Teknik Pengintegralan

Teknik Substitusi

ditunkan

↳ Misal $u = g(x)$ maka $du = g'(x)\,dx$

"paling rumit"

Jika $$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

Contoh Soal :

$$\int \underline{x}\,\underbrace{(x^2+2)^5}_{u^5}\,\underline{dx}$$

Misal $u = x^2 + 2$ maka $du = 2x\,dx$

$x\,dx = \frac{du}{2}$

Diperoleh

$$\int x(x^2+2)^5\,dx = \int u^5 \cdot \frac{du}{2}$$

$$= \frac{1}{2} \cdot \frac{1}{5+1} u^{5+1} + C$$

$$= \frac{1}{12} u^6 + C$$

$$= \frac{1}{12} (x^2+2)^6 + C$$

Integral Parsial

$$\left.\begin{aligned} \int u\,dv &= uv - \int v\,du \\ \int_a^b u\,dv &= [uv]_a^b - \int_a^b v\,du \end{aligned}\right\} \int uv' = uv - \int vu'$$

Contoh Soal :

$\int \underset{\smile}{x} . \underbrace{e^{x+2}} dx$

↗ $u \to u'$ lebih sederhana

↘ $v' \to v$ lebih mudah diintegralkan

Misal $u = x$ $\quad v' = e^{x+2}$

$u' = 1$ $\quad v = e^{x+2}$

Memisalkan u :

I LATE → eksponen

→ Trigonometri

→ Aljabar x^n

inverse

logaritma log / ln

$\int x e^{x+2} dx$

$= x . e^{x+2} - \int e^{x+2} dx$

$= x . e^{x+2} - e^{x+2} + c$

$\int e^{x+2} dx$

$u = x + 2 \to du = dx$

$\int e^{u} du = e^{u} + c$

$= e^{x+2} + C$

Integral Trigonometri

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\boxed{\begin{aligned}
2 \sin x \cos y &= \sin (x+y) + \sin (x-y) \\
2 \cos x . \sin y &= \sin (x+y) - \sin (x-y) \\
2 \cos x . \cos y &= \cos (x+y) + \cos (x-y) \\
-2 \sin x . \sin y &= \cos (x+y) - \cos (x-y)
\end{aligned}}$$

$$\int \sin^n x\, dx \qquad \int \cos^n x\, dx$$

n genap $\left\{ \begin{array}{l} \sin^2 x = \dfrac{1-\cos 2x}{2} \\ \cos^2 x = \dfrac{1+\cos 2x}{2} \end{array} \right.$

n ganjil $\rightarrow \int \underbrace{\sin^{n-1} x}_{\sin^2 x = 1-\cos^2 x} \cdot \underbrace{\sin x \cdot dx}_{\text{misal}}$

$u = \cos x$

$du = -\sin x\, dx$

n genap $\rightarrow \sin^n x = (\sin^2 x)^{\frac{n}{2}} = \left(\dfrac{1-\cos 2x}{2}\right)^{n/2}$

$\rightarrow \cos^n x = (\cos^2 x)^{n/2} = \left(\dfrac{1+\cos 2x}{2}\right)^{n/2}$

contoh:

genap

$$\int \sin^4 x\, dx = \int (\sin^2 x)^2\, dx$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \int \frac{1}{4} - \frac{\cos 2x}{2} + \frac{\cos^2 2x}{4}\, dx$$

$$= \int \frac{1}{4} - \frac{\cos 2x}{2} + \frac{1}{4}\left(\frac{1+\cos 4x}{2}\right) dx$$

$$= \int \frac{1}{4} - \frac{\cos 2x}{2} + \frac{1}{8} + \frac{1}{8}\cos 4x\, dx$$

$$= \frac{3}{8}x - \frac{1}{2}\,\frac{\sin 2x}{2} + \frac{1}{8}\cdot\frac{\sin 4x}{4} + C$$

$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

$$\frac{d}{dx}\sin Ax = A\cos Ax$$

$$\int \sin Ax\, dx = \frac{-\cos Ax}{A} + C$$

$$\int \cos^3 x\, dx = \int \underbrace{\cos^2 x}_{1-\sin^2 x} \cdot \cos x \cdot dx$$

$u = \sin x$

$du = \cos x\, dx$

$$= \int (1-\sin^2 x) \cos x \,.\, dx$$

Dengan $u = \sin x$ maka $du = \cos x \, dx$

diperoleh $\int (1-u^2)\, du$

$$= u - \frac{1}{3}u^3 + C$$

$$= \sin x - \frac{1}{3}\sin^3 x + C$$

* Bentuk $\int \sin^m x \cos^n x \, dx$

1) m ganjil

$$\int \sin^{\overset{\text{genap}}{m-1}} x \,.\, \cos^n x \,.\, \sin x \, dx$$

Misalkan $u = \cos x$

$du = -\sin x \, dx$

$$\left(\sin^2 x\right)^{\frac{m-1}{2}}$$

$$= \left(1-\cos^2 x\right)^{\frac{m-1}{2}}$$

2) n ganjil

$$\int \sin^m x \,.\, \cos^{\overset{\text{genap}}{n-1}} x \,.\, \cos x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$$\left(\cos^2 x\right)^{\frac{n-1}{2}}$$

$$= \left(1-\sin^2 x\right)^{\frac{n-1}{2}}$$

3) m & n genap

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

* Bentuk $\int \tan^n x\, dx$ dan $\int \cot^n x\, dx$

Turunan
$\cot x \to -\csc^2 x$
$\tan \to \sec^2 x$

$\hookrightarrow \tan^n x = \tan^{n-2} x \cdot \tan^2 x$

$= \tan^{n-2} x \cdot (\sec^2 x - 1)$

$= \underline{\tan^{n-2} x \cdot \sec^2 x} - \underbrace{\tan^{n-2} x}_{\text{direduksi}}$

$\hookrightarrow \cot^n x = \cot^{n-2} x \cdot \cot^2 x$

$= \cot^{n-2} x \cdot (\csc^2 x - 1)$

$= \underline{\cot^{n-2} x \cdot \csc^2 x} - \underbrace{\cot^{n-2} x}_{\text{direduksi}}$

Contoh :

$$\int \tan^2 x\, dx = \int \sec^2 x - 1\, dx$$
$$= \tan x - x + C$$

* Bentuk $\boxed{\int \tan^m x \cdot \sec^n x\, dx}$ dan $\boxed{\int \cot^m x \cdot \csc^n x\, dx}$

$\hookrightarrow$ n genap

$$\int \tan^m x \cdot \sec^n x\, dx = \int \underline{\tan^m x} \cdot \underbrace{\sec^{n-2} x}_{} \cdot \underbrace{\sec^2 x\, dx}_{du}$$

$(\sec^2 x)^{\frac{n-2}{2}}$

$= (1 + \tan^2 x)^{\frac{n-2}{2}}$

$u = \tan x$
$du = \sec^2 x\, dx$

$$\int \cot^m x \cdot \csc^n x\, dx = \int \cot^m x \cdot \underbrace{\csc^{n-2} x}_{} \cdot \underbrace{\csc^2 x\, dx}_{}$$

$(\csc^2 x)^{\frac{n-2}{2}} = (1 + \cot^2 x)^{\frac{n-2}{2}}$

$u = \cot x$
$du = -\csc^2 x\, dx$

↳ m ganjil

$$\int \tan^m x \cdot \sec^n x \, dx = \int \underbrace{\tan^{m-1} x}_{\tan^2 x = \sec^2 x - 1} \cdot \sec^{n-1} x \cdot \underbrace{\tan x \cdot \sec x \, dx}$$

$u = \sec x$

$du = \tan x \cdot \sec x \, dx$

$$\int \cot^m x \cdot \csc^n x \, dx = \int \underbrace{\cot^{m-1} x}_{\cot^2 x = \csc^2 x - 1} \cdot \underbrace{\csc^{n-1} x} \cdot \underbrace{\cot x \cdot \csc x \, dx}$$

$u = \csc x$

$du = -\csc x \cdot \cot x$

Contoh :

ganjil

$$\int \cot^3 x \cdot \csc^2 x \, dx = \int \underbrace{\cot^2 x}_{\csc^2 x - 1} \cdot \csc x \cdot \cot x \cdot \csc x \, dx$$

$u = \csc x$

$-du = \cot x \cdot \csc x \, dx$

$$= \int (\csc^2 x - 1) \csc x \cdot \cot x \cdot \csc x \, dx$$

Misal $u = \csc x$ maka $-du = \cot x \cdot \csc x \, dx$

diperoleh $\int (u^2 - 1) \, u \, (-du)$

$$= \int (1 - u^2) u \, du$$

$$= \int u - u^3 \, du$$

$$= \frac{1}{2} u^2 - \frac{1}{4} u^4 + C$$

$$= \frac{1}{2} \csc^2 x - \frac{1}{4} \csc^4 x + C$$

Contoh :

Jika $\int \tan^{13} x \, dx = A$, maka $\int \tan^{15} x \, dx$ adalah ....

Jawab :

$$\int \tan^{15} x = \int \tan^{13} x \cdot \tan^2 x \, dx$$

$$= \int \tan^{13} x \cdot (\sec^2 x - 1) \, dx$$

$$= \int \tan^{13} x \cdot \sec^2 x \, dx - \underbrace{\int \tan^{13} x \, dx}_{A}$$

$u = \tan x$

$du = \sec^2 x \, dx$

Misal $u = \tan x$ maka

$$\int \tan^{15} x = \int u^{13} \, du - A$$

$$= \frac{1}{14} u^{14} - A + C$$

↳ konstanta

$$= \frac{1}{14} \tan^{14} x - A + C$$

X menjadi +

contoh :

$$\int \sin 6x \cdot \cos 3x \, dx = \int \frac{1}{2} \left[\sin 9x + \sin 3x \right] dx$$

$$= \int \frac{1}{2} \sin 9x \, dx + \int \frac{1}{2} \sin 3x \, dx$$

$$= \frac{1}{2} \cdot \frac{(-\cos 9x)}{9} + \frac{1}{2} \frac{(-\cos 3x)}{3} + C$$

$$= \frac{-\cos 9x}{18} - \frac{\cos 3x}{6} + C$$

Integral Trigonometri

- A $\int \sin^n x \, dx$ / $\int \cos^n x \, dx$
  - n genap
  - n ganjil
- B $\int \sin^m x \cdot \cos^n x \, dx$
  - m ganjil
  - n ganjil
  - m & n genap
- C $\int \tan^n x \, dx$ / $\int \cot^n x \, dx$
- D $\int \tan^m x \cdot \sec^n x \, dx$ / $\int \cot^m x \cdot \csc^n x \, dx$
  - n genap
  - m ganjil
- E perkalian → penjumlahan