Quantitative Risk Management
Assignment 5 Solutions

Question 1: The cumulative distribution function (CDF) of X is:

$$\mathbb{P}(X \le v) = \mathbb{P}(\mu + \sqrt{W}\, Z \le v) = \sum_{i=1}^{n} \mathbb{P}(\mu + \sqrt{W}\, Z \le v \mid W = k_i)\, \mathbb{P}(W = k_i)$$

$$= \sum_{i=1}^{n} \mathbb{P}(\mu + \sqrt{k_i}\, Z \le v)\, p_i = \sum_{i=1}^{n} \mathbb{P}\left(Z \le \frac{v-\mu}{\sqrt{k_i}}\right) p_i = \sum_{i=1}^{n} \Phi\left(\frac{v-\mu}{\sqrt{k_i}}\right) p_i,$$

where $\Phi(\cdot)$ denotes the standard normal distribution function.

By definition, the $\mathrm{VaR}_\alpha(X)$ satisfies $\mathbb{P}(X \le v_0) = \alpha$. Hence, we define the function

$$\boxed{f(v) = \sum_{i=1}^{n} p_i\, \Phi\left(\frac{v-\mu}{\sqrt{k_i}}\right) - \alpha.}$$

Then the value $v_0 = \mathrm{VaR}_\alpha(X)$ is the unique root of $f(v)$.

Uniqueness of the root. Each function $v \mapsto \Phi((v-\mu)/\sqrt{k_i})$ is strictly increasing in v, and therefore their weighted sum with positive weights p_i is also strictly increasing. Furthermore,

$$\lim_{v \to -\infty} f(v) = -\alpha, \qquad \lim_{v \to \infty} f(v) = 1 - \alpha.$$

By the Intermediate Value Theorem, f has at least one root. Since f is strictly increasing, this root is unique.

Conclusion. The VaR of X can be obtained as the unique solution of

$$f(v) = \sum_{i=1}^{n} p_i\, \Phi\left(\frac{v-\mu}{\sqrt{k_i}}\right) - \alpha = 0.$$

Because $f(v)$ increases smoothly from $-\alpha$ to $1-\alpha$, it crosses zero exactly once.

Question 2: There are many examples of random variables that are uncorrelated but not independent. One convenient construction is as follows.

Let

$$X \sim \mathcal{N}(0,1),$$

and let A be an independent random variable taking values

$$A = \begin{cases} 1, & \text{with probability } p, \\ -1, & \text{with probability } 1-p. \end{cases}$$

Define

$$Y = AX.$$

Step 1. Compute means and covariance.

Since $\mathbb{E}[X] = 0$ and $\mathbb{E}[A] = 2p - 1$, we have

$$\mathbb{E}[Y] = \mathbb{E}[AX] = \mathbb{E}[A]\,\mathbb{E}[X] = 0.$$

Hence both X and Y are centered. The covariance is

$$\mathrm{Cov}(X, Y) = \mathbb{E}[XY] = \mathbb{E}[AX^2] = \mathbb{E}[A]\,\mathbb{E}[X^2] = (2p - 1) \cdot 1 = 2p - 1.$$

If we choose $p = \frac{1}{2}$, then $\mathbb{E}[A] = 0$ and therefore

$$\mathrm{Cov}(X, Y) = 0,$$

so X and Y are *uncorrelated*.

Step 2. Show that X and Y are not independent.
When $p = \frac{1}{2}$, the random variable A takes values ± 1 with equal probability. Then $Y = X$ with probability $\frac{1}{2}$ and $Y = -X$ with probability $\frac{1}{2}$.
Let us check joint probabilities. For any $h > 0$,

$$\begin{aligned}
\mathbb{P}(X > h,\, Y > h) &= \mathbb{P}(X > h,\, AX > h) \\
&= \tfrac{1}{2}\,\mathbb{P}(X > h,\, X > h) + \tfrac{1}{2}\,\mathbb{P}(X > h,\, -X > h) \\
&= \tfrac{1}{2}\,\mathbb{P}(X > h) + \tfrac{1}{2}\,\mathbb{P}(X > h,\, X < -h) \\
&= \tfrac{1}{2}\,\mathbb{P}(X > h),
\end{aligned}$$

since the second term is zero for $h > 0$. Hence

$$\mathbb{P}(X > h,\, Y > h) = \tfrac{1}{2}\,[1 - \Phi(h)].$$

On the other hand, if X and Y were independent, we would have

$$\mathbb{P}(X > h)\,\mathbb{P}(Y > h) = [1 - \Phi(h)]^2.$$

These two expressions are not equal for general $h > 0$ (e.g., for $h = 1$), which shows that X and Y are *not independent*.

Conclusion. For $p = \frac{1}{2}$, the pair (X, Y) satisfies

$$\mathrm{Cov}(X, Y) = 0, \quad \text{but} \quad X \text{ and } Y \text{ are not independent.}$$

This example illustrates that uncorrelatedness does not imply independence.

Question 3: We can write the vector $X = (X_1, X_2)^\top$ in matrix form:

$$X = \sqrt{W} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}.$$

Then

$$X = \sqrt{W}\, AZ.$$

Step 1. Normal variance mixture representation.
Since $Z \sim \mathcal{N}_2(0, I_2)$ and $W \geq 0$ is independent of Z, the random vector X is of the form

$$X = \sqrt{W}\, AZ,$$

which is exactly the structure of a *normal variance mixture*. Conditional on W, we have

$$X \mid W \sim \mathcal{N}_2(0,\, W\, AA^\top).$$

Step 2. Compute the unconditional covariance.
The unconditional covariance matrix of X is

$$\mathrm{Cov}(X) = \mathbb{E}[W]\, AA^\top,$$

since $\mathbb{E}[ZZ^\top] = I_2$ and W is independent of Z.
For the Pareto(θ) distribution with $F_W(w) = 1 - w^{-\theta}$ ($w \geq 1$), we have

$$\mathbb{E}[W] = \frac{\theta}{\theta - 1}, \qquad \text{for } \theta > 1.$$

Compute

$$AA^\top = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^\top = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2I_2.$$

Hence,

$$\mathrm{Cov}(X) = \mathbb{E}[W]\, AA^\top = \frac{\theta}{\theta - 1} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Step 3. Interpretation.
The covariance matrix is diagonal, which implies that

$$\mathrm{Cov}(X_1, X_2) = 0.$$

Thus, X_1 and X_2 are *uncorrelated*.
However, because they share the same random scale factor $\sqrt{W}$, they are *not independent*: conditional on W, they are Gaussian and independent, but unconditionally they exhibit tail dependence induced by the common mixing variable W.

Conclusion. The vector $X = (X_1, X_2)^\top$ follows a *normal variance mixture* distribution with mixing variable $W \sim \text{Pareto}(\theta)$ and covariance matrix

$$\mathrm{Cov}(X) = \frac{\theta}{\theta - 1} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

The components X_1 and X_2 are uncorrelated but not independent.

Question 4: Part 1): The expanded matrices $\mathcal{X}$ and $\mathcal{F}$ are constructed from the data as demonstrated in the lecture slides. The regression is performed to compute $\hat{\mathcal{B}}$ as:

$$\hat{\mathcal{B}} = (\mathcal{F}^T \mathcal{F})^{-1} \mathcal{F}^T \mathcal{X}.$$

This results in the following numerical values for the vector $\hat{\boldsymbol{a}}$ and factor loadings $\hat{\boldsymbol{B}}$:

$$\hat{\boldsymbol{a}} = \begin{bmatrix} -0.342 \times 10^{-3} \\ 0.121 \times 10^{-3} \\ 0.193 \times 10^{-3} \\ 0.021 \times 10^{-3} \end{bmatrix}, \qquad \hat{\boldsymbol{B}} = \begin{bmatrix} 0.813 \\ 0.978 \\ 0.583 \\ 0.522 \end{bmatrix}.$$

The components of $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{B}}$ correspond to the ordering of ticker symbols: IBM, MMM, MCD, WMT.

Part 2): The residual errors are computed as:

$$\hat{\mathcal{E}} = \mathcal{X} - \mathcal{F}\hat{\mathcal{B}}.$$

The sample correlation matrix of both the error sequence $\hat{\boldsymbol{\epsilon}}_1, \ldots, \hat{\boldsymbol{\epsilon}}_n$ and the original returns $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ is computed. The corresponding values are summarized in the following two matrices:

$$\hat{\boldsymbol{\rho}}_\epsilon = \begin{bmatrix} 1.000 & 0.002 & 0.063 & -0.004 \\ 0.002 & 1.000 & 0.023 & 0.030 \\ 0.063 & 0.023 & 1.000 & 0.122 \\ -0.004 & 0.030 & 0.122 & 1.000 \end{bmatrix}, \qquad \hat{\boldsymbol{\rho}}_X = \begin{bmatrix} 1.000 & 0.528 & 0.426 & 0.320 \\ 0.528 & 1.000 & 0.498 & 0.420 \\ 0.426 & 0.498 & 1.000 & 0.382 \\ 0.320 & 0.420 & 0.382 & 1.000 \end{bmatrix}.$$

Note that the correlations between the residual errors are significantly lower than the correlations between the original returns. This is to be expected since all of the utilized stocks are included in the S&P500 index.

Question 5: Part 1): Based off of the given realizations of the loss, $VaR_{0.95}$ is found to be 10.42.

Part 2): The matrix $\bar{\boldsymbol{\Sigma}}$ is computed as:

$$\bar{\boldsymbol{\Sigma}} = \begin{bmatrix} 1.73 & 1.74 & -1.74 & 1.73 \\ 1.74 & 3.44 & 1.67 & 0.09 \\ -1.74 & 1.67 & 23.62 & -0.13 \\ 1.73 & 0.09 & -0.13 & 6.65 \end{bmatrix}.$$

The matrix of eigenvectors is given by:

$$\bar{\boldsymbol{\Gamma}} = \begin{bmatrix} 0.07 & 0.34 & 0.41 & -0.84 \\ -0.07 & 0.18 & 0.85 & 0.48 \\ -0.99 & 0.02 & -0.04 & -0.10 \\ 0.01 & 0.92 & -0.32 & 0.21 \end{bmatrix}.$$

The first column of $\bar{\boldsymbol{\Gamma}}$ is the eigenvector corresponding to the first principal component. Note that the magnitude of the third component of this vector is very close to 1. This indicates that most of the variability in the first principal component (and original data) is solely due to the third component of the data. This is further exemplified by the fact that the variance of the third component is 23.62, significantly larger than the variance of any other component.

Part 3): In this part, we approximate $\boldsymbol{X}$ with a factor model where the factors are the first two principal components of $\boldsymbol{X}$. Performing the principal component transform on $\boldsymbol{X}$ gives us the 10,000 realizations of the principal components $\boldsymbol{Y}$:

$$\boldsymbol{Y}_k = \bar{\boldsymbol{\Gamma}}^T(\boldsymbol{X}_k - \bar{\boldsymbol{\mu}}).$$

Setting the error terms to zero, the factor model becomes:

$$\boldsymbol{X} = \bar{\boldsymbol{\mu}} + \begin{bmatrix} \boldsymbol{\gamma}_1 & \boldsymbol{\gamma}_2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

We can recompute $VaR_{0.95}$ to be equal to 9.87. This is close to, but lower than the original value of 10.42, as should be expected because the first two principal components account for 88% of the variability in this data.