$

Proof. It follows from the standard change of variables formula. Indeed, for any $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ measurable and bounded,

$$\begin{aligned} \int_{\mathbb{R}^d} \phi \circ T(x) f(x) dx &= \int_{\mathbb{R}^d} \phi(y) g(y) dy \\ &= \int_{\mathbb{R}^d} \phi \circ T(x) g(T(x)) |\det J_{T(x)}| dx, \end{aligned}$$

which concludes because ϕ was arbitrary and T was bijective. \square

Definition 1.3. We will call *canonical projections* the functions

$$\begin{aligned}\pi_X : \mathbb{R}^d \times \mathbb{R}^d &\rightarrow \mathbb{R}^d, & \pi_X[(x, y)] &= x \\ \pi_Y : \mathbb{R}^d \times \mathbb{R}^d &\rightarrow \mathbb{R}^d, & \pi_Y[(x, y)] &= y\end{aligned}$$

Sometimes we will abuse of this notation without specifying the projections' domain and codomain, but it will always be clear from the context.

Definition 1.4. Given $\mu_0 \in \mathcal{P}$ and $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$, we denote the set of *couplings* as

$$\Gamma(\mu_0, \mu_1) := \{\gamma \in \mathcal{P} \times \mathcal{P} : \pi_X \# \gamma = \mu_0, \pi_Y \# \gamma = \mu_1\}.$$

We have the machinery to define the fundamental optimization problems.

Definition 1.5. Given $\mu_0, \mu_1 \in \mathcal{P}$ and a cost function $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, we define the *Monge's problem* as

$$(\text{MP}) = \inf_{T: \mathbb{R}^d \rightarrow \mathbb{R}^d} \left\{ \int_{\mathbb{R}^d} c(T(x), x) d\mu_0(x) : T \# \mu_0 = \mu_1 \right\}$$

This formally translates that we aim to move a probability measure to another one by minimizing a given cost function. We can relax the need of a transport map by only requiring a restriction in terms of couplings.

Definition 1.6. Given $\mu_0, \mu_1 \in \mathcal{P}$ and a cost function $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, we define the *Kantorovich's problem* as

$$(\text{KP}) = \inf_{\gamma \in \mathcal{P} \times \mathcal{P}} \left\{ \int_{\mathbb{R}^d} c(x, y) d\gamma(x, y) : \gamma \in \Gamma(\mu_0, \mu_1) \right\}.$$

Remarkably the set of couplings is never empty: it is enough to consider $\mu_0 \otimes \mu_1 \in \mathcal{P} \times \mathcal{P}$. Moreover, due to Prokhorov and Banach-Alaoglu's theorems, the infimum is actually reached, besides very pathological situations [3].

Furthermore any transport map T such that $T \# \mu_0 = \mu_1$ induces a coupling $\gamma_T := (id, T) \# \mu_0 \in \Gamma(\mu_0, \mu_1)$, as can be checked into [3].

In the following we are going to work only with $c(x, y) = \|x - y\|^2$, as this

choice is the most natural and the most studied, and furthermore we can anticipate that this choice will enable us to define the tangent space (in a fixed point of our Wasserstein space) as a Hilbert space. Accordingly to this choice, to avoid issues with $+\infty$, we will restrict ourselves to \mathcal{P}_2 .

We have the important theorem, due to Brenier.

Theorem 1.7. Given $\mu_0 \in \mathcal{P}_2^{ac}(\lambda)$ and $\mu_1 \in \mathcal{P}_2$, then there exists a unique optimizer $\bar{\gamma}$ in (KP). In addition, there exists $T = \nabla\phi$, with $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ convex, such that $\bar{\gamma} = (id, T)\#\mu_0$ and T is the unique optimizer in (MP).

2 Wasserstein Spaces

In this section we are going to consider the metric induced by (KP) on the space of measures and study its properties. In particular, we will show that the geometry induced by this metric lifts the geometry of the underlying space, as desired.

It turns out that this space also has a natural differential structure that leads not only to defining geodesics, but also to notions of tangent spaces and gradients. These, in turn, allow for the definition of Wasserstein gradient flows, which are tools of fundamental importance in statistical applications.

2.1 Wasserstein geometry

As hinted in the previous section, the inf in (KP) is really a min, besides pathological situations. Recalling that we restricted to the quadratic cost, we can thus define the following.

Definition 2.1. Given two probability measures $\mu_0, \mu_1 \in \mathcal{P}_2$, we define the *Wasserstein distance* between them as

$$\mathcal{W}_2(\mu_0, \mu_1) := \min_{\gamma} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|_2^2 d\gamma(x, y) \mid \gamma \in \Gamma(\mu_0, \mu_1) \right)^{\frac{1}{2}}.$$

This is indeed a metric, as the next result shows.

Theorem 2.2. $\mathcal{W}_2 : \mathcal{P}_2 \times \mathcal{P}_2 \rightarrow \mathbb{R}$ is a metric on \mathcal{P}_2 .

The proof can be found in [1]. Remarkably, when the optimal coupling is induced (uniquely) by an optimal map, the distance simplifies to

$$\mathcal{W}(\mu_0, \mu_1) = \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} \|T_{\mu_0 \rightarrow \mu_1}(x) - x\|_2^2 d\mu_0(x) \right)^{\frac{1}{2}}.$$

From now on we are going to stick with $\mathcal{P}_2^{ac}(\lambda)$, the family of measure with finite second moment and absolutely continuous with respect to Lebesgue, so that we can always assume the existence of such representation, thanks to Brenier's theorem.

Recalling our definitions from the first section, we can show that $(\mathcal{P}_2, \mathcal{W}_2)$ is a geodesic space by exhibiting geodesics between any two fixed $\mu_0, \mu_1 \in \mathcal{P}_2^{ac}(\lambda)$.

Proposition 2.3. Given any $\mu_0, \mu_1 \in \mathcal{P}_2$, the constant speed geodesic with respect to the Wasserstein distance is $\mu_t := T_t \# \mu_0$, where $T_t(x) := (1-t)x + tT_{\mu_0 \rightarrow \mu_1}(x)$.

The proof can be found in [1].

Moreover, we can see that the underlying space's geometry is preserved by looking at the isometry $(x, \|\cdot\|) \mapsto (\delta_x, \mathcal{W}_2)$:

Proposition 2.4. Given any $x_0, x_1 \in \mathbb{R}^d$, we have that

$$\|x_1 - x_0\|_2 = \mathcal{W}_2(\delta_{x_0}, \delta_{x_1}).$$

Proof. Obviously we notice that $\Gamma(\delta_{x_0}, \delta_{x_1}) = \delta_{(x_0, x_1)}$, and the cost induced by the coupling (which is actually induced by the transport map that sends $x \mapsto x_1$, $\forall x \in \mathbb{R}^d$), is

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^2 d\delta_{(x_0, x_1)} = \|x_0 - x_1\|^2,$$

which concludes our argument. \square

Moreover, the transport structure of the Wasserstein space induces not only a metric space, but a manifold-like geometry, referred to in the literature as *Otto calculus*. In order to fix ideas, let us take as a toy example the usual Riemannian geometry framework on \mathbb{S}^2 .

Example 2.5. Let us consider a generic $x_0 \in \mathbb{S}^2$. Now, for any $x_1 \in \mathbb{S}^2$ (not equal and not antipodal to x_0), there exists a unique constant speed geodesic ω , i.e. the smaller section of the maximal circumference through x_0 and x_1 . By smoothness, the vector $\omega'(0)$ which tells a moving particle starting in x_0 where to (infinitesimally) move if it wants to follow ω , is well defined and unique.

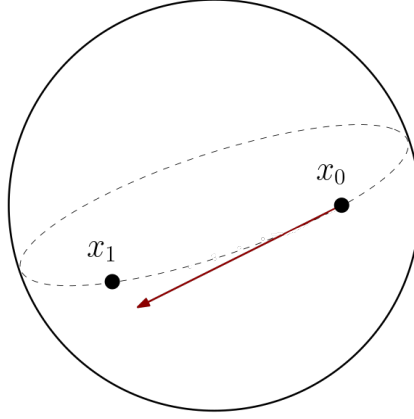


Figure 3: The vector $\omega'(0)$ is in red.

Now, for fixed x_0 , if we take the linear combination of the vectors $\omega'(0)$ as x_1 varies in \mathbb{S}^2 , we obtain a plane. We denote such plane as the tangent space $\mathcal{T}_{x_0}\mathbb{S}^2$. We are going to use the same exact idea to lift the concept of tangent space to our measures space.

Under our assumptions, for fixed $\mu_0 \in \mathcal{P}_2$, we can identify uniquely any $\mu_1 \in \mathcal{P}_2$ with $T_{\mu_0 \rightarrow \mu_1}$. Furthermore, since we showed that $(\mathcal{P}_2, \mathcal{W}_2)$ is a geodesic space, we are motivated to define a *tangent space* like structure in the following way accordingly to our example:

Definition 2.6. Given $\mu_0 \in \mathcal{P}_2$, we define the *tangent space* at μ_0 as

$$\mathcal{T}_{\mu_0}\mathcal{P}_2 := \overline{\{\zeta(T_{\mu_0 \rightarrow \mu_1} - id) : \mu_1 \in \mathcal{P}_2; \zeta > 0\}}^{L^2(\mu_0)}$$

The geometric intuition is straightforward: in this metric space, the role of the tangent vector is played by $\frac{dT_t}{dt} = \frac{d}{dt} [(1-t)id + tT_{\mu_0 \rightarrow \mu_1}] = T_{\mu_0 \rightarrow \mu_1} - id$. Then, by our assumption on finiteness of second moments, every such map satisfies $T_{\mu_0 \rightarrow \mu_1} - id \in L_2(\mu_0)$. We hence take the closure with respect to

$$\|\cdot\|_{L_2(\mu_0)}.$$

Though not obvious, an equivalent (and often useful) definition is the following:

$$\mathcal{T}_{\mu_0}\mathcal{P}_2 = \overline{\{\nabla\phi \mid \phi : \mathbb{R}^d \rightarrow \mathbb{R} \text{ compactly supported and smooth}\}}^{L_2(\mu_0)} \quad (1)$$

which is given in [2]. As a subset of $L_2(\mu_0)$, the tangent space inherits the inner product:

$$\langle f, g \rangle_{\mu_0} = \int_{\mathbb{R}^d} f(x)g(x) d\mu_0(x), \quad f, g \in L_2(\mu_0),$$

and this makes it clear why we chose to work with \mathcal{W}_2 instead of any other $\mathcal{W}_p, p \in [1, +\infty]$. Though not obvious from the definition, $\mathcal{T}_{\mu_0}\mathcal{P}_2$ is a linear space, as shown in [1]. Motivated by this differential structure, we can define the following tools:

Definition 2.7. Given $\mu_0 \in \mathcal{P}_2$ we define a formal *exponential map* as

$$\exp_{\mu_0} : \mathcal{T}_{\mu_0}\mathcal{P}_2 \rightarrow \mathcal{P}_2, \quad \exp_{\mu_0}(T) = (T + id) \# \mu_0.$$

Moreover, we define a *logarithm map* as its left inverse, projecting onto $\mathcal{T}_{\mu_0}\mathcal{P}_2$

In particular if $\mu_0 \ll \lambda$, Brenier's theorem yields that

$$\log_{\mu_0}(\mu_1) = T_{\mu_0 \rightarrow \mu_1} - id.$$

In general, for fixed $\mu_0 \in \mathcal{P}_2$, the exponential map takes a generic transformation as input and interprets it as a *tangent vector* to μ_0 , and by our identification $\mu_1 \longleftrightarrow T_{\mu_0 \rightarrow \mu_1}$ it outputs the unique measure that induces the constant speed geodesic that produces that tangent vector. The logarithm map does the inverse.

2.2 Evolution of measures

Now that we have described the geometry of this space, we are interested in studying the evolution of probability measures when a family of time-dependent vector fields is acting on the underlying space. More precisely let $v_t : \mathbb{R}^d \rightarrow \mathbb{R}^d$, where $t \geq 0$ represents time. The motion of particles in the

underlying space is described by the following Ordinary Differential Equation (ODE):

$$\dot{X}_t = v_t(X_t), \quad t \geq 0, \quad (2)$$

where X_t represents the position of a particle at time t .

The distribution μ_t of X_t solving (2) evolves over time according to the action of the vector field, and in the context of this setup, we will show that this evolution is governed by the *continuity equation*, which expresses the conservation of probability as the underlying particles move through the space.

Preliminarily, we define $\partial_t \mu_t$ as the measure that satisfies $\int_{\mathbb{R}^d} g d(\partial_t \mu_t) = \partial_t \mathbb{E}[g(X_t)]$ for any test function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ (where test means *smooth and compactly supported*).

Theorem 2.8. Let $(v_t)_{t \geq 0} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a time dependent vector field with $v_t \in L_1(\mathbb{R}^d)$ for any $t \geq 0$, and suppose that particles evolve according to (2). Then $X_t \sim \mu_t$, where μ_t satisfies

$$\int_{\mathbb{R}^d} g d(\partial_t \mu_t) = \int_{\mathbb{R}^d} \langle \nabla g, v_t \rangle_2 d\mu_t, \quad (3)$$

for every test function g .

The proof can be found in [1].

Whenever $d\mu_t = f_t d\lambda$, with $f_t \in C^1(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$, then (3) is equivalent to

$$\partial_t f_t + \langle \nabla, (f_t v_t) \rangle_2 = 0 \quad (4)$$

in weak sense. To see this, fix a test function g , and notice that

$$\begin{aligned} \int_{\mathbb{R}^d} g d(\partial_t \mu_t)(x) &= \partial_t \int_{\mathbb{R}^d} g(x) f_t(x) d(x) \text{ by definition} \\ &= \int_{\mathbb{R}^d} g(x) \partial_t f_t(x) d(x). \text{ by dominated convergence} \end{aligned}$$

On the other side,

$$\begin{aligned} \int_{\mathbb{R}^d} \langle \nabla g(x), v_t(x) \rangle_2 d\mu_t(x) &= \int_{\mathbb{R}^d} \langle \nabla g(x), v_t(x) \rangle_2 f_t(x) dx \text{ by definition} \\ &= - \int_{\mathbb{R}^d} g(x) \langle \nabla, (f_t(x) v_t(x)) \rangle_2 dx, \text{ integration by parts} \end{aligned}$$

which implies, by identification, that $\partial_t f_t + \langle \nabla, (f_t v_t) \rangle_2 = 0$.

In this flavor, up to defining $\langle \nabla, (\mu_t v_t) \rangle_2$ as the distribution that satisfies

$$\int_{\mathbb{R}^d} g(x) d(\langle \nabla, \mu_t v_t \rangle_2)(x) = - \int_{\mathbb{R}^d} \langle \nabla g(x), v_t(x) \rangle_2 d\mu_t(x),$$

for any test function g ; even when μ_t does not admit density with respect to the Lebesgue measure, we say that (3) is equivalent to the weak generalized continuity equation (or weak continuity equation for brevity)

$$\partial_t \mu_t + \langle \nabla, (\mu_t v_t) \rangle_2 = 0. \quad (5)$$

Every *nice* curve of probability measures can be interpreted as a fluid moving along a time varying vector field. And by *nice* we mean:

Definition 2.9. A curve $t \mapsto \mu_t \in \mathcal{P}_2$ is said to be *absolutely continuous* if at every time, the metric derivative is finite, i.e., if

$$\text{for all } t, |\dot{\mu}|(t) := \lim_{s \rightarrow t} \frac{\mathcal{W}_2(\mu_s, \mu_t)}{|s - t|} < +\infty.$$

Theorem 2.10. Let $t \mapsto \mu_t$ be an absolutely continuous curve of measures. Then:

1. For any vector field (\tilde{v}) that satisfies the weak continuity equation, we have

$$|\dot{\mu}_t| \leq \|\tilde{v}_t\|_{L_2(\mu_t)}.$$

2. Conversely, there exists a unique choice of vector field $(v_t)_{t \geq 0}$ that satisfies the weak continuity equation and

$$\|v_t\|_{L_2(\mu_t)} \leq |\dot{\mu}_t|.$$

Moreover, $v_t = \nabla \psi_t$ for $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ and

$$v_t = \lim_{\delta \rightarrow 0} \frac{T_{\mu_t \rightarrow \mu_{t+\delta}} - id}{\delta}.$$

For a proof, see [1].

The theorem just states that starting from an (absolutely continuous) curve $t \mapsto \mu_t$ we can identify any infinitesimal displacement as a constant speed geodesic between μ_t and $\mu_{t+\delta}$ and accordingly find uniquely the vector field v_t that induces it.

2.3 First variation of functionals

The goal of this and the following subsections is to understand how functionals $\mathcal{F} : \mathcal{P}_2 \rightarrow \mathbb{R}$, $\mu_t \mapsto \mathcal{F}(\mu_t)$ evolve as the underlying space is subject to a family of time dependent vector fields. This will lead us to a notion of gradient flow in $(\mathcal{P}_2, \mathcal{W}_2)$, which will be our main tool for applications.

This subsection is specifically devoted to an informal description of the tools required to work with functionals. Although its abstract nature may momentarily disrupt the flow of the exposition, we have chosen to include it here rather than in the preliminaries, so as to provide sufficient motivation drawn from the surrounding context.

We want to work our way towards a rigorous definition of *differential operator* associated to \mathcal{F} .

The first obstacle comes from the non-linearity of probability measures (the sum of two probability measures is no longer a measure), so that the direct differentiation does not make sense. Indeed, we typically understand differentiation of a smooth function f on a Euclidean space as:

$$f(x+h) - f(x) = [(\delta f)(x)](h) + o(h), \quad h \rightarrow 0, \quad x, h \in \mathbb{R}^d,$$

where $[(\delta f)(x)]$ is a linear and bounded functional on \mathbb{R}^d , i.e., a matrix.

However, say the functional \mathcal{F} admits an extension over the space of *signed measures* \mathcal{M} (on \mathbb{R}^d). This is a linear space, so that differentiation is more easily understood in the usual way, in that the (extended) functional \mathcal{F} is differentiable if there exists a continuous linear functional $[\delta \mathcal{F}(\mu)]$ on \mathcal{M} such that:

$$\mathcal{F}(\mu + \epsilon \chi) - \mathcal{F}(\mu) = \epsilon [\delta \mathcal{F}(\mu)](\chi) + o(\epsilon), \quad \mu, \chi \in \mathcal{M}, \quad \epsilon \rightarrow 0. \quad (6)$$

It turns out that every such continuous linear functional has a particular form. This result, known as *Kantorovich-Rubinstein duality*, states that the dual space of \mathcal{M} can be identified with the space of bounded continuous functions:

$$\mathcal{M}^* \simeq C_b(\mathbb{R}^d),$$

in the sense that for any linear functional G acting on measures defined on \mathcal{M} , there exists $g \in C_b(\mathbb{R}^d)$ such that:

$$G(\chi) = \int_{\mathbb{R}^d} g d\chi, \quad \forall \chi \in \mathcal{M}.$$

Therefore, with a slight abuse of notation, i.e. identifying the map $[\delta\mathcal{F}(\mu)]$ with its Riesz representative living in $C_b(\mathbb{R}^d)$, we can rewrite (6) as:

$$\mathcal{F}(\mu + \epsilon\chi) - \mathcal{F}(\mu) = \epsilon \int_{\mathbb{R}^d} [\delta\mathcal{F}(\mu)] d\chi + o(\epsilon), \quad \mu, \chi \in \mathcal{M}, \quad \epsilon \rightarrow 0. \quad (7)$$

2.4 Wasserstein gradient flows

Coming back to probability measures, by Theorem 2.10, we know that given a regular enough flow μ_t we can associate to it a unique vector field v_t such that (3) (a.k.a. the weak continuity equation) holds. Furthermore we can write in weak sense

$$\mu_t = \mu_0 + t \partial_t \mu_t + o(t), \quad t \rightarrow 0,$$

and by substituting this into (7), we get

$$\lim_{t \rightarrow 0} \frac{\mathcal{F}(\mu_t) - \mathcal{F}(\mu_0)}{t} = \int_{\mathbb{R}^d} [\delta\mathcal{F}(\mu_0)] d(\partial_t \mu_t),$$

Now, assuming vanishing boundary conditions (which will be the case in functional of interest), we get:

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\mathcal{F}(\mu_t) - \mathcal{F}(\mu_0)}{t} &= \int_{\mathbb{R}^d} [\delta\mathcal{F}(\mu_0)] d(\partial_t \mu_t) \\ &= \int_{\mathbb{R}^d} \langle (\nabla[\delta\mathcal{F}(\mu_0)])(x), v_t(x) \rangle_2 d\mu_t(x) = \langle (\nabla[\delta\mathcal{F}(\mu_0)]), v_t \rangle_{L_2(\mu_t)}, \end{aligned}$$

by the same steps of Theorem 2.8.

Moreover, since $\nabla[\delta\mathcal{F}(\mu)]$ is the gradient of a bounded function, by the characterization in (1), we know that $\nabla[\delta\mathcal{F}(\mu_0)] \in \mathcal{T}_{\mu_0} \mathcal{P}_2$.

We have informally proven the following¹:

¹For a rigorous proof, take a look at [1].

Theorem 2.11. Let $\mathcal{F} : \mathcal{P}_2 \rightarrow \mathbb{R}$ be a functional with bounded first variation. Then, the Wasserstein gradient of \mathcal{F} is the vector field defined by:

$$\nabla_{\mathcal{W}} \mathcal{F}(\mu_0) = \nabla[\delta \mathcal{F}(\mu_0)],$$

where $[\delta \mathcal{F}(\mu_0)] \in C_b(\mathbb{R}^d)$ is tacitly the Riesz representative of the first variation of \mathcal{F} at μ_0 , while ∇ is the usual Euclidean gradient.

Example 2.12. Given a potential $V : \mathbb{R}^d \rightarrow \mathbb{R}$, we can define the *potential energy* as

$$\mathcal{V}(\mu) := \int_{\mathbb{R}^d} V d\mu, \quad \text{for any } \mu \in \mathcal{P}_2.$$

Then,

$$\partial_t \mathcal{V}(\mu_t) = \int_{\mathbb{R}^d} V d(\partial_t \mu_t),$$

and thus we can identify $\delta \mathcal{V}(\mu) = V$, for any $\mu \in \mathcal{P}_2$. Therefore,

$$\nabla_{\mathcal{W}_2} \mathcal{V}(\mu) = \nabla V, \quad \text{for any } \mu \in \mathcal{P}_2.$$

Example 2.13. Given $\mu \in \mathcal{P}_2^{ac}$, with $d\mu = f d\lambda$, we can define the *entropy functional* as

$$\text{Ent}(\mu) := \int_{\mathbb{R}^d} f \log(f) d\lambda.$$

Then,

$$\partial_t \text{Ent}(\mu_t) = \int_{\mathbb{R}^d} (\partial_t f_t \log(f_t) + \partial_t f_t) d\lambda = \int_{\mathbb{R}^d} \partial_t f_t (\log(f_t) + 1) d\lambda,$$

and therefore we can identify $\delta \text{Ent}(\mu) = \log(f) + 1$. Then we can write

$$\nabla_{\mathcal{W}_2} \text{Ent}(\mu) = \nabla \log f.$$

Now, Theorem 2.11 is showing that the first variation of \mathcal{F} naturally leads to a gradient structure in the Wasserstein space, indeed we can expand $\mathcal{F}(\mu_t)$ to the first order:

$$\mathcal{F}(\mu_{t+h}) = \mathcal{F}(\mu_t) + h \langle \nabla_{\mathcal{W}} \mathcal{F}(\mu_t), v_t \rangle_{L_2(\mu_t)} + o(h), \quad h \rightarrow 0.$$

We can now finally define the Wasserstein gradient flow of a functional.

Informally, a gradient flow in the Wasserstein space is a curve of measures $(\mu_t)_{t \geq 0}$ such that the tangent vector to the curve at t equals $-\nabla_{\mathcal{W}_2} \mathcal{F}(\mu_t)$. Recalling that the tangent vector governs the evolution of $(\mu_t)_{t \geq 0}$ via the continuity equation (3), we arrive at the following definition.

Definition 2.14. Let $\mathcal{F} : \mathcal{P}_2 \rightarrow \mathbb{R}$ a functional. Then $(\mu_t)_{t \geq 0}$ is called the *Wasserstein gradient flow* of \mathcal{F} if it solves the following PDE in weak sense:

$$\partial_t \mu_t = \langle \nabla, (\mu_t \nabla_{\mathcal{W}_2} \mathcal{F}(\mu_t)) \rangle_2. \quad (8)$$

Unsurprisingly, this gradient flow yields a principled approach for dynamically, smoothly evolving a probability measure in the Wasserstein space, with the aim of minimizing the objective functional \mathcal{F} :

$$\partial_t \mathcal{F}(\mu_t) = \langle \nabla_{\mathcal{W}_2} \mathcal{F}(\mu_t), v_t \rangle_{L_2(\mu_t)} = -\|\nabla_{\mathcal{W}_2} \mathcal{F}(\mu_t)\|_{L_2(\mu_t)}^2.$$

Clearly, in order to have theoretical and quantitative guarantees about the convergence to a minima we need to address the convexity of the functional of interest. This will be studied in the next section for a specific case of interest.

3 KL Divergence

We start by properly defining our notion of divergence between measures.

Definition 3.1. Given two probability measures $\mu, \pi \ll \lambda$ on \mathbb{R}^d , with $d\mu = g d\lambda$, $d\pi = f d\lambda$, the *Kullback-Leibler divergence* is defined as:

$$\mathcal{D}_{KL}(\mu \| \pi) := \int_{\mathbb{R}^d} \log \left(\frac{g(x)}{f(x)} \right) g(x) dx = \int_{\mathbb{R}^d} \log \left(\frac{g}{f} \right) g d\lambda.$$

The KL divergence quantifies how much the measure μ differs from the target measure π . Let us start by stating a natural but nonetheless important property of our chosen divergence.

Theorem 3.2. Given two probability measures $\mu, \pi \ll \lambda$ on \mathbb{R}^d , with $d\mu = g d\lambda$, $d\pi = f d\lambda$, we have that $\mathcal{D}_{KL}(\mu \| \pi) \geq 0$, and $\mathcal{D}_{KL}(\mu \| \pi) = 0$ if and only if $f = g$ λ -a.e.

The proof follows from Jensen's inequality.

Unfortunately, in general $\mathcal{D}_{KL}(\nu \| \pi) \neq \mathcal{D}_{KL}(\pi \| \nu)$, which already shows that the KL divergence is not a metric.

From now on, we are going to focus on the situation in which $\pi = f d\lambda$

with $f(x) \propto e^{-V(x)}$. This is very common in applications such as Bayesian statistics, complex systems and statistical mechanics.

As we hinted in the introduction, the optimization problem

$$\mu^* = \arg \min_{\mu \in \mathcal{P}_2^{ac}(\lambda)} \mathcal{D}_{KL}(\mu \parallel \pi) \quad (9)$$

where $d\mu = g d\lambda$, $d\pi = f d\lambda = k \tilde{f} d\lambda = k d\tilde{\pi}$ (and we have direct access to \tilde{f}), does not require us to compute the normalizing constant k . Indeed, a direct calculation yields

$$\mathcal{D}_{KL}(\mu \parallel \pi) = \int_{\mathbb{R}^d} \log \left(\frac{g}{k \tilde{f}} \right) g d\lambda = \int_{\mathbb{R}^d} \left[\log \left(\frac{g}{\tilde{f}} \right) - \log(k) \right] g d\lambda.$$

Separating the terms:

$$\mathcal{D}_{KL}(\mu \parallel \pi) = \int_{\mathbb{R}^d} \log \left(\frac{g}{\tilde{f}} \right) g d\lambda - \log(k) \int_{\mathbb{R}^d} g d\lambda,$$

but $\int_{\mathbb{R}^d} g d\lambda = 1$, and thus:

$$\mathcal{D}_{KL}(\mu \parallel \pi) = \mathcal{D}_{KL}(\mu \parallel \tilde{\pi}) - \log(k)$$

which implies

$$\arg \min_{\mu \in \mathcal{P}_2^{ac}(\lambda)} \mathcal{D}_{KL}(\mu \parallel \pi) = \arg \min_{\mu \in \mathcal{P}_2^{ac}(\lambda)} \mathcal{D}_{KL}(\mu \parallel \tilde{\pi}),$$

as wanted to prove.

We will show that whenever V is convex, then $\mathcal{D}_{KL}(\cdot \parallel \pi)$ is *geodesically convex* (in $(\mathcal{P}_2^{ac}(\lambda), \mathcal{W}_2)$): a property that brings important results, as the next subsection will show.

3.1 Geodesic convexity of KL divergence

Let us recall some general knowledge about convex functions from analysis.

Definition 3.3. A function $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is said to be *convex* if

$$V((1-t)x_0 + tx_1) \leq (1-t)V(x_0) + tV(x_1), \quad \forall x_0, x_1 \in \mathbb{R}^d, \forall t \in [0, 1].$$

In short, a function is convex if for any two fixed points in the domain, if we project the segment joining them to the epigraph the curve we obtain lies uniformly below the convex interpolation of the images of the point.

We can strength this definition to requiring a *uniform rate of convexity*:

Definition 3.4. A function $V : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be α -convex (or α -strongly convex) for some $\alpha > 0$ if

$$V((1-t)x_0 + tx_1) \leq (1-t)V(x_0) + tV(x_1) - \frac{\alpha}{2}t(1-t)\|x_1 - x_0\|^2, \quad \forall x_0, x_1 \in \mathbb{R}^d, \forall t \in [0, 1].$$

We have the following useful characterization.

Proposition 3.5. Given a function $V : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$, we have

$$\begin{aligned} V \text{ is } \alpha\text{-convex} &\iff V - \alpha\|\cdot\|^2 \text{ is convex} \\ &\iff D^2V \geq \alpha Id_d. \end{aligned}$$

The proof follows from standard multivariate calculus.

In order to extend the concept of convexity to functionals we have to use the same strategy of the first section: instead of segments, we look at geodesics.

Definition 3.6. Let (\mathcal{X}, d) be a geodesic space. A functional $\mathcal{F} : \mathcal{X} \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be *geodesically convex* (or *displacement convex*) if for every pair of points $x_0, x_1 \in \mathcal{X}$, and every constant-speed geodesic $(x_t)_{t \in [0, 1]}$ joining them, the map $t \mapsto V(x_t)$ is convex. That is,

$$V(x_t) \leq (1-t)V(x_0) + tV(x_1), \quad \forall t \in [0, 1].$$

Definition 3.7. Let (\mathcal{X}, d) be a geodesic space. A functional $\mathcal{F} : \mathcal{X} \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be α -geodesically convex for some $\alpha > 0$ if for every pair of points $x_0, x_1 \in \mathcal{X}$, and every constant-speed geodesic $(x_t)_{t \in [0, 1]}$ joining them, the map $t \mapsto V(x_t)$ satisfies

$$V(x_t) \leq (1-t)V(x_0) + tV(x_1) - \frac{\alpha}{2}t(1-t)\|x_1 - x_0\|^2, \quad \forall t \in [0, 1].$$

For us, the geodesic space upon which the functionals are acting will be $(\mathcal{P}_2, \mathcal{W}_2)$.

We can already show that the entropy functional, introduced in [Example 2.12](#), is geodesically convex.

Proposition 3.8. $\text{Ent}(\mu)$ is geodesically convex.

Proof. Let $\mu_0 = f_0 d\lambda$ and $\mu_1 = f_1 d\lambda$ be two absolutely continuous probability measures in \mathcal{P}_2 . Consider the Wasserstein geodesic $(\mu_t)_{t \in [0,1]}$ defined by

$$\mu_t := ((1-t)\text{id} + tT_{\mu_0 \rightarrow \mu_1})_{\#} \mu_0.$$

Clearly $\mu_t = f_t d\lambda$ is absolutely continuous for each $t \in [0,1]$. Using the change of variables formula from Theorem 1.2, the density f_t satisfies:

$$f_t((1-t)\text{id} + tT_{\mu_0 \rightarrow \mu_1})(x) = \frac{f_0(x)}{\det((1-t)Id_d + t\nabla T_{\mu_0 \rightarrow \mu_1})(x))},$$

and we can define $y = g(x) := ((1-t)\text{id} + tT_{\mu_0 \rightarrow \mu_1})(x)$.

Clearly id is differentiable and bijective, and since $T_{\mu_0 \rightarrow \mu_1}$ is the gradient of a convex function by Brenier's, it is also differentiable almost everywhere, thanks to Alexandrov theorem [3]. On top of that, as a corollary to Brenier's, it can be shown that the inverse of $T_{\mu_0 \rightarrow \mu_1}$ is well defined almost surely, so that it is bijective.

Finally, the weighted sum of two (almost everywhere) differentiable and bijective functions is itself (almost everywhere) differentiable and bijective, so that our change of variables $f_t(g(x)) = \frac{f_0(x)}{\det(\nabla g(x))}$ is well justified.

The entropy at time t is:

$$\begin{aligned} \text{Ent}(\mu_t) &= \int_{\mathbb{R}^d} f_t(y) \log f_t(y) dy \\ &= \int_{\mathbb{R}^d} f_t(g(x)) \log f_t(g(x)) \det(\nabla g(x)) dx \\ &= \int_{\mathbb{R}^d} \frac{f_0(x)}{\det(\nabla g(x))} \log \left(\frac{f_0(x)}{\det(\nabla g(x))} \right) \det(\nabla g(x)) dx \\ &= \int_{\mathbb{R}^d} f_0(x) \log \left(\frac{f_0(x)}{\det((1-t)Id_d + t\nabla T_{\mu_0 \rightarrow \mu_1})(x))} \right) dx, \end{aligned}$$

This can be split as:

$$\text{Ent}(\mu_t) = \int_{\mathbb{R}^d} f_0(x) \log f_0(x) dx - \int_{\mathbb{R}^d} f_0(x) \log \det((1-t)Id_d + t\nabla T_{\mu_0 \rightarrow \mu_1})(x) dx.$$

The first term is constant in t , and the second term is concave in t , since $A \mapsto \log \det A$ is concave on the space of positive definite matrices, and $t \mapsto (1-t)\text{id} + t\nabla T_{\mu_0 \rightarrow \mu_1})(x)$ is affine in t .

Therefore, $t \mapsto \text{Ent}(\mu_t)$ is convex, and we conclude:

$$\text{Ent}(\mu_t) \leq (1-t)\text{Ent}(\mu_0) + t\text{Ent}(\mu_1),$$

proving that the entropy functional is geodesically convex. \square

Additionally, we have an important result regarding the potential energy functional $\mathcal{V}(\mu) = \int_{\mathbb{R}^d} V d\mu$ defined in Example 2.13.

Theorem 3.9. V is α -convex if and only if \mathcal{V} is α -geodesically convex.

Proof. (\Rightarrow) We claim that if $\mu_0, \mu_1 \in \mathcal{P}_2$, and $\bar{\gamma} \in \Gamma(\mu_0, \mu_1)$ is the optimal coupling between the two, then $t \mapsto \mu_t$ is convex, where μ_t is the constant speed geodesic.

$$\begin{aligned} \int_{\mathbb{R}^d} V d\mu_t &= \int_{\mathbb{R}^d \times \mathbb{R}^d} V((1-t)x_0 + tx_1) d\bar{\gamma}(x_0, x_1) \\ &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \left[(1-t)V(x_0) + tV(x_1) - \frac{\alpha}{2}t(1-t)\|x_1 - x_0\|^2 \right] d\bar{\gamma}(x_0, x_1) \\ &= (1-t) \int_{\mathbb{R}^d} V d\mu_0 + t \int_{\mathbb{R}^d} V d\mu_1 - \frac{\alpha}{2}t(1-t)\mathcal{W}_2^2(\mu_0, \mu_1). \end{aligned}$$

(\Leftarrow) For the other direction, it is sufficient to apply the α -geodesically convexity of \mathcal{V} to δ_{x_0} and δ_{x_1} to get the α -convexity of V . \square

We can finally show the main result for this subsection.

Theorem 3.10. If $\pi = f d\lambda$, with $f = ke^{-V}$ for some normalizing constant k , and $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is convex, then $\mathcal{D}_{KL}(\cdot \| \pi)$ is geodesically convex. Accordingly, if V is α -convex, then $\mathcal{D}_{KL}(\cdot \| \pi)$ is α -geodesically convex.

Proof. Let $\pi = f d\lambda$ with $f = ke^{-V}$ for some normalizing constant $k > 0$, and assume $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is convex. For any absolutely continuous probability measure $\mu \ll \lambda$ with density $g = \frac{d\mu}{d\lambda}$, we can write the Kullback–Leibler divergence as:

$$\mathcal{D}_{KL}(\mu \| \pi) = \int_{\mathbb{R}^d} g \log \left(\frac{g}{f} \right) d\lambda = \int_{\mathbb{R}^d} g \log g d\lambda + \int_{\mathbb{R}^d} Vg d\lambda + \log k.$$

That is,

$$\mathcal{D}_{KL}(\mu \parallel \pi) = \text{Ent}(\mu) + \mathcal{V}(\mu) + \log k,$$

where $\text{Ent}(\mu)$ is the (negative) entropy relative to the Lebesgue measure; and $\mathcal{V}(\mu) := \int V d\mu$ is the potential energy. Now observe the following:

1. The entropy functional $\mu \mapsto \text{Ent}(\mu)$ is geodesically convex by Proposition 3.8.
2. From Theorem 3.9, if V is convex, then $\mathcal{V}(\mu)$ is also geodesically convex.

Since the sum of two geodesically convex functionals remains geodesically convex, we conclude that $\mathcal{D}_{KL}(\cdot \parallel \pi)$ is geodesically convex.

Moreover, if V is α -convex, then \mathcal{V} is α -geodesically convex (again by Theorem 3.9), and therefore $\mathcal{D}_{KL}(\cdot \parallel \pi)$ is α -geodesically convex as well. \square

The main point of this section is that if the family \mathcal{Q} is geodesically convex (i.e. whenever $\mu_0, \mu_1 \in \mathcal{Q}$, then the constant speed geodesic between them is also in \mathcal{Q}), and \mathcal{V} is α -geodesically convex then the solution to (9) (in which we optimize restricted to $\mathcal{Q} \subseteq \mathcal{P}_2^{ac}(\lambda)$) is unique.

To see this, suppose for the sake of contradiction that there exist two distinct minimizers $\mu_0^*, \mu_1^* \in \mathcal{Q}$ such that $\mathcal{V}(\mu_0^*) = \mathcal{V}(\mu_1^*) = \inf_{\mu \in \mathcal{Q}} \mathcal{V}(\mu)$. Since \mathcal{Q} is geodesically convex, the constant-speed geodesic $(\mu_t^*)_{t \in [0,1]}$ connecting μ_0^* and μ_1^* is contained in \mathcal{Q} .

By α -geodesic convexity of \mathcal{V} , we have for all $t \in (0, 1)$:

$$\mathcal{V}(\mu_t^*) \leq (1-t)\mathcal{V}(\mu_0^*) + t\mathcal{V}(\mu_1^*) - \frac{\alpha}{2}t(1-t)W_2^2(\mu_0^*, \mu_1^*).$$

Since both μ_0^* and μ_1^* are minimizers, the right-hand side equals $\inf_{\mu \in \mathcal{Q}} \mathcal{V}(\mu)$ minus a strictly positive term (unless $\mu_0^* = \mu_1^*$). Therefore:

$$\mathcal{V}(\mu_t^*) < \inf_{\mu \in \mathcal{Q}} \mathcal{V}(\mu),$$

which contradicts the minimality of μ_0^* and μ_1^* .

We conclude that the minimizer is unique.

Recall that by what we computed in the previous section (examples [2.12](#), [2.13](#)), we can write the Wasserstein gradient flow as

$$\nabla_{\mathcal{W}_2} \mathcal{D}_{KL}(\cdot \| \pi) = \nabla V + \nabla \log f. \quad (10)$$

On top of uniqueness of the optimizer in [\(9\)](#), we have the Poljak–Łojasiewicz inequality that gives us a rate of convergence to the optimal solution when the optimization is addressed via Wasserstein gradient flows.

Theorem 3.11. Let $\pi \propto \exp(-V)$ be a density on \mathbb{R}^d , where V is α -convex. Let $\mathcal{Q} \subseteq \mathcal{P}_2^{ac}(\lambda)(\mathbb{R}^d)$ be geodesically convex. Then, the Wasserstein gradient flow $(\mu_t)_{t \geq 0}$ of $\text{KL}(\cdot \| \pi)$ constrained to lie in \mathcal{Q} satisfies

$$\mathcal{D}_{KL}(\mu_t \| \pi) - \mathcal{D}_{KL}(\mu^* \| \pi) \leq e^{-2\alpha t} [\mathcal{D}_{KL}(\mu_0 \| \pi) - \mathcal{D}_{KL}(\mu^* \| \pi)].$$

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