EPFL

Gradient Flows in Wasserstein Spaces

Variational Inference and Sampling

Student: L.Raffo

PostDoc. L.V.Santoro

motivation

consider the Euclidean space $\mathbb{R}^d$.

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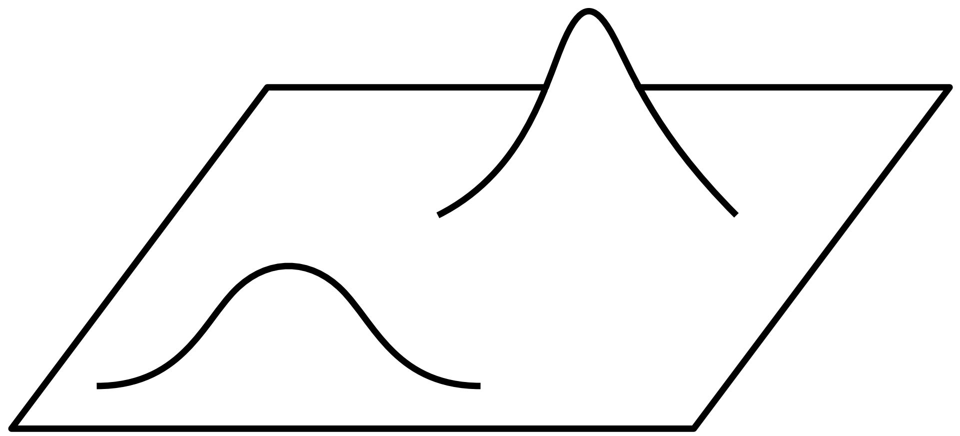
look at the family of probability measures $\mathcal{P}$ on $\mathbb{R}^d$.



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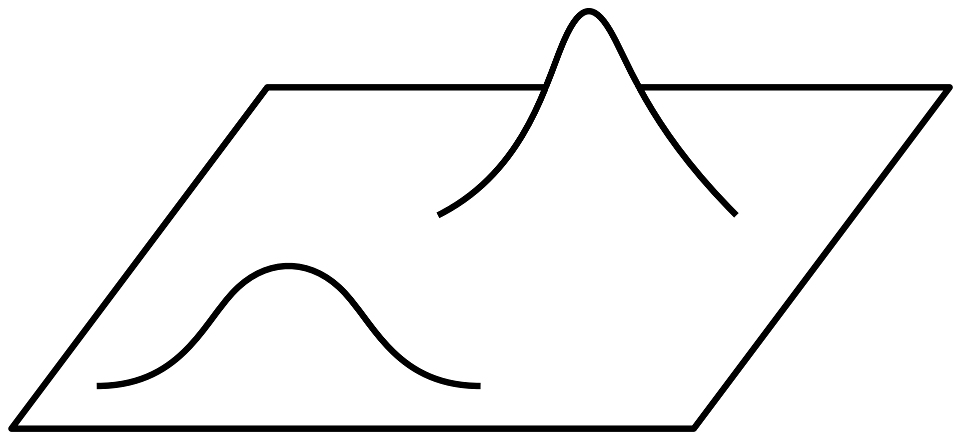

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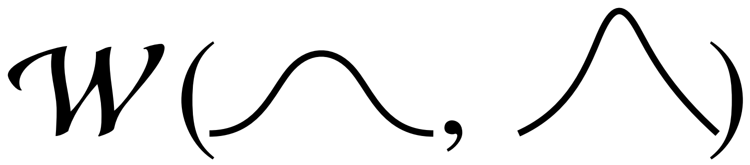

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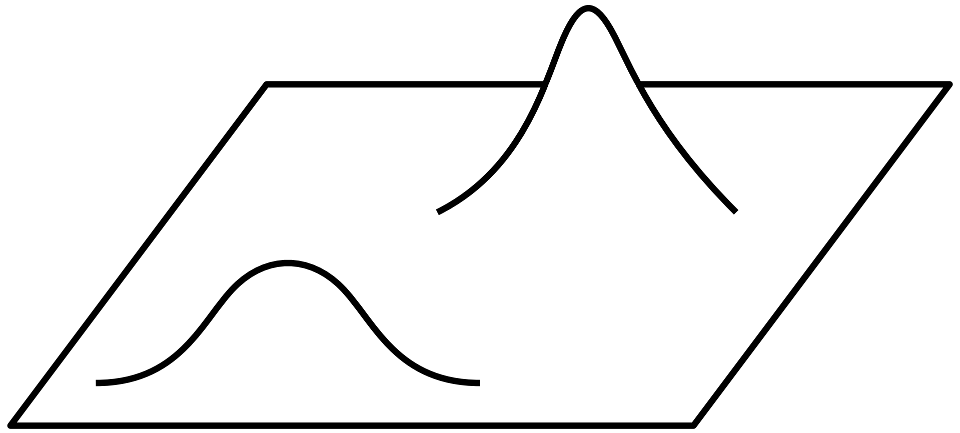

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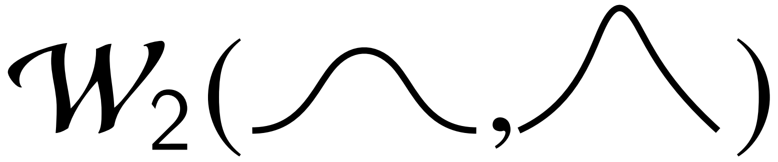

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5. sampling. Langevin diffusion as a gradient flow.

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moreover, we can choose $I_2 = [0, 1]$.

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if $(\mathcal{S}, d)$ is a geodesic space, for any $x_0, x_1 \in \mathcal{S}$ we can define the *midpoint* as $\omega_{\frac{1}{2}}$, where $\omega_{\frac{1}{2}} = \omega(\frac{1}{2})$ and ω is a constant speed geodesic between x_0 and x_1.

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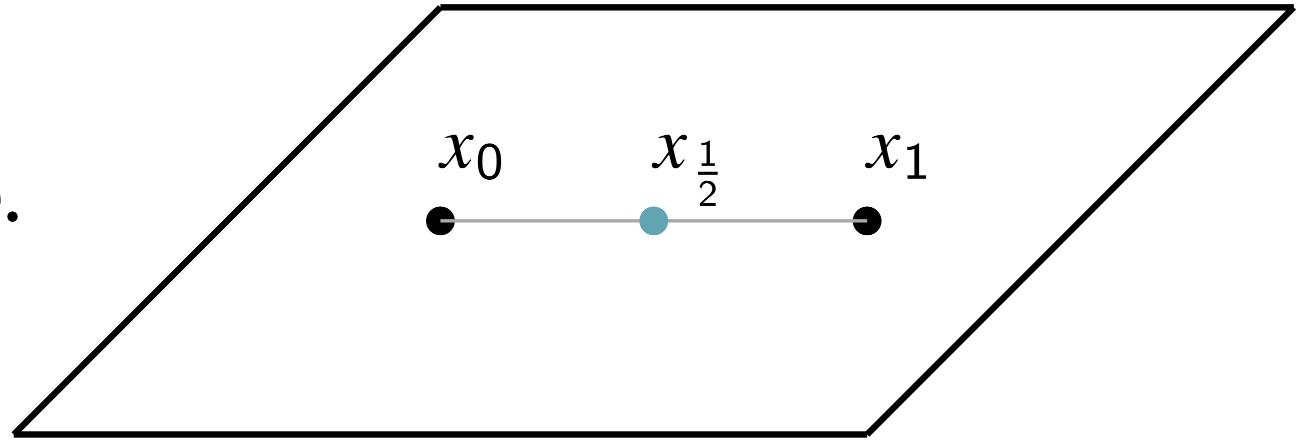

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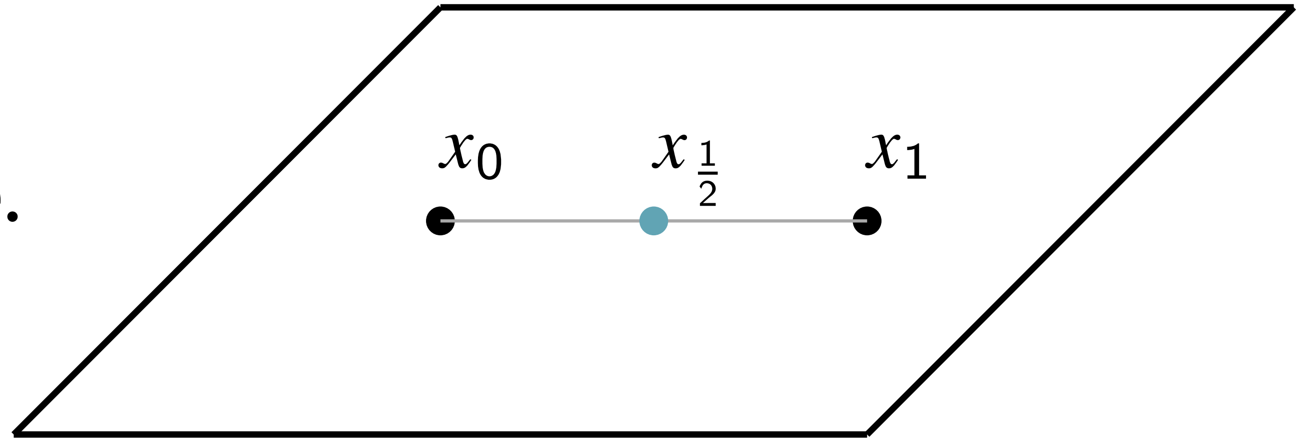

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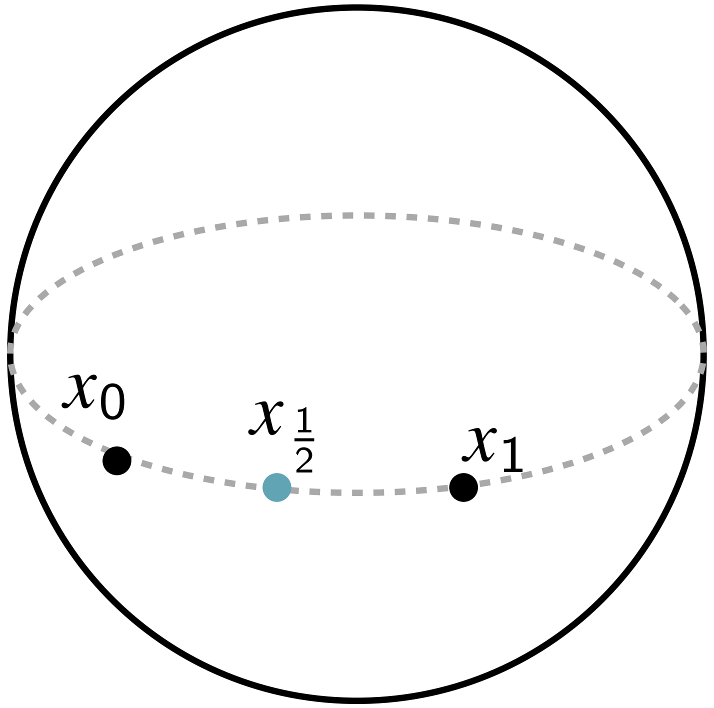

$\|\cdot\|_2$ is induced by the inner product on tangent spaces of $\mathbb{S}^2$ induced by the Euclidean one.