Quantitative Risk Management
Assignment 10 Solutions

Question 1:

Our goal is to determine whether the logistic distribution lies in the maximum domain of attraction of some GEV distribution H_ξ, and to identify the corresponding normalizing sequences $a_n > 0$ and b_n.
A standard approach is to begin by analyzing the tail of the distribution, since the tail behaviour determines the shape parameter ξ in the GEV limit.

Step 1: Study the upper tail of the logistic distribution.
The logistic CDF is

$$F_X(x) = \frac{1}{1+\exp(-(x-\mu)/\sigma)}.$$

Thus the survival function (tail probability) is

$$1 - F_X(x) = \frac{1}{1+\exp((x-\mu)/\sigma)}.$$

We will use the asymptotic notation

$$f(x) \sim g(x) \quad (x \to \infty) \quad \iff \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

This means: *"f and g are asymptotically equal; they behave the same for large x."*
Now note that as $x \to \infty$,

$$e^{(x-\mu)/\sigma} \to \infty, \qquad \text{so } \ 1 + e^{(x-\mu)/\sigma} \sim e^{(x-\mu)/\sigma}.$$

Therefore:

$$1 - F_X(x) = \frac{1}{1+e^{(x-\mu)/\sigma}} \sim \frac{1}{e^{(x-\mu)/\sigma}} = e^{-(x-\mu)/\sigma}.$$

This shows that the logistic tail is *exponential*, which identifies the distribution as belonging to the *Gumbel* maximum domain of attraction:

$$F_X \in \mathrm{MDA}(H_0).$$

Step 2: Find the centering sequence b_n.
In extreme value theory, the maximum

$$M_n = \max(X_1, \ldots, X_n)$$

typically lies near the level b_n that satisfies

$$1 - F_X(b_n) \sim \frac{1}{n}.$$

This corresponds to choosing the level that is exceeded with probability $1/n$, so that among n observations we expect about one exceedance. This ensures that $F_X(b_n)^n$ converges to a non-degenerate limit.
Using the exact expression for the tail,

$$1 - F_X(x) = \frac{1}{1+\exp((x-\mu)/\sigma)},$$

we solve

$$1 - F_X(b_n) = \frac{1}{n}.$$

Thus

$$\frac{1}{1 + \exp((b_n - \mu)/\sigma)} = \frac{1}{n},$$

which gives

$$1 + \exp((b_n - \mu)/\sigma) = n, \qquad \exp((b_n - \mu)/\sigma) = n - 1.$$

Therefore,

$$b_n = \mu + \sigma \log(n-1) \sim \mu + \sigma \log n.$$

Step 3: Find the scaling sequence a_n.
We now evaluate the tail at $b_n + a_n x$:

$$1 - F_X(b_n + a_n x) = \frac{1}{1 + \exp((b_n + a_n x - \mu)/\sigma)}.$$

Since $b_n - \mu = \sigma \log n$, this becomes

$$1 - F_X(b_n + a_n x) = \frac{1}{1 + n\,\exp(a_n x/\sigma)}.$$

Rewrite this as

$$1 - F_X(b_n + a_n x) = \frac{1}{n} \cdot \frac{1}{\exp(a_n x/\sigma) + 1/n}.$$

As $n \to \infty$,

$$\exp(a_n x/\sigma) \gg 1/n,$$

so

$$1 - F_X(b_n + a_n x) \sim \frac{1}{n}\,\exp\Big(-\frac{a_n}{\sigma}x\Big).$$

Thus

$$F_X(b_n + a_n x) = 1 - \frac{1}{n}e^{-(a_n/\sigma)x} + o(1/n).$$

Then

$$F_X(b_n + a_n x)^n \;\longrightarrow\; \exp\Big(-e^{-(a_n/\sigma)x}\Big).$$

To match the Gumbel limit $H_0(x) = \exp(-e^{-x})$, we must have

$$\frac{a_n}{\sigma} = 1, \quad \Longrightarrow \quad a_n = \sigma.$$

Final result:
With

$$a_n = \sigma, \qquad b_n = \mu + \sigma \log n,$$

we obtain

$$F_X^n(b_n + a_n x) \;\longrightarrow\; \exp\big(-e^{-x}\big) = H_0(x),$$

so the logistic distribution is in the Gumbel maximum domain of attraction.

$$\boxed{F_X \in \mathrm{MDA}(H_0), \qquad a_n = \sigma,\;\; b_n = \mu + \sigma \log n.}$$

Question 2:

a) Since F_X is differentiable, X has a density. Differentiating,

$$f_X(x) = \frac{d}{dx}\left(\frac{1}{1 + e^{-x}}\right) = \frac{e^{-x}}{(1 + e^{-x})^2}.$$

b) For large x,

$$1 - F_X(x) = \frac{1}{1+e^x} \sim e^{-x}, \qquad F_X(-x) \sim e^{-x},$$

so the logistic distribution has exponentially decaying tails on both sides.

Since for every $k \in \mathbb{N}$,

$$\int_0^\infty x^k e^{-x}\, dx = \Gamma(k+1) = k! < \infty,$$

any exponential tail dominates any polynomial growth. Hence all moments $\mathbb{E}|X|^k$ are finite.

Alternative argument: Once part (c) is established, we know that $F_X \in \mathrm{MDA}(H_0)$, the Gumbel domain. A standard result (seen in the lecture) is that every distribution in the Gumbel domain has finite moments of all orders. Thus, the conclusion $\mathbb{E}|X|^k < \infty$ for all $k \in \mathbb{N}$ also follows directly from the domain-of-attraction classification.

c) We first examine the upper tail:

$$1 - F_X(x) = \frac{1}{1+e^x} \sim e^{-x} \qquad (x \to \infty).$$

This shows that the tail is *exponential*, hence the distribution is in the Gumbel domain of attraction:

$$F_X \in \mathrm{MDA}(H_0).$$

To determine normalizing sequences $a_n > 0$ and b_n, we use the standard rule:

$$1 - F_X(b_n) \sim \frac{1}{n}.$$

Solving

$$\frac{1}{1+e^{b_n}} = \frac{1}{n} \quad \Longrightarrow \quad e^{b_n} = n-1,$$

gives

$$b_n = \log(n-1) \sim \log n.$$

Next, evaluate the tail at $b_n + a_n x$:

$$1 - F_X(b_n + a_n x) = \frac{1}{1 + e^{\,b_n + a_n x}} = \frac{1}{1 + n\, e^{a_n x}}.$$

As $n \to \infty$,

$$1 - F_X(b_n + a_n x) \sim \frac{1}{n}\, e^{-a_n x}.$$

Thus

$$F_X(b_n + a_n x) = 1 - \frac{1}{n} e^{-a_n x} + o(1/n),$$

so

$$F_X(b_n + a_n x)^n \to \exp\big(-e^{-a_n x}\big).$$

To match the Gumbel limit $H_0(x) = \exp(-e^{-x})$, we require

$$a_n = 1.$$

Hence the normalizing sequences are:

$$a_n = 1, \qquad b_n = \log n.$$

d) For $x \geq 0$,

$$F_u(x) = \mathbb{P}(X - u \leq x \mid X > u) = \frac{F_X(u+x) - F_X(u)}{1 - F_X(u)}.$$

Using $F_X(t) = 1/(1+e^{-t})$,

$$\begin{aligned} F_u(x) &= \frac{\frac{1}{1+e^{-(u+x)}} - \frac{1}{1+e^{-u}}}{\frac{e^{-u}}{1+e^{-u}}} \\ &= \frac{e^{-u} - e^{-(u+x)}}{e^{-u}(1+e^{-(u+x)})} \\ &= \frac{1-e^{-x}}{1+e^{-x-u}}. \end{aligned}$$

e) To determine whether the excess distribution converges to a generalized Pareto distribution, we use the Pickands–Balkema–de Haan theorem. It states that

$$\lim_{u \to \infty} \sup_{x>0} \left| F_u(x) - G_{\xi,\beta(u)}(x) \right| = 0 \quad \text{if and only if} \quad F_X \in \mathrm{MDA}(H_\xi).$$

From part (c), we know that $F_X \in \mathrm{MDA}(H_0)$, i.e., the Gumbel case. Therefore, if a limit exists, it must be a GPD with shape parameter $\xi = 0$, i.e., an exponential distribution.

Recall the excess distribution:

$$F_u(x) = \frac{F_X(u+x) - F_X(u)}{1 - F_X(u)} = \frac{1-e^{-x}}{1+e^{-x-u}}, \qquad x \geq 0.$$

Evaluate the pointwise limit.
As $u \to \infty$, we have $e^{-u} \to 0$, so

$$F_u(x) = \frac{1-e^{-x}}{1+e^{-x-u}} \longrightarrow 1 - e^{-x} = G_{0,1}(x).$$

Thus the excess distribution converges pointwise to an exponential $\mathrm{GPD}(0,1)$.

Show the uniform convergence.
To apply the theorem, we must also check uniform convergence over $x > 0$:

$$\lim_{u \to \infty} \sup_{x>0} \left| F_u(x) - (1 - e^{-x}) \right| = 0.$$

Compute the difference:

$$\left| F_u(x) - (1 - e^{-x}) \right| = (1 - e^{-x}) \left| \frac{1}{1+e^{-x-u}} - 1 \right|.$$

Simplify the inner term:

$$\frac{1}{1+e^{-x-u}} - 1 = \frac{1 - (1+e^{-x-u})}{1+e^{-x-u}} = -\frac{e^{-x-u}}{1+e^{-x-u}}.$$

Thus,

$$\left| F_u(x) - (1 - e^{-x}) \right| \leq \frac{e^{-x-u}}{1+e^{-x-u}},$$

since $0 < 1 - e^{-x} < 1$.

Let $y = e^{-x} \in (0,1]$. Then

$$\frac{e^{-x-u}}{1+e^{-x-u}} = \frac{ye^{-u}}{1+ye^{-u}} \leq ye^{-u} \leq e^{-u}.$$

Taking the supremum over $x > 0$,

$$\sup_{x>0} \left| F_u(x) - (1 - e^{-x}) \right| \le e^{-u} \longrightarrow 0.$$

Conclusion:
We have shown uniform convergence to $G_{0,1}$ on $(0, \infty)$. Therefore,

$$\boxed{\lim_{u\to\infty} \sup_{x>0} \left| F_u(x) - G_{0,1}(x) \right| = 0, \qquad \xi = 0,\ \beta(u) \equiv 1.}$$

So the limiting generalized Pareto distribution is GPD$(0, 1)$.

Question 3:

a) For any transition matrix of a Markov chain, each row must sum to one:

$$\sum_j P_{ij} = 1.$$

In credit–rating applications, the default state D is an *absorbing state*: once a borrower defaults, they remain in state D with probability one. Thus the last row must be:

$$(0,\ 0,\ 0,\ 1).$$

We now fill in the missing probabilities row by row:

- Row A:
$$0.95 + ? + 0 + 0 = 1 \quad \Rightarrow \quad P_{AB} = 0.05.$$
- Row B:
$$0.05 + ? + 0.1 + 0.05 = 1 \quad \Rightarrow \quad P_{BB} = 0.8.$$
- Row C:
$$0 + 0.2 + 0.5 + ? = 1 \quad \Rightarrow \quad P_{CD} = 0.3.$$
- Row D: absorbing:
$$(0, 0, 0, 1).$$

Thus the completed one–year transition matrix is:

$$P = \begin{pmatrix} 0.95 & 0.05 & 0 & 0 \\ 0.05 & 0.8 & 0.1 & 0.05 \\ 0 & 0.2 & 0.5 & 0.3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

b) In a stationary Markov chain, the t–step transition probabilities are given by:

$$P^t = \underbrace{P \cdot P \cdots P}_{t \text{ times}}.$$

The probability that a borrower starting in rating i is in default after t years is:

$$\Pr(X_t = D \mid X_0 = i) = (P^t)_{iD}, \qquad i \in \{A, B, C\}.$$

We compute the powers P^t for $t = 1, \ldots, 20$ and extract the last column. This yields the default-probability paths for the three initial ratings. A typical result is shown in Figure 1: default risk increases with time and is highest for the lowest initial rating.

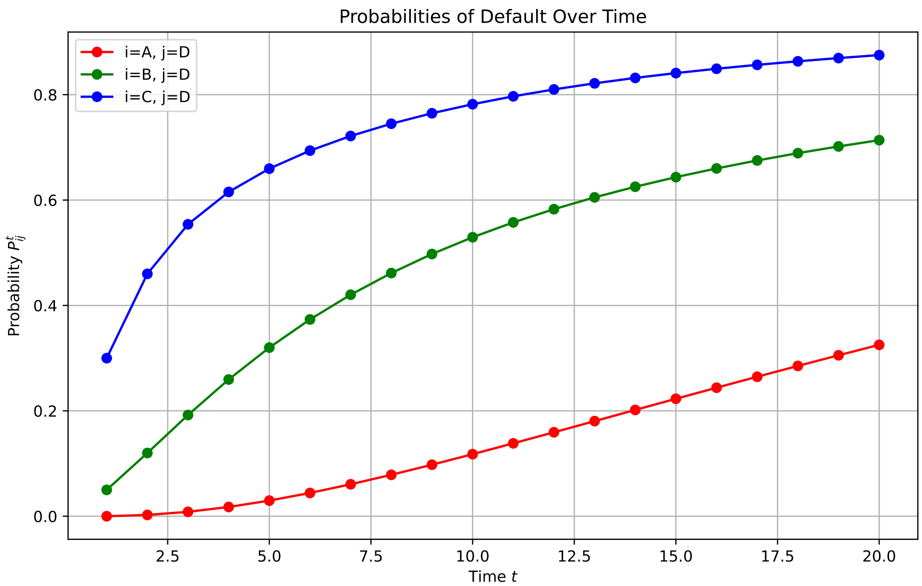

Figure 1: Multi-year default probabilities $(P^t)_{iD}$ for $i = A, B, C$.

Question 4:

a) To verify that v is an eigenvector of P, we check whether

$$Pv = \lambda v$$

for some scalar λ.

For $v_1 = (1, 1, 1)^\top$*:*

$$Pv_1 = \begin{pmatrix} 2/5 + 2/5 + 1/5 \\ 1/5 + 3/5 + 1/5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 \cdot v_1.$$

Hence $\lambda_1 = 1$.

For $v_2 = (1, 1, 0)^\top$*:*

$$Pv_2 = \begin{pmatrix} 2/5 + 2/5 \\ 1/5 + 3/5 \\ 0 \end{pmatrix} = \frac{4}{5} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{4}{5} v_2.$$

Hence $\lambda_2 = 4/5$.

For $v_3 = (-2, 1, 0)^\top$*:*

$$Pv_3 = \begin{pmatrix} -4/5 + 2/5 \\ -2/5 + 3/5 \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{5} v_3.$$

Hence $\lambda_3 = 1/5$.

Thus the eigenvalues of P are

$$\boxed{\lambda_1 = 1, \quad \lambda_2 = \tfrac{4}{5}, \quad \lambda_3 = \tfrac{1}{5}}.$$

b) Since P has three distinct eigenvalues, it is diagonalizable. Let A be the matrix with columns equal to the eigenvectors:

$$A = (\, v_1 \; v_2 \; v_3 \,) = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We are given A^{-1} and the diagonal matrix

$$\Delta = \mathrm{diag}\big(1, \tfrac{4}{5}, \tfrac{1}{5}\big).$$

It is straightforward to verify that

$$P = A\Delta A^{-1}.$$

c) For a time-homogeneous Markov chain,

$$P^{20} = A\,\Delta^{20}\,A^{-1}.$$

Since

$$\Delta^{20} = \mathrm{diag}\Big(1, \big(\tfrac{4}{5}\big)^{20}, \big(\tfrac{1}{5}\big)^{20}\Big),$$

the long-run behavior is dominated by the eigenvalue 1, corresponding to the absorbing default state. The probability that a borrower starting in Rating (1) is in default after 20 years is the $(1,3)$ entry of P^{20}:

$$(P^{20})_{1,3} = 1 - \left(\frac{4}{5}\right)^{20} \approx 0.988.$$

Thus the default probability over 20 years for an initially highly rated borrower is

$$\boxed{\Pr(X_{20} = D \mid X_0 = 1) \approx 98.8\%}.$$

Question 5:

a) The real–world expected payoff is:

$$\mathbb{E}[X_T] = p \cdot 1 + (1-p) \cdot 0.4 = 0.99 \cdot 1 + 0.01 \cdot 0.4 = 0.994.$$

Discounting at the risk–free rate:

$$\text{Expected discounted payoff} = \frac{0.994}{1.025} \approx 0.97.$$

Since $0.97 > p_1(0,1) = 0.961$, the bond appears to offer a risk premium for bearing default risk.

b) Under the risk–neutral measure, the discounted expected payoff must equal the market price:

$$p_1(0,1) = \frac{(1-q) \cdot 1 + q \cdot 0.4}{1.025}.$$

Solving for the risk–neutral default probability q:

$$0.961 \cdot 1.025 = 1 - 0.6q \quad \Rightarrow \quad q = 0.025.$$

Thus the risk–neutral default probability is significantly larger than the real–world probability (2.5% vs. 1%), again reflecting compensation for credit risk.

c) A stylized CDS pays 1 if default occurs by time T, and 0 otherwise. Its risk–neutral price at time 0 is therefore:

$$V_0 = \frac{q}{1.025} = \frac{0.025}{1.025} \approx 0.0244.$$

To replicate this payoff using:

- ξ_1 units of the defaultable bond,
- ξ_2 units of the default–free zero–coupon bond,

we match the CDS payoff in both scenarios:

$$\begin{cases} \text{Default:} & 0.4\xi_1 + 1.025\xi_2 = 1, \\ \text{No default:} & 1.0\xi_1 + 1.025\xi_2 = 0. \end{cases}$$

Subtracting the two equations:

$$(1 - 0.4)\xi_1 = -1 \quad \Rightarrow \quad \xi_1 = -\frac{1}{0.6} = -1.6667.$$

Substitute into the no-default equation:

$$-1.6667 + 1.025\xi_2 = 0 \quad \Rightarrow \quad \xi_2 = 1.6260.$$

Thus the replicating strategy is:

$$\boxed{\xi_1 = -1.6667 \quad \text{(short defaultable bond)}, \qquad \xi_2 = 1.6260 \quad \text{(long risk-free bond)}.}$$

This portfolio produces exactly the CDS payoff in both states and therefore must have the same price.