Quantitative Risk Management
Assignment 9 Solutions

Question 1:

1) Exponential distribution:
Let $c_n = \frac{\log n}{\beta}$ and $d_n = \frac{1}{\beta}$. Then

$$\begin{aligned}\mathbb{P}\left(\frac{M_n - c_n}{d_n} \le x\right) &= \mathbb{P}(M_n \le d_n x + c_n) = F^n(d_n x + c_n) \\ &= \left(1 - e^{-\beta(d_n x + c_n)}\right)^n \\ &= \left(1 - e^{-\beta(x/\beta + \log n/\beta)}\right)^n \\ &= \left(1 - \frac{1}{n}e^{-x}\right)^n.\end{aligned}$$

Taking the limit as $n \to \infty$ gives

$$\left(1 - \frac{1}{n}e^{-x}\right)^n \longrightarrow \exp\left(-e^{-x}\right),$$

which is the Gumbel distribution $H_0(x)$. Hence $F \in \mathrm{MDA}(H_0)$.

2) Pareto distribution:
Let $c_n = \kappa n^{1/\alpha} - \kappa$ and $d_n = \frac{\kappa n^{1/\alpha}}{\alpha}$. Then

$$\mathbb{P}\left(\frac{M_n - c_n}{d_n} \le x\right) = F^n(d_n x + c_n).$$

We compute the argument of F:

$$\begin{aligned}\kappa + d_n x + c_n &= \kappa + \frac{\kappa n^{1/\alpha}}{\alpha}x + \kappa n^{1/\alpha} - \kappa \\ &= \kappa n^{1/\alpha}\left(1 + \frac{x}{\alpha}\right).\end{aligned}$$

Thus,

$$\begin{aligned}F(d_n x + c_n) &= 1 - \left(\frac{\kappa}{\kappa n^{1/\alpha}(1 + x/\alpha)}\right)^\alpha \\ &= 1 - \frac{1}{n}\left(1 + \frac{x}{\alpha}\right)^{-\alpha}.\end{aligned}$$

Hence

$$\mathbb{P}\left(\frac{M_n - c_n}{d_n} \le x\right) = \left(1 - \frac{1}{n}\left(1 + \frac{x}{\alpha}\right)^{-\alpha}\right)^n.$$

Taking $n \to \infty$:

$$\left(1 - \frac{1}{n}a\right)^n \longrightarrow e^{-a}, \qquad a = \left(1 + \frac{x}{\alpha}\right)^{-\alpha}.$$

Thus the limit distribution is

$$H(x) = \exp\left(-(1 + x/\alpha)^{-\alpha}\right),$$

which is the GEV distribution with shape parameter $\xi = 1/\alpha$. Hence $F \in \mathrm{MDA}(H_{1/\alpha})$.

Question 2: If X has excess distribution $F_u = G_{\xi,\beta}$ over the threshold u, then

$$F_u(x) = \mathbb{P}(X - u \le x \mid X > u) = G_{\xi,\beta}(x).$$

We compute the excess distribution over a higher threshold $v \ge u$:

$$\begin{aligned} F_v(x) &= \mathbb{P}(X - v \le x \mid X > v) \\ &= 1 - \mathbb{P}(X - v > x \mid X > v) \\ &= 1 - \frac{\mathbb{P}(X > v + x)}{\mathbb{P}(X > v)}. \end{aligned}$$

We now express these terms in excesses above u:

$$\mathbb{P}(X > v + x) = \mathbb{P}(X - u > (v - u) + x), \qquad \mathbb{P}(X > v) = \mathbb{P}(X - u > v - u).$$

Thus,

$$\begin{aligned} F_v(x) &= 1 - \frac{\mathbb{P}(X - u > (v - u) + x)}{\mathbb{P}(X - u > v - u)} \\ &= 1 - \frac{1 - F_u((v - u) + x)}{1 - F_u(v - u)} \\ &= 1 - \frac{1 - G_{\xi,\beta}(x + v - u)}{1 - G_{\xi,\beta}(v - u)}. \end{aligned}$$

Using the explicit form of the GPD,

$$G_{\xi,\beta}(t) = 1 - \left(1 + \frac{\xi t}{\beta}\right)^{-1/\xi},$$

we obtain

$$\begin{aligned} F_v(x) &= 1 - \frac{\left(1 + \frac{\xi(x+v-u)}{\beta}\right)^{-1/\xi}}{\left(1 + \frac{\xi(v-u)}{\beta}\right)^{-1/\xi}} \\ &= 1 - \left(\frac{\beta + \xi(x + v - u)}{\beta + \xi(v - u)}\right)^{-1/\xi} \\ &= 1 - \left(1 + \frac{\xi x}{\beta + \xi(v - u)}\right)^{-1/\xi}. \end{aligned}$$

This is the GPD with scale parameter

$$\beta' = \beta + \xi(v - u).$$

Hence,

$$F_v(x) = G_{\xi,\, \beta + (v-u)\xi}(x),$$

as was to be shown.

Question 3:

1) Negative log returns and sample mean excess plot:
Let P_t denote the MSFT closing price. We compute the negative log returns

$$X_t = -\log\left(\frac{P_t}{P_{t-1}}\right),$$

and evaluate the empirical mean excess function

$$e_n(v) = \frac{\sum_{t=1}^{n} (X_t - v)\, \mathbf{1}_{\{X_t > v\}}}{\sum_{t=1}^{n} \mathbf{1}_{\{X_t > v\}}}.$$

Plotting $e_n(v)$ as a function of v gives the sample mean excess plot shown in Figure 1.

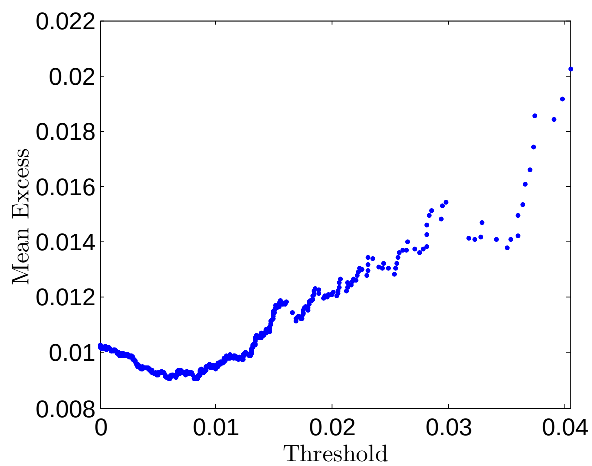

Figure 1: Sample mean excess plot for MSFT negative log returns.

2) GPD fit above the threshold $u = 0.01$:
We consider exceedances

$$\hat{X}_i = X_i - u, \qquad X_i > u,$$

and denote by N_u the number of exceedances. We assume that

$$\hat{X}_1, \ldots, \hat{X}_{N_u} \sim \mathrm{GPD}(\xi, \beta),$$

and estimate the parameters by maximizing the log-likelihood

$$\ell(\xi, \beta) = \sum_{i=1}^{N_u} \log g_{\xi,\beta}(\hat{X}_i),$$

where $g_{\xi,\beta}$ is the GPD density.
The maximum likelihood estimates are:

$$\hat{\xi} = 0.2171,$$
$$\hat{\beta} = 0.0075.$$

Question 4:
We first analyze the tail of F. For $x \geq 0$,

$$1 - F(x) = \left(\frac{\kappa}{\kappa + x^\theta}\right)^\lambda = \left(\frac{1}{1 + x^\theta/\kappa}\right)^\lambda.$$

Rewrite this expression by factoring out $x^{\theta\lambda}$:

$$\begin{aligned} 1 - F(x) &= x^{-\theta\lambda}\left(\frac{\kappa}{1 + \kappa/x^\theta}\right)^\lambda \\ &= x^{-\theta\lambda} L(x), \end{aligned}$$

where

$$L(x) = \left(\frac{\kappa}{1 + \kappa/x^\theta}\right)^\lambda.$$

We now verify that L is slowly varying at infinity. For any fixed $t > 0$,

$$\frac{L(tx)}{L(x)} = \left(\frac{\kappa}{1 + \kappa/(tx)^\theta} \Big/ \frac{\kappa}{1 + \kappa/x^\theta} \right)^\lambda = \left(\frac{1 + \kappa/x^\theta}{1 + \kappa/(t^\theta x^\theta)} \right)^\lambda .$$

Since $x^\theta \to \infty$,

$$\lim_{x\to\infty} \frac{1 + \kappa/x^\theta}{1 + \kappa/(t^\theta x^\theta)} = 1,$$

and therefore

$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1.$$

Thus, L is slowly varying at infinity.
Hence the tail is regularly varying with index

$$-(\theta\lambda).$$

By Gnedenko's theorem, a distribution with the survival function

$$1 - F(x) = x^{-1/\xi} L(x)$$

belongs to $\mathrm{MDA}(H_\xi)$ with

$$\frac{1}{\xi} = \theta\lambda \qquad \Longrightarrow \qquad \xi = \frac{1}{\theta\lambda}.$$

$$\boxed{F \in \mathrm{MDA}(H_{1/(\theta\lambda)})}$$