Quantitative Risk Management
Assignment 7 Solutions

Question 1.

Part 1): We have the AR(1) process

$$Y_t = \phi Y_{t-1} + \epsilon_t, \qquad |\phi| < 1, \qquad (\epsilon_t) \sim WN(0,1).$$

Taking variance:

$$\mathrm{Var}(Y_t) = \mathbb{E}[Y_t^2] = \mathbb{E}[(\phi Y_{t-1} + \epsilon_t)^2] = \phi^2 \,\mathrm{Var}(Y_{t-1}) + 1 \;\Rightarrow\; \mathrm{Var}(Y_t) = \frac{1}{1-\phi^2}.$$

Now for the covariance:

$$\mathrm{Cov}(Y_t, Y_{t+h}) = \mathbb{E}[Y_t Y_{t+h}] = \mathbb{E}[Y_t(\phi^h Y_t + \text{terms independent of } Y_t)] = \phi^h \mathbb{E}[Y_t^2].$$

Hence

$$\gamma(h) = \frac{\phi^{|h|}}{1-\phi^2}, \qquad \rho(h) = \frac{\gamma(h)}{\gamma(0)} = \phi^{|h|}.$$

Part 2): We start from

$$X_t = \sigma_t Z_t, \qquad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2,$$

and decompose:

$$X_t^2 = \sigma_t^2 + \sigma_t^2(Z_t^2 - 1).$$

The term $\sigma_t^2(Z_t^2-1)$ is a martingale difference sequence (MDS) because:

- Z_t is independent of $\mathcal{F}_{t-1}$ and of σ_t,
- $\mathbb{E}[Z_t^2 - 1] = 0$, and
- $\mathbb{E}[|\sigma_t^2(Z_t^2-1)|] < \infty$ if $\mathbb{E}[X_t^4] < \infty$.

Thus

$$\mathbb{E}[\sigma_t^2(Z_t^2-1) \mid \mathcal{F}_{t-1}] = \sigma_t^2 \mathbb{E}[Z_t^2 - 1] = 0.$$

Define

$$\epsilon_t := \sigma_t^2(Z_t^2 - 1).$$

Then (ϵ_t) is white noise, and

$$X_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \epsilon_t.$$

To center the process, subtract its mean $c = \mathbb{E}[X_t^2] = \frac{\alpha_0}{1-\alpha_1}$, giving:

$$X_t^2 - c = \alpha_1(X_{t-1}^2 - c) + \epsilon_t.$$

This is an AR(1) process with mean zero and coefficient $\phi = \alpha_1$.
The condition $\mathbb{E}[X_t^4] < \infty$ ensures that $\mathbb{E}[\epsilon_t^2] < \infty$, so the MDS ϵ_t has finite variance — necessary for covariance stationarity.

Part 3): From Part 1), the autocorrelation function for $(X_t^2 - c)_{t\in\mathbb{Z}}$ is

$$\rho(h) = \alpha_1^{|h|}.$$

Hence, even though the ARCH(1) process (X_t) itself is white noise (uncorrelated), its squared process (X_t^2) exhibits positive autocorrelation whenever $\alpha_1 > 0$.

Interpretation:

- (X_t) shows no serial correlation — returns are unpredictable.
- (X_t^2) (or volatility) shows persistence — large shocks tend to be followed by large shocks.
- The persistence of volatility decays geometrically at rate $|\alpha_1|$.

Question 2.

Let (X_t) be an $ARCH(1)$ process:

$$X_t = \sigma_t Z_t, \qquad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2,$$

with $\mathbb{E}[Z_t] = 0$, $\mathbb{E}[Z_t^2] = 1$, and finite $\mathbb{E}[Z_t^4] < \infty$. Assume the process admits a finite fourth moment $\mathbb{E}[X_t^4] < \infty$.

Step 1: Expand the fourth power. Start from $X_t^2 = \sigma_t^2 Z_t^2$ and substitute for σ_t^2:

$$X_t^2 = (\alpha_0 + \alpha_1 X_{t-1}^2) Z_t^2.$$

Squaring both sides gives

$$X_t^4 = (\alpha_0 + \alpha_1 X_{t-1}^2)^2 Z_t^4 = (\alpha_0^2 + 2\alpha_0\alpha_1 X_{t-1}^2 + \alpha_1^2 X_{t-1}^4) Z_t^4.$$

Step 2: Take expectations. Using independence of Z_t and X_{t-1}, we have

$$\mathbb{E}[X_t^4] = \mathbb{E}[Z_t^4]\big(\alpha_0^2 + 2\alpha_0\alpha_1 \mathbb{E}[X_{t-1}^2] + \alpha_1^2 \mathbb{E}[X_{t-1}^4]\big).$$

By strict stationarity, $\mathbb{E}[X_{t-1}^4] = \mathbb{E}[X_t^4]$.

Step 3: Substitute known moments. From the covariance-stationarity of $ARCH(1)$, we know

$$\mathbb{E}[X_t^2] = \frac{\alpha_0}{1-\alpha_1}.$$

Substituting this gives

$$\mathbb{E}[X_t^4] = \mathbb{E}[Z_t^4]\left(\alpha_0^2 + 2\alpha_0\alpha_1 \frac{\alpha_0}{1-\alpha_1} + \alpha_1^2 \mathbb{E}[X_t^4]\right).$$

Step 4: Solve for $\mathbb{E}[X_t^4]$. Rearrange to isolate $\mathbb{E}[X_t^4]$:

$$\mathbb{E}[X_t^4]\big(1 - \alpha_1^2 \mathbb{E}[Z_t^4]\big) = \mathbb{E}[Z_t^4]\left(\alpha_0^2 + \frac{2\alpha_0^2\alpha_1}{1-\alpha_1}\right).$$

Simplify:

$$\mathbb{E}[X_t^4] = \frac{\alpha_0^2 \mathbb{E}[Z_t^4](1+\alpha_1)}{(1-\alpha_1)(1-\alpha_1^2\mathbb{E}[Z_t^4])}.$$

Remarks:

- The denominator requires $1 - \alpha_1^2 \mathbb{E}[Z_t^4] > 0$ for $\mathbb{E}[X_t^4]$ to exist — this gives a fourth-moment stationarity condition.
- When $Z_t \sim \mathcal{N}(0,1)$, $\mathbb{E}[Z_t^4] = 3$, so finiteness requires $\alpha_1 < 1/\sqrt{3} \approx 0.577$.
- This illustrates how heavy-tailed innovations (large $\mathbb{E}[Z_t^4]$) tighten the stability bound on α_1.

Question 3.

We show that every d-dimensional copula C satisfies the Fréchet bounds:

$$\max\left\{1-d+\sum_{i=1}^{d} u_i,\, 0\right\} \;\le\; C(u_1,\dots,u_d) \;\le\; \min\{u_1,\dots,u_d\}, \qquad (u_1,\dots,u_d)\in[0,1]^d.$$

(a) Lower bound.

$$C(u_1,\dots,u_d) = \Pr(U_1 \le u_1,\dots,U_d \le u_d) = 1-\Pr\left(\bigcup_{i=1}^{d}\{U_i > u_i\}\right).$$

By the union bound,

$$\Pr\left(\bigcup_{i=1}^{d}\{U_i > u_i\}\right) \le \sum_{i=1}^{d}\Pr(U_i > u_i) = \sum_{i=1}^{d}(1-u_i),$$

hence

$$C(u_1,\dots,u_d) \ge 1-\sum_{i=1}^{d}(1-u_i) = 1-d+\sum_{i=1}^{d}u_i.$$

Because C is a cdf, it must also satisfy $C(u_1,\dots,u_d)\ge 0$. Combining both inequalities gives

$$C(u_1,\dots,u_d) \ge \max\left\{1-d+\sum_{i=1}^{d}u_i,\, 0\right\}.$$

(b) Upper bound. Since the joint probability of all events cannot exceed the probability of any one of them,

$$C(u_1,\dots,u_d) = \Pr(U_1 \le u_1,\dots,U_d \le u_d) \le \Pr(U_i \le u_i) = u_i, \quad \forall i,$$

hence

$$C(u_1,\dots,u_d) \le \min\{u_1,\dots,u_d\}.$$

(c) Case $d=2$: the countermonotonic copula. Let $(U_1,U_2)=(U,1-U)$ with $U\sim\mathrm{Unif}(0,1)$. Then

$$C(u_1,u_2) = \Pr(U_1\le u_1,\ U_2\le u_2) = \Pr(U\le u_1,\ 1-U\le u_2) = \Pr(1-u_2\le U\le u_1).$$

This probability equals $\max\{u_1+u_2-1,\,0\}$, which coincides with the Fréchet *lower bound* in dimension two. Hence $(U,1-U)$ follows the countermonotonic copula.

(d) Example attaining the upper bound. For perfect positive dependence, take $(U_1,\dots,U_d)=(U,\dots,U)$ with $U\sim\mathrm{Unif}(0,1)$. Then

$$C(u_1,\dots,u_d) = \Pr(U\le\min\{u_1,\dots,u_d\}) = \min\{u_1,\dots,u_d\},$$

which equals the Fréchet *upper bound*. This copula is called the comonotonic copula.

Intuition. The Fréchet bounds describe the extreme possible dependence structures between uniformly distributed marginals:

- the *upper bound* corresponds to perfect positive dependence (all U_i move together);
- the *lower bound* corresponds to perfect negative dependence (U_i move in opposite directions);
- any real copula lies between these two extremes.

Question 4:
Let $\log X_1 \sim \mathcal{N}(0,1)$ and $\log X_2 \sim \mathcal{N}(0,\sigma^2)$. Write

$$X_1 = e^{Z_1}, \qquad X_2 = e^{\sigma Z_2}, \qquad Z_1, Z_2 \overset{iid}{\sim} \mathcal{N}(0,1).$$

Recall the moment formula for a standard normal: for any $a \in \mathbb{R}$,

$$\mathbb{E}\big[e^{aZ}\big] \;=\; e^{\frac{1}{2}a^2}, \qquad Z \sim \mathcal{N}(0,1).$$

Means and variances.

$$\mathbb{E}[X_1] = e^{\frac{1}{2}}, \quad \mathbb{E}[X_1^2] = e^2, \quad \mathrm{Var}(X_1) = e^2 - e,$$
$$\mathbb{E}[X_2] = e^{\frac{1}{2}\sigma^2}, \quad \mathbb{E}[X_2^2] = e^{2\sigma^2}, \quad \mathrm{Var}(X_2) = e^{2\sigma^2} - e^{\sigma^2}.$$

Maximal correlation (comonotonic): take $Z_1 = Z_2 = Z$. Then $X_1 = e^Z$, $X_2 = e^{\sigma Z}$, so

$$\mathbb{E}[X_1 X_2] = \mathbb{E}\Big[e^{(1+\sigma)Z}\Big] = e^{\frac{1}{2}(1+\sigma)^2}.$$

Hence

$$\mathrm{Cov}(X_1, X_2) = e^{\frac{1}{2}(1+\sigma)^2} - e^{\frac{1}{2}} e^{\frac{1}{2}\sigma^2} = e^{\frac{1}{2}(1+\sigma^2)}(e^{\sigma} - 1).$$

Therefore,

$$\rho_{\max} = \frac{e^{\frac{1}{2}(1+\sigma^2)}(e^{\sigma} - 1)}{\sqrt{(e^2 - e)\,(e^{2\sigma^2} - e^{\sigma^2})}} = \frac{e^{\sigma} - 1}{\sqrt{(e-1)\,(e^{\sigma^2} - 1)}}.$$

Minimal correlation (countermonotonic): take $Z_1 = -Z_2 = Z$. Then $X_1 = e^Z$, $X_2 = e^{-\sigma Z}$, so

$$\mathbb{E}[X_1 X_2] = e^{\frac{1}{2}(1-\sigma)^2}, \quad \mathrm{Cov}(X_1, X_2) = e^{\frac{1}{2}(1+\sigma^2)}(e^{-\sigma} - 1),$$

and

$$\rho_{\min} = \frac{e^{-\sigma} - 1}{\sqrt{(e-1)\,(e^{\sigma^2} - 1)}}.$$

Summary (attainable bounds):

$$\boxed{\rho_{\max}(\sigma) = \frac{e^{\sigma} - 1}{\sqrt{(e-1)\,(e^{\sigma^2} - 1)}}, \qquad \rho_{\min}(\sigma) = \frac{e^{-\sigma} - 1}{\sqrt{(e-1)\,(e^{\sigma^2} - 1)}}}.$$

Notes:

- The comonotonic case corresponds to the Fréchet upper bound (perfect positive dependence).
- The countermonotonic case corresponds to the Fréchet lower bound (perfect negative dependence).
- Because of the nonlinear (exponential) transformation, even perfect dependence in (Z_1, Z_2) yields $|\rho| < 1$ for (X_1, X_2) unless $\sigma = 1$. The asymmetry of the lognormal distribution limits attainable correlation.
- As $\sigma \to \infty$, both $\rho_{\max}$ and $\rho_{\min}$ tend to 0. Intuitively, $X_2 = e^{\sigma Z}$ becomes extremely volatile relative to $X_1 = e^Z$, so even under perfect (co- or counter-)monotonic dependence, the (linear) correlation collapses to zero — consistent with the flattening of both curves in the plot below.
- As $\sigma \to 0^+$, $X_2 = e^{\sigma Z_2} \to 1$ (nearly constant), while $X_1 = e^{Z_1}$ remains random. Thus, the correlation at $\sigma = 0$ is undefined (since $\mathrm{Var}(X_2) = 0$), but the *limit* exists and equals

$$\lim_{\sigma \to 0^+} \rho_{\max} = \frac{1}{\sqrt{e-1}} \approx 0.763, \qquad \lim_{\sigma \to 0^+} \rho_{\min} = -\frac{1}{\sqrt{e-1}} \approx -0.763.$$

  This matches the endpoints in the figure below.

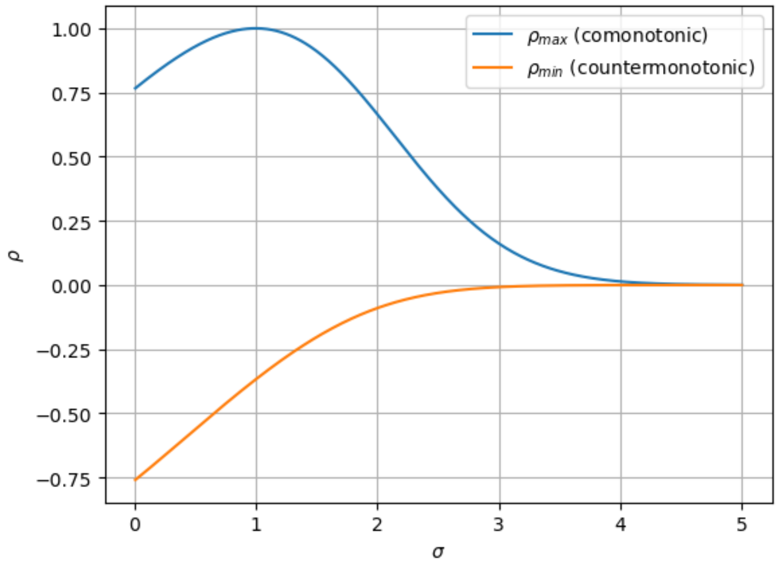

Figure 1: Attainable correlations for a pair of lognormal variables.

Python code to reproduce the figure:

```
import numpy as np
import matplotlib.pyplot as plt

sigma = np.linspace(0.01, 5, 500)
rho_max = (np.exp(sigma) - 1) / np.sqrt((np.e - 1) * (np.exp(sigma**2) - 1))
rho_min = (np.exp(-sigma) - 1) / np.sqrt((np.e - 1) * (np.exp(sigma**2) - 1))

plt.plot(sigma, rho_max, label=r"$\rho_{max}$ (comonotonic)")
plt.plot(sigma, rho_min, label=r"$\rho_{min}$ (countermonotonic)")
plt.xlabel(r"$\sigma$")
plt.ylabel(r"$\rho$")
plt.legend()
plt.grid(True)
plt.show()
```