Analysis of a mission with refueling

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1 Introduction

In this document, we aim to derive performance metrics in terms of payload mass, mission time, and fuel consumption for a mission with refueling.

In Section 2, we start by considering a simple mission without refueling. The initial analysis is based on the lecture notes of the Space Propulsion lectures from the MIT. In Section 3 we generalize the expressions derived in Section 2 to include refueling. Then, in Section 4 we add penalties to the expressions derived in Section 3 to account for the time and Δv spent to achieve refueling.

2 Mission without refueling

The total wet mass of the spacecraft is:

$$m_0 = m_{ps} + m_p + m_s + m_{pay}\,, \tag{1}$$

where m_{ps} is the mass of the power and propulsion systems, m_p is the mass of the propellant, m_s is the mass of the structure, and m_{pay} is the payload mass.

For a given value of Δv required by the mission, and for a fixed exhaust velocity c, the mass of the propellant is given by the rocket equation as:

$$m_p = m_0\left(1 - e^{-\Delta v/c}\right). \tag{2}$$

We define the specific mass per unit power α for the propulsion system as:

$$\alpha = \frac{m_{ps}}{P}\,. \tag{3}$$

For chemical propulsion: $\alpha_{chem} \in [0.002, 0.2]$ kg/kW.
For electric propulsion: $\alpha_{elec} \in [20, 200]$ kg/kW.

We define the efficiency η of the propulsion system as:

$$\eta = \frac{\text{Jet power}}{\text{Total source power}} = \frac{\frac{1}{2}\dot{m}c^2}{P} \tag{4}$$

where $\dot{m}$ is the mass flow rate of the exhaust and c is the exhaust velocity.

We assume a constant mass flow rate $\dot{m} = m_p/t_m$, where t_m is the mission time. Therefore, we can rewrite the expression of m_{ps} as:

$$m_{ps} = \alpha P = \alpha \frac{\dot{m}c^2}{2\eta} = \alpha \frac{m_p}{t_m} \frac{c^2}{2\eta} = \frac{\alpha c^2}{2\eta m} m_p = \frac{\alpha c^2}{2\eta m} m_0 \left(1 - e^{-\Delta v/c}\right) . \tag{5}$$

We define the Stuhlinger velocity v_{ch} as:

$$v_{ch} = \sqrt{\frac{2\eta t_m}{\alpha}} . \tag{6}$$

The meaning of the Stuhlinger velocity v_{ch} is that, if the propulsion/power mass were to be accelerated by converting all of the electrical energy generated during time t_m, it would then reach the velocity v_{ch}. This is therefore the upper limit of Δv that can be achieved with an electric propulsion system. The effects of efficiency, propulsion/power specific mass and mission time are all lumped into the parameter v_{ch}.

Therefore, the mass of the propulsion and power systems m_{ps} can be written as:

$$m_{ps} = \left(\frac{c}{v_{ch}}\right)^2 m_0 \left[1 - e^{-\Delta v/c}\right] . \tag{7}$$

The balance of Eq. 1 can be written as:

$$\begin{aligned} H = \frac{m_{pay} + m_s}{m_0} &= \frac{m_0 - m_p - m_{ps}}{m_0} \\ &= 1 - \frac{m_p}{m_0} - \frac{m_{ps}}{m_0} \\ &= 1 - 1 + e^{-\Delta v/c} - \left(\frac{c}{v_{ch}}\right)^2 \left[1 - e^{-\Delta v/c}\right] \\ &= e^{-\Delta v/c} - \left(\frac{c}{v_{ch}}\right)^2 \left[1 - e^{-\Delta v/c}\right] \end{aligned}$$

We normalize the velocities Δv and c by v_{ch} to obtain the final expression:

$$H = \exp\left(\frac{-\Delta v/v_{ch}}{c/v_{ch}}\right) - \left(\frac{c}{v_{ch}}\right)^2 \left[1 - \exp\left(\frac{-\Delta v/v_{ch}}{c/v_{ch}}\right)\right] \tag{8}$$

We define the normalized quantities:

$$x = \frac{c}{v_{ch}} \text{ and } y = \frac{\Delta v}{v_{ch}}$$

The objective function can be written as:

$$H = e^{-y/x} - x^2 \left(1 - e^{-y/x}\right) \tag{9}$$

The value of H as a function of c/v_{ch} for different values of $\Delta v/v_{ch}$ is shown in Fig. 1. The figure shows that the maximum Δv for which a positive payload can be carried is of the order of $0.8\, v_{ch}$.

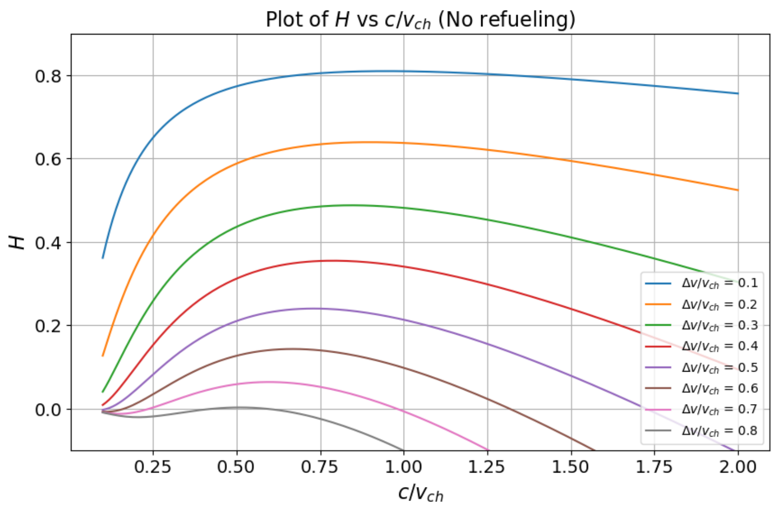

Figure 1: Normalized payload as a function of normalized specific impulse.

The maximum payload mass fraction H and its corresponding propellant mass fraction m_p/m_0, and power/propulsion system mass fraction m_{ps}/m_0 are shown in Fig. 2.

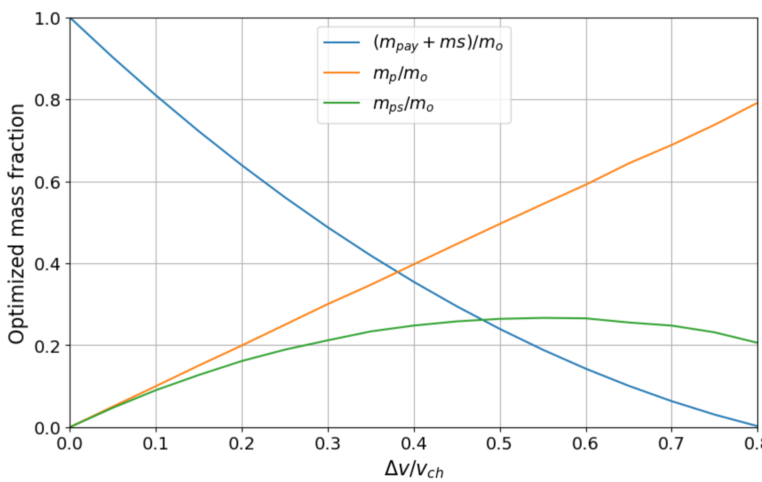

Figure 2: Optimized mass fractions as a function of $\Delta v/v_{ch}$.

3 Mission with refueling - excluding penalties

We consider a mission going from an initial orbit r_0 to a final orbit r_n requiring a given Δv. We also consider the availability of $n-1$ refueling stations as shown in Fig. 3. Between 2 refuelings, the spacecraft has to achieve $\beta_i \Delta v$, such as:

$$\sum_1^n \beta_i = 1\,, \tag{10}$$

which can also be written as:

$$\sum_1^n \beta_i \Delta v = \Delta v\,. \tag{11}$$

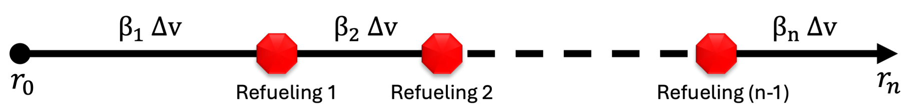

Figure 3: Diagram of the mission with refueling stations.

Mass consideration:

Between two refuelings, the total mass is:

$$m_{0,i} = m_{ps} + m_{p,i} + m_s + m_{pay,i}\,, \tag{12}$$

and the mass of the propellant is:

$$m_{p,i} = m_{0,i}\,(1 - e^{-\beta_i \Delta v/c})\,. \tag{13}$$

Therefore, the total mass of the propellant that should be used to accomplish the mission is:

$$m_p = \sum_1^n m_{p,i} = \sum_1^n m_{0,i}\,(1 - e^{-\beta_i \Delta v/c})\,. \tag{14}$$

Under the assumption of a constant mass flow rate $\dot{m} = m_p/t_m$, the mass of the power and propulsion system is:

$$\begin{aligned} m_{ps} &= \alpha P = \alpha \frac{\dot{m} c^2}{2\eta} \\ &= \frac{\alpha c^2}{2\eta}\frac{m_p}{t_m} \\ &= \frac{\alpha c^2}{2\eta}\frac{\sum_1^n m_{0,i}\,[1 - \exp(-\beta_i \Delta v/c)]}{t_m} \end{aligned} \tag{15}$$

Another simplifying assumption that could safely be made is that the total mass of the spacecraft after each refueling $m_{0,i}$ is constant, i.e.:

$$\begin{aligned} \forall i,j;\ m_{0,i} &= m_{0,j} \\ \forall i,j;\ m_{ps}+m_{p,i}+m_s+m_{pay,i} &= m_{ps}+m_{p,j}+m_s+m_{pay,j} \end{aligned} \tag{16}$$

This assumption means that the mass of the propellant and the payload at the beginning of 2 different sections of the mission i and j, if different, is not significant compared to the total mass of the spacecraft. This will allow us to rewrite equations 13, 14 and 15 as:

$$m_{p,i} = m_0\,(1-e^{-\beta_i \Delta v/c})\,, \tag{17}$$

$$m_p = \sum_1^n m_{p,i} = m_0 \sum_1^n (1-e^{-\beta_i \Delta v/c})\,, \tag{18}$$

and

$$\begin{aligned} m_{ps} &= \frac{\alpha c^2}{2\eta t_m} m_0 \sum_1^n \left[1-\exp(-\beta_i \Delta v/c)\right] \\ &= \left(\frac{c}{v_{vh}}\right)^2 m_0 \sum_1^n \left[1-\exp(-\beta_i \Delta v/c)\right] \end{aligned} \tag{19}$$

Time consideration:

In this part, we want to compare the mission time of the mission with refueling t_m with the mission time of the mission without refueling t_m^*. For the rest of the document, the superscript '$*$' will stand for the quantities corresponding to the mission without refueling.

We assume that the mass flow rate of the exhaust is the same for the mission without refueling and the mission with refueling. Thus, we can write:

$$\dot{m} = \left(\frac{m_p^*}{t_m^*}\right)_{\text{No refuel}} = \left(\frac{m_p}{t_m}\right)_{\text{refuel}}\,. \tag{20}$$

The time of the mission with refueling t_m is

$$\begin{aligned} t_m = \frac{m_p}{m_p^*}\,t_m^* &= \frac{m_0 \sum_1^n (1-e^{-\beta_i \Delta v/c})}{m_0\,(1-e^{-\Delta v/c})}\,t_m^* \\ &= \frac{\sum_1^n (1-e^{-\beta_i \Delta v/c})}{1-e^{-\Delta v/c}}\,t_m^*\,. \end{aligned} \tag{21}$$

Finally, we can write t_m as:

$$t_m = \tau\, t_m^*\,, \tag{22}$$

where
$$\tau = \frac{\sum_1^n (1 - e^{-\beta_i \Delta v/c})}{1 - e^{-\Delta v/c}} \,. \tag{23}$$

The Stuhlinger velocity v_{ch} can be written as:

$$\begin{aligned}
v_{ch} &= \sqrt{\frac{2\eta t_m}{\alpha}} \\
&= \sqrt{\frac{2\eta\, \tau t_m^*}{\alpha}} \\
&= \sqrt{\tau}\sqrt{\frac{2\eta t_m^*}{\alpha}} \\
v_{ch} &= \sqrt{\tau}\; v_{ch}^* \,,
\end{aligned} \tag{24}$$

where v_{ch}^* is the Stuhlinger velocity for a mission without refueling.

We notice that:
$$\tau \geq 1 \implies v_{ch} \geq v_{ch}^* \,. \tag{25}$$

This means that the upper limit of Δv that can be achieved with an electric propulsion system in a mission with refueling is greater than that of a mission without refueling.

We reconsider the expressions of m_{ps} in equation 19 to include v_{ch}^*:

$$\begin{aligned}
m_{ps} &= \left(\frac{c}{v_{vh}}\right)^2 m_0 \sum_1^n \left[1 - \exp(-\beta_i \Delta v/c)\right] \\
&= \left(\frac{c}{\sqrt{\tau} v_{vh}^*}\right)^2 m_0 \sum_1^n \left[1 - \exp(-\beta_i \Delta v/c)\right] \\
&= \frac{1}{\tau} \left(\frac{c}{v_{vh}^*}\right)^2 m_0 \sum_1^n \left[1 - \exp(-\beta_i \Delta v/c)\right] \\
&= \left(\frac{c}{v_{vh}^*}\right)^2 m_0 \left[1 - \exp(-\Delta v/c)\right] \\
&= \left(\frac{c}{v_{vh}^*}\right)^2 m_0 \left[1 - \exp\left(-\frac{\Delta v/v_{ch}^*}{c/v_{ch}^*}\right)\right] \,.
\end{aligned} \tag{26}$$

The new objective function is:

$$\begin{aligned}
H_i &= \frac{m_{pay,i} + m_s}{m_0} = \frac{m_0 - m_{p,i} - m_{ps}}{m_0} \\
&= \exp\left(-\beta_i \frac{\Delta v/v_{ch}^*}{c/v_{ch}^*}\right) - \left(\frac{c}{v_{vh}^*}\right)^2 \left[1 - \exp\left(-\frac{\Delta v/v_{ch}^*}{c/v_{ch}^*}\right)\right] \,.
\end{aligned} \tag{27}$$

We define the normalized quantities:

$$x = \frac{c}{v^*_{ch}} \quad \text{and } y = \frac{\Delta v}{v^*_{ch}} \ .$$

The objective function can be written as:

$$H_i = \exp\left(-\beta_i \, \frac{y}{x}\right) - x^2 \, \left[1 - \exp\left(-\frac{y}{x}\right)\right] \tag{28}$$

The value of H_i varies at each section of the mission. However, the actual payload of the mission H_m is the same across all the mission, and therefore,

$$H_m = \min(H_i) \tag{29}$$

Thus, we can write

$$\begin{aligned} H_m &= \min\left\{\exp\left(-\beta_i \, \frac{y}{x}\right) - x^2 \, \left[1 - \exp\left(-\frac{y}{x}\right)\right]\right\} \\ &= \min\left\{\exp\left(-\beta_i \, \frac{y}{x}\right)\right\} - x^2 \, \left[1 - \exp\left(-\frac{y}{x}\right)\right] \\ &= \exp\left(-\max\{\beta_i\} \, \frac{y}{x}\right) - x^2 \, \left[1 - \exp\left(-\frac{y}{x}\right)\right] \end{aligned} \tag{30}$$

Equation 30 shows that the maximum payload mass of the mission is limited by the maximum value of β_i (i.e.: the longest section of the mission).

We can also rewrite the time coefficient τ in terms of x and y as:

$$\tau = \frac{\sum_1^n (1 - e^{-\beta_i y/x})}{1 - e^{-y/x}} \ . \tag{31}$$

Fuel consumption consideration

In this part, we want to derive an expression for the specific fuel consumption per payload mass.

$$\begin{aligned} f &= \frac{m_p/m_0}{(m_{pay,m} + m_s)/m_0} \\ &= \frac{\sum_1^n (1 - e^{-\beta_i \Delta v/c})}{min(H_i)} \\ &= \frac{\sum_1^n (1 - e^{-\beta_i \, y/x})}{\exp\left(-\max\{\beta_i\} \, \frac{y}{x}\right) - x^2 \, \left[1 - \exp\left(-\frac{y}{x}\right)\right]} \end{aligned} \tag{32}$$

Graphical representation of the results

The values of H as a function of c/v^*_{ch} for $\Delta v/v^*_{ch} = 0.8$ and different values of β are shown in Fig. 4. The figure shows that for a mission without refueling

($\beta = 1$), the payload mass is negative ($H \leq 0$) because the limit of Δv is $0.8 v^*_{ch}$ as mentioned in Section 2. However, if the spacecraft could achieve smaller portions of the total Δv, then the maximum payload mass increases. For example, if the total Δv is completed through 4 refuelings, where the spacecraft achieves $5 \times 0.2 \Delta v$, then the maximum value of H would be about 0.54.

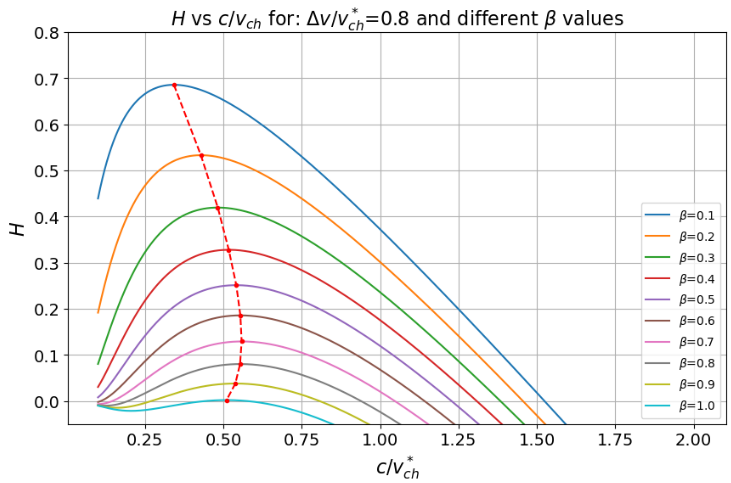

Figure 4: H as a function of c/v^*_{ch} for $\Delta v/v^*_{ch} = 0.6$ and different values of β. The dashed red line represents the maximum value of H for different values of β.

The maximum value of $\Delta v/v^*_{ch}$ as a function of β is shown in Fig. 5. This figure demonstrates that the maximum value of $\Delta v/v^*_{ch}$ for which a positive payload can be carried increases when the spacecraft is allowed to achieve smaller portions of the total Δv in series instead of accomplishing the whole mission in one single effort.

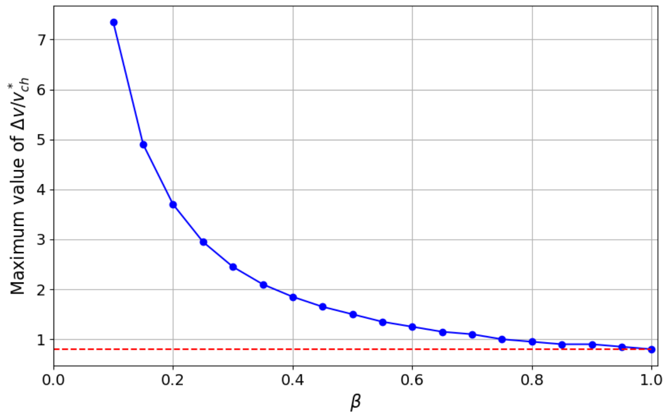

Figure 5: Maximum possible value of $\Delta v/v^*_{ch}$
as a function of β.

The optimal value of c/v^*_{ch} that maximizes the objective function H as a function of β is shown in Fig. 6. The figure shows the results for different values of $\Delta v/v^*_{ch}$.

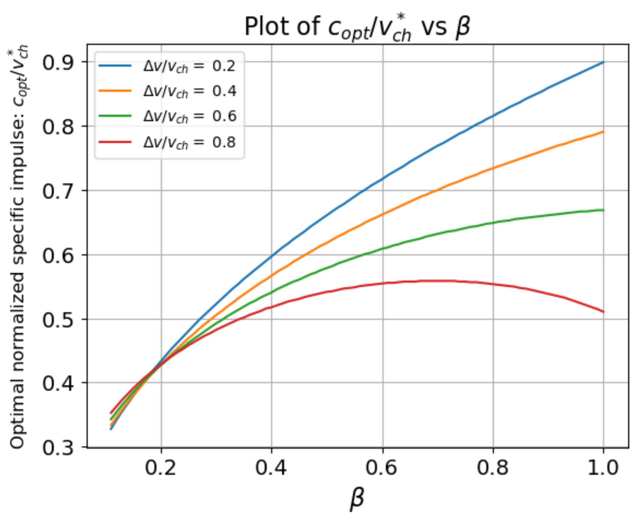

Figure 6: Optimal value of c/v^*_{ch} that maximizes H as a function of β.

Figure. 7 shows a contour plot of the magnitude of the objective function H as a function of $\Delta v/v^*_{ch}$ and c/v^*_{ch}, and for different refueling sequences. The figure shows that the limit of the payload mass is determined by the longest portion of the mission (i.e.: $\max(\beta_i)$). **For example, a refueling sequence of $\{0.6, 0.4\}$ will have the same maximum payload mass as a refueling sequence of $\{0.6, 0.2, 0.2\}$.**

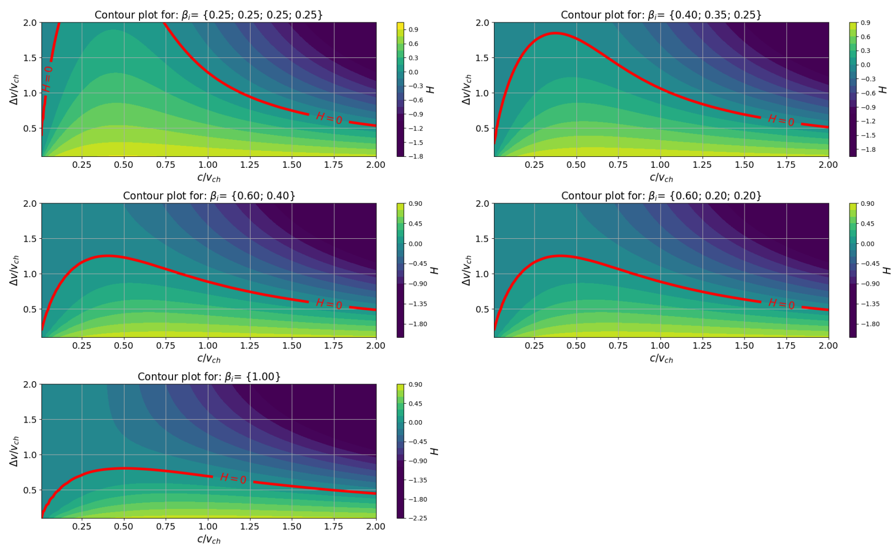

Figure 7: Contour plot of the magnitude of H as a function of $\Delta v/v^*_{ch}$ and c/v^*_{ch} for different refueling sequences.

The time coefficient τ as a function of c/v^*_{ch} for $\Delta v/v^*_{ch} = 0.6$ is shown in Fig. 8 for different refueling sequences. The figure shows a longer mission time for an increasing number of refuelings. For example, even though the two refueling sequences $\{0.6, 0.4\}$ and $\{0.6, 0.2, 0.2\}$ have the same payload mass H, but the sequence $\{0.6, 0.2, 0.2\}$ takes longer because it requires an additional refueling.

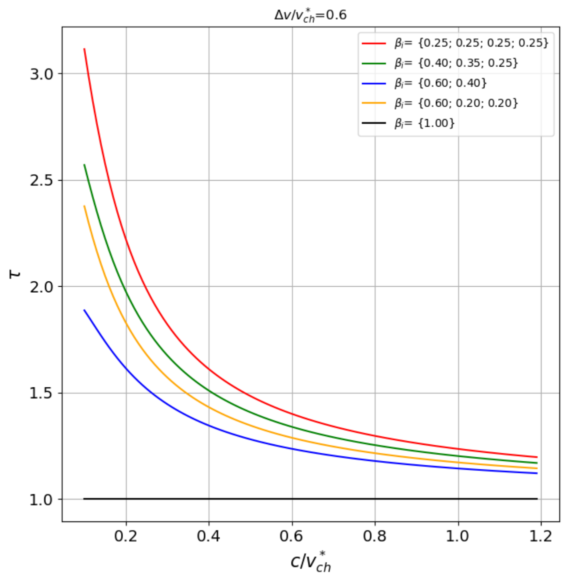

Figure 8: Time coefficient τ vs c/v^*_{ch} for $\Delta v/v^*_{ch} = 0.6$ for different refueling sequences.

The specific fuel consumption f as a function of c/v^*_{ch} for a fixed value of $\Delta v/v^*_{ch}$ equal to 0.6 is shown in Fig. 9 for different refueling sequences. The figure shows a decreasing fuel consumption per payload mass for missions with shorter sections (i.e.: smaller values of β_i).

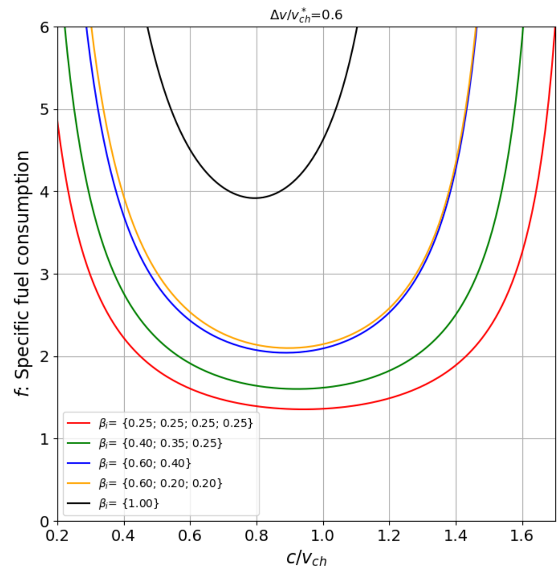

Figure 9: Specific fuel consumption f vs c/vch^* for $\Delta v/v^*_{ch} = 0.6$ for different refueling sequences.

4 Mission with refueling - including penalties

In this section, we add Delta-v penalties $\Delta v_{p,i}$ to account for the extra manoeuvres that the spacecraft has to make in order to refuel. The extra Δv is quantified as a portion of the total Δv as:

$$\Delta v_{p,i} = \gamma_i \Delta v\,. \tag{33}$$

The penalties are only added to the first $n-1$ sections because the n^{th} section ends at the final orbit r_n without having to refuel again. The new expression of Δv_i is:

$$\Delta v_i = \begin{cases} \beta_i \Delta v + \gamma_i \Delta v = (\beta_i + \gamma_i)\Delta v & ; 1 \le i \le n-1 \\ \beta_n \Delta v & ; i = n \end{cases} \tag{34}$$

The new expression of the propellant mass $m_{p,i}$ is:

$$m_{p,i} = \begin{cases} m_0\,(1 - e^{-(\beta_i+\gamma_i)\Delta v/c}) & ; 1 \le i \le n-1 \\ m_0\,(1 - e^{-\beta_n \Delta v/c}) & ; i = n \end{cases} \tag{35}$$

The new expression of the objective function H_i is:

$$H_i = \begin{cases} \exp\left[-(\beta_i + \gamma_i)\,\frac{y}{x}\right] - x^2\,\left[1 - \exp\left(-\frac{y}{x}\right)\right] & ; 1 \le i \le n-1 \\ \exp\left(-\beta_n\,\frac{y}{x}\right) - x^2\,\left[1 - \exp\left(-\frac{y}{x}\right)\right] & ; i = n \end{cases} \tag{36}$$

With $H_m = \min(H_i)$.
The new expression of the time coefficient τ is:

$$\tau = \frac{\sum_1^{n-1}(1 - e^{-(\beta_i+\gamma_i)y/x}) + 1 - e^{-\beta_n y/x}}{1 - e^{-y/x}}\,. \tag{37}$$

The new expression of the specific fuel consumption f is:

$$f = \frac{\sum_1^{n-1}(1 - e^{-(\beta_i+\gamma_i)y/x}) + 1 - e^{-\beta_n y/x}}{\min(H_i)}\,. \tag{38}$$

Graphical representation of the results

The values of normalized payload mass H, the time coefficient τ, and the specific fuel consumption f as a function of c/v^*_{ch} for different refueling sequences including penalties are respectively shown in Figures 10, 11, and 12 . The results without penalty ($\gamma = 0$) are shown in solid lines, whereas the results with penalty ($\gamma_i = \gamma = 0.05$) are shown in dashed lines.

The added penalties lead to a decrease in the maximum payload mass and an increase in the mission time and an increase in the specific fuel consumption.

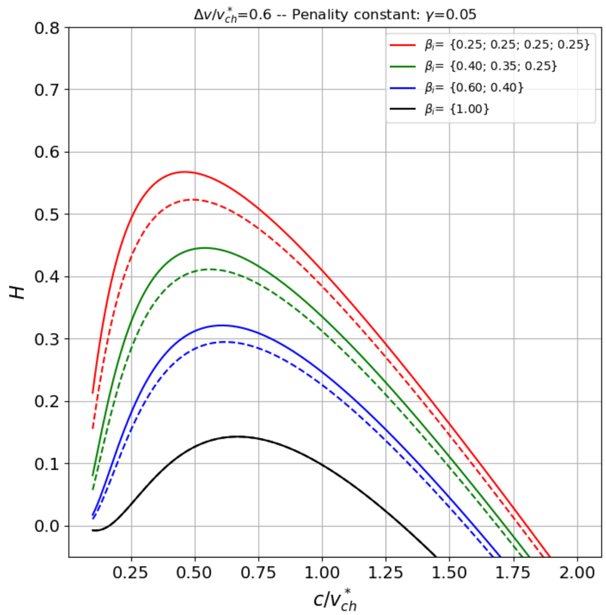

Figure 10: H as a function of c/v^*_{ch} for $\Delta v/v^*_{ch} = 0.6$ and different values of β. Solid lines represent the case of no penalties (γ=0) and the dashed lines represent the case with $\gamma = 0.5$.

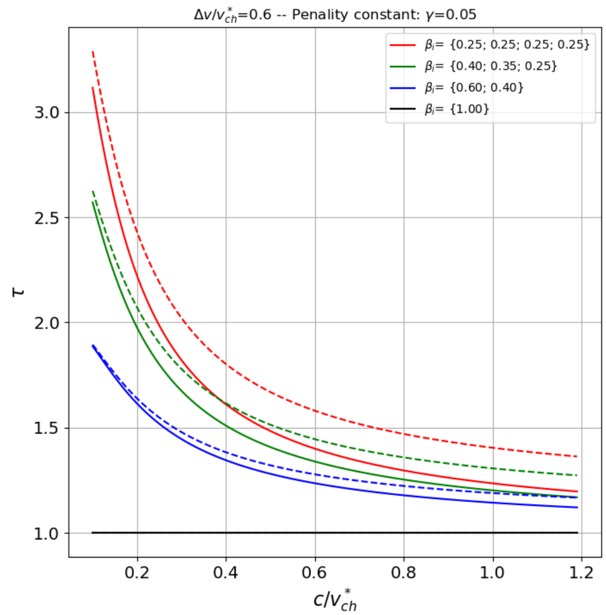

Figure 11: Time coefficient τ as a function of c/v^*_{ch} for $\Delta v/v^*_{ch} = 0.6$ and different values of β. Solid lines represent the case of no penalties (γ=0) and the dashed lines represent the case with $\gamma = 0.5$.

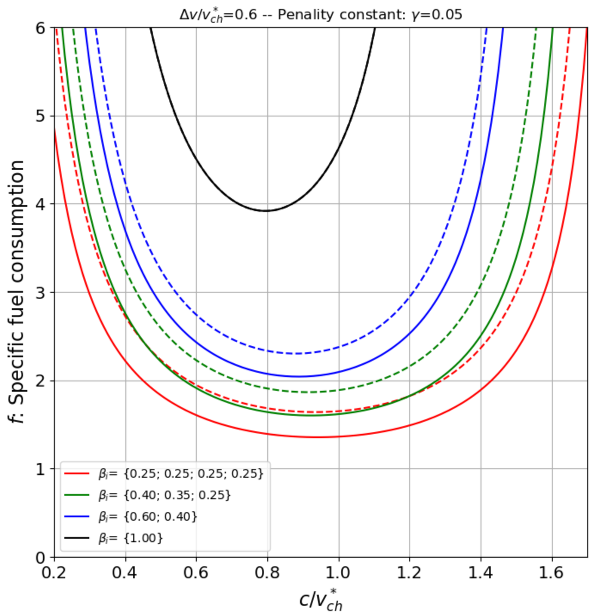

Figure 12: Specific fuel consumption f as a function of c/v_{ch}^* for $\Delta v/v_{ch}^* = 0.6$ and different values of β. Solid lines represent the case of no penalties (γ=0) and the dashed lines represent the case with $\gamma = 0.5$.