Week - 5:

Understanding Eigenvectors and Eigenvalues (Geometric Interpretation)

What is an Eigenproblem?

The word **"eigen"** comes from German, meaning **"characteristic"**.
An **eigenproblem** is about finding the **characteristic directions and scalings** of a transformation.

The Big Idea (Geometrically)

Imagine applying a **linear transformation** (like scaling, rotating, or shearing) to a shape in space.
A simple way to picture it:

1. Draw a **square** centered at the origin.
2. Apply the transformation.
3. Observe how the square changes (stretch, skew, rotate, etc.).

We're interested in vectors that behave **specially** under this transformation.

What Makes Eigenvectors Special?

When transforming the whole space, **most vectors**:

- **Change direction** and
- **Change length**.

But **eigenvectors** are different:

- They **do not change direction** (stay on the same line) may be **opposite** directions..
- They might get **longer or shorter**, but they **don’t rotate**.

These are called **eigenvectors**, and the amount they stretch/shrink is the **eigenvalue**.

Example 1: Vertical Scaling

Transformation: **Scale vertically by 2**

- ✅ **Vertical vector**: Points in the same direction, length doubled → **eigenvector**, eigenvalue = **2**
- ✅ **Horizontal vector**: Same direction, length unchanged → **eigenvector**, eigenvalue = **1**
- ❌ **Diagonal vector**: Direction changed → **not** an eigenvector

---

Example 2: Pure Shear

Transformation: **Shift top of square sideways**

- ✅ **Horizontal vector**: Same direction → **eigenvector**
- ❌ Others: Direction changed → **not** eigenvectors

Example 3: Rotation

Transformation: **Rotate the whole shape**

- ❌ **All vectors** change direction → **no eigenvectors**
    - (Unless the rotation is 0° or 180°)

Generalization

Even in **3D or higher dimensions**, the idea stays the same:

- **Eigenvectors** point in directions that are **unchanged**.

- **Eigenvalues** measure how much those directions are **stretched/shrunk**.

✅ Summary Table

Concept	Meaning
Eigenvector	A vector that doesn't change direction under a transformation
Eigenvalue	The factor by which the eigenvector is stretched/shrunk
No eigenvectors	Happens when all vectors rotate (like pure rotation)

Constructing Eigenvalues and Eigenvectors

- **Eigenvectors**: Vectors that stay along the same direction after a transformation (they may change in length or direction but not in their span).
- **Eigenvalues**: Scalars that describe how much the eigenvectors are stretched or compressed during the transformation.

The algebraic expression for eigenvectors and eigenvalues is:

$$\mathbf{Ax = \lambda x}$$

Where:

- A is the transformation matrix.
- x is the eigenvector.
- λ is the eigenvalue.

To find eigenvalues, rearrange this as:

$$\mathbf{(A - \lambda I)\, x = 0}$$

Where I is the identity matrix. The determinant of $\mathbf{(A-\lambda I)}$ must be zero:

Now, for this equation to hold true, either:

- The matrix $\mathbf{(A-\lambda I)}$ must be singular (non-invertible), which means it has a determinant of **zero**, or
- The vector **x** must be the zero vector, but we discard this trivial solution

det(A−λI)=0

This gives the **characteristic polynomial**, which you solve for λ.

Example: 2x2 Matrix

Consider a matrix A given by:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

To find the eigenvalues:

- We calculate the determinant of $A - \lambda I$, where I is the identity matrix:

$$\det(A - \lambda I) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = (a - \lambda)(d - \lambda) - bc$$

- Expanding this gives the **characteristic polynomial:**

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

- Solving this quadratic equation gives the eigenvalues λ.

Example 1: Vertical Scaling

Matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$:

- Eigenvalues: $\lambda = 1$, $\lambda = 2$.
- Eigenvectors for $\lambda = 1$: Any vector along the x-axis $\begin{pmatrix} t \\ 0 \end{pmatrix}$.
- Eigenvectors for $\lambda = 2$: Any vector along the y-axis $\begin{pmatrix} 0 \\ t \end{pmatrix}$.

Example 2: 90-Degree Rotation

Matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$:

- The characteristic equation is $\lambda^2 + 1 = 0$, leading to complex eigenvalues $\lambda = \pm i$.
- No real eigenvectors exist for this transformation.