Quantitative Risk Management
Assignment 3 Solutions

Question 1: The value of the portfolio at time 0 is equal to:

$$V_0 = C^{BS}(0, T, S_0, K, r, \sigma_0) + \lambda S_0$$

where the position taken in the stock is equal to:

$$\lambda = -\frac{\partial C^{BS}}{\partial S}(0, T, S_0, K, r, \sigma_0)$$

The position in the stock is held constant over the time period Δ, independent of changes in S. With this in mind note that a derivative of V_0 with respect to S_0 is equal to zero, hence the portfolio is delta neutral. The value of the portfolio at time Δ is:

$$V_\Delta = C^{BS}(\Delta, T, S_0 e^{X_{1,\Delta}}, K, r, \sigma_0 + X_{2,\Delta}) + \lambda S_0 e^{X_{1,\Delta}}$$

where the vector $(X_{1,\Delta}, X_{2,\Delta})$ has the distribution indicated in the question. Thus the loss is equal to:

$$L_\Delta = -C^{BS}(\Delta, T, S_0 e^{X_{1,\Delta}}, K, r, \sigma_0 + X_{2,\Delta}) + C^{BS}(0, T, S_0, K, r, \sigma_0) - \lambda S_0(e^{X_{1,\Delta}} - 1)$$

The general formula for the linearized loss is:

$$L^\delta_\Delta = -\left(\partial_t f(t, \mathbf{Z}_t)\Delta + \sum_{i=1}^{d} \partial_{Z_i} f(t, \mathbf{Z}_t) X_{i,t+\Delta}\right)$$

which in this specific case becomes:

$$L^\delta_\Delta = -\left(\frac{\partial C^{BS}}{\partial t}(0, T, S_0, K, r, \sigma_0)\Delta + \frac{\partial C^{BS}}{\partial S}(0, T, S_0, K, r, \sigma_0) S_0 X_{1,\Delta} + \lambda S_0 X_{1,\Delta} + \frac{\partial C^{BS}}{\partial \sigma}(0, T, S_0, K, r, \sigma_0) X_{2,\Delta}\right)$$

Since $\lambda = -\frac{\partial C^{BS}}{\partial S}(0, T, S_0, K, r, \sigma_0)$, the dependence on $X_{1,\Delta}$ vanishes:

$$L^\delta_\Delta = -\frac{\partial C^{BS}}{\partial t}(0, T, S_0, K, r, \sigma_0)\Delta - \frac{\partial C^{BS}}{\partial \sigma}(0, T, S_0, K, r, \sigma_0) X_{2,\Delta}$$

To use the variance-covariace method to compute VaR_α, VaR^{mean}_α, and ES_α, we require the mean and variance of L^δ_Δ. These are clearly given by:

$$\mathbb{E}[L^\delta_\Delta] = -\frac{\partial C^{BS}}{\partial t}(0, T, S_0, K, r, \sigma_0)\Delta$$
$$\mathbb{V}[L^\delta_\Delta] = \left(\frac{\partial C^{BS}}{\partial \sigma}(0, T, S_0, K, r, \sigma_0)\right)^2 \mathbb{V}[X_{2,\Delta}]$$

Thus, the following expressions hold for the variance-covariance method:

$$VaR_\alpha = \mathbb{E}[L^\delta_\Delta] + \sqrt{\mathbb{V}[L^\delta_\Delta]}\Phi^{-1}(\alpha)$$
$$VaR^{mean}_\alpha = \sqrt{\mathbb{V}[L^\delta_\Delta]}\Phi^{-1}(\alpha)$$
$$ES_\alpha = \mathbb{E}[L^\delta_\Delta] + \sqrt{\mathbb{V}[L^\delta_\Delta]}\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$$

	$\alpha = 0.95$			$\alpha = 0.99$		
	VaR_α	VaR_α^{mean}	ES_α	VaR_α	VaR_α^{mean}	ES_α
Monte-Carlo	0.0364	0.0044	0.0376	0.0383	0.0063	0.0392
Linearized Monte-Carlo	0.0366	0.0044	0.0377	0.0385	0.0063	0.0395
Variance-Covariace	0.0367	0.0045	0.0378	0.0386	0.0064	0.0395

The table above summarizes the numerical results for each method.

Question 2: Let L have the Student t distribution with ν degrees of freedom. The probability density function of L is:

$$g_\nu(x) = K\left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

where K is a constant which normalizes the integral of $g_\nu(x)$. Let $t_\nu(x)$ be the cumulative distribution function of L, which is continuous and strictly increasing. Assume $\nu > 1$ such that $ES_\alpha(L)$ exists. Then:

$$\begin{aligned}
ES_\alpha(L) &= \mathbb{E}[L|L > VaR_\alpha] \\
&= \frac{1}{1-\alpha}\int_{VaR_\alpha}^{\infty} x g_\nu(x)dx \\
&= \frac{K}{1-\alpha}\int_{t_\nu^{-1}(\alpha)}^{\infty} x\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} dx \\
&= \frac{K}{2(1-\alpha)}\int_{t_\nu^{-1}(\alpha)^2}^{\infty} \left(1+\frac{u}{\nu}\right)^{-\frac{\nu+1}{2}} du \\
&= \frac{K}{2(1-\alpha)}\cdot\frac{2\nu}{1-\nu}\left(1+\frac{u}{\nu}\right)^{-\frac{\nu-1}{2}}\Bigg|_{u=t_\nu^{-1}(\alpha)^2}^{\infty} \\
&= \frac{K}{1-\alpha}\frac{\nu}{\nu-1}\left(1+\frac{t_\nu^{-1}(\alpha)^2}{\nu}\right)^{-\frac{\nu-1}{2}} \\
&= \frac{K}{1-\alpha}\frac{\nu}{\nu-1}\left(1+\frac{t_\nu^{-1}(\alpha)^2}{\nu}\right)^{-\frac{\nu+1}{2}}\left(1+\frac{t_\nu^{-1}(\alpha)^2}{\nu}\right) \\
&= \left(\frac{g_\nu(t_\nu^{-1}(\alpha))}{1-\alpha}\right)\left(\frac{\nu + t_\nu^{-1}(\alpha)^2}{\nu-1}\right)
\end{aligned}$$

Question 3: The values of $VaR_{0.95}$ and $VaR_{0,99}$ are plotted in Figure 1 along with the losses that were realized on those particular days. The days on which VaR_α was breached are indicated with a marker.

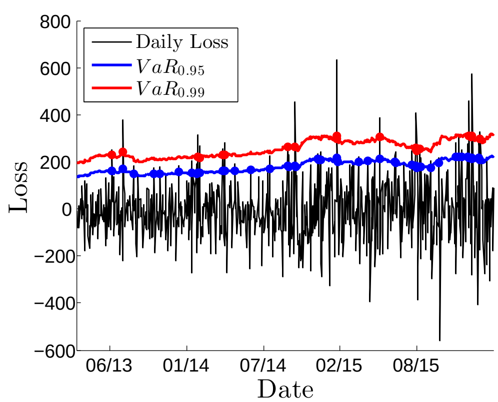

Figure 1: Daily VaR_α estimates and losses from March 11, 2013 to March 10, 2016.

The total number of days on which VaR_α is computed and then corresponding losses observed is 751. Of those days, $VaR_{0.95}$ was breached 48 times and $VaR_{0.99}$ was breached 18 times. If the methods are accurate, the probabilities of observing this many or more breaches in each case are given by:

$$\begin{aligned} 1 - Bino^{-1}(47, 751, 0.05) &= 0.05185 \\ &= 5.185\% \\ 1 - Bino^{-1}(17, 751, 0.01) &= 7.5 \cdot 10^{-4} \\ &= 0.075\% \end{aligned}$$

The probability corresponding to $\alpha = 0.95$ is small, whereas not completely unreasonable to observe, but the result for $\alpha = 0.99$ shows that we saw an extremely unlikely large number of breaches. It is likely that this method of computing VaR should be supplemented with other methods. A simple visual analysis of the stock prices over the five year period of consideration shows much more volatility in the last two years than in the time leading up to it.

Question 4: Part 1) The geometric distribution with 0 included in the support has probability mass function $\mathbb{P}(L = k) = (1-p)^k p$. For $p = 0.5$, a table of values of the cumulative distribution function is given by:

$\mathbb{P}(L \le 0)$	0.5
$\mathbb{P}(L \le 1)$	0.75
$\mathbb{P}(L \le 2)$	0.875
$\mathbb{P}(L \le 3)$	0.935
$\mathbb{P}(L \le 4)$	0.96875

We see that the CDF changes from less than 0.95 to greater than 0.95 as the index k changes from 3 to 4. This means that $VaR_{0.95} = 4$ as can be checked from the definition of VaR_α:

$$VaR_{0.95} = \inf\{x \in \mathbb{R} : F_L(x) \ge 0.95\} = 4$$

Figure 2 plots VaR_α as a function of α for this particular distribution. Note that it has discontinuities due to the discreteness of the random variable L.

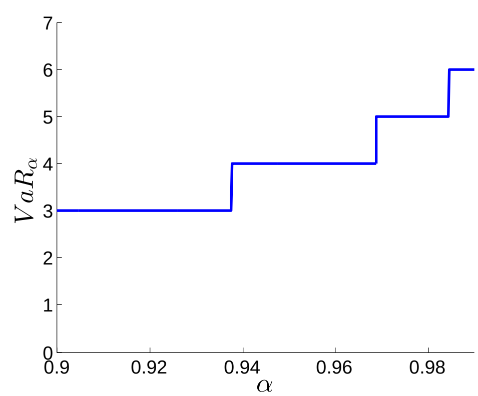

Figure 2: VaR_α as a function of α for the geometric distribution with $p = 0.5$.

Part 2) If X and Y are independent Poisson with parameters λ_X and λ_Y, then $L = X + Y$ is Poisson with parameter $\lambda_Z = \lambda_X + \lambda_Y$. Values of VaR_α for each random variable can be found via the CDF as in the previous part of this question. The resulting values are shown in Figure 3.

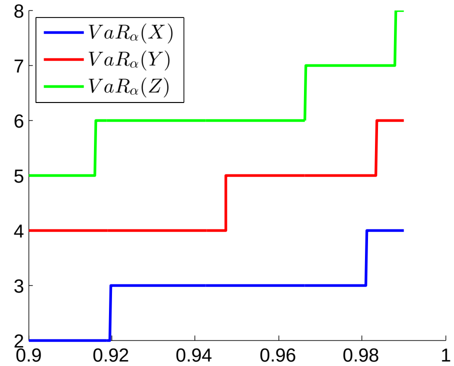

Figure 3: VaR_α as a function of α for the Poisson distributed random variables.