

# Quantitative Risk Management

## Assignment 11 Solutions

**Question 1:** A possible way to profit from an expected deterioration in the credit quality of issuer A is to take a position in credit default swaps (CDS) referencing A.

Strategy: The investor enters a *protection buyer* position at today's CDS spread  $s_0$ . If the credit quality of A worsens, the CDS spread widens to  $s_1 > s_0$ . The investor may then close the position by *selling protection* at the higher spread  $s_1$ . The profit arises from the present value of the spread difference, which reflects the increase in the default probability of A.

Risks:

- *Market (view) risk:* If the credit quality of A does not deteriorate, or even improves, the CDS spread will tighten instead of widening, resulting in a loss when the position is closed.
- *Counterparty risk:* The payoff of a CDS depends on the protection seller's ability to perform in case of default of A. Although collateralization mitigates this, it does not eliminate the risk entirely.
- *Liquidity and timing risk:* The spread may widen only after the investor's horizon, or market liquidity may be insufficient to unwind the position at a fair price.
- *Model/pricing risk:* The mark-to-market valuation of CDS positions depends on assumptions about recovery rates, default intensities, and discounting.

In summary, buying protection and later selling it at a higher spread is a standard strategy to benefit from a deterioration in credit quality, but it involves several market and counterparty risks.

**Question 2:** In Merton's model, default at time  $T$  occurs if  $V_T < B$ . Under the assumption that the asset value follows a geometric Brownian motion, the default probability is

$$p = \Phi(I),$$

where

$$I = \frac{\log(B/V_0) - (\mu_V - \frac{1}{2}\sigma_V^2)T}{\sigma_V\sqrt{T}} = \frac{\log(B/V_0) - \mu_V T}{\sigma_V\sqrt{T}} + \frac{1}{2}\sigma_V\sqrt{T}.$$

Since  $\Phi$  is strictly increasing, the default probability is increasing in  $\sigma_V$  if and only if

$$\frac{\partial I}{\partial \sigma_V} > 0.$$

*Step 1:* Compute the derivative.

$$\frac{\partial I}{\partial \sigma_V} = -\frac{\log(B/V_0) - \mu_V T}{\sigma_V^2\sqrt{T}} + \frac{1}{2}\sqrt{T} = \frac{\mu_V T + \log(V_0/B)}{\sigma_V^2\sqrt{T}} + \frac{1}{2}\sqrt{T}.$$

*Step 2:* Sign of the derivative.

The derivative contains two terms:

$$\frac{\mu_V T + \log(V_0/B)}{\sigma_V^2\sqrt{T}} \quad \text{and} \quad \frac{1}{2}\sqrt{T}.$$

Both are non-negative under the assumptions:

$$\begin{aligned} V_0 > B &\implies \log(V_0/B) > 0, \\ \mu_V &> 0. \end{aligned}$$

Hence,

$$\frac{\partial I}{\partial \sigma_V} > 0,$$

which implies that the default probability

$$p = \Phi(I)$$

is strictly increasing in the asset volatility  $\sigma_V$ .

*Economic interpretation:* Higher asset volatility increases the uncertainty around the future asset value. Even if the expected growth rate is positive and the firm is currently solvent ( $V_0 > B$ ), greater dispersion increases the probability that the terminal value  $V_T$  falls below the debt level  $B$ . This is the classic “volatility increases default risk” result in the Merton model.

*Remark:* The conditions  $V_0 > B$  and  $\mu_V > 0$  are *sufficient* for  $\partial I / \partial \sigma_V > 0$ , but not *necessary*. For some parameter choices with  $V_0 < B$  or  $\mu_V < 0$  the derivative can still be positive, but in other cases it may become negative.

**Question 3:** We work under the Merton model, where the asset value follows under the physical measure  $\mathbb{P}$  the SDE

$$dV_t = V_t \mu_V dt + V_t \sigma_V dW_t,$$

with solution

$$V_T = V_0 \exp\left(\left(\mu_V - \frac{1}{2}\sigma_V^2\right)T + \sigma_V \sqrt{T} Z\right), \quad Z \sim N(0, 1).$$

Default at  $T = 1$  occurs when  $V_1 < B$ .

(a) Physical and risk-neutral default probabilities (for  $T = 1$ ):

Taking logarithms of the default event,

$$V_1 < B \iff \log V_1 < \log B.$$

Using the lognormal solution above, this becomes

$$Z < \frac{\log(B/V_0) - (\mu_V - \frac{1}{2}\sigma_V^2)}{\sigma_V} =: I_{\mathbb{P}}(\sigma_V).$$

Thus, the *physical* default probability is

$$p(\sigma_V) = \mathbb{P}(V_1 < B) = \Phi(I_{\mathbb{P}}(\sigma_V)),$$

where

$$I_{\mathbb{P}}(\sigma_V) = \frac{\log(B/V_0) - \mu_V + \frac{1}{2}\sigma_V^2}{\sigma_V}.$$

Under the risk-neutral measure  $\mathbb{Q}$ , the drift of  $V_t$  becomes  $r$ :

$$dV_t = V_t r dt + V_t \sigma_V dW_t^{\mathbb{Q}},$$

since the market price of risk is

$$\theta = \frac{\mu_V - r}{\sigma_V}.$$

Repeating the same calculation with  $\mu_V$  replaced by  $r$ , the risk-neutral default probability is

$$q(\sigma_V) = \mathbb{Q}(V_1 < B) = \Phi(I_{\mathbb{Q}}(\sigma_V)),$$

where

$$I_{\mathbb{Q}}(\sigma_V) = \frac{\log(B/V_0) - r + \frac{1}{2}\sigma_V^2}{\sigma_V}.$$

Thus both  $p(\sigma_V)$  and  $q(\sigma_V)$  are explicit functions of  $\sigma_V$ .

(b) Plots for  $r = 5\%$  and  $\theta = -1, 0, 1$ :

For  $r = 0.05$ ,  $\theta = (\mu_V - r)/\sigma_V$  implies

$$\mu_V = r + \theta\sigma_V.$$

Substituting this into  $p(\sigma_V)$  yields the physical default curve for each value of  $\theta$ .

The three figures below display:

$$p(\sigma_V), \quad q(\sigma_V), \quad p(\sigma_V) - q(\sigma_V), \quad \sigma_V \in (0, 1).$$

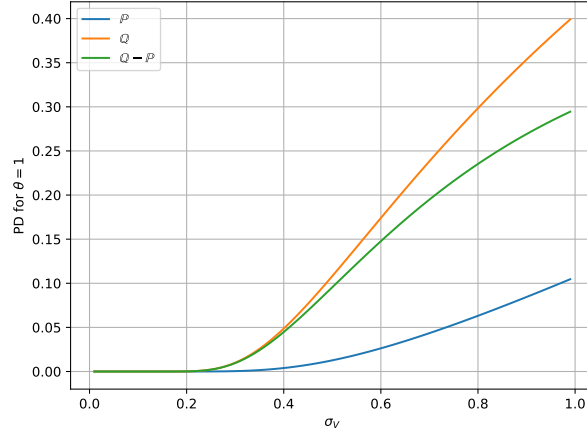


Figure 1:  $\theta = 1$ .

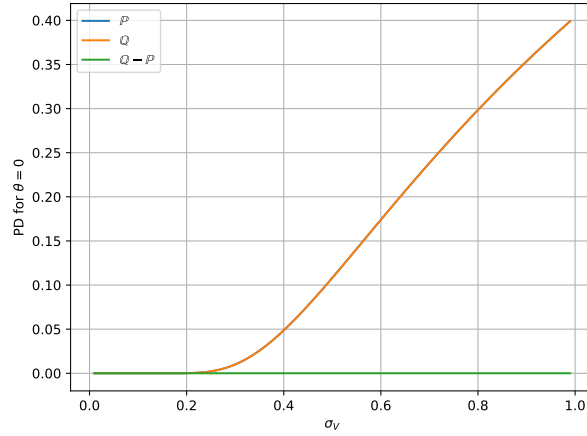


Figure 2:  $\theta = 0$ .

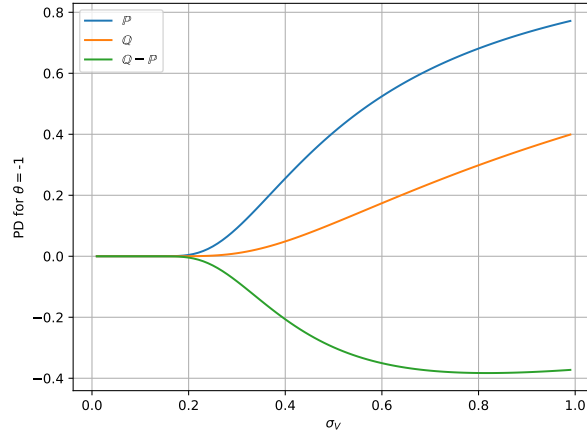


Figure 3:  $\theta = -1$ .

Interpretation:

- Case  $\theta = 1$  (positive market price of risk): A higher  $\sigma_V$  increases expected asset return under  $\mathbb{P}$  (because  $\mu_V = r + \theta\sigma_V$  increases). Thus the physical default probability increases more slowly than the risk-neutral one, and the gap  $p(\sigma_V) - q(\sigma_V)$  becomes *larger and positive*. Intuitively: risky firms earn a premium under  $\mathbb{P}$ , but not under  $\mathbb{Q}$ .
- Case  $\theta = 0$  (no risk premium): Then  $\mu_V = r$ , and physical and risk-neutral dynamics coincide. Hence

$$p(\sigma_V) = q(\sigma_V), \quad p - q \equiv 0.$$

- Case  $\theta = -1$  (negative risk premium): More volatile firms have *lower* expected returns under  $\mathbb{P}$ . This causes  $p(\sigma_V)$  to increase faster than  $q(\sigma_V)$  for large  $\sigma_V$ . The difference  $p(\sigma_V) - q(\sigma_V)$  may change sign depending on parameter values, giving a non-monotone shape.

Economic insight: Risk-neutral default probabilities reflect *pricing* of default risk, not real-world likelihoods. When  $\theta > 0$ , the market requires compensation for bearing volatility, so risk-neutral probabilities are distorted upward relative to physical ones. When  $\theta < 0$ , the opposite distortion occurs.

#### Question 4.

a) Equity and debt values at  $t = 0$  under the Merton model:

In the Merton model, the total firm value at time  $t$  is denoted by  $V_t$ , and the firm has a single zero-coupon debt obligation of face value  $B$  maturing at time  $T$ . Equity holders receive the residual value of the firm after debt is paid, so their payoff at  $T$  is

$$S_T = (V_T - B)^+.$$

Thus equity is a European call option on the firm value with strike  $B$ . We denote the equity value at time 0 by

$$S_0 := \text{value of equity at time 0} = C_0,$$

where  $C_0$  is the Black–Scholes call price.

Equity value:

Using the Black–Scholes formula,

$$S_0 = C_0 = V_0 \Phi(d_1) - Be^{-rT} \Phi(d_2),$$

with

$$d_1 = \frac{\ln(V_0/B) + (r + \frac{1}{2}\sigma_V^2)T}{\sigma_V \sqrt{T}}, \quad d_2 = d_1 - \sigma_V \sqrt{T}.$$

For  $V_0 = 100$ ,  $B = 50$ ,  $r = 0.05$ ,  $\sigma_V = 0.25$ , and  $T = 5$ :

$$d_1 \simeq 1.97, \quad d_2 \simeq 1.41.$$

Thus

$$S_0 = 100 \Phi(1.97) - 50e^{-0.25} \Phi(1.41) \simeq 61.7.$$

Debt value:

The debtholders receive at maturity

$$D_T = \min(V_T, B) = B - (B - V_T)^+.$$

Discounting under  $\mathbb{Q}$  gives

$$D_0 = e^{-rT} \mathbb{E}_{\mathbb{Q}}[D_T] = Be^{-rT} - P_0,$$

where  $P_0$  is the value of a put on the assets with strike  $B$ .

Using European put–call parity,

$$C_0 - P_0 = V_0 - Be^{-rT},$$

we solve for the put price:

$$P_0 = C_0 - V_0 + Be^{-rT}.$$

Substituting into  $D_0 = Be^{-rT} - P_0$ :

$$D_0 = Be^{-rT} - (C_0 - V_0 + Be^{-rT}) = V_0 - C_0.$$

Using  $C_0 = V_0\Phi(d_1) - Be^{-rT}\Phi(d_2)$ :

$$D_0 = V_0 - C_0 = V_0 - (V_0\Phi(d_1) - Be^{-rT}\Phi(d_2)).$$

Rearranging,

$$D_0 = V_0(1 - \Phi(d_1)) + Be^{-rT}\Phi(d_2) = V_0\Phi(-d_1) + Be^{-rT}\Phi(d_2).$$

This is the standard Merton closed-form value for debt.

Numerically:

$$D_0 = 100\Phi(-1.97) + 50e^{-0.25}\Phi(1.41) \simeq 38.3.$$

Verification of balance sheet identity:

$$S_0 + D_0 = 61.7 + 38.3 = 100 = V_0.$$

This confirms the Merton identity

$$\text{firm value} = \text{equity} + \text{debt}.$$

b) Credit spread surfaces for different leverage levels:

The credit spread for maturity  $T$  is defined as

$$c(0, T) = -\frac{1}{T} \log \left( \frac{D_0}{Be^{-rT}} \right),$$

i.e., the yield spread over the risk-free zero-coupon bond.

We compute  $c(0, T)$  for:

$$L = \frac{B}{V_0} \in \{0.3, 0.6, 0.9\}, \quad \sigma_V \in (0, 1), \quad T \in (0, 10),$$

and plot the resulting surfaces below.

Interpretation:

- *Low leverage* ( $L = 0.3$ ). Default is unlikely, so credit spreads are small and increase smoothly in both  $T$  and  $\sigma_V$ . For longer maturities, spreads eventually flatten: additional time adds little risk once the probability of default saturates.
- *Medium leverage* ( $L = 0.6$ ). The firm is riskier, spreads increase more sharply with  $\sigma_V$ . The surface shows stronger curvature in  $T$ , reflecting the interplay between default probability growth and discounting.
- *High leverage* ( $L = 0.9$ ). The firm is close to the default boundary. Even a small increase in  $\sigma_V$  produces a large jump in spreads, especially for short maturities—this is the well-known “volatility spike” for highly levered firms.

These patterns illustrate the key insight of the Merton model: credit spreads are driven primarily by leverage and asset volatility, and their relationship becomes highly nonlinear as the firm approaches the default threshold.

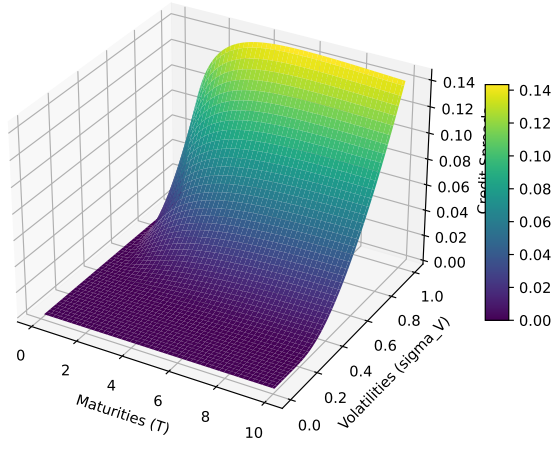


Figure 4:  $L = 0.3$

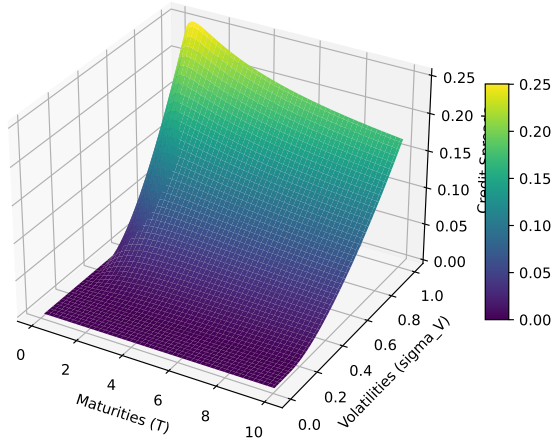


Figure 5:  $L = 0.6$

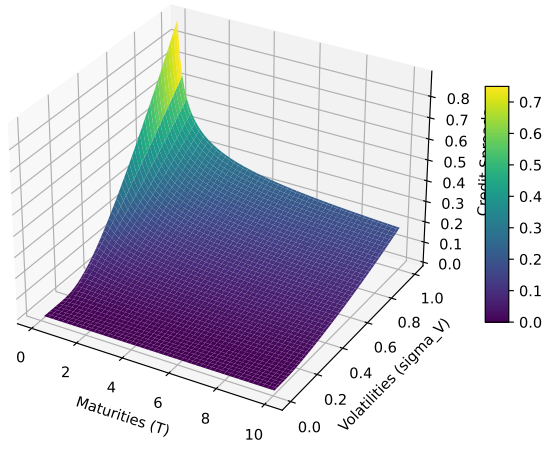


Figure 6:  $L = 0.9$