Quantitative Risk Management
Assignment 7

Question 1: Recall that $(X_t)_{t\in\mathbb{Z}}$ is an $ARCH(1)$ process if it is strictly stationary and satisfies:

$$\begin{aligned} X_t &= \sigma_t Z_t \\ \sigma_t^2 &= \alpha_0 + \alpha_1 X_{t-1}^2, \end{aligned}$$

for some $(Z_t)_{t\in\mathbb{Z}}$ that is $SWN(0,1)$, $\alpha_0 > 0$, and $\alpha_1 \geq 0$. The process $(X_t)_{t\in\mathbb{Z}}$ is white noise if and only if $\alpha_1 < 1$. In this case $(X_t)_{t\in\mathbb{Z}}$ has no serial correlation. This question shows that $(X_t^2)_{t\in\mathbb{Z}}$ does have serial correlation.

1. Consider an $AR(1)$ process $(Y_t)_{t\in\mathbb{Z}}$ of the form $Y_t = \phi Y_{t-1} + \epsilon_t$, where $|\phi| < 1$ and $(\epsilon_t)_{t\in\mathbb{Z}}$ is white noise. Compute the autocorrelation function $\rho(h)$ of $(Y_t)_{t\in\mathbb{Z}}$.
2. Assume that the $ARCH(1)$ process $(X_t)_{t\in\mathbb{Z}}$ satisfies $\mathbb{E}[X_t^4] < \infty$. Show that for some $c \in \mathbb{R}$ the process $(X_t^2 - c)_{t\in\mathbb{Z}}$ is an $AR(1)$ process with zero mean. Why is the condition $\mathbb{E}[X_t^4] < \infty$ needed?

   Hint: Start with $X_t^2 = \sigma_t^2 + \sigma_t^2(Z_t^2 - 1)$ and prove that $\sigma_t^2(Z_t^2-1)$ is a martingale difference sequence.
3. Find the autocorrelation function $\rho(h)$ of $(X_t^2)_{t\in\mathbb{Z}}$. State what you can conclude.

Question 2: Let $(X_t)_{t\in\mathbb{Z}}$ be an $ARCH(1)$ process and assume it has finite fourth moments. Compute $\mathbb{E}[X_t^4]$ in terms of α_0, α_1, and $\mathbb{E}[Z_t^4]$.

Question 3: Show that the Fréchet bounds hold:

$$\max\left\{1 - d + \sum_{i=1}^{d} u_i, 0\right\} \leq C(u_1, \dots, u_d) \leq \min\{u_1, \dots, u_d\}.$$

Show that, for $d = 2$, the Fréchet lower bound is the distribution of $(U, 1-U)$, where $U \sim Unif(0,1)$: this is called the counter-monotonicity copula.
Provide an example (e.g., for $d = 2$) for which the Fréchet upper bound is attained.

Question 4: Let X_1 and X_2 have the following marginal lognormal distributions: $\log(X_1) \sim \mathcal{N}(0,1)$ and $\log(X_2) \sim \mathcal{N}(0,\sigma^2)$. Find the minimum and maximum attainable correlations between X_1 and X_2. Plot these values as a function of σ for $\sigma \in (0,5)$.