IT- 24631

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Assignment - 03

Prove that the set of rational number Q, equipped with the two binary operations of addition and multiplication, forms a field.

$\rightarrow$ we take the rational number Q to be the set of equivalence classes of ordered pair (a,b) with $a,b \in Z$ and $b \neq 0$ where $(a,b) \sim (a',b')$ iff $ab' = a'b$. we identify the class (a,b) with the usual fraction $\frac{a}{b}$. Define addition and multiplication in the usual way:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}, \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

for $b \neq 0$, $d \neq 0$. Below we show these operation makes Q a field.

1. The operation are well defined

we must check that if $\frac{a}{b} = \frac{a'}{b'}$ and $\frac{c}{d} = \frac{c'}{d'}$ then $\frac{ad+bc}{bd} = \frac{a'd'+b'c'}{b'd'}$

IT-24631

and $\frac{ac}{bd} = \frac{a'c'}{b'd'}$ from $\frac{a}{b} = \frac{a'}{b'}$

and $\frac{c}{d} = \frac{c'}{d'}$ we have $ab' = a'b$ and $cd' = c'd$. Compute $(ad+bc)b'd' = (ab')(dd') + (bc)(b'd') = (a'b)(dd') + (bc)(b'd')$

and similarly expand the right hand numerator times $bdb'd'$. Rearranging and using $ab' = a'b$, $cd' = c'd$ shows both cross products are equal, therefore the sums represent the same equivalence class. So addition and multiplication are well defined

2. $(Q, +)$ is an abelian group:

Take any $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in Q$

* closure: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ is a rational number since $bd \neq 0$

* associativity: follows from associativity

IT- 24631

of ~~integer~~ integer addition:

$$\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f}=\frac{ad+bc}{bd}+\frac{e}{f}$$

$$=\frac{(ad+bc)f+e(bd)}{(bd)f}$$

and a similar expansion for $\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)$ both give the same numeration by associativity / commutativity of integer operations.

* Identity: $0=\frac{0}{1}$ satisfies $\frac{a}{b}+0=\frac{a}{b}$

* Inverse: additive inverse of $\frac{a}{b}$ is $-\frac{a}{b}$ $=\frac{-a}{b}$ because $\frac{a}{b}+\frac{-a}{b}=\frac{0}{b}=0$

* Commutativity: $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}=\frac{bc+ad}{db}$ $=\frac{c}{d}+\frac{a}{b}$

3. Multiplication on $Q/\{0\}$ is an abelian group

* Closure: product $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$ is rational

IT- 24631

Since $bd \neq 0$

✓ Associativity and commutativity: follow from associativity and commutativity integer mul^n.

$$\left(\frac{a}{b}\cdot\frac{c}{d}\right)\cdot\frac{e}{f} = \frac{ac}{bd}\cdot\frac{e}{f} = \frac{(ac)e}{(bd)f}$$

$$= \frac{a(ce)}{b(df)} = \frac{a}{b}\cdot\frac{ce}{df} = \frac{a}{b}\cdot\left(\frac{c}{d}\cdot\frac{e}{f}\right)$$

✓ Multiplicative identity: $1 = \frac{1}{1}$ satisfy

$$\frac{a}{b}\cdot 1 = \frac{a}{b}$$

✓ Distributivity: for addition and mul^n

$$\frac{a}{b}\cdot\left(\frac{c}{d}+\frac{e}{f}\right) = \frac{a}{b}\cdot\frac{cf+ed}{df} = \frac{a(cf+ed)}{bdf}$$

$$= \frac{acf+aed}{bdf} = \frac{ac}{bd}+\frac{ae}{bf} = \frac{a}{b}\cdot\frac{c}{d}+$$

$\frac{a}{b}\cdot\frac{e}{f}$ using distributivity.

So, Q is a commutative ring with unity 1.