Quantitative Risk Management
Assignment 5

Question 1: Suppose $X = \mu + \sqrt{W}\,Z$, where $Z \sim \mathcal{N}(0,1)$ is independent of $W \geq 0$. The random variable W takes discrete values $W \in \{k_1, \ldots, k_n\}$ with probabilities

$$\mathbb{P}(W = k_i) = p_i, \qquad i = 1, \ldots, n.$$

1. Construct a function of a real variable v whose root v_0 satisfies $v_0 = \mathrm{VaR}_\alpha(X)$.
2. Argue that this function has a unique root.

Question 2: Construct two random variables with zero correlation that are not independent. Prove that they satisfy these requirements.

Question 3: Let W have a Pareto distribution with parameter $\theta > 1$, so that $F_W(w) = 1 - w^{-\theta}$, $w \geq 1$. Consider in addition two i.i.d. random variables $Z_i \sim \mathcal{N}(0,1)$, $i = 1, 2$, for which $Z = (Z_1, Z_2)$ is independent of W. Consider

$$X_1 = \sqrt{W}(Z_1 + Z_2),$$
$$X_2 = \sqrt{W}(Z_1 - Z_2).$$

Show that vector $X = (X_1, X_2)$ has a normal variance mixture distribution and compute its covariance matrix. Is it possible to state that X_1 and X_2 are uncorrelated?

Question 4: Download 5 years of historical prices of the following companies: International Business Machines Corporation (IBM), McDonald's Corp. (MCD), 3M Company (MMM), and Wal-Mart Stores Inc. (WMT). Moreover, download 5 years of data of the S&P500 index (this can be found by entering the Stock Market Indexes menu on WRDS and selecting the variable "Level on S&P Composite Index"). Use 5 years of data between March 17, 2011 and March 17, 2016.

1. Let $\boldsymbol{X}_i$ represent log returns of each of the stocks and let F_i represent the corresponding log returns of the S&P index. Perform a regression analysis to estimate a 1-factor model by finding $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{B}}$ from

$$\boldsymbol{X}_i = \boldsymbol{a} + \boldsymbol{B} F_i + \boldsymbol{\epsilon}_i.$$

2. Construct the matrix of residual errors $\hat{\mathcal{E}} = \mathcal{X} - \mathcal{F}\hat{\mathcal{B}}$ and compute the sample correlation matrix of these errors. Compare it to the sample correlation matrix of the original returns $\mathcal{X}$ and comment on the results (report both matrices).

Question 5: Copy the following code into the beginning of a Matlab script, or reproduce it in Python:

```
rng(1);
N = 10000;
A = [ 1  0 0 0;
      1  1 0 0;
     -1  2 3 0;
      1 -1 1 1];
x = trnd(5,N,4);
X = (A*x')';
```

This code generates an array `X`, which consists of 10,000 rows and 4 columns. Each row represents a single data point observation of a 4-dimensional random vector. In this problem, assume that the loss of a portfolio is equal to $L = \sum_{k=1}^{4} X_k$.

1. Based on the 10,000 observations in $\boldsymbol{X}$, compute $VaR_\alpha(L)$ for $\alpha = 0.95$.
2. What is the eigenvector corresponding to the first principal component of $\boldsymbol{X}$? Can you find a link between the magnitude of some component of this vector and some component in the covariance matrix of $\boldsymbol{X}$? Which of the four components of $\boldsymbol{X}$ would you expect to contribute most to the 1st principle component?
3. Approximate $\boldsymbol{X}$ by using its first two principal components as factors (i.e., set the error terms to zero). Recompute $VaR_\alpha(L)$ and compare it with its previous estimate.

The following Matlab functions may be useful for this problem: `cov`, `eig`, `pca`. Be sure to read the documentation on these functions before using them. Matlab may use different conventions for eigenvalue ordering depending on the context.