Quantitative Risk Management
Assignment 2 Solutions

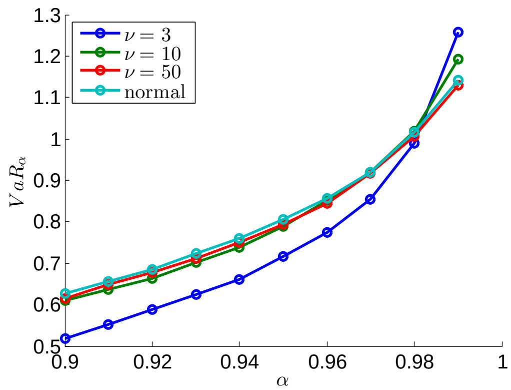
Question 1. For a holding period of one day, the Gaussian VaR is approximately $\sqrt{0.12} \cdot \Phi^{-1}(0.99)/\sqrt{250} =$ 5.1% (assuming de-annualization), while the method of historical simulation yields 5.427%.

Question 2. One computes the losses for each scenario as

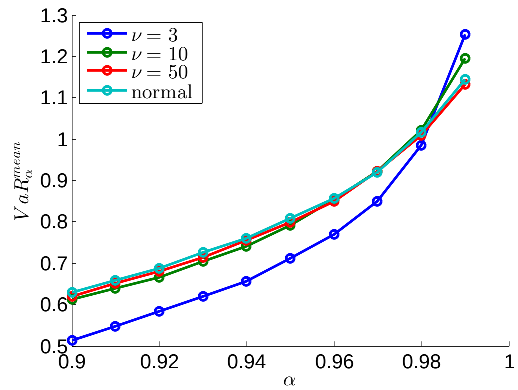
$$L = -400 r_A - 600 r_B.$$

The Value at Risk is then 12, while the ES (using the formula on the lecture slide 'Expected Shortfall for a Discrete Distribution') is 23.84.

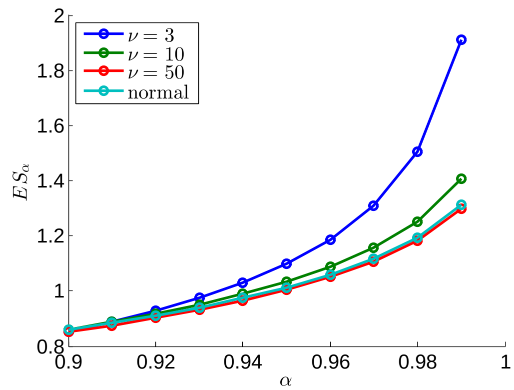
Question 3. The values of VaR_α, VaR_α^{mean} and ES_α are plotted with respect to α below for each distribution of interest.



(a) VaR_α



(b) VaR_α^{mean}



(c) ES_α

Figure 1: Risk versus confidence level for various distributions.

To compare the resulting values of VaR_α and VaR_α^{mean}, we plot the difference below:

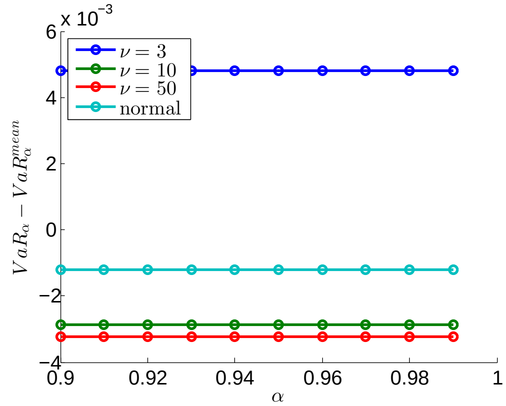


Figure 2: Difference between VaR_α and VaR_α^{mean}.

The difference between VaR_α and VaR_α^{mean} is constant with respect to α for each distribution because the same random variables were used across all values of α. Note that the difference in each case is close to zero. This is because all of our risk factors have a mean of zero and a small variance, so the expected losses in each case are very close to zero (but not identically zero because the loss is not linear with respect to risk factor changes).

In Figure 1a) above, the value of VaR_α is shown for each distribution. The results may be considered surprising because the t-distribution with 3 degrees of freedom yields the lowest value of VaR_α for most confidence levels, whereas the normal yields the highest. Not until $\alpha = 0.99$ does the typical "fat tail" behaviour of the t-distribution take over and the expected ordering is seen. This is because VaR is only a frequency-based risk measure.

In Figure 1c) above, we see an expected behavior of ES_α for each distribution: the t-distribution with $\nu = 3$ has the largest value of ES_α for each confidence level, as expected from its "fat tail" characteristic. This is because ES is both a frequency- and severity-based risk measure.

Question 4. There are of course an infinite number of counterexamples to the subadditivity of VaR_α. We consider the following one. Let L_1 and L_2 be independent Bernoulli random variables with success parameter $1 - \alpha - \epsilon$ where $0 \leq \epsilon < 1 - \alpha$. Then

$$\begin{aligned}\mathbb{P}(L_i = 0) &= \alpha + \epsilon \\ \mathbb{P}(L_i = 1) &= 1 - \alpha - \epsilon\end{aligned}$$

Clearly, $VaR_\alpha(L_i) = 0$. The distribution of $L = L_1 + L_2$ is Binomial with parameters $n = 2$ and $p = 1 - \alpha - \epsilon$. This random variable takes on three possible values with the following probabilities:

$$\begin{aligned}\mathbb{P}(L = 0) &= (\alpha + \epsilon)^2 \\ \mathbb{P}(L = 1) &= 2(\alpha + \epsilon)(1 - \alpha - \epsilon) \\ \mathbb{P}(L = 2) &= (1 - \alpha - \epsilon)^2\end{aligned}$$

If ϵ is chosen such that $(\alpha + \epsilon)^2 < \alpha$, then $VaR_\alpha(L) > 0 = VaR_\alpha(L_1) + VaR_\alpha(L_2)$, showing that VaR_α is not subadditive.