Quantitative Risk Management
Assignment 4 Solutions

Question 1. Part 1): The current price of bond i is 1000 CHF, meaning $V_{i,0} = 1000$ CHF. If company i does not default after one year, then the value of the bond will be $V_{i,1} = 1050$ CHF. If company i does default, then the bond will have a value of $V_{i,1} = 0$ CHF. The loss L_i as a function of I_i is then:

$$\begin{aligned} L_i &= -(V_{i,1} - V_{i,0}) \\ &= -(1050(1 - I_i) - 1000) \\ &= 1050I_i - 50 \end{aligned}$$

Part 2): The probability distribution is given by the following:

$$\mathbb{P}(L_i = -50) = 0.98$$
$$\mathbb{P}(L_i = 1000) = 0.02$$

Part 3): The value of portfolio V_a is a multiple of a single bond so that $L_a = 100(1050I_i - 50)$. Thus, the probability distribution of L_a is:

$$\mathbb{P}(L_a = -5000) = 0.98$$
$$\mathbb{P}(L_a = 100000) = 0.02$$

The portfolio V_b is the sum of 100 independent copies of a single bond, so the losses are given by:

$$\begin{aligned} L_b &= \sum_{i=1}^{100} 1050I_i - 100 \cdot 50 \\ &= 1050 \sum_{i=1}^{100} I_i - 100 \cdot 50 \\ &= 1050N - 100 \cdot 50, \end{aligned}$$

where N counts the number of defaults. Since each I_i is Bernoulli and they are independent, N has the Binomial distribution with parameters $n = 100$ and $p = 0.02$. For portfolio V_a, VaR_α takes the values:

$$VaR_{0.95}(L_a) = -5000$$
$$VaR_{0.99}(L_a) = 100000$$

For portfolio V_b, we can first compute the quantiles of N:

$$Bino^{-1}(0.95, 100, 0.02) = 5$$
$$Bino^{-1}(0.99, 100, 0.02) = 6$$

Positive homogeneity and translation invariance of value-at-risk give us:

$$VaR_{0.95}(L_b) = 1050 \cdot 5 - 100 \cdot 50 = 250$$
$$VaR_{0.99}(L_b) = 1050 \cdot 6 - 100 \cdot 50 = 1300$$

Question 2. Given the joint pdf

$$f(x_1,x_2)=\theta(\theta+1)(x_1+x_2-1)^{-(\theta+2)}, \qquad x_1,x_2\ge 1,\ \theta>0,$$

we derive the joint and marginal distribution functions and compute the correlation between X_1 and X_2.

Step 1: A convenient change of variables.
The joint density depends only on the combination x_1+x_2-1. Hence it is natural to define

$$s=x_1+x_2-1, \qquad u=x_1.$$

Then $x_2=s-u+1$. The Jacobian of the transformation $(x_1,x_2)\mapsto(u,s)$ is

$$J=\begin{pmatrix}\frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial s}\\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial s}\end{pmatrix}=\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}, \qquad |\det J|=1.$$

Therefore, $dx_1\,dx_2=du\,ds$ and the support becomes

$$1\le u\le s, \qquad s\ge 1.$$

This change is convenient because the pdf becomes a simple function of s only:

$$f(u,s)=\theta(\theta+1)\,s^{-(\theta+2)}.$$

Step 2: Marginal distribution of X_1.
Integrate out x_2:

$$\begin{aligned} f_{X_1}(x_1)&=\int_1^\infty \theta(\theta+1)(x_1+x_2-1)^{-(\theta+2)}\,dx_2\\ &=\theta(\theta+1)\int_{x_1}^\infty t^{-(\theta+2)}\,dt=\theta x_1^{-(\theta+1)}, \qquad x_1\ge 1.\end{aligned}$$

Thus each margin follows a Pareto$(1,\theta)$ distribution with

$$F_{X_1}(x)=1-x^{-\theta}, \qquad F_{X_2}(x)=1-x^{-\theta}, \qquad x\ge 1.$$

Step 3: Joint CDF.
Use inclusion–exclusion:

$$F(x_1,x_2)=1-\Pr(X_1>x_1)-\Pr(X_2>x_2)+\Pr(X_1>x_1,X_2>x_2).$$

We already know $\Pr(X_i>x_i)=x_i^{-\theta}$. Compute the joint survival function:

$$\begin{aligned}\Pr(X_1>x_1,X_2>x_2)&=\int_{x_1}^\infty\int_{x_2}^\infty \theta(\theta+1)(u+v-1)^{-(\theta+2)}\,dv\,du\\ &=\theta(\theta+1)\int_{x_1}^\infty \frac{(u+x_2-1)^{-(\theta+1)}}{\theta+1}\,du=(x_1+x_2-1)^{-\theta}.\end{aligned}$$

Therefore,

$$\boxed{F(x_1,x_2)=1-x_1^{-\theta}-x_2^{-\theta}+(x_1+x_2-1)^{-\theta}, \qquad x_1,x_2\ge 1.}$$

Step 4: Moments and correlation.
Each marginal is Pareto$(1,\theta)$, so

$$\mathbb{E}[X_i]=\frac{\theta}{\theta-1}, \quad \operatorname{Var}(X_i)=\frac{\theta}{(\theta-1)^2(\theta-2)}, \qquad (\theta>2).$$

Compute the cross-moment:

$$\begin{aligned}\mathbb{E}[X_1X_2]&=\iint_{[1,\infty)^2} x_1x_2\,\theta(\theta+1)(x_1+x_2-1)^{-(\theta+2)}\,dx_2\,dx_1\\ &=\theta(\theta+1)\int_1^\infty\left[\int_1^s u(s-u+1)\,du\right]s^{-(\theta+2)}\,ds,\end{aligned}$$

where we used $(x_1, x_2) \to (u, s)$ as before. The inner integral is polynomial:

$$\int_1^s u(s-u+1)\,du = \tfrac{1}{6}s^3 + \tfrac{1}{2}s^2 - \tfrac{1}{2}s - \tfrac{1}{6}.$$

Hence, for $\theta > 2$,

$$\mathbb{E}[X_1X_2] = \theta(\theta+1)\left[\tfrac{1}{6}\int_1^\infty s^{1-\theta}ds + \tfrac{1}{2}\int_1^\infty s^{-\theta}ds - \tfrac{1}{2}\int_1^\infty s^{-(\theta+1)}ds - \tfrac{1}{6}\int_1^\infty s^{-(\theta+2)}ds\right].$$

Evaluating the integrals:

$$\int_1^\infty s^{1-\theta}ds = \frac{1}{\theta-2}, \qquad \int_1^\infty s^{-\theta}ds = \frac{1}{\theta-1}, \qquad \int_1^\infty s^{-(\theta+1)}ds = \frac{1}{\theta}, \qquad \int_1^\infty s^{-(\theta+2)}ds = \frac{1}{\theta+1}.$$

After simplification,

$$\mathbb{E}[X_1X_2] = \frac{\theta^2-\theta-1}{(\theta-2)(\theta-1)}, \qquad \theta > 2.$$

Thus

$$\mathrm{Cov}(X_1X_2) = \mathbb{E}[X_1X_2] - \mathbb{E}[X_1]\,\mathbb{E}[X_2] = \frac{1}{(\theta-2)(\theta-1)^2},$$

$$\mathrm{Var}(X_i) = \frac{\theta}{(\theta-2)(\theta-1)^2},$$

and consequently

$$\boxed{\rho = \frac{\mathrm{Cov}(X_1X_2)}{\mathrm{Var}(X_1)} = \frac{1}{\theta}, \qquad \text{for } \theta > 2.}$$

Interpretation: As θ increases, the correlation $\rho = 1/\theta$ decreases, meaning that X_1 and X_2 become less dependent (approaching independence for large θ). The transformation $s = x_1 + x_2 - 1$ was useful throughout because the joint density depends on x_1 and x_2 only through this sum, simplifying all double integrals into powers of s.

Question 3. The VaR is given by $\mu_1 + \mu_2 + \sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}\Phi^{-1}(\alpha)$, denoting μ_i and σ_i as the mean and the standard deviation of X_i, for $i \in \{1, 2\}$.

Question 4. Let $(X_1, X_2) \sim \mathcal{N}(\mathbf{0}, I_2)$ (so X_1, X_2 are i.i.d. $\mathcal{N}(0,1)$), and let

$$(Y_1, Y_2) = (X_1, VX_1),$$

where $V \in \{-1, 1\}$ with $\mathbb{P}(V=1) = \mathbb{P}(V=-1) = \frac{1}{2}$, and $V \perp X_1$.

(i) Same marginal distributions. Clearly $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. For $Y_1 = X_1$ we have $Y_1 \sim \mathcal{N}(0,1)$. For $Y_2 = VX_1$:

$$\mathbb{P}(Y_2 \le y) = \tfrac{1}{2}\,\mathbb{P}(X_1 \le y) + \tfrac{1}{2}\,\mathbb{P}(-X_1 \le y) = \tfrac{1}{2}\,\Phi(y) + \tfrac{1}{2}\,(1 - \Phi(-y)) = \Phi(y),$$

so $Y_2 \sim \mathcal{N}(0,1)$. Thus each coordinate of (X_1, X_2) and (Y_1, Y_2) is $\mathcal{N}(0,1)$.

(ii) Uncorrelated components. For (X_1, X_2), independence implies $\mathrm{Cov}(X_1, X_2) = 0$. For (Y_1, Y_2):

$$\mathbb{E}[Y_1Y_2] = \mathbb{E}[X_1 \cdot VX_1] = \mathbb{E}[V]\,\mathbb{E}[X_1^2] = 0 \cdot 1 = 0$$

since $V \perp X_1$ and $\mathbb{E}[V] = \frac{1}{2}(1) + \frac{1}{2}(-1) = 0$. Because $\mathbb{E}[Y_1] = \mathbb{E}[Y_2] = 0$, it follows $\mathrm{Cov}(Y_1, Y_2) = 0$. *Remark:* Y_1 and Y_2 are *not* independent (indeed $Y_2 = \pm Y_1$), but they are uncorrelated.

(iii) CDFs of $X_1 + X_2$ and $Y_1 + Y_2$.

- $X_1 + X_2 \sim \mathcal{N}(0,2)$ by independence, so

$$F_{X_1+X_2}(x) = \Phi\left(\frac{x}{\sqrt{2}}\right).$$

- For $S_Y := Y_1 + Y_2 = X_1 + VX_1 = (1+V)X_1$:

$$S_Y = \begin{cases} 2X_1, & \text{with prob. } \frac{1}{2}\ (V=1), \\ 0, & \text{with prob. } \frac{1}{2}\ (V=-1). \end{cases}$$

Thus S_Y is a mixture: $\frac{1}{2}\,\delta_0 + \frac{1}{2}\,\mathcal{N}(0,4)$. Hence its CDF is

$$F_{S_Y}(y) = \begin{cases} \frac{1}{2}\,\Phi\left(\frac{y}{2}\right), & y < 0, \\ \frac{1}{2} + \frac{1}{2}\,\Phi\left(\frac{y}{2}\right), & y \geq 0. \end{cases}$$

Equivalently, $F_{S_Y}(y) = \frac{1}{2}\,\Phi(y/2) + \frac{1}{2}\,\mathbf{1}_{\{y \geq 0\}}$, and you can see the point mass at 0: $F_{S_Y}(0^-) = \frac{1}{4}$, $F_{S_Y}(0) = \frac{3}{4}$.

(iv) Value-at-Risk.

- For $X_1 + X_2 \sim \mathcal{N}(0,2)$:

$$\boxed{VaR_\alpha(X_1 + X_2) = \sqrt{2}\,\Phi^{-1}(\alpha).}$$

- For S_Y:

$$F_{S_Y}(y) = \begin{cases} \frac{1}{2}\,\Phi(y/2), & y < 0, \\ \frac{1}{2} + \frac{1}{2}\,\Phi(y/2), & y \geq 0. \end{cases}$$

Solve $F_{S_Y}(y) = \alpha$:

  - If $0 < \alpha < \frac{1}{4}$, then $y < 0$ and

$$\tfrac{1}{2}\,\Phi(y/2) = \alpha \;\Rightarrow\; y = 2\,\Phi^{-1}(2\alpha)\ (< 0).$$

  - If $\frac{1}{4} \leq \alpha \leq \frac{3}{4}$, the jump at 0 implies

$$VaR_\alpha(S_Y) = 0 \quad (\text{since } F_{S_Y}(0^-) = \tfrac{1}{4},\ F_{S_Y}(0) = \tfrac{3}{4}).$$

  - If $\frac{3}{4} < \alpha < 1$, then $y > 0$ and

$$\tfrac{1}{2} + \tfrac{1}{2}\,\Phi(y/2) = \alpha \;\Rightarrow\; y = 2\,\Phi^{-1}(2\alpha - 1)\ (> 0).$$

So

$$\boxed{VaR_\alpha(Y_1 + Y_2) = \begin{cases} 2\,\Phi^{-1}(2\alpha), & 0 < \alpha < \frac{1}{4}, \\ 0, & \frac{1}{4} \leq \alpha \leq \frac{3}{4}, \\ 2\,\Phi^{-1}(2\alpha - 1), & \frac{3}{4} < \alpha < 1. \end{cases}}$$

Comments:

- (Y_1, Y_2) has standard normal *margins* but a very different *joint* law than (X_1, X_2).
- Y_1 and Y_2 are uncorrelated (covariance 0) but not independent; dependence shows up in the sum via a point mass at 0.
- The VaR curve for $Y_1 + Y_2$ is flat on $[\frac{1}{4}, \frac{3}{4}]$ because of that mass.

Question 5. Elliptical X admits the representation $X = \mu + AY$, where Y is spherical with $\mathbb{E}[Y] = 0$ and $\mathrm{Cov}(Y) = \mathrm{Var}(Y_1)\, I_d$, and $\Sigma := \mathrm{Cov}(X) = \mathrm{Var}(Y_1)\, AA^\top$. Define the portfolio losses $L_w = -w^\top X$ and $L_v = -v^\top X$. The equal-mean-return assumption $w^\top \mu = v^\top \mu$ is equivalent to equal expected losses $\mathbb{E}[L_w] = \mathbb{E}[L_v]$.

Step 1: Loss decomposition.
Using $X = \mu + AY$ and spherical symmetry ($b^\top Y \stackrel{d}{=} \|b\|\, Y_1$),

$$L_w = -w^\top \mu \;-\; \|A^\top w\|\, Y_1, \qquad L_v = -v^\top \mu \;-\; \|A^\top v\|\, Y_1.$$

Since $w^\top \mu = v^\top \mu$, both losses share the same mean term $-w^\top \mu$.

Step 2: ES properties (translation invariance & positive homogeneity).
For any constant c and $a > 0$,

$$ES_\alpha(Z + c) = ES_\alpha(Z) + c, \qquad ES_\alpha(aZ) = a\, ES_\alpha(Z).$$

Applying to L_w and L_v,

$$ES_\alpha(L_w) = -w^\top \mu \;+\; \|A^\top w\|\, ES_\alpha(-Y_1), \qquad ES_\alpha(L_v) = -v^\top \mu \;+\; \|A^\top v\|\, ES_\alpha(-Y_1).$$

Because Y_1 is symmetric around 0, for $\alpha > 1/2$ the tail of $-Y_1$ is positive and $ES_\alpha(-Y_1) > 0$, a constant that depends only on the base family and α.
Hence, with equal means,

$$ES_\alpha(L_w) \le ES_\alpha(L_v) \quad \Longleftrightarrow \quad \|A^\top w\| \le \|A^\top v\|.$$

Step 3: Relate scale to variance of losses.
Since $\mathrm{Cov}(X) = \mathrm{Var}(Y_1)\, AA^\top$,

$$\mathrm{Var}(L_w) = \mathrm{Var}(w^\top X) = w^\top \Sigma\, w = \mathrm{Var}(Y_1)\, \|A^\top w\|^2, \qquad \mathrm{Var}(L_v) = \mathrm{Var}(Y_1)\, \|A^\top v\|^2.$$

As $\mathrm{Var}(Y_1) > 0$, the norm ordering is equivalent to the variance ordering:

$$\|A^\top w\| \le \|A^\top v\| \quad \Longleftrightarrow \quad \mathrm{Var}(L_w) \le \mathrm{Var}(L_v).$$

Step 4:
Combining the steps,

$$\boxed{ES_\alpha(L_w) \;\le\; ES_\alpha(L_v) \iff \mathrm{Var}(L_w) \;\le\; \mathrm{Var}(L_v), \quad \text{for } \alpha > 1/2 \text{ (and whenever } ES_\alpha(-Y_1) \text{ exists).}}$$

Remarks.

- Intuitively, among portfolios with the same expected return, the one with smaller variance also has smaller Expected Shortfall.
- The result holds for any elliptical family (Gaussian, Student-t, etc.). In the Gaussian case, $ES_\alpha(L_w) = -w^\top \mu + \sigma_w \cdot \frac{\phi(z_\alpha)}{1-\alpha}$ with $\sigma_w = \sqrt{\mathrm{Var}(L_w)}$ and $z_\alpha = \Phi^{-1}(\alpha)$.
- $ES_\alpha(-Y_1)$ is a positive constant reflecting the tail loss of the base spherical distribution.
- The factor $\mathrm{Var}(Y_1)$ acts as a common scale parameter; some authors normalize Y such that $\mathrm{Var}(Y_1) = 1$, in which case $\Sigma = AA^\top$.