Quantitative Risk Management
Assignment 1 Solutions

Question 1: In each case, the loss is equal to:

$$L(t, t + \Delta) = -\lambda S(e^X - 1) \tag{1}$$

Since standard programming languages only simulate from a standard t-distribution with ν degrees of freedom, we have to multiply the simulated variables by a constant to achieve the correct variance. Let T be a standard t-distributed variable with ν degrees of freedom, and let $X = mT$. Then:

$$\begin{aligned}
\mathbb{V}[X] &= m^2 \mathbb{V}[T] \\
m &= \sqrt{\frac{\mathbb{V}[X]}{\mathbb{V}[T]}} \\
m &= \frac{0.01}{\sqrt{\frac{\nu}{\nu-2}}}
\end{aligned}$$

For each distribution of interest X, the empirical distribution of losses is shown below. The red curve represents the probability density function of a normal distribution with mean and variance equal to the empirical mean and variance of the loss.
For larger values of ν, the empirical distribution appears to be closer to Gaussianity. When ν is small, we see that the normal distribution underestimates the weight in the center of the distribution and the heaviness of the tails. In none of these cases is the distribution of $L(t, t+\Delta)$ actually a normal distribution. From (1), we see that $L(t, t + \Delta)$ is bounded above by λS. A normal random variable is unbounded, and so $L(t, t + \Delta)$ is not normal.
The linearized loss is equal to:

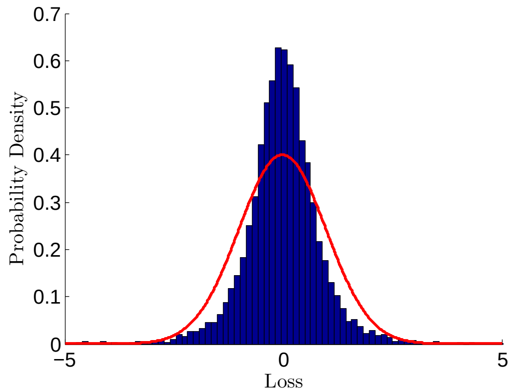
$$L^\delta(t, t + \Delta) = -\lambda S X$$

The mean and variance of $L^\delta(t, t + \Delta)$ are computed easily:

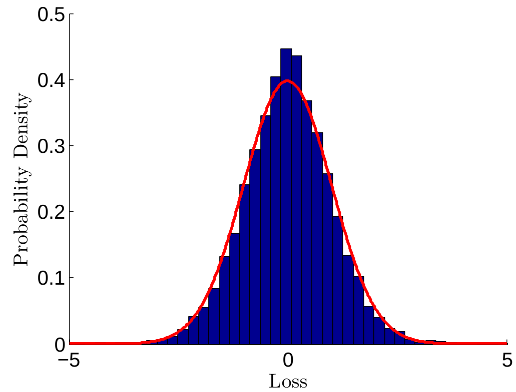
$$\begin{aligned}
\mathbb{E}[L^\delta(t, t + \Delta)] &= -\lambda S \mathbb{E}[X] \\
&= 0 \\
\mathbb{V}[L^\delta(t, t + \Delta)] &= \lambda^2 S^2 \mathbb{V}[X] \\
&= 1^2 \cdot 100^2 \cdot 0.01^2 \\
&= 1
\end{aligned}$$

The exact distributions of $L^\delta(t, t + \Delta)$ corresponding to each distribution of X are the following:

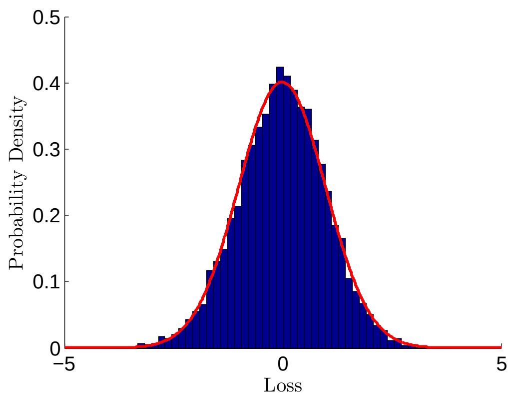
1. A scaled Student's t-distribution with 3 degrees of freedom, mean zero, and standard deviation 1.
2. A scaled Student's t-distribution with 10 degrees of freedom, mean zero, and standard deviation 1.
3. A scaled Student's t-distribution with 50 degrees of freedom, mean zero, and standard deviation 1.
4. A standard normal distribution.



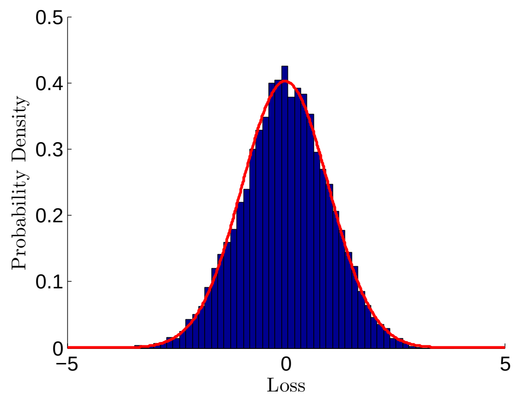
(a) Student's t - $\nu = 3$



(b) Student's t - $\nu = 10$



(c) Student's t - $\nu = 50$



(d) Gaussian

Figure 1: Loss distribution for various distributions of risk factor changes X.

Question 2: For the given parameters, the empirical distribution of $L(t, t + \Delta)$ is:

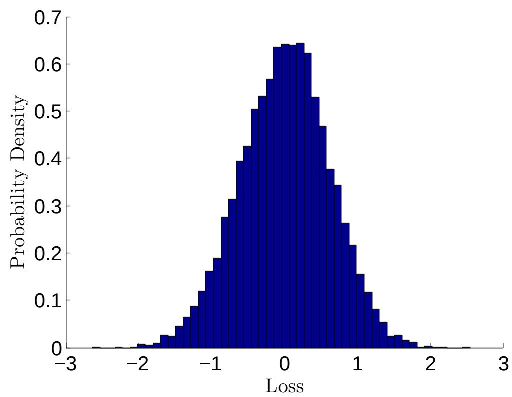


Figure 2: Loss distribution for the European call option.

When simulating the values of $\sigma_{t+\Delta}$, we should generally require that the volatility remains positive. Since the changes are assumed to be normally distributed, there is a small chance that the corresponding volatility becomes negative. Hence, we can model the future volatility with the absolute value of the simulated quantities. Note that the negative values would actually happen very rarely and this approximate fix will not have a significant effect on the overall distribution.

The linearized loss is equal to the following:

$$L^{\delta}(t, t + \Delta) = -\frac{\partial C^{BS}}{\partial t}\Delta - \frac{\partial C^{BS}}{\partial S} S X_1 - \frac{\partial C^{BS}}{\partial r} X_2 - \frac{\partial C^{BS}}{\partial \sigma} X_3 \tag{2}$$

These partial derivatives have closed-form expressions:

$$\begin{aligned}
\frac{\partial C^{BS}}{\partial t} &= -\frac{S\phi(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}\Phi(d_2) \\
\frac{\partial C^{BS}}{\partial S} &= \Phi(d_1) \\
\frac{\partial C^{BS}}{\partial r} &= KTe^{-rT}\Phi(d_2) \\
\frac{\partial C^{BS}}{\partial \sigma} &= S\phi(d_1)\sqrt{T}
\end{aligned}$$

where

$$\begin{aligned}
d_1 &= \frac{\log\frac{S}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T}
\end{aligned}$$

Using the same simulated values of $\boldsymbol{X}$ and substituting into (2) gives the following empirical distribution:

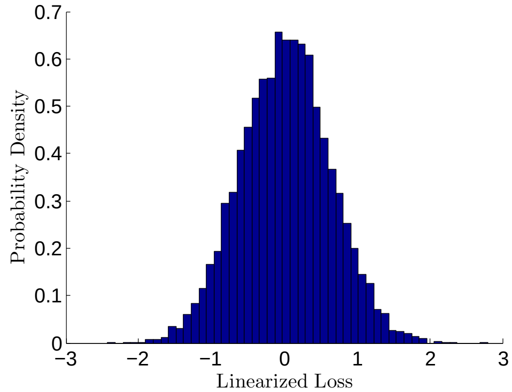


Figure 3: Linearized loss distribution for the European call option.

There are three risk factor changes that contribute to the linearized loss, each of them multiplied by a scaling factor. To asses which of these contributes the most to the linearized loss, we compute the variance of each of the three terms:

$$
\begin{aligned}
\mathbb{V}\left[\frac{\partial C^{BS}}{\partial S} S X_1\right] &= \left(\frac{\partial C^{BS}}{\partial S}\right)^2 S^2 \mathbb{V}[X_1] \\
&= 0.64^2 \cdot 100^2 \cdot 0.01^2 \\
&= 0.41 \\
\mathbb{V}\left[\frac{\partial C^{BS}}{\partial r} X_2\right] &= \left(\frac{\partial C^{BS}}{\partial r}\right)^2 \mathbb{V}[X_2] \\
&= 53.23^2 \cdot 10^{-8} \\
&= 2.8 \cdot 10^{-5} \\
\mathbb{V}\left[\frac{\partial C^{BS}}{\partial \sigma} X_3\right] &= \left(\frac{\partial C^{BS}}{\partial \sigma}\right)^2 \mathbb{V}[X_3] \\
&= 37.52^2 \cdot 10^{-6} \\
&= 1.4 \cdot 10^{-3}
\end{aligned}
$$

These results show that the majority of the randomness in $L^\delta(t, t + \Delta)$ is caused by the changes in the underlying stock price.

Question 3: $X \sim \mathcal{N}(\mu, \sigma^2)$.

$$
\begin{aligned}
\mathbb{E}[e^X] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} e^x dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x^2-2\mu x+\mu^2)}{2\sigma^2}} e^x dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x^2-2(\mu+\sigma^2)x+\mu^2)}{2\sigma^2}} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x^2-2(\mu+\sigma^2)x)}{2\sigma^2}} e^{-\frac{\mu^2}{2\sigma^2}} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{x^2-2(\mu+\sigma^2)x+(\mu+\sigma^2)^2-(\mu+\sigma^2)^2}{2\sigma^2}} e^{-\frac{\mu^2}{2\sigma^2}} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu-\sigma^2)^2}{2\sigma^2}} e^{\frac{(\mu+\sigma^2)^2}{2\sigma^2}} e^{-\frac{\mu^2}{2\sigma^2}} dx \\
&= \frac{e^{\frac{(\mu+\sigma^2)^2}{2\sigma^2}} e^{-\frac{\mu^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x'-\mu)^2}{2\sigma^2}} dx', \qquad \text{using } x' = x - \sigma^2 \\
&= e^{\frac{(\mu+\sigma^2)^2}{2\sigma^2}} e^{-\frac{\mu^2}{2\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x'-\mu)^2}{2\sigma^2}} dx' \\
&= e^{\frac{\mu^2+2\sigma^2\mu+\sigma^4}{2\sigma^2}} e^{-\frac{\mu^2}{2\sigma^2}} \\
&= e^{\mu+\frac{1}{2}\sigma^2}
\end{aligned}
$$

Question 4:

1. The value of the portfolio at time t is the sum of the positions in the two stocks in EUR

$$V_t = \lambda_1 S_{t,1} + \lambda_2 S_{t,2} e_t = \lambda_1 e^{Z_{t,1}} + \lambda_2 e^{Z_{t,2}} e^{Z_{t,3}} = f(t, Z_t),$$

   with $Z_t = (Z_{t,1}, Z_{t,2}, Z_{t,3})^\top$.

2. The loss of the portfolio at time $t+1$ is given by

$$L_{t,t+1} = -\lambda_1 S_{t,1} \left(e^{X_{t+1,1}} - 1\right) - \lambda_2 S_{t,2} e_t \left(e^{X_{t+1,2}} e^{X_{t+1,3}} - 1\right).$$

3. The linearized loss of the portfolio at time $t+1$ is given by

$$L^\delta_{t,t+1} = -\lambda_1 S_{t,1} X_{t+1,1} - \lambda_2 S_{t,2} e_t (X_{t+1,2} + X_{t+1,3}).$$

4. We define

$$w_1 = \frac{\lambda_1 S_{t,1}}{V_t}$$

   and

$$w_2 = \frac{\lambda_2 S_{t,2} e_t}{V_t},$$

   so that

$$L^\delta_{t,t+1} = -V_t \left(w_1 X_{t+1,1} + w_2 (X_{t+1,2} + X_{t+1,3})\right).$$