Quantitative Risk Management
Assignment 6 Solutions

Question 1. We recall that a (second–order) covariance stationary process requires: (i) $\mathbb{E}[X_t] = \mu$ constant in t, (ii) $\mathrm{Var}(X_t) = \gamma(0)$ constant in t, and (iii) $\mathrm{Cov}(X_t, X_{t+h}) = \gamma(h)$ depends only on the lag h (not on t).

(a) Process with parity switch.

$$X_t = \begin{cases} \epsilon_t, & t \text{ odd}, \\ \epsilon_{t+1}, & t \text{ even}, \end{cases} \qquad (\epsilon_t) \sim WN(0, \sigma^2).$$

Mean and variance are constant:

$$\mathbb{E}[X_t] = 0, \qquad \mathrm{Var}(X_t) = \sigma^2 \quad \text{for all } t.$$

However, the lag-1 covariance depends on the *time index*:

$$\mathrm{Cov}(X_t, X_{t+1}) = \begin{cases} \sigma^2, & t \text{ even (since } X_t = \epsilon_{t+1},\ X_{t+1} = \epsilon_{t+1}), \\ 0, & t \text{ odd (since } X_t = \epsilon_t,\ X_{t+1} = \epsilon_{t+2}), \end{cases}$$

so it is not a function of the lag only. (For $|h| \geq 2$ one gets 0.) Hence the autocovariance is not time-invariant $\Rightarrow$ the process is not covariance stationary.

(b) Partial-sum (random walk) process.

$$X_t = \sum_{j=1}^{t} \epsilon_j, \qquad (\epsilon_t) \sim WN(0, \sigma^2).$$

We have $\mathbb{E}[X_t] = 0$ and

$$\mathrm{Var}(X_t) = \mathrm{Var}\left(\sum_{j=1}^{t} \epsilon_j\right) = \sum_{j=1}^{t} \mathrm{Var}(\epsilon_j) = t\,\sigma^2,$$

which depends on t and diverges with t. (More generally, $\mathrm{Cov}(X_t, X_s) = \min(t, s)\,\sigma^2$ depends on the calendar times, not just on $|t - s|$.) Therefore the process is not covariance stationary.

Question 2. Consider

$$X_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}, \qquad (\epsilon_t) \sim WN(0, \sigma^2).$$

Causality and covariance stationarity. This process can be written as a finite weighted sum of current and past shocks:

$$X_t = 1 \cdot \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}.$$

That means X_t only depends on the current innovation ϵ_t and a few past ones $(\epsilon_{t-1}, \epsilon_{t-2})$, but never on future shocks. Therefore, the process is called causal — its present value can be generated recursively from past information only.
Because it is an MA(2) process (a finite linear combination of white noise), its mean and variance are constant over time, and the covariance depends only on the lag between observations. Hence it is covariance stationary.

First and second moments.

$$\mathbb{E}[X_t] = 0, \qquad \mathrm{Var}(X_t) = \gamma(0) = \sigma^2 \sum_{i=0}^{2} \theta_i^2 = \sigma^2\,(1 + \theta_1^2 + \theta_2^2).$$

Autocovariance function. Using the general MA formula (with $\theta_0 = 1$):

$$\gamma(h) = \mathrm{Cov}(X_t, X_{t-h}) = \sigma^2 \sum_{i=0}^{2-|h|} \theta_i \theta_{i+|h|}, \quad |h| \le 2,$$

and $\gamma(h) = 0$ for $|h| \ge 3$. Explicitly,

$$\gamma(1) = \sigma^2(\theta_1 + \theta_1\theta_2), \qquad \gamma(2) = \sigma^2\theta_2,$$

and by symmetry $\gamma(-h) = \gamma(h)$.
Autocorrelation function. With $\gamma(0) = \sigma^2(1 + \theta_1^2 + \theta_2^2)$:

$$\rho(h) = \begin{cases} \dfrac{\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2}, & |h| = 1, \\ \dfrac{\theta_2}{1 + \theta_1^2 + \theta_2^2}, & |h| = 2, \\ 0, & |h| \ge 3. \end{cases}$$

Question 3. Let

$$X_t = \sum_{i=0}^{\infty} \psi_i\, \epsilon_{t-i}, \qquad (\epsilon_t) \sim WN(0, \sigma_\epsilon^2), \qquad \sum_{i=0}^{\infty} |\psi_i| < \infty.$$

Absolute summability implies square summability, so $\sum_i \psi_i^2 < \infty$ and all second moments below are finite. Moreover, absolute convergence justifies interchanging sums (Fubini/Tonelli).

Variance. Since the white-noise terms are uncorrelated across time,

$$\gamma(0) := \mathrm{Var}(X_t) = \mathbb{E}[X_t^2] = \mathbb{E}\left[\Big(\sum_{i\ge 0} \psi_i \epsilon_{t-i}\Big)^2\right] = \sigma_\epsilon^2 \sum_{i=0}^{\infty} \psi_i^2.$$

Autocovariance. For $h \ge 0$,

$$\gamma(h) := \mathrm{Cov}(X_t, X_{t-h}) = \mathbb{E}[X_t X_{t-h}] = \mathbb{E}\left[\left(\sum_{i\ge 0} \psi_i \epsilon_{t-i}\right)\left(\sum_{j\ge 0} \psi_j \epsilon_{t-h-j}\right)\right].$$

Using $\mathbb{E}[\epsilon_{t-i}\epsilon_{t-h-j}] = 0$ unless $t - i = t - h - j$ (i.e., $j = i - h$), and $\mathbb{E}[\epsilon_{t-i}^2] = \sigma_\epsilon^2$, only the matching index pairs contribute:

$$\gamma(h) = \sigma_\epsilon^2 \sum_{i=h}^{\infty} \psi_i\, \psi_{i-h} = \sigma_\epsilon^2 \sum_{k=0}^{\infty} \psi_{k+h}\, \psi_k, \qquad h \ge 0,$$

where we reindexed with $k = i - h$. By symmetry of covariance, $\gamma(-h) = \gamma(h)$.

Autocorrelation. With $\gamma(0) = \sigma_\epsilon^2 \sum_{i\ge 0} \psi_i^2$, the ACF is

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \frac{\sum_{k=0}^{\infty} \psi_k\, \psi_{k+|h|}}{\sum_{i=0}^{\infty} \psi_i^2}, \qquad h \in \mathbb{Z}.$$

Because $\gamma(-h) = \gamma(h)$ (covariance symmetry), the ACF depends only on the *absolute value* of the lag.

Remarks. (i) The formula shows covariance stationarity: $\gamma(h)$ depends only on the lag h.
(ii) For an MA(q), $\psi_i = 0$ for $i > q$, so $\rho(h)$ cuts off at $|h| > q$.

Question 4. The $ARMA(1,1)$ process

$$X_t - \phi X_{t-1} = \epsilon_t + \theta\epsilon_{t-1}, \qquad (\epsilon_t) \sim WN(0, \sigma_\epsilon^2),$$

is causal if $|\phi| < 1$ and $\phi \neq -\theta$. Then it admits the infinite moving–average representation

$$X_t = \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i}, \qquad \text{with } \sum_{i=0}^{\infty} |\psi_i| < \infty.$$

The coefficients satisfy

$$\sum_{i=0}^{\infty} \psi_i z^i = \frac{1+\theta z}{1-\phi z}, \qquad |z| < 1.$$

Step 1: Expand as a power series. Since $(1-\phi z)^{-1} = \sum_{k=0}^{\infty} \phi^k z^k$ for $|z| < 1$, we have

$$\frac{1+\theta z}{1-\phi z} = (1+\theta z)\sum_{k=0}^{\infty} \phi^k z^k = \sum_{k=0}^{\infty} \phi^k z^k + \theta \sum_{k=0}^{\infty} \phi^k z^{k+1}.$$

Collecting powers of z gives

$$\psi_0 = 1, \qquad \psi_i = \phi^{i-1}(\phi+\theta), \quad i \geq 1.$$

Hence the infinite–MA representation is

$$X_t = \epsilon_t + (\phi+\theta)\epsilon_{t-1} + (\phi+\theta)\phi\,\epsilon_{t-2} + (\phi+\theta)\phi^2\epsilon_{t-3} + \cdots$$

which shows how past shocks contribute with geometrically decaying weights.

Step 2: Compute the autocorrelation function. For any causal $ARMA$ process,

$$\rho(h) = \frac{\sum_{i=0}^{\infty} \psi_i \psi_{i+|h|}}{\sum_{i=0}^{\infty} \psi_i^2}.$$

We compute numerator and denominator separately.

Denominator (variance term).

$$D = \sum_{i=0}^{\infty} \psi_i^2 = 1 + (\phi+\theta)^2 \sum_{i=1}^{\infty} \phi^{2(i-1)} = 1 + \frac{(\phi+\theta)^2}{1-\phi^2}.$$

Numerator (covariance at lag h). For $h \geq 1$,

$$\sum_{i=0}^{\infty} \psi_i \psi_{i+h} = \psi_0 \psi_h + \sum_{i=1}^{\infty} \psi_i \psi_{i+h}.$$

Using $\psi_0 = 1$ and $\psi_i = \phi^{i-1}(\phi+\theta)$ for $i \geq 1$:
- The first term is $\psi_0\psi_h = \phi^{h-1}(\phi+\theta)$.
- The remaining sum is

$$\sum_{i=1}^{\infty} \psi_i \psi_{i+h} = (\phi+\theta)^2 \sum_{i=1}^{\infty} \phi^{2i+h-2} = (\phi+\theta)^2 \phi^h \sum_{i=1}^{\infty} \phi^{2(i-1)} = (\phi+\theta)^2 \phi^h \frac{1}{1-\phi^2}.$$

Thus

$$N(h) = \sum_{i=0}^{\infty} \psi_i \psi_{i+h} = \phi^{h-1}(\phi+\theta) + (\phi+\theta)^2 \phi^h \frac{1}{1-\phi^2}.$$

Step 3: Simplify the fraction. We can factor $N(h)$ as

$$N(h) = \phi^{h-1}(\phi+\theta)\left[1 + \frac{(\phi+\theta)\phi}{1-\phi^2}\right] = \phi^{h-1}(\phi+\theta)\frac{1-\phi^2+\phi(\phi+\theta)}{1-\phi^2} = \phi^{h-1}(\phi+\theta)\frac{1+\phi\theta}{1-\phi^2}.$$

Hence

$$\rho(h) = \frac{N(h)}{D} = \frac{\phi^{|h|-1}(\phi+\theta)\dfrac{1+\phi\theta}{1-\phi^2}}{1+\dfrac{(\phi+\theta)^2}{1-\phi^2}} = \frac{\phi^{|h|-1}(\phi+\theta)(1+\phi\theta)}{(1-\phi^2)+(\phi+\theta)^2}.$$

Since $(1-\phi^2)+(\phi+\theta)^2 = 1+\theta^2+2\phi\theta$, the final closed-form expression is

$$\boxed{\rho(h) = \frac{\phi^{|h|-1}(\phi+\theta)(1+\phi\theta)}{1+\theta^2+2\phi\theta}\,, \quad h \neq 0; \qquad \rho(0) = 1.}$$

Step 4: Interpretation.

- $\rho(h)$ depends on $\phi^{|h|-1}$, so correlations decay geometrically at rate $|\phi|$:

  $$\frac{|\rho(h+1)|}{|\rho(h)|} = |\phi|.$$

  Thus the autoregressive parameter controls the persistence of dependence over time.
- When $\theta = 0$, the process reduces to $AR(1)$, giving $\rho(h) = \phi^{|h|}$.
- When $\phi = 0$, it reduces to $MA(1)$, giving $\rho(1) = \theta/(1+\theta^2)$ and $\rho(h) = 0$ for $h > 1$.