Quantitative Risk Management
Assignment 10

Question 1: Let X have a logistic distribution with CDF

$$F_X(x) = \left(1 + \exp\left(-\frac{x-\mu}{\sigma}\right)\right)^{-1}, \qquad \mu \in \mathbb{R},\ \sigma > 0.$$

Show that $F_X \in \text{MDA}(H_\xi)$ for some ξ, and determine the normalizing sequences $a_n > 0$ and b_n such that

$$F_X^n(a_n x + b_n) \ \to \ H_\xi(x).$$

Question 2: Let X be a random variable with cdf $F_X(x) = \frac{1}{1+\exp(-x)}$.

a) Does X have a density? If yes, derive it.

b) Find all $k \in \mathbb{N}$ such that
$$\mathbb{E}|X|^k < \infty.$$

c) Does F_X belong to $\text{MDA}(H_\xi)$ for a generalized extreme value distribution H_ξ? If yes, what is H_ξ and what are the normalizing sequences?

d) Calculate the excess distribution function $F_u(x) = \mathbb{P}(X - u \leq x | X > u), \quad x \geq 0$.

e) Does there exist a parameter $\xi \in \mathbb{R}$ and a positive measurable function β such that
$$\lim_{u\to\infty} \sup_{x>0} \left|F_u(x) - G_{\xi,\beta(u)}(x)\right| = 0$$
for a generalized Pareto distribution $G_{\xi,\beta}$? If yes, for which ξ and β does this hold?

Question 3: A bank uses a basic internal rating system with three non-default ratings A, B, and C, plus a default state D. The one–year rating transition matrix contains missing values:

$$P = \begin{pmatrix} 0.95 & ? & 0 & 0 \\ 0.05 & ? & 0.1 & 0.05 \\ 0 & 0.2 & 0.5 & ? \\ ? & ? & ? & ? \end{pmatrix},$$

where rows correspond to current ratings (A, B, C, D) and columns correspond to future ratings.

a) Fill in the missing transition probabilities.

b) Assuming the rating system follows a stationary Markov chain (i.e., the one–year transition matrix is the same in each period), compute and plot the t–year default probabilities
$$\Pr(X_t = D \mid X_0 = A), \qquad \Pr(X_t = D \mid X_0 = B), \qquad \Pr(X_t = D \mid X_0 = C),$$
for $t = 1, \ldots, 20$.

Question 4: Consider a stationary Markov chain with three rating states:

$$(1)\ \text{Good}, \qquad (2)\ \text{Medium}, \qquad (3)\ \text{Default}.$$

The one–year transition matrix is

$$P = \begin{pmatrix} \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ 0 & 0 & 1 \end{pmatrix}.$$

a) Verify that the vectors

$$v_1 = (1,1,1)^\top, \qquad v_2 = (1,1,0)^\top, \qquad v_3 = (-2,1,0)^\top$$

are eigenvectors of P, and compute their corresponding eigenvalues.

b) Show that P is diagonalizable and can be written as

$$P = A\,\Delta\,A^{-1},$$

where

$$\Delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{4}{5} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{3} & \frac{2}{3} & -1 \\ -\frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}.$$

Compute the corresponding matrix A.

c) Using the diagonalization in part (b), compute the probability that a borrower starting in rating 1 will be in default within 20 years.

Question 5: Consider a one–period model for a defaultable zero–coupon bond with maturity $T = 1$ year and face value 1. The bond pays:

$$\text{Payoff at } T = \begin{cases} 1, & \text{if no default,} \\ 1-\delta = 0.4, & \text{if default.} \end{cases}$$

The physical (real–world) default probability is $1-p = 0.01$. The risk–free simple interest rate is $r = 0.025$. The current price of the defaultable bond is $p_1(0,1) = 0.961$.

a) Compute the expected payoff of the bond under the real–world measure.

b) Determine the risk–neutral default probability implied by the bond price.

c) Let τ denote the default time of the bond. In this one–period model,

$$\tau = \begin{cases} 0, & \text{if default occurs within the period,} \\ T, & \text{otherwise.} \end{cases}$$

A stylized credit default swap (CDS) pays $\mathbf{1}_{\{\tau \le T\}}$, i.e., it pays 1 if default occurs before maturity. Compute its price and construct a replicating portfolio using the defaultable bond and a default-free bond.