What is Linear Algebra?

Linear Algebra is the branch of mathematics that deals with **vectors**, **matrices**, and **linear transformations** between vector spaces.

What is a Vector?

A **vector** is an **ordered list of numbers**, which can represent things like:

- Direction and magnitude in space (physics)
- Data points (machine learning)
- Coordinates (math/geometry)

Mathematically, a vector is written as:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad \text{or} \quad \vec{v} = [v_1, v_2, \ldots, v_n]$$

Examples:

- In 2D: $\vec{v} = [3, 4]$
- In 3D: $\vec{v} = [1, -2, 5]$

Vector Types:

1. **Row Vector**: 1 × n matrix → [v1,v2,…,vn]
2. **Column Vector**: n × 1 matrix → same as above but vertically stacked
3. **Zero Vector**: All elements are 0 → [0,0,…,0]
4. **Unit Vector**: Magnitude is 1

Vector Operations:

Vector Operations

1. Addition

$$\vec{a} + \vec{b} = [a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n]$$

2. Scalar Multiplication

$$c \cdot \vec{v} = [c \cdot v_1, c \cdot v_2, \ldots, c \cdot v_n]$$

3. Dot Product

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \Rightarrow \text{scalar}$$

4. Magnitude (Length)

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

5. Unit Vector

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

6. Angle Between Vectors

$$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

Geometric Interpretation:

In 2D/3D, a vector represents an arrow:

- **Direction** = direction of the arrow
- **Magnitude** = length of the arrow

For example, [3,4] points to the position (3,4) from the origin and has length.

For example, $[3, 4]$ points to the position (3,4) from the origin and has length:

$$|\vec{v}| = \sqrt{3^2 + 4^2} = 5$$

Linear Combination:

You can combine vectors using scalars:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

This is the basis for span, basis vectors, and vector spaces.

Inner Product (Dot Product):

The **inner product** is a generalization of the **dot product**, which measures similarity between two vectors.

For two **n-dimensional vectors**:

$$\vec{a} = [a_1, a_2, \ldots, a_n], \quad \vec{b} = [b_1, b_2, \ldots, b_n]$$

The **dot product** is defined as:

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

📌 Example:

For $\vec{a} = [1, 2, 3]$ and $\vec{b} = [4, -5, 6]$:

$$\vec{a} \cdot \vec{b} = (1)(4) + (2)(-5) + (3)(6) = 4 - 10 + 18 = 12$$

🔶 Geometric Interpretation: Angle Between Vectors

The dot product is related to the **angle θ** between two vectors:

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \cdot ||\vec{b}|| \cdot \cos(\theta)$$

$$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}|| \cdot ||\vec{b}||}$$

📌 Example:

Find the angle between $\vec{a} = [1, 0]$ and $\vec{b} = [0, 1]$:

$$\vec{a} \cdot \vec{b} = (1)(0) + (0)(1) = 0$$

Since **cos(θ) = 0**, we get:

$$\theta = 90^\circ$$

So the vectors are **perpendicular** (orthogonal).

Properties of the Dot Product:

1️⃣ Commutativity

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

✅ Order does not matter.

2️⃣ Distributivity

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

3️⃣ Scalar Multiplication

$$(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b})$$

✅ Scaling one vector scales the dot product.

4️⃣ Zero Vector Property

$$\vec{a} \cdot \vec{0} = 0$$

✅ Any vector dotted with the **zero vector** is **zero**.

6️⃣ Relation to Magnitude

$$\vec{a} \cdot \vec{a} = ||\vec{a}||^2$$

✅ The dot product of a vector with itself gives its **squared magnitude**.

7️⃣ Angle Between Vectors

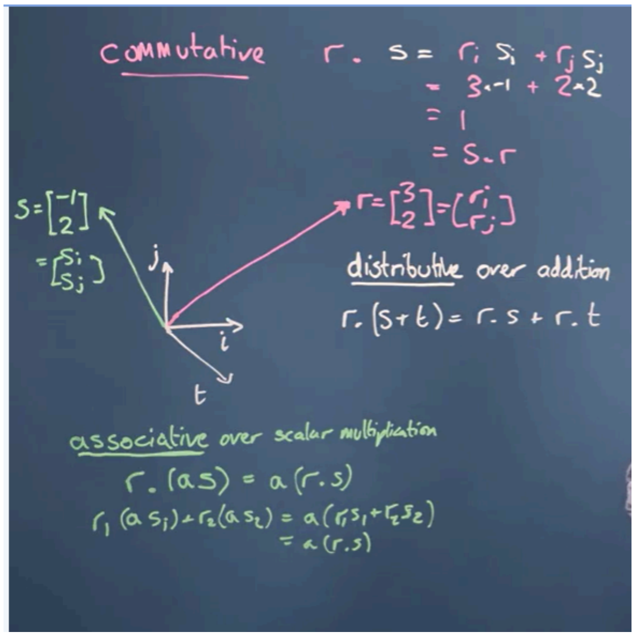
$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \cdot ||\vec{b}|| \cdot \cos(\theta)$$

✅ Defines the **angle θ** between vectors.

8️⃣ Triangle Inequality

$$|\vec{a} \cdot \vec{b}| \leq ||\vec{a}|| \cdot ||\vec{b}||$$

✅ The absolute dot product is **never greater than the product of magnitudes**.



Dot Product of a Vector with Itself:

For a vector **a** in n-dimensional space:

$$\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + \cdots + a_n^2$$

This is simply the **sum of the squares of its components**.

◆ Relation to Magnitude (Norm)

The dot product of a vector with itself gives its **squared magnitude**:

$$\vec{a} \cdot \vec{a} = ||\vec{a}||^2$$

where:

$$||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

is the **Euclidean norm** (length) of the vector.

Special case:

1 Zero Vector:
If $\vec{a} = [0, 0, \ldots, 0]$, then:

$$\vec{a} \cdot \vec{a} = 0$$

2 Unit Vector:
If $\vec{a}$ is a **unit vector** (i.e., $||\vec{a}|| = 1$), then:

$$\vec{a} \cdot \vec{a} = 1$$

Projection:

In **linear algebra**, a **projection** is the process of mapping a vector onto another vector or a subspace. The result is a new vector that lies on the target vector or subspace, representing the component of the original vector in that direction.

Projection Using $\cos\theta$ Formula

We know from the dot product formula:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

Rearrange to find $\cos\theta$:

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

The **projection of $\mathbf{a}$ onto $\mathbf{b}$** using the $\cos\theta$ formula is:

$$\text{proj}_{\mathbf{b}}\mathbf{a} = |\mathbf{a}|\cos\theta$$

Expanding $\cos\theta$:

$$\text{proj}_{\mathbf{b}}\mathbf{a} = |\mathbf{a}|\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Canceling $|\mathbf{a}|$:

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$$

This gives the **scalar projection** (length of the projection).

For the **vector projection**, multiply by the unit vector of $\mathbf{b}$:

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$$

Example:

Let:

$$\mathbf{a} = (3,4), \quad \mathbf{b} = (1,2)$$

1. **Compute dot product:**

$$\mathbf{a}\cdot\mathbf{b} = (3)(1) + (4)(2) = 3 + 8 = 11$$

2. **Find magnitude of $\mathbf{b}$:**

$$|\mathbf{b}| = \sqrt{1^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$$

3. **Compute scalar projection:**

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{11}{\sqrt{5}}$$

4. **Compute vector projection:**

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{11}{5} \cdot (1, 2) = \left(\frac{11}{5}, \frac{22}{5}\right)$$

The **scalar projection** of vector $\mathbf{s}$ onto $\mathbf{r}$ is given by:

$$\text{Scalar Projection} = \frac{\mathbf{s} \cdot \mathbf{r}}{|\mathbf{r}|}$$

Step 1: Compute the Dot Product $\mathbf{s} \cdot \mathbf{r}$

$$\mathbf{s} \cdot \mathbf{r} = (10 \times 3) + (5 \times -4) + (-6 \times 0)$$
$$= 30 - 20 + 0 = 10$$

Step 2: Compute the Magnitude of $\mathbf{r}$

$$|\mathbf{r}| = \sqrt{3^2 + (-4)^2 + 0^2}$$

Step 3: Compute the Scalar Projection

$$\frac{10}{5} = 2$$

Final Answer:

$$\boxed{2}$$

Vector Projection Definition

The **vector projection** of a vector $\mathbf{s}$ onto another vector $\mathbf{r}$ is the component of $\mathbf{s}$ that lies in the direction of $\mathbf{r}$.

The formula for the **vector projection** of $\mathbf{s}$ onto $\mathbf{r}$ is:

$$\text{proj}_{\mathbf{r}}\mathbf{s} = \frac{\mathbf{s} \cdot \mathbf{r}}{\mathbf{r} \cdot \mathbf{r}}\mathbf{r}$$

Example Calculation:

Given vectors:

$$\mathbf{r} = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} 10 \\ 5 \\ -6 \end{bmatrix}$$

Step 1: Compute the Dot Product $\mathbf{s} \cdot \mathbf{r}$

$$\mathbf{s} \cdot \mathbf{r} = (10 \times 3) + (5 \times -4) + (-6 \times 0)$$

$$= 30 - 20 + 0 = 10$$

Step 2: Compute $\mathbf{r} \cdot \mathbf{r}$

$$\mathbf{r} \cdot \mathbf{r} = (3 \times 3) + (-4 \times -4) + (0 \times 0)$$

$$= 9 + 16 + 0 = 25$$

Step 3: Compute Vector Projection

$$\text{proj}_{\mathbf{r}}\mathbf{s} = \frac{10}{25}\mathbf{r} = \frac{2}{5}\mathbf{r}$$

$$= \frac{2}{5}\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ -\frac{8}{5} \\ 0 \end{bmatrix}$$

Final Answer:

$$\boxed{\begin{bmatrix} \frac{6}{5} \\ -\frac{8}{5} \\ 0 \end{bmatrix}}$$

Changing Basis:

Changing basis means expressing a vector or a system of vectors in terms of a new set of basis vectors instead of the original ones.

In simpler terms, it's like translating coordinates from one language (basis) to another while preserving the meaning (vector itself).

Mathematical Explanation

1. Standard Basis Representation

Any vector $\mathbf{v}$ in $\mathbb{R}^n$ can be represented in terms of the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, where:

$$\mathbf{v} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

Here, x_i are the coordinates in the **standard basis**.

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2. New Basis Representation

Suppose we introduce a new basis $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$, where each basis vector is different from the standard basis. Now, we express $\mathbf{v}$ in terms of B:

$$\mathbf{v} = y_1\mathbf{b}_1 + y_2\mathbf{b}_2 + \cdots + y_n\mathbf{b}_n$$

Here, y_i are the new coordinates in the **new basis**.

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3. Transformation Matrix

To convert coordinates from the old basis to the new basis, we use a **change of basis matrix**.

If P is a matrix whose columns are the new basis vectors:

$$P = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \dots \quad \mathbf{b}_n]$$

Then, the relation between the standard basis coordinates x and the new basis coordinates y is:

$$x = Py \quad \Rightarrow \quad y = P^{-1}x$$

where:

- x = coordinates in the standard basis.
- y = coordinates in the new basis.
- P = matrix of new basis vectors.
- P^{-1} = inverse of P, used to transform from old basis to new basis.

Example of basis change:

$$\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

in the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, and we want to express it in the new basis:

$$\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The change of basis matrix is:

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

To find new coordinates y:

$$y = P^{-1}x$$

Vector space:

A **vector space** is a collection of vectors that can be added together and multiplied by scalars while following certain mathematical rules. It consists of:

- A set of vectors
- A set of scalars (from a field, e.g., real numbers R\mathbb{R}R)
- Two operations: **vector addition** and **scalar multiplication**

Example of a Vector Space:

- The set of all 2D vectors R2\mathbb{R}^2R2
- The set of all polynomials of degree ≤ 2

For example, in R^2 vectors like

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

can be added together, and multiplied by a scalar, and the result will still be in $\mathbb{R}^2$.

Basis :

2. Basis

A **basis** of a vector space is a **minimal set of linearly independent vectors** that can be used to express every other vector in that space.

Properties of a Basis:

1. **Spanning**: Any vector in the space can be written as a combination of the basis vectors.
2. **Linear Independence**: No vector in the basis can be written as a combination of the others.

Example of a Basis in $\mathbb{R}^2$:

The **standard basis** is:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Any vector in $\mathbb{R}^2$, like $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, can be written as:

$$\mathbf{v} = 3\mathbf{e}_1 + 2\mathbf{e}_2$$

Other bases are possible, for example:

$$\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

As long as they **span the space** and are **linearly independent**, they form a basis.

Linear Independence:

A set of vectors is **linearly independent** if none of them can be written as a combination of the others.

Mathematical Definition

Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are **linearly independent** if the only solution to:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$$

is when $c_1 = c_2 = \ldots = c_n = 0$.

Example of Linearly Independent Vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

To check independence, solve:

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This leads to two equations:

$$c_1 + 3c_2 = 0$$

$$2c_1 + 4c_2 = 0$$

Solving this gives $c_1 = 0, c_2 = 0$, meaning they are **linearly independent**.

Example of Linearly Dependent Vectors

If we have:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Since $\mathbf{v}_2 = 2\mathbf{v}_1$, these vectors are **linearly dependent**.

Relationship Between These Concepts

1. **A basis is a set of linearly independent vectors** that span a vector space.
2. **Linear independence ensures that the basis vectors do not overlap** in representation.
3. **A vector space can have many different bases**, but the number of basis vectors (dimension) remains the same.

Transformation:

Matrices act as **transformations** that modify geometric spaces by rotating, scaling, reflecting, shearing, or projecting vectors. When a matrix A is applied to a vector v, the result is a new transformed vector:

$$A\mathbf{v} = \mathbf{v}'$$

This means that every point or vector in space is mapped to a new location, effectively transforming the entire space.

Types of Transformations by Matrices:

Types of Transformations by Matrices

1. Scaling

A **diagonal matrix** scales a vector along coordinate axes.

$$A = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

Effect:

- If $s_x, s_y > 1$, the space expands.
- If $0 < s_x, s_y < 1$, the space shrinks.
- If s_x or s_y is negative, it also reflects across an axis.

Example: Scaling by 2

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

2. Rotation

A **rotation matrix** rotates vectors by an angle θ.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Effect: Rotates vectors counterclockwise by θ.

Example: 90° rotation

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

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3. Reflection

A reflection flips points across a given line.

Reflection across the **x-axis**:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Effect:

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ -y \end{bmatrix}$$

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4. Shear Transformation

Shearing skews the space in a given direction.

Horizontal shear:

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Effect: The x-coordinate shifts by a factor of k times the y-coordinate.

5. Projection

A projection matrix maps vectors onto a line or plane.

Projection onto the **x-axis**:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Effect:

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ 0 \end{bmatrix}$$

Geometric Interpretation

Each transformation **preserves some properties while changing others**:

- **Rotations** preserve distances and angles.
- **Scaling** changes sizes but preserves parallel lines.
- **Reflections** flip orientations but maintain distances.
- **Shearing** distorts angles while keeping areas.
- **Projections** reduce dimensionality.

Composition and Combination of Matrix Transformations:

In **linear algebra**, we can apply multiple transformations sequentially using **matrix multiplication**. This process is called the **composition of transformations**. When two or more transformations are applied to a vector or space, their combined effect is represented by the **product of their transformation matrices**.

1. Composition of Transformations (Sequential Application)

If we have two transformations:

- **First transformation**: A
- **Second transformation**: B

Then, applying A **first, then** B to a vector $\mathbf{v}$ gives:

$$\mathbf{v'} = B(A\mathbf{v})$$

By associativity of matrix multiplication:

$$\mathbf{v'} = (BA)\mathbf{v}$$

This means the **overall transformation** is given by the **product matrix** BA.

> **Key point: The order matters!**
> - **"BA is generally NOT the same as AB."**
> - **"This means transformations are not alway[illegible]mmutative."**

2. Example: Rotation Followed by Scaling

Step 1: Define Transformations

- Rotation by θ:

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- Scaling by factors s_x, s_y:

$$S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

Step 2: Find Combined Transformation

Applying scaling **after rotation**:

$$T = SR = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Gaussian Elimination Method:

Gaussian elimination is a method for solving systems of linear equations. It systematically transforms a system into an upper triangular or row-echelon form using elementary row operations, making it easier to solve.

Purpose

It is used for:

- Solving linear equations
- Finding the inverse of a matrix
- Determining the rank of a matrix
- Finding the determinant of a matrix

Steps in Gaussian Elimination

1. **Convert the system into an augmented matrix**
2. **Use row operations to get an upper triangular (row-echelon) form**
   - Swap rows (if necessary)
   - Multiply a row by a scalar
   - Subtract multiples of a row from another row to eliminate variables

Problem:

Suppose we have the following system of equations representing the price of apples and bananas:

$$2a + 3b = 8$$

$$4a + 5b = 14$$

where:

- a represents the price of an apple
- b represents the price of a banana

We will use **Gaussian elimination** to solve for a and b.

Step 1: Write the System as an Augmented Matrix

The system:

$$\begin{aligned} 2a + 3b &= 8 \\ 4a + 5b &= 14 \end{aligned}$$

becomes the augmented matrix:

$$\begin{bmatrix} 2 & 3 & |8 \\ 4 & 5 & |14 \end{bmatrix}$$

Step 2: Convert to Row Echelon Form

We perform **row operations** to create a triangular form.

Step 2.1: Make the first pivot (top-left) a 1

Divide **Row 1** by **2**:

$$R_1 \leftarrow \frac{1}{2}R_1$$

$$\begin{bmatrix} 1 & 1.5 & |4 \\ 4 & 5 & |14 \end{bmatrix}$$

Step 2.2: Eliminate the first element in Row 2

Make the first element in Row 2 zero by subtracting **4 × Row 1** from **Row 2**:

$$R_2 \leftarrow R_2 - 4R_1$$

$$\begin{bmatrix} 1 & 1.5 & |4 \\ 0 & -1 & |-2 \end{bmatrix}$$

Step 3: Convert to Reduced Row Echelon Form

Step 3.1: Make the second pivot a 1

Divide **Row 2** by **-1**:

$$R_2 \leftarrow -R_2$$

$$\begin{bmatrix} 1 & 1.5 & |4 \\ 0 & 1 & |2 \end{bmatrix}$$

Step 3.2: Make the second column in Row 1 zero

Subtract **1.5 × Row 2** from **Row 1**:

$$R_1 \leftarrow R_1 - 1.5R_2$$

$$\begin{bmatrix} 1 & 0 & |1 \\ 0 & 1 & |2 \end{bmatrix}$$

Step 4: Read the Solution

From the final matrix:

$$a = 1$$
$$b = 2$$

Thus, the price of:

- **One apple is 1**
- **One banana is 2**

Row Echelon Form:

The **row echelon form (REF)** of a matrix is achieved by performing **row operations** to get a triangular-like structure with leading ones (pivots).

Steps to Convert a Matrix to Row Echelon Form

1. **Identify the leftmost nonzero column (Pivot Column).**
2. **Swap rows** (if necessary) to ensure the first row has a nonzero entry in the pivot column.
3. **Make the leading coefficient 1 (Pivot = 1).**
   - If the pivot is not 1, divide the row by the pivot value.
4. **Make all entries below the pivot zero** using row operations.
   - Subtract multiples of the pivot row from lower rows.
5. **Move to the next column and repeat** for the remaining submatrix.

Example: Converting a Matrix to Row Echelon Form

Given matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

Step 1: Ensure the first pivot is 1 (Already in position).

Step 2: Make entries below the first pivot (first column) zero

Perform:

- $R_2 \leftarrow R_2 - 3R_1$
- $R_3 \leftarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -1 & 0 \end{bmatrix}$$

Step 3: Make the second pivot 1

Multiply row 2 by -1:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 0 \end{bmatrix}$$

Step 4: Make entries below the second pivot zero

Perform:

- $R_3 \leftarrow R_3 + R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Step 5: Make the third pivot 1

Divide R_3 by 2:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

This is the **row echelon form (REF)**! 🚀

Use Cases of Row Echelon Form (REF)

Row Echelon Form (REF) is a fundamental technique in **linear algebra** used in many real-world applications. Below some application are given:

1️⃣ Solving Systems of Linear Equations (Gaussian Elimination)

We solve:

$$\begin{aligned} a + b + c &= 15 \\ 3a + 2b + c &= 28 \\ 2a + b + 2c &= 23 \end{aligned}$$

Convert to augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & |15 \\ 3 & 2 & 1 & |28 \\ 2 & 1 & 2 & |23 \end{bmatrix}$$

Step 1: Make the first column a leading 1

Already **1** in row 1, so proceed.

Step 2: Make lower entries in column 1 zero

- $R_2 \rightarrow R_2 - 3R_1$
- $R_3 \rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 & |15 \\ 0 & -1 & -2 & |-17 \\ 0 & -1 & 0 & |-7 \end{bmatrix}$$

Step 3: Make the second column a leading 1

Multiply R_2 by **-1** to make a leading **1**:

$$\begin{bmatrix} 1 & 1 & 1 & |15 \\ 0 & 1 & 2 & |17 \\ 0 & -1 & 0 & |-7 \end{bmatrix}$$

Step 4: Make lower entries in column 2 zero

- $R_3 \rightarrow R_3 + R_2$

$$\begin{bmatrix} 1 & 1 & 1 & |15 \\ 0 & 1 & 2 & |17 \\ 0 & 0 & 2 & |10 \end{bmatrix}$$

Step 5: Make third column a leading 1

Divide R_3 by **2**:

$$\begin{bmatrix} 1 & 1 & 1 & |15 \\ 0 & 1 & 2 & |17 \\ 0 & 0 & 1 & |5 \end{bmatrix}$$

Step 6: Back-substitution

- **From R_3:** $c = 5$
- **From R_2:** $b + 2(5) = 17 \rightarrow b = 7$
- **From R_1:** $a + 7 + 5 = 15 \rightarrow a = 3$

✅ **Solution:**

$$a = 3, \quad b = 7, \quad c = 5$$

2 Finding Matrix Inverses

Find A^{-1} for:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

Augment with identity matrix:

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]$$

Step 1: Make first column a leading 1

Divide R_2 by **2**:

$$\left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array}\right]$$

Step 2: Make lower entries in column 1 zero

$R_2 \rightarrow R_2 - 2R_1$:

$$\left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & -5 & 1 & -2 \end{array}\right]$$

Step 3: Make second column a leading 1

Divide R_2 by **-5**:

$$\left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & 1 & -1/5 & 2/5 \end{array}\right]$$

Step 4: Make upper entries in column 2 zero

$R_1 \to R_1 - 4R_2$:

$$\left[\begin{array}{cc|cc} 1 & 0 & 4/5 & -3/5 \\ 0 & 1 & -1/5 & 2/5 \end{array}\right]$$

✅ **Inverse Matrix:**

$$A^{-1} = \begin{bmatrix} 4/5 & -3/5 \\ -1/5 & 2/5 \end{bmatrix}$$

3 Determining Linear Independence

Given:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Perform row operations:

- $R_2 \to R_2 - 2R_1$
- $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there are **zero rows**, vectors are **linearly dependent**.

If a matrix A satisfies the condition:

$$A^T A = I$$

where I is the identity matrix, then A is called an **orthonormal matrix** (if it's square) or an **orthonormal set of vectors** (if it's not square). This property arises from **orthonormality** in linear algebra.

Why Does $A^T A = I$?

Let's break it down step by step:

Step 1: Definition of Transpose and Multiplication

- The **transpose** of a matrix A, denoted as A^T, is obtained by flipping its rows into columns.
- When you multiply A^T by A, each entry in the resulting matrix is a **dot product** between two column vectors of A.

Step 2: Orthonormal Columns

If the **columns** of A are **orthonormal**, it means:

- Each column has a **unit length** (i.e., dot product with itself is 1).
- Different columns are **perpendicular** (i.e., dot product between two different columns is 0).

Mathematically, if A has column vectors $v_1, v_2, \ldots, v_n$:

$$A = [v_1 \quad v_2 \quad \ldots \quad v_n]$$

Then,

$$A^T A = \begin{bmatrix} v_1^T v_1 & v_1^T v_2 & \ldots & v_1^T v_n \\ v_2^T v_1 & v_2^T v_2 & \ldots & v_2^T v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n^T v_1 & v_n^T v_2 & \ldots & v_n^T v_n \end{bmatrix}$$

Since $v_i^T v_j = 0$ for $i \neq j$ (orthogonal columns), and $v_i^T v_i = 1$ (unit norm), we get:

$$A^T A = I$$

Eigenvalues and Eigenvectors :

Eigenvalues and eigenvectors are fundamental concepts in **linear algebra**, commonly used in areas such as **machine learning, quantum mechanics, and computer vision**.

What is an Eigenvector?

An **eigenvector** of a square matrix A is a **nonzero vector** v that **does not change its direction** when multiplied by A. Instead, it only gets **scaled** by some scalar λ:

$$Av = \lambda v$$

where:

- A is an $n \times n$ matrix.
- v is a **nonzero vector** (i.e., $v \neq 0$).
- λ is a **scalar**, called the **eigenvalue**.

What is Eigenvalue?

2. What is an Eigenvalue?

An **eigenvalue** λ is the **scaling factor** by which an eigenvector v is stretched or shrunk when transformed by A.

If $\lambda > 1$, the eigenvector gets stretched.
If $0 < \lambda < 1$, the eigenvector gets shrunk.
If $\lambda < 0$, the eigenvector flips direction.

To find eigenvalues, solve the **characteristic equation**:

$$\det(A - \lambda I) = 0$$

where I is the **identity matrix**.

Eigenvalue and Eigenvector only possible in square matrix?

Yes, **eigenvectors and eigenvalues are only defined for square matrices** (i.e., $n \times n$ matrices).

To find eigenvalues, we solve:

$$\det(A - \lambda I) = 0$$

where:

- A is an $n \times n$ matrix,
- I is the $n \times n$ identity matrix,
- λ is a scalar.

Since the **determinant** is only defined for **square matrices**, eigenvalues and eigenvectors are only applicable to square matrices.

Done ↓

What About Non-Square Matrices?

For **rectangular matrices** (e.g., $m \times n$ where $m \neq n$), we cannot compute eigenvalues and eigenvectors in the traditional sense.

However, alternatives exist:

1. **Singular Value Decomposition (SVD)**:
   - Any $m \times n$ matrix can be decomposed into singular values and singular vectors.
   - Similar to eigenvectors but works for non-square matrices.
2. **Left and Right Eigenvectors** (for non-square operators in functional analysis).

3. Example Calculation

Consider the matrix:

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

Step 1: Compute Characteristic Equation

Solve:

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} = 0$$

$$(4-\lambda)(3-\lambda) - (2 \times 1) = 0$$

$$(12 - 4\lambda - 3\lambda + \lambda^2) - 2 = 0$$

$$\lambda^2 - 7\lambda + 10 = 0$$

Factorize:

$$(\lambda - 5)(\lambda - 2) = 0$$

So, the **eigenvalues** are:

$$\lambda_1 = 5, \quad \lambda_2 = 2$$

Step 2: Find Eigenvectors

For $\lambda_1 = 5$:

Solve:

$$(A - 5I)v = 0$$

$$\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the first row:

$$-1x + 2y = 0 \Rightarrow x = 2y$$

Let $y = 1$, then $x = 2$.

Eigenvector:

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 2$:

Solve:

$$(A - 2I)v = 0$$

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the first row:

$$2x + 2y = 0 \Rightarrow x = -y$$

Let $y = 1$, then $x = -1$.

Eigenvector:

$$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Final Answer

- **Eigenvalues:** $5, 2$
- **Eigenvectors:**
  - $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (for $\lambda_1 = 5$)
  - $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (for $\lambda_2 = 2$)

4. Why Are Eigenvalues and Eigenvectors Useful?

- **Principal Component Analysis (PCA)**: Used for dimensionality reduction in machine learning.
- **Markov Chains**: Helps analyze steady-state behavior.
- **Differential Equations**: Used in solving systems of linear differential equations.
- **Quantum Mechanics**: Defines quantum states in physics.
- **Graph Theory**: Used in Google's **PageRank algorithm**.

* * *

Recall that for a matrix A, the eigenvectors of the matrix are vectors for which applying the matrix transformation is the same as scaling by some constant.

What is a Basis?

A **basis** of a vector space is a set of **linearly independent** vectors that **span** the entire space.

Key Properties of a Basis

1. **Spanning**: Any vector in the space can be written as a **linear combination** of the basis vectors.
2. **Linear Independence**: No basis vector can be written as a combination of the others.
3. **Uniqueness of Representation**: Every vector in the space has a **unique** representation using basis vectors.
4. **Dimension**: The number of vectors in the basis is the **dimension** of the space.

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1. Standard Basis in $\mathbb{R}^2$

In 2D space, the standard basis is:

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Any vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$ can be written as:

$$v = xe_1 + ye_2$$

2. Basis of $\mathbb{R}^3$

Standard basis in 3D:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Changing to the Eigenbasis

Changing to an **eigenbasis** means expressing a vector or a transformation in terms of the eigenvectors of a matrix. This is useful in diagonalization, simplifying linear transformations, and solving differential equations.

Steps to Change to the Eigenbasis

1. Find the Eigenvalues and Eigenvectors

For a square matrix A:

- Solve $\det(A - \lambda I) = 0$ to find eigenvalues λ_i.
- Solve $(A - \lambda_i I)x = 0$ to get eigenvectors.

2. Form the Eigenvector Matrix P and Diagonal Matrix D

- Let P be the matrix whose columns are eigenvectors.
- Let D be the diagonal matrix with eigenvalues on the diagonal.

3. Convert Coordinates to the Eigenbasis

- A vector v in the standard basis can be written in the **eigenbasis** as:

$$v' = P^{-1}v$$

  where v' represents the coordinates in the new basis.

Eigenvector:

What is an Eigenvector?

An **eigenvector** of a matrix A is a nonzero vector v that only gets **scaled** (not rotated or changed in direction) when multiplied by A.

Mathematically, it satisfies the equation:

$$Av = \lambda v$$

where:

- A is an $n \times n$ **square matrix**
- v is an **eigenvector**
- λ is the **eigenvalue** (the scaling factor)

Example: Changing to Eigenbasis

Consider the matrix:

$$A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$$

1. **Find eigenvalues** by solving $\det(A - \lambda I) = 0$:

$$\begin{vmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

Expanding,

$$(4-\lambda)(1-\lambda) + 2 = 0$$

$$4 - 4\lambda - \lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

Solving $(\lambda - 3)(\lambda - 2) = 0$,

- Eigenvalues: $\lambda_1 = 3, \lambda_2 = 2$.

2. **Find eigenvectors**
   - For $\lambda_1 = 3$, solve $(A - 3I)x = 0$:

$$\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

     - Solution: $x_1 = 2x_2$, so eigenvector is $v_1 = [2, 1]$.

   - For $\lambda_2 = 2$, solve $(A - 2I)x = 0$:

$$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

     - Solution: $x_1 = x_2$, so eigenvector is $v_2 = [1, 1]$.

3. **Construct P and D**

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

4. **Convert to Eigenbasis**
   Given a vector $v = [x, y]$, its coordinates in the eigenbasis are:

$$v' = P^{-1}v$$

   where P^{-1} is computed as:

$$P^{-1} = \frac{1}{(2 \cdot 1 - 1 \cdot 1)} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Applying this, v' is computed as:

$$v' = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$