Quantitative Risk Management
Assignment 8 Solutions

Question 1.
Let the joint cdf of (X, Y) be

$$F_{X,Y}(x, y) = \frac{1}{1 + e^{-x} + e^{-y}}, \qquad (x, y) \in \mathbb{R}^2.$$

(a) *Marginal distributions.*

Compute the marginal of X by taking the limit as $y \to \infty$:

$$F_X(x) = \lim_{y\to\infty} F_{X,Y}(x, y) = \lim_{y\to\infty} \frac{1}{1 + e^{-x} + e^{-y}} = \frac{1}{1 + e^{-x}}.$$

Similarly,

$$F_Y(y) = \lim_{x\to\infty} F_{X,Y}(x, y) = \frac{1}{1 + e^{-y}}.$$

Hence both marginals are standard logistic distributions:

$$F_X(x) = F_Y(y) = \frac{1}{1 + e^{-x}}, \qquad f_X(x) = f_Y(y) = \frac{e^{-x}}{(1 + e^{-x})^2}.$$

(b) *Find the copula.*

By Sklar's theorem, for continuous marginals we have

$$C(u, v) = F_{X,Y}\big(F_X^{-1}(u),\, F_Y^{-1}(v)\big).$$

Compute the quantile (inverse cdf) of the logistic distribution:

$$u = \frac{1}{1 + e^{-x}} \quad\Longrightarrow\quad \frac{1}{u} - 1 = e^{-x} \quad\Longrightarrow\quad x = -\log\left(\frac{1}{u} - 1\right) = \log\left(\frac{u}{1-u}\right).$$

Therefore,

$$F_X^{-1}(u) = \log\left(\frac{u}{1-u}\right), \qquad F_Y^{-1}(v) = \log\left(\frac{v}{1-v}\right).$$

Now substitute into $F_{X,Y}(x, y)$:

$$\begin{aligned}
C(u, v) &= F_{X,Y}\left(\log\frac{u}{1-u},\, \log\frac{v}{1-v}\right) \\
&= \frac{1}{1 + e^{-\log\left(\frac{u}{1-u}\right)} + e^{-\log\left(\frac{v}{1-v}\right)}} \\
&= \frac{1}{1 + \frac{1-u}{u} + \frac{1-v}{v}} \\
&= \frac{1}{\frac{uv + u(1-v) + v(1-u)}{uv}} = \frac{uv}{u + v - uv}.
\end{aligned}$$

Hence the copula is

$$\boxed{C(u, v) = \frac{uv}{u + v - uv}}, \qquad (u, v) \in [0, 1]^2.$$

(c) *Verification that C is a valid copula.*

We check the three defining properties:

(i) *Groundedness and uniform marginals:*

$$C(0,v)=C(u,0)=0, \qquad C(u,1)=u, \quad C(1,v)=v.$$

Indeed, $C(u,1)=\frac{u\cdot 1}{u+1-u}=u$ and similarly for $C(1,v)$.

(ii) *2-increasing property:* We show that $\frac{\partial^2 C(u,v)}{\partial u\,\partial v}\geq 0$ for all $(u,v)\in(0,1)^2$.
First compute:

$$\frac{\partial C}{\partial u}=\frac{v(u+v-uv)-uv(1-v)}{(u+v-uv)^2}=\frac{v^2}{(u+v-uv)^2}.$$

Then

$$\frac{\partial^2 C}{\partial v\,\partial u}=\frac{2v(u+v-uv)^2-v^2\cdot 2(u+v-uv)(1-u)}{(u+v-uv)^4}=\frac{2v(u+v-uv)[(u+v-uv)-(1-u)v]}{(u+v-uv)^4}.$$

Since all terms are nonnegative for $(u,v)\in[0,1]^2$, this derivative is positive, implying that C is 2-increasing.
(An easier argument: $C(u,v)$ is an Archimedean copula with generator $\varphi(t)=\frac{t}{1-t}$, which is known to produce valid copulas.)

(iii) *Range:* $C(u,v)\in[0,1]$ for $(u,v)\in[0,1]^2$, since numerator and denominator are positive and $C(1,1)=1$.

Conclusion: $C(u,v)=\dfrac{uv}{u+v-uv}$ is a valid (Archimedean) copula known as the *Ali–Mikhail–Haq copula* with parameter 1. It represents a mild form of positive dependence between U and V.

Question 2. Recall that the lower tail dependence coefficient is defined as

$$\lambda_l(X_1,X_2)=\lim_{q\to 0^+}\frac{C(q,q)}{q}.$$

We now derive the analogous expression for the *upper* tail dependence coefficient $\lambda_u(X_1,X_2)$.

By definition:

$$\begin{aligned}\lambda_u(X_1,X_2)&=\lim_{q\to 1^-}\Pr\big(X_2>F_2^{\leftarrow}(q)\,\big|\,X_1>F_1^{\leftarrow}(q)\big)\\&=\lim_{q\to 1^-}\frac{\Pr\big(X_2>F_2^{\leftarrow}(q),\,X_1>F_1^{\leftarrow}(q)\big)}{\Pr\big(X_1>F_1^{\leftarrow}(q)\big)}.\end{aligned}$$

Rewriting the numerator:

$$\begin{aligned}\Pr(X_2>F_2^{\leftarrow}(q),\,X_1>F_1^{\leftarrow}(q))&=1-\Pr(X_1\leq F_1^{\leftarrow}(q))-\Pr(X_2\leq F_2^{\leftarrow}(q))\\&\quad+\Pr(X_1\leq F_1^{\leftarrow}(q),\,X_2\leq F_2^{\leftarrow}(q))\\&=1-2q+C(q,q),\end{aligned}$$

where we used that $\Pr(X_i\leq F_i^{\leftarrow}(q))=q$ and the joint term equals the copula $C(q,q)$.

Since $\Pr(X_1>F_1^{\leftarrow}(q))=1-q$, we obtain:

$$\boxed{\lambda_u(X_1,X_2)=\lim_{q\to 1^-}\frac{1-2q+C(q,q)}{1-q}.}$$

Remark. This expression mirrors the lower-tail case, except the limit is taken as $q\to 1^-$ and the numerator corresponds to the survival probabilities in the joint upper tail.

Question 3. We compute λ_l and λ_u for two important Archimedean copulas.

The Gumbel copula is

$$C_\theta^{Gu}(u,v) = \exp\Big\{ -\big((-\log u)^\theta + (-\log v)^\theta\big)^{1/\theta}\Big\}, \qquad \theta \geq 1.$$

Lower tail dependence. Set $u = v = q$. Then

$$C_\theta^{Gu}(q,q) = \exp\{-(2(-\log q)^\theta)^{1/\theta}\} = \exp\{-2^{1/\theta}(-\log q)\} = q^{\,2^{1/\theta}}.$$

Hence

$$\lambda_l = \lim_{q\to 0^+} \frac{q^{2^{1/\theta}}}{q} = \lim_{q\to 0^+} q^{\,2^{1/\theta}-1} = 0,$$

since $2^{1/\theta} > 1$.

Upper tail dependence. By definition,

$$\lambda_u = \lim_{q\to 1^-} \frac{1 - 2q + C_\theta^{Gu}(q,q)}{1-q}.$$

From the above,

$$C_\theta^{Gu}(q,q) = q^{\,2^{1/\theta}}.$$

Thus

$$\lambda_u = \lim_{q\to 1^-} \frac{1-2q+q^{2^{1/\theta}}}{1-q}.$$

Rewrite the numerator:

$$1 - 2q + q^A = (1-q) - (q - q^A), \qquad A = 2^{1/\theta}.$$

Divide by $(1-q)$:

$$\lambda_u = 1 - \lim_{q\to 1^-} \frac{q-q^A}{1-q}.$$

The limit has the form $0/0$, so we apply L'Hôpital's rule:

$$\lim_{q\to 1^-} \frac{q-q^A}{1-q} = \lim_{q\to 1^-} \frac{1-Aq^{A-1}}{-1} = A - 1.$$

Hence

$$\lambda_u = 1 - (A-1) = 2 - A = 2 - 2^{1/\theta}.$$

$$\boxed{\lambda_l = 0, \qquad \lambda_u = 2 - 2^{1/\theta}.}$$

The Clayton copula is

$$C_\theta^{Cl}(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}, \qquad \theta > 0.$$

Lower tail dependence. Set $u = v = q$. Then

$$C_\theta^{Cl}(q,q) = (2q^{-\theta} - 1)^{-1/\theta} = q\,(2 - q^\theta)^{-1/\theta}.$$

Hence

$$\lambda_l = \lim_{q\to 0^+} \frac{q(2-q^\theta)^{-1/\theta}}{q} = \lim_{q\to 0^+} (2-q^\theta)^{-1/\theta} = 2^{-1/\theta}.$$

Upper tail dependence. The upper tail dependence for the Clayton copula is

$$\lambda_u = \lim_{q\to 1^-} \frac{1-2q+C_\theta^{Cl}(q,q)}{1-q}.$$

Since $C_\theta^{Cl}(1,1) = 1$, both numerator and denominator tend to 0, so L'Hôpital's rule applies.

Let $N(q) = 1 - 2q + C_\theta^{Cl}(q,q)$ and $D(q) = 1 - q$. Then

$$\lambda_u = \lim_{q \to 1^-} \frac{N(q)}{D(q)} = \lim_{q \to 1^-} \frac{N'(q)}{D'(q)} = -\lim_{q \to 1^-} N'(q),$$

because $D'(q) = -1$.
Write

$$C_\theta^{Cl}(q,q) = (2q^{-\theta} - 1)^{-1/\theta} =: f(q).$$

Using the chain rule with $h(q) = 2q^{-\theta} - 1$ and $g(x) = x^{-1/\theta}$,

$$f'(q) = g'(h(q))h'(q) = \Big(-\frac{1}{\theta}\Big)\big(2q^{-\theta} - 1\big)^{-1/\theta - 1} \cdot \big(-2\theta q^{-\theta-1}\big) = 2q^{-\theta-1}\big(2q^{-\theta} - 1\big)^{-1/\theta-1}.$$

Hence

$$N'(q) = -2 + f'(q) = -2 + 2q^{-\theta-1}\big(2q^{-\theta} - 1\big)^{-1/\theta-1}.$$

As $q \to 1^-$, $q^{-\theta-1} \to 1$ and $2q^{-\theta} - 1 \to 1$, so

$$\lim_{q \to 1^-} N'(q) = -2 + 2 \cdot 1 \cdot 1 = 0.$$

Therefore

$$\lambda_u = -\lim_{q \to 1^-} N'(q) = 0.$$

The Clayton copula has no upper tail dependence.

$$\boxed{\lambda_l = 2^{-1/\theta}, \qquad \lambda_u = 0.}$$

Question 4. We take the first two return series from the Fama–French international dataset and transform them into a pseudo-sample of copula data by applying the probability integral transform componentwise:

$$\hat{U}_{it} = \hat{F}_i(X_{it}), \qquad i = 1, 2,$$

where $\hat{F}_i$ is the empirical CDF of margin i. The resulting pseudo-observations $(\hat{U}_{1t}, \hat{U}_{2t})$ lie in $(0,1)^2$ and retain the dependence structure of the data. A scatter plot is shown in Figure 1.

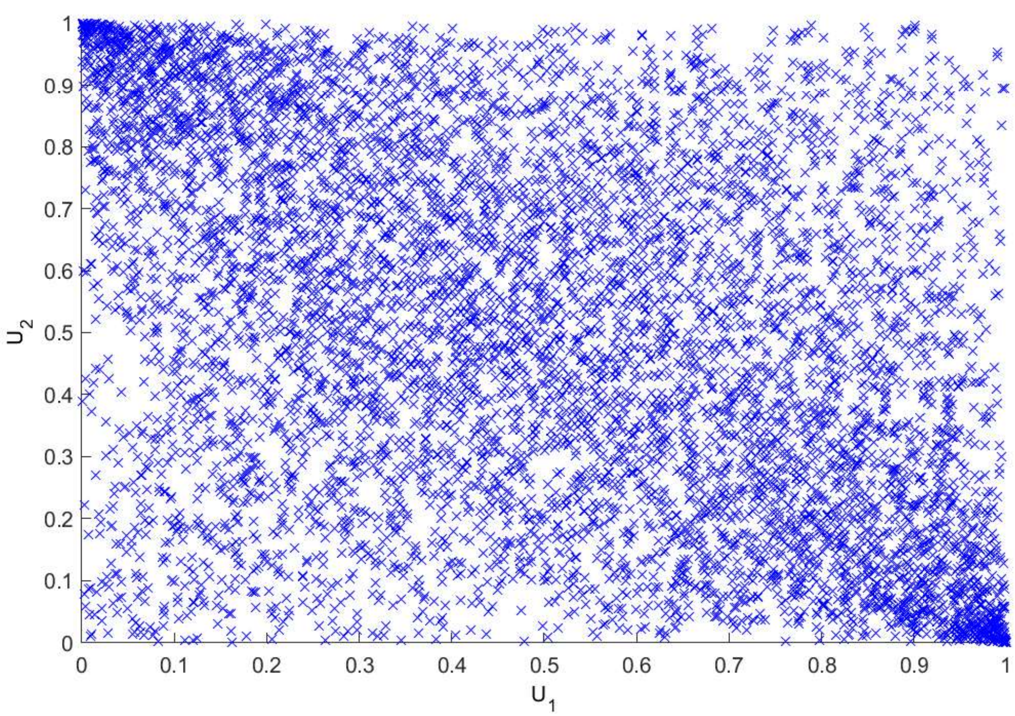

Figure 1: Pseudo-sample obtained by the empirical distribution transform.

Using this pseudo-sample, we estimate the parameter θ for three parametric copula families (Gumbel, Clayton, Frank) by maximum likelihood. Given a candidate θ, the log-likelihood is

$$\ell(\theta; \hat{\mathbf{U}}) \;=\; \sum_{t=1}^{n} \log c_\theta(\hat{U}_{1t}, \hat{U}_{2t}),$$

where c_θ is the corresponding copula density. The MATLAB function `copulapdf` is used to evaluate c_θ, and the optimization is performed using `fmincon`. The estimated parameters and maximum log-likelihood values are reported in Table 1.

Copula Family	$\hat{\theta}$	$\log L(\hat{\theta})$
Gumbel	1.00	-0.0059
Clayton	0.00	-0.0032
Frank	-3.13	825.73

Table 1: Maximum likelihood estimates for three copula families.

Interpretation. Among the three copula families considered, the Frank copula achieves by far the highest log-likelihood. This suggests that, within this limited parametric set, the Frank copula provides the best fit to the empirical dependence structure of the data. Of course, with real financial data there is no "true" generating copula; the result merely indicates which model describes the observed dependence most adequately among the candidates considered.

Question 5. Let $X \sim \mathcal{N}(0,1)$ and $Y = ZX$, where Z is independent of X with $\mathbb{P}(Z = 1) = p$ and $\mathbb{P}(Z = -1) = 1 - p$.

Step 1: Identify the marginal distributions.
Since $Y = ZX$ and Z only flips the sign of X,

$$Y \overset{d}{=} X \sim \mathcal{N}(0,1).$$

Thus, the Probability Integral Transform (PIT) variables are

$$U_1 = F(X), \qquad U_2 = F(Y),$$

where F is the standard normal CDF.

Step 2: Use the mixture structure.
Condition on the two possible values of Z:

$$C(u_1,u_2) = \mathbb{P}(U_1 \le u_1,\, U_2 \le u_2) = p\,\mathbb{P}(U_1 \le u_1, U_2 \le u_2 \mid Z = 1) + (1-p)\,\mathbb{P}(U_1 \le u_1, U_2 \le u_2 \mid Z = -1).$$

Case 1: $Z = 1$.
Then $Y = X$, so $U_2 = F(Y) = F(X) = U_1$. Thus

$$\mathbb{P}(U_1 \le u_1, U_1 \le u_2) = \min(u_1, u_2).$$

Case 2: $Z = -1$.
Then $Y = -X$, so

$$U_2 = F(Y) = F(-X) = 1 - F(X) = 1 - U_1.$$

Thus

$$\mathbb{P}(U_1 \le u_1,\, U_2 \le u_2) = \mathbb{P}(U_1 \le u_1,\, 1 - U_1 \le u_2) = \mathbb{P}(1 - u_2 \le U_1 \le u_1) = \max(u_1 + u_2 - 1,\, 0).$$

This is the countermonotone copula.

Step 3: Combine the two cases.
Therefore,

$$\boxed{C(u_1,u_2) = p\,\min(u_1,u_2) + (1-p)\,\max(u_1+u_2-1, 0).}$$

This shows that (X,Y) has a copula equal to a convex combination of the comonotonic and countermonotonic copulas:

$$C = p\, C^{\text{Comonotone}} + (1-p)\, C^{\text{Countermonotone}}.$$

Question 6. We want to compute the joint CDF

$$\mathbb{P}(U_1 \le u_1, \dots, U_d \le u_d).$$

Step 1: Condition on V. Using the law of total probability,

$$\mathbb{P}(U_1 \le u_1, \dots, U_d \le u_d) = \int_0^\infty \mathbb{P}(U_1 \le u_1, \dots, U_d \le u_d \mid V = v)\, dG(v).$$

Step 2: Use conditional independence. Given $V = v$, the variables $U_1, \dots, U_d$ are independent, hence

$$\mathbb{P}(U_1 \le u_1, \dots, U_d \le u_d \mid V = v) = \prod_{i=1}^d F_{U_i|V}(u_i; v).$$

Substituting the given conditional CDF,

$$\prod_{i=1}^d F_{U_i|V}(u_i; v) = \prod_{i=1}^d \exp\Big(-v\, \hat{G}^{-1}(u_i)\Big) = \exp\left(-v \sum_{i=1}^d \hat{G}^{-1}(u_i)\right).$$

Step 3: Identify the Laplace–Stieltjes transform.
Thus

$$\mathbb{P}(U_1 \le u_1, \dots, U_d \le u_d) = \int_0^\infty \exp\left(-v \sum_{i=1}^d \hat{G}^{-1}(u_i)\right) dG(v).$$

But by definition of the Laplace–Stieltjes transform,

$$\hat{G}(t) = \int_0^\infty e^{-tv}\, dG(v).$$

So by setting $t = \sum_{i=1}^d \hat{G}^{-1}(u_i)$, we obtain

$$\boxed{\mathbb{P}(U_1 \le u_1, \dots, U_d \le u_d) = \hat{G}\Big(\hat{G}^{-1}(u_1) + \cdots + \hat{G}^{-1}(u_d)\Big).}$$

This is the standard mixture representation of an Archimedean copula.