Quantitative Risk Management
Assignment 4

Question 1: Consider $d = 100$ defaultable corporate bonds each with a face value of 1000 CHF, an annual coupon of 5%, and a time to maturity of one year. Suppose that the current price of each bond is 1000 CHF (i.e., they all trade at par). Assume that defaults of different bonds are independent. The default probability is identical for all bonds and is equal to 0.02. Denote with L_i the loss on a bond of company i over the next year and with I_i the default indicator of firm i (i.e., $I_i = 1$ if firm i defaults).

1. Write L_i in terms of risk factor I_i.
2. What is the probability distribution of L_i?
3. Compare the following two portfolios, each worth 100,000 CHF: V_a consists of 100 units of a single bond and V_b consists of 1 unit of each bond (with $d = 100$). Write L_a and L_b in terms of the risk factors. What are the distributions of L_a and L_b? Compute VaR_α for both portfolios with $\alpha = 0.95$ and $\alpha = 0.99$. Comment on the results.

Question 2: Given the joint density function of random variables X_1 and X_2:

$$f(x_1, x_2) = \theta(\theta + 1)(x_1 + x_2 - 1)^{-(\theta+2)}, \quad x_1, x_2 \geq 1,\, \theta > 0,$$

1. derive an expression for the joint CDF;
2. derive an expression for the marginal CDFs;
3. compute the correlation between X_1 and X_2.

Question 3: Consider $Y = (X_1, X_2)$ to be jointly normally distributed with correlation ρ. Compute $VaR_\alpha(X_1 + X_2)$.

Question 4: Consider $(X_1, X_2) \sim \mathcal{N}(\mathbf{0}, I_2)$ and $(Y_1, Y_2) = (X_1, VX_1)$, with V a random variable attaining values 1 and -1, each with probability 1/2. Assuming independence between V and X_1,

1. show that the two random vectors have the same marginal distributions;
2. show that the two random vectors consist of uncorrelated random variables;
3. derive the CDFs of $X_1 + X_2$ and $Y_1 + Y_2$;
4. compute $VaR_\alpha(X_1 + X_2)$ and $VaR_\alpha(Y_1 + Y_2)$.

Question 5. Let $X = (X_1, \dots, X_d)$ be an elliptically distributed return vector with mean $\mu = \mathbb{E}[X]$. For a portfolio $w \in \mathbb{R}^d$, define the *one-day loss* $L_w := -w^\top X$. Assume two portfolios w, v satisfy the same expected return, i.e., $w^\top \mu = v^\top \mu$ (equivalently, the same expected loss $\mathbb{E}[L_w] = \mathbb{E}[L_v]$).
Show that for suitable levels α,

$$ES_\alpha(L_w) \;\leq\; ES_\alpha(L_v) \quad \iff \quad \mathrm{Var}(L_w) \;\leq\; \mathrm{Var}(L_v).$$