Quantitative Risk Management
Assignment 8

Question 1: Consider a bivariate random vector (X, Y) with a joint distribution given by:

$$F_{X,Y}(x, y) = \frac{1}{1 + e^{-x} + e^{-y}}.$$

1. Compute the marginal distributions of X and Y.
2. Find an expression for the copula $C(u, v)$ of the two random variables.
3. Check that it is a valid copula.

Question 2: Recall that $\lambda_l(X_1, X_2) = \lim_{q\to 0^+} \frac{C(q,q)}{q}$. Find the corresponding expression for $\lambda_u(X_1, X_2)$.

Question 3: Compute $\lambda_u(X_1, X_2)$ and $\lambda_l(X_1, X_2)$ in terms of θ for the Gumbel and Clayton copulas.

Question 4: Download the spreadsheet posted on the course website. This is Fama-French data on international portfolios from `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/`
Take the first two columns of data. Transform this data to get a pseudo-sample of the copula by assuming the marginal distributions are the empirical distributions of this data. Show a scatter plot of the pseudo-sample and perform a maximum likelihood estimation for the Gumbel, Clayton, and Frank copulas on the pseudo-sample. Based on the maximum-likelihood fits, which copula family appears to provide the best description of the dependence structure in the data?
Hint: You can use the Matlab functions `copulapdf` and `fmincon`.

Question 5: Let $X \sim \mathcal{N}(0, 1)$ and define $Y = ZX$, where Z is independent of X and

$$Z = \begin{cases} 1 & \text{with probability } p, \\ -1 & \text{with probability } 1 - p. \end{cases}$$

Determine the copula of the bivariate vector (X, Y).

Question 6: Let G be a distribution function on $\mathbb{R}^+$ with $G(0) = 0$, and let

$$\hat{G}(t) := \int_0^\infty e^{-tv}\, dG(v), \qquad t \geq 0,$$

be its Laplace–Stieltjes transform (with the convention $\hat{G}(\infty) = 0$). Let $V \sim G$, and let $U_1, \ldots, U_d$ be conditionally independent given V, with

$$F_{U_i|V}(u_i; v) = \mathbb{P}(U_i \leq u_i \mid V = v) = \exp\bigl(-v\, \hat{G}^{-1}(u_i)\bigr), \qquad u_i \in [0, 1].$$

Show that the joint distribution of $(U_1, \ldots, U_d)$ is

$$\mathbb{P}(U_1 \leq u_1, \ldots, U_d \leq u_d) = \hat{G}\Bigl(\hat{G}^{-1}(u_1) + \cdots + \hat{G}^{-1}(u_d)\Bigr).$$