1 Ex7 : Task2:

Find a Fourier series for $f(x) = x^2$ and use it to derive another series for π^2 and a series for π^4.

1.1 Fourier Series for $f(x) = x^2$

$$f(-x) = f(x) \Rightarrow \text{even} \Rightarrow b_n = 0 \forall n$$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx && (1)\\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx dx && (2)\\
&= \frac{1}{\pi} \left[\frac{n^2x^2 - 2\sin nx + 2nx \cos nx}{n^3} \right]_{-\pi}^{\pi} && (3)\\
&= \frac{1}{\pi} \left[\frac{2n\pi \cos n\pi}{n^3} + \frac{2n\pi \cos n\pi}{n^3} \right] && (4)\\
&= \frac{1}{\pi n^3} 4n\pi \qquad \underbrace{\cos n\pi}_{+1\text{for n even};-1\text{for n odd}} && (5)\\
&= (-1)^n \frac{4}{n^2} && (6)
\end{aligned}$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx && (7)\\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx && (8)\\
&= \frac{1}{\pi} \left[\frac{1}{3} x^3 \right]_{-\pi}^{\pi} && (9)\\
&= \frac{2\pi^2}{3} && (10)
\end{aligned}$$

$$\Rightarrow f(x) = \frac{2\pi^2}{3} + 4 \sum_{n \in N} (-1)^n \frac{\cos nx}{n^2} \qquad (11)$$

1.2 Series for π^2:

$$\begin{aligned}
f(0) &= 0 && (12)\\
&= \frac{2\pi^2}{3} + 4 \sum_{n \in N} \frac{(-1)^n}{n^2} && (13)
\end{aligned}$$

$$\Rightarrow \frac{2\pi^2}{3} = -4 \sum_{n\in N} \frac{(-1)^n}{n^2} \tag{14}$$

$$\Rightarrow \pi^2 = 6 \sum_{n\in N} \frac{(-1)^{(n+1)}}{n^2} \tag{15}$$

$$\Rightarrow \pi^2 = \frac{6}{1} - \frac{6}{4} + \frac{6}{9} - \frac{6}{16} \pm \dots \tag{16}$$

1.3 Series for π^4:

using Parseval relation

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{17}$$

$$\begin{aligned}
\Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx &= \frac{1}{\pi}\int_{-\pi}^{\pi} (x^2)^2 dx && (18)\\
&= \frac{1}{\pi}\left[\frac{x^5}{5}\right]_{-\pi}^{\pi} && (19)\\
&= \frac{2\pi^4}{5} && (20)\\
&\overbrace{=}^{(17)} \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + \underbrace{b_n^2}_{=0}) && (21)\\
&= \frac{1}{2}\left(\frac{2\pi^2}{3}\right)^2 + \sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{n^2}\right)^2 && (22)\\
&= \frac{2\pi^4}{9} + \sum_{n=1}^{\infty}\frac{16}{n^4} && (23)
\end{aligned}$$

$$\Rightarrow \frac{2\pi^4}{5} = \frac{2\pi^4}{9} + \sum_{n=1}^{\infty}\frac{16}{n^4} \tag{24}$$

$$\Rightarrow \frac{8\pi^4}{45} = \sum_{n=1}^{\infty}\frac{16}{n^4} \tag{25}$$

$$\Rightarrow \pi^4 = 90\sum_{n=1}^{\infty}\frac{1}{n^4} \tag{26}$$

$$\Rightarrow \pi^4 = 90\sum_{n=1}^{\infty}\frac{1}{n^4} \tag{27}$$