Задача 1) Дадени са матриците:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ -1 & -1 & 0 \end{pmatrix}$$

а) $\det A = ?$

б) матрицата A^{-1}, ако съществува

в) решението на матричното уравнение $AX = B^T$

г) рангът на матрицата B

Задача 2) Докажете, че множеството

$P = \{(x, y, z) \mid x + 2y + z = 0,\ x - z = 0\}$ е векторно пространство. Определете размерността му.

Задача 3) Относно декартова координатна система са дадени точките $A(4, 2)$, $B(2, -2)$, $C(0, 2)$.

Да се намерят:

а) уравнението на правата AB

б) уравнението на височината h, минаваща през точка C и перпендикулярна на AB.

в) уравнението на медианата през върха C на $\triangle ABC$.

Задача 4) Нека $OABC$ е тетраедър, където $\overrightarrow{OA} = \vec{a} \times \vec{b}$, $\overrightarrow{OB} = \vec{a} \times (\vec{a} \times \vec{b})$, $\overrightarrow{OC} = \vec{b} \times (\vec{a} \times \vec{b})$

$\vec{a}(2, -1, 2)$, $\vec{b}(0, 1, -1)$

а) Намерете височината на тетраедъра, спусната от върха C.

Задача 1 а)

$$\det A = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix} \begin{matrix} 1 & -1 \\ -1 & 2 \\ 2 & 2 \end{matrix} = 6-2-(2+3) = -1$$

б) $\det A \neq 0 \Rightarrow$ матрицата A е неособена $\Rightarrow \exists A^{-1}$

$$A_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 2.3 - 1.2 = 4$$

$$A_{12} = (-1)^{1+2} \cdot \begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix} = -[3.(-1) - 1.2] = 5$$

$$A_{13} = (-1)^{1+3} \cdot \begin{vmatrix} -1 & 2 \\ 2 & 2 \end{vmatrix} = (-1).2 - 4 = -6$$

$$A_{21} = (-1)^{2+1} \cdot \begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} = -[(-1).3 - 0.2] = 3$$

$$A_{22} = (-1)^{2+2} \cdot \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 3$$

$$A_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} = -[1.2 - (2.(-1))] = -4$$

$$A_{31} = (-1)^{3+1} \cdot \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = (-1).1 = -1$$

$$A_{32} = (-1)^{3+2} \cdot \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = -1$$

$$A_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 1.2 - ((-1).(-1)) = 1$$

$$A^{-1} = \frac{1}{\det A} \cdot \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T = \frac{1}{-1} \begin{pmatrix} 4 & 5 & -6 \\ 3 & 3 & -4 \\ -1 & -1 & 1 \end{pmatrix}^T = -\begin{pmatrix} 4 & 3 & -1 \\ 5 & 3 & -1 \\ -6 & -4 & 1 \end{pmatrix}^T$$

$$\Rightarrow A^{-1} = \begin{pmatrix} -4 & -3 & 1 \\ -5 & -3 & 1 \\ 6 & 4 & -1 \end{pmatrix}$$

в) $AX = B^T \Rightarrow \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} X = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ -1 & -1 & 0 \end{pmatrix}^T$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} X = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & -1 \\ 0 & -2 & 0 \end{pmatrix}$$

$AX = B \Rightarrow X = A^{-1} \cdot B$

$$X = \underset{(3\times3)}{\begin{pmatrix} -4 & -3 & 1 \\ -5 & -3 & 1 \\ 6 & 4 & -1 \end{pmatrix}} \underset{(3\times3)}{\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & -1 \\ 0 & -2 & 0 \end{pmatrix}} = \underset{(3\times3)}{\begin{pmatrix} -1 & -5 & 7 \\ -2 & -5 & 8 \\ 2 & 6 & -10 \end{pmatrix}}$$

$(-4).1 + (-3).(-1) + 1.0 = -1$
$(-4).0 + (-3).1 + 1.(-2) = -5$
$(-4).(-1) + (-3).(-1) + 1.0 = 7$

$\begin{cases} -5.1 + (-3).(-1) + 1.0 = -2 \\ -5.0 + (-3).1 + 1.(-2) = -5 \\ (-5).(-1) + (-3).(-1) + 1.0 = 8 \end{cases}$

$\begin{cases} 6.1 + 4.(-1) + (-1).0 = 2 \\ 6.0 + 4.1 + (-1).(-2) = 6 \\ 6.(-1) + 4.(-1) + (-1).0 = -10 \end{cases}$

г) $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ -1 & -1 & 0 \end{pmatrix} \cdot 1 \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix} \cdot 2 \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & -4 \end{pmatrix}$

$\Rightarrow rg B = 3$

Задача 2

$\begin{vmatrix} x + 2y + z = 0 \\ x - z = 0 \end{vmatrix} \Rightarrow \begin{vmatrix} z + 2y + z = 0 \\ x = z \end{vmatrix} \Rightarrow y = -z$

$(x, y, z) = (z, -z, z)$

Нека $\vec{a}(z_1, -z_1, z_1)$ и $\vec{b}(z_2, -z_2, z_2)$

1) $\vec{a} + \vec{b} = (z_1, -z_1, z_1) + (z_2, -z_2, z_2) = (z_1 + z_2, -(z_1 + z_2), z_1 + z_2)$
$\Rightarrow \vec{a} + \vec{b} \in P$

2) $\lambda \in R$, $\vec{a}(z_1, -z_1, z_1)$

$\lambda\vec{a} = \lambda(z_1, -z_1, z_1) = (\lambda z_1, -\lambda z_1, \lambda z_1) \in P$

От 1) и 2) ⇒, че P е векторно пространство

$(x, y, z) = (z, -z, z) = z(1, -1, 1)$

$\{\vec{e_1}(1, -1, 1)\}$

$\lambda_1 \cdot e_1 = \vec{0} \Rightarrow \lambda_1(1, -1, 1) = (0, 0, 0)$

$\left| \begin{array}{l} \lambda_1 = 0 \\ -\lambda_1 = 0 \\ \lambda_1 = 0 \end{array} \right. \Rightarrow \lambda_1 = 0 \Rightarrow \{\vec{e_1}\}$ образува линейно независима система ⇒ база ⇒ $\dim P = 1$

Задача 3

а) $AB: \begin{cases} \text{з т. } A(4,2) \\ \text{з т. } B(2,-2) \end{cases}$ $\quad \vec{AB} = B - A = (2,-2) - (4,2) = (-2,-4)$

$AB: \dfrac{x-4}{-2} = \dfrac{y-2}{-4}$

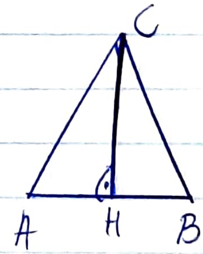
$AB: (-4)(x-4) = (-2).(y-2)$

$AB: -4x + 16 = -2y + 4$

$AB: -4x + 2y + 12 = 0$

б) $h: \begin{cases} \text{з т. } C(0,2) \\ \perp AB: -4x + 2y + 12 = 0 \end{cases} \Rightarrow \vec{N_{AB}}(-4, 2)$

$\vec{N_h}(-2, -4)$

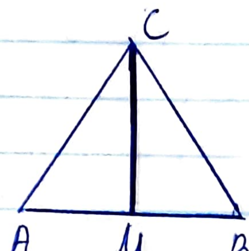


$h: Ax + By + C = 0$

$-2x - 4y + C = 0$

$-2.0 - 4.2 + C = 0 \Rightarrow \boxed{C = 8} \Rightarrow h: -2x - 4y + 8 = 0$

в) $CM: \begin{cases} \text{з т. } C(0,2) \\ \text{з т. } M \end{cases}$



Нека т.M - среда на AB. т.$M(x,y)$

$(x, y) = \left(\dfrac{4+2}{2}, \dfrac{2+(-2)}{2}\right) = (3, 0) \Rightarrow$ т. $M(3,0)$

$\vec{CM} = M - C = (3,0) - (0,2) = (3,-2)$

$CM: \dfrac{x-0}{3} = \dfrac{y-2}{-2} \Rightarrow CM: \dfrac{x}{3} = \dfrac{y-2}{-2}$

Задача 4 а) $\vec{a}(2,-1,2)$

$\vec{b}(0,1,-1)$

$$\vec{OA}=\vec{a}\times\vec{b}=\left(\begin{vmatrix}-1 & 2\\ 1 & -1\end{vmatrix};-\begin{vmatrix}2 & 2\\ 0 & -1\end{vmatrix};\begin{vmatrix}2 & -1\\ 0 & 1\end{vmatrix}\right)=(-1,2,2)$$

$\vec{a}(2,-1,2)$

$\vec{a}\times\vec{b}(-1,2,2)$

$$\vec{OB}=\vec{a}\times(\vec{a}\times\vec{b})=\left(\begin{vmatrix}-1 & 2\\ 2 & 2\end{vmatrix};-\begin{vmatrix}2 & 2\\ -1 & 2\end{vmatrix};\begin{vmatrix}2 & -1\\ -1 & 2\end{vmatrix}\right)=$$

$$=(-6,-6,3)\parallel(-2,-2,1)$$

$\vec{b}(0,1,-1)$

$\vec{a}\times\vec{b}(-1,2,2)$

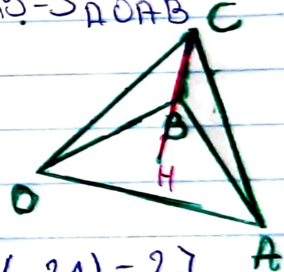
$$\vec{OC}=\vec{b}\times(\vec{a}\times\vec{b})=\left(\begin{vmatrix}1 & -1\\ 2 & 2\end{vmatrix};-\begin{vmatrix}0 & -1\\ -1 & 2\end{vmatrix};\begin{vmatrix}0 & 1\\ -1 & 2\end{vmatrix}\right)=$$

$$=(4,1,1)$$

$V_{OABC}=\frac{1}{6}\left|\vec{OA}\,\vec{OB}\,\vec{OC}\right|$

$V_{OABC}=\frac{B.h}{3}$, където $B=S_{\Delta OAB}$

$S_{\Delta OAB}=\frac{1}{2}\left|\vec{OA}\times\vec{OB}\right|$



$$\vec{OA}\,\vec{OB}\,\vec{OC}=\begin{vmatrix}-1 & 2 & 2\\ -2 & -2 & 1\\ 4 & 1 & 1\end{vmatrix}\begin{matrix}-1 & 2\\ -2 & -2\\ 4 & 1\end{matrix}=2+8-4-(-16-1-4)=6-(-21)=27$$

$$\Rightarrow V_{OABC}=\frac{1}{6}\cdot|27|=\frac{1}{\cancel{6}}\cdot\cancel{27}=\frac{9}{2}$$

$$\vec{OA}\times\vec{OB}=\left(\begin{vmatrix}2 & 2\\ -2 & 1\end{vmatrix};-\begin{vmatrix}-1 & 2\\ -2 & 1\end{vmatrix};\begin{vmatrix}-1 & 2\\ -2 & -2\end{vmatrix}\right)=(6,-3,6)\parallel(2,-1,2)$$

$\vec{OA}(-1,2,2)$

$\vec{OB}(-2,-2,1)$

$$\left|\vec{OA}\times\vec{OB}\right|=\sqrt{2^2+(-1)^2+2^2}=\sqrt{9}=3\Rightarrow B=\frac{1}{2}.3=\frac{3}{2}$$

$$\Rightarrow V=\frac{B.h}{3}\Rightarrow\frac{9}{2}=\frac{\frac{3}{2}\cdot h}{3}\Rightarrow 27=\cancel{2}.\frac{3}{\cancel{2}}.h\Rightarrow\boxed{h=9}$$