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(1) Let G be a group order pq, where p and q are distinct primes. Prove that G is abelian.

Ans: False

Explanation: The symmetric group S_3, which has order $6 = 2\times 3$. S_3 is not abelian as it contains non-commuting elements (12) & (123) permutations.

(2) Prove that if G is a group of order p^2 where p is prime, then G is abelian if and only if it has p+1 subgroups of order p.

Answer: ~~Flase~~ False

Explanation: if $|G| = p^2$ then G is abelian iff it has p+1 subgroups of order p.

All groups of order p^2 are abelian but there are two abelian types: C_{p^2} (has 1 subgroup of order p) and

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$C_p \times C_p$ (has $p+1$ subgroups) so,

abelian does not force $p+1$ subgroups:

(3) Let G be a finite group & H be a proper subgroup of G. Prove that the union of all ~~congu~~ conjugates of H can't be equal to G.

Ans: True

Ex: This is a standard result in Group theory. The union of all conjugates of a proper subgroup H is a proper subset of G. This can be shown using the formula for the number of conjugates & the fact that the intersection of conjugates has index at least 2, leading to a size contradiction if the union were equal to G.

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(4) Let G be a group & N be a normal subgroup of G. ~~If~~ G/N is cyclic & N is acyclic, prove that G is abelian.

Ans: False

Ex: Let N be the alternating subgroups A_3 which is acyclic of order 3. The G/N is cyclic order 2. How ever S_3 is not abelian, showing that the conditions do not guarantee G is abelian.

(5) Prove that in any group G, the set of elements of finite order forms a subgroup of G.

Answer: False.

Exp: In the infinite dihedral group D_∞. the element of order are the reflections. but the product of two distinct reflections. is ~~an~~ a translation. which has infinite order. Thus, the set of elements of finite order is not closed under multiplication & is not a subgroup.

(6) Let G be a finite group & p be the smallest prime diving $|G|$. Prove that any subgroup of index p in G is normal.

Answer: True

Ext. If H is a subgroup of index p in G & p is the smallest prime diving $|G|$ then H is normal.

This can be proven using the action of G on the cosets of H and considering the homo-morphism into the symmetric group S_p.