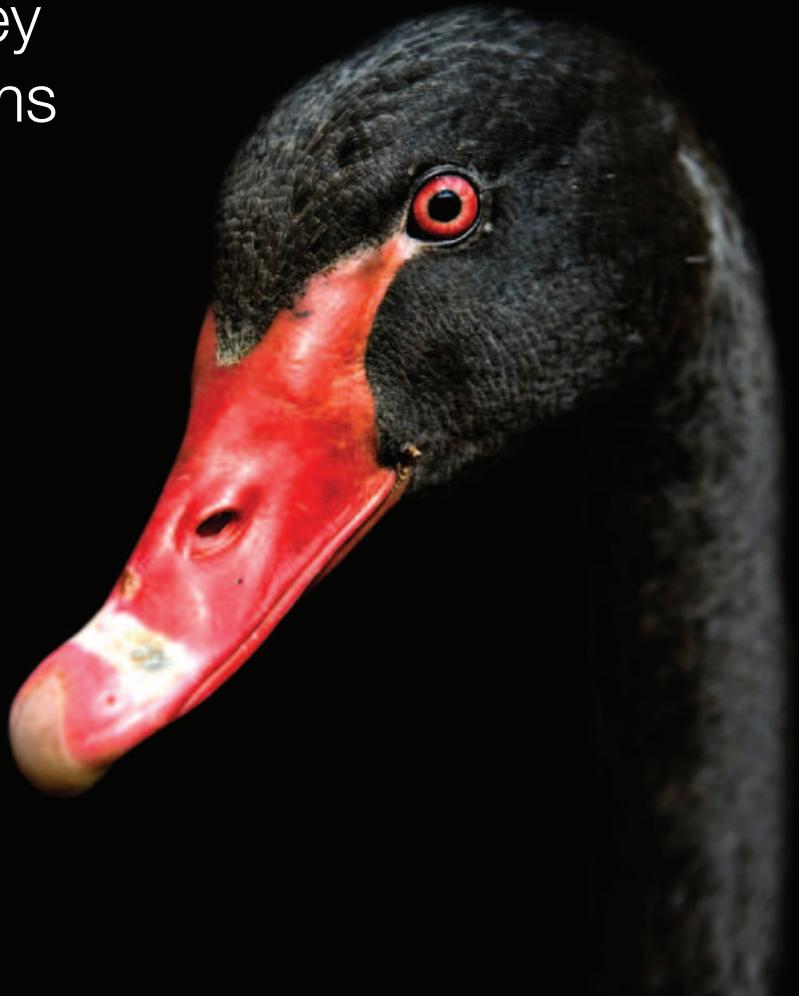


Third edition

# Statistics for Business and Economics

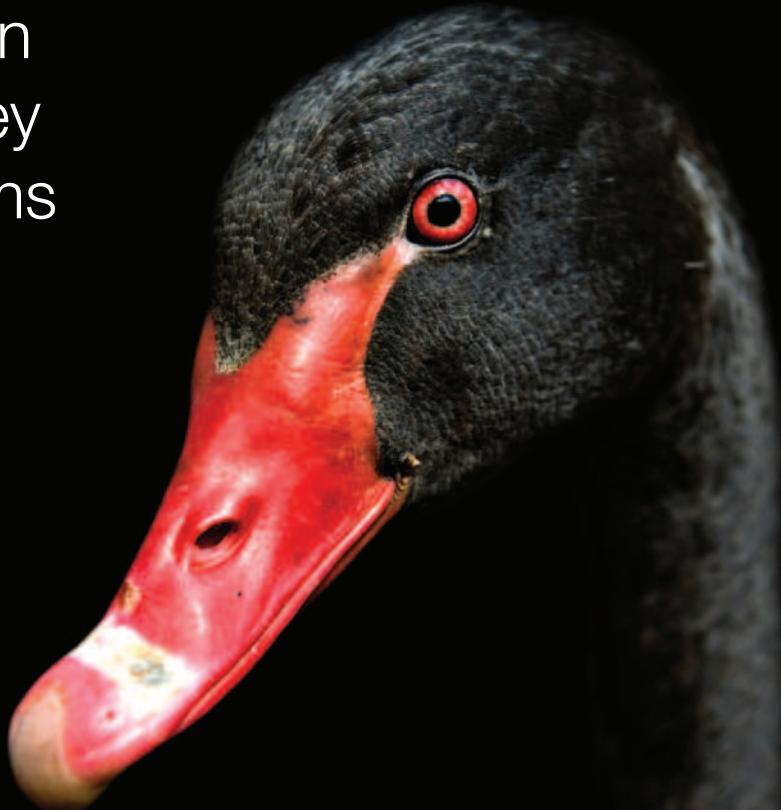
David R. Anderson  
Dennis J. Sweeney  
Thomas A. Williams  
Jim Freeman  
Eddie Shoesmith



Third edition

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David R. Anderson  
Dennis J. Sweeney  
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Jim Freeman  
Eddie Shoesmith



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Third Edition**

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Thomas A. Williams, Jim Freeman and  
Eddie Shoesmith

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# DEDICATION



*'To the memory of my grandparents, Lizzie and Hal-sey'*

**JIM FREEMAN**

*'To all my family, past, present and future'*

**EDDIE SHOESMITH**



# PREFACE

The purpose of *Statistics for Business and Economics* is to give students, primarily those in the fields of business, management and economics, a conceptual introduction to the field of statistics and its many applications. The text is applications oriented and written with the needs of the non-mathematician in mind. The mathematical prerequisite is knowledge of algebra.

Applications of data analysis and statistical methodology are an integral part of the organization and presentation of the material in the text. The discussion and development of each technique are presented in an application setting, with the statistical results providing insights to problem solution and decision-making.

Although the book is applications oriented, care has been taken to provide sound methodological development and to use notation that is generally accepted for the topic being covered. Hence, students will find that this text provides good preparation for the study of more advanced statistical material. A revised and updated bibliography to guide further study is included as an appendix.

The online platform introduces the student to the software packages MINITAB 16, SPSS 21 and Microsoft® Office EXCEL 2010, and emphasizes the role of computer software in the application of statistical analysis. MINITAB and SPSS are illustrated as they are two of the leading statistical software packages for both education and statistical practice. EXCEL is not a statistical software package, but the wide availability and use of EXCEL makes it important for students to understand the statistical capabilities of this package. MINITAB, SPSS and EXCEL procedures are provided on the dedicated online platform so that instructors have the flexibility of using as much computer emphasis as desired for the course.

## THE EMEA EDITION

This is the 3rd EMEA edition of *Statistics for Business and Economics*. It is based on the 2nd EMEA edition and the 11th United States (US) edition. The US editions have a distinguished history and deservedly high reputation for clarity and soundness of approach, and we maintained the presentation style and readability of those editions in preparing the international edition. We have replaced many of the US-based examples, case studies and exercises with equally interesting and appropriate ones sourced from a wider geographical base, particularly the UK, Ireland, continental Europe, South Africa and the Middle East. We have also streamlined the book by moving four non-mandatory chapters, the software section and exercise answers to the associated online platform. Other notable changes in this 3rd EMEA edition are summarized here.

## CHANGES IN THE 3RD EMEA EDITION

- **Self-test exercises** Certain exercises are identified as self-test exercises. Completely worked-out solutions for those exercises are provided on the online platform that accompanies the text. Students can attempt the self-test exercises and immediately check the solution to evaluate their understanding of the concepts presented in the chapter.

- **Other content revisions** The following additional content revisions appear in the new edition:
  - New examples of times series data are provided in Chapter 1.
  - Chapter 9 contains a revised introduction to hypothesis testing, with a better set of guidelines for identifying the null and alternative hypotheses.
  - Chapter 13 makes much more explicit the linkage between Analysis of Variance and experimental design.
  - Chapter 17 now includes coverage of the popular Holt's linear exponential smoothing methodology.
  - The treatment of non-parametric methods in Chapter 18 has been revised and updated.
  - Chapter 19 on index numbers (on the online platform) has been updated with current index numbers.
  - A number of case problems have been added or updated. These are in the chapters on Descriptive Statistics, Discrete Probability Distributions, Inferences about Population Variances, Tests of Goodness of Fit and Independence, Simple Linear Regression, Multiple Regression, Regression Analysis: Model Building, Non-Parametric Methods, Index Numbers and Decision Analysis. These case problems provide students with the opportunity to analyze somewhat larger data sets and prepare managerial reports based on the results of the analysis.
  - Each chapter begins with a Statistics in Practice article that describes an application of the statistical methodology to be covered in the chapter. New to this edition are Statistics in Practice articles for Chapters 2, 9, 10 and 11, with several other articles substantially updated and revised for this new edition.
  - New examples and exercises have been added throughout the book, based on real data and recent reference sources of statistical information. We believe that the use of real data helps generate more student interest in the material and enables the student to learn about both the statistical methodology and its application.
  - To accompany the new exercises and examples, data files are available on the online platform. The data sets are available in MINITAB, SPSS and EXCEL formats. Data set logos are used in the text to identify the data sets that are available on the online platform. Data sets for all case problems as well as data sets for larger exercises are included.
- **Software sections** In the 3rd EMEA edition, we have updated the software sections to provide step-by-step instructions for the latest versions of the software packages: MINITAB 16, SPSS 21 and Microsoft® Office EXCEL 2010. The software sections have been relocated to the online platform.



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He has published more than 30 articles in the area of management science and statistics. The National Science Foundation, IBM, Procter & Gamble, Federated Department Stores, Kroger and Cincinnati Gas & Electric have funded his research, which has been published in *Management Science*, *Operations Research*, *Mathematical Programming*, *Decision Sciences* and other journals. Professor Sweeney has co-authored ten textbooks in the areas of statistics, management science, linear programming and production and operations management.

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# WALK-THROUGH TOUR

## 2 Descriptive Statistics: Tabular and Graphical Presentations

### CHAPTER CONTENTS

Statistics in practice L'Oréal, the statistical graphics are worth it  
2.1 Summarising qualitative data  
2.2 Summarising quantitative data  
2.3 Cross-tabulations and scatter diagrams

### LEARNING OBJECTIVES

After studying this chapter and doing the exercises, you should be able to construct and interpret several types of tabular and graphical data summaries.

- 1 For single qualitative variables: frequency, relative frequency and percentage frequency distributions; bar charts and pie charts.
- 2 For single quantitative variables: frequency, relative frequency and percentage frequency distributions; cumulative frequency, relative cumulative frequency and percentage cumulative frequency distributions; dot plots, stem-and-leaf plots, histograms and cumulative distribution plots (ogives).
- 3 For pairs of qualitative and quantitative data: cross-tabulations, with row and column percentages; scatter diagrams.
- 4 For pairs of quantitative variables: scatter diagrams.
- 5 You should be able to give an example of Simpson's paradox and explain the relevance of this paradox to the cross-tabulation of variables.

A<sup>\*</sup> s explained in Chapter 1, data can be classified as either qualitative or quantitative. Qualitative data use labels or names to identify categories of items. Quantitative data are numerical values that indicate how much or how many.

This chapter introduces methods and graphical methods commonly used to summarise both qualitative and quantitative data. Everyone is exposed to these types of presentation in annual reports (see Statistics in Practice), newspaper articles and television news. It is important to understand how they are prepared and how they are interpreted. We begin with methods for summarising single variables. Section 2.3 introduces methods for summarising the relationship between two variables.

Modern spreadsheet and statistical software packages provide extensive capabilities for summarising data and preparing graphical presentations. EXCEL, IBM SPSS and MINITAB are three widely available packages. There are guides to some of their capabilities on the companion website.

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### 20 CHAPTER 2 DESCRIPTIVE STATISTICS: TABULAR AND GRAPHICAL PRESENTATIONS

#### STATISTICS IN PRACTICE

Marks & Spencer: not just any statistical graphics

Marks & Spencer has a company history going back to 1884. The group is based in London, but has offices across the UK as well as overseas. Marks & Spencer is well known for its promotional activities and its advertising slogan 'Your M&S'. Marks & Spencer advertisements have featured many famous celebrities including Twiggy, Erin O'Connor, David Beckham, Claudia Schiffer, Ross Huntington-Whiteley and Antonio Banderas.

Marks & Spencer's shares are traded on the London Stock Exchange and it is a constituent of the FTSE 100 Index. Like all public companies, Marks & Spencer publishes an annual report. In the

annual report, alongside many photographs of its annual general meeting, there are pictures of its different nature statistical charts showing in particular the financial performance of the company. The examples here are from Marks and Spencer's 2013 Annual Report. First is a chart showing Marks and Spencer's sales by country. Next is a chart showing the breakdown of Marks & Spencer's international revenue, and finally a line graph showing Marks & Spencer's share price over time.

We are exposed to statistical charts of this type almost daily: in newspapers and magazines, on TV, online and in annual reports such as the Marks & Spencer Annual Report. In this chapter, you will learn about tabular and graphical methods of descriptive statistics, such as frequency distributions, bar charts, histograms, stem-and-leaf displays, cross-tabulations and others. The goal of these methods is to summarise data so that they can be easily understood and interpreted.

A window display showing an array of personalities who have modelled for Marks & Spencer

**Learning Objectives** We have set out clear learning objectives at the start of each chapter in the text, as is now common in texts in the UK and elsewhere. These objectives summarize the core content of each chapter in a list of key points.

**Statistics in Practice** Each chapter begins with a Statistics in Practice article that describes an application of the statistical methodology to be covered in the chapter.

### 24 CHAPTER 2 DESCRIPTIVE STATISTICS: TABULAR AND GRAPHICAL PRESENTATIONS

FIGURE 2.2 Pie chart of new car purchases (produced in EXCEL)

#### EXERCISES

##### Methods

1. The response to a question has three alternatives: A, B and C. A sample of 120 responses provides 60 A, 24 B and 36 C. Construct the frequency and relative frequency distributions.
2. A partial relative frequency distribution is given below.

| Class | Relative frequency |
|-------|--------------------|
| A     | 0.22               |
| B     | 0.38               |
| C     | 0.40               |
| D     |                    |

3. What is the relative frequency of class D?  
a. 0.22  
b. 0.38  
c. Construct the frequency distribution.  
d. Construct the percentage frequency distribution.

3. A questionnaire provides 58 Yes, 42 No and 20 No opinion answers.

- a. In the construction of a pie chart, how many degrees would be in the sector of the pie showing the Yes answers?

- b. How many degrees would be in the sector of the pie showing the No answers?

- c. Construct a pie chart.

- d. Construct a bar chart.

##### Application

4. CESAMobile is a customer experience management company based in Finland. The company does extensive market research in the mobile telecommunications field. Their research shows that the four most popular mobile operating systems in Nordic countries are Apple iOS, Symbian OS, Android and Nokia Symbian. The following table gives the percentage of users using each of these systems.

|         |         |         |         |         |         |         |         |         |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| Android | Apple   | Android | Symbian | Apple   | Apple   | Apple   | Apple   | Android |
| Android | Apple   | Symbian | Apple   | Apple   | Nokia   | Android | Apple   | Apple   |
| Nokia   | Symbian | Symbian | Apple   | Nokia   | Symbian | Android | Nokia   | Android |
| Android | Apple   | Android |
| Apple   | Nokia   | Symbian | Symbian | Android | Android | Android | Symbian | Android |

**Exercises** The exercises are split into two parts: Methods and Applications. The Methods exercises require students to use the formulae and make the necessary computations. The Applications exercises require students to use the chapter material in real-world situations. Thus, students first focus on the computational 'nuts and bolts', then move on to the subtleties of statistical application and interpretation. Answers to even-numbered exercises are provided on the online platform, while a full set of answers are provided in the lecturers' Solutions Manual. Supplementary exercises are provided on the textbook's online platform. Self-test exercises are highlighted throughout by the 'COMPLETE SOLUTIONS' icon and contain fully-worked solutions on the online platform.

**COMPLETE SOLUTIONS**

**Sampling from an infinite population**

In some situations, the population is either infinite, or so large that for practical purposes it must be treated as infinite. For example, suppose that a fast-food restaurant would like to obtain a profile of its customers by selecting a simple random sample of customers and asking each customer to complete a short questionnaire. The process of customers visiting the restaurant can be viewed as being an ongoing process that makes listing or counting every element in the population impossible. The definition of a simple random sample from an infinite population follows.

**Simple random sample (infinite population)**

A simple random sample from an infinite population is a sample selected such that the following conditions are satisfied:

1. Each element selected comes from the population.
2. Each element is selected independently.

For the example of a simple random sample of customers at a fast-food restaurant, any customer who comes into the restaurant will satisfy the first requirement. The second requirement will be satisfied if a sample selection procedure is devised to select the items independently and thereby avoid any selection bias that gives higher selection probabilities to certain types of customers. Selection bias would occur if, for example, the procedure for selecting the sample from the population caused the analyst to expect these customers to exhibit similar profiles. Selection bias can be avoided by ensuring that the selection of a particular customer does not influence the selection of any other customer. In other words, the customers must be selected independently.

Infinite populations are often associated with an ongoing process that operates continuously over time. For example, parts being manufactured on a production line, transactions occurring at a bank telephone, or arriving at a technical support center, and customers visiting stores may all be viewed as coming from an infinite population. In such cases, an effective sampling procedure will ensure that no selection bias occurs and that the sample elements are selected independently.

**EXERCISES****Methods**

1. Consider a finite population with five elements labeled A, B, C, D and E. Ten possible simple random samples of size 2 could be selected:
  - a. If the first sample begins with B, AC and so on.
  - b. Using simple random sampling, what is the probability that each sample of size 2 is selected?
  - c. Assume random number 1 corresponds to A, random number 2 corresponds to B, and so on. List the simple random sample of size 2 that will be selected by using the random digits 8 6 5 7 5 3 2.

**Notes** Recent US editions have included marginal and end-of-chapter notes.

We have not adopted this layout, but have included the important material in the text itself.

value of ten. How have a point value of one or 11. A 52-card deck contains 16 cards with a point value of ten (jacks, queens, kings and tens) and four aces.

- a. What is the probability that both cards dealt are aces or ten-point cards?
- b. What is the probability that both of the cards are aces?
- c. What is the probability that the first card dealt has a point value of ten?

d. A blackjack is a ten-point card and an ace for a value of 21. Use your answers to parts (a), (b) and (c) to determine the probability that a player is dealt a blackjack. Hint: Part (d) is not a hypothesis test.

36. A company plans to select a team of six students from Gust University for a business game competition from a pool of 18 undergraduates. Nine are from the second year management course, five are from the first year management and the remainder are from outside the management school. What is the probability that:
 

- a. All five team members are second year management?
- b. No students from outside the management school are selected?

37. Manufacturers are switching in lots of 15 items. Four items are randomly drawn from each lot and tested. If the lot is considered acceptable if no defectives are among the four tested, what is the probability that the shipment will be rejected?



For the data files, additional online summary questions, answers, and the software section for this chapter, go to the online platform.

**SUMMARY**

A random variable provides a numerical description of the outcome of an experiment. The probability distribution for a random variable describes how the probabilities are distributed over the values the random variable can assume. A variety of examples are used to distinguish between discrete and continuous random variables. The probability distribution for a discrete random variable is defined by a probability function, denoted by  $p(x) = P(X = x)$ , which provides the probability associated with each value of the random variable. From the probability function, the expected value, variance and standard deviation for the random variable can be computed and relevant interpretations of these terms are provided.

Perticular attention was devoted to the binomial distribution, which can be used to determine the probability of a success in a sequence of independent trials. The binomial distribution has the following properties:

1. The experiment consists of a sequence of identical trials.
2. Two outcomes are possible on each trial, one called success and the other failure.
3. The probability of a success  $\pi$  does not change from trial to trial. Consequently, the probability of failure,  $1 - \pi$ , does not change from trial to trial.
4. The trials are independent.

**Summaries** Each chapter includes a summary to remind students of what they have learnt so far and offer a useful way to review for exams.



For the data files and additional online resources for Chapter 1, go to the accompanying online platform. (See the 'About the Digital Resources' page in the front of the book for more information on access.)

**SUMMARY**

Statistics is the art and science of collecting, analysing, presenting and interpreting data. Nearly every college student majoring in business or economics is required to take a course in statistics. We began the chapter by describing typical statistical applications for business and economics.

Data consist of the facts and figures that are collected and analyzed. A set of measurements obtained on a single element is called a data item. A collection of data items obtained on a particular variable include nominal, ordinal, interval and ratio. The scale of measurement for a variable is nominal when the data use labels or names to identify an attribute of an element. The scale is ordinal when the data indicate the rank or position of the data and the order or rank of the data is meaningful. The interval and ratio scales of measurement are based on the concept of distance. The interval between values is expressed in terms of a fixed unit of measure. Finally, the scale of measurement is ratio when the data show all the properties of interval data and the ratio of two values is meaningful.

For categorical data, the data items can be labeled or named to identify an attribute of each element. Categorical data use either the nominal or ordinal scale of measurement and may be nonmetric or numeric. Quantitative data are based on the concept of distance and the ratio scale of measurement. Quantitative data are either the data are quantitative. Therefore, statistical computations used for quantitative data are not always appropriate for qualitative data.

In Sections 1.4 and 1.5 we introduced the topics of descriptive statistics and statistical inference. Definitions of the population and sample were provided and different types of descriptive statistics – tabular, graphic and summary – were described. The basic concepts of statistical inference were also introduced. A sample is used to make inferences or test hypotheses about the characteristics of a population.

The last two sections of the chapter provide information on the role of computers in statistical analysis and a brief overview of the relative new field of data mining.

**KEY TERMS**

|                        |                       |
|------------------------|-----------------------|
| Categorical data       | Ordinal scale         |
| Categorical variable   | Population            |
| Census                 | Quantitative data     |
| Cross-sectional data   | Quantitative variable |
| Data                   | Ratio scale           |
| Data mining            | Sample                |
| Data set               | Sample survey         |
| Descriptive statistics | Statistical inference |
| Elements               | Statistics            |
| Interval scale         | Time series data      |
| Nominal scale          | Variable              |
| Observation            |                       |

**Data sets accompany text** Over 200 data sets are available on the online platform that accompanies the text. The data sets are available in MINITAB, SPSS and EXCEL formats. Data set logos are used in the text to identify the data sets that are available online. Data sets for all case problems as well as data sets for larger exercises are also included on the online platform.



**HYPERGEOMETRIC PROBABILITY DISTRIBUTION** 27

Formulas were also presented for the probability function, mean and variance of the binomial distribution.

The Poisson distribution can be used to determine the probability of obtaining  $x$  occurrences over an interval of time or space. The necessary assumptions for the Poisson distribution to apply in a given situation are that:

1. The probability of an occurrence of the event is the same for any two intervals of equal length;
2. The occurrence or non-occurrence of the event in any interval is independent of the occurrence or non-occurrence of the event in any other interval.

A third type of probability distribution, called the multinomial, was introduced in Section 5.6. Like the binomial, it is used to compute the probability of  $x$  successes in  $n$  trials. But, in contrast to the binomial, the probability of success changes from trial to trial.

**KEY TERMS**

|   |                                     |
|---|-------------------------------------|
| Binomial experiment                       | Hypergeometric probability function |
| Binomial probability distribution         | Poisson probability distribution    |
| Binomial probability function             | Poisson probability function        |
| Continuous random variable                | Probability distribution            |
| Discrete random variable                  | Probability function                |
| Discrete uniform probability distribution | Random variable                     |
| Expected value                            | Standard deviation                  |
| Hypogeometric probability distribution    | Variance                            |

**KEY FORMULAE**

Discrete uniform probability function

$$P(x) = \frac{1}{n} \quad (5.3)$$

where  $n$  = the number of values the random variable may assume

Expected value of a discrete random variable

$$E(X) = \mu = \sum x p(x) \quad (5.4)$$

Variance of a discrete random variable

$$\text{Var}(X) = \sigma^2 = \sum (x - \mu)^2 p(x) \quad (5.5)$$

Number of experimental outcomes providing exactly  $x$  successes in  $n$  trials

$$\binom{n}{x} = \frac{n!}{x!(n-x)!} \quad (5.6)$$

**CHAPTER 10 STATISTICAL INFERENCE ABOUT MEANS AND PROPORTIONS WITH TWO POPULATIONS**

**KEY TERMS**

|                     |                           |
|---------------------|---------------------------|
| Independent samples | Pooled estimator of $\pi$ |
| Matched samples     |                           |

**KEY FORMULAE**

Point estimator of the difference between two population means

$$\bar{x}_1 - \bar{x}_2 \quad (10.10)$$

Standard error of  $\bar{x}_1 - \bar{x}_2$

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \quad (10.2)$$

Interval estimate of the difference between two population means:  $\sigma_1$  and  $\sigma_2$  known

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \quad (10.4)$$

Test statistic for hypothesis tests about  $\mu_1 - \mu_2$ :  $\sigma_1$  and  $\sigma_2$  known

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \quad (10.5)$$

Interval estimate of the difference between two population means:  $\sigma_1$  and  $\sigma_2$  unknown

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \quad (10.6)$$

Degrees of freedom for the  $t$  distribution using two independent random samples

$$df = \frac{\left( \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} \right)^2}{\left( \frac{1}{n_1-1} \right) \left( \frac{\sigma_1^2}{n_1} \right) + \left( \frac{1}{n_2-1} \right) \left( \frac{\sigma_2^2}{n_2} \right)} \quad (10.7)$$

Test statistic for hypothesis tests about  $\mu_1 - \mu_2$ :  $\sigma_1$  and  $\sigma_2$  unknown

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \quad (10.8)$$

**Key terms** Key terms are highlighted in the text, listed at the end of each chapter and given a full definition in the Glossary at the end of the textbook.

**Key formulae** Key formulae are listed at the end of each chapter for easy reference.

**CASE PROBLEM 2**

**Chocolate Perfection Website Transactions**

Chocolate Perfection manufactures and sells quality chocolate products in Dubai. Two years ago the company developed a website and began selling its products over the Internet. Website sales have exceeded expectations and management at the company is now considering strategies to increase sales even further. To learn more about the website's performance, a sample of 50 Chocolate Perfection transactions was selected from the previous month's sales. Data showing the day of the week each transaction occurred, the type of browser the customer used, the time spent on the website, the number of website pages viewed and the amount spent by each of the 50 customers are contained in the file named "Shoppers". Amount spent is in United Arab Emirates dirham (AED). (One Euro is around five AED.) A screenshot of a data is shown below.

| Customer | Day | Browser           | Time (min) | Pages Viewed | Amount Spent (AED) |
|----------|-----|-------------------|------------|--------------|--------------------|
| 1        | Mon | Internet Explorer | 12.0       | 4            | 200.09             |
| 2        | Wed | Other             | 19.5       | 6            | 348.28             |
| 3        | Mon | Internet Explorer | 8.5        | 4            | 98.70              |
| 4        | Tue | Firefox           | 11.4       | 2            | 164.16             |
| 5        | Wed | Internet Explorer | 11.3       | 4            | 243.21             |
| 6        | Fri | Firefox           | 10.5       | 6            | 193.93             |
| 7        | Sun | Internet Explorer | 11.4       | 2            | 132.27             |
| 8        | Fri | Firefox           | 11.4       | 3            | 205.37             |
| 9        | Wed | Firefox           | 12.7       | 3            | 209.39             |

Chocolate Perfection would like to use the sample data to analyse the relationship between the day of the week, the type of browser used, the time spent on the website and view more pages after spending more money during their visit to the website. The company would like to know if there is any relationship between the day of the week and the type of browser has on sales.

**Managerial report**

Use the following six descriptive statistics to learn about the customers who visit the Chocolate Perfection website. Include the following in your report:

1. Prepare a scatter diagram and compute the simple correlation coefficient to explore the relationship between the total amount spent and the number of pages viewed for each type of browser. What observations can you make about Chocolate Perfection's business, based on these numerical summaries?
2. Summarize the frequency, the total amount spent and the mean amount spent for each day of the week. What observations can you make about Chocolate Perfection's business based on the results of the analysis?
3. Summarize the frequency, the total amount spent and the mean amount spent for each type of browser for each type of browser. What observations can you make about Chocolate Perfection's business, based on the results of the analysis?
4. Construct a scatter diagram and compute the simple correlation coefficient to explore the relationship between the time spent on the website and the amount spent. Use the horizontal axis for the time spent on the website. Discuss.
5. Construct a scatter diagram and compute the simple correlation coefficient to explore the relationship between the total amount spent and the number of pages viewed. Use the horizontal axis for the number of pages viewed. Discuss.
6. Construct a scatter diagram and compute the simple correlation coefficient to explore the relationship between the time spent on the website and the number of pages viewed. Use the horizontal axis to represent the number of pages viewed. Discuss.

**Case problems** The end-of-chapter case problems provide students with the opportunity to analyse somewhat larger data sets and prepare managerial reports based on the results of the analysis.

# DIGITAL RESOURCES

## Dedicated Instructor Resources

To discover the dedicated instructor online support resources accompanying this textbook, instructors should register here for access:

<http://login.cengage.com>

Resources include:

- Solutions Manual
- ExamView Testbank
- PowerPoint slides



### Instructor access

Instructors can access the online student platform by registering at <http://login.cengage.com> or by speaking to their local Cengage Learning EMEA representative.

### Instructor resources

Instructors can use the integrated Engagement Tracker to track students' preparation and engagement. The tracking tool can be used to monitor progress of the class as a whole, or for individual students.

### Student access

Students can access the online platform using the unique personal access card included in the front of the book.

### Student resources

The platform offers a range of interactive learning tools tailored to the third edition of *Statistics for Business and Economics*, including:

- Interactive eBook
- Data files referred to in the text
- Answers to in-text exercises
- Software section
- Four additional chapters for further study
- Glossary, flashcards and more

# 1 Data and Statistics



## CHAPTER CONTENTS

Statistics in Practice *The Economist*

- 1.1 Applications in business and economics
- 1.2 Data
- 1.3 Data sources
- 1.4 Descriptive statistics
- 1.5 Statistical inference
- 1.6 Computers and statistical analysis
- 1.7 Data mining

**LEARNING OBJECTIVES** After reading this chapter and doing the exercises, you should be able to:

- 1 Appreciate the breadth of statistical applications in business and economics.
- 2 Understand the meaning of the terms elements, variables and observations, as they are used in statistics.
- 3 Understand the difference between qualitative, quantitative, cross-sectional and time series data.
- 4 Find out about data sources available for statistical analysis both internal and external to the firm.
- 5 Appreciate how errors can arise in data.
- 6 Understand the meaning of descriptive statistics and statistical inference.
- 7 Distinguish between a population and a sample.
- 8 Understand the role a sample plays in making statistical inferences about the population.

**F**requently, we see the following kinds of statements in newspaper and magazine articles:

- The Ifo World Economic Climate Index fell again substantially in January 2009. The climate indicator stands at 50.1 (1995 = 100); its historically lowest level since introduction in the early 1980s (*CESifo*, April 2009).
- The IMF projected the global economy would shrink 1.3 per cent in 2009 (*Fin24*, 23 April 2009).
- The Foothsie finished the week on a winning streak despite shock figures that showed the economy has contracted by almost 2 per cent already in 2009 (*This is Money*, 25 April 2009).
- China's growth rate fell to 6.1 per cent in the year to the first quarter (*The Economist*, 16 April 2009).

- GM receives further \$2bn in loans (*BBC News*, 24 April 2009).
- Handset shipments to drop by 20 per cent (*In-Stat*, 2009).

The numerical facts in the preceding statements (50.1, 1.3 per cent, 2 per cent, 6.1 per cent, \$2bn, 20 per cent) are called statistics. Thus, in everyday usage, the term *statistics* refers to numerical facts. However, the field, or subject, of statistics involves much more than numerical facts. In a broad sense, **statistics** is the art and science of collecting, analyzing, presenting and interpreting data. Particularly in business and economics, the information provided by collecting, analyzing, presenting and interpreting data gives managers and decision-makers a better understanding of the business and economic environment and thus enables them to make more informed and better decisions. In this text, we emphasize the use of statistics for business and economic decision-making.

Chapter 1 begins with some illustrations of the applications of statistics in business and economics. In Section 1.2 we define the term *data* and introduce the concept of a data set. This section also introduces key terms such as *variables* and *observations*, discusses the difference between quantitative and categorical data, and illustrates the uses of cross-sectional and time series data. Section 1.3 discusses how data can be obtained from existing sources or through survey and experimental studies designed to obtain new data. The important role that the Internet now plays in obtaining data is also highlighted. The use of data in developing descriptive statistics and in making statistical inferences is described in Sections 1.4 and 1.5. The last two sections of Chapter 1 outline respectively the role of computers in statistical analysis and introduce the relatively new field of data mining.



## STATISTICS IN PRACTICE

The Economist

**F**ounded in 1843, *The Economist* is an international weekly news and business magazine written for top-level business executives and political decision-makers. The publication aims to provide readers with in-depth analyses of international politics, business news and trends, global economics and culture.



*The Economist* is published by the Economist Group – an international company employing nearly 1000 staff worldwide – with offices in London, Frankfurt, Paris and Vienna; in New York, Boston and Washington, DC; and in Hong Kong, mainland China, Singapore and Tokyo.

Between 1998 and 2008 the magazine's worldwide circulation grew by 100 per cent – recently exceeding 180 000 in the UK, 230 000 in continental Europe, 780 000 plus copies in North America and nearly 130 000 in the Asia-Pacific region. It is read in more than 200 countries and with a readership of four million, is one of the world's most influential business publications. Along with the *Financial Times*, it is arguably one of the two most successful print publications to be introduced in the US market during the past decade.

Complementing *The Economist* brand within the Economist Brand family, the Economist Intelligence Unit provides access to a comprehensive database of worldwide indicators and forecasts covering more than 200 countries, 45 regions and eight key industries. The Economist Intelligence Unit aims to help executives make informed business decisions through dependable intelligence delivered online, in print, in customized research as well as through conferences and peer interchange.

Alongside the Economist Brand family, the Group manages and runs the CFO and Government brand families for the benefit of senior finance executives and government decision-makers (in Brussels and Washington respectively).

## 1.1 APPLICATIONS IN BUSINESS AND ECONOMICS

In today's global business and economic environment, anyone can access vast amounts of statistical information. The most successful managers and decision-makers understand the information and know how to use it effectively. In this section, we provide examples that illustrate some of the uses of statistics in business and economics.

### Accounting

Public accounting firms use statistical sampling procedures when conducting audits for their clients. For instance, suppose an accounting firm wants to determine whether the amount of accounts receivable shown on a client's balance sheet fairly represents the actual amount of accounts receivable. Usually the large number of individual accounts receivable makes reviewing and validating every account too time-consuming and expensive. As common practice in such situations, the audit staff selects a subset of the accounts called a sample. After reviewing the accuracy of the sampled accounts, the auditors draw a conclusion as to whether the accounts receivable amount shown on the client's balance sheet is acceptable.

### Finance

Financial analysts use a variety of statistical information to guide their investment recommendations. In the case of stocks, the analysts review a variety of financial data including price/earnings ratios and dividend yields. By comparing the information for an individual stock with information about the stock market averages, a financial analyst can begin to draw a conclusion as to whether an individual stock is over- or under-priced. Similarly, historical trends in stock prices can provide a helpful indication on when investors might consider entering (or re-entering) the market. For example, *Money Week* (3 April 2009) reported a Goldman Sachs analysis that indicated, because stocks were unusually cheap at the time, real average returns of up to 6 per cent in the US and 7 per cent in Britain might be possible over the next decade – based on long-term cyclically adjusted price/earnings ratios.

### Marketing

Electronic scanners at retail checkout counters collect data for a variety of marketing research applications. For example, data suppliers such as ACNielsen purchase point-of-sale scanner data from grocery stores, process the data and then sell statistical summaries of the data to manufacturers. Manufacturers spend vast amounts per product category to obtain this type of scanner data. Manufacturers also purchase data and statistical summaries on promotional activities such as special pricing and the use of in-store displays. Brand managers can review the scanner statistics and the promotional activity statistics to gain a better understanding of the relationship between promotional activities and sales. Such analyses often prove helpful in establishing future marketing strategies for the various products.

### Production

Today's emphasis on quality makes quality control an important application of statistics in production. A variety of statistical quality control charts are used to monitor the output of a production process. In particular, an  $x$ -bar chart can be used to monitor the average output. Suppose, for example, that a machine fills containers with 330g of a soft drink. Periodically, a production worker selects a sample of containers and computes the average number of grams in the sample. This average, or  $x$ -bar value, is plotted on an  $x$ -bar chart. A plotted value above the chart's upper control limit indicates overfilling, and a plotted value below the chart's lower control limit indicates underfilling. The process is termed 'in control' and allowed to continue as long as the plotted  $x$ -bar values fall between the chart's upper and lower control limits. Properly interpreted, an  $x$ -bar chart can help determine when adjustments are necessary to correct a production process.

## Economics

Economists frequently provide forecasts about the future of the economy or some aspect of it. They use a variety of statistical information in making such forecasts. For instance, in forecasting inflation rates, economists use statistical information on such indicators as the Producer Price Index, the unemployment rate and manufacturing capacity utilization. Often these statistical indicators are entered into computerized forecasting models that predict inflation rates.

Applications of statistics such as those described in this section are an integral part of this text. Such examples provide an overview of the breadth of statistical applications. To supplement these examples, chapter-opening Statistics in Practice articles obtained from a variety of topical sources are used to introduce the material covered in each chapter. These articles show the importance of statistics in a wide variety of business and economic situations.

## 1.2 DATA

**Data** are the facts and figures collected, analyzed and summarized for presentation and interpretation. All the data collected in a particular study are referred to as the **data set** for the study. Table 1.1 shows a data set summarizing information for equity (share) trading at the 22 European Stock Exchanges in March 2009.

**TABLE 1.1** European stock exchange monthly statistics domestic equity trading (electronic order book transactions) March 2009

| Exchange          | Total             |                  |
|-------------------|-------------------|------------------|
|                   | Trades            | Turnover         |
| Athens            | 599 192           | 2 009.8          |
| Borsa Italiana    | 5 921 099         | 44 385.9         |
| Bratislava        | 111               | 0.1              |
| Bucharest         | 79 921            | 45.3             |
| Budapest          | 298 871           | 1 089.6          |
| Bulgarian         | 14 040            | 64.4             |
| Cyprus            | 31 167            | 76.1             |
| Deutsche Börse    | 7 642 241         | 86 994.5         |
| Euronext          | 15 282 996        | 116 488          |
| Irish             | 79 973            | 549.8            |
| Ljubljana         | 11 172            | 35.6             |
| London            | 16 539 588        | 114 283.6        |
| Luxembourg        | 1 152             | 125              |
| Malta             | 638               | 1.9              |
| NASDAQ OMX Nordic | 4 550 073         | 40 927.4         |
| Oslo Bars         | 981 362           | 9 755.1          |
| Prague            | 65 153            | 1 034.8          |
| SIX Swiss         | 440 578           | 2 667.1          |
| Spanish (BME)     | 2 799 329         | 60 387.6         |
| SWX Europe        | n/a               | n/a              |
| Warsaw            | 1 155 379         | 2 468.6          |
| Wiener Borse      | 433 545           | 2 744            |
| <b>TOTAL</b>      | <b>56 927 580</b> | <b>486 021.7</b> |

Source: European Stock Exchange monthly statistics ([www.fese.be/en/?inc=art&id=3](http://www.fese.be/en/?inc=art&id=3))



EXCHANGES  
2009

## Elements, variables and observations

**Elements** are the entities on which data are collected. For the data set in Table 1.1, each individual European exchange is an element; the element names appear in the first column. With 22 exchanges, the data set contains 22 elements.

A **variable** is a characteristic of interest for the elements. The data set in Table 1.1 includes the following three variables:

- *Exchange*: at which the equities were traded.
- *Trades*: number of trades during the month.
- *Turnover*: value of trades (€m) during the month.

Measurements collected on each variable for every element in a study provide the data. The set of measurements obtained for a particular element is called an **observation**. Referring to Table 1.1, we see that the set of measurements for the first observation (Athens Exchange) is 599 192 and 2009.8. The set of measurements for the second observation (Borsa Italiana) is 5 921 099 and 44 385.9; and so on. A data set with 22 elements contains 22 observations.

## Scales of measurement

Data collection requires one of the following scales of measurement: nominal, ordinal, interval or ratio. The scale of measurement determines the amount of information contained in the data and indicates the most appropriate data summarization and statistical analyses.

When the data for a variable consist of labels or names used to identify an attribute of the element, the scale of measurement is considered a **nominal scale**. For example, referring to the data in Table 1.1, we see that the scale of measurement for the exchange variable is nominal because Athens Exchange, Borsa Italiana ... Wiener Börse are labels used to identify where the equities are traded. In cases where the scale of measurement is nominal, a numeric code as well as non-numeric labels may be used. For example, to facilitate data collection and to prepare the data for entry into a computer database, we might use a numeric code by letting 1, denote the Athens Exchange, 2, the Borsa Italiana ... and 22, Wiener Börse. In this case the numeric values 1, 2, ... 22 provide the labels used to identify where the stock is traded. The scale of measurement is nominal even though the data appear as numeric values.

The scale of measurement for a variable is called an **ordinal scale** if the data exhibit the properties of nominal data and the order or rank of the data is meaningful. For example, Eastside Automotive sends customers a questionnaire designed to obtain data on the quality of its automotive repair service. Each customer provides a repair service rating of excellent, good or poor. Because the data obtained are the labels – excellent, good or poor – the data have the properties of nominal data. In addition, the data can be ranked, or ordered, with respect to the service quality. Data recorded as excellent indicate the best service, followed by good and then poor. Thus, the scale of measurement is ordinal. Note that the ordinal data can also be recorded using a numeric code. For example, we could use 1 for excellent, 2 for good and 3 for poor to maintain the properties of ordinal data. Thus, data for an ordinal scale may be either non-numeric or numeric.

The scale of measurement for a variable becomes an **interval scale** if the data show the properties of ordinal data and the interval between values is expressed in terms of a fixed unit of measure. Interval data are always numeric. Graduate Management Admission Test (GMAT) scores are an example of interval-scaled data. For example, three students with GMAT scores of 620 550 and 470 can be ranked or ordered in terms of best performance to poorest performance. In addition, the differences between the scores are meaningful. For instance, student one scored  $620 - 550 = 70$  points more than student two, while student two scored  $550 - 470 = 80$  points more than student three.

The scale of measurement for a variable is a **ratio scale** if the data have all the properties of interval data and the ratio of two values is meaningful. Variables such as distance, height, weight and time use the ratio scale of measurement. This scale requires that a zero value be included to indicate that nothing exists for the variable at the zero point. For example, consider the cost of a car. A zero value for the cost would

indicate that the car has no cost and is free. In addition, if we compare the cost of €30 000 for one car to the cost of €15 000 for a second car, the ratio property shows that the first car is  $\text{€}30\,000/\text{€}15\,000 = \text{two times}$ , or twice, the cost of the second car.

## Categorical and quantitative data

Data can be further classified as either categorical or quantitative. **Categorical data** include labels or names used to identify an attribute of each element. Categorical data use either the nominal or ordinal scale of measurement and may be non-numeric or numeric. **Quantitative data** require numeric values that indicate how much or how many. Quantitative data are obtained using either the interval or ratio scale of measurement.

A **categorical variable** is a variable with categorical data, and a **quantitative variable** is a variable with quantitative data. The statistical analysis appropriate for a particular variable depends upon whether the variable is categorical or quantitative. If the variable is categorical, the statistical analysis is rather limited. We can summarize categorical data by counting the number of observations in each category or by computing the proportion of the observations in each category. However, even when the categorical data use a numeric code, arithmetic operations such as addition, subtraction, multiplication and division do not provide meaningful results. Section 2.1 discusses ways for summarizing categorical data.

On the other hand, arithmetic operations often provide meaningful results for a quantitative variable. For example, for a quantitative variable, the data may be added and then divided by the number of observations to compute the average value. This average is usually meaningful and easily interpreted. In general, more alternatives for statistical analysis are possible when the data are quantitative. Section 2.2 and Chapter 3 provide ways of summarizing quantitative data.

## Cross-sectional and time series data

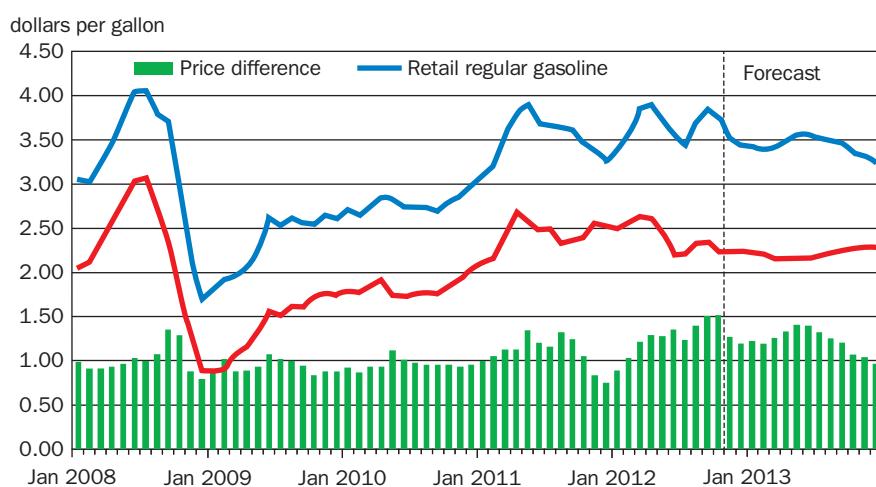
For purposes of statistical analysis, distinguishing between cross-sectional data and time series data is important. **Cross-sectional data** are data collected at the same or approximately the same point in time. The data in Table 1.1 are cross-sectional because they describe the two variables for the 22 exchanges at the same point in time. **Time series data** are data collected over several time periods. For example, Figure 1.1 provides a graph of the wholesale price (US\$) of crude oil per gallon for the period January 2008 and January 2012. It shows that starting around July 2008 the average price dipped sharply to less than \$2 per gallon. However, by November 2011 it had recovered to \$3 per gallon since when it has mostly hovered between \$3.50 and \$4 per gallon. Most of the statistical methods presented in this text apply to cross-sectional rather than time series data.

Quantitative data that measure how many are discrete. Quantitative data that measure how much are continuous because no separation occurs between the possible data values.

**FIGURE 1.1**

Wholesale price of  
crude oil per gallon  
(US\$) 2008–2012  
EIA ([www.eia.doe.gov/](http://www.eia.doe.gov/))

**U.S. Gasoline and Crude Oil Prices**



Crude oil price is composite refiner acquisition cost. Retail prices include state and federal

Source: Short-Term Energy Outlook, November 2012

## 1.3 DATA SOURCES

Data can be obtained from existing sources or from surveys and experimental studies designed to collect new data.

### Existing sources

In some cases, data needed for a particular application already exist. Companies maintain a variety of databases about their employees, customers and business operations. Data on employee salaries, ages and years of experience can usually be obtained from internal personnel records. Other internal records contain data on sales, advertising expenditures, distribution costs, inventory levels and production quantities. Most companies also maintain detailed data about their customers. Table 1.2 shows some of the data commonly available from internal company records.

Organizations that specialize in collecting and maintaining data make available substantial amounts of business and economic data. Companies access these external data sources through leasing arrangements or by purchase. Dun & Bradstreet, Bloomberg and the Economist Intelligence Unit are three sources that provide extensive business database services to clients. ACNielsen built successful businesses collecting and processing data that they sell to advertisers and product manufacturers.

Data are also available from a variety of industry associations and special interest organizations. The European Tour Operators, Association and European Travel Commission provide information on tourist trends and travel expenditures by visitors to and from countries in Europe. Such data would be of interest to firms and individuals in the travel industry. The Graduate Management Admission Council maintains data on test scores, student characteristics and graduate management education programmes. Most of the data from these types of sources are available to qualified users at a modest cost.

The Internet continues to grow as an important source of data and statistical information. Almost all companies maintain websites that provide general information about the company as well as data on sales, number of employees, number of products, product prices and product specifications. In addition, a number of companies now specialize in making information available over the Internet. As a result, one can obtain access to stock quotes, meal prices at restaurants, salary data and an almost infinite variety of information. Government agencies are another important source of existing data. For instance, Eurostat maintains considerable data on employment rates, wage rates, size of the labour force and union membership. Table 1.3 lists selected governmental agencies and some of the data they provide. Most government agencies that collect and process data also make the results available through a website. For instance, the Eurostat has a wealth of data at its website, <http://ec.europa.eu/eurostat>. Figure 1.2 shows the homepage for the Eurostat.

**TABLE 1.2** Examples of data available from internal company records

| Source             | Some of the data typically available  |
|--------------------|---|
| Employee records   | Name, address, social security number, salary, number of vacation days, number of sick days and bonus         |
| Production records | Part or product number, quantity produced, direct labour cost and materials cost                              |
| Inventory records  | Part or product number, number of units on hand, reorder level, economic order quantity and discount schedule |
| Sales records      | Product number, sales volume, sales volume by region and sales volume by customer type                        |
| Credit records     | Customer name, address, phone number, credit limit and accounts receivable balance                            |
| Customer profile   | Age, gender, income level, household size, address and preferences  |

**TABLE 1.3** Examples of data available from selected European sources

| Source   | Some of the data available  |
|--|---|
| Europa rates ( <a href="http://europa.eu">http://europa.eu</a> )                               | Travel, VAT (value added tax), euro exchange  |
| Eurostat ( <a href="http://epp.eurostat.ec.europa.eu/">http://epp.eurostat.ec.europa.eu/</a> ) | employment, population and social conditions  |
| European Central Bank ( <a href="http://www.ecb.int/">www.ecb.int/</a> )                       | Education and training, labour market, living<br>conditions and welfare   |
|  | Monetary, financial markets, interest rate and<br>balance of payments statistics, unit labour costs,<br>compensation per employee, labour productivity,<br>consumer prices, construction prices |

The screenshot shows the Eurostat website interface. At the top, there's a navigation bar with links for Home, Statistics, Publications, About Eurostat, and Help. The main content area features several sections:

- Latest news releases:** A list of recent news items with dates and descriptions, such as "Industrial production down by 2.0% in euro area" (13.05.2009) and "Around 20 000 asylum applicants registered each month in EU27" (08.05.2009).
- Statistics Database:** A sidebar listing "Most popular database tables" including GDP per capita in PPS, Real GDP growth rate, Total population, Unemployment rate, Employment rate, Inflation (monthly), and Inflation rate (annual).
- Selected Statistics:** A sidebar listing Structural Indicators, Euroindicators/PEEs, Sustainable development indicators, Government finance, and Prices (HICP).
- Selected Publications:** A sidebar listing Eurostat Yearbook, European Business, and Regional Yearbook.
- Country profiles:** A map of Europe where countries are color-coded (green, yellow, red) and linked to their profiles.
- Business Cycle Clock:** A diagram showing four phases of the business cycle: (1) Expansion, (2) Peak, (3) Decline, and (4) Trough.
- In the spotlight:** A section featuring a yellow bird icon and the text "Financial Turmoil".
- News:** A section with a blue background containing a "Starting March 2009" heading and a link to "The new website demo".

**FIGURE 1.2**

Eurostat homepage

## Statistical studies

Sometimes the data needed for a particular application are not available through existing sources. In such cases, the data can often be obtained by conducting a statistical study. Statistical studies can be classified as either *experimental* or *observational*.

In an experimental study, a variable of interest is first identified. Then one or more other variables are identified and controlled so that data can be obtained about how they influence the variable of interest. For example, a pharmaceutical firm might be interested in conducting an experiment to learn about how a new drug affects blood pressure. Blood pressure is the variable of interest in the study. The dosage level of the new drug is another variable that is hoped to have a causal effect on blood pressure. To obtain data about the effect of the new drug, researchers select a sample of individuals. The dosage level of the new drug is controlled, as different groups of individuals are given different dosage levels. Before and after data on blood pressure are collected for each group. Statistical analysis of the experimental data can help determine how the new drug affects blood pressure.

Non-experimental, or observational, statistical studies make no attempt to control the variables of interest. A survey is perhaps the most common type of observational study. For instance, in a personal interview survey, research questions are first identified. Then a questionnaire is designed and administered to a sample of individuals. Some restaurants use observational studies to obtain data about their customers' opinions of the quality of food, service, atmosphere and so on. A questionnaire used by the Lobster Pot Restaurant in Limerick City, Ireland, is shown in Figure 1.3. Note that the customers completing the questionnaire are asked to provide ratings for five variables: food quality, friendliness of service, promptness of service, cleanliness and management. The response categories of excellent, good, satisfactory and unsatisfactory provide ordinal data that enable Lobster Pot's managers to assess the quality of the restaurant's operation.

Managers wanting to use data and statistical analyses as an aid to decision-making must be aware of the time and cost required to obtain the data. The use of existing data sources is desirable when data must be obtained in a relatively short period of time.



We are happy you stopped by the Lobster Pot Restaurant and want to make sure you will come back. So, if you have a little time, we will really appreciate it if you will fill out this card. Your comments and suggestions are extremely important to us. Thank you!

Server's Name \_\_\_\_\_

|                  | Excellent                | Good                     | Satisfactory             | Unsatisfactory           |
|------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| Food Quality     | <input type="checkbox"/> | <input type="checkbox"/> | <input type="checkbox"/> | <input type="checkbox"/> |
| Friendly Service | <input type="checkbox"/> | <input type="checkbox"/> | <input type="checkbox"/> | <input type="checkbox"/> |
| Prompt Service   | <input type="checkbox"/> | <input type="checkbox"/> | <input type="checkbox"/> | <input type="checkbox"/> |
| Cleanliness      | <input type="checkbox"/> | <input type="checkbox"/> | <input type="checkbox"/> | <input type="checkbox"/> |
| Management       | <input type="checkbox"/> | <input type="checkbox"/> | <input type="checkbox"/> | <input type="checkbox"/> |
| Comments _____   |                          |                          |                          |                          |

What prompted your visit to us? \_\_\_\_\_

Please drop in suggestion box at entrance. Thank you.

**FIGURE 1.3**

Customer opinion questionnaire used by the Lobster Pot Restaurant, Limerick City, Ireland

If important data are not readily available from an existing source, the additional time and cost involved in obtaining the data must be taken into account. In all cases, the decision-maker should consider the contribution of the statistical analysis to the decision-making process. The cost of data acquisition and the subsequent statistical analysis should not exceed the savings generated by using the information to make a better decision.

## Data acquisition errors

Managers should always be aware of the possibility of data errors in statistical studies. Using erroneous data can be worse than not using any data at all. An error in data acquisition occurs whenever the data value obtained is not equal to the true or actual value that would be obtained with a correct procedure. Such errors can occur in a number of ways. For example, an interviewer might make a recording error, such as a transposition in writing the age of a 24-year-old person as 42, or the person answering an interview question might misinterpret the question and provide an incorrect response.

Experienced data analysts take great care in collecting and recording data to ensure that errors are not made. Special procedures can be used to check for internal consistency of the data. For instance, such procedures would indicate that the analyst should review the accuracy of data for a respondent shown to be 22 years of age but reporting 20 years of work experience. Data analysts also review data with unusually large and small values, called outliers, which are candidates for possible data errors. In Chapter 3 we present some of the methods statisticians use to identify outliers.

Errors often occur during data acquisition. Blindly using any data that happen to be available or using data that were acquired with little care can result in misleading information and bad decisions. Thus, taking steps to acquire accurate data can help ensure reliable and valuable decision-making information.

## 1.4 DESCRIPTIVE STATISTICS

Most of the statistical information in newspapers, magazines company reports and other publications consists of data that are summarized and presented in a form that is easy for the reader to understand. Such summaries of data, which may be tabular, graphical or numerical, are referred to as **descriptive statistics**.

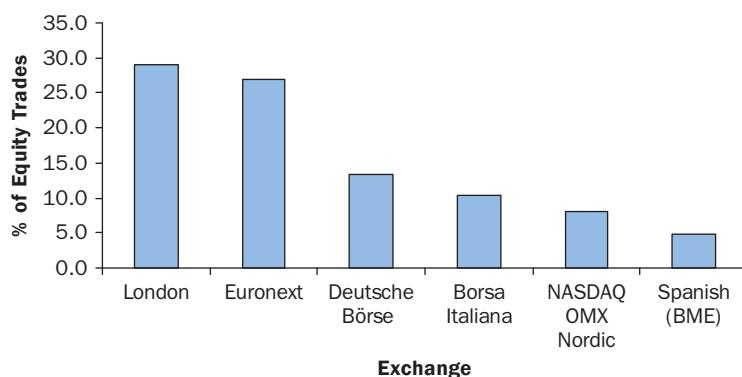
Refer again to the data set in Table 1.1 showing data on 22 European stock exchanges. Methods of descriptive statistics can be used to provide summaries of the information in this data set. For example, a tabular summary of the data for the six busiest exchanges by trade for the categorical variable exchange is shown in Table 1.4. A graphical summary of the same data, called a bar graph, is shown in Figure 1.4. These types of tabular and graphical summaries generally make the data easier to interpret. Referring to Table 1.4 and Figure 1.4, we can see easily that the majority of trades are for the London exchange (covering trading in Paris, Brussels, Amsterdam and Lisbon). On a percentage basis, 29.1 per cent of all trades for the 22 European stock exchanges occur through London. Similarly 26.8 per cent occur for Euronext and 13.4 per cent for Deutsche Börse. Note from Table 1.4 that 93 per cent of all trades take place in just six of the 22 European exchanges.

**TABLE 1.4** Per cent frequencies for six busiest exchanges by trades

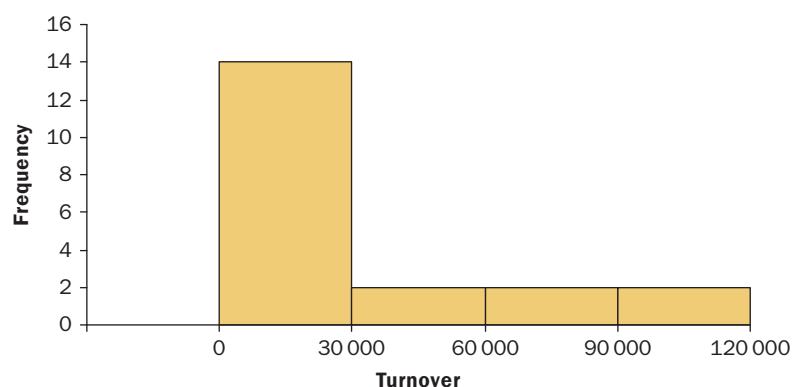
| Exchange          | % of Trades |
|-------------------|-------------|
| London            | 29.1        |
| Euronext          | 26.8        |
| Deutsche Börse    | 13.4        |
| Borsa Italiana    | 10.4        |
| NASDAQ OMX Nordic | 8.0         |
| Spanish (BME)     | 4.9         |
| <b>TOTAL</b>      | <b>92.6</b> |

**FIGURE 1.4**

Bar graph for the exchange variable

**FIGURE 1.5**

Histogram of turnover (€m)



A graphical summary of the data for the quantitative variable turnover for the exchanges, called a histogram, is provided in Figure 1.5. The histogram makes it easy to see that the turnover ranges from €0.0 to €120 000m, with the highest concentrations between €0 and €30 000m.

In addition to tabular and graphical displays, numerical descriptive statistics are used to summarize data. The most common numerical descriptive statistic is the average, or mean. Using the data on the variable turnover for the exchanges in Table 1.1, we can compute the average turnover by adding the turnover for the 21 exchanges where turnover has been declared and dividing the sum by 21. Doing so provides an average turnover of €23 144 million. This average demonstrates a measure of the central tendency, or central location, of the data for that variable.

In a number of fields, interest continues to grow in statistical methods that can be used for developing and presenting descriptive statistics. Chapters 1 and 3 devote attention to the tabular, graphical and numerical methods of descriptive statistics.

## 1.5 STATISTICAL INFERENCE

Many situations require data for a large group of elements (individuals, companies, voters, households, products, customers and so on). Because of time, cost and other considerations, data can be collected from only a small portion of the group. The larger group of elements in a particular study is called the **population**, and the smaller group is called the **sample**. Formally, we use the following definitions.

### Population

A *population* is the set of all elements of interest in a particular study.

### Sample

A *sample* is a subset of the population.

The process of conducting a survey to collect data for the entire population is called a **census**. The process of conducting a survey to collect data for a sample is called a **sample survey**. As one of its major contributions, statistics uses data from a sample to make estimates and test hypotheses about the characteristics of a population through a process referred to as **statistical inference**.

As an example of statistical inference, let us consider the study conducted by Electronica Nieves. Nieves manufactures a high-intensity light bulb used in a variety of electrical products. In an attempt to increase the useful life of the light bulb, the product design group developed a new light bulb filament. In this case, the population is defined as all light bulbs that could be produced with the new filament. To evaluate the advantages of the new filament, 200 bulbs with the new filament were manufactured and tested. Data collected from this sample showed the number of hours each light bulb operated before the filament burned out or the bulb failed. See Table 1.5.

Suppose Nieves wants to use the sample data to make an inference about the average hours of useful life for the population of all light bulbs that could be produced with the new filament. Adding the 200 values in Table 1.5 and dividing the total by 200 provides the sample average lifetime for the light bulbs: 76 hours. We can use this sample result to estimate that the average lifetime for the light bulbs in the population is 76 hours. Figure 1.6 provides a graphical summary of the statistical inference process for Electronica Nieves.

**TABLE 1.5** Hours until failure for a sample of 200 light bulbs for the Electronica Nieves example

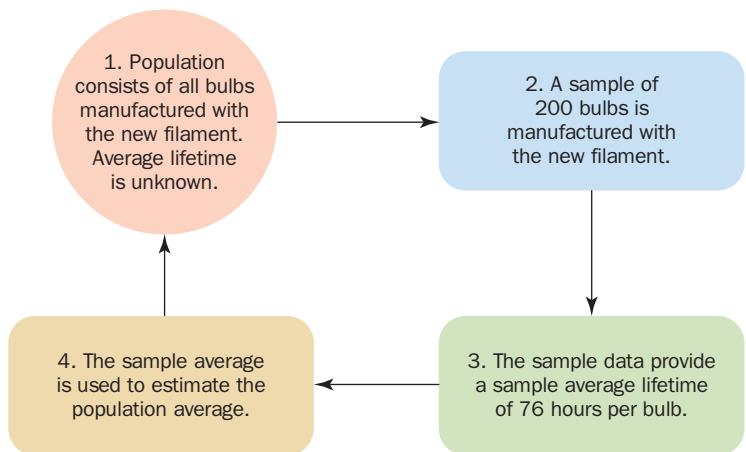
|     |     |    |     |     |    |    |    |    |     |
|-----|-----|----|-----|-----|----|----|----|----|-----|
| 107 | 73  | 68 | 97  | 76  | 79 | 94 | 59 | 98 | 57  |
| 54  | 65  | 71 | 70  | 84  | 88 | 62 | 61 | 79 | 98  |
| 66  | 62  | 79 | 86  | 68  | 74 | 61 | 82 | 65 | 98  |
| 62  | 116 | 65 | 88  | 64  | 79 | 78 | 79 | 77 | 86  |
| 74  | 85  | 73 | 80  | 68  | 78 | 89 | 72 | 58 | 69  |
| 92  | 78  | 88 | 77  | 103 | 88 | 63 | 68 | 88 | 81  |
| 75  | 90  | 62 | 89  | 71  | 71 | 74 | 70 | 74 | 70  |
| 65  | 81  | 75 | 62  | 94  | 71 | 85 | 84 | 83 | 63  |
| 81  | 62  | 79 | 83  | 93  | 61 | 65 | 62 | 92 | 65  |
| 83  | 70  | 70 | 81  | 77  | 72 | 84 | 67 | 59 | 58  |
| 78  | 66  | 66 | 94  | 77  | 63 | 66 | 75 | 68 | 76  |
| 90  | 78  | 71 | 101 | 78  | 43 | 59 | 67 | 61 | 71  |
| 96  | 75  | 64 | 76  | 72  | 77 | 74 | 65 | 82 | 86  |
| 66  | 86  | 96 | 89  | 81  | 71 | 85 | 99 | 59 | 92  |
| 68  | 72  | 77 | 60  | 87  | 84 | 75 | 77 | 51 | 45  |
| 85  | 67  | 87 | 80  | 84  | 93 | 69 | 76 | 89 | 75  |
| 83  | 68  | 72 | 67  | 92  | 89 | 82 | 96 | 77 | 102 |
| 74  | 91  | 76 | 83  | 66  | 68 | 61 | 73 | 72 | 76  |
| 73  | 77  | 79 | 94  | 63  | 59 | 62 | 71 | 81 | 65  |
| 73  | 63  | 63 | 89  | 82  | 64 | 85 | 92 | 64 | 73  |



NIEVES

**FIGURE 1.6**

The process of statistical inference for the Electronica Nieves example



Whenever statisticians use a sample to estimate a population characteristic of interest, they usually provide a statement of the quality, or precision, associated with the estimate. For the Nieves example, the statistician might state that the point estimate of the average lifetime for the population of new light bulbs is 76 hours with a margin of error of  $\pm$  four hours. Thus, an interval estimate of the average lifetime for all light bulbs produced with the new filament is 72 hours to 80 hours. The statistician can also state how confident he or she is that the interval from 72 hours to 80 hours contains the population average.

## 1.6 COMPUTERS AND STATISTICAL ANALYSIS

Because statistical analysis typically involves large amounts of data, analysts frequently use computer software for this work. For instance, computing the average lifetime for the 200 light bulbs in the Electronica Nieves example (see Table 1.5) would be quite tedious without a computer. To facilitate computer usage, the larger data sets in this book are available on the website that accompanies the text. A logo in the left margin of the text (e.g. Nieves) identifies each of these data sets. The data files are available in MINITAB, SPSS and EXCEL formats. In addition, we provide instructions on the website for carrying out many of the statistical procedures using MINITAB, SPSS and EXCEL.

## 1.7 DATA MINING

With the aid of magnetic card readers, bar code scanners, and point-of-sale terminals, most organizations obtain large amounts of data on a daily basis. And, even for a small local restaurant that uses touch screen monitors to enter orders and handle billing, the amount of data collected can be significant. For large retail companies, the sheer volume of data collected is hard to conceptualize, and determining how to effectively use these data to improve profitability is a challenge. For example, mass retailers such as Wal-Mart capture data on 20 to 30 million transactions every day, telecommunication companies such as Vodafone generated in 2011 an average of a billion call records per day, and Visa processes 6800 payment transactions per second or approximately 600 million transactions per day. Storing and managing the transaction data is a significant undertaking.

The term data warehousing is used to refer to the process of capturing, storing and maintaining the data. Computing power and data collection tools have reached the point where it is now feasible to store and retrieve extremely large quantities of data in seconds. Analysis of the data in the warehouse may result in decisions that will lead to new strategies and higher profits for the organization.

The subject of **data mining** deals with methods for developing useful decision-making information from large data bases. Using a combination of procedures from statistics, mathematics and computer science, analysts 'mine the data' in the warehouse to convert it into useful information, hence the name

data mining. Data mining systems that are the most effective use automated procedures to extract information from the data using only the most general or even vague queries by the user. And data mining software automates the process of uncovering hidden predictive information that in the past required hands-on analysis.

The major applications of data mining have been made by companies with a strong consumer focus, such as retail businesses, financial organizations and communication companies. Data mining has been successfully used to help retailers such as Amazon and Barnes & Noble determine one or more related products that customers who have already purchased a specific product are also likely to purchase. Then, when a customer logs on to the company's website and purchases a product, the website uses pop-ups to alert the customer about additional products that the customer is likely to purchase. In another application, data mining may be used to identify customers who are likely to spend more than €20 on a particular shopping trip. These customers may then be identified as the ones to receive special email or regular mail discount offers to encourage them to make their next shopping trip before the discount termination date.

Data mining is a technology that relies heavily on methodology such as statistics, clustering, decision trees and rule induction. But it takes a creative integration of all these methods and computer science technologies involving artificial intelligence and machine learning to make data mining effective. A significant investment in time and money is required to implement commercial data mining software packages developed by firms such as IBM SPSS and SAS. The statistical concepts introduced in this text will be helpful in understanding the statistical methodology used by data mining software packages and enable you to better understand the statistical information that is developed.

Because statistical models play an important role in developing predictive models in data mining, many of the concerns that statisticians deal with in developing statistical models are also applicable. For instance, a concern in any statistical study involves the issue of model reliability. Finding a statistical model that works well for a particular sample of data does not necessarily mean that it can be reliably applied to other data. One of the common statistical approaches to evaluating model reliability is to divide the sample data set into two parts: a training data set and a test data set. If the model developed using the training data is able to accurately predict values in the test data, we say that the model is reliable. One advantage that data mining has over classical statistics is that the enormous amount of data available allows the data mining software to partition the data set so that a model developed for the training data set may be tested for reliability on other data. In this sense, the partitioning of the data set allows data mining to develop models and relationships and then quickly observe if they are repeatable and valid with new and different data. On the other hand, a warning for data mining applications is that with so much data available, there is a danger of over-fitting the model to the point that misleading associations and cause/effect conclusions appear to exist. Careful interpretation of data mining results and additional testing will help avoid this pitfall.

Although statistical methods play an important role in data mining, both in terms of discovering relationships in the data and predicting future outcomes, a thorough coverage of the topic is outside the scope of this text.

## EXERCISES



COMPLETE  
SOLUTIONS

1. Discuss the differences between statistics as numerical facts and statistics as a discipline or field of study.
2. Every year *Condé Nast Traveler* conducts an annual survey of subscribers to determine the best new places to stay throughout the world. Table 1.6 shows the ten hotels that were most highly ranked in their 2006 'hot list' survey. Note that (daily) rates quoted are for double rooms and are variously expressed in US dollars, British pounds or euros.
  - a. How many elements are in this data set?
  - b. How many variables are in this data set?

- c. Which variables are categorical and which variables are quantitative?  
d. What type of measurement scale is used for each of the variables?
3. Refer to Table 1.6:

- What is the average number of rooms for the ten hotels?
- If €1 = US\$1.3149 = £0.8986 compute the average room rate in euros.

**TABLE 1.6** The ten best new hotels to stay in, in the world

| Hot list ranking | Name of property                                   | Country   | Room rate | Number of rooms |
|------------------|--|-----------|-----------|-----------------|
| 1                | Amangalla, Galle                                   | Sri Lanka | US\$574   | 30              |
| 2                | Amanwella, Tangalle                                | Sri Lanka | US\$275   | 30              |
| 3                | Bairro Alto Hotel, Lisbon                          | Portugal  | €180      | 55              |
| 4                | Basico, Playa Del Carmen                           | Mexico    | US\$166   | 15              |
| 5                | Beit Al Mamlouka                                   | Syria     | £75       | 8               |
| 6                | Brown's Hotel, London                              | England   | £347      | 117             |
| 7                | Byblos Art Hotel Villa Amista, Verona              | Italy     | €270      | 60              |
| 8                | Cavas Wine Lodge, Mendoza                          | Argentina | US\$375   | 14              |
| 9                | Convento Do Espinheiro Heritage Hotel & Spa, Evora | Portugal  | €213      | 59              |
| 10               | Cosmopolitan, Toronto                              | Canada    | £150      | 97              |

Source: Condé Nast Traveler, May 2006 ([www.cntraveler.com/magazine/the-hot-list-2006](http://www.cntraveler.com/magazine/the-hot-list-2006))

- What is the percentage of hotels located in Portugal?
  - What is the percentage of hotels with 20 rooms or fewer?
4. Audio systems are typically made up of an MP3 player, a mini disk player, a cassette player, a CD player and separate speakers. The data in Table 1.7 show the product rating and retail price range for a popular selection of systems. Note that the code Y is used to confirm when a player is included in the system, N when it is not. Output power (watts) details are also provided (Kelkoo Electronics 2006).
- How many elements does this data set contain?
  - What is the population?
  - Compute the average output power for the sample.
5. Consider the data set for the sample of eight audio systems in Table 1.7.
- How many variables are in the data set?
  - Which of the variables are quantitative and which are categorical?
  - What percentage of the audio systems has a four star rating or higher?
  - What percentage of the audio systems includes an MP3 player?



HOTELS



COMPLETE  
SOLUTIONS

**TABLE 1.7** A sample of eight audio systems

| Brand and model    | Product rating (# of stars) | Price (£) | MP3 player | Mini disk player | Cassette player | CD (watts) player | Output |
|--------------------|-----------------------------|-----------|------------|------------------|-----------------|-------------------|--------|
| Technics SCEH790   | 1                           | 320–400   | Y          | N                | Y               | Y                 | 360    |
| Yamaha M170        | 3                           | 162–290   | N          | N                | N               | Y                 | 50     |
| Panasonic SCPM29   | 5                           | 188       | Y          | N                | Y               | Y                 | 70     |
| Pure Digital DMX50 | 3                           | 180–230   | N          | N                | N               | Y                 | 80     |
| Sony CMTNEZ3       | 5                           | 60–100    | Y          | N                | Y               | Y                 | 30     |
| Philips FWM589     | 4                           | 143–200   | Y          | N                | N               | Y                 | 400    |
| Philips MCM9       | 5                           | 93–110    | Y          | N                | Y               | Y                 | 100    |
| Samsung MM-C6      | 5                           | 100–130   | Y          | N                | N               | Y                 | 40     |

Source: Kelkoo (<http://audiovisual.kelkoo.co.uk>)

AUDIO-SYSTEMS



COMPLETE SOLUTIONS

6. State whether each of the following variables is categorical or quantitative and indicate its measurement scale.
- Annual sales.
  - Soft drink size (small, medium, large).
  - Occupational classification (SOC 2000).
  - Earnings per share.
  - Method of payment (cash, cheque, credit card).
7. The Health & Wellbeing Survey ran over a three-week period (ending 19 October 2007) and 389 respondents took part. The survey asked the respondents to respond to the statement, 'How would you describe your own physical health at this time?' (<http://inform.glam.ac.uk/news/2007/10/24/health-wellbeing-staff-survey-results/>). Response categories were strongly agree, agree, neither agree or disagree, disagree and strongly disagree.
- What was the sample size for this survey?
  - Are the data categorical or quantitative?
  - Would it make more sense to use averages or percentages as a summary of the data for this question?
  - Of the respondents, 57 per cent agreed with the statement. How many individuals provided this response?
8. State whether each of the following variables is categorical or quantitative and indicate its measurement scale.
- Age.
  - Gender.
  - Class rank.
  - Make of car.
  - Number of people favouring closer European integration.

- 9.** Figure 1.7 provides a bar chart summarizing the actual earnings for Volkswagen for the years 2000 to 2008 (Source: *Volkswagen AG Annual Reports 2001–2008*).
- Are the data categorical or quantitative?
  - Are the data times series or cross-sectional?
  - What is the variable of interest?
  - Comment on the trend in Volkswagen's earnings over time. Would you expect to see an increase or decrease in 2009?

**FIGURE 1.7**

Volkswagen's  
earnings (€m)  
1998–2009



- 10.** The Hawaii Visitors' Bureau collects data on visitors to Hawaii. The following questions were among 16 asked in a questionnaire handed out to passengers during incoming airline flights.
- This trip to Hawaii is my: 1st, 2nd, 3rd, 4th, etc.
  - The primary reason for this trip is: (ten categories including vacation, convention, honeymoon).
  - Where I plan to stay: (11 categories including hotel, apartment, relatives, camping).
  - Total days in Hawaii.
- What is the population being studied?
  - Is the use of a questionnaire a good way to reach the population of passengers on incoming airline flights?
  - Comment on each of the four questions in terms of whether it will provide categorical or quantitative data.
- 11.** A manager of a large corporation recommends a \$10 000 raise be given to keep a valued subordinate from moving to another company. What internal and external sources of data might be used to decide whether such a salary increase is appropriate?
- 12.** In a recent study of causes of death in men 60 years of age and older, a sample of 120 men indicated that 48 died as a result of some form of heart disease.
- Develop a descriptive statistic that can be used as an estimate of the percentage of men 60 years of age or older who die from some form of heart disease.
  - Are the data on cause of death categorical or quantitative?
  - Discuss the role of statistical inference in this type of medical research.
- 13.** In 2007, 75.4 per cent of *Economist* readers had stayed in a hotel on business in the previous 12 months with 32.4 per cent of readers using first business class for travel.
- What is the population of interest in this study?
  - Is class of travel a categorical or quantitative variable?
  - If a reader had stayed in a hotel on business in the previous 12 months, would this be classed as a categorical or quantitative variable?
  - Does this study involve cross-sectional or time series data?
  - Describe any statistical inferences *The Economist* might make on the basis of the survey.



## ONLINE RESOURCES

For the data files and additional online resources for Chapter 1, go to the accompanying online platform. (See the 'About the Digital Resources' page in the front of the book for more information on access.)

## SUMMARY

Statistics is the art and science of collecting, analyzing, presenting and interpreting data. Nearly every college student majoring in business or economics is required to take a course in statistics. We began the chapter by describing typical statistical applications for business and economics.

Data consist of the facts and figures that are collected and analyzed. A set of measurements obtained for a particular element is an observation. Four scales of measurement used to obtain data on a particular variable include nominal, ordinal, interval and ratio. The scale of measurement for a variable is nominal when the data use labels or names to identify an attribute of an element. The scale is ordinal if the data demonstrate the properties of nominal data and the order or rank of the data is meaningful. The scale is interval if the data demonstrate the properties of ordinal data and the interval between values is expressed in terms of a fixed unit of measure. Finally, the scale of measurement is ratio if the data show all the properties of interval data and the ratio of two values is meaningful.

For purposes of statistical analysis, data can be classified as categorical or quantitative.

Categorical data use labels or names to identify an attribute of each element. Categorical data use either the nominal or ordinal scale of measurement and may be non-numeric or numeric. Quantitative data are numeric values that indicate how much or how many. Quantitative data use either the interval or ratio scale of measurement. Ordinary arithmetic operations are meaningful only if the data are quantitative. Therefore, statistical computations used for quantitative data are not always appropriate for categorical data.

In Sections 1.4 and 1.5 we introduced the topics of descriptive statistics and statistical inference. Definitions of the population and sample were provided and different types of descriptive statistics – tabular, graphical and numerical – used to summarize data. The process of statistical inference uses data obtained from a sample to make estimates or test hypotheses about the characteristics of a population.

The last two sections of the chapter provide information on the role of computers in statistical analysis and a brief overview of the relative new field of data mining.

## KEY TERMS

**Categorical data**

**Categorical variable**

**Census**

**Cross-sectional data**

**Data**

**Data mining**

**Data set**

**Descriptive statistics**

**Elements**

**Interval scale**

**Nominal scale**

**Observation**

**Ordinal scale**

**Population**

**Quantitative data**

**Quantitative variable**

**Ratio scale**

**Sample**

**Sample survey**

**Statistical inference**

**Statistics**

**Time series data**

**Variable**

# 2 Descriptive Statistics: Tabular and Graphical Presentations



## CHAPTER CONTENTS

Statistics in Practice Marks and Spencer: not just any statistical graphics

- 2.1 Summarizing qualitative data
- 2.2 Summarizing quantitative data
- 2.3 Cross-tabulations and scatter diagrams

**LEARNING OBJECTIVES** After studying this chapter and doing the exercises, you should be able to construct and interpret several different types of tabular and graphical data summaries.

- 1 For single qualitative variables: frequency, relative frequency and percentage frequency distributions; bar charts and pie charts.
- 2 For single quantitative variables: frequency, relative frequency and percentage frequency distributions; cumulative frequency, relative cumulative frequency and percentage cumulative frequency distributions; dot plots, stem-and-leaf plots, histograms and cumulative distribution plots (ogives).
- 3 For pairs of qualitative and quantitative data: cross-tabulations, with row and column percentages.
- 4 For pairs of quantitative variables: scatter diagrams.
- 5 You should be able to give an example of Simpson's paradox and explain the relevance of this paradox to the cross-tabulation of variables.

**A**s explained in Chapter 1, data can be classified as either qualitative or quantitative. **Qualitative data** use labels or names to identify categories of like items. **Quantitative data** are numerical values that indicate how much or how many.

This chapter introduces tabular and graphical methods commonly used to summarize both qualitative and quantitative data. Everyone is exposed to these types of presentation in annual reports (see Statistics in Practice), newspaper articles and research studies. It is important to understand how they are prepared and how they should be interpreted. We begin with methods for summarizing single variables. Section 2.3 introduces methods for summarizing the relationship between two variables.

Modern spreadsheet and statistical software packages provide extensive capabilities for summarizing data and preparing graphical presentations. EXCEL, IBM SPSS and MINITAB are three widely available packages. There are guides to some of their capabilities on the associated online platform.





## STATISTICS IN PRACTICE

Marks & Spencer: not just any statistical graphics

Marks & Spencer has a company history going back to 1884. The group is based in London, but has offices across the UK as well as overseas. Most people are likely to have come across its promotional activities and its advertising slogan 'Your M&S'. Marks & Spencer advertisements have featured a long list of well-known faces, including Twiggy, Erin O'Connor, David Beckham, Claudia Schiffer, Rosie Huntington-Whiteley and Antonio Banderas.

Marks & Spencer's shares are traded on the London Stock Exchange and it is a constituent of the FTSE 100 Index. Like all public companies, Marks & Spencer publishes an annual report. In the

annual report, alongside many photographs of its ambassadors and models, there are pictures of a different nature: statistical charts illustrating in particular the financial performance of the company. The examples here are from Marks and Spencer's 2013 Annual Report. First is a chart showing Marks & Spencer's governance framework, then a bar chart showing the breakdown of Marks & Spencer's international revenue, and finally a line graph showing mystery shopper feedback.

We are exposed to statistical charts of this type almost daily: in newspapers and magazines, on TV, online and in business reports such as the Marks & Spencer Annual Report. In this chapter, you will learn about tabular and graphical methods of descriptive statistics such as frequency distributions, bar charts, histograms, stem-and-leaf displays, cross-tabulations and others. The goal of these methods is to summarize data so that they can be easily understood and interpreted.



A window display showing an array of personalities who have modelled for Marks & Spencer

**Our Committees and Committee Chairmen**

For more on our Governance framework go to [marksandspencer.com/the company](http://marksandspencer.com/the company)

**International revenue**

|                   |              |                   |              |                   |
|-------------------|--------------|-------------------|--------------|-------------------|
| <b>£1,075.4 m</b> | <b>11/12</b> | <b>£1,066.1 m</b> | <b>12/13</b> | <b>£1,075.4 m</b> |
| <b>↑4.5%</b>      | <b>10/11</b> | <b>£1,007.3 m</b> | <b>11/12</b> | <b>£1,066.1 m</b> |
|                   | <b>09/10</b> | <b>£968.7 m</b>   | <b>10/11</b> | <b>£1,077.3 m</b> |
|                   |              |                   | <b>09/10</b> | <b>£968.7 m</b>   |

**Analysis** We are continuing to transform M&S into a more internationally focused business and are making progress against our target of increasing international sales by £300 m to £500 m by 2013/14.

**UK Mystery Shopping scores**

| Period | 11/12 | 12/13 |
|--------|-------|-------|
| Apr    | ~78   | ~79   |
| May    | ~81   | ~81   |
| Jun    | ~80   | ~80   |
| Jul    | ~82   | ~82   |
| Aug    | ~81   | ~82   |
| Sep    | ~83   | ~83   |
| Oct    | ~82   | ~83   |
| Nov    | ~83   | ~84   |
| Dec    | ~81   | ~84   |
| Jan    | ~83   | ~85   |
| Feb    | ~82   | ~85   |
| Mar    | ~83   | ~85   |

Average score **81%**

**Analysis** Mystery Shop scores remained high this year at 81%. However, to help us be more in touch with customers we plan to replace our monthly Mystery Shop programme with a more regular, in-depth customer satisfaction survey.

**Annual space growth**

**2.8%**

**Analysis** As consumer's shopping habits change, we continue to evolve our space selectively. We expect the planned opening of new space will add c.2% to the UK in 2013/14.

## 2.1 SUMMARIZING QUALITATIVE DATA

### Frequency distribution

We begin with a definition.

#### Frequency distribution

A frequency distribution is a tabular summary of data showing the number (frequency) of items in each of several non-overlapping classes.

The following example demonstrates the construction and interpretation of a **frequency distribution** for qualitative data. Audi, BMW, Mercedes, Opel and VW are five popular brands of car in Germany. The data in Table 2.1 are for a sample of 50 new car purchases of these five brands.

To construct a frequency distribution, we count the number of times each brand appears in Table 2.1. VW appears 19 times, Mercedes appears 13 times and so on. These counts are summarized in the frequency distribution in Table 2.2. The summary offers more insight than the original data. We see that VW is the leader, Mercedes is second, Audi is third. Opel and BMW are tied for fourth.

**TABLE 2.1** Data from a sample of 50 new car purchases

|          |          |          |      |          |
|----------|----------|----------|------|----------|
| VW       | BMW      | Mercedes | Audi | VW       |
| VW       | Mercedes | Audi     | VW   | Audi     |
| VW       | VW       | VW       | Audi | Mercedes |
| VW       | VW       | Opel     | Opel | BMW      |
| VW       | Audi     | Mercedes | Audi | Mercedes |
| VW       | Mercedes | Mercedes | VW   | Mercedes |
| VW       | VW       | Mercedes | Opel | Mercedes |
| Mercedes | BMW      | VW       | VW   | VW       |
| BMW      | Opel     | Audi     | Opel | Mercedes |
| VW       | Mercedes | BMW      | VW   | Audi     |

**TABLE 2.2** Frequency distribution of new car purchases

| Brand    | Frequency |
|----------|-----------|
| Audi     | 8         |
| BMW      | 5         |
| Mercedes | 13        |
| Opel     | 5         |
| VW       | 19        |
| Total    | 50        |

### Relative frequency and percentage frequency distributions

A frequency distribution shows the number (frequency) of items in each of several non-overlapping classes. We are often interested in the proportion, or percentage, of items in each class. The *relative frequency* of a class is the fraction or proportion of items belonging to a class. For a data set with  $n$  observations, the relative frequency of each class is:



CAR BRANDS

### Relative frequency

$$\text{Relative frequency of a class} = \frac{\text{Frequency of the class}}{n} \quad (2.1)$$

The *percentage frequency* of a class is the relative frequency multiplied by 100.

A **relative frequency distribution** is a tabular summary showing the relative frequency for each class. A **percentage frequency distribution** summarizes the percentage frequency for each class. Table 2.3 shows these distributions for the car purchase data. The relative frequency for VW is  $19/50 = 0.38$ , the relative frequency for Mercedes is  $13/50 = 0.26$  and so on. From the percentage frequency distribution, we see that 38 per cent of the purchases were VW, 26 per cent were Mercedes and so on. We can also note, for example, that  $38 + 26 = 64$  per cent of the purchases were of the top two car brands.

**TABLE 2.3** Relative and percentage frequency distributions of new car purchases

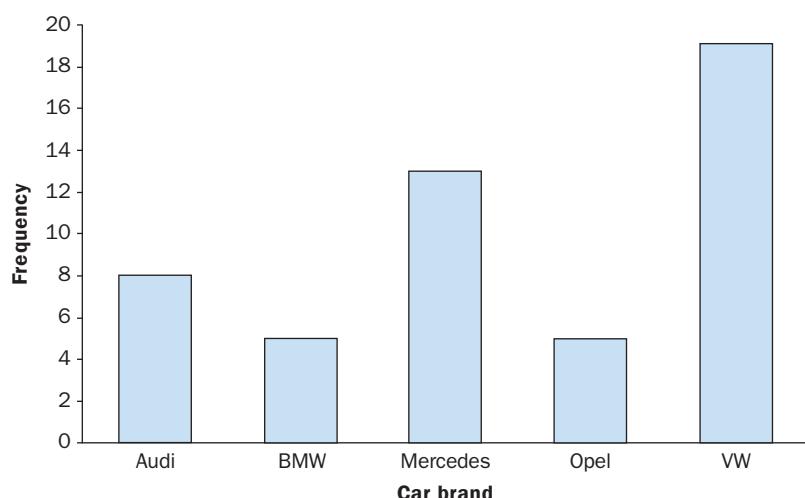
| Brand    | Relative frequency | Percentage frequency |
|----------|--------------------|----------------------|
| Audi     | 0.16               | 16                   |
| BMW      | 0.10               | 10                   |
| Mercedes | 0.26               | 26                   |
| Opel     | 0.10               | 10                   |
| VW       | 0.38               | 38                   |
| Total    | 1.00               | 100                  |

### Bar charts and pie charts

A **bar chart**, or **bar graph**, is a pictorial representation of a frequency, relative frequency, or percentage frequency distribution. On one axis of the chart (usually the horizontal), we specify the labels for the classes (categories) of data. A frequency, relative frequency or percentage frequency scale can be used for the other axis of the chart (usually the vertical). Then, using a bar of fixed width drawn above each class label, we make the length of the bar equal the frequency, relative frequency or percentage frequency of the class. For qualitative data, the bars should be separated to emphasize the fact that each class is separate. Figure 2.1 shows a bar chart of the frequency distribution for the 50 new car purchases.

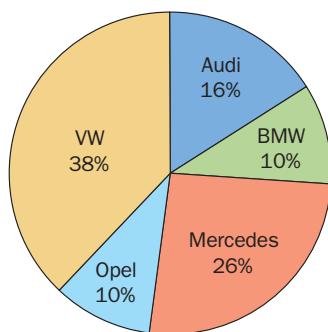
**FIGURE 2.1**

Bar chart of new car purchases



**FIGURE 2.2**

Pie chart of new car purchases



A **pie chart** is another way of presenting relative frequency and percentage frequency distributions. We first draw a circle to represent all the data. Then we subdivide the circle into sectors that correspond to the relative frequency for each class. For example, because a circle contains 360 degrees and VW shows a relative frequency of 0.38, the sector of the pie chart labelled VW consists of  $0.38(360) = 136.8$  degrees. The sector of the pie chart labelled Mercedes consists of  $0.26(360) = 93.6$  degrees. Similar calculations for the other classes give the pie chart in Figure 2.2. The numerical values shown for each sector can be frequencies, relative frequencies or percentage frequencies.

Often the number of classes in a frequency distribution is the same as the number of categories in the data, as for the car purchase data in this section. Data that included all car brands would require many categories, most of which would have a small number of purchases. Classes with smaller frequencies can be grouped into an aggregate class labelled ‘other’. Classes with frequencies of 5 per cent or less would most often be treated in this way.

In quality control applications, bar charts are used to summarize the most important causes of problems. When the bars are arranged in descending order of height from left to right with the most frequently occurring cause appearing first, the bar chart is called a *Pareto diagram*, named after its founder, Vilfredo Pareto, an Italian economist.

## EXERCISES

### Methods

1. The response to a question has three alternatives: A, B and C. A sample of 120 responses provides 60 A, 24 B and 36 C. Construct the frequency and relative frequency distributions.
2. A partial relative frequency distribution is given below.

| Class | Relative frequency |
|-------|--------------------|
| A     | 0.22               |
| B     | 0.18               |
| C     | 0.40               |
| D     |                    |

- a. What is the relative frequency of class D?
  - b. The total sample size is 200. What is the frequency of class D?
  - c. Construct the frequency distribution.
  - d. Construct the percentage frequency distribution.
3. A questionnaire provides 58 Yes, 42 No and 20 No-opinion answers.
    - a. In the construction of a pie chart, how many degrees would be in the sector of the pie showing the Yes answers?



COMPLETE  
SOLUTIONS

- b. How many degrees would be in the sector of the pie showing the No answers?
- c. Construct a pie chart.
- d. Construct a bar chart.

### Applications

4. CEM4Mobile is a customer experience management company based in Finland. The company does extensive market research in the mobile telecommunications field. Its research shows that the four most popular mobile operating systems in Nordic countries are Apple iOS, Symbian OS, Android and Nokia OS. A sample of 50 page loads from mobile browsing services follows.

|         |         |         |         |         |         |         |         |         |         |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| Android | Android | Android | Symbian | Apple   | Apple   | Symbian | Apple   | Apple   | Android |
| Android | Symbian | Android | Apple   | Nokia   | Android | Apple   | Apple   | Apple   | Nokia   |
| Nokia   | Apple   | Symbian | Apple   | Nokia   | Symbian | Android | Nokia   | Android | Apple   |
| Android | Symbian | Symbian | Apple   | Android | Android | Apple   | Android | Android | Apple   |
| Apple   | Nokia   | Symbian | Symbian | Android | Android | Apple   | Symbian | Symbian | Android |

- a. Are these data qualitative or quantitative?
- b. Construct frequency and percentage frequency distributions.
- c. Construct a bar chart and a pie chart.
- d. On the basis of the sample, which mobile operating system was the most popular?  
Which one was second?



NORDIC OS

5. A Wikipedia article listed the six most common last names in Belgium as (in alphabetical order): Jacobs, Janssens, Maes, Mertens, Peeters and Willems. A sample of 50 individuals with one of these last names provided the following data.

|         |         |         |          |          |          |         |          |          |          |
|---------|---------|---------|----------|----------|----------|---------|----------|----------|----------|
| Peeters | Peeters | Willems | Janssens | Janssens | Peeters  | Jacobs  | Maes     | Janssens | Mertens  |
| Jacobs  | Maes    | Peeters | Willems  | Jacobs   | Maes     | Peeters | Janssens | Maes     | Maes     |
| Peeters | Maes    | Peeters | Maes     | Janssens | Janssens | Mertens | Jacobs   | Jacobs   | Peeters  |
| Mertens | Maes    | Peeters | Janssens | Willems  | Willems  | Peeters | Janssens | Willems  | Mertens  |
| Jacobs  | Willems | Peeters | Janssens | Mertens  | Janssens | Peeters | Mertens  | Mertens  | Janssens |

Summarize the data by constructing the following:

- a. Relative and percentage frequency distributions.
  - b. A bar chart.
  - c. A pie chart.
  - d. Based on these data, what are the three most common last names?
6. The flextime system at Electronics Associates allows employees to begin their working day at 7:00, 7:30, 8:00, 8:30 or 9:00 a.m. The following data represent a sample of the starting times selected by the employees.



BELGIUM NAMES



COMPLETE SOLUTIONS

|      |      |      |      |      |      |      |      |      |      |
|------|------|------|------|------|------|------|------|------|------|
| 7:00 | 8:30 | 9:00 | 8:00 | 7:30 | 7:30 | 8:30 | 8:30 | 7:30 | 7:00 |
| 8:30 | 8:30 | 8:00 | 8:00 | 7:30 | 8:30 | 7:00 | 9:00 | 8:30 | 8:00 |

Summarize the data by constructing the following:

- a. A frequency distribution.
  - b. A percentage frequency distribution.
  - c. A bar chart.
  - d. A pie chart.
  - e. What do the summaries tell you about employee preferences in the flextime system?
7. A Merrill Lynch Client Satisfaction Survey asked clients to indicate how satisfied they were with their financial consultant. Client responses were coded 1 to 7, with 1 indicating 'not at all satisfied' and

7 indicating ‘extremely satisfied’. The following data are from a sample of 60 responses for a particular financial consultant.

|   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|
| 5 | 7 | 6 | 6 | 7 | 5 | 5 | 7 | 3 | 6 |
| 7 | 7 | 6 | 6 | 6 | 5 | 5 | 6 | 7 | 7 |
| 6 | 6 | 4 | 4 | 7 | 6 | 7 | 6 | 7 | 6 |
| 5 | 7 | 5 | 7 | 6 | 4 | 7 | 5 | 7 | 6 |
| 6 | 5 | 3 | 7 | 7 | 6 | 6 | 6 | 6 | 5 |
| 5 | 6 | 6 | 7 | 7 | 5 | 6 | 4 | 6 | 6 |

- a. Construct a frequency distribution and a relative frequency distribution for the data.
- b. Construct a bar chart.
- c. On the basis of your summaries, comment on the clients’ overall evaluation of the financial consultant.

## 2.2 SUMMARIZING QUANTITATIVE DATA

### Frequency distribution

As defined in Section 2.1, a frequency distribution is a tabular summary of data showing the number (frequency) of items in each of several non-overlapping classes. This definition holds for quantitative as well as qualitative data. However, with quantitative data there is usually more work involved in defining the non-overlapping classes.

Consider the quantitative data in Table 2.4. These data show the time in days required to complete year-end audits for a sample of 20 clients of Sanderson and Clifford, a small accounting firm. The data are rounded to the nearest day. The three steps necessary to define the classes for a frequency distribution with quantitative data are:

- 1 Determine the number of non-overlapping classes.
- 2 Determine the width of each class.
- 3 Determine the class limits.

We demonstrate these steps using the audit time data in Table 2.4.

### Number of classes

Classes are formed by specifying ranges that will be used to group the data. As a general guideline, we recommend using between 5 and 20 classes. For a small sample of data, as few as five or six classes may be used to summarize the data. For larger samples, more classes are usually required. The aim is to use enough classes to show the pattern of variation in the data, but not so many classes that some contain very few data points. Because the sample in Table 2.4 is relatively small ( $n = 20$ ), we chose to construct a frequency distribution with five classes.

**TABLE 2.4** Year-end audit times (in days)

|    |    |    |    |    |    |    |    |    |    |
|----|----|----|----|----|----|----|----|----|----|
| 12 | 14 | 19 | 18 | 15 | 15 | 18 | 17 | 20 | 27 |
| 22 | 23 | 22 | 21 | 33 | 28 | 14 | 18 | 16 | 13 |



AUDIT

## Width of the classes

The second step is to choose a width for the classes. As a general guideline, we recommend using the same width for each class. This reduces the chance of inappropriate interpretations. The choices for the number and the width of classes are not independent decisions. More classes means a smaller class width and vice versa. To determine an approximate class width, we identify the largest and smallest data values. Then we can use the following expression to determine the approximate class width.

### Approximate class width

$$\frac{\text{Largest data value} - \text{Smallest data value}}{\text{Number of classes}} \quad (2.2)$$

The approximate width given by equation (2.2) can be rounded to a more convenient value. For example, an approximate class width of 9.28 might be rounded to 10.

For the year-end audit times, the largest value is 33 and the smallest value is 12. We decided to summarize the data with five classes, so equation (2.2) provides an approximate class width of  $(33 - 12)/5 = 4.2$ . We decided to round up and use a class width of five days.

In practice, the number of classes and the appropriate class width are determined by trial and error. Once a possible number of classes is chosen, equation (2.2) is used to find the approximate class width. The process can be repeated for a different number of classes. Ultimately, the analyst uses judgement to determine the number of classes and class width that provide a good summary of the data. Different people may construct different, but equally acceptable, frequency distributions. The goal is to reveal the natural grouping and variation in the data.

For the audit time data, after deciding to use five classes, each with a width of five days, the next task is to specify the class limits for each of the classes.

## Class limits

Class limits must be chosen so that each data item belongs to one and only one class. The *lower class limit* identifies the smallest possible data value assigned to the class. The *upper class limit* identifies the largest possible data value assigned to the class. In constructing frequency distributions for qualitative data, we did not need to specify class limits because each data item naturally fell into a separate class (category). But with quantitative data, class limits are necessary to determine where each data value belongs.

Using the audit time data, we selected ten days as the lower class limit and 14 days as the upper class limit for the first class. This class is denoted 10–14 in Table 2.5. The smallest data value, 12, is included in the 10–14 class. We then selected 15 days as the lower class limit and 19 days as the upper class limit of the next class. We continued defining the lower and upper class limits to obtain a total of five classes: 10–14, 15–19, 20–24, 25–29 and 30–34. The largest data value, 33, is included in the 30–34 class. The difference between the lower class limits of adjacent classes is the class width. Using the first two lower class limits of 10 and 15, we see that the class width is  $15 - 10 = 5$ .

A frequency distribution can now be constructed by counting the number of data values belonging to each class. For example, the data in Table 2.5 show that four values (12, 14, 14 and 13) belong to the 10–14 class. The frequency for the 10–14 class is 4. Continuing this counting process for the 15–19, 20–24, 25–29 and 30–34 classes provides the frequency distribution in Table 2.5. Using this frequency distribution, we can observe that:

- 1** The most frequently occurring audit times are in the class 15–19 days. Eight of the 20 audit times belong to this class.
- 2** Only one audit required 30 or more days.

Other comments are possible, depending on the interests of the person viewing the frequency distribution. The value of a frequency distribution is that it provides insights about the data not easily obtained from the data in their original unorganized form.

**TABLE 2.5** Frequency distribution for the audit time data

| Audit time (days) | Frequency |
|-------------------|-----------|
| 10–14             | 4         |
| 15–19             | 8         |
| 20–24             | 5         |
| 25–29             | 2         |
| 30–34             | 1         |
| Total             | 20        |

The appropriate values for the class limits with quantitative data depend on the level of accuracy of the data. For instance, with the audit time data, the limits used were integer values because the data were rounded to the nearest day. If the data were rounded to the nearest one-tenth of a day (e.g. 12.3, 14.4), the limits would be stated in tenths of days. For example, the first class would be 10.0–14.9. If the data were rounded to the nearest one-hundredth of a day (e.g. 12.34, 14.45), the limits would be stated in hundredths of days, e.g. the first class would be 10.00–14.99.

An *open-ended* class requires only a lower class limit or an upper class limit. For example, in the audit time data, suppose two of the audits had taken 58 and 65 days. Rather than continuing with classes 35–39, 40–44, 45–49 and so on, we could simplify the frequency distribution to show an open-ended class of ‘35 or more’. This class would have a frequency count of 2. Most often the open-ended class appears at the upper end of the distribution. Sometimes an open-ended class appears at the lower end of the distribution and occasionally such classes appear at both ends.

### Class midpoint

In some applications, we want to know the midpoints of the classes in a frequency distribution for quantitative data. The **class midpoint** is the value halfway between the lower and upper class limits. For the audit time data, the five class midpoints are 12, 17, 22, 27 and 32.

## Relative frequency and percentage frequency distributions

We define the relative frequency and percentage frequency distributions for quantitative data in the same way as for qualitative data. The relative frequency is simply the proportion of the observations belonging to a class. With  $n$  observations,

$$\text{Relative frequency of a class} = \frac{\text{Frequency of the class}}{n}$$

The percentage frequency of a class is the relative frequency multiplied by 100.

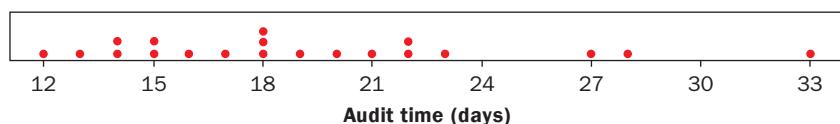
Based on the class frequencies in Table 2.5 and with  $n = 20$ , Table 2.6 shows the relative frequency and percentage frequency distributions for the audit time data. Note that 0.40 of the audits, or 40 per cent, required from 15 to 19 days. Only 0.05 of the audits, or 5 per cent, required 30 or more days. Again, additional interpretations and insights can be obtained by using Table 2.6.

**TABLE 2.6** Relative and percentage frequency distributions for the audit time data

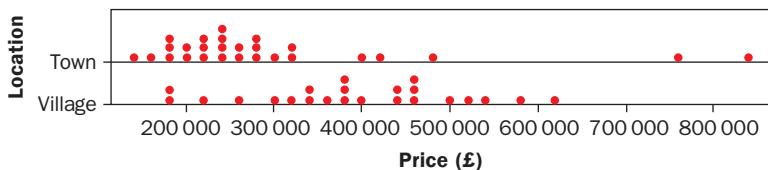
| Audit time (days) | Relative frequency | Percentage frequency |
|-------------------|--------------------|----------------------|
| 10–14             | 0.20               | 20                   |
| 15–19             | 0.40               | 40                   |
| 20–24             | 0.25               | 25                   |
| 25–29             | 0.10               | 10                   |
| 30–34             | 0.05               | 5                    |
| Total             | 1.00               | 100                  |

**FIGURE 2.3**

Dot plot for the audit time data

**FIGURE 2.4**

Dot plot comparing selling prices for houses in town and village locations



## Dot plot

One of the simplest graphical summaries of data is a **dot plot**. A horizontal axis shows the range of values for the observations. Each data value is represented by a dot placed above the axis. Figure 2.3 is a dot plot produced in MINITAB for the audit time data in Table 2.4. The three dots located above 18 on the horizontal axis indicate that three audit times of 18 days occurred.

Dot plots show the details of the data and are useful for comparing data distributions for two or more samples. For example, Figure 2.4 shows a MINITAB dot plot comparing the selling prices of houses for two samples of houses: one in town locations and the other in village locations.

## Histogram

A **histogram** is a chart showing quantitative data previously summarized in a frequency, relative frequency or percentage frequency distribution. The variable of interest is placed on the horizontal axis and the frequency, relative frequency or percentage frequency on the vertical axis. The frequency, relative frequency or percentage frequency of each class is shown by drawing a rectangle whose base is determined by the class limits on the horizontal axis and whose height is the corresponding frequency, relative frequency or percentage frequency.

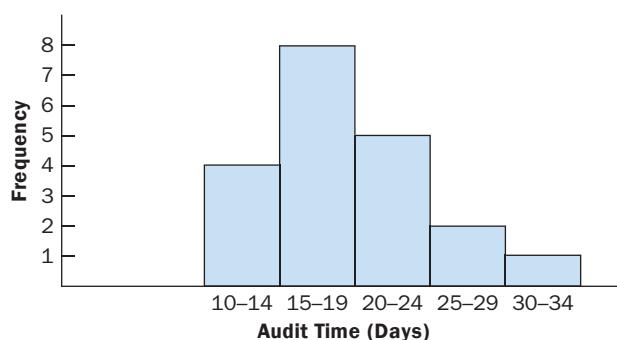
Figure 2.5 is a histogram for the audit time data. The class with the greatest frequency is shown by the rectangle above the class 15–19 days. The height of the rectangle shows that the frequency of this class is 8. A histogram for the relative or percentage frequency distribution of this data would look the same as the histogram in Figure 2.5 except that the vertical axis would be labelled with relative or percentage frequency values.

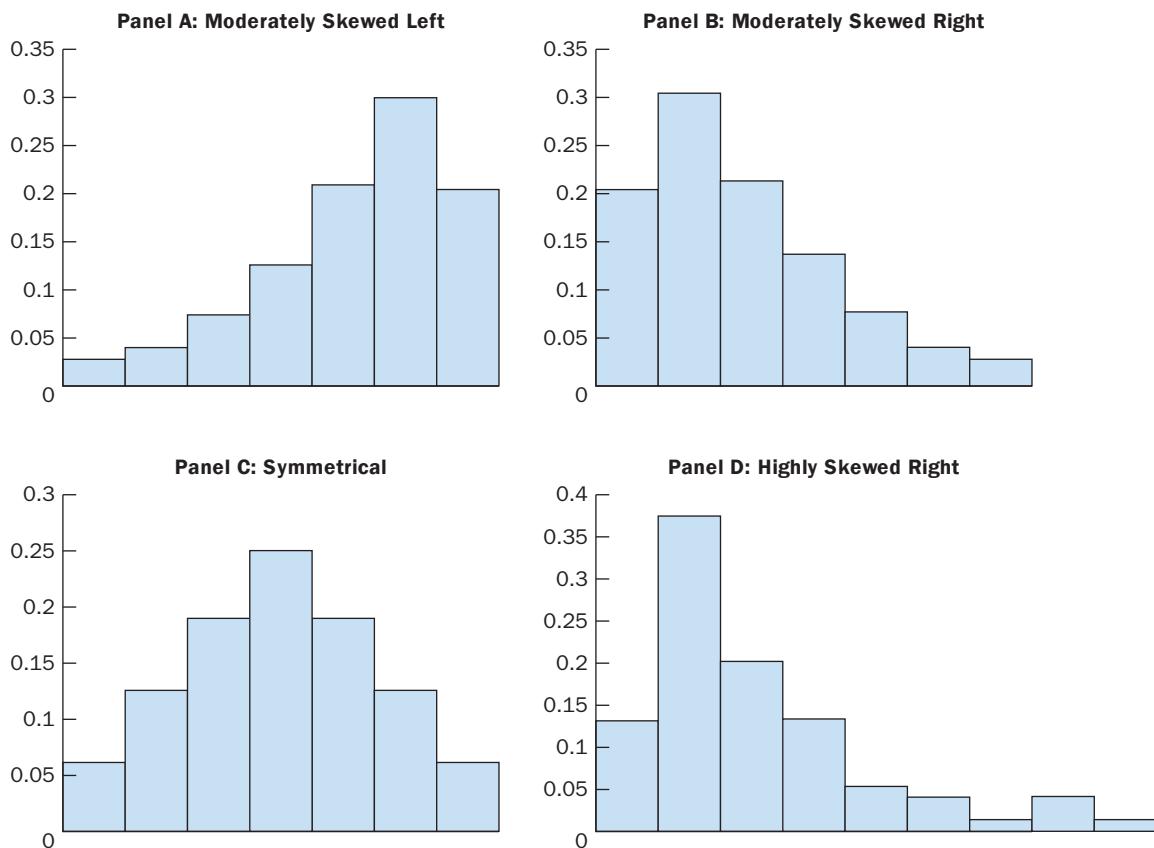
As Figure 2.5 shows, the adjacent rectangles of a histogram touch one another. This is the usual convention for a histogram, unlike a bar chart. Because the classes for the audit time data are stated as 10–14, 15–19, 20–24 and so on, one-unit spaces of 14 to 15, 19 to 20, etc. would seem to be needed between the classes. Eliminating the spaces in the histogram for the audit-time data helps show that, even though the data are rounded to the nearest full day, all values between the lower limit of the first class and the upper limit of the last class are possible.

One of the most important uses of a histogram is to provide information about the shape, or form, of a distribution. Figure 2.6 contains four histograms constructed from relative frequency distributions. Panel A shows the histogram for a set of data moderately skewed to the left. A histogram is skewed to the left, or

**FIGURE 2.5**

Histogram for the audit time data



**FIGURE 2.6**

Histograms showing differing levels of skewness

negatively skewed, if its tail extends further to the left. A histogram like this might be seen for scores from a relatively simple test. There are no scores above 100 per cent, most of the scores are above 70 per cent and only a few really low scores occur. Panel B shows the histogram for a set of data moderately skewed to the right. A histogram is skewed to the right, or positively skewed, if its tail extends further to the right. An example of this type of histogram would be for data such as house values. A relatively small number of expensive homes create the skewness in the right tail.

Panel C shows a symmetrical histogram. In a symmetrical histogram, the left tail mirrors the shape of the right tail. Histograms for real data are never perfectly symmetrical, but for many applications may be roughly symmetrical. Data for IQ scores, heights and weights of people and so on, lead to histograms that are roughly symmetrical. Panel D shows a histogram highly skewed to the right (positively skewed). This histogram was constructed from data on the amount of customer purchases over one day at a women's clothing store. Data from applications in business and economics often lead to histograms that are skewed to the right: for instance, data on wealth, salaries, purchase amounts and so on.

## Cumulative distributions

A variation of the frequency distribution that provides another tabular summary of quantitative data is the **cumulative frequency distribution**. The cumulative frequency distribution uses the number of classes, class widths and class limits adopted for the frequency distribution. However, rather than showing the frequency of each class, the cumulative frequency distribution shows the number of data items with values *less than or equal to the upper class limit* of each class. The first two columns of Table 2.7 show the cumulative frequency distribution for the audit time data.

**TABLE 2.7** Cumulative frequency, cumulative relative frequency and cumulative percentage frequency distributions for the audit time data

| Audit time (days)        | Cumulative frequency | Cumulative relative frequency | Cumulative percentage frequency |
|--------------------------|----------------------|-------------------------------|---------------------------------|
| Less than or equal to 14 | 4                    | 0.20                          | 20                              |
| Less than or equal to 19 | 12                   | 0.60                          | 60                              |
| Less than or equal to 24 | 17                   | 0.85                          | 85                              |
| Less than or equal to 29 | 19                   | 0.95                          | 95                              |
| Less than or equal to 34 | 20                   | 1.00                          | 100                             |

Consider the class with the description 'less than or equal to 24'. The cumulative frequency for this class is simply the sum of the frequencies for all classes with data values less than or equal to 24. For the frequency distribution in Table 2.5, the sum of the frequencies for classes 10–14, 15–19 and 20–24 indicates that  $4 + 8 + 5 = 17$  data values are less than or equal to 24. The cumulative frequency distribution in Table 2.7 also shows that four audits were completed in 14 days or less and 19 audits were completed in 29 days or less.

A **cumulative relative frequency distribution** shows the proportion of data items and a **cumulative percentage frequency distribution** shows the percentage of data items with values less than or equal to the upper limit of each class. The cumulative relative frequency distribution can be computed either by summing the relative frequencies in the relative frequency distribution, or by dividing the cumulative frequencies by the total number of items. Using the latter approach, we found the cumulative relative frequencies in column 3 of Table 2.7 by dividing the cumulative frequencies in column 2 by the total number of items ( $n = 20$ ). The cumulative percentage frequencies were computed by multiplying the cumulative relative frequencies by 100.

The cumulative relative and percentage frequency distributions show that 0.85 of the audits, or 85 per cent, were completed in 24 days or less; 0.95 of the audits, or 95 per cent, were completed in 29 days or less and so on.

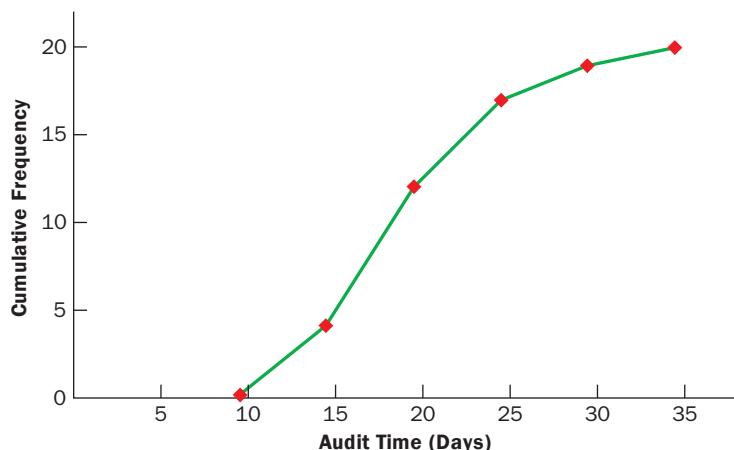
The last entry in a cumulative frequency distribution always equals the total number of observations. The last entry in a cumulative relative frequency distribution always equals 1.00 and the last entry in a cumulative percentage frequency distribution always equals 100.

## Cumulative distribution plot (ogive)

A graph of a cumulative distribution, called an **ogive**, shows data values on the horizontal axis and either the cumulative frequencies, the cumulative relative frequencies, or the cumulative percentage frequencies on the vertical axis. Figure 2.7 illustrates a cumulative distribution plot or ogive for the cumulative frequencies of the audit time data.

**FIGURE 2.7**

Ogive for the audit time data



The ogive is constructed by plotting a point corresponding to the cumulative frequency of each class. Because the classes for the audit time data are 10–14, 15–19, 20–24 and so on, one-unit gaps appear from 14 to 15, 19 to 20 and so on. These gaps are eliminated by plotting points halfway between the class limits. So, 14.5 is used for the 10–14 class, 19.5 is used for the 15–19 class and so on. The ‘less than or equal to 14’ class with a cumulative frequency of four is shown on the ogive in Figure 2.7 by the point located at 14.5 on the horizontal axis and 4 on the vertical axis. The ‘less than or equal to 19’ class with a cumulative frequency of 12 is shown by the point located at 19.5 on the horizontal axis and 12 on the vertical axis. Note that one additional point is plotted at the left end of the ogive. This point starts the ogive by showing that no data values fall below the 10–14 class. It is plotted at 9.5 on the horizontal axis and 0 on the vertical axis. The plotted points are connected by straight lines to complete the ogive.

## Exploratory data analysis: stem-and-leaf display

**Exploratory data analysis** techniques consist of simple arithmetic and easy-to-draw graphs that can be used to summarize data quickly. One technique – referred to as a **stem-and-leaf display** – can be used to show both the rank order and shape of a data set simultaneously. To illustrate the stem-and-leaf display, consider the data in Table 2.8. These came from a 150-question aptitude test given to 50 individuals recently interviewed for a position at Hawkins Manufacturing. The data indicate the number of questions answered correctly.

To construct a stem-and-leaf display, we first arrange the leading digits of each data value to the left of a vertical line. To the right of the vertical line, on the line corresponding to the appropriate first digit, we record the last digit for each data value as we pass through the observations in the order they were recorded.

|    |                       |
|----|-----------------------|
| 6  | 9 8                   |
| 7  | 2 3 6 3 6 5           |
| 8  | 6 2 3 1 1 0 4 5       |
| 9  | 7 2 2 6 2 1 5 8 8 5 4 |
| 10 | 7 4 8 0 2 6 6 0 6     |
| 11 | 2 8 5 9 3 5 9         |
| 12 | 6 8 7 4               |
| 13 | 2 4                   |
| 14 | 1                     |

Sorting the digits on each line into rank order is now relatively simple. This leads to the stem-and-leaf display shown here.

|    |                       |
|----|-----------------------|
| 6  | 8 9                   |
| 7  | 2 3 3 5 6 6           |
| 8  | 0 1 1 2 3 4 5 6       |
| 9  | 1 2 2 2 4 5 5 6 7 8 8 |
| 10 | 0 0 2 4 6 6 6 7 8     |
| 11 | 2 3 5 5 8 9 9         |
| 12 | 4 6 7 8               |
| 13 | 2 4                   |
| 14 | 1                     |

The numbers to the left of the vertical line (6, 7, ..., 14) form the *stem*, and each digit to the right of the vertical line is a *leaf*. For example, the first row has a stem value of 6 and leaves of 8 and 9. It indicates that two data values have a first digit of six. The leaves show that the data values are 68 and 69. Similarly, the second row indicates that six data values have a first digit of 7. The leaves show that the data values are 72, 73, 73, 75, 76 and 76. Rotating the page counter-clockwise onto its side provides a picture of the data that is similar to a histogram with classes of 60–69, 70–79, 80–89 and so on.

**TABLE 2.8** Number of questions answered correctly on an aptitude test

|     |     |     |     |     |     |     |     |     |     |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 112 | 72  | 69  | 97  | 107 | 73  | 92  | 76  | 86  | 73  |
| 126 | 128 | 118 | 127 | 124 | 82  | 104 | 132 | 134 | 83  |
| 92  | 108 | 96  | 100 | 92  | 115 | 76  | 91  | 102 | 81  |
| 95  | 141 | 81  | 80  | 106 | 84  | 119 | 113 | 98  | 75  |
| 68  | 98  | 115 | 106 | 95  | 100 | 85  | 94  | 106 | 119 |

Although the stem-and-leaf display may appear to offer the same information as a histogram, it has two primary advantages.

- 1 The stem-and-leaf display is easier to construct by hand for small data sets.
- 2 Within a class interval, the stem-and-leaf display provides more information than the histogram because the stem-and-leaf shows the actual data.

Just as a frequency distribution or histogram has no absolute number of classes, neither does a stem-and-leaf display have an absolute number of rows or stems. If we believe that our original stem-and-leaf display condensed the data too much, we can stretch the display by using two stems for each leading digit (using five stems for each leading digit is also a possibility). Using two stems for each leading digit, we would place all data values ending in 0, 1, 2, 3 and 4 in one row and all values ending in 5, 6, 7, 8 and 9 in a second row. The following display illustrates this approach. This stretched stem-and-leaf display is similar to a frequency distribution with intervals of 65–69, 70–74, 75–79 and so on.

|    |   |           |
|----|---|-----------|
| 6  | 8 | 9         |
| 7  | 2 | 3 3       |
| 7  | 5 | 6 6       |
| 8  | 0 | 1 1 2 3 4 |
| 8  | 5 | 6         |
| 9  | 1 | 2 2 2 4   |
| 9  | 5 | 5 6 7 8 8 |
| 10 | 0 | 0 2 4     |
| 10 | 6 | 6 6 7 8   |
| 11 | 2 | 3         |
| 11 | 5 | 5 8 9 9   |
| 12 | 4 |           |
| 12 | 6 | 7 8       |
| 13 | 2 | 4         |
| 14 | 1 |           |

The preceding example shows a stem-and-leaf display for data with three digits. Stem-and-leaf displays for data with more than three digits are possible. For example, consider the following data on the number of burgers sold by a fast-food restaurant for each of 15 weeks.

|      |      |      |      |      |      |      |      |
|------|------|------|------|------|------|------|------|
| 1565 | 1852 | 1644 | 1766 | 1888 | 1912 | 2044 | 1812 |
| 1790 | 1679 | 2008 | 1852 | 1967 | 1954 | 1733 |      |

A stem-and-leaf display of these data follows.

Leaf unit = 10

|    |         |
|----|---------|
| 15 | 6       |
| 16 | 4 7     |
| 17 | 3 6 9   |
| 18 | 1 5 5 8 |
| 19 | 1 5 6   |
| 20 | 0 4     |

A single digit is used to define each leaf, and only the first three digits of each observation have been used to construct the display. At the top of the display we have specified leaf unit = 10. Consider the first stem (15) and its associated leaf (6). Combining these numbers gives 156. To reconstruct an approximation of the original data value, we must multiply this number by 10, the value of the *leaf unit*:  $156 \times 10 = 1560$ . Although it is not possible to reconstruct the exact data value from the display, using a single digit for each leaf enables stem-and-leaf displays to be constructed for data having a large number of digits. Leaf units may be 100, 10, 1, 0.1 and so on. Where the leaf unit is not shown on the display, it is assumed to equal 1.

## EXERCISES

### Methods

- 8.** Consider the following data.

|    |    |    |    |    |    |    |    |    |    |
|----|----|----|----|----|----|----|----|----|----|
| 14 | 21 | 23 | 21 | 16 | 19 | 22 | 25 | 16 | 16 |
| 24 | 24 | 25 | 19 | 16 | 19 | 18 | 19 | 21 | 12 |
| 16 | 17 | 18 | 23 | 25 | 20 | 23 | 16 | 20 | 19 |
| 24 | 26 | 15 | 22 | 24 | 20 | 22 | 24 | 22 | 20 |

- a. Construct a frequency distribution using classes of 12–14, 15–17, 18–20, 21–23 and 24–26.
  - b. Construct a relative frequency distribution and a percentage frequency distribution using the classes in (a).
- 9.** Consider the following frequency distribution. Construct a cumulative frequency distribution and a cumulative relative frequency distribution.

| Class | Frequency |
|-------|-----------|
| 10–19 | 10        |
| 20–29 | 14        |
| 30–39 | 17        |
| 40–49 | 7         |
| 50–59 | 2         |

- 10.** Construct a histogram and an ogive for the data in Exercise 9.

- 11.** Consider the following data.

|     |      |      |      |      |      |      |      |      |      |
|-----|------|------|------|------|------|------|------|------|------|
| 8.9 | 10.2 | 11.5 | 7.8  | 10.0 | 12.2 | 13.5 | 14.1 | 10.0 | 12.2 |
| 6.8 | 9.5  | 11.5 | 11.2 | 14.9 | 7.5  | 10.0 | 6.0  | 15.8 | 11.5 |

- a. Construct a dot plot.
- b. Construct a frequency distribution.
- c. Construct a percentage frequency distribution.

- 12.** Construct a stem-and-leaf display for the following data.

70    72    75    64    58    83    80    82    76    75    68    65    57    78    85    72

- 13.** Construct a stem-and-leaf display for the following data.

11.3    9.6    10.4    7.5    8.3    10.5    10.0    9.3    8.1    7.7    7.5    8.4    6.3    8.8



FREQUENCY



COMPLETE  
SOLUTIONS



COMPLETE  
SOLUTIONS

### Applications

- 14.** A doctor's office staff studied the waiting times for patients who arrive at the office with a request for emergency service. The following data with waiting times in minutes were collected over a one-month period.

2 5 10 12 4 4 5 17 11 8 9 8 12 21 6 8 7 13 18 3

Use classes of 0–4, 5–9 and so on in the following:

- Show the frequency distribution.
- Show the relative frequency distribution.
- Show the cumulative frequency distribution.
- Show the cumulative relative frequency distribution.
- What proportion of these patients wait nine minutes or less?

- 15.** Data for the numbers of units produced by a production employee during the most recent 20 days are shown here.

|     |     |     |     |     |     |     |     |     |     |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 160 | 170 | 181 | 156 | 176 | 148 | 198 | 179 | 162 | 150 |
| 162 | 156 | 179 | 178 | 151 | 157 | 154 | 179 | 148 | 156 |

Summarize the data by constructing the following:

- A frequency distribution.
- A relative frequency distribution.
- A cumulative frequency distribution.
- A cumulative relative frequency distribution.
- A cumulative distribution plot (ogive).

- 16.** The closing prices of 40 company shares (in Kuwaiti dinar) follow.

|        |        |        |        |        |        |        |        |
|--------|--------|--------|--------|--------|--------|--------|--------|
| 44.00  | 0.80   | 69.00  | 226.00 | 68.00  | 51.00  | 265.00 | 130.00 |
| 172.00 | 202.00 | 52.00  | 134.00 | 81.00  | 50.00  | 550.00 | 28.50  |
| 13.00  | 435.00 | 218.00 | 270.00 | 52.00  | 108.00 | 248.00 | 0.45   |
| 188.00 | 800.00 | 59.00  | 65.00  | 355.00 | 410.00 | 102.00 | 174.00 |
| 136.00 | 34.00  | 64.00  | 660.00 | 122.00 | 62.00  | 290.00 | 90.00  |



SHARES

- Construct frequency and relative frequency distributions.
- Construct cumulative frequency and cumulative relative frequency distributions.
- Construct a histogram.
- Using your summaries, make comments and observations about the price of shares.

- 17.** The table below shows the estimated 2013 mid-year population of Kenya, by age group, rounded to the nearest thousand (from the US Census Bureau International Data Base).

| Age group | Population (000s) |
|-----------|-------------------|
| 0–9       | 13 310            |
| 10–19     | 9 601             |
| 20–29     | 7 904             |
| 30–39     | 5 975             |
| 40–49     | 3 273             |
| 50–59     | 2 076             |
| 60–69     | 1 171             |
| 70–79     | 555               |
| 80+       | 171               |

- Construct a percentage frequency distribution.
- Construct a cumulative percentage frequency distribution.



COMPUTER

COMPLETE  
SOLUTIONS

- c. Construct a cumulative distribution plot (ogive).
- d. Using the ogive, estimate the age that divides the population into halves (you will learn in Chapter 3 that this is called the *median*).

- 18.** The *Nielsen Home Technology Report* provided information about home technology and its usage by individuals aged 12 and older. The following data are the hours of personal computer usage during one week for a sample of 50 individuals.

|      |     |     |     |     |     |     |      |      |     |      |      |      |
|------|-----|-----|-----|-----|-----|-----|------|------|-----|------|------|------|
| 4.1  | 1.5 | 5.9 | 3.4 | 5.7 | 1.6 | 6.1 | 3.0  | 3.7  | 3.1 | 4.8  | 2.0  | 3.3  |
| 11.1 | 3.5 | 4.1 | 4.1 | 8.8 | 5.6 | 4.3 | 7.1  | 10.3 | 6.2 | 7.6  | 10.8 | 0.7  |
| 4.0  | 9.2 | 4.4 | 5.7 | 7.2 | 6.1 | 5.7 | 5.9  | 4.7  | 3.9 | 3.7  | 3.1  | 12.1 |
| 14.8 | 5.4 | 4.2 | 3.9 | 4.1 | 2.8 | 9.5 | 12.9 | 6.1  | 3.1 | 10.4 |      |      |

Summarize the data by constructing the following:

- a. A frequency distribution (use a class width of three hours).
  - b. A relative frequency distribution.
  - c. A histogram.
  - d. A cumulative distribution plot (ogive).
  - e. Comment on what the data indicate about personal computer usage at home.
- 19.** The daily high and low temperatures (in degrees Celsius) for 20 cities on one particular day follow.

| City         | High | Low | City           | High | Low |
|--------------|------|-----|----------------|------|-----|
| Athens       | 24   | 12  | Melbourne      | 19   | 10  |
| Bangkok      | 33   | 23  | Montreal       | 18   | 11  |
| Cairo        | 29   | 14  | Paris          | 25   | 13  |
| Copenhagen   | 18   | 4   | Rio de Janeiro | 27   | 16  |
| Dublin       | 18   | 8   | Rome           | 27   | 12  |
| Havana       | 30   | 20  | Seoul          | 18   | 10  |
| Hong Kong    | 27   | 22  | Singapore      | 32   | 24  |
| Johannesburg | 16   | 10  | Sydney         | 20   | 13  |
| London       | 23   | 9   | Tokyo          | 26   | 15  |
| Manila       | 34   | 24  | Vancouver      | 14   | 6   |

- a. Prepare a stem-and-leaf display for the high temperatures.
- b. Prepare a stem-and-leaf display for the low temperatures.
- c. Compare the stem-and-leaf displays from parts (a) and (b), and comment on the differences between daily high and low temperatures.
- d. Use the stem-and-leaf display from part (a) to determine the number of cities having a high temperature of 25 degrees or above.
- e. Provide frequency distributions for both high and low temperature data.

## 2.3 CROSS-TABULATIONS AND SCATTER DIAGRAMS

So far in this chapter, we have focused on methods for summarizing *one variable at a time*. Often a manager or decision-maker requires tabular and graphical methods that will assist in the understanding of the *relationship between two variables*. Cross-tabulation and scatter diagrams are two such methods.

### Cross-tabulation

A **cross-tabulation** is a tabular summary of data for two variables. Consider the following data from a consumer restaurant review, based on a sample of 300 restaurants in a large European city. Table 2.9 shows the data for the first five restaurants: the restaurant's quality rating and typical meal price. Quality

**TABLE 2.9** Quality rating and meal price for 300 restaurants

| Restaurant | Quality rating | Meal price (€) |
|------------|----------------|----------------|
| 1          | Disappointing  | 18             |
| 2          | Good           | 22             |
| 3          | Disappointing  | 28             |
| 4          | Excellent      | 38             |
| 5          | Good           | 33             |
| .          | .              | .              |
| .          | .              | .              |



RESTAURANT

**TABLE 2.10** Cross-tabulation of quality rating and meal price for 300 restaurants

| Quality rating | Meal price |        |        |        |     | Total |
|----------------|------------|--------|--------|--------|-----|-------|
|                | €10–19     | €20–29 | €30–39 | €40–49 |     |       |
| Disappointing  | 42         | 40     | 2      | 0      | 84  |       |
| Good           | 34         | 64     | 46     | 6      | 150 |       |
| Excellent      | 2          | 14     | 28     | 22     | 66  |       |
| Total          | 78         | 118    | 76     | 28     | 300 |       |

rating is a qualitative variable with categories ‘disappointing’, ‘good’ and ‘excellent’. Meal price is a quantitative variable that ranges from €10 to €49.

A cross-tabulation of the data is shown in Table 2.10. The left and top margin labels define the classes for the two variables. In the left margin, the row labels (disappointing, good and excellent) correspond to the three classes of the quality rating variable. In the top margin, the column labels (€10–19, €20–29, €30–39 and €40–49) correspond to the four classes of the meal price variable. Each restaurant in the sample provides a quality rating and a meal price, and so is associated with a cell in one of the rows and one of the columns of the cross-tabulation. For example, restaurant 5 has a good quality rating and a meal price of €33. This restaurant belongs to the cell in row 2 and column 3 of Table 2.10. In constructing a cross-tabulation, we simply count the number of restaurants that belong to each of the cells in the cross-tabulation.

We see that the greatest number of restaurants in the sample (64) have a good rating and a meal price in the €20–29 range. Only two restaurants have an excellent rating and a meal price in the €10–19 range. In addition, note that the right and bottom margins of the cross-tabulation provide the frequency distributions for quality rating and meal price separately. From the frequency distribution in the right margin, we see the quality rating data showing 84 disappointing restaurants, 150 good restaurants and 66 excellent restaurants.

Dividing the totals in the right margin by the total for that column provides relative and percentage frequency distributions for the quality rating variable.

| Quality rating | Relative frequency | Percentage frequency |
|----------------|--------------------|----------------------|
| Disappointing  | 0.28               | 28                   |
| Good           | 0.50               | 50                   |
| Excellent      | 0.22               | 22                   |
| Total          | 1.00               | 100                  |

We see that 28 per cent of the restaurants were rated disappointing, 50 per cent were rated good and 22 per cent were rated excellent.

Dividing the totals in the bottom row of the cross-tabulation by the total for that row provides relative and percentage frequency distributions for the meal price variable. In this case the values do not add exactly to 100, because the values being summed are rounded. From the percentage frequency distribution we quickly see that 26 per cent of the meal prices are in the lowest price class (€10–19), 39 per cent are in the next higher class and so on.

| Meal price | Relative frequency | Percentage frequency |
|------------|--------------------|----------------------|
| €10–19     | 0.26               | 26                   |
| €20–29     | 0.39               | 39                   |
| €30–39     | 0.25               | 25                   |
| €40–49     | 0.09               | 9                    |
| Total      | 1.00               | 100                  |

The frequency and relative frequency distributions constructed from the margins of a cross-tabulation provide information about each of the variables individually, but they do not shed any light on the relationship between the variables. The primary value of a cross-tabulation lies in the insight it offers about this relationship. Converting the entries in a cross-tabulation into row percentages or column percentages can provide the insight.

For row percentages, the results of dividing each frequency in Table 2.10 by its corresponding row total are shown in Table 2.11. Each row of Table 2.11 is a percentage frequency distribution of meal price for one of the quality rating categories. Of the restaurants with the lowest quality rating (disappointing), we see that the greatest percentages are for the less expensive restaurants (50.0 per cent have €10–19 meal prices and 47.6 per cent have €20–29 meal prices). Of the restaurants with the highest quality rating (excellent), we see that the greatest percentages are for the more expensive restaurants (42.4 per cent have €30–39 meal prices and 33.4 per cent have €40–49 meal prices). Hence, the cross-tabulation reveals that higher meal prices are associated with the higher quality restaurants, and the lower meal prices are associated with the lower quality restaurants.

Cross-tabulation is widely used for examining the relationship between two variables. The final reports for many statistical studies include a large number of cross-tabulations. In the restaurant survey, the cross-tabulation is based on one qualitative variable (quality rating) and one quantitative variable (meal price). Cross-tabulations can also be constructed when both variables are qualitative and when both variables are quantitative. When quantitative variables are used, we must first create classes for the values of the variable. For instance, in the restaurant example we grouped the meal prices into four classes (€10–19, €20–29, €30–39 and €40–49).

## Simpson's paradox

In many cases, a summary cross-tabulation showing how two variables are related has in effect been aggregated across a third variable (or across more than one variable). If so, we must be careful in drawing conclusions about the relationship between the two variables in the aggregated cross-tabulation. In some cases the conclusions based upon the aggregated cross-tabulation can be completely reversed if we look at the non-aggregated data, something known as **Simpson's paradox**. To provide an illustration, we consider an example involving the analysis of sales success for two sales executives in a mobile telephone company.

**TABLE 2.11** Row percentages for each quality rating category

| Quality rating | Meal Price |        |        |        | Total |
|----------------|------------|--------|--------|--------|-------|
|                | €10–19     | €20–29 | €30–39 | €40–49 |       |
| Disappointing  | 50.0       | 47.6   | 2.4    | 0.0    | 100   |
| Good           | 22.7       | 42.7   | 30.6   | 4.0    | 100   |
| Excellent      | 3.0        | 21.2   | 42.4   | 33.4   | 100   |

The two sales executives are Aaron and Theo. They handle enquiries for renewal of two types of mobile telephone agreement: pre-pay contracts and pay-as-you-go (PAYG) agreements. The cross-tabulation below shows the outcomes for 200 enquiries each for Aaron and Theo, aggregated across the two types of agreement. The cross-tabulation involves two variables: outcome (sale or no sale) and sales executive (Aaron or Theo). It shows the number of sales and the number of no-sales for each executive, along with the column percentages in parentheses next to each value.

|                 |  | <i>Sales executive</i> |             |              |
|-----------------|--|------------------------|-------------|--------------|
|                 |  | <i>Aaron</i>           | <i>Theo</i> | <i>Total</i> |
| <i>Sales</i>    |  | 82 (41%)               | 102 (51%)   | 184          |
| <i>No-sales</i> |  | 118 (59%)              | 98 (49%)    | 216          |
| <i>Total</i>    |  | 200 (100%)             | 200 (100%)  | 400          |

The column percentages indicate that Aaron's overall sales success rate was 41 per cent, compared with Theo's 51 per cent success rate, suggesting that Theo has the better sales performance. A problem arises with this conclusion, however. The following cross-tabulations show the enquiries handled by Aaron and Theo for the two types of agreement separately.

| <i>Pre-pay</i>  |              |             | <i>PAYG</i>  |             |              |
|-----------------|--------------|-------------|--------------|-------------|--------------|
|                 | <i>Aaron</i> | <i>Theo</i> | <i>Aaron</i> | <i>Theo</i> | <i>Total</i> |
| <i>Sales</i>    | 56 (35%)     | 18 (30%)    | 74           | 26 (65%)    | 84 (60%)     |
| <i>No-sales</i> | 104 (65%)    | 42 (70%)    | 146          | 14 (35%)    | 56 (40%)     |
| <i>Total</i>    | 160 (100%)   | 60 (100%)   | 220          | 40 (100%)   | 140 (100%)   |
|                 |              |             |              |             | 180          |

We see that Aaron achieved a 35 per cent success rate for pre-pay contracts and 65 per cent for PAYG agreements. Theo had a 30 per cent success rate for pre-pay and 60 per cent for PAYG. This comparison suggests that Aaron has a better success rate than Theo for both types of agreement, a result that contradicts the conclusion reached when the data were aggregated across the two types of agreement. This example illustrates Simpson's paradox.

Note that for both sales executives the sales success rate was much higher for PAYG than for pre-pay contracts. Because Theo handled a much higher proportion of PAYG enquiries than Aaron, the aggregated data favoured Theo. When we look at the cross-tabulations for the two types of agreement separately, however, Aaron shows the better record. Hence, for the original cross-tabulation, we see that the *type of agreement* is a hidden variable that should not be ignored when evaluating the records of the sales executives.

Because of Simpson's paradox, we need to be especially careful when drawing conclusions using aggregated data. Before drawing any conclusions about the relationship between two variables shown for a cross-tabulation – or, indeed, any type of display involving two variables (like the scatter diagram illustrated in the next section) – you should consider whether any hidden variable or variables could affect the results.

## Scatter diagram and trend line

A **scatter diagram** is a graphical presentation of the relationship between two quantitative variables, and a **trend line** is a line that provides an approximation of the relationship. Consider the advertising/sales relationship for a hi-fi equipment store. On ten occasions during the past three months, the store used weekend television commercials to promote sales at its stores. The managers want to investigate whether a relationship exists between the number of commercials shown and sales at the store during the following week. Sample data for the ten weeks with sales in thousands of euros (€000s) are shown in Table 2.12.



STEREO

**TABLE 2.12** Sample data for the hi-fi equipment store

| Week | Number of commercials | Sales in €000s |
|------|-----------------------|----------------|
| 1    | 2                     | 50             |
| 2    | 5                     | 57             |
| 3    | 1                     | 41             |
| 4    | 3                     | 54             |
| 5    | 4                     | 54             |
| 6    | 1                     | 38             |
| 7    | 5                     | 63             |
| 8    | 3                     | 48             |
| 9    | 4                     | 59             |
| 10   | 2                     | 46             |

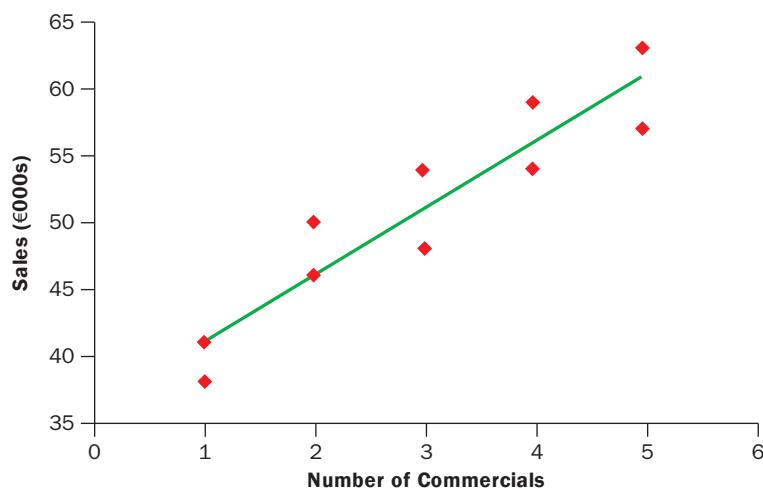
Figure 2.8 shows the scatter diagram and the trend line\* for the data in Table 2.12. The number of commercials ( $x$ ) is shown on the horizontal axis and the sales ( $y$ ) are shown on the vertical axis. For week 1,  $x = 2$  and  $y = 50$ . A point with those coordinates is plotted on the scatter diagram. Similar points are plotted for the other nine weeks. Note that during two of the weeks one commercial was shown, during two of the weeks two commercials were shown and so on.

The completed scatter diagram in Figure 2.8 indicates a positive relationship between the number of commercials and sales. Higher sales are associated with a higher number of commercials. The relationship is not perfect in that all points are not on a straight line. However, the general pattern of the points and the trend line suggest that the overall relationship is positive.

Some general scatter diagram patterns and the types of relationships they suggest are shown in Figure 2.9. The top left panel depicts a positive relationship similar to the one for the number of commercials and sales example. In the top right panel, the scatter diagram shows no apparent relationship between the variables. The bottom panel depicts a negative relationship where  $y$  tends to decrease as  $x$  increases.

**FIGURE 2.8**

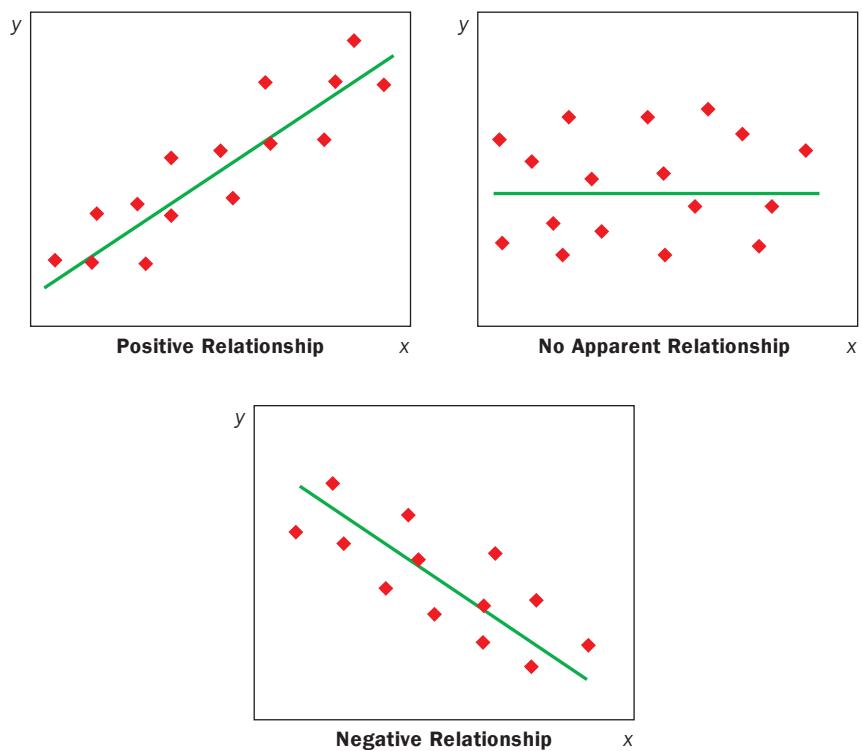
Scatter diagram and trend line for the hi-fi equipment store



\*The equation of the trend line is  $y = 4.95x + 36.15$ . The slope of the trend line is 4.95 and the  $y$ -intercept (the point where the line intersects the  $y$  axis) is 36.15. We will discuss in detail the interpretation of the slope and  $y$ -intercept for a linear trend line in Chapter 14 when we study simple linear regression.

**FIGURE 2.9**

Types of relationships depicted by scatter diagrams



## EXERCISES

### Methods

- 20.** The following data are for 30 observations involving two qualitative variables,  $X$  and  $Y$ . The categories for  $X$  are A, B and C; the categories for  $Y$  are 1 and 2.

| Observation | $X$ | $Y$ | Observation | $X$ | $Y$ |
|-------------|-----|-----|-------------|-----|-----|
| 1           | A   | 1   | 16          | B   | 2   |
| 2           | B   | 1   | 17          | C   | 1   |
| 3           | B   | 1   | 18          | B   | 1   |
| 4           | C   | 2   | 19          | C   | 1   |
| 5           | B   | 1   | 20          | B   | 1   |
| 6           | C   | 2   | 21          | C   | 2   |
| 7           | B   | 1   | 22          | B   | 1   |
| 8           | C   | 2   | 23          | C   | 2   |
| 9           | A   | 1   | 24          | A   | 1   |
| 10          | B   | 1   | 25          | B   | 1   |
| 11          | A   | 1   | 26          | C   | 2   |
| 12          | B   | 1   | 27          | C   | 2   |
| 13          | C   | 2   | 28          | A   | 1   |
| 14          | C   | 2   | 29          | B   | 1   |
| 15          | C   | 2   | 30          | B   | 2   |



CROSSTAB



**COMPLETE  
SOLUTIONS**



**SCATTER**

- Construct a cross-tabulation for the data, with  $X$  as the row variable and  $Y$  as the column variable.
- Calculate the row percentages.
- Calculate the column percentages.
- What is the relationship, if any, between  $X$  and  $Y$ ?

**21.** The following 20 observations are for two quantitative variables.

| Observation | $X$ | $Y$ | Observation | $X$ | $Y$ |
|-------------|-----|-----|-------------|-----|-----|
| 1           | -22 | 22  | 11          | -37 | 48  |
| 2           | -33 | 49  | 12          | 34  | -29 |
| 3           | 2   | 8   | 13          | 9   | -18 |
| 4           | 29  | -16 | 14          | -33 | 31  |
| 5           | -13 | 10  | 15          | 20  | -16 |
| 6           | 21  | -28 | 16          | -3  | 14  |
| 7           | -13 | 27  | 17          | -15 | 18  |
| 8           | -23 | 35  | 18          | 12  | 17  |
| 9           | 14  | -5  | 19          | -20 | -11 |
| 10          | 3   | -3  | 20          | -7  | -22 |

- Construct a scatter diagram for the relationship between  $X$  and  $Y$ .
- What is the relationship, if any, between  $X$  and  $Y$ ?

### Applications

**22.** Recently, management at Oak Tree Golf Course received a few complaints about the condition of the greens. Several players complained that the greens are too fast. Rather than react to the comments of just a few, the Golf Association conducted a survey of 100 male and 100 female golfers. The survey results are summarized here.

| Handicap   | Male golfers     |      | Female golfers   |          |      |
|------------|------------------|------|------------------|----------|------|
|            | Greens condition |      | Greens condition |          |      |
|            | Too fast         | Fine | Handicap         | Too fast | Fine |
| Under 15   | 10               | 40   | Under 15         | 1        | 9    |
| 15 or more | 25               | 25   | 15 or more       | 39       | 51   |

- Combine these two cross-tabulations into one with 'male', 'female' as the row labels and the column labels 'too fast' and 'fine'. Which group shows the highest percentage saying that the greens are too fast?
  - Refer to the initial cross-tabulations. For those players with low handicaps (better players), which group (male or female) shows the highest percentage saying the greens are too fast?
  - Refer to the initial cross-tabulations. For those players with higher handicaps, which group (male or female) shows the highest percentage saying the greens are too fast?
  - What conclusions can you draw about the preferences of men and women concerning the speed of the greens? Are the conclusions you draw from part (a) as compared with parts (b) and (c) consistent? Explain any apparent inconsistencies.
- 23.** The file 'House Sales' on the online platform contains data for a sample of 50 houses advertised for sale in a regional UK newspaper. The first five rows of data are shown for illustration below.



**COMPLETE  
SOLUTIONS**

| Price   | Location | House type    | Bedrooms | Reception rooms | Bedrooms + receptions | Garage capacity |
|---------|----------|---------------|----------|-----------------|-----------------------|-----------------|
| 234 995 | Town     | Detached      | 4        | 2               | 6                     | 1               |
| 319 000 | Town     | Detached      | 4        | 2               | 6                     | 1               |
| 154 995 | Town     | Semi-detached | 2        | 1               | 3                     | 0               |
| 349 950 | Village  | Detached      | 4        | 2               | 6                     | 2               |
| 244 995 | Town     | Detached      | 3        | 2               | 5                     | 1               |



HOUSE SALES

- a. Prepare a cross-tabulation using sale price (rows) and house type (columns). Use classes of 100 000–199 999, 200 000–299 999, etc. for sale price.
- b. Compute row percentages and comment on any relationship between the variables.
24. Refer to the data in Exercise 23.
- a. Prepare a cross-tabulation using number of bedrooms and house type.
- b. Prepare a frequency distribution for number of bedrooms.
- c. Prepare a frequency distribution for house type.
- d. How has the cross-tabulation helped in preparing the frequency distributions in parts (b) and (c)?
25. The file ‘OECD 2012’ on the online platform contains data for 33 countries taken from the website of the Organization for Economic Cooperation & Development in mid-2012. The two variables are the Gini coefficient for each country and the percentage of children in the country estimated to be living in poverty. The Gini coefficient is a widely used measure of income inequality. It varies between 0 and 1, with higher coefficients indicating more inequality. The first five rows of data are shown for illustration below.

| Country        | Child poverty (%) | Income inequality |
|----------------|-------------------|-------------------|
| Australia      | 14.0              | 0.336             |
| Austria        | 7.9               | 0.261             |
| Belgium        | 11.3              | 0.259             |
| Canada         | 15.1              | 0.324             |
| Czech Republic | 8.4               | 0.256             |



OECD 2012

## ONLINE RESOURCES

For the data files, online summary, additional questions and answers, and software section for Chapter 2, go to the accompanying online platform.



## SUMMARY

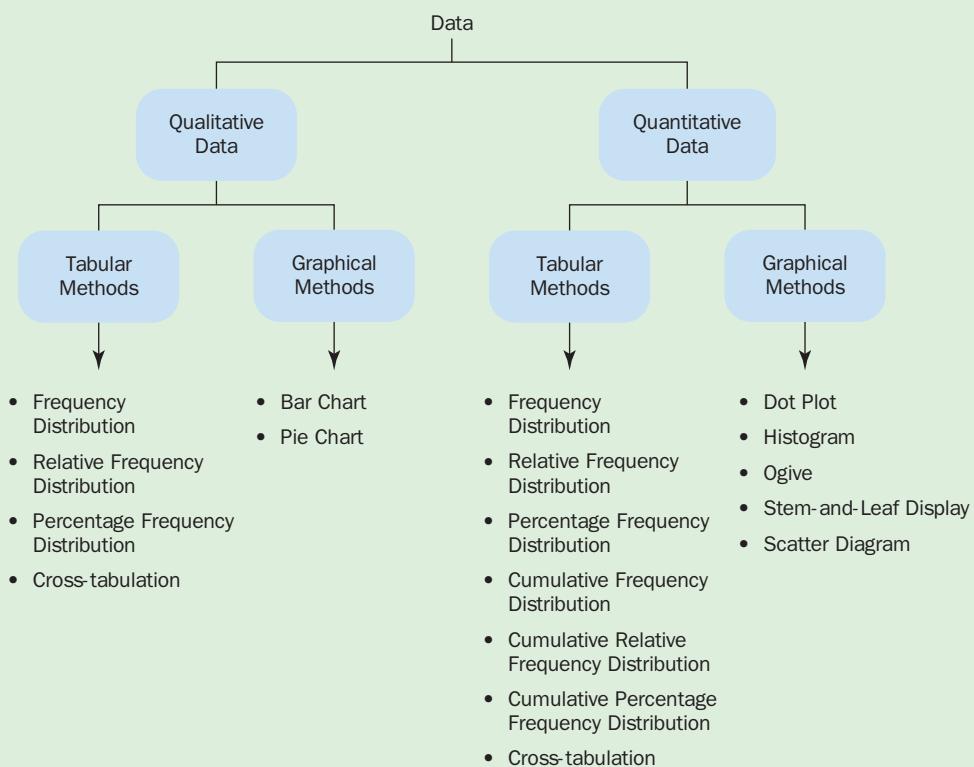
A set of data, even if modest in size, is often difficult to interpret directly in the form in which it is gathered. Tabular and graphical methods provide procedures for organizing and summarizing data so that patterns are revealed and the data are more easily interpreted.

Figure 2.10 shows the tabular and graphical methods presented in this chapter.

Frequency distributions, relative frequency distributions, percentage frequency distributions, bar charts and pie charts were presented as tabular and graphical procedures for summarizing qualitative data. Frequency distributions, relative frequency distributions, percentage frequency distributions, dot plots, histograms, cumulative frequency distributions, cumulative relative frequency distributions, cumulative percentage frequency distributions and cumulative distribution plots (ogives) were presented as ways of summarizing quantitative data. A stem-and-leaf display provides an exploratory data analysis technique that can be used to summarize quantitative data.

Cross-tabulation was presented as a tabular method for summarizing data for two variables. An example of Simpson's paradox was set out, to illustrate the care that must be taken when interpreting relationships between two variables using aggregated data. The scatter diagram was introduced as a graphical method for showing the relationship between two quantitative variables.

With large data sets, computer software packages are essential in constructing tabular and graphical summaries of data. The software guides on the online platform show how EXCEL, IBM SPSS and MINITAB can be used for this purpose.



**FIGURE 2.10**

Tabular and graphical methods for summarizing data

## KEY TERMS

**Bar chart**

**Bar graph**

**Class midpoint**

**Cross-tabulation**

**Cumulative frequency distribution**

**Cumulative percentage frequency distribution**

**Cumulative relative frequency distribution**

**Dot plot**

**Exploratory data analysis**

**Frequency distribution**

**Histogram**

**Ogive**

Percentage frequency distribution  
 Pie chart  
 Qualitative data  
 Quantitative data  
 Relative frequency distribution

Scatter diagram  
 Simpson's paradox  
 Stem-and-leaf display  
 Trend line

## KEY FORMULAE

### Relative frequency

$$\text{Relative frequency of a class} = \frac{\text{Frequency of the class}}{n} \quad (2.1)$$

### Approximate class width

$$\frac{\text{Largest data value} - \text{Smallest data value}}{\text{Number of classes}} \quad (2.2)$$

## CASE PROBLEM



### In The Mode Fashion Stores

In The Mode is a chain of women's fashion stores. The chain recently ran a promotion in which discount coupons were sent to customers. Data collected for a sample of 100 in-store credit card transactions during a single day following the promotion are contained in the file 'Mode' on the online platform. A portion of the data set is shown below. A non-zero amount for the discount variable indicates that the customer brought in the promotional coupons and used them. For a very few customers, the discount amount is actually greater than the sales amount (see, for example, customer 4). In The Mode's management would like to use this sample data to learn about its customer base and to evaluate the promotion involving discount coupons.

### Managerial report

Use tables and charts to help management develop a customer profile and to evaluate the promotional campaign. At a minimum, your report should include the following.

1. Percentage frequency distributions for key variables.
2. A bar chart or pie chart showing the percentage of customer purchases possibly attributable to the promotional campaign.
3. A cross-tabulation of type of customer (regular or promotional) versus sales. Comment on any similarities or differences present.
4. A scatter diagram of sales versus discount for only those customers responding to the promotion. Comment on any relationship apparent between sales and discount.
5. A scatter diagram to explore the relationship between sales and customer age.



| Customer | Method of payment | Items | Discount | Sales  | Gender | Marital status | Age |
|----------|-------------------|-------|----------|--------|--------|----------------|-----|
| 1        | Visa Debit        | 1     | 0.00     | 39.50  | Male   | Married        | 32  |
| 2        | Store Card        | 1     | 25.60    | 102.40 | Female | Married        | 36  |
| 3        | Store Card        | 1     | 0.00     | 22.50  | Female | Married        | 32  |
| 4        | Store Card        | 5     | 121.10   | 100.40 | Female | Married        | 28  |
| 5        | Mastercard        | 2     | 0.00     | 54.00  | Female | Married        | 34  |
| 6        | Mastercard        | 1     | 0.00     | 44.50  | Female | Married        | 44  |
| 7        | Store Card        | 2     | 19.50    | 78.00  | Female | Married        | 30  |
| 8        | Visa              | 1     | 0.00     | 22.50  | Female | Married        | 40  |
| 9        | Store Card        | 2     | 22.48    | 56.52  | Female | Married        | 46  |
| 10       | Store Card        | 1     | 0.00     | 44.50  | Female | Married        | 36  |



# 3 Descriptive Statistics: Numerical Measures



## CHAPTER CONTENTS

Statistics in Practice TV audience measurement

- 3.1 Measures of location
- 3.2 Measures of variability
- 3.3 Measures of distributional shape, relative location and detecting outliers
- 3.4 Exploratory data analysis
- 3.5 Measures of association between two variables
- 3.6 The weighted mean and working with grouped data

**LEARNING OBJECTIVES** After studying this chapter and doing the exercises, you should be able to calculate and interpret the following statistical measures that help to describe the central location, variability and shape of data sets.

- 1 The mean, median and mode.
- 2 Percentiles (including quartiles), the range, the interquartile range, the variance, the standard deviation and the coefficient of variation.
- 3 You should understand the concept of skewness of distribution. You should be able to calculate  $z$ -scores and understand their role in identifying data outliers.
- 4 You should understand the role of Chebyshev's theorem and of the empirical rule in estimating the spread of data sets.
- 5 Five-number summaries and box plots.
- 6 Scatter diagrams, covariance and Pearson's correlation coefficient.
- 7 Weighted means.
- 8 Estimates of mean and standard deviation for grouped data.

In Chapter 2 we discussed tabular and graphical data summaries. In this chapter, we present several numerical measures for summarizing data.

We start with numerical summary measures for a single variable. When a data set contains more than one variable, the same numerical measures can be computed separately for each variable. However, in the two-variable case we shall also examine measures of the relationship between the variables.



## STATISTICS IN PRACTICE

TV audience measurement

Television audience levels and audience share are important issues for advertisers, sponsors and, in the case of public service broadcasting, governments. In recent years in many countries, the number of TV channels available has increased substantially because of the use of digital, satellite and cable services. The Broadcasters' Audience Research Board (BARB) in the UK, for example, lists over 250 channels in its 'multi-channel viewing summary'. Technology also now allows viewers to 'time-shift' their viewing. Accurate audience measurement thereby becomes a more difficult task.

The *Handbook on Radio and Television Audience Research*<sup>\*</sup> has a section on data analysis, in which



the author makes the point 'most audience research is quantitative'. He then goes on to describe the various measures that are commonly used in this field, including: 'ratings', 'gross rating points', 'viewing share', 'viewing hours' and 'reach'. Many of the measures involve the use of averages: for example, 'average weekly viewing per person'.

BARB publishes viewing figures on its website, [www.barb.co.uk](http://www.barb.co.uk). Figures for the week ending 22 July 2012, for example, a week before the start of the 2012 London Olympics, showed that 'average daily reach' for the lead public broadcasting channel BBC1 was just over 26 million viewers. This represented about 45 per cent of the potential viewing audience. Average weekly viewing for BBC1 was estimated at slightly under five hours per person. Two weeks later, with the 2012 Olympics in full swing and TV coverage being provided by the BBC, average daily reach for BBC1 had risen to 32 million viewers, and average viewing time had more than doubled to over ten hours per person.

In this chapter, you will learn how to compute and interpret some of the statistical measures used in reports such as those presented by BARB. You will learn about the mean, median and mode, and about other descriptive statistics such as the range, variance, standard deviation, percentiles and correlation. These numerical measures will assist in the understanding and interpretation of data.

\* *Handbook on Radio and Television Audience Research*, by Graham Mytton, published by UNICEF/UNESCO/BBC World Service Training Trust, web edition (2007).

We introduce numerical measures of location, dispersion, shape and association. If they are computed for sample data, they are called **sample statistics**. If they are computed for data for a whole population, they are called **population parameters**. In statistical inference, a sample statistic is referred to as the **point estimator** of the corresponding population parameter. In Chapter 7 we shall discuss in more detail the process of point estimation. In the guides on the associated online platform, we show how EXCEL, IBM SPSS and MINITAB can be used to compute many of the numerical measures described in the chapter.

### 3.1 MEASURES OF LOCATION

#### Mean

The most commonly used measure of location is the **mean**. The mean provides a measure of central location for the data. If the data are from a sample, the mean is denoted by putting a bar over the data symbol, e.g.  $\bar{x}$ . If the data are from a population, the Greek letter  $\mu$  (mu) is used to denote the mean. When people refer to the 'average' value, they are usually referring to the mean value.

In statistical formulae, it is customary to denote the value of variable  $X$  for the first sample observation by  $x_1$ , for the second sample observation by  $x_2$  and so on. In general, the value of variable  $X$  for the  $i$ th observation is denoted by  $x_i$ . (As we shall see in Chapters 5 and 6, a common convention in statistics is to *name* variables using capital letters, e.g.  $X$ , but to refer to specific values of those variables using small letters, e.g.  $x$ .) For a sample with  $n$  observations, the formula for the sample mean is as follows:

### Sample mean

$$\bar{x} = \frac{\sum x_i}{n} \quad (3.1)$$

In equation (3.1), the numerator is the sum of the values of the  $n$  observations. That is:

$$\sum x_i = x_1 + x_2 + \dots + x_n$$

The Greek letter  $\Sigma$  (upper case sigma) is the summation sign.

To illustrate the computation of a sample mean, consider the following class size data for a sample of five university classes.

46 54 42 46 32

We use the notation  $x_1, x_2, x_3, x_4, x_5$  to represent the number of students in each of the five classes.

$$x_1 = 46 \quad x_2 = 54 \quad x_3 = 42 \quad x_4 = 46 \quad x_5 = 32$$

To compute the sample mean, we can write:

$$\bar{x} = \frac{\sum x_i}{n} = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{n} = \frac{46 + 54 + 42 + 46 + 32}{5} = 44$$

The sample mean class size is 44 students.

Here is a second illustration. Suppose a university careers office has sent a questionnaire to a sample of business school graduates requesting information on monthly starting salaries. Table 3.1 shows the data collected. The mean monthly starting salary for the sample of 12 business school graduates is computed as:

$$\bar{x} = \frac{\sum x_i}{n} = \frac{x_1 + x_2 + \dots + x_{12}}{12} = \frac{2020 + 2075 + \dots + 2040}{12} = \frac{24840}{12} = 2070$$

Equation (3.1) shows how the mean is computed for a sample with  $n$  observations. The formula for computing the mean of a population remains the same, but we use different notation to indicate that we are working with the entire population. We denote the number of observations in a population by  $N$ , and the population mean as  $\mu$ .



SALARY

### Population mean

$$\mu = \frac{\sum x_i}{N} \quad (3.2)$$

## Median

The **median** is another measure of central location for a variable. The median is the value in the middle when the data are arranged in ascending order (smallest value to largest value).

### Median

Arrange the data in ascending order.

- For an odd number of observations, the median is the middle value.
- For even number of observations, the median is the average of the two middle values.

Let us compute the median class size for the sample of five university classes. We first arrange the data in ascending order.

32 42 46 46 54

Because  $n = 5$  is odd, the median is the middle value. This data set contains two observations with values of 46 (the 3rd and 4th ordered observations). Each observation is treated separately when we arrange the data in ascending order. The median class size is 46 students (the 3rd ordered observation).

Suppose we also compute the median starting salary for the 12 business school graduates in Table 3.1. We first arrange the data in ascending order.

1955 1980 2020 2040 2040 2050 2060 2070 2075 2125 2165 2260  


Because  $n = 12$  is even, we identify the middle two values: 2050 and 2060. The median is the average of these values:

$$\text{Median} = \frac{2050 + 2060}{2} = 2055$$

Although the mean is the more commonly used measure of central location, in some situations the median is preferred. The mean is influenced by extremely small and large data values. For example, suppose one of the graduates (see Table 3.1) had a starting salary of €5000 per month (perhaps his/her family owns the company). If we change the highest monthly starting salary in Table 3.1 from €2260 to €5000, the sample mean changes from €2070 to €2298. The median of €2055, however, is unchanged, because €2050 and €2060 are still the middle two values. With the extremely high starting salary included, the median provides a more robust measure of central location than the mean. When a data set contains extreme values, the median is often the preferred measure of central location.\*

**TABLE 3.1** Monthly starting salaries for a sample of 12 business school graduates

| Graduate | Monthly starting salary (€) | Graduate | Monthly starting salary (€) |
|----------|-----------------------------|----------|-----------------------------|
| 1        | 2020                        | 7        | 2050                        |
| 2        | 2075                        | 8        | 2165                        |
| 3        | 2125                        | 9        | 2070                        |
| 4        | 2040                        | 10       | 2260                        |
| 5        | 1980                        | 11       | 2060                        |
| 6        | 1955                        | 12       | 2040                        |

\*Another measure sometimes used when extreme values are present is the *trimmed mean*. A percentage of the smallest and largest values are removed from a data set, and the mean of the remaining values is computed. For example, to get the 5 per cent trimmed mean, the smallest 5 per cent and the largest 5 per cent of the data values are removed, and the mean of the remaining values is computed. Using the sample with  $n = 12$  starting salaries,  $0.05(12) = 0.6$ . Rounding this value to 1 indicates that the 5 per cent trimmed mean would remove the smallest data value and the largest data value. The 5 per cent trimmed mean using the 10 remaining observations is 2062.5.

## Mode

A third measure of location is the **mode** (although the mode does not necessarily measure *central* location). The mode is defined as follows.

### Mode

The mode is the value that occurs with the greatest frequency.

To illustrate the identification of the mode, consider the sample of five class sizes.

The only value that occurs more than once is 46. This value occurs twice and consequently is the mode. In the sample of starting salaries for the business school graduates, the only monthly starting salary that occurs more than once is €2040, and therefore this value is the mode for that data set.

Situations can arise for which the greatest frequency occurs at two or more different values. In these instances more than one mode exists. If the data contain exactly two modes, we say that the data are *bimodal*. If data contain more than two modes, we say that the data are *multimodal*. In multimodal cases the modes are almost never reported, because listing three or more modes would not be particularly helpful in describing a central location for the data.

The mode is an important measure of location for qualitative data. For example, the qualitative data set in Table 2.2 resulted in the following frequency distribution for new car purchases.

| <i>Car brand</i> | <i>Frequency</i> |
|------------------|------------------|
| Audi             | 8                |
| BMW              | 5                |
| Mercedes         | 13               |
| Opel             | 8                |
| VW               | 19               |
| Total            | 50               |

The mode, or most frequently purchased car brand, is VW. For this type of data it obviously makes no sense to speak of the mean or median. The mode provides the information of interest, the most frequently purchased car brand.

## Percentiles

A **percentile** provides information about how the data are spread over the interval from the smallest value to the largest value. For data that do not contain numerous repeated values, the  $p$ th percentile divides the data into two parts: approximately  $p$  per cent of the observations have values less than the  $p$ th percentile; approximately  $(100 - p)$  per cent of the observations have values greater than the  $p$ th percentile. The  $p$ th percentile is formally defined as follows.

### Percentile

The  $p$ th percentile is a value such that *at least*  $p$  per cent of the observations are less than or equal to this value and *at least*  $(100 - p)$  per cent of the observations are greater than or equal to this value.

Colleges and universities sometimes report admission test scores in terms of percentiles. For instance, suppose an applicant obtains a raw score of 54 on the verbal portion of an admission test. It may not be readily apparent how this student performed in relation to other students taking the same test. However,

if the raw score of 54 corresponds to the 70th percentile, we know that approximately 70 per cent of the students scored lower than this individual and approximately 30 per cent of the students scored higher than this individual.

The following procedure can be used to compute the  $p$ th percentile.

### Calculating the $p$ th percentile

1. Arrange the data in ascending order (smallest value to largest value).

2. Compute an index  $i$

$$i = \left( \frac{p}{100} \right) n$$

where  $p$  is the percentile of interest and  $n$  is the number of observations.

3. a. If  $i$  is not an integer, round up. The next integer greater than  $i$  denotes the position of the  $p$ th percentile.
- b. If  $i$  is an integer, the  $p$ th percentile is the average of the values in positions  $i$  and  $i + 1$ .

As an illustration, consider the 85th percentile for the starting salary data in Table 3.1.

1. Arrange the data in ascending order.

1955 1980 2020 2040 2040 2050 2060 2070 2075 2125 2165 2260

2

$$i = \left( \frac{p}{100} \right) n = \left( \frac{85}{100} \right) 12 = 10.2$$

3. Because  $i$  is not an integer, round up. The position of the 85th percentile is the next integer greater than 10.2: the 11th position.

Returning to the data, we see that the 85th percentile is the data value in the 11th position, or 2165.

As another illustration of this procedure, consider the calculation of the 50th percentile for the starting salary data. Applying step 2, we obtain:

$$i = \left( \frac{p}{100} \right) n = \left( \frac{50}{100} \right) 12 = 6$$

Because  $i$  is an integer, step 3(b) states that the 50th percentile is the average of the sixth and seventh data values; that is  $(2050 + 2060)/2 = 2055$ . Note that the 50th percentile is also the median.

## Quartiles

For the purposes of describing data distribution, it is often useful to consider the values that divide the data set into four parts, with each part containing approximately one-quarter (25 per cent) of the observations. Figure 3.1 shows a data distribution divided into four parts. The division points are referred to as the **quartiles** and are defined as:

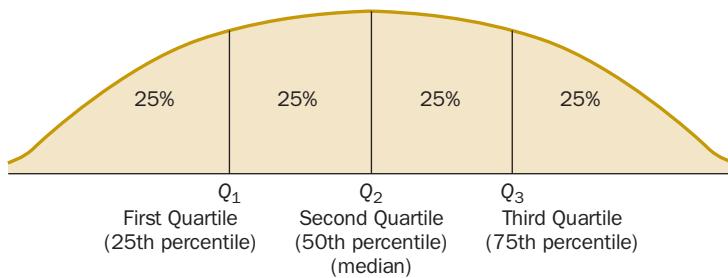
$Q_1$  = first quartile, or 25th percentile

$Q_2$  = second quartile, or 50th percentile (also the median)

$Q_3$  = third quartile, or 75th percentile

**FIGURE 3.1**

Location of the quartiles



The starting salary data are again arranged in ascending order. We have already identified  $Q_2$ , the second quartile (median), as 2055.

1955 1980 2020 2040 2040 2050 2060 2070 2075 2125 2165 2260

The computations of quartiles  $Q_1$  and  $Q_3$  use the rule for finding the 25th and 75th percentiles.

For  $Q_1$ :

$$i = \left( \frac{P}{100} \right) n = \left( \frac{25}{100} \right) 12 = 3$$

Because  $i$  is an integer, step 3(b) indicates that the first quartile, or 25th percentile, is the average of the third and fourth data values; hence:

$$Q_1 = (2020 + 2040)/2 = 2030$$

For  $Q_3$ :

$$i = \left( \frac{P}{100} \right) n = \left( \frac{75}{100} \right) 12 = 9$$

Again, because  $i$  is an integer, step 3(b) indicates that the third quartile, or 75th percentile, is the average of the ninth and tenth data values; hence:

$$Q_3 = (2075 + 2125)/2 = 2100$$

1955 1980 2020 | 2040 2040 2050 | 2060 2070 2075 | 2125 2165 2260  
 $Q_1 = 2030$                              $Q_2 = 2055$                              $Q_3 = 2100$   
(Median)

We defined the quartiles as the 25th, 50th and 75th percentiles. Hence we computed the quartiles in the same way as percentiles. However, other conventions are sometimes used to compute quartiles. The actual values reported for quartiles may vary slightly depending on the convention used. Nevertheless, the objective in all cases is to divide the data into four approximately equal parts.

## EXERCISES

### Methods

1. Consider a sample with data values of 10, 20, 12, 17 and 16. Compute the mean and median.
2. Consider a sample with data values of 10, 20, 21, 17, 16 and 12. Compute the mean and median.
3. Consider a sample with data values of 27, 25, 20, 15, 30, 34, 28 and 25. Compute the 20th, 25th, 65th and 75th percentiles.



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4. Consider a sample with data values of 53, 55, 70, 58, 64, 57, 53, 69, 57, 68 and 53. Compute the mean, median and mode.

### Applications

5. A sample of 30 Irish engineering graduates had the following starting salaries. Data are in thousands of euros.

|      |      |      |      |      |      |      |      |      |      |
|------|------|------|------|------|------|------|------|------|------|
| 36.8 | 34.9 | 35.2 | 37.2 | 36.2 | 35.8 | 36.8 | 36.1 | 36.7 | 36.6 |
| 37.3 | 38.2 | 36.3 | 36.4 | 39.0 | 38.3 | 36.0 | 35.0 | 36.7 | 37.9 |
| 38.3 | 36.4 | 36.5 | 38.4 | 39.4 | 38.8 | 35.4 | 36.4 | 37.0 | 36.4 |

- What is the mean starting salary?
- What is the median starting salary?
- What is the mode?
- What is the first quartile?
- What is the third quartile?

6. The following data were obtained for the number of minutes spent listening to recorded music for a sample of 30 individuals on one particular day.

|      |      |      |      |      |      |       |      |      |      |
|------|------|------|------|------|------|-------|------|------|------|
| 88.3 | 4.3  | 4.6  | 7.0  | 9.2  | 0.0  | 99.2  | 34.9 | 81.7 | 0.0  |
| 85.4 | 0.0  | 17.5 | 45.0 | 53.3 | 29.1 | 28.8  | 0.0  | 98.9 | 64.5 |
| 4.4  | 67.9 | 94.2 | 7.6  | 56.6 | 52.9 | 145.6 | 70.4 | 65.1 | 63.6 |

- Compute the mean.
- Compute the median.
- Compute the first and third quartiles.
- Compute and interpret the 40th percentile.

7. miniRank ([www.minirank.com](http://www.minirank.com)) rates the popularity of websites in most countries of the world, using a points system. The 25 most popular sites in South Africa as listed in July 2012 were as follows (the points scores have been rounded to one decimal place):

| Website  | Points | Website  | Points |
|--|--------|--|--------|
| <a href="http://www.intoweb.co.za">http://www.intoweb.co.za</a>          | 253.1  | <a href="http://www.dweb.co.za">www.dweb.co.za</a>                 | 118.2  |
| <a href="http://www.weathersa.co.za">http://www.weathersa.co.za</a>      | 252.3  | <a href="http://dweb.co.za">dweb.co.za</a>                         | 108.5  |
| <a href="http://www.traffic.co.za">www.traffic.co.za</a>                 | 212.4  | <a href="http://www.webworx.org.za">www.webworx.org.za</a>         | 107.6  |
| <a href="http://www.gov.za">www.gov.za</a>                               | 167.0  | <a href="http://www.bacchus.co.za">www.bacchus.co.za</a>           | 105.2  |
| <a href="http://www.intowebtraining.co.za">www.intowebtraining.co.za</a> | 164.6  | <a href="http://www.services.gov.za">www.services.gov.za</a>       | 103.3  |
| <a href="http://www.capewebdesign.co.za">www.capewebdesign.co.za</a>     | 161.7  | <a href="http://www.info.gov.za">www.info.gov.za</a>               | 102.2  |
| <a href="http://www.sawebweather.co.za">www.sawebweather.co.za</a> ,     | 153.3  | <a href="http://www.sars.co.za">www.sars.co.za</a>                 | 95.6   |
| <a href="http://www.web-inn.co.za">www.web-inn.co.za</a>                 | 136.8  | <a href="http://www.sars.gov.za">www.sars.gov.za</a>               | 93.8   |
| <a href="http://www.searchengine.co.za">www.searchengine.co.za</a>       | 136.1  | <a href="http://www.mwebbusiness.co.za">www.mwebbusiness.co.za</a> | 93.6   |
| <a href="http://www.sawebweather.co.za">www.sawebweather.co.za</a>       | 133.6  | <a href="http://www.dti.gov.za">www.dti.gov.za,</a>                | 84.0   |
| <a href="http://www.iol.co.za">www.iol.co.za</a>                         | 132.5  | <a href="http://www.jdconsulting.co.za">www.jdconsulting.co.za</a> | 82.2   |
| <a href="http://www.tradeplace.co.za">www.tradeplace.co.za</a>           | 128.6  | <a href="http://www.linx.co.za">www.linx.co.za</a>                 | 81.0   |
| <a href="http://www.proudlysa.co.za">www.proudlysa.co.za</a>             | 126.9  |  |        |

- Compute the mean and median.
- Do you think it would be better to use the mean or the median as the measure of central location for these data? Explain.
- Compute the first and third quartiles.
- Compute and interpret the 85th percentile.



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8. Following is a sample of age data for individuals working from home by ‘telecommuting’.

|    |    |    |    |    |    |    |    |    |    |
|----|----|----|----|----|----|----|----|----|----|
| 18 | 54 | 20 | 46 | 25 | 48 | 53 | 27 | 26 | 37 |
| 40 | 36 | 42 | 25 | 27 | 33 | 28 | 40 | 45 | 25 |

- Compute the mean and the mode.
- Suppose the median age of the population of all adults is 35.5 years. Use the median age of the preceding data to comment on whether the at-home workers tend to be younger or older than the population of all adults.
- Compute the first and third quartiles.
- Compute and interpret the 32nd percentile.

## 3.2 MEASURES OF VARIABILITY

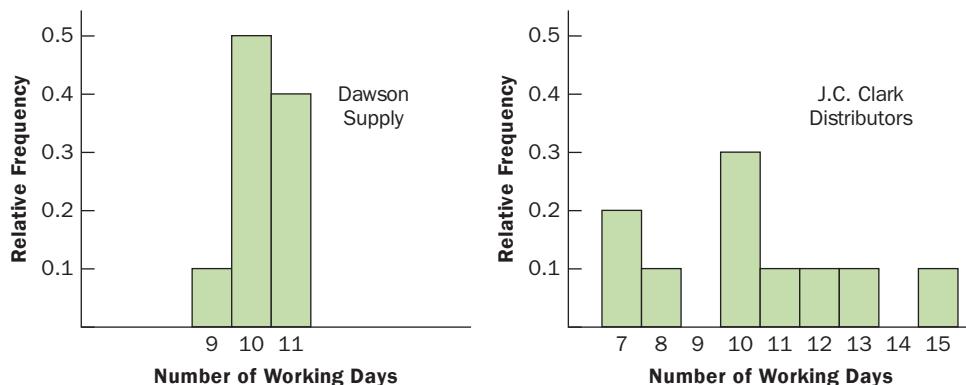
In addition to measures of location, it is often desirable to consider measures of variability, or dispersion. For example, suppose you are a purchaser for a large manufacturing firm and you regularly place orders with two different suppliers. After several months of operation, you find that the mean number of days required to fill orders is ten days for both of the suppliers. The histograms summarizing the number of working days required to fill orders from the suppliers are shown in Figure 3.2. Although the mean number of days is ten for both suppliers, do the two suppliers demonstrate the same degree of reliability in terms of making deliveries on schedule? Note the dispersion, or variability, in delivery times indicated by the histograms. Which supplier would you prefer?

For most firms, receiving materials and supplies on schedule is important. The seven- or eight-day deliveries shown for J.C. Clark Distributors might be viewed favourably. However, a few of the slow 13- to 15-day deliveries could be disastrous in terms of keeping a workforce busy and production on schedule. This example illustrates a situation in which the variability in the delivery times may be an overriding consideration in selecting a supplier. For most purchasing agents, the lower variability shown for Dawson Supply would make Dawson the preferred supplier.

We turn now to a discussion of some commonly used measures of variability.

### Range

The simplest measure of variability is the **range**.



**FIGURE 3.2**

Historical data showing the number of days required to fill orders

**Range**

$$\text{Range} = \text{Largest value} - \text{Smallest value}$$

Refer to the data on starting salaries for business school graduates in Table 3.1. The largest starting salary is 2260 and the smallest is 1955. The range is  $2260 - 1955 = 305$ .

Although the range is the easiest of the measures of variability to compute, it is seldom used as the only measure. The range is based on only two of the observations and so is highly influenced by extreme values. Suppose one of the graduates received a starting salary of €5000 per month. In this case, the range would be  $5000 - 1955 = 3045$  rather than 305. This would not be especially descriptive of the variability in the data because 11 of the 12 starting salaries are relatively closely grouped between 1955 and 2165.

**Interquartile range**

A measure of variability that overcomes the dependency on extreme values is the **interquartile range (IQR)**. This measure of variability is simply the difference between the third quartile,  $Q_3$ , and the first quartile,  $Q_1$ . In other words, the interquartile range is the range for the middle 50 per cent of the data.

**Interquartile range**

$$IQR = Q_3 - Q_1$$

(3.3)

For the data on monthly starting salaries, the quartiles are  $Q_3 = 2100$  and  $Q_1 = 2030$ . The interquartile range is  $2100 - 2030 = 70$ .

**Variance**

The **variance** is a measure of variability that uses all the data. The variance is based on the difference between the value of each data value and the mean. This difference is called a *deviation about the mean*. For a sample, a deviation about the mean is written  $(x_i - \bar{x})$ . For a population, it is written  $(x_i - \mu)$ . In the computation of the variance, the deviations about the mean are *squared*.

If the data are for a population, the average of the squared deviations is called the *population variance*. The population variance is denoted by the Greek symbol  $\sigma^2$  (sigma squared). For a population of  $N$  observations and with  $\mu$  denoting the population mean, the definition of the population variance is as follows:

**Population variance**

$$\sigma^2 = \frac{\sum(x_i - \mu)^2}{N} \quad (3.4)$$

In most statistical applications, the data being analyzed are for a sample. When we compute a sample variance, we are often interested in using it to estimate the population variance  $\sigma^2$ . Although a detailed explanation is beyond the scope of this text, it can be shown that if the sum of the squared deviations about the sample mean is divided by  $n - 1$ , not by  $n$ , the resulting sample variance provides an unbiased estimate of the population variance (a formal definition of unbiasedness is given in Chapter 7).

**TABLE 3.2** Computation of deviations and squared deviations about the mean for the class size data

| Number of students<br>in class ( $x_i$ ) | Mean class size<br>( $\bar{x}$ ) | Deviation about the<br>mean ( $x_i - \bar{x}$ ) | Squared deviation about<br>the mean ( $x_i - \bar{x}$ ) <sup>2</sup> |
|--|----------------------------------|---|--|
| 46                                       | 44                               | 2   | 4  |
| 54                                       | 44                               | 10  | 100  |
| 42                                       | 44                               | -2  | 4  |
| 46                                       | 44                               | 2   | 4  |
| 32                                       | 44                               | -12   | 144  |
| <b>Totals</b>                            |                                  | <b>0</b>  | <b>256</b>   |
|  |                                  | $\sum(x_i - \bar{x})$                           | $\sum(x_i - \bar{x})^2$  |

For this reason, the *sample variance*, denoted by  $s^2$ , is defined as follows:

### Sample variance

$$s^2 = \frac{\sum(x_i - \bar{x})^2}{n - 1} \quad (3.5)$$

Consider the data on class size for the sample of five university classes (Section 3.1). A summary of the data, including the computation of the deviations about the mean and the squared deviations about the mean, is shown in Table 3.2. The sum of squared deviations about the mean is  $\sum(x_i - \bar{x})^2 = 256$ . Hence, with  $n - 1 = 4$ , the sample variance is:

$$s^2 = \frac{\sum(x_i - \bar{x})^2}{n - 1} = \frac{256}{4} = 64$$

The units associated with the sample variance can cause confusion. Because the values summed in the variance calculation,  $(x_i - \bar{x})^2$ , are squared, the units associated with the sample variance are also *squared*. For instance, the sample variance for the class size data is  $s^2 = 64$  (students)<sup>2</sup>. The squared units make it difficult to obtain an intuitive understanding and interpretation of the variance. We recommend that you think of the variance as a measure useful in comparing the amount of variability for two or more comparable variables. The one with the larger variance will show the greater variability.

As another illustration, consider the starting salaries in Table 3.1 for the 12 business school graduates. In Section 3.1, we showed that the sample mean starting salary was 2070. The computation of the sample variance ( $s^2 = 6754.5$ ) is shown in Table 3.3.

In Tables 3.2 and 3.3 we show both the sum of the deviations about the mean and the sum of the squared deviations about the mean. Note that in both tables,  $\sum(x_i - \bar{x}) = 0$ . The positive deviations and negative deviations cancel each other, causing the sum of the deviations about the mean to equal zero. For any data set, the sum of the deviations about the mean will *always equal zero*.

An alternative formula for the computation of the sample variance:

$$s^2 = \frac{\sum x_i^2 - n\bar{x}^2}{n - 1}$$

where:

$$\sum x_i^2 = x_1^2 + x_2^2 + \dots + x_n^2$$

## Standard deviation

The **standard deviation** is defined to be the positive square root of the variance. Following the notation we adopted for a sample variance and a population variance, we use  $s$  to denote the sample standard deviation and  $\sigma$  to denote the population standard deviation.

**TABLE 3.3** Computation of the sample variance for the starting salary data

| Monthly salary ( $x_i$ ) | Sample mean ( $\bar{x}$ ) | Deviation about the mean ( $x_i - \bar{x}$ ) | Squared deviation about the mean ( $x_i - \bar{x}$ ) <sup>2</sup> |
|--------------------------|---------------------------|--|---|
| 2020                     | 207                       | -50  | 2 500   |
| 2075                     | 207                       | 5  | 25  |
| 2125                     | 207                       | 55   | 3 025   |
| 2040                     | 207                       | -30  | 900   |
| 1980                     | 207                       | -90  | 8 100   |
| 1955                     | 207                       | -115   | 13 225  |
| 2050                     | 207                       | -20  | 400   |
| 2165                     | 207                       | 95   | 9 025   |
| 2070                     | 207                       | 0  | 0   |
| 2260                     | 207                       | 190  | 36 100  |
| 2060                     | 207                       | -10  | 100   |
| 2040                     | 207                       | -30  | 900   |
| <b>Totals</b>            |                           | <b>0</b>                                     | <b>74 300</b>   |

Using equation (3.5)

$$s^2 = \frac{\sum(x_i - \bar{x})^2}{n-1} = \frac{74\,300}{11} = 6754.5$$

The standard deviation is derived from the variance as shown in equations (3.6) and (3.7).

### Standard deviation

$$\text{Population standard deviation} = \sigma = \sqrt{\sigma^2} \quad (3.6)$$

$$\text{Sample standard deviation} = s = \sqrt{s^2} \quad (3.7)$$

Recall that the sample variance for the sample of class sizes in five university classes is  $s^2 = 64$ . Hence the sample standard deviation is:

$$s = \sqrt{64} = 8$$

For the data on starting salaries, the sample standard deviation is:

$$s = \sqrt{6754.5} = 82.2$$

What is gained by converting the variance to its corresponding standard deviation? Recall that the units associated with the variance are squared. For example, the sample variance for the starting salary data of business school graduates is  $s^2 = 6754.5$  (€)<sup>2</sup>. Because the standard deviation is the square root of the variance, the units are euros for the standard deviation,  $s = €82.2$ . In other words, the standard deviation is measured in the same units as the original data. The standard deviation is therefore more easily compared to the mean and other statistics measured in the same units as the original data.

### Coefficient of variation

In some situations we may be interested in a descriptive statistic that indicates how large the standard deviation is relative to the mean. This measure is called the **coefficient of variation** and is usually expressed as a percentage.

**Coefficient of variation**

$$\left( \frac{\text{Standard deviation}}{\text{Mean}} \times 100 \right) \% \quad (3.8)$$

For the class size data, we found a sample mean of 44 and a sample standard deviation of 8. The coefficient of variation is  $(8/44) \times 100\% = 18.2\%$ . The coefficient of variation tells us that the sample standard deviation is 18.2 per cent of the value of the sample mean. For the starting salary data with a sample mean of 2070 and a sample standard deviation of 82.2, the coefficient of variation,  $(82.2/2070) \times 100\% = 4.0\%$ , tells us the sample standard deviation is only 4.0 per cent of the value of the sample mean. In general, the coefficient of variation is a useful statistic for comparing the variability of variables that have different standard deviations and different means.

**EXERCISES****Methods**

9. Consider a sample with data values of 10, 20, 12, 17 and 16. Calculate the range and interquartile range.
10. Consider a sample with data values of 10, 20, 12, 17 and 16. Calculate the variance and standard deviation.
11. Consider a sample with data values of 27, 25, 20, 15, 30, 34, 28 and 25. Calculate the range, interquartile range, variance and standard deviation.



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**Applications**

12. The goals scored in six handball matches were 41, 34, 42, 45, 35 and 37. Using these data as a sample, compute the following descriptive statistics.
  - a. Range.
  - b. Variance.
  - c. Standard deviation.
  - d. Coefficient of variation.
13. Dinner bill amounts for set menus at a Dubai restaurant, Al Khayam, show the following frequency distribution. The amounts are in AED (Emirati Dirham). Compute the mean, variance and standard deviation.



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| Dinner bill (AED) | Frequency |
|-------------------|-----------|
| 30                | 2         |
| 40                | 6         |
| 50                | 4         |
| 60                | 4         |
| 70                | 2         |
| 80                | 2         |
| <b>Total</b>      | <b>20</b> |



CRIME

- 14.** The following data were used to construct the histograms of the number of days required to fill orders for Dawson Supply and for J.C. Clark Distributors (see Figure 3.2).

Dawson Supply days for delivery:      11    10    9    10    11    11    11    10    11    10    10  
 Clark Distributors days for delivery:    8    10    13    7    10    11    10    7    15    12

Use the range and standard deviation to support the previous observation that Dawson Supply provides the more consistent and reliable delivery times.

- 15.** Police records show the following numbers of daily crime reports for a sample of days during the winter months and a sample of days during the summer months.

|         |    |    |    |    |    |    |    |    |    |    |
|---------|----|----|----|----|----|----|----|----|----|----|
| Winter: | 18 | 20 | 15 | 16 | 21 | 20 | 12 | 16 | 19 | 20 |
| Summer: | 28 | 18 | 24 | 32 | 18 | 29 | 23 | 38 | 28 | 18 |

- Compute the range and interquartile range for each period.
- Compute the variance and standard deviation for each period.
- Compute the coefficient of variation for each period.
- Compare the variability of the two periods.

- 16.** A production department uses a sampling procedure to test the quality of newly produced items. The department employs the following decision rule at an inspection station: if a sample of 14 items has a variance of more than 0.005, the production line must be shut down for repairs. Suppose the following data have just been collected:

|      |      |      |      |      |      |      |
|------|------|------|------|------|------|------|
| 3.43 | 3.45 | 3.43 | 3.48 | 3.52 | 3.50 | 3.39 |
| 3.48 | 3.41 | 3.38 | 3.49 | 3.45 | 3.51 | 3.50 |

Should the production line be shut down? Why or why not?

### 3.3 MEASURES OF DISTRIBUTIONAL SHAPE, RELATIVE LOCATION AND DETECTING OUTLIERS

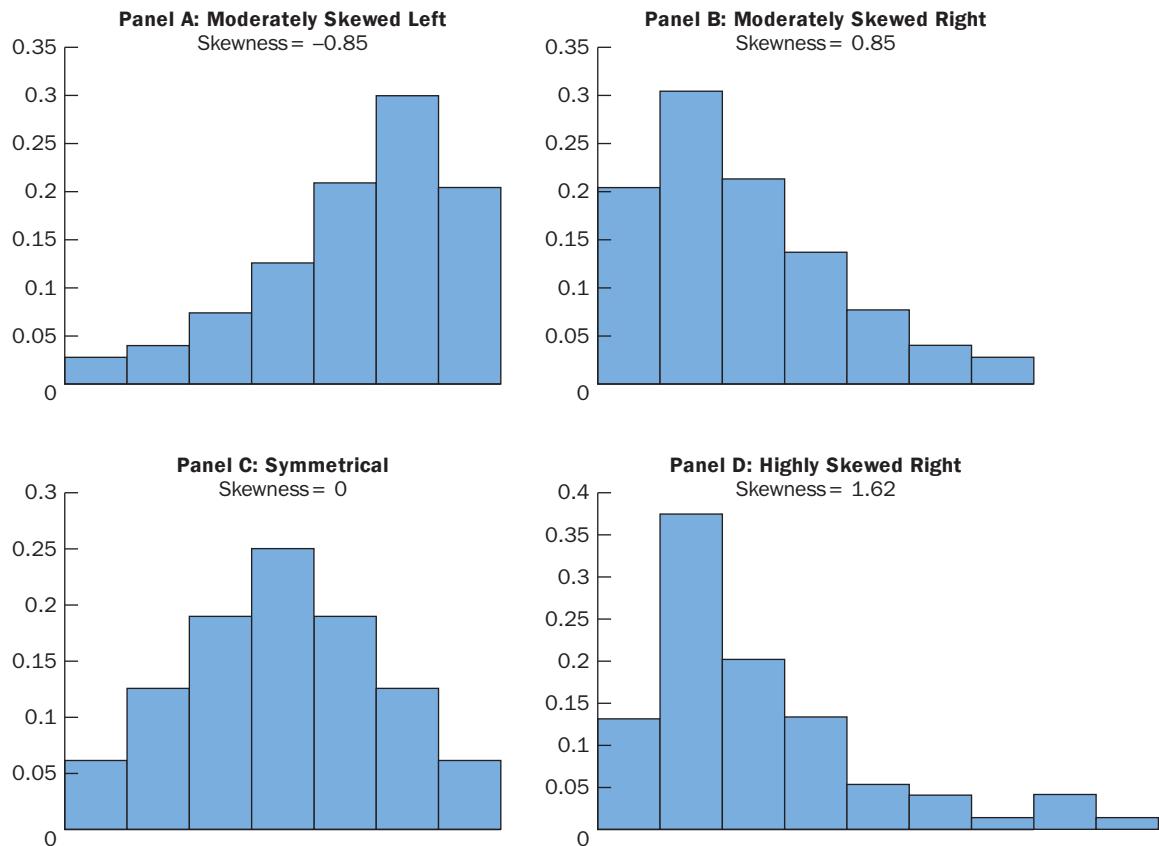
We described several measures of location and variability for data distributions. It is also often important to have a measure of the shape of a distribution. In Chapter 2 we noted that a histogram offers an excellent graphical display showing the shape of a distribution. An important numerical measure of the shape of a distribution is **skewness**.

#### Distributional shape

Four histograms constructed from relative frequency distributions are shown in Figure 3.3. The histograms in Panels A and B are moderately skewed. The one in Panel A is skewed to the left: its skewness is  $-0.85$  (negative skewness). The histogram in Panel B is skewed to the right: its skewness is  $+0.85$  (positive skewness). The histogram in Panel C is symmetrical: its skewness is zero. The histogram in Panel D is highly skewed to the right: its skewness is  $1.62$ . The formula used to compute skewness is somewhat complex.\* However, the skewness can be easily computed using statistical software (see the software guides on the online platform).

\*The formula for the skewness of sample data is:

$$\text{Skewness} = \frac{n}{(n-1)(n-2)} \sum \left( \frac{x_i - \bar{x}}{s} \right)^3$$

**FIGURE 3.3**

Histograms showing the skewness for four distributions

For a symmetrical distribution, the mean and the median are equal. When the data are positively skewed, the mean will usually be greater than the median. When the data are negatively skewed, the mean will usually be less than the median. The data used to construct the histogram in Panel D are customer purchases at a women's fashion store. The mean purchase amount is €77.60 and the median purchase amount is €59.70. The few large purchase amounts pull up the mean, but the median remains unaffected. The median provides a better measure of typical values when the data are highly skewed.

## **z-Scores**

In addition to measures of location, variability and shape for a data set, we are often also interested in the relative location of data items within a data set. Such measures can help us determine whether a particular item is close to the centre of a data set or far out in one of the tails.

By using both the mean and standard deviation, we can determine the relative location of any observation. Suppose we have a sample of  $n$  observations, with the values denoted by  $x_1, x_2, \dots, x_n$ . Assume the sample mean  $\bar{x}$ , and the sample standard deviation  $s$  are already computed. Associated with each value  $x_i$  is a value called its **z-score**. Equation (3.9) shows how the z-score is computed for each  $x_i$ .

### **z-score**

$$z_i = \frac{x_i - \bar{x}}{s} \quad (3.9)$$

where  $z_i$  = the z-score for  $x_i$ ;  $\bar{x}$  = the sample mean,  $s$  = the sample standard deviation.

**TABLE 3.4** z-scores for the class size data

| Number of students<br>in class ( $x_i$ ) | Deviation about the mean<br>( $x_i - \bar{x}$ ) | $\text{z-score} = \frac{x_i - \bar{x}}{s}$ |
|--|---|--|
| 46                                       | 2   | $2/8 = 0.25$                               |
| 54                                       | 10  | $10/8 = 1.25$                              |
| 42                                       | -2  | $-2/8 = -0.25$                             |
| 46                                       | 2   | $2/8 = 0.25$                               |
| 32                                       | -12   | $-12/8 = -1.50$                            |

The z-score is often called the *standardized value* or the *standard score*. The z-score,  $z_i$ , represents the *number of standard deviations*  $x_i$  is from the mean  $\bar{x}$ . For example,  $z_1 = 1.2$  would indicate that  $x_1$  is 1.2 standard deviations higher than the sample mean. Similarly,  $z_2 = -0.5$  would indicate that  $x_2$  is 0.5, or  $1/2$ , standard deviation lower than the sample mean. Data values above the mean have a z-score greater than zero. Data values below the mean have a z-score less than zero. A z-score of zero indicates that the data value is equal to the sample mean.

The z-score is a measure of the relative location of the observation in a data set. Hence, observations in two different data sets with the same z-score can be said to have the same relative location in terms of being the same number of standard deviations from the mean.

The z-scores for the class size data are computed in Table 3.4. Recall the previously computed sample mean,  $\bar{x} = 44$ , and sample standard deviation,  $s = 8$ . The z-score of  $-1.50$  for the fifth observation shows it is farthest from the mean: it is 1.50 standard deviations below the mean.

## Chebyshev's theorem

**Chebyshev's theorem** enables us to make statements about the proportion of data values that lie within a specified number of standard deviations of the mean.

### Chebyshev's theorem

At least  $(1 - 1/z^2) \times 100\%$  of the data values must be within  $z$  standard deviations of the mean, where  $z$  is any value greater than 1.

Some of the implications of this theorem, with  $z = 2, 3$  and  $4$  standard deviations, follow:

- At least 75 per cent of the data values must be within  $z = 2$  standard deviations of the mean.
- At least 89 per cent of the data values must be within  $z = 3$  standard deviations of the mean.
- At least 94 per cent of the data values must be within  $z = 4$  standard deviations of the mean.

Suppose that the mid-term test scores for 100 students in a university business statistics course had a mean of 70 and a standard deviation of 5. How many students had test scores between 60 and 80? How many students had test scores between 58 and 82?

For the test scores between 60 and 80, we note that 60 is two standard deviations below the mean and 80 is two standard deviations above the mean. Using Chebyshev's theorem, we see that at least 75 per cent of the observations must have values within two standard deviations of the mean. Hence, at least 75 per cent of the students must have scored between 60 and 80.

For the test scores between 58 and 82, we see that  $(58 - 70)/5 = -2.4$ , i.e. 58 is 2.4 standard deviations below the mean. Similarly,  $(82 - 70)/5 = +2.4$ , so 82 is 2.4 standard deviations above the mean. Applying Chebyshev's theorem with  $z = 2.4$ , we have:

$$\left(1 - \frac{1}{z^2}\right) = \left(1 - \frac{1}{(2.4)^2}\right) = 0.826$$

At least 82.6 per cent of the students must have test scores between 58 and 82.

## Empirical rule

Chebyshev's theorem applies to any data set, regardless of the shape of the distribution. It could be used, for example, with any of the skewed distributions in Figure 3.3. In many practical applications, however, data sets exhibit a symmetrical mound-shaped or bell-shaped distribution like the one shown in Figure 3.4. When the data are believed to approximate this distribution, the **empirical rule** can be used to determine the percentage of data values that must be within a specified number of standard deviations of the mean. The empirical rule is based on the normal probability distribution, which will be discussed in Chapter 6. The normal distribution is used extensively throughout this book.

### Empirical rule

For data with a bell-shaped distribution:

- Approximately 68 per cent of the data values will be within one standard deviation of the mean.
- Approximately 95 per cent of the data values will be within two standard deviations of the mean.
- Almost all of the data values will be within three standard deviations of the mean.

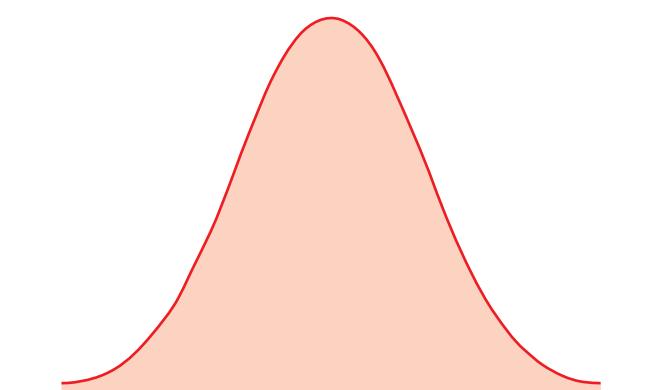
For example, the empirical rule allows us to say that *approximately* 95 per cent of the data values will be within two standard deviations of the mean (Chebyshev's theorem allows us to conclude only that at least 75 per cent of the data values will be in that interval).

Consider liquid detergent cartons being filled automatically on a production line. Filling weights frequently have a bell-shaped distribution. If the mean filling weight is 500 grams and the standard deviation is 7 grams, we can use the empirical rule to draw the following conclusions:

- Approximately 68 per cent of the filled cartons will have weights between 493 and 507 grams (that is, within one standard deviation of the mean).
- Approximately 95 per cent of the filled cartons will have weights between 486 and 514 grams (that is, within two standard deviations of the mean).
- Almost all filled cartons will have weights between 479 and 521 grams (that is, within three standard deviations of the mean).

**FIGURE 3.4**

A symmetrical mound-shaped or bell-shaped distribution



## Detecting outliers

Sometimes a data set will have one or more observations with unusually large or unusually small values. These extreme values are called **outliers**. Experienced statisticians take steps to identify outliers and then review each one carefully. An outlier may be a data value that has been incorrectly recorded. If so, it can be corrected before further analysis. An outlier may also be from an observation that was incorrectly included in the data set. If so, it can be removed. Finally, an outlier may be an unusual data value that has been recorded correctly and belongs in the data set. In such cases it should remain.

Standardized values ( $z$ -scores) can be used to identify outliers. The empirical rule allows us to conclude that, for data with a bell-shaped distribution, almost all the data values will be within three standard deviations of the mean. Hence, we recommend treating any data value with a  $z$ -score less than  $-3$  or greater than  $+3$  as an outlier, if the sample is small or moderately sized. Such data values can then be reviewed for accuracy and to determine whether they belong in the data set.

Refer to the  $z$ -scores for the class size data in Table 3.4. The  $z$ -score of  $-1.50$  shows the fifth class size is furthest from the mean. However, this standardized value is well within the  $-3$  to  $+3$  guideline for outliers. Hence, the  $z$ -scores give no indication that outliers are present in the class size data.

## EXERCISES

### Methods



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17. Consider a sample with data values of 10, 20, 12, 17 and 16. Calculate the  $z$ -score for each of the five observations.
18. Consider a sample with a mean of 500 and a standard deviation of 100. What are the  $z$ -scores for the following data values: 520, 650, 500, 450 and 280?
19. Consider a sample with a mean of 30 and a standard deviation of 5. Use Chebyshev's theorem to determine the percentage of the data within each of the following ranges.
  - a. 20 to 40
  - b. 15 to 45
  - c. 22 to 38
  - d. 18 to 42
  - e. 12 to 48
20. Suppose the data have a bell-shaped distribution with a mean of 30 and a standard deviation of 5. Use the empirical rule to determine the percentage of data within each of the following ranges.
  - a. 20 to 40
  - b. 15 to 45
  - c. 25 to 35

### Applications

21. The results of a survey of 1154 adults showed that on average adults sleep 6.9 hours per day during the working week. Suppose that the standard deviation is 1.2 hours.
  - a. Use Chebyshev's theorem to calculate the percentage of individuals who sleep between 4.5 and 9.3 hours per day.
  - b. Use Chebyshev's theorem to calculate the percentage of individuals who sleep between 3.9 and 9.9 hours per day.
  - c. Assume that the number of hours of sleep follows a bell-shaped distribution. Use the empirical rule to calculate the percentage of individuals who sleep between 4.5 and 9.3 hours per day. How does this result compare to the value that you obtained using Chebyshev's theorem in part (a)?

- 22.** Suppose that IQ scores have a bell-shaped distribution with a mean of 100 and a standard deviation of 15.
- What percentage of people have an IQ score between 85 and 115?
  - What percentage of people have an IQ score between 70 and 130?
  - What percentage of people have an IQ score of more than 130?
  - A person with an IQ score greater than 145 is considered a genius. Does the empirical rule support this statement? Explain.
- 23.** Suppose the average charge for a seven-day hire of an economy-class car in Kuwait City is KWD (Kuwaiti dinar) 75.00, and the standard deviation is KWD20.00.
- What is the z-score for a seven-day hire charge of KWD56.00?
  - What is the z-score for a seven-day hire charge of KWD153.00?
  - Interpret the z-scores in parts (a) and (b). Comment on whether either should be considered an outlier.
- 24.** Consumer Review posts reviews and ratings of a variety of products on the Internet. The following is a sample of 20 speaker systems and their ratings, on a scale of 1 to 5, with 5 being best.

| Speaker               | Rating | Speaker                | Rating |
|-----------------------|--------|------------------------|--------|
| Infinity Kappa 6.1    | 4.00   | ACI Sapphire III       | 4.67   |
| Allison One           | 4.12   | Bose 501 Series        | 2.14   |
| Cambridge Ensemble II | 3.82   | DCM KX-212             | 4.09   |
| Dynaudio Contour 1.3  | 4.00   | Eosone RSF1000         | 4.17   |
| Hsu Rsch. HRSW12V     | 4.56   | Joseph Audio RM7si     | 4.88   |
| Legacy Audio Focus    | 4.32   | Martin Logan Aerius    | 4.26   |
| 26 Mission 73li       | 4.33   | Omni Audio SA 12.3     | 2.32   |
| PSB 400i              | 4.50   | Polk Audio RT12        | 4.50   |
| Snell Acoustics D IV  | 4.64   | Sunfire True Subwoofer | 4.17   |
| Thiel CS1.5           | 4.20   | Yamaha NS-A636         | 2.17   |

- Compute the mean and the median.
- Compute the first and third quartiles.
- Compute the standard deviation.
- The skewness of this data is 1.67. Comment on the shape of the distribution.
- What are the z-scores associated with Allison One and Omni Audio?
- Do the data contain any outliers? Explain.



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SPEAKERS

## 3.4 EXPLORATORY DATA ANALYSIS

In Chapter 2 we introduced the stem-and-leaf display as an exploratory data analysis technique. In this section we continue exploratory data analysis by considering five-number summaries and box plots.

### Five-number summary

In a **five-number summary** the following five numbers are used to summarize the data.

- Smallest value (minimum).
- First quartile ( $Q_1$ ).
- Median ( $Q_2$ ).

- 4 Third quartile ( $Q_3$ ).
- 5 Largest value (maximum).

The easiest way to construct a five-number summary is to first place the data in ascending order. Then it is easy to identify the smallest value, the three quartiles and the largest value. The monthly starting salaries shown in Table 3.1 for a sample of 12 business school graduates are repeated here in ascending order.

|              |      |      |      |      |              |      |      |      |              |      |      |
|--------------|------|------|------|------|--------------|------|------|------|--------------|------|------|
| 1955         | 1980 | 2020 | 2040 | 2040 | 2050         | 2060 | 2070 | 2075 | 2125         | 2165 | 2260 |
| $Q_1 = 2030$ |      |      |      |      | $Q_2 = 2055$ |      |      |      | $Q_3 = 2100$ |      |      |
|              |      |      |      |      | (Median)     |      |      |      |              |      |      |

The median of 2055 and the quartiles  $Q_1 = 2030$  and  $Q_3 = 2100$  were computed in Section 3.1. The smallest value is 1955 and the largest value is 2260. Hence the five-number summary for the salary data is 1955, 2030, 2055, 2100, 2260. Approximately one-quarter, or 25 per cent, of the observations are between adjacent numbers in a five-number summary.

## Box plot

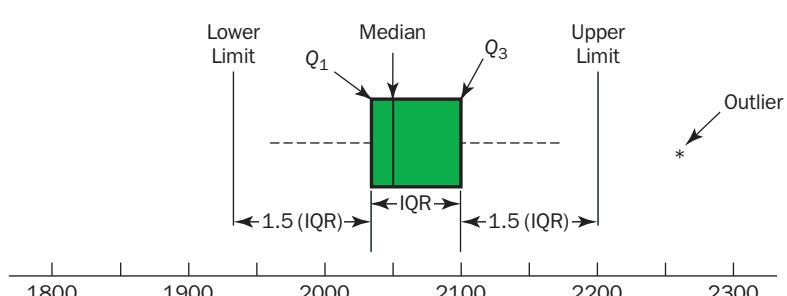
A **box plot** is a slightly elaborated graphical version of the five-number summary. Figure 3.5 shows the construction of a box plot for the monthly starting salary data.

- 1 A box is drawn with the ends of the box located at the first and third quartiles. For the salary data,  $Q_1 = 2030$  and  $Q_3 = 2100$ . This box contains the middle 50 per cent of the data.
- 2 A vertical line is drawn in the box at the location of the median (2055 for the salary data).
- 3 By using the interquartile range,  $IQR = Q_3 - Q_1$ , *limits* are located. The limits for the box plot are 1.5(IQR) below  $Q_1$  and 1.5(IQR) above  $Q_3$ . For the salary data,  $IQR = Q_3 - Q_1 = 2100 - 2030 = 70$ . Hence, the limits are  $2030 - 1.5(70) = 1925$  and  $2100 + 1.5(70) = 2205$ . Data outside these limits are considered *outliers*.
- 4 The dashed lines in Figure 3.5 are called *whiskers*. The whiskers are drawn from the ends of the box to the smallest and largest values *inside the limits* computed in step 3. Hence the whiskers end at salary values of 1955 and 2165.
- 5 Finally, the location of each outlier is shown with a symbol, often \*. In Figure 3.5 we see one outlier, 2260. (Note that box plots do not necessarily identify the same outliers as identifying  $z$ -scores less than  $-3$  or greater than  $+3$ .)

Figure 3.5 includes the upper and lower limits, to show how these limits are computed and where they are located for the salary data. Although the limits are always computed, they are not generally drawn on the box plots. The MINITAB box plots in Figure 3.6 illustrate the usual appearance, and also demonstrate that box plots are an excellent graphical tool for making comparisons amongst two or more groups.

**FIGURE 3.5**

Box plot of the starting salary data with lines showing the lower and upper limits



**FIGURE 3.6**

Box plot of monthly salary

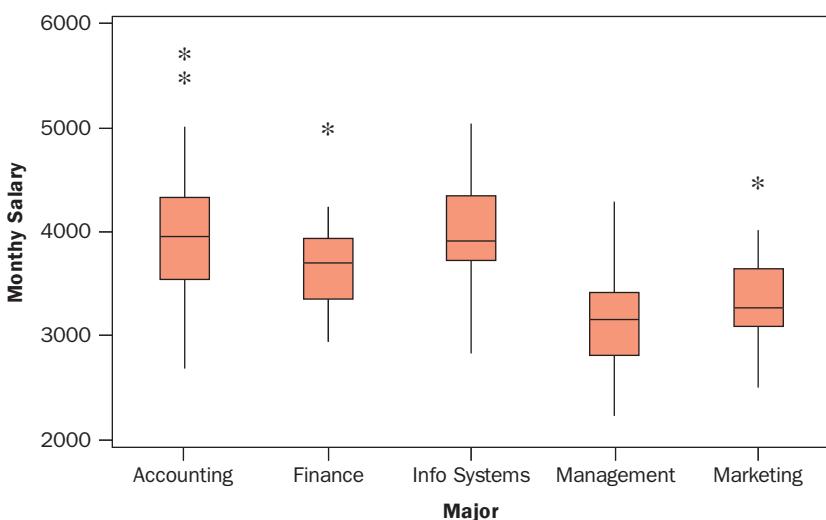


Figure 3.6 compares monthly starting salaries for a sample of 111 graduates, by major discipline. The major is shown on the horizontal axis and each box plot is arranged vertically above the relevant major label. The box plots in Figure 3.6 indicate that, for example:

- The highest median salary is in Accounting, the lowest in Management.
- Accounting salaries show the highest variation.
- There are high salary outliers for Accounting, Finance and Marketing.

## EXERCISES

### Methods

25. Consider a sample with data values of 27, 25, 20, 15, 30, 34, 28 and 25. Provide the five-number summary for the data.
26. Construct a box plot for the data in Exercise 25.
27. Prepare the five-number summary and the box plot for the following data: 5, 15, 18, 10, 8, 12, 16, 10, 6.
28. A data set has a first quartile of 42 and a third quartile of 50. Compute the lower and upper limits for the corresponding box plot. Should a data value of 65 be considered an outlier?

### Applications

29. Annual sales, in millions of dollars, for 21 pharmaceutical companies follow.

|        |       |       |       |       |        |       |
|--------|-------|-------|-------|-------|--------|-------|
| 8 408  | 1 374 | 1 872 | 8 879 | 2 459 | 11 413 | 608   |
| 14 138 | 6 452 | 1 850 | 2 818 | 1 356 | 10 498 | 7 478 |
| 4 019  | 4 341 | 739   | 2 127 | 3 653 | 5 794  | 8 305 |



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EQUITY



STOCK 2008

- a. Provide a five-number summary.
- b. Compute the lower and upper limits (for the box plot).
- c. Do the data contain any outliers?
- d. Johnson & Johnson's sales are the largest on the list at \$14 138 million. Suppose a data entry error (a transposition) had been made and the sales had been entered as \$41 138 million. Would the method of detecting outliers in part (c) identify this problem and allow for correction of the data entry error?
- e. Construct a box plot.

- 30.** A goal of management is to help their company earn as much as possible relative to the capital invested. One measure of success is return on equity – the ratio of net income to stockholders' equity. Return on equity percentages are shown here for 25 companies.

|      |      |      |      |      |      |      |      |      |      |
|------|------|------|------|------|------|------|------|------|------|
| 9.0  | 19.6 | 22.9 | 41.6 | 11.4 | 15.8 | 52.7 | 17.3 | 12.3 | 5.1  |
| 17.3 | 31.1 | 9.6  | 8.6  | 11.2 | 12.8 | 12.2 | 14.5 | 9.2  | 16.6 |
| 5.0  | 30.3 | 14.7 | 19.2 | 6.2  |      |      |      |      |      |

- a. Provide a five-number summary.
- b. Compute the lower and upper limits (for the box plot).
- c. Do the data contain any outliers? How would this information be helpful to a financial analyst?
- d. Construct a box plot.

- 31.** In 2008, stock markets around the world lost value. The website [www.owneverystock.com](http://www.owneverystock.com) listed the following percentage falls in stock market indices between the start of the year and the beginning of October.

| Country        | % Fall | Country     | % Fall |
|----------------|--------|-------------|--------|
| New Zealand    | 27.05  | Brazil      | 39.59  |
| Canada         | 27.30  | Japan       | 39.88  |
| Switzerland    | 28.42  | Sweden      | 40.35  |
| Mexico         | 29.99  | Egypt       | 41.57  |
| Australia      | 31.95  | Singapore   | 41.60  |
| Korea          | 32.18  | Italy       | 42.88  |
| United Kingdom | 32.37  | Belgium     | 43.70  |
| Spain          | 32.69  | India       | 44.16  |
| Malaysia       | 32.86  | Hong Kong   | 44.52  |
| Argentina      | 36.83  | Netherlands | 44.61  |
| France         | 37.71  | Norway      | 46.98  |
| Israel         | 37.84  | Indonesia   | 47.13  |
| Germany        | 37.85  | Austria     | 50.06  |
| Taiwan         | 38.79  | China       | 60.24  |

- a. What are the mean and median percentage changes for these countries?
- b. What are the first and third quartiles?
- c. Do the data contain any outliers? Construct a box plot.
- d. What percentile would you report for Belgium?

## 3.5 MEASURES OF ASSOCIATION BETWEEN TWO VARIABLES

We have examined numerical methods used to summarize *one variable at a time*. Often a manager or decision-maker is interested in the *relationship between two variables*. In this section we present covariance and correlation as descriptive measures of the relationship between two variables.

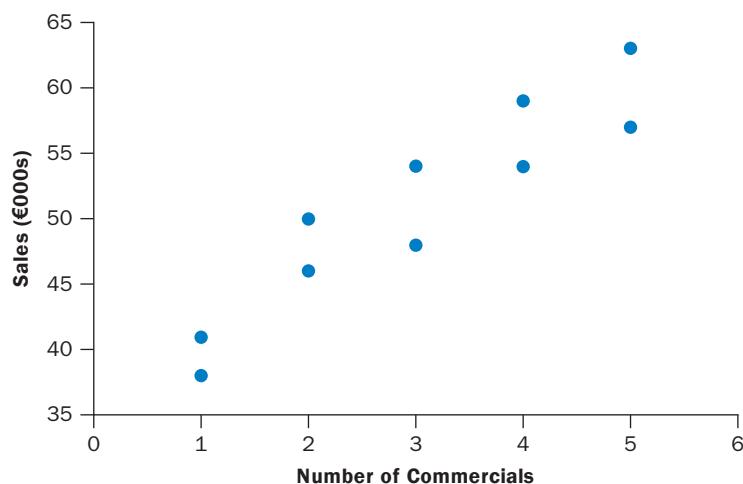
We begin by reconsidering the hi-fi equipment store discussed in Section 2.3. The store's manager wants to determine the relationship between the number of weekend television commercials shown and the sales at the store during the following week. Sample data with sales expressed in €000s were given in Table 2.12, and are repeated here in the first three columns of Table 3.5. It shows ten observations ( $n = 10$ ), one for each week.

The scatter diagram in Figure 3.7 shows a positive relationship, with higher sales (vertical axis) associated with a greater number of commercials (horizontal axis). The scatter diagram suggests that a straight line could be used as an approximation of the relationship. In the following discussion, we introduce **covariance** as a descriptive measure of the linear association between two variables.

**TABLE 3.5** Calculations for the sample covariance

| Week          | Number of commercials $x_i$ | Sales volume (€000s) $y_i$ | $x_i - \bar{x}$ | $y_i - \bar{y}$ | $(x_i - \bar{x})(y_i - \bar{y})$ |
|---------------|-----------------------------|----------------------------|-----------------|-----------------|----------------------------------|
| 1             | 2                           | 50                         | -1              | -1              | 1                                |
| 2             | 5                           | 57                         | 2               | 6               | 12                               |
| 3             | 1                           | 41                         | -2              | -10             | 20                               |
| 4             | 3                           | 54                         | 0               | 3               | 0                                |
| 5             | 4                           | 54                         | 1               | 3               | 3                                |
| 6             | 1                           | 38                         | -2              | -13             | 26                               |
| 7             | 5                           | 63                         | 2               | 12              | 24                               |
| 8             | 3                           | 48                         | 0               | -3              | 0                                |
| 9             | 4                           | 59                         | 1               | 8               | 8                                |
| 10            | 2                           | 46                         | -1              | -5              | 5                                |
| <b>Totals</b> | <b>30</b>                   | <b>510</b>                 | <b>0</b>        | <b>0</b>        | <b>99</b>                        |

**FIGURE 3.7**  
Scatter diagram for the hi-fi equipment store



## Covariance

For a sample of size  $n$  with the observations  $(x_1, y_1), (x_2, y_2)$  and so on, the sample covariance is defined as follows:

### Sample covariance

$$s_{XY} = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{n-1} \quad (3.10)$$

This formula pairs each  $x_i$  with a corresponding  $y_i$ . We then sum the products obtained by multiplying the deviation of each  $x_i$  from its sample mean  $\bar{x}$  by the deviation of the corresponding  $y_i$  from its sample mean  $\bar{y}$ . This sum is then divided by  $n - 1$ .

To measure the strength of the linear relationship between the number of commercials  $X$  and the sales volume  $Y$  in the hi-fi equipment store problem, we use equation (3.10) to compute the sample covariance. The calculations in Table 3.5 show the computation of  $\sum(x_i - \bar{x})(y_i - \bar{y})$ . Note that  $\bar{x} = 30/10 = 3$  and  $\bar{y} = 510/10 = 51$ . Using equation (3.10), we obtain a sample covariance of:

$$s_{XY} = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{n-1} = \frac{99}{10-1} = 11$$

The formula for computing the covariance of a population of size  $N$  is similar to equation (3.10), but we use different notation to indicate that we are working with the entire population.

### Population covariance

$$\sigma_{XY} = \frac{\sum(x_i - \mu_X)(y_i - \mu_Y)}{N} \quad (3.11)$$

In equation (3.11) we use the notation  $\mu_X$  for the population mean of  $X$  and  $\mu_Y$  for the population mean of  $Y$ . The population covariance  $\sigma_{XY}$  is defined for a population of size  $N$ .

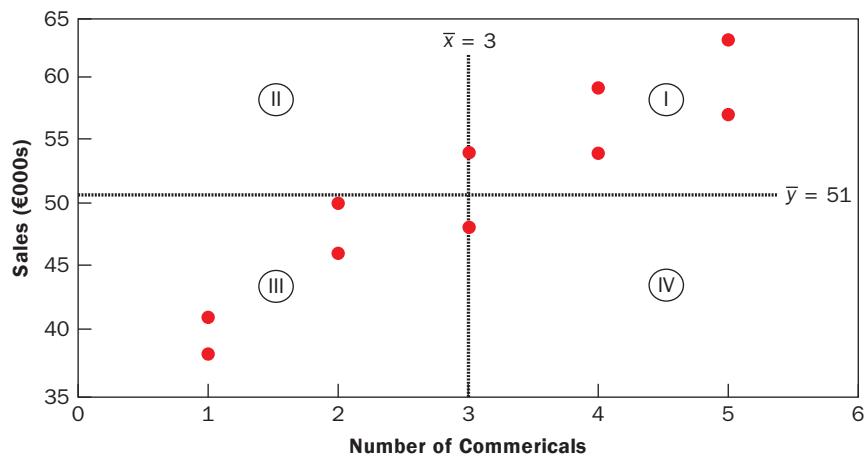
## Interpretation of the covariance

To aid in the interpretation of the sample covariance, consider Figure 3.8. It is the same as the scatter diagram of Figure 3.7 with a vertical dashed line at  $\bar{x} = 3$  and a horizontal dashed line at  $\bar{y} = 51$ . The lines divide the graph into four quadrants. Points in quadrant I correspond to  $x_i$  greater than  $\bar{x}$  and  $y_i$  greater than  $\bar{y}$ . Points in quadrant II correspond to  $x_i$  less than  $\bar{x}$  and  $y_i$  greater than  $\bar{y}$  and so on. Hence, the value of  $(x_i - \bar{x})(y_i - \bar{y})$  is positive for points in quadrants I and III, negative for points in quadrants II and IV.

If the value of  $s_{XY}$  is positive, the points with the greatest influence on  $s_{XY}$  are in quadrants I and III. Hence, a positive value for  $s_{XY}$  indicates a positive linear association between  $X$  and  $Y$ ; that is, as the value of  $X$  increases, the value of  $Y$  increases. If the value of  $s_{XY}$  is negative, however, the points with the greatest influence are in quadrants II and IV. Hence, a negative value for  $s_{XY}$  indicates a negative linear association between  $X$  and  $Y$ ; that is, as the value of  $X$  increases, the value of  $Y$  decreases. Finally, if the points are evenly distributed across all four quadrants, the value  $s_{XY}$  will be close to zero, indicating no linear association between  $X$  and  $Y$ . Figure 3.9 shows the values of  $s_{XY}$  that can be expected with three different types of scatter diagrams.

**FIGURE 3.8**

Partitioned scatter diagram for the hi-fi equipment store



Referring again to Figure 3.8, we see that the scatter diagram for the hi-fi equipment store follows the pattern in the top panel of Figure 3.9. As we expect, the value of the sample covariance indicates a positive linear relationship with  $s_{XY} = 11$ .

From the preceding discussion, it might appear that a large positive value for the covariance indicates a strong positive linear relationship and that a large negative value indicates a strong negative linear relationship. However, one problem with using covariance as a measure of the strength of the linear relationship is that the value of the covariance depends on the units of measurement for  $X$  and  $Y$ . For example, suppose we are interested in the relationship between height  $X$  and weight  $Y$  for individuals. Clearly the strength of the relationship should be the same whether we measure height in metres or centimetres (or feet). Measuring the height in centimetres, however, gives us much larger numerical values for  $(x_i - \bar{x})$  than when we measure height in metres. Hence, with height measured in centimetres, we would obtain a larger value for the numerator  $\sum(x_i - \bar{x})(y_i - \bar{y})$  in equation (3.10) – and hence a larger covariance – when in fact the relationship does not change. The **correlation coefficient** is measure of the relationship between two variables that is not affected by the units of measurement for  $X$  and  $Y$ .

## Correlation coefficient

For sample data, the Pearson product moment correlation coefficient is defined as follows:

### Pearson product moment correlation coefficient: sample data

$$r_{XY} = \frac{s_{XY}}{s_X s_Y} \quad (3.12)$$

where:

$r_{XY}$  = sample correlation coefficient

$s_{XY}$  = sample covariance

$s_X$  = sample standard deviation of  $X$

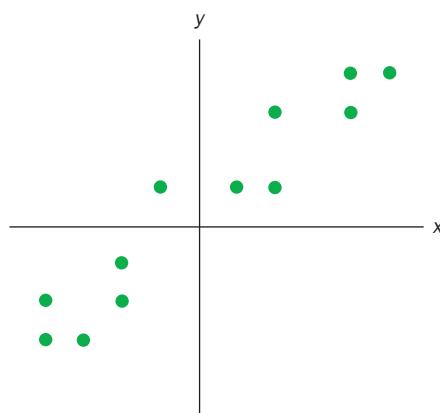
$s_Y$  = sample standard deviation of  $Y$

Equation (3.12) shows that the Pearson product moment correlation coefficient for sample data (commonly referred to more simply as the *sample correlation coefficient*) is computed by dividing the sample covariance by the product of the sample standard deviation of  $X$  and the sample standard deviation of  $Y$ .

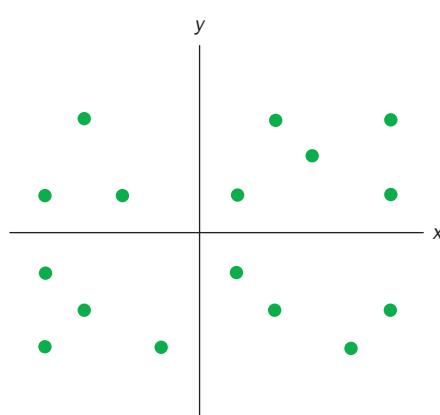
**FIGURE 3.9**

Interpretation of sample covariance

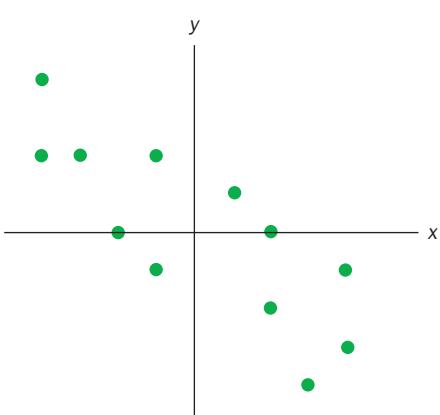
$s_{XY}$  positive:  
 $(X$  and  $Y$  are positively  
 linearly related)



$s_{XY}$  approximately 0:  
 $(X$  and  $Y$  are not  
 linearly related)



$s_{XY}$  negative:  
 $(X$  and  $Y$  are negatively  
 linearly related)



Let us now compute the sample correlation coefficient for the hi-fi equipment store. Using the data in Table 3.5, we can compute the sample standard deviations for the two variables.

$$s_X = \sqrt{\frac{\sum(x_i - \bar{x})^2}{n-1}} = \sqrt{\frac{20}{9}} = 1.49$$

$$s_Y = \sqrt{\frac{\sum(y_i - \bar{y})^2}{n-1}} = \sqrt{\frac{566}{9}} = 7.93$$

Now, because  $s_{XY} = 11$ , the sample correlation coefficient equals:

$$r_{XY} = \frac{s_{XY}}{s_X s_Y} = \frac{11}{(1.49)(7.93)} = +0.93$$

The formula for computing the correlation coefficient for a population, denoted by the Greek letter  $\rho_{XY}$  ( $\rho$  is rho, pronounced 'row', to rhyme with 'go'), follows.

### Pearson product moment correlation coefficient: population data

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \quad (3.13)$$

where:

$\rho_{XY}$  = population correlation coefficient

$\sigma_{XY}$  = population covariance

$\sigma_X$  = population standard deviation for  $X$

$\sigma_Y$  = population standard deviation for  $Y$

The sample correlation coefficient  $r_{XY}$  provides an estimate of the population correlation coefficient  $\rho_{XY}$ .

### Interpretation of the correlation coefficient

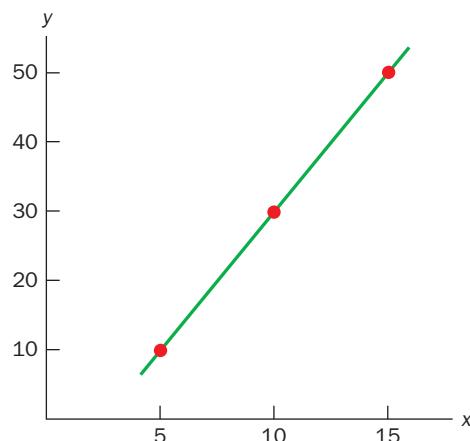
First let us consider a simple example that illustrates the concept of a perfect positive linear relationship. The scatter diagram in Figure 3.10 depicts the relationship between  $X$  and  $Y$  based on the following sample data.

| $x_i$ | $y_i$ |
|-------|-------|
| 5     | 10    |
| 10    | 30    |
| 15    | 50    |

The straight line drawn through the three points shows a perfect linear relationship between  $X$  and  $Y$ . In order to apply equation (3.12) to compute the sample correlation we must first compute  $s_{XY}$ ,  $s_X$  and  $s_Y$ . Some of the computations are shown in Table 3.6.

**FIGURE 3.10**

Scatter diagram depicting a perfect positive linear relationship



**TABLE 3.6** Computations used in calculating the sample correlation coefficient

|               | $x_i$          | $y_i$          | $x_i - \bar{x}$ | $(x_i - \bar{x})^2$ | $y_i - \bar{y}$ | $(y_i - \bar{y})^2$ | $(x_i - \bar{x})(y_i - \bar{y})$ |
|---------------|----------------|----------------|-----------------|---------------------|-----------------|---------------------|----------------------------------|
|               | 5              | 10             | -5              | 25                  | -20             | 400                 | 100                              |
|               | 10             | 30             | 0               | 0                   | 0               | 0                   | 0                                |
|               | 15             | 50             | 5               | 25                  | 20              | 400                 | 100                              |
| <b>Totals</b> | <b>30</b>      | <b>90</b>      | <b>0</b>        | <b>50</b>           | <b>0</b>        | <b>800</b>          | <b>200</b>                       |
|               | $\bar{x} = 10$ | $\bar{y} = 10$ |                 |                     |                 |                     |                                  |

Using the results in Table 3.6, we find:

$$s_{XY} = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{n-1} = \frac{200}{2} = 100$$

$$s_X = \sqrt{\frac{\sum(x_i - \bar{x})^2}{n-1}} = \sqrt{\frac{50}{2}} = 5$$

$$s_Y = \sqrt{\frac{\sum(y_i - \bar{y})^2}{n-1}} = \sqrt{\frac{800}{2}} = 20$$

$$r_{XY} = \frac{s_{XY}}{s_X s_Y} = \frac{100}{5 \times 20} = +1$$

We see that the value of the sample correlation coefficient is +1.

In general, it can be shown that if all the points in a data set fall on a positively sloped straight line, the value of the sample correlation coefficient is +1. That is, a sample correlation coefficient of +1 corresponds to a perfect positive linear relationship between  $X$  and  $Y$ . If the points in the data set fall on a straight line with a negative slope, the value of the sample correlation coefficient is -1. That is, a sample correlation coefficient of -1 corresponds to a perfect negative linear relationship between  $X$  and  $Y$ .

Suppose that a data set indicates a positive linear relationship between  $X$  and  $Y$  but that the relationship is not perfect. The value of  $r_{XY}$  will be less than 1, indicating that the points in the scatter diagram are not all on a straight line. As the points deviate more and more from a perfect positive linear relationship, the value of  $r_{XY}$  becomes closer and closer to zero. A value of  $r_{XY}$  equal to zero indicates no linear relationship between  $X$  and  $Y$ , and values of  $r_{XY}$  near zero indicate a weak linear relationship.

For the data involving the hi-fi equipment store, recall that  $r_{XY} = +0.93$ . Therefore, we conclude that a strong positive linear relationship occurs between the number of commercials and sales. More specifically, an increase in the number of commercials is associated with an increase in sales.

In closing, we note that correlation provides a measure of linear association and not necessarily causation. A high correlation between two variables does not mean that one variable causes the other. For instance, we may find that a restaurant's quality rating and its typical meal price are positively correlated. However, increasing the meal price will not cause quality to increase.

## EXERCISES

### Methods

32. Five observations taken for two variables follow.

|       |    |    |    |    |    |
|-------|----|----|----|----|----|
| $x_i$ | 4  | 6  | 11 | 3  | 16 |
| $y_i$ | 50 | 50 | 40 | 60 | 30 |

- a. Construct a scatter diagram with the  $x_i$  values on the horizontal axis.
- b. What does the scatter diagram developed in part (a) indicate about the relationship between the two variables?
- c. Compute and interpret the sample covariance.
- d. Compute and interpret the sample correlation coefficient.

**33.** Five observations taken for two variables follow.

|       |   |    |    |    |    |
|-------|---|----|----|----|----|
| $x_i$ | 6 | 11 | 15 | 21 | 27 |
| $y_i$ | 6 | 9  | 6  | 17 | 12 |

- a. Construct a scatter diagram for these data.
- b. What does the scatter diagram indicate about a relationship between  $X$  and  $Y$ ?
- c. Compute and interpret the sample covariance.
- d. Compute and interpret the sample correlation coefficient.

### Applications

**34.** Below are return on investment figures (%) and current ratios (current assets/current liabilities) for 15 German companies, for the year 2011 (file G\_Comp on the online platform).

| Company          | Return on investment (%) | Current ratio |
|------------------|--------------------------|---------------|
| Adidas           | 8.15                     | 1.50          |
| BASF             | 14.66                    | 1.64          |
| Bayer            | 6.37                     | 1.50          |
| BMW              | 5.98                     | 1.04          |
| Continental      | 7.15                     | 1.06          |
| Daimler          | 5.70                     | 1.11          |
| Deutsche Bank    | 0.25                     | 0.82          |
| Deutsche Telekom | 2.46                     | 0.65          |
| Fresenius        | 9.10                     | 1.34          |
| Henkel           | 9.16                     | 1.58          |
| Linde            | 5.60                     | 0.89          |
| SAP              | 20.53                    | 1.54          |
| Siemens          | 8.87                     | 1.21          |
| Tui              | 1.53                     | 0.65          |
| Volkswagen       | 7.46                     | 1.05          |



G\_COMP

- a. Construct a scatter diagram with current ratio on the horizontal axis.
- b. Is there any relationship between return on investment and current ratio? Explain.
- c. Compute and interpret the sample covariance.
- d. Compute and interpret the sample correlation coefficient.
- e. What does the sample correlation coefficient tell you about the relationship between return on investment and current ratio?

**35.** Stock markets across the Eurozone tend to have mutual influences on each other. The index levels of the German DAX index and the French CAC 40 index for ten weeks beginning with 4 June 2012 are shown below (file 'DAX\_CAC' on the online platform).

| Date      | DAX     | CAC 40  |
|-----------|---------|---------|
| 04-Jun-12 | 6130.82 | 3051.69 |
| 11-Jun-12 | 6229.41 | 3087.62 |
| 18-Jun-12 | 6263.25 | 3090.90 |



DAX\_CAC

| Date      | DAX     | CAC 40  |
|-----------|---------|---------|
| 25-Jun-12 | 6416.28 | 3196.65 |
| 02-Jul-12 | 6410.11 | 3168.79 |
| 09-Jul-12 | 6557.10 | 3180.81 |
| 16-Jul-12 | 6630.02 | 3193.89 |
| 23-Jul-12 | 6689.40 | 3280.19 |
| 30-Jul-12 | 6865.66 | 3374.19 |
| 06-Aug-12 | 6967.95 | 3453.28 |

- a. Compute the sample correlation coefficient for these data.
- b. Are they poorly correlated, or do they have a close association?

## 3.6 THE WEIGHTED MEAN AND WORKING WITH GROUPED DATA

In Section 3.1, we presented the mean as one of the most important measures of central location. The formula for the mean of a sample with  $n$  observations is re-stated as follows.

$$\bar{x} = \frac{\sum x_i}{n} = \frac{x_1 + x_2 + \dots + x_n}{n} \quad (3.14)$$

In this formula, each  $x_i$  is given equal importance or weight. Although this practice is most common, in some instances the mean is computed by giving each observation a weight that reflects its importance. A mean computed in this manner is referred to as a **weighted mean**. The weighted mean is computed as follows:

### Weighted mean

$$\bar{x} = \frac{\sum w_i x_i}{\sum w_i} \quad (3.15)$$

where:

$x_i$  = value of observation  $i$   
 $w_i$  = weight for observation  $i$

For sample data, equation (3.15) provides the weighted sample mean. For population data,  $\mu$  replaces  $\bar{x}$  and equation (3.15) provides the weighted population mean.

As an example of the need for a weighted mean, consider the following sample of five purchases of a raw material over the past three months. Note that the cost per kilogram has varied from €2.80 to €3.40 and the quantity purchased has varied from 500 to 2750 kilograms.

| Purchase | Cost per kilogram (€) | Number of kilograms |
|----------|-----------------------|---------------------|
| 1        | 3.00                  | 1200                |
| 2        | 3.40                  | 500                 |
| 3        | 2.80                  | 2750                |
| 4        | 2.90                  | 1000                |
| 5        | 3.25                  | 800                 |

Suppose a manager asked for information about the mean cost per kilogram of the raw material. Because the quantities ordered vary, we must use the formula for a weighted mean. The five cost-per-kilogram values are  $x_1 = 3.00$ ,  $x_2 = 3.40$ , ... etc. The weighted mean cost per kilogram is found by weighting each cost by its corresponding quantity. The weights are  $w_1 = 1200$ ,  $w_2 = 500$ , ... etc. Using equation (3.15), the weighted mean is calculated as follows:

$$\bar{x} = \frac{\sum w_i x_i}{\sum w_i} = \frac{1200(3.00) + 500(3.40) + 2750(2.80) + 1000(2.90) + 800(3.25)}{1200 + 500 + 2750 + 1000 + 800}$$

$$= \frac{18500}{6250} = 2.96$$

The weighted mean computation shows that the mean cost per kilogram for the raw material is €2.96. Note that using equation (3.14) rather than the weighted mean formula would have provided misleading results. In this case, the mean of the five cost-per-kilogram values is  $(3.00 + 3.40 + 2.80 + 2.90 + 3.25)/5 = 15.35/5 = €3.07$ , which overstates the actual mean cost per kilogram purchased.

When observations vary in importance, the analyst must choose the weight that best reflects the importance of each observation in the determination of the mean, in the context of the particular application.

## Grouped data

In most cases, measures of location and variability are computed by using the individual data values. Sometimes, however, data are available only in a grouped or frequency distribution form. We show how the weighted mean formula can be used to obtain approximations of the mean, variance and standard deviation for **grouped data**.

Recall from Section 2.2 the frequency distribution of times in days required to complete year-end audits for the small accounting firm of Sanderson and Clifford. It is shown again in the first two columns of Table 3.7 ( $n = 20$  clients). Based on this frequency distribution, what is the sample mean audit time?

To compute the mean using only the grouped data, we treat the midpoint of each class as being representative of the items in the class. Let  $M_i$  denote the midpoint for class  $i$  and let  $f_i$  denote the frequency of class  $i$ . The weighted mean formula (3.15) is then used with the data values denoted as  $M_i$  and the weights given by the frequencies  $f_i$ . In this case, the denominator of equation (3.15) is the sum of the frequencies, which is the sample size  $n$ . That is,  $\sum f_i = n$ .

Hence, the equation for the sample mean for grouped data is as follows in equation (3.16).

### Sample mean for grouped data

$$\bar{x} = \frac{\sum f_i M_i}{n} \quad (3.16)$$

where

- $M_i$  = the midpoint for class  $i$
- $f_i$  = the frequency for class  $i$
- $n$  = the sample size

With the class midpoints,  $M_i$ , halfway between the class limits, the first class of 10–14 in Table 3.7 has a midpoint at  $(10 + 14)/2 = 12$ . The five class midpoints and the weighted mean computation for the audit time data are summarized in Table 3.7. The sample mean audit time is 19 days.

To compute the variance for grouped data, we use a slightly altered version of the formula for the variance given in equation (3.5). The squared deviations of the data about the sample mean  $\bar{x}$  were written  $(x_i - \bar{x})^2$ . However, with grouped data, the values are not known. In this case, we treat the class midpoint,  $M_i$ , as being representative of the  $x_i$  values in the corresponding class.

**TABLE 3.7** Computation of the sample mean audit time for grouped data

| Audit time (days) | Frequency ( $f_i$ ) | Class midpoint ( $M_i$ )  | $f_i M_i$  |
|-------------------|---------------------|---|------------|
| 10–14             | 4                   | 12  | 48         |
| 15–19             | 8                   | 17  | 136        |
| 20–24             | 5                   | 22  | 110        |
| 25–29             | 2                   | 27  | 54         |
| 30–34             | 1                   | 32  | 32         |
| <b>Totals</b>     | <b>20</b>           |   | <b>380</b> |
|                   |                     | Sample mean $\bar{x} = \frac{\sum f_i M_i}{n} = \frac{180}{20} = 19$ days |            |

The squared deviations about the sample mean,  $(x_i - \bar{x})^2$ , are replaced by  $(M_i - \bar{x})^2$ . Then, just as we did with the sample mean calculations for grouped data, we weight each value by the frequency of the class,  $f_i$ . The sum of the squared deviations about the mean for all the data is approximated by  $\sum f_i (M_i - \bar{x})^2$ .

The term  $n - 1$  rather than  $n$  appears in the denominator in order to make the sample variance an unbiased estimator of the population variance. The following formula is used to obtain the sample variance for grouped data.

#### Sample variance for grouped data

$$s^2 = \frac{\sum f_i (M_i - \bar{x})^2}{n-1} \quad (3.17)$$

The calculation of the sample variance for audit times based on the grouped data from Table 3.7 is shown in Table 3.8. The sample variance is 30. The standard deviation for grouped data is simply the square root of the variance for grouped data. For the audit time data, the sample standard deviation is  $s = \sqrt{30} = 5.48$ .

Note that formulae (3.16) and (3.17) are for a sample. Population summary measures are computed similarly in equations (3.18) and (3.19).

**TABLE 3.8** Computation of the sample variance of audit times for grouped data

| Audit time (days) | Class midpoint ( $M_i$ ) | Frequency ( $f_i$ ) | Deviation ( $M_i - \bar{x}$ ) | Squared deviation ( $(M_i - \bar{x})^2$ )  | $f_i (M_i - \bar{x})^2$      |
|-------------------|--------------------------|---------------------|-------------------------------|--|------------------------------|
| 10–14             | 12                       | 4                   | -7                            | 49   | 196                          |
| 15–19             | 17                       | 8                   | -2                            | 4  | 32                           |
| 20–24             | 22                       | 5                   | 3                             | 9  | 45                           |
| 25–29             | 27                       | 2                   | 8                             | 64   | 128                          |
| 30–34             | 32                       | 1                   | 13                            | 169  | 169                          |
| <b>Total</b>      |                          | <b>20</b>           |                               |  | <b>570</b>                   |
|                   |                          |                     |                               | Sample variance $= \frac{\sum f_i (M_i - \bar{x})^2}{n-1} = \frac{570}{19} = 30$ | $\sum f_i (M_i - \bar{x})^2$ |

**Population mean for grouped data**

$$\mu = \frac{\sum f_i M_i}{N} \quad (3.18)$$

**Population variance for grouped data**

$$\sigma^2 = \frac{\sum f_i (M_i - \mu)^2}{N} \quad (3.19)$$

**EXERCISES****Methods**

- 36.** Consider the following data and corresponding weights.

| $x_i$ | Weight |
|-------|--------|
| 3.2   | 6      |
| 2.0   | 3      |
| 2.5   | 2      |
| 5.0   | 8      |

- a. Compute the weighted mean.
  - b. Compute the sample mean of the four data values without weighting. Note the difference in the results provided by the two computations.
- 37.** Consider the sample data in the following frequency distribution.

| Class | Midpoint | Frequency |
|-------|----------|-----------|
| 3–7   | 5        | 4         |
| 8–12  | 10       | 7         |
| 13–17 | 15       | 9         |
| 18–22 | 20       | 5         |

- a. Compute the sample mean.
- b. Compute the sample variance and sample standard deviation.

**Applications**

- 38.** The assessment for a statistics module comprises a multiple-choice test, a data analysis project, an EXCEL test and a written examination. Scores for Jil and Ricardo on the four components are shown below.

| Assessment            | Jil | Ricardo |
|-----------------------|-----|---------|
| Multiple-choice test  | 80% | 48%     |
| Data analysis project | 60% | 78%     |
| EXCEL test            | 62% | 60%     |
| Written examination   | 57% | 53%     |



COMPLETE  
SOLUTIONS

- a. Calculate weighted mean scores (%) for Jil and Ricardo assuming the respective weightings for the four components are 20, 20, 30, 30.
- b. Calculate weighted mean scores (%) for Jil and Ricardo assuming the respective weightings for the four components are 10, 25, 15, 50.
39. A petrol station recorded the following frequency distribution for the number of litres of petrol sold per car in a sample of 680 cars.

| Petrol (litres) | Frequency  |
|-----------------|------------|
| 1–15            | 74         |
| 16–30           | 192        |
| 31–45           | 280        |
| 46–60           | 105        |
| 61–75           | 23         |
| 76–90           | 6          |
| <b>Total</b>    | <b>680</b> |

Compute the mean, variance and standard deviation for these grouped data. If the petrol station expects to serve petrol to about 120 cars on a given day, estimate the total number of litres of petrol that will be sold.



## ONLINE RESOURCES

For the data files, online summary, additional questions and answers, and software section for Chapter 3, go to the online platform.

## SUMMARY

In this chapter we introduced several descriptive statistics that can be used to summarize the location, variability and shape of a data distribution. The measures introduced in this chapter summarize the data in terms of numerical values. When the numerical values obtained are for a sample, they are called sample statistics. When the numerical values obtained are for a population, they are called population parameters. In statistical inference, the sample statistic is referred to as the point estimator of the population parameter. Some of the notation used for sample statistics and population parameters follow.

|                    | Sample statistic | Population parameter |
|--------------------|------------------|----------------------|
| Mean               | $\bar{x}$        | $\mu$                |
| Variance           | $s^2$            | $\sigma^2$           |
| Standard deviation | $s$              | $\sigma$             |
| Covariance         | $s_{XY}$         | $\rho_{XY}$          |
| Correlation        | $r_{XY}$         | $\sigma_{XY}$        |

As measures of central location, we defined the mean, median and mode. Then the concept of percentiles was used to describe other locations in the data set. Next, we presented the range, interquartile range, variance, standard deviation and coefficient of variation as measures of variability or dispersion. Our primary measure of the shape of a data distribution was the skewness. Negative values indicate a data distribution skewed to the left. Positive values indicate a data distribution

skewed to the right. We showed how to calculate z-scores, and indicated how they can be used to identify outlying observations. We then described how the mean and standard deviation could be used, applying Chebyshev's theorem and the empirical rule, to provide more information about the distribution of data and to identify outliers.

In Section 3.4 we showed how to construct a five-number summary and a box plot to provide simultaneous information about the location, variability and shape of the distribution.

Section 3.5 introduced covariance and the correlation coefficient as measures of association between two variables.

In Section 3.6, we showed how to compute a weighted mean and how to calculate a mean, variance and standard deviation for grouped data.

## KEY TERMS

**Box plot**  
**Chebyshev's theorem**  
**Coefficient of variation**  
**Correlation coefficient**  
**Covariance**  
**Empirical rule**  
**Five-number summary**  
**Grouped data**  
**Interquartile range (IQR)**  
**Mean**  
**Median**  
**Mode**

**Outlier**  
**Percentile**  
**Point estimator**  
**Population parameter**  
**Quartiles**  
**Range**  
**Sample statistic**  
**Skewness**  
**Standard deviation**  
**Variance**  
**Weighted mean**  
**z-score**

## KEY FORMULAE

### Sample mean

$$\bar{x} = \frac{\sum x_i}{n} \quad (3.1)$$

### Population mean

$$\mu = \frac{\sum x_i}{N} \quad (3.2)$$

### Interquartile range

$$IQR = Q_3 - Q_1 \quad (3.3)$$

**Population variance**

$$\sigma^2 = \frac{\sum(x_i - \mu)^2}{N} \quad (3.4)$$

**Sample variance**

$$s^2 = \frac{\sum(x_i - \bar{x})^2}{n-1} \quad (3.5)$$

**Standard deviation**

$$\text{Population standard deviation} = \sigma = \sqrt{\sigma^2} \quad (3.6)$$

$$\text{Sample standard deviation} = s = \sqrt{s^2} \quad (3.7)$$

**Coefficient of variation**

$$\left( \frac{\text{Standard deviation}}{\text{Mean}} \times 100 \right) \% \quad (3.8)$$

**z-score**

$$z_i = \frac{x_i - \bar{x}}{s} \quad (3.9)$$

**Sample covariance**

$$s_{XY} = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{n-1} \quad (3.10)$$

**Population covariance**

$$\sigma_{XY} = \frac{\sum(x_i - \mu_X)(y_i - \mu_Y)}{N} \quad (3.11)$$

**Pearson product moment correlation coefficient: sample data**

$$r_{XY} = \frac{s_{XY}}{s_X s_Y} \quad (3.12)$$

**Pearson product moment correlation coefficient: population data**

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \quad (3.13)$$

**Weighted mean**

$$\bar{x} = \frac{\sum w_i x_i}{\sum w_i} \quad (3.15)$$

**Sample mean for grouped data**

$$\bar{x} = \frac{\sum f_i M_i}{n} \quad (3.16)$$

**Sample variance for grouped data**

$$s^2 = \frac{\sum f_i (M_i - \bar{x})^2}{n-1} \quad (3.17)$$

**Population mean for grouped data**

$$\mu = \frac{\sum f_i M_i}{N} \quad (3.18)$$

**Population variance for grouped data**

$$\sigma^2 = \frac{\sum f_i (M_i - \mu)^2}{N} \quad (3.19)$$



## CASE PROBLEM 1



### Company Profiles

The file 'Companies 2012' on the online platform contains a data set compiled mid-year 2012. It comprises figures relating to samples of companies whose shares are traded on the stock exchanges in Germany, France, South Africa and Israel. The data contained in the file are:

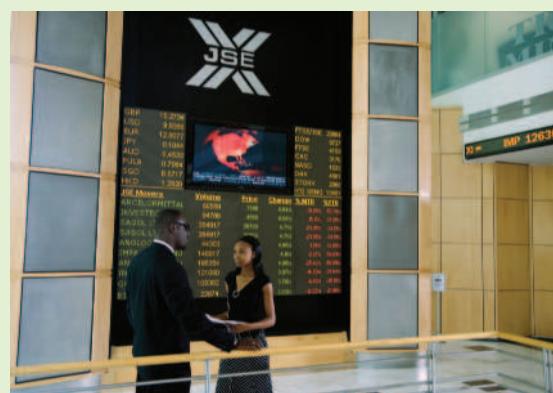
- Name of company.
- Country of stock exchange where the shares are traded.
- Return on shareholders' funds in 2011 (%).
- Profit margin in 2011 (%).
- Return on total assets in 2011 (%).
- Current ratio, 2011.
- Solvency ratio, 2011.
- Price/earnings ratio, 2011.

The first few rows of data are shown below.

| Company name     | Country | Return on share holders' funds, 2011 (%) |        | Profit margin 2011 (%) | Return on total assets 2011 (%) |                     |                      | Current ratio, 2011 | Solvency ratio, 2011 | Price/earnings ratio, 2011 |
|------------------|---------|--|--------|------------------------|---------------------------------|---------------------|----------------------|---------------------|----------------------|----------------------------|
|                  |         | Germany                                  | France |                        | Assets                          | Current ratio, 2011 | Solvency ratio, 2011 |                     |                      |                            |
| Adidas AG        | Germany | 17.40                                    | 6.85   | 8.15                   | 1.50                            | 46.81               | 15.72                |                     |                      |                            |
| Allianz SE       | Germany | 10.79                                    | 6.99   | 0.77                   |                                 | 7.15                | 11.92                |                     |                      |                            |
| Altana AG        | Germany | 3.32                                     | 3.28   | 2.28                   | 2.40                            | 68.77               | 200.13               |                     |                      |                            |
| BASF SE          | Germany | 37.16                                    | 11.90  | 14.66                  | 1.64                            | 39.46               | 7.96                 |                     |                      |                            |
| Bayer AG         | Germany | 17.50                                    | 9.04   | 6.37                   | 1.50                            | 36.41               | 16.47                |                     |                      |                            |
| BWW AG           | Germany | 27.31                                    | 10.69  | 5.98                   | 1.04                            | 21.91               | 6.52                 |                     |                      |                            |
| Commerz bank     | Germany | 2.04                                     | 4.09   | 0.08                   | 0.41                            | 3.75                | 8.92                 |                     |                      |                            |
| Continental AG   | Germany | 26.05                                    | 6.06   | 7.15                   | 1.06                            | 27.44               | 7.71                 |                     |                      |                            |
| Daimler AG       | Germany | 21.32                                    | 7.84   | 5.70                   | 1.11                            | 26.75               | 6.35                 |                     |                      |                            |
| Deutsche Bank AG | Germany | 9.86                                     | 16.16  | 0.25                   | 0.82                            | 2.53                | 6.23                 |                     |                      |                            |

### Managerial report

1. Produce summaries for each of the numerical variables in the file using suitable descriptive statistics. For each variable, identify outliers as well as summarizing the overall characteristics of the data distribution.
2. Investigate whether there are any differences between countries in average profit margin. Similarly, investigate whether there are differences between countries in average current ratio and in average price/earnings ratio.
3. Investigate whether there is any relationship between return on investment and current ratio. Similarly, investigate whether there is any relationship between return on investment and price/earnings ratio.



The Johannesburg Stock Exchange

## CASE PROBLEM 2



### **Chocolate Perfection Website Transactions**

Chocolate Perfection manufactures and sells quality chocolate products in Dubai. Two years ago the company developed a website and began selling its products over the Internet. Website sales have exceeded the company's expectations, and management is now considering strategies to increase sales even further. To learn more about the website customers, a sample of 50 Chocolate Perfection transactions was selected from the previous month's sales. Data showing the day of the week each transaction was made, the type of browser the customer used, the time spent on the website, the number of website pages viewed and the amount



spent by each of the 50 customers are contained in the file named 'Shoppers'. Amount spent is in United Arab Emirates dirham (AED). (One Euro is around five AED.) A portion of the data is shown below.

| Customer | Day | Browser           | Time (min) | Pages Viewed | Amount Spent (AED) |
|----------|-----|-------------------|------------|--------------|--------------------|
| 1        | Mon | Internet Explorer | 12.0       | 4            | 200.09             |
| 2        | Wed | Other             | 19.5       | 6            | 348.28             |
| 3        | Mon | Internet Explorer | 8.5        | 4            | 97.92              |
| 4        | Tue | Firefox           | 11.4       | 2            | 164.16             |
| 5        | Wed | Internet Explorer | 11.3       | 4            | 243.21             |
| 6        | Sat | Firefox           | 10.5       | 6            | 248.83             |
| 7        | Sun | Internet Explorer | 11.4       | 2            | 132.27             |
| 8        | Fri | Firefox           | 4.3        | 6            | 205.37             |
| 9        | Wed | Firefox           | 12.7       | 3            | 260.35             |

Chocolate Perfection would like to use the sample data to determine if online shoppers who spend more time and view more pages also spend more money during their visit to the website. The company would also like to investigate the effect that the day of the week and the type of browser has on sales.

#### **Managerial report**

Use the methods of descriptive statistics to learn about the customers who visit the Chocolate Perfection website. Include the following in your report:

- Graphical and numerical summaries for the length of time the shopper spends on the website, the number of pages viewed and the mean amount spent per transaction. Discuss what you learn about Chocolate Perfection's online shoppers from these numerical summaries.
- Summarize the frequency, the total amount spent and the mean amount spent per transaction for each day of week. What observations can you make about Chocolate Perfection's business based on the day of the week? Discuss.

- Summarize the frequency, the total amount spent and the mean amount spent per transaction for each type of browser. What observations can you make about Chocolate Perfection's business, based on the type of browser? Discuss.
- Construct a scatter diagram and compute the sample correlation coefficient to explore the relationship between the time spent on the website and the amount spent. Use the horizontal axis for the time spent on the website. Discuss.
- Construct a scatter diagram and compute the sample correlation coefficient to explore the relationship between the number of website pages viewed and the amount spent. Use the horizontal axis for the number of website pages viewed. Discuss.
- Construct a scatter diagram and compute the sample correlation coefficient to explore the relationship between the time spent on the website and the number of pages viewed. Use the horizontal axis to represent the number of pages viewed. Discuss.



SHOPPERS



# 4

# Introduction To Probability

## CHAPTER CONTENTS

Statistics in Practice Combating junk email

- 4.1 Experiments, counting rules and assigning probabilities
- 4.2 Events and their probabilities
- 4.3 Some basic relationships of probability
- 4.4 Conditional probability
- 4.5 Bayes' theorem

**LEARNING OBJECTIVES** After reading this chapter and doing the exercises, you should be able to:

- 1 Appreciate the role probability information plays in the decision-making process.
- 2 Understand probability as a numerical measure of the likelihood of occurrence.
- 3 Appreciate the three methods commonly used for assigning probabilities and understand when they should be used.
- 4 Use the laws that are available for computing the probabilities of events.
- 5 Understand how new information can be used to revise initial (prior) probability estimates using Bayes' theorem.

**M**anagers often base their decisions on an analysis of uncertainties such as the following:

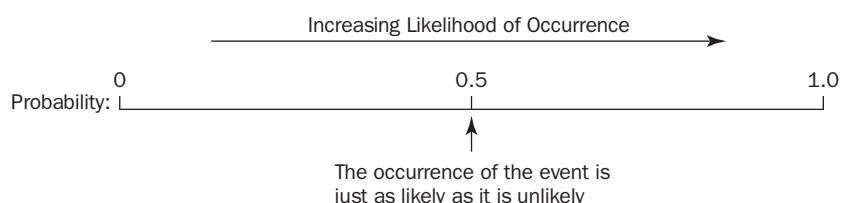
- 1 What are the chances that sales will decrease if we increase prices?
- 2 What is the likelihood a new assembly method will increase productivity?
- 3 How likely is it that the project will be finished on time?
- 4 What is the chance that a new investment will be profitable?

**Probability** is a numerical measure of the likelihood that an event will occur. Thus, probabilities can be used as measures of the degree of uncertainty associated with the four events previously listed. If probabilities are available, we can determine the likelihood of each event occurring.

Probability values are always assigned on a scale from 0 to 1. A probability near zero indicates an event is unlikely to occur; a probability near 1 indicates an event is almost certain to occur. Other probabilities between 0 and 1 represent degrees of likelihood that an event will occur. For example, if we consider the event ‘rain tomorrow’, we understand that when the weather report indicates ‘a near-zero probability of rain’, it means almost no chance of rain. However, if a 0.90 probability of rain is reported, we know that rain is likely to occur. A 0.50 probability indicates that rain is just as likely to occur as not. Figure 4.1 depicts the view of probability as a numerical measure of the likelihood of an event occurring.

**FIGURE 4.1**

Probability as a numerical measure of the likelihood of an event occurring

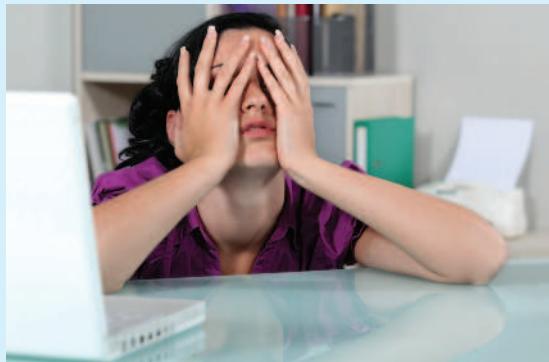


## STATISTICS IN PRACTICE

### Combating junk email

Junk email remains a major Internet scourge. In April 2012 it was estimated 77.2 per cent of electronic mail worldwide was spam (unsolicited commercial email).<sup>1</sup> In 2011, India, Russia and Vietnam accounted for more than 30 per cent of it.<sup>2</sup> In the past, spam has been inextricably linked to the spread of malware on the Web and indeed a significant proportion of spam was botnet-generated. Spam is often associated with porn – the notorious Facebook attack in 2011 being entirely in keeping with this growing phenomenon.<sup>3</sup> Spam is time-consuming to deal with and an increasing brake on further email take-up and usage.

Various initiatives have been undertaken to help counter the problem. However, determining which messages are ‘good’ and which are ‘spam’ is difficult to establish even with the most sophisticated spam filters (spam-busters). One of the earliest and most effective



techniques for dealing with spam is the adaptive Naïve Bayes’ method which exploits the probability relationship

$$P(\text{spam} \mid \text{message}) = \frac{P(\text{message} \mid \text{spam}) \ P(\text{spam})}{P(\text{message})}$$

$$\text{where } P(\text{message}) = P(\text{message} \mid \text{spam}) \ P(\text{spam}) + P(\text{message} \mid \text{good}) \ P(\text{good})$$

Here:

$P(\text{spam})$  is the prior probability a message is spam based on past experience,

$P(\text{message} \mid \text{spam})$  is estimated from a training corpus (a set of messages known to be good or spam) on the (naïve) assumption that every word in the message is independent of every other so that:

$$P(\text{message} \mid \text{spam}) = P(\text{first word} \mid \text{spam})$$

$$P(\text{second word} \mid \text{spam}) \dots$$

$$P(\text{last word} \mid \text{spam})$$

Similarly:

$$P(\text{message} \mid \text{good}) = P(\text{first word} \mid \text{good})$$

$$P(\text{second word} \mid \text{good}) \dots$$

$$P(\text{last word} \mid \text{good})$$

Advantages of Naïve Bayes are its simplicity and ease of implementation. Indeed it is often found to be very effective – even compared to methods based on more complex modelling procedures.

<sup>1</sup>[www.kaspersky.co.uk/about/news/spam/2012/Spam\\_in\\_April\\_2012\\_Junk\\_Mail\\_Gathers\\_Pace\\_in\\_the\\_US](http://www.kaspersky.co.uk/about/news/spam/2012/Spam_in_April_2012_Junk_Mail_Gathers_Pace_in_the_US)

<sup>2</sup>[www.cisco.com/en/US/prod/collateral/vpndevc/security\\_annual\\_report\\_2011.pdf](http://www.cisco.com/en/US/prod/collateral/vpndevc/security_annual_report_2011.pdf)

<sup>3</sup><http://mashable.com/2011/11/15/facebook-spam-porn/>

## 4.1 EXPERIMENTS, COUNTING RULES AND ASSIGNING PROBABILITIES

We define an **experiment** as a process that generates well-defined outcomes. On any single repetition of an experiment, one and only one of the possible experimental outcomes will occur. Several examples of experiments and their associated outcomes follow.

| <i>Experiment</i>            | <i>Experimental outcomes</i> |
|------------------------------|------------------------------|
| Toss a coin                  | Head, tail                   |
| Select a part for inspection | Defective, non-defective     |
| Conduct a sales call         | Purchase, no purchase        |
| Role a die                   | 1, 2, 3, 4, 5, 6             |
| Play a football game         | Win, lose, draw              |

By specifying all possible experimental outcomes, we identify the **sample space** for an experiment.

### Sample space

The sample space for an experiment is the set of all experimental outcomes.

An experimental outcome is also called a **sample point** to identify it as an element of the sample space. Consider the first experiment in the preceding table – tossing a coin. The upward face of the coin – a head or a tail – determines the experimental outcomes (sample points). If we let  $S$  denote the sample space, we can use the following notation to describe the sample space.

$$S = \{\text{Head, Tail}\}$$

The sample space for the second experiment in the table – selecting a part for inspection – can be described as follows

$$S = \{\text{Defective, Non-defective}\}$$

Both of the experiments just described have two experimental outcomes (sample points). However, suppose we consider the fourth experiment listed in the table – rolling a die. The possible experimental outcomes, defined as the number of dots appearing on the upward face of the die, are the six points in the sample space for this experiment.

$$S = \{1, 2, 3, 4, 5, 6\}$$

## Counting rules, combinations and permutations

Being able to identify and count the experimental outcomes is a necessary step in assigning probabilities. We now discuss three useful counting rules.

### Multiple-step experiments

The first counting rule applies to multiple-step experiments. Consider the experiment of tossing two coins. Let the experimental outcomes be defined in terms of the pattern of heads and tails appearing on the upward faces of the two coins. How many experimental outcomes are possible for this experiment?

The experiment of tossing two coins can be thought of as a two-step experiment in which step 1 is the tossing of the first coin and step 2 is the tossing of the second coin. If we use  $H$  to denote a head and  $T$  to denote a tail,  $(H, H)$  indicates the experimental outcome with a head on the first coin and a head on the second coin. Continuing this notation, we can describe the sample space ( $S$ ) for this coin-tossing experiment as follows:

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

Thus, we see that four experimental outcomes are possible. In this case, we can easily list all of the experimental outcomes.

The counting rule for multiple-step experiments makes it possible to determine the number of experimental outcomes without listing them.

### A counting rule for multiple-step experiments

If an experiment can be described as a sequence of  $k$  steps with  $n_1$  possible outcomes on the first step,  $n_2$  possible outcomes on the second step and so on, then the total number of experimental outcomes is given by:

$$n_1 \times n_2 \times \dots \times n_k$$

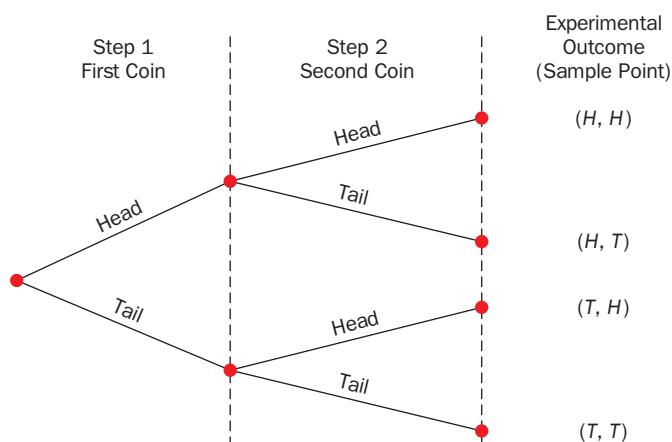
Viewing the experiment of tossing two coins as a sequence of first tossing one coin ( $n_1 = 2$ ) and then tossing the other coin ( $n_2 = 2$ ), we can see from the counting rule that there are  $2 \times 2 = 4$  distinct experimental outcomes. They are  $S = \{(H, H), (H, T), (T, H), (T, T)\}$ . The number of experimental outcomes in an experiment involving tossing six coins is  $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$ .

A **tree diagram** is a graphical representation that helps in visualizing a multiple-step experiment. Figure 4.2 shows a tree diagram for the experiment of tossing two coins. The sequence of steps moves from left to right through the tree. Step 1 corresponds to tossing the first coin, and step 2 corresponds to tossing the second coin. For each step, the two possible outcomes are head or tail. Note that for each possible outcome at step 1 two branches correspond to the two possible outcomes at step 2. Each of the points on the right end of the tree corresponds to an experimental outcome. Each path through the tree from the leftmost node to one of the nodes at the right side of the tree corresponds to a unique sequence of outcomes.

Let us now see how the counting rule for multiple-step experiments can be used in the analysis of a capacity expansion project for Kristof Projects Limited (KPL). KPL is starting a project designed to increase the generating capacity of one of its plants in southern Norway. The project is divided into two sequential stages or steps: stage 1 (design) and stage 2 (construction). Even though each stage will be scheduled and controlled as closely as possible, management cannot predict beforehand the exact time required to complete each stage of the project. An analysis of similar construction projects revealed possible completion times for the design stage of two, three or four months and possible completion times for the construction stage of six, seven or eight months.

**FIGURE 4.2**

Tree diagram for the experiment of tossing two coins



**TABLE 4.1** Experimental outcomes (sample points) for the KPL project

| Completion time (months) |                         |                                      |   |
|--------------------------|-------------------------|--------------------------------------|---|
| Stage 1<br>Design        | Stage 2<br>Construction | Notation for<br>experimental outcome | Total project<br>completion time (months) |
| 2                        | 6                       | (2, 6)                               | 8   |
| 2                        | 7                       | (2, 7)                               | 9   |
| 2                        | 8                       | (2, 8)                               | 10  |
| 3                        | 6                       | (3, 6)                               | 9   |
| 3                        | 7                       | (3, 7)                               | 10  |
| 3                        | 8                       | (3, 8)                               | 11  |
| 4                        | 6                       | (4, 6)                               | 10  |
| 4                        | 7                       | (4, 7)                               | 11  |
| 4                        | 8                       | (4, 8)                               | 12  |

In addition, because of the critical need for additional electrical power, management set a goal of ten months for the completion of the entire project.

Because this project has three possible completion times for the design stage (step 1) and three possible completion times for the construction stage (step 2), the counting rule for multiple-step experiments can be applied here to determine a total of  $3 \times 3 = 9$  experimental outcomes. To describe the experimental outcomes, we use a two-number notation; for instance, (2, 6) indicates that the design stage is completed in two months and the construction stage is completed in six months. This experimental outcome results in a total of  $2 + 6 = 8$  months to complete the entire project. Table 4.1 summarizes the nine experimental outcomes for the KPL problem. The tree diagram in Figure 4.3 shows how the nine outcomes (sample points) occur.

The counting rule and tree diagram help the project manager identify the experimental outcomes and determine the possible project completion times. We see that the project will be completed in 8 to 12 months, with six of the nine experimental outcomes providing the desired completion time of ten months or less. Even though identifying the experimental outcomes may be helpful, we need to consider how probability values can be assigned to the experimental outcomes before making an assessment of the probability that the project will be completed within the desired ten months.

### Combinations

A second useful counting rule allows one to count the number of experimental outcomes when the experiment involves selecting  $n$  objects from a (usually larger) set of  $N$  objects. It is called the counting rule for combinations.

#### Counting rule for combinations

The number of combinations of  $N$  objects taken  $n$  at a time is:

$${}^N C_n = \binom{N}{n} = \frac{N!}{n!(N-n)!}$$

where:

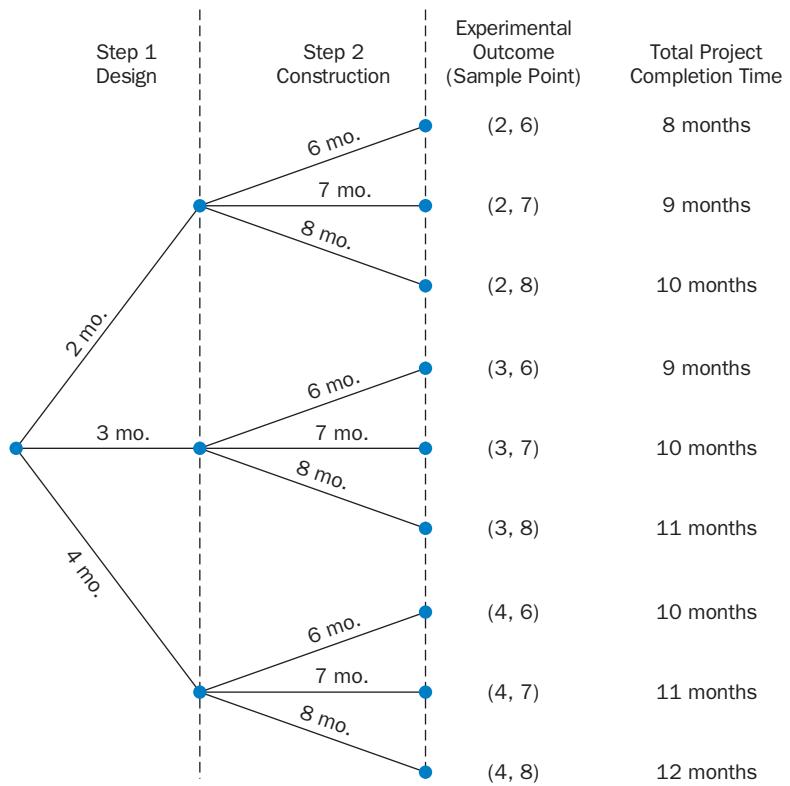
$$\begin{aligned} N! &= N \times (N - 1) \times (N - 2) \times \dots \times (2) \times (1) \\ n! &= n \times (n - 1) \times (n - 2) \times \dots \times (2) \times (1) \end{aligned}$$

and, by definition:

$$0! = 1$$

**FIGURE 4.3**

Tree diagram for the KPL project



The notation ! means *factorial*: for example, 5 factorial is  $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ .

Consider a quality control procedure in which an inspector randomly selects two of five parts to test for defects. In a group of five parts, how many combinations of two parts can be selected? The counting rule in equation (4.1) shows that with  $N = 5$  and  $n = 2$ , we have:

$${}^5C_2 = \binom{5}{2} = \frac{5 \times 4 \times 3 \times 2 \times 1}{(2 \times 1) \times (3 \times 2 \times 1)} = \frac{120}{12} = 10 \quad (4.1)$$

Thus, ten outcomes are possible for the experiment of randomly selecting two parts from a group of five. If we label the five parts as A, B, C, D and E, the ten combinations or experimental outcomes can be identified as AB, AC, AD, AE, BC, BD, BE, CD, CE and DE.

As another example, consider that the Spanish Lotto 6-49 system uses the random selection of six integers from a group of 49 to determine the weekly lottery winner. The counting rule for combinations, equation (4.1), can be used to determine the number of ways six different integers can be selected from a group of 49.

$$\binom{49}{6} = \frac{49!}{6!(49-6)!} = \frac{49!}{6!43!} = \frac{49 \times 48 \times 47 \times 46 \times 45 \times 44}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 13\,983\,816$$

The counting rule for combinations tells us that more than 13 million experimental outcomes are possible in the lottery drawing. An individual who buys a lottery ticket has one chance in 13 983 816 of winning.

### Permutations

A third counting rule that is sometimes useful is the counting rule for permutations. It allows one to compute the number of experimental outcomes when  $n$  objects are to be selected from a set of  $N$  objects where the order of selection is important. The same  $n$  objects selected in a different order is considered a different experimental outcome.

### Counting rule for permutations

The number of permutations of  $N$  objects taken at  $n$  is given by:

$${}^N P_n = n! \binom{N}{n} = \frac{N!}{(N-n)!}$$

The counting rule for permutations closely relates to the one for combinations; however, an experiment results in more permutations than combinations for the same number of objects because every selection of  $n$  objects can be ordered in  $n!$  different ways.

As an example, consider again the quality control process in which an inspector selects two of five parts to inspect for defects. How many permutations may be selected? The counting rule in equation (4.2) shows that with  $N = 5$  and  $n = 2$ , we have:

$${}^5 P_2 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{120}{6} = 20 \quad (4.2)$$

Thus, 20 outcomes are possible for the experiment of randomly selecting two parts from a group of five when the order of selection must be taken into account. If we label the parts A, B, C, D and E, the 20 permutations are AB, BA, AC, CA, AD, DA, AE, EA, BC, CB, BD, DB, BE, EB, CD, DC, CE, EC, DE and ED.

## Assigning probabilities

Now let us see how probabilities can be assigned to experimental outcomes. The three approaches most frequently used are the classical, relative frequency and subjective methods. Regardless of the method used, two **basic requirements for assigning probabilities** must be met.

### Basic requirements for assigning probabilities

1. The probability assigned to each experimental outcome must be between 0 and 1, inclusively. If we let  $E_i$  denote the  $i$ th experimental outcome and  $P(E_i)$  its probability, then this requirement can be written as:

$$0 \leq P(E_i) \leq 1 \text{ for all } i \quad (4.3)$$

2. The sum of the probabilities for all the experimental outcomes must equal 1.0. For  $n$  experimental outcomes, this requirement can be written as:

$$P(E_1) + P(E_2) + \dots + P(E_n) = 1 \quad (4.4)$$

The **classical method** of assigning probabilities is appropriate when all the experimental outcomes are equally likely. If  $n$  experimental outcomes are possible, a probability of  $1/n$  is assigned to each experimental outcome. When using this approach, the two basic requirements for assigning probabilities are automatically satisfied.

For example, consider the experiment of tossing a fair coin: the two experimental outcomes – head and tail – are equally likely. Because one of the two equally likely outcomes is a head, the probability of observing a head is  $1/2$  or 0.50. Similarly, the probability of observing a tail is also  $1/2$  or 0.50.

As another example, consider the experiment of rolling a die. It would seem reasonable to conclude that the six possible outcomes are equally likely, and hence each outcome is assigned a probability of  $1/6$ . If  $P(1)$  denotes the probability that one dot appears on the upward face of the die, then  $P(1) = 1/6$ . Similarly,  $P(2) = 1/6$ ,  $P(3) = 1/6$ ,  $P(4) = 1/6$ ,  $P(5) = 1/6$  and  $P(6) = 1/6$ . Note that these probabilities

satisfy the two basic requirements of equations (4.3) and (4.4) because each of the probabilities is greater than or equal to zero and they sum to 1.0.

The **relative frequency method** of assigning probabilities is appropriate when data are available to estimate the proportion of the time the experimental outcome will occur if the experiment is repeated a large number of times. As an example, consider a study of waiting times in the X-ray department for a local hospital. A clerk recorded the number of patients waiting for service at 9:00 a.m. on 20 successive days, and obtained the following results.

| Number waiting | Number of days outcome<br>occurred |
|----------------|------------------------------------|
| 0              | 2                                  |
| 1              | 5                                  |
| 2              | 6                                  |
| 3              | 4                                  |
| 4              | 3                                  |
| Total = 20     |                                    |

These data show that on two of the 20 days, zero patients were waiting for service; on five of the days, one patient was waiting for service and so on. Using the relative frequency method, we would assign a probability of  $2/20 = 0.10$  to the experimental outcome of zero patients waiting for service,  $5/20 = 0.25$  to the experimental outcome of one patient waiting,  $6/20 = 0.30$  to two patients waiting,  $4/20 = 0.20$  to three patients waiting and  $3/20 = 0.15$  to four patients waiting. As with the classical method, using the relative frequency method automatically satisfies the two basic requirements of equations (4.3) and (4.4).

The **subjective method** of assigning probabilities is most appropriate when one cannot realistically assume that the experimental outcomes are equally likely and when little relevant data are available. When the subjective method is used to assign probabilities to the experimental outcomes, we may use any information available, such as our experience or intuition. After considering all available information, a probability value that expresses our *degree of belief* (on a scale from 0 to 1) that the experimental outcome will occur, is specified. Because subjective probability expresses a person's degree of belief, it is personal. Using the subjective method, different people can be expected to assign different probabilities to the same experimental outcome.

The subjective method requires extra care to ensure that the two basic requirements of equations (4.3) and (4.4) are satisfied. Regardless of a person's degree of belief, the probability value assigned to each experimental outcome must be between 0 and 1, inclusive, and the sum of all the probabilities for the experimental outcomes must equal 1.0.

Consider the case in which Tomas and Margit Elsbernd make an offer to purchase a house. Two outcomes are possible:

$$E_1 = \text{their offer is accepted}$$

$$E_2 = \text{their offer is rejected}$$

Margit believes that the probability their offer will be accepted is 0.8; thus, Margit would set  $P(E_1) = 0.8$  and  $P(E_2) = 0.2$ . Tomas, however, believes that the probability that their offer will be accepted is 0.6; hence, Tomas would set  $P(E_1) = 0.6$  and  $P(E_2) = 0.4$ . Note that Tomas' probability estimate for  $E_1$  reflects a greater pessimism that their offer will be accepted.

Both Margit and Tomas assigned probabilities that satisfy the two basic requirements. The fact that their probability estimates are different emphasizes the personal nature of the subjective method.

Even in business situations where either the classical or the relative frequency approach can be applied, managers may want to provide subjective probability estimates. In such cases, the best probability estimates often are obtained by combining the estimates from the classical or relative frequency approach with subjective probability estimates.

## Probabilities for the KPL project

To perform further analysis on the KPL project, we must develop probabilities for each of the nine experimental outcomes listed in Table 4.1. On the basis of experience and judgement, management concluded that the experimental outcomes were not equally likely. Hence, the classical method of assigning probabilities could not be used. Management then decided to conduct a study of the completion times for similar projects undertaken by KPL over the past three years. The results of a study of 40 similar projects are summarized in Table 4.2.

After reviewing the results of the study, management decided to employ the relative frequency method of assigning probabilities. Management could have provided subjective probability estimates, but felt that the current project was quite similar to the 40 previous projects. Thus, the relative frequency method was judged best.

In using the data in Table 4.2 to compute probabilities, we note that outcome (2, 6) – stage 1 completed in two months and stage 2 completed in six months – occurred six times in the 40 projects. We can use the relative frequency method to assign a probability of  $6/40 = 0.15$  to this outcome. Similarly, outcome (2, 7) also occurred in six of the 40 projects, providing a  $6/40 = 0.15$  probability. Continuing in this manner, we obtain the probability assignments for the sample points of the KPL project shown in Table 4.3.

**TABLE 4.2** Completion results for 40 KPL projects

| Completion times (months) |                      | Sample point | Number of past projects having these completion times |
|---------------------------|----------------------|--------------|---|
| Stage 1 Design            | Stage 2 Construction |              |   |
| 2                         | 6                    | (2, 6)       | 6   |
| 2                         | 7                    | (2, 7)       | 6   |
| 2                         | 8                    | (2, 8)       | 2   |
| 3                         | 9                    | (3, 6)       | 4   |
| 3                         | 7                    | (3, 7)       | 8   |
| 3                         | 8                    | (3, 8)       | 2   |
| 4                         | 6                    | (4, 6)       | 2   |
| 4                         | 7                    | (4, 7)       | 4   |
| 4                         | 8                    | (4, 8)       | 6   |
|                           |                      |              | <b>Total = 40</b>                                     |

**TABLE 4.3** Probability assignments for the KPL project based on the relative frequency method

| Sample point | Project completion time | Probability of sample point |
|--------------|-------------------------|-----------------------------|
| (2, 6)       | 8 months                | $P(2, 6) = 6/40 = 0.15$     |
| (2, 7)       | 9 months                | $P(2, 7) = 6/40 = 0.15$     |
| (2, 8)       | 10 months               | $P(2, 8) = 2/40 = 0.05$     |
| (3, 6)       | 9 months                | $P(3, 6) = 4/40 = 0.10$     |
| (3, 7)       | 10 months               | $P(3, 7) = 8/40 = 0.20$     |
| (3, 8)       | 11 months               | $P(3, 8) = 2/40 = 0.05$     |
| (4, 6)       | 10 months               | $P(4, 6) = 2/40 = 0.05$     |
| (4, 7)       | 11 months               | $P(4, 7) = 4/40 = 0.10$     |
| (4, 8)       | 12 months               | $P(4, 8) = 6/40 = 0.15$     |
|              |                         | <b>Total</b> <b>1.00</b>    |

Note that  $P(2, 6)$  represents the probability of the sample point  $(2, 6)$ ,  $P(2, 7)$  represents the probability of the sample point  $(2, 7)$  and so on.

## EXERCISES

### Methods

1. An experiment has three steps with three outcomes possible for the first step, two outcomes possible for the second step and four outcomes possible for the third step. How many experimental outcomes exist for the entire experiment?
2. How many ways can three items be selected from a group of six items? Use the letters A, B, C, D, E and F to identify the items, and list each of the different combinations of three items.
3. How many permutations of three items can be selected from a group of six? Use the letters A, B, C, D, E and F to identify the items, and list each of the permutations of items B, D and F.
4. Consider the experiment of tossing a coin three times.
  - a. Develop a tree diagram for the experiment.
  - b. List the experimental outcomes.
  - c. What is the probability for each experimental outcome?
5. Suppose an experiment has five equally likely outcomes:  $E_1, E_2, E_3, E_4, E_5$ . Assign probabilities to each outcome and show that the requirements in equations (4.3) and (4.4) are satisfied. What method did you use?
6. An experiment with three outcomes has been repeated 50 times, and it was learned that  $E_1$  occurred 20 times,  $E_2$  occurred 13 times and  $E_3$  occurred 17 times. Assign probabilities to the outcomes. What method did you use?
7. A decision-maker subjectively assigned the following probabilities to the four outcomes of an experiment:  $P(E_1) = 0.10$ ,  $P(E_2) = 0.15$ ,  $P(E_3) = 0.40$  and  $P(E_4) = 0.20$ . Are these probability assignments valid? Explain.
8. Applications for zoning changes in a large metropolitan city go through a two-step process: a review by the planning commission and a final decision by the city council. At step 1 the planning commission reviews the zoning change request and makes a positive or negative recommendation concerning the change. At step 2 the city council reviews the planning commission's recommendation and then votes to approve or to disapprove the zoning change. Suppose the developer of an apartment complex submits an application for a zoning change. Consider the application process as an experiment.
  - a. How many sample points are there for this experiment? List the sample points.
  - b. Construct a tree diagram for the experiment.
9. A total of 11 Management students, four International Management and American Business Studies (IMABS) and eight International Management and French Studies (IMF) students have volunteered to take part in an inter-university tournament.
  - a. How many different ways can a team consisting of eight Management students, two IMABS and five IMF students be selected?
  - b. If after the team has been selected, one Management, one IMABS and two IMF students are found to be suffering from glandular fever and are unable to play, what is the probability that the team will not have to be changed?



COMPLETE  
SOLUTIONS



COMPLETE  
SOLUTIONS



COMPLETE  
SOLUTIONS



- 10.** A company that franchises coffee houses conducted taste tests for a new coffee product. Four blends were prepared, then randomly chosen individuals were asked to taste the blends and state which one they liked best. Results of the taste test for 100 individuals are given.

| Blend | Number choosing |
|-------|-----------------|
| 1     | 20              |
| 2     | 30              |
| 3     | 35              |
| 4     | 15              |

- a. Define the experiment being conducted. How many times was it repeated?
  - b. Prior to conducting the experiment, it is reasonable to assume preferences for the four blends are equal. What probabilities would you assign to the experimental outcomes prior to conducting the taste test? What method did you use?
  - c. After conducting the taste test, what probabilities would you assign to the experimental outcomes? What method did you use?
- 11.** A company that manufactures toothpaste is studying five different package designs. Assuming that one design is just as likely to be selected by a consumer as any other design, what selection probability would you assign to each of the package designs? In an actual experiment, 100 consumers were asked to pick the design they preferred. The following data were obtained. Do the data confirm the belief that one design is just as likely to be selected as another? Explain.

| Design times | Number of preferred |
|--------------|---------------------|
| 1            | 5                   |
| 2            | 15                  |
| 3            | 30                  |
| 4            | 40                  |
| 5            | 10                  |

## 4.2 EVENTS AND THEIR PROBABILITIES

In the introduction to this chapter we used the term *event* much as it would be used in everyday language. Then, in Section 4.1 we introduced the concept of an experiment and its associated experimental outcomes or sample points. Sample points and events provide the foundation for the study of probability. We must now introduce the formal definition of an **event** as it relates to sample points. Doing so will provide the basis for determining the probability of an event.

### Event

An event is a collection of sample points.

For example, let us return to the KPL project and assume that the project manager is interested in the event that the entire project can be completed in ten months or less. Referring to Table 4.3, we see that

six sample points – (2, 6), (2, 7), (2, 8), (3, 6), (3, 7) and (4, 6) – provide a project completion time of ten months or less. Let  $C$  denote the event that the project is completed in ten months or less; we write

$$C = \{(2, 6), (2, 7), (2, 8), (3, 6), (3, 7), (4, 6)\}$$

Event  $C$  is said to occur if *any one* of these six sample points appears as the experimental outcome.

Other events that might be of interest to KPL management include the following:

$L$  = The event that the project is completed in *less* than ten months

$M$  = The event that the project is completed in *more* than ten months

Using the information in Table 4.3, we see that these events consist of the following sample points:

$$L = \{(2, 6), (2, 7), (3, 6)\}$$

$$M = \{(3, 8), (4, 7), (4, 8)\}$$

A variety of additional events can be defined for the KPL project, but in each case the event must be identified as a collection of sample points for the experiment.

Given the probabilities of the sample points shown in Table 4.3, we can use the following definition to compute the probability of any event that KPL management might want to consider.

### Probability of an event

The probability of any event is equal to the sum of the probabilities of the sample points for the event.

Using this definition, we calculate the probability of a particular event by adding the probabilities of the sample points (experimental outcomes) that make up the event. We can now compute the probability that the project will take ten months or less to complete. Because this event is given by  $C = \{(2, 6), (2, 7), (2, 8), (3, 6), (3, 7), (4, 6)\}$ , the probability of event  $C$ , denoted  $P(C)$ , is given by:

$$\begin{aligned} P(C) &= P(2, 6) + P(2, 7) + P(2, 8) + P(3, 6) + P(3, 7) + P(4, 6) \\ &= 0.15 + 0.15 + 0.05 + 0.10 + 0.20 + 0.05 = 0.70 \end{aligned}$$

Similarly, because the event that the project is completed in less than ten months is given by  $L = \{(2, 6), (2, 7), (3, 6)\}$ , the probability of this event is given by:

$$\begin{aligned} P(L) &= P(2, 6) + P(2, 7) + P(3, 6) \\ &= 0.15 + 0.15 + 0.10 = 0.40 \end{aligned}$$

Finally, for the event that the project is completed in more than ten months, we have  $M = \{(3, 8), (4, 7), (4, 8)\}$  and thus:

$$\begin{aligned} P(M) &= P(3, 8) + P(4, 7) + P(4, 8) \\ &= 0.05 + 0.10 + 0.15 = 0.30 \end{aligned}$$

Using these probability results, we can now tell KPL management that there is a 0.70 probability that the project will be completed in ten months or less, a 0.40 probability that the project will be completed in less than ten months, and a 0.30 probability that the project will be completed in more than ten months. This procedure of computing event probabilities can be repeated for any event of interest to the KPL management.

Any time that we can identify all the sample points of an experiment and assign probabilities to each, we can compute the probability of an event using the definition. However, in many experiments the large number of sample points makes the identification of the sample points, as well as the determination of their associated probabilities, extremely cumbersome, if not impossible. In the remaining sections of this chapter, we present some basic probability relationships that can be used to compute the probability of an event without knowledge of all the sample point probabilities.

## EXERCISES

### Methods

- 12.** An experiment has four equally likely outcomes:  $E_1$ ,  $E_2$ ,  $E_3$  and  $E_4$ .
- What is the probability that  $E_2$  occurs?
  - What is the probability that any two of the outcomes occur (e.g.  $E_1$  or  $E_3$ )?
  - What is the probability that any three of the outcomes occur (e.g.  $E_1$  or  $E_2$  or  $E_4$ )?
- 13.** Consider the experiment of selecting a playing card from a deck of 52 playing cards. Each card corresponds to a sample point with a  $1/52$  probability.
- List the sample points in the event an ace is selected.
  - List the sample points in the event a club is selected.
  - List the sample points in the event a face card (jack, queen or king) is selected.
  - Find the probabilities associated with each of the events in parts (a), (b) and (c).
- 14.** Consider the experiment of rolling a pair of dice. Suppose that we are interested in the sum of the face values showing on the dice.
- How many sample points are possible? (*Hint:* Use the counting rule for multiple-step experiments.)
  - List the sample points.
  - What is the probability of obtaining a value of 7?
  - What is the probability of obtaining a value of 9 or greater?
  - Because each roll has six possible even values (2, 4, 6, 8, 10 and 12) and only five possible odd values (3, 5, 7, 9 and 11), the dice should show even values more often than odd values. Do you agree with this statement? Explain.
  - What method did you use to assign the probabilities requested?



COMPLETE  
SOLUTIONS

### Applications

- 15.** Refer to the KPL sample points and sample point probabilities in Tables 4.2 and 4.3.
- The design stage (stage 1) will run over budget if it takes four months to complete. List the sample points in the event the design stage is over budget.
  - What is the probability that the design stage is over budget?
  - The construction stage (stage 2) will run over budget if it takes eight months to complete.  
List the sample points in the event the construction stage is over budget.
  - What is the probability that the construction stage is over budget?
  - What is the probability that both stages are over budget?
- 16.** Suppose that a manager of a large apartment complex provides the following subjective probability estimates about the number of vacancies that will exist next month.

| Vacancies | Probability |
|-----------|-------------|
| 0         | 0.10        |
| 1         | 0.15        |
| 2         | 0.30        |
| 3         | 0.20        |
| 4         | 0.15        |
| 5         | 0.10        |



COMPLETE  
SOLUTIONS

Provide the probability of each of the following events.

- No vacancies.
- At least four vacancies.
- Two or fewer vacancies.

- 17.** When three marksmen take part in a shooting contest, their chances of hitting the target are  $1/2$ ,  $1/3$  and  $1/4$  respectively. If all three marksmen fire at it simultaneously
- What is the chance that one and only one bullet will hit the target?
  - What is the chance that two marksmen will hit the target (and therefore one will not)?
  - What is the chance that all three marksmen will hit the target?

## 4.3 SOME BASIC RELATIONSHIPS OF PROBABILITY

### Complement of an event

Given an event  $A$ , the **complement of  $A$**  is defined to be the event consisting of all sample points that are *not* in  $A$ . The complement of  $A$  is denoted by  $\bar{A}$ . Figure 4.4 is a diagram, known as a **Venn diagram**, which illustrates the concept of a complement. The rectangular area represents the sample space for the experiment and as such contains all possible sample points. The circle represents event  $A$  and contains only the sample points that belong to  $A$ . The shaded region of the rectangle contains all sample points not in event  $A$ , and is by definition the complement of  $A$ .

In any probability application, either event  $A$  or its complement  $\bar{A}$  must occur. Therefore, we have:

$$P(A) + P(\bar{A}) = 1$$

Solving for  $P(A)$ , we obtain the following result.

#### Computing probability using the complement

$$P(A) = 1 - P(\bar{A}) \quad (4.5)$$

Equation (4.5) shows that the probability of an event  $A$  can be computed easily if the probability of its complement,  $P(\bar{A})$ , is known.

As an example, consider the case of a sales manager who, after reviewing sales reports, states that 80 per cent of new customer contacts result in no sale. By allowing  $A$  to denote the event of a sale and  $\bar{A}$  to denote the event of no sale, the manager is stating that  $P(\bar{A}) = 0.80$ . Using equation (4.5), we see that:

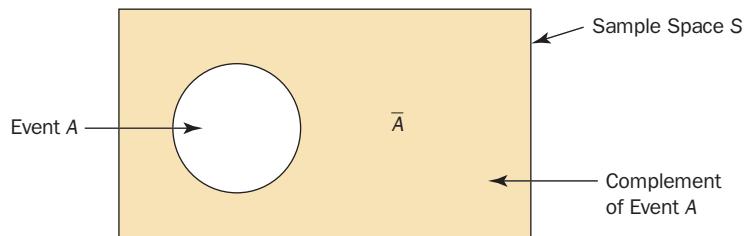
$$P(A) = 1 - P(\bar{A}) = 1 - 0.80 = 0.20$$

We can conclude that a new customer contact has a 0.20 probability of resulting in a sale.

In another example, a purchasing agent states a 0.90 probability that a supplier will send a shipment that is free of defective parts. Using the complement, we can conclude that there is a  $1 - 0.90 = 0.10$  probability that the shipment will contain defective parts.

**FIGURE 4.4**

Complement of event  $A$  is shaded



## Addition law

The addition law is helpful when we are interested in knowing the probability that at least one of two events occurs. That is, with events  $A$  and  $B$  we are interested in knowing the probability that event  $A$  or event  $B$  or both occur.

Before we present the addition law, we need to discuss two concepts related to the combination of events: the *union* of events and the *intersection* of events. Given two events  $A$  and  $B$ , the **union of  $A$  and  $B$**  is defined as follows.

### Union of two events

The *union* of  $A$  and  $B$  is the event containing *all* sample points belonging to  $A$  or  $B$  or both. The union is denoted by  $A \cup B$ .

The Venn diagram in Figure 4.5 depicts the union of events  $A$  and  $B$ . Note that the two circles contain all the sample points in event  $A$  as well as all the sample points in event  $B$ .

The fact that the circles overlap indicates that some sample points are contained in both  $A$  and  $B$ . The definition of the **intersection of  $A$  and  $B$**  follows.

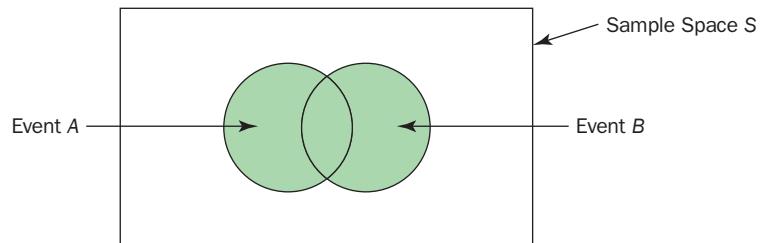
### Intersection of two events

Given two events  $A$  and  $B$ , the *intersection* of  $A$  and  $B$  is the event containing the sample points belonging to *both*  $A$  and  $B$ . The intersection is denoted by  $A \cap B$ .

The Venn diagram depicting the intersection of events  $A$  and  $B$  is shown in Figure 4.6. The area where the two circles overlap is the intersection; it contains the sample points that are in both  $A$  and  $B$ .

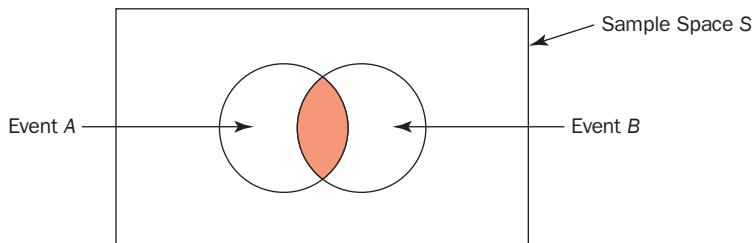
**FIGURE 4.5**

Union of events  $A$  and  $B$  is shaded



**FIGURE 4.6**

Intersection of events  $A$  and  $B$  is shaded



The **addition law** provides a way to compute the probability that event  $A$  or event  $B$  or both occur. In other words, the addition law is used to compute the probability of the union of two events. The addition law is written as follows in equation (4.6).

### Addition law

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad (4.6)$$

To understand the addition law intuitively, note that the first two terms in the addition law,  $P(A) + P(B)$ , account for all the sample points in  $A \cup B$ . However, because the sample points in the intersection  $A \cap B$  are in both  $A$  and  $B$ , when we compute  $P(A) + P(B)$ , we are in effect counting each of the sample points in  $A \cup B$  twice. We correct for this over-counting by subtracting  $P(A \cap B)$ .

As an example of an application of the addition law, consider the case of a small assembly plant with 50 employees. Each worker is expected to complete work assignments on time and in such a way that the assembled product will pass a final inspection. On occasion, some of the workers fail to meet the performance standards by completing work late or assembling a defective product. At the end of a performance evaluation period, the production manager found that five of the 50 workers completed work late, six of the 50 workers assembled a defective product and two of the 50 workers both completed work late *and* assembled a defective product.

Let:

$L$  = the event that the work is completed

$D$  = the event that the assembled product is defective

The relative frequency information leads to the following probabilities:

$$P(L) = \frac{5}{50} = 0.10$$

$$P(D) = \frac{6}{50} = 0.12$$

$$P(L \cap D) = \frac{2}{50} = 0.04$$

After reviewing the performance data, the production manager decided to assign a poor performance rating to any employee whose work was either late or defective; thus the event of interest is  $L \cup D$ . What is the probability that the production manager assigned an employee a poor performance rating?

Using equation (4.6), we have:

$$\begin{aligned} P(L \cup D) &= P(L) + P(D) - P(L \cap D) \\ &= 0.10 + 0.12 - 0.04 = 0.18 \end{aligned}$$

This calculation tells us that there is a 0.18 probability that a randomly selected employee received a poor performance rating.

As another example of the addition law, consider a recent study conducted by the personnel manager of a major computer software company. The study showed that 30 per cent of the employees who left the firm within two years did so primarily because they were dissatisfied with their salary, 20 per cent left because they were dissatisfied with their work assignments and 12 per cent of the former employees indicated dissatisfaction with *both* their salary and their work assignments. What is the probability that an employee who leaves within two years does so because of dissatisfaction with salary, dissatisfaction with the work assignment or both?

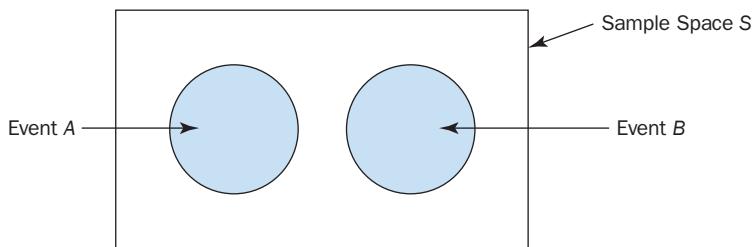
Let:

$S$  = the event that the employee leaves because of salary

$W$  = the event that the employee leaves because of work assignment

**FIGURE 4.7**

Mutually exclusive events



We have  $P(S) = 0.30$ ,  $P(W) = 0.20$  and  $P(S \cap W) = 0.12$ . Using equation (4.6) we have

$$P(S) + P(W) - P(S \cap W) = 0.30 + 0.20 - 0.12 = 0.38$$

We find a 0.38 probability that an employee leaves for salary or work assignment reasons.

Before we conclude our discussion of the addition law, let us consider a special case that arises for **mutually exclusive events**.

### Mutually exclusive events

Two events are said to be mutually exclusive if the events have no sample points in common.

Events  $A$  and  $B$  are mutually exclusive if, when one event occurs, the other cannot occur. Thus, a requirement for  $A$  and  $B$  to be mutually exclusive is that their intersection must contain no sample points. The Venn diagram depicting two mutually exclusive events  $A$  and  $B$  is shown in Figure 4.7. In this case  $P(A \cap B) = 0$  and the addition law can be written as follows.

### Addition law for mutually exclusive events

$$P(A \cup B) = P(A) + P(B)$$

## EXERCISES

### Methods

- 18.** Suppose that we have a sample space with five equally likely experimental outcomes:  $E_1, E_2, E_3, E_4, E_5$ . Let:

$$\begin{aligned}A &= \{E_1, E_2\} \\B &= \{E_3, E_4\} \\C &= \{E_2, E_3, E_5\}\end{aligned}$$

- Find  $P(A)$ ,  $P(B)$  and  $P(C)$ .
- Find  $P(A \cup B)$ . Are  $A$  and  $B$  mutually exclusive?
- Find  $\bar{A}$ ,  $\bar{C}$ ,  $P(\bar{A})$  and  $P(\bar{C})$ .
- Find  $A \cup \bar{B}$  and  $P(A \cup \bar{B})$ .
- Find  $P(B \cup C)$ .

- 19.** Suppose that we have a sample space  $S = \{E_1, E_2, E_3, E_4, E_5, E_6, E_7\}$ , where  $E_1, E_2, \dots, E_7$  denote the sample points. The following probability assignments apply:  $P(E_1) = 0.05$ ,  $P(E_2) = 0.20$ ,  $P(E_3) = 0.20$ ,  $P(E_4) = 0.25$ ,  $P(E_5) = 0.15$ ,  $P(E_6) = 0.10$ , and  $P(E_7) = 0.05$ . Let:

$$\begin{aligned}A &= \{E_1, E_2\} \\B &= \{E_3, E_4\} \\C &= \{E_2, E_3, E_5\}\end{aligned}$$

- a. Find  $P(A)$ ,  $P(B)$ , and  $P(C)$ .
- b. Find  $A \cup B$  and  $P(A \cup B)$ .
- c. Find  $A \cap B$  and  $P(A \cap B)$ .
- d. Are events  $A$  and  $C$  mutually exclusive?
- e. Find  $\bar{B}$  and  $P(\bar{B})$ .

### Applications

- 20.** A survey of magazine subscribers showed that 45.8 per cent rented a car during the past 12 months for business reasons, 54 per cent rented a car during the past 12 months for personal reasons and 30 per cent rented a car during the past 12 months for both business and personal reasons.
- a. What is the probability that a subscriber rented a car during the past 12 months for business or personal reasons?
  - b. What is the probability that a subscriber did not rent a car during the past 12 months for either business or personal reasons?

## 4.4 CONDITIONAL PROBABILITY

Often, the probability of an event is influenced by whether a related event already occurred. Suppose we have an event  $A$  with probability  $P(A)$ . If we obtain new information and learn that a related event, denoted by  $B$ , already occurred, we will want to take advantage of this information by calculating a new probability for event  $A$ . This new probability of event  $A$  is called a **conditional probability** and is written  $P(A | B)$ . We use the notation  $|$  to indicate that we are considering the probability of event  $A$  given the condition that event  $B$  has occurred. Hence, the notation  $P(A | B)$  reads ‘the probability of  $A$  given  $B$ ’.

Consider the situation of the promotion status of male and female police officers of a regional police force in France. The police force consists of 1200 officers: 960 men and 240 women. Over the past two years, 324 officers on the police force received promotions. The specific breakdown of promotions for male and female officers is shown in Table 4.4.

After reviewing the promotion record, a committee of female officers raised a discrimination case on the basis that 288 male officers had received promotions but only 36 female officers had received promotions.

**TABLE 4.4** Promotion status of police officers over the past two years

|              | Men | Women | Total |
|--------------|-----|-------|-------|
| Promoted     | 288 | 36    | 324   |
| Not promoted | 672 | 204   | 876   |
| Totals       | 960 | 240   | 1200  |

The police administration argued that the relatively low number of promotions for female officers was due not to discrimination, but to the fact that relatively few females are members of the police force. Let us show how conditional probability could be used to analyze the discrimination charge.

Let:

$$\begin{aligned}M &= \text{event an officer is a man} \\W &= \text{event an officer is a woman} \\A &= \text{event an officer is promoted} \\\bar{A} &= \text{event an officer is not promoted}\end{aligned}$$

Dividing the data values in Table 4.4 by the total of 1200 officers enables us to summarize the available information with the following probability values.

$$\begin{aligned}P(M \cap A) &= 288/1200 = 0.24 = \text{probability that a randomly selected officer is a man and is promoted} \\P(M \cap \bar{A}) &= 672/1200 = 0.56 = \text{probability that a randomly selected officer is a man and not promoted} \\P(W \cap A) &= 36/1200 = 0.03 = \text{probability that a randomly selected officer is a woman and is promoted} \\P(W \cap \bar{A}) &= 204/1200 = 0.17 = \text{probability that a randomly selected officer is a woman and is not promoted}\end{aligned}$$

Because each of these values gives the probability of the intersection of two events, the probabilities are called **joint probabilities**. Table 4.5 is referred to as a *joint probability table*.

The values in the margins of the joint probability table provide the probabilities of each event separately. That is,  $P(M) = 0.80$ ,  $P(W) = 0.20$ ,  $P(A) = 0.27$  and  $P(\bar{A}) = 0.73$ . These probabilities are referred to as **marginal probabilities** because of their location in the margins of the joint probability table. We note that the marginal probabilities are found by summing the joint probabilities in the corresponding row or column of the joint probability table. For instance, the marginal probability of being promoted is  $P(A) = P(M \cap A) + P(W \cap A) = 0.24 + 0.03 = 0.27$ . From the marginal probabilities, we see that 80 per cent of the force is male, 20 per cent of the force is female, 27 per cent of all officers received promotions and 73 per cent were not promoted.

Consider the probability that an officer is promoted given that the officer is a man. In conditional probability notation, we are attempting to determine  $P(A | M)$ . By definition,  $P(A | M)$  tells us that we are concerned only with the promotion status of the 960 male officers. Because 288 of the 960 male officers received promotions, the probability of being promoted given that the officer is a man is  $288/960 = 0.30$ . In other words, given that an officer is a man, that officer has a 30 per cent chance of receiving a promotion over the past two years.

**TABLE 4.5** Joint probability table for promotions

|                  |      | Men ( $M$ ) | Women ( $W$ ) | Totals |
|------------------|------|-------------|---------------|--------|
| Promoted ( $A$ ) | 0.24 | 0.03        | 0.27          |        |
|                  | 0.56 | 0.17        | 0.73          |        |
| Totals           | 0.80 | 0.20        | 1.00          |        |

Joint probabilities appear in the body of the table.

Marginal probabilities appear in the margins of the table.

This procedure was easy to apply because the values in Table 4.4 show the number of officers in each category. We now want to demonstrate how conditional probabilities such as  $P(A | M)$  can be computed directly from related event probabilities rather than the frequency data of Table 4.4.

We have shown that  $P(A | M) = 288/960 = 0.30$ . Let us now divide both the numerator and denominator of this fraction by 1200, the total number of officers in the study.

$$P(A|M) = \frac{288}{960} = \frac{288/1200}{960/1200} = \frac{0.24}{0.80} = 0.30$$

We now see that the conditional probability  $P(A | M)$  can be computed as 0.24/0.80. Refer to the joint probability table (Table 4.5). Note in particular that 0.24 is the joint probability of A and M; that is,  $P(A \cap M) = 0.24$ . Also note that 0.80 is the marginal probability that a randomly selected officer is a man; that is,  $P(M) = 0.80$ . Thus, the conditional probability  $P(A | M)$  can be computed as the ratio of the joint probability  $P(A \cap M)$  to the marginal probability  $P(M)$ .

$$P(A | M) = \frac{P(A \cap M)}{P(M)} = \frac{0.24}{0.80} = 0.30$$

The fact that conditional probabilities can be computed as the ratio of a joint probability to a marginal probability provides the following general formula (equations (4.7) and (4.8)) for conditional probability calculations for two events A and B.

### Conditional probability

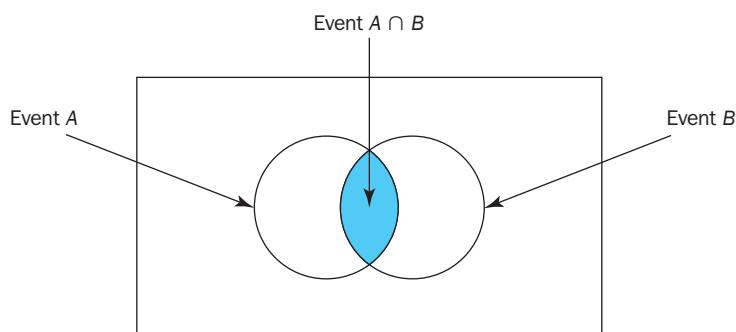
$$P(A | B) = \frac{P(A \cap B)}{P(B)} \quad (4.7)$$

or

$$P(B | A) = \frac{P(A \cap B)}{P(A)} \quad (4.8)$$

The Venn diagram in Figure 4.8 is helpful in obtaining an intuitive understanding of conditional probability. The circle on the right shows that event B has occurred; the portion of the circle that overlaps with event A denotes the event  $(A \cap B)$ . We know that, once event B has occurred, the only way that we can also observe event A is for the event  $(A \cap B)$  to occur. Thus, the ratio  $P(A \cap B)/P(B)$  provides the conditional probability that we will observe event A given that event B has already occurred.

**FIGURE 4.8**  
Conditional probability



Let us return to the issue of discrimination against the female officers. The marginal probability in row 1 of Table 4.5 shows that the probability of promotion of an officer is  $P(A) = 0.27$  (regardless of whether that officer is male or female). However, the critical issue in the discrimination case involves the two conditional probabilities  $P(A | M)$  and  $P(A | W)$ . That is, what is the probability of a promotion *given* that the officer is a man, and what is the probability of a promotion *given* that the officer is a woman? If these two probabilities are equal, a discrimination argument has no basis because the chances of a promotion are the same for male and female officers. However, a difference in the two conditional probabilities will support the position that male and female officers are treated differently in promotion decisions.

We already determined that  $P(A | M) = 0.30$ . Let us now use the probability values in Table 4.5 and the basic relationship of conditional probability in equation (4.7) to compute the probability that an officer is promoted given that the officer is a woman; that is,  $P(A | W)$ . Using equation (4.7), with  $W$  replacing  $B$ , we obtain:

$$P(A|W) = \frac{P(A \cap W)}{P(W)} = \frac{0.03}{0.20} = 0.15$$

What conclusion do you draw? The probability of a promotion given that the officer is a man is 0.30, twice the 0.15 probability of a promotion given that the officer is a woman. Although the use of conditional probability does not in itself prove that discrimination exists in this case, the conditional probability values support the argument presented by the female officers.

## Independent events

In the preceding illustration,  $P(A) = 0.27$ ,  $P(A | M) = 0.30$  and  $P(A | W) = 0.15$ . We see that the probability of a promotion (event  $A$ ) is affected or influenced by whether the officer is a man or a woman. Particularly, because  $P(A | M) \neq P(A)$ , we would say that events  $A$  and  $M$  are dependent events. That is, the probability of event  $A$  (promotion) is altered or affected by knowing that event  $M$  (the officer is a man) exists. Similarly, with  $P(A | W) \neq P(A)$ , we would say that events  $A$  and  $W$  are *dependent events*. However, if the probability of event  $A$  is not changed by the existence of event  $M$  – that is,  $P(A | M) = P(A)$  – we would say that events  $A$  and  $M$  are **independent events**. This situation leads to the following definition of the independence of two events.

### Independent events

Two events  $A$  and  $B$  are independent if

$$P(A | B) = P(A) \tag{4.9}$$

or

$$P(B | A) = P(B) \tag{4.10}$$

## Multiplication law

Whereas the addition law of probability is used to compute the probability of a union of two events, the multiplication law is used to compute the probability of the intersection of two events. The multiplication law is based on the definition of conditional probability. Using equations (4.7) and (4.8) and solving for  $P(A \cap B)$ , we obtain the **multiplication law**, as in equations (4.11) and (4.12).

**Multiplication law**

$$P(A \cap B) = P(A)P(B | A) \quad (4.11)$$

or

$$P(A \cap B) = P(B)P(A | B) \quad (4.12)$$

To illustrate the use of the multiplication law, consider a newspaper circulation department where it is known that 84 per cent of the households in a particular neighbourhood subscribe to the daily edition of the paper. If we let  $D$  denote the event that a household subscribes to the daily edition,  $P(D) = 0.84$ . In addition, it is known that the probability that a household that already holds a daily subscription also subscribes to the Sunday edition (event  $S$ ) is 0.75; that is,  $P(S | D) = 0.75$ .

What is the probability that a household subscribes to both the Sunday and daily editions of the newspaper? Using the multiplication law, we compute the desired  $P(S \cap D)$  as

$$P(S \cap D) = P(D)P(S | D) = 0.84 \times 0.75 = 0.63$$

We now know that 63 per cent of the households subscribe to both the Sunday and daily editions.

Before concluding this section, let us consider the special case of the multiplication law when the events involved are independent. Recall that events  $A$  and  $B$  are independent whenever  $P(A | B) = P(A)$  or  $P(B | A) = P(B)$ . Hence, using equations (4.11) and (4.12) for the special case of independent events, we obtain the following multiplication law (equation (4.13)).

**Multiplication law for independent events**

$$P(A \cap B) = P(A)P(B) \quad (4.13)$$

To compute the probability of the intersection of two independent events, we simply multiply the corresponding probabilities. Note that the multiplication law for independent events provides another way to determine whether  $A$  and  $B$  are independent. That is, if  $P(A \cap B) = P(A)P(B)$ , then  $A$  and  $B$  are independent; if  $P(A \cap B) \neq P(A)P(B)$ , then  $A$  and  $B$  are dependent.

As an application of the multiplication law for independent events, consider the situation of a service station manager who knows from past experience that 80 per cent of the customers use a credit card when they purchase petrol. What is the probability that the next two customers purchasing petrol will each use a credit card? If we let

$A$  = the event that the first customer uses a credit card

$B$  = the event that the second customer uses a credit card

then the event of interest is  $A \cap B$ . Given no other information, we can reasonably assume that  $A$  and  $B$  are independent events. Thus

$$P(A \cap B) = P(A)P(B) = 0.80 \times 0.80 = 0.64$$

To summarize this section, we note that our interest in conditional probability is motivated by the fact that events are often related. In such cases, we say the events are dependent and the conditional probability formulae in equations (4.7) and (4.8) must be used to compute the event probabilities. If two events are not related, they are independent; in this case neither event's probability is affected by whether the other event occurred.

## EXERCISES

### Methods

- 21.** Suppose that we have two events,  $A$  and  $B$ , with  $P(A) = 0.50$ ,  $P(B) = 0.60$  and  $P(A \cap B) = 0.40$ .
- Find  $P(A | B)$ .
  - Find  $P(B | A)$ .
  - Are  $A$  and  $B$  independent? Why or why not?
- 22.** Assume that we have two events,  $A$  and  $B$ , that are mutually exclusive. Assume further that we know  $P(A) = 0.30$  and  $P(B) = 0.40$ .
- What is  $P(A \cap B)$ ?
  - What is  $P(A | B)$ ?
  - A student in statistics argues that the concepts of mutually exclusive events and independent events are really the same, and that if events are mutually exclusive they must be independent. Do you agree with this statement? Use the probability information in this problem to justify your answer.
  - What general conclusion would you make about mutually exclusive and independent events given the results of this problem?

### Applications

- 23.** A Paris nightclub obtains the following data on the age and marital status of 140 customers.

| Age        | Marital status |         |
|------------|----------------|---------|
|            | Single         | Married |
| Under 30   | 77             | 14      |
| 30 or over | 28             | 21      |

- Develop a joint probability table for these data.
  - Use the marginal probabilities to comment on the age of customers attending the club.
  - Use the marginal probabilities to comment on the marital status of customers attending the club.
  - What is the probability of finding a customer who is single and under the age of 30?
  - If a customer is under 30, what is the probability that he or she is single?
  - Is marital status independent of age? Explain, using probabilities.
- 24.** A slot machine in Melbourne has a hold facility. A gambler experiments with this to see if their success rate is higher when they use ‘hold’ compared to when they do not.

The results from 120 plays can be summarized as follows.

|          | Win | Lose |
|----------|-----|------|
| Hold     | 14  | 36   |
| Not hold | 10  | 60   |

What is the probability that the gambler:

- Holds?
- Wins?
- Wins given that they held?
- Held and lost?
- Held given that they won?

25. A sample of convictions and compensation orders issued at a number of Manx courts was followed up to see whether the offender had paid the compensation to the victim. Details by gender of offender are as follows:

| Offender<br>gender | Payment outcome |           |              |
|--------------------|-----------------|-----------|--------------|
|                    | Paid in full    | Part paid | Nothing paid |
| Male               | 754             | 62        | 61           |
| Female             | 157             | 7         | 6            |

- a. What is the probability that no compensation was paid?
  - b. What is the probability that the offender was not male given that compensation was part paid?
26. A purchasing agent in Haifa placed rush orders for a particular raw material with two different suppliers, *A* and *B*. If neither order arrives in four days, the production process must be shut down until at least one of the orders arrives. The probability that supplier *A* can deliver the material in four days is 0.55. The probability that supplier *B* can deliver the material in four days is 0.35.
- a. What is the probability that both suppliers will deliver the material in four days? Because two separate suppliers are involved, we are willing to assume independence.
  - b. What is the probability that at least one supplier will deliver the material in four days?
  - c. What is the probability that the production process will be shut down in four days because of a shortage of raw material (that is, both orders are late)?



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## 4.5 BAYES' THEOREM

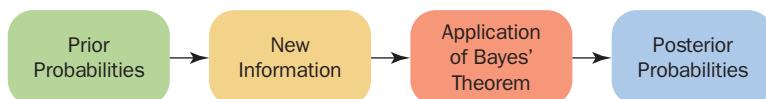
In the discussion of conditional probability, we indicated that revising probabilities when new information is obtained is an important phase of probability analysis. Often, we begin the analysis with initial or **prior probability** estimates for specific events of interest. Then, from sources such as a sample, a special report or a product test, we obtain additional information about the events. Given this new information, we update the prior probability values by calculating revised probabilities, referred to as **posterior probabilities**. **Bayes' theorem** provides a means for making these probability calculations. The steps in this probability revision process are shown in Figure 4.9.

As an application of Bayes' theorem, consider a manufacturing firm that receives shipments of parts from two different suppliers. Let  $A_1$  denote the event that a part is from supplier 1 and  $A_2$  denote the event that a part is from supplier 2. Currently, 65 per cent of the parts purchased by the company are from supplier 1 and the remaining 35 per cent are from supplier 2. Hence, if a part is selected at random, we would assign the prior probabilities  $P(A_1) = 0.65$  and  $P(A_2) = 0.35$ .

The quality of the purchased parts varies with the source of supply. Historical data suggest that the quality ratings of the two suppliers are as shown in Table 4.6.

**FIGURE 4.9**

Probability revision  
using Bayes'  
theorem



**TABLE 4.6** Historical quality levels of two suppliers

|            | Percentage good parts | Percentage bad parts |
|------------|-----------------------|----------------------|
| Supplier 1 | 98                    | 2                    |
| Supplier 2 | 95                    | 5                    |

If we let  $G$  denote the event that a part is good and  $B$  denote the event that a part is bad, the information in Table 4.6 provides the following conditional probability values:

$$\begin{aligned} P(G | A_1) &= 0.98 & P(B | A_1) &= 0.02 \\ P(G | A_2) &= 0.95 & P(B | A_2) &= 0.05 \end{aligned}$$

The tree diagram in Figure 4.10 depicts the process of the firm receiving a part from one of the two suppliers and then discovering that the part is good or bad as a two-step experiment. We see that four experimental outcomes are possible: two correspond to the part being good and two correspond to the part being bad.

Each of the experimental outcomes is the intersection of two events, so we can use the multiplication rule to compute the probabilities. For instance:

$$P(A_1, G) = P(A_1 \cap G) = P(A_1)P(G | A_1) = 0.05$$

The process of computing these joint probabilities can be depicted in what is called a probability tree (see Figure 4.11). From left to right through the tree, the probabilities for each branch at step 1 are prior probabilities and the probabilities for each branch at step 2 are conditional probabilities. To find the probabilities of each experimental outcome, we simply multiply the probabilities on the branches leading to the outcome. Each of these joint probabilities is shown in Figure 4.11 along with the known probabilities for each branch.

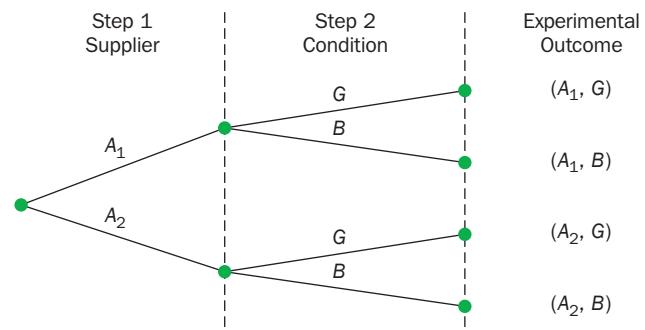
Suppose now that the parts from the two suppliers are used in the firm's manufacturing process and that a machine breaks down because it attempts to process a bad part. Given the information that the part is bad, what is the probability that it came from supplier 1 and what is the probability that it came from supplier 2? With the information in the probability tree (Figure 4.11), Bayes' theorem can be used to answer these questions.

Letting  $B$  denote the event that the part is bad, we are looking for the posterior probabilities  $P(A_1 | B)$  and  $P(A_2 | B)$ . From the law of conditional probability, we know that:

$$P(A_1 | B) = \frac{P(A_1 \cap B)}{P(B)} \quad (4.14)$$

**FIGURE 4.10**

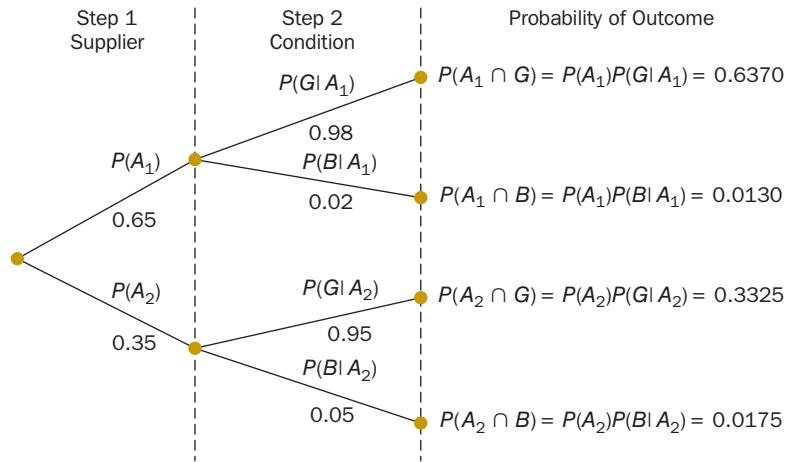
Tree diagram for two-supplier example



Note: Step 1 shows that the part comes from one of two suppliers, and Step 2 shows whether the part is good or bad.

**FIGURE 4.11**

Probability tree for two-supplier example



Referring to the probability tree, we see that:

$$P(A_1 \cap B) = P(A_1)P(B | A_1) \quad (4.15)$$

To find  $P(B)$ , we note that event  $B$  can occur in only two ways:  $(A_1 \cap B)$  and  $(A_2 \cap B)$ . Therefore, we have:

$$\begin{aligned} P(B) &= P(A_1 \cap B) + P(A_2 \cap B) \\ &= P(A_1)P(B | A_1) + P(A_2)P(B | A_2) \end{aligned} \quad (4.16)$$

Substituting from equations (4.15) and (4.16) into equation (4.14) and writing a similar result for  $P(A_2 | B)$ , we obtain Bayes' theorem for the case of two events.

#### Bayes' theorem (two-event case)

$$P(A_1 | B) = \frac{P(A_1)P(B | A_1)}{P(A_1)P(B | A_1) + P(A_2)P(B | A_2)} \quad (4.17)$$

$$P(A_2 | B) = \frac{P(A_2)P(B | A_2)}{P(A_1)P(B | A_1) + P(A_2)P(B | A_2)} \quad (4.18)$$

Using equation (4.17) and the probability values provided in the example, we have

$$\begin{aligned} P(A_1 | B) &= \frac{P(A_1)P(B | A_1)}{P(A_1)P(B | A_1) + P(A_2)P(B | A_2)} \\ &= \frac{0.65 \times 0.02}{0.65 \times 0.02 + 0.35 \times 0.05} = \frac{0.0130}{0.0130 + 0.0175} \\ &= \frac{0.0130}{0.0305} = 0.4262 \end{aligned}$$

In addition, using equation (4.18), we find  $P(A_2 | B)$ .

$$\begin{aligned} P(A_2 | B) &= \frac{0.35 \times 0.05}{0.65 \times 0.02 + 0.35 \times 0.05} \\ &= \frac{0.0175}{0.0130 + 0.0175} = \frac{0.0175}{0.0305} = 0.5738 \end{aligned}$$

Note that in this application we started with a probability of 0.65 that a part selected at random was from supplier 1. However, given information that the part is bad, the probability that the part is from supplier 1 drops to 0.4262. In fact, if the part is bad, it has better than a 50–50 chance that it came from supplier 2; that is,  $P(A_2 | B) = 0.5738$ .

Bayes' theorem is applicable when the events for which we want to compute posterior probabilities are mutually exclusive and their union is the entire sample space.\* For the case of  $n$  mutually exclusive events  $A_1, A_2, \dots, A_n$ , whose union is the entire sample space, Bayes' theorem can be used to compute any posterior probability  $P(A_i | B)$  as shown in equation (4.19).

### Bayes' theorem

$$P(A_i | B) = \frac{P(A_i)P(B | A_i)}{P(A_1)P(B | A_1) + P(A_2)P(B | A_2) + \dots + P(A_n)P(B | A_n)} \quad (4.19)$$

With prior probabilities  $P(A_1), P(A_2), \dots, P(A_n)$  and the appropriate conditional probabilities  $P(B | A_1), P(B | A_2), \dots, P(B | A_n)$ , equation (4.19) can be used to compute the posterior probability of the events  $A_1, A_2, \dots, A_n$ .

### Tabular approach

A tabular approach is helpful in conducting the Bayes' theorem calculations. Such an approach is shown in Table 4.7 for the parts supplier problem. The computations shown there are done in the following steps.

**Step 1** Prepare the following three columns:

Column 1 – The mutually exclusive events  $A_i$  for which posterior probabilities are desired.

Column 2 – The prior probabilities  $P(A_i)$  for the events.

Column 3 – The conditional probabilities  $P(B | A_i)$  of the new information  $B$  given each event.

**Step 2** In column 4, compute the joint probabilities  $P(A_i \cap B)$  for each event and the new information  $B$  by using the multiplication law. These joint probabilities are found by multiplying the prior probabilities in column 2 by the corresponding conditional probabilities in column 3; that is,  $P(A_i \cap B) = P(A_i)P(B | A_i)$ .

**TABLE 4.7** Tabular approach to Bayes' theorem calculations for the two-supplier problem

| (1)<br>Events<br>$A_i$ | (2)<br>Prior<br>probabilities<br>$P(A_i)$ | (3)<br>Conditional<br>probabilities<br>$P(B   A_i)$ | (4)<br>Joint<br>probabilities<br>$P(A_i \cap B)$ | (5)<br>Posterior<br>probabilities<br>$P(A_i   B)$ |
|------------------------|---|---|--|---|
| $A_1$                  | 0.65                                      | 0.02  | 0.0130   | $0.0130/0.0305 = 0.4262$                          |
| $A_2$                  | 0.35                                      | 0.05  | $\underline{0.0175}$                             | $0.0175/0.0305 = \underline{0.5738}$              |
|                        |   |   | $P(B) = 0.0305$                                  | 1.0000  |

\* If the union of events is the entire sample space, the events are said to be collectively exhaustive.

**Step 3** Sum the joint probabilities in column 4. The sum is the probability of the new information,  $P(B)$ . Thus we see in Table 4.7 that there is a 0.0130 probability that the part came from supplier 1 and is bad and a 0.0175 probability that the part came from supplier 2 and is bad. Because these are the only two ways in which a bad part can be obtained, the sum  $0.0130 + 0.0175$  shows an overall probability of 0.0305 of finding a bad part from the combined shipments of the two suppliers.

**Step 4** In column 5, compute the posterior probabilities using the basic relationship of conditional probability.

$$P(A_i | B) = \frac{P(A_i \cap B)}{P(B)}$$

Note that the joint probabilities  $P(A_i \cap B)$  are in column (4) and the probability  $P(B)$  is the sum of column (4).

## EXERCISES

### Methods

- 27.** The prior probabilities for events  $A_1$  and  $A_2$  are  $P(A_1) = 0.40$  and  $P(A_2) = 0.60$ . It is also known that  $P(A_1 \cap A_2) = 0$ . Suppose  $P(B | A_1) = 0.20$  and  $P(B | A_2) = 0.05$ .
- Are  $A_1$  and  $A_2$  mutually exclusive? Explain.
  - Compute  $P(A_1 \cap B)$  and  $P(A_2 \cap B)$ .
  - Compute  $P(B)$ .
  - Apply Bayes' theorem to compute  $P(A_1 | B)$  and  $P(A_2 | B)$ .
- 28.** The prior probabilities for events  $A_1$ ,  $A_2$  and  $A_3$  are  $P(A_1) = 0.20$ ,  $P(A_2) = 0.50$  and  $P(A_3) = 0.30$ . The conditional probabilities of event  $B$  given  $A_1$ ,  $A_2$  and  $A_3$  are  $P(B | A_1) = 0.50$ ,  $P(B | A_2) = 0.40$  and  $P(B | A_3) = 0.30$ .
- Compute  $P(B \cap A_1)$ ,  $P(B \cap A_2)$  and  $P(B \cap A_3)$ .
  - Apply Bayes' theorem, equation (4.19), to compute the posterior probability  $P(A_2 | B)$ .
  - Use the tabular approach to applying Bayes' theorem to compute  $P(A_1 | B)$ ,  $P(A_2 | B)$  and  $P(A_3 | B)$ .

### Applications

- 29.** Records show that for every 100 items produced in a factory during the day shift, two are defective and for every 100 items produced during the night shift, four are defective. What is the prior probability of the bid being successful (that is, prior to the request for additional information)?
- If during a 24-hour period, 2000 items are produced during the day and 800 at night, what is the probability that an item picked at random from the output over 24 hours came from the night shift if it was defective?
- 30.** A company is about to sell to a new client. It knows from past experience that there is a real possibility that the client may default on payment. As a precaution the company checks with a consultant on the likelihood of the client defaulting in this case and is given an estimate of 20 per cent. Sometimes the consultant gets it wrong. Your own experience of the consultant is that he is correct 70 per cent of the time when he predicts that the client will default but that 20 per cent of clients who he believes will not default actually do.
- What is the probability that the new client will not default?



COMPLETE  
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**31.** In 2011, there were 1901 fatalities recorded on Britain's roads, 60 of which were for children (Department of Transport, 2012). Correspondingly, serious injuries totalled 23 122 of which 20 770 were for adults.

- What is the probability of a serious injury given the victim was a child?
- What is the probability that the victim was an adult given a fatality occurred?

**32.** The following cross-tabulation shows industry type and price/earnings (P/E) ratio for 100 companies in the consumer products and banking industries.

| Industry | P/E ratio |       |       |       |       | Total |
|----------|-----------|-------|-------|-------|-------|-------|
|          | 5–9       | 10–14 | 15–19 | 20–24 | 25–29 |       |
| Consumer | 4         | 10    | 18    | 10    | 8     | 50    |
| Banking  | 14        | 14    | 12    | 6     | 4     | 50    |
| Total    | 18        | 24    | 30    | 16    | 12    | 100   |

- What is the probability that a company had a P/E greater than 9 and belonged to the consumer industry?
- What is the probability that a company with a P/E in the range 15–19 belonged to the banking industry?

**33.** A large investment advisory service has a number of analysts who prepare detailed studies of individual companies. On the basis of these studies the analysts make 'buy' or 'sell' recommendations on the companies' shares. The company classes an excellent analyst as one who will be correct 80 per cent of the time, a good analyst as who will be correct 60 per cent of the time and a poor analyst who will be correct 40 per cent of the time.

Two years ago, the advisory service hired Mr Smith who came with considerable experience from the research department of another firm. At the time he was hired it was thought that the probability was 0.90 that he was an excellent analyst, 0.09 that he was a good analyst and 0.01 that he was a poor analyst. In the past two years he has made ten recommendations of which only three have been correct.

Assuming that each recommendation is an independent event what probability would you assign to Mr Smith being:

- An excellent analyst?
- A good analyst?
- A poor analyst?

**34.** An electronic component is produced by four production lines in a manufacturing operation. The components are costly, are quite reliable and are shipped to suppliers in 50-component lots. Because testing is destructive, most buyers of the components test only a small number before deciding to accept or reject lots of incoming components. All four production lines usually only produce 1 per cent defective components which are randomly dispersed in the output.

Unfortunately, production line 1 suffered mechanical difficulty and produced 10 per cent defectives during the month of April. This situation became known to the manufacturer after the components had been shipped. A customer received a lot in April and tested five components.

Two failed. What is the probability that this lot came from production line 1?



## ONLINE RESOURCES

For the data files, additional online summary, questions and answers for Chapter 4, visit the online platform.

## SUMMARY

In this chapter we introduced basic probability concepts and illustrated how probability analysis can be used to provide helpful information for decision-making. We described how probability can be interpreted as a numerical measure of the likelihood that an event will occur and reviewed classical, relative frequency and subjective methods for deriving probabilities. In addition, we saw that the probability of an event can be computed either by summing the probabilities of the experimental outcomes (sample points) comprising the event or by using the relationships established by the addition, conditional probability, and multiplication laws of probability. For cases in which new information is available, we showed how Bayes' theorem can be used to obtain revised or posterior probabilities.

## KEY TERMS

**Addition law**

**Basic requirements for assigning probabilities**

**Bayes' theorem**

**Classical method**

**Complement of A**

**Conditional probability**

**Event**

**Experiment**

**Independent events**

**Intersection of A and B**

**Joint probability**

**Marginal probability**

**Multiplication law**

**Mutually exclusive events**

**Posterior probabilities**

**Prior probabilities**

**Probability**

**Relative frequency method**

**Sample point**

**Sample space**

**Subjective method**

**Tree diagram**

**Union of A and B**

**Venn diagram**

## KEY FORMULAE

### Counting rule for combinations

$${}^N C_n = \binom{N}{n} = \frac{N!}{n!(N-n)!} \quad (4.1)$$

### Counting rule for permutations

$${}^N P_n = n! \binom{N}{n} = \frac{N!}{(N-n)!} \quad (4.2)$$

### Computing probability using the complement

$$P(A) = 1 - P(\bar{A}) \quad (4.5)$$

### Addition law

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad (4.6)$$

**Conditional probability**

$$P(A | B) = \frac{P(A \cap B)}{P(B)} \quad (4.7)$$

$$P(B | A) = \frac{P(A \cap B)}{P(A)} \quad (4.8)$$

**Multiplication law**

$$P(A \cap B) = P(B)P(A | B) \quad (4.11)$$

$$P(A \cap B) = P(A)P(B | A) \quad (4.12)$$

**Multiplication law for independent events**

$$P(A \cap B) = P(A)P(B) \quad (4.13)$$

**Bayes' theorem**

$$P(A_i | B) = \frac{P(A_i)P(B | A_i)}{P(A_1)P(B | A_1) + P(A_2)P(B | A_2) + \dots + P(A_n)P(B | A_n)} \quad (4.19)$$

**CASE PROBLEM****BAC and the Alcohol Test**

In 2005, 6.7 per cent of accidents with injuries in Austria were caused by drunk drivers. The police in Wachau, Austria, a region which is famous for its wine production, is interested in buying equipment for testing drivers' blood alcohol levels. The law in Austria requires that the driver's licence be withdrawn if the driver is found to have more than 0.05 per cent BAC (blood alcohol concentration).

Due to the large number of factors that come into play regarding the consumption and reduction (burn off) rates of different people, there is no blood alcohol calculator that is 100 per cent accurate. Factors include the sex (male/female) of the drinker, differing

metabolism rates, various health issues and the combination of medications being taken, drinking frequency, amount of food in the stomach and small intestine and when it was eaten, elapsed time and many others. The best that can be done is a rough estimate of the BAC level based on known inputs.

There are three types of equipment available with the following conditions:

- 1.** The Saliva Screen is a disposable strip which can be used once – this is the cheapest method.
- 2.** The Alcometer™ is an instrument attached to a container into which the driver breathes, with the Alcometer™ then measuring the BAC concentration through an analysis of the driver's breath. The draw-back to the Alcometer™ is that it can only detect the alcohol level correctly if it is used within two hours of alcohol consumption. It is less effective if used beyond this two-hour period.

| Type          | <i>False positive</i> | <i>False negative</i> |
|---------------|-----------------------|-----------------------|
| Saliva Screen | 0.020                 | 0.03                  |
| Alcometer™    | 0.015                 | 0.02                  |
| Intoximeter   | 0.020                 | 0.01                  |

### Managerial report

Carry out an appropriate probability analysis of this information on behalf of the police and advise them accordingly. (Note that it would be particularly helpful if you could assess the effectiveness of the different equipment types separately for weekdays and

3. The Intoximeter is the most expensive of the three and it works through a blood sample of the driver. The advantage for this is that it can test the BAC up to 12 hours after alcohol consumption. False positive is the situation where the test indicates a high BAC level in a driver that actually does not have such a level. The false negative is when the test indicates a low level of BAC when the driver is actually highly intoxicated.

Police records show that the percentage of drivers (late night) that drink heavily and drive, ranges between 6 per cent on weekdays and 10 per cent on the weekend.



Case problem provided by Dr Ibrahim Wazir, Webster University, Vienna



# 5

# Discrete Probability Distributions

## CHAPTER CONTENTS

Statistics in Practice Improving the performance reliability of combat aircraft

- 5.1 Random variables
- 5.2 Discrete probability distributions
- 5.3 Expected value and variance
- 5.4 Binomial probability distribution
- 5.5 Poisson probability distribution
- 5.6 Hypergeometric probability distribution

**LEARNING OBJECTIVES** After reading this chapter and doing the exercises you should be able to:

- 1 Understand the concepts of a random variable and a probability distribution.
- 2 Distinguish between discrete and continuous random variables.
- 3 Compute and interpret the expected value, variance and standard deviation for a discrete random variable.
- 4 Compute and work with probabilities involving a binomial probability distribution.
- 5 Compute and work with probabilities involving a Poisson probability distribution.
- 6 Know when and how to use the hypergeometric probability distribution.

In this chapter we continue the study of probability by introducing the concepts of random variables and probability distributions. The focus of this chapter is discrete probability distributions. Three special discrete probability distributions – the binomial, Poisson and hypergeometric – are covered.

## 5.1 RANDOM VARIABLES

In Chapter 4 we defined the concept of an experiment and its associated experimental outcomes. A **random variable** provides a means for describing experimental outcomes using numerical values. Random variables must assume numerical values.

**Random variable**

A random variable is a numerical description of the outcome of an experiment.

**STATISTICS IN PRACTICE**

Improving the performance reliability of combat aircraft

**M**odern combat aircraft are expensive to acquire and maintain. In today's post-Cold War world the emphasis is therefore on deploying as few aircraft

would double. Another strategy is to build 'redundancy' into the design. In practice this would involve the aircraft carrying additional engines which would only come into use if one of the operational engines failed. To determine the number of additional engines required, designers have relied on the Poisson distribution. Calculations based on this distribution show that an aircraft with two engines would need at least four redundant engines to achieve a target



as are required and for these to be made to perform as reliably as possible in conflict and peace-keeping situations. Different strategies have been considered by manufacturers for improving the performance reliability of aircraft. One such is to reduce the incidence of faults per flying hour to improve the aircraft's survival time. For example, the Tornado averages 800 faults per 1000 flying hours but if this rate could be halved, the mean operational time between faults

maintenance-free operating period (MFOP) of 150 hours. Given that each engine weighs over a tonne, occupies a space of at least  $2\text{m}^3$  and costs some €3m, clearly this has enormous implications for those wishing to pursue this solution further.

Source: Kumar U D, Knezivc J and Crocker (1999) Maintenance-free operating period –an alternative measure to MTBF and failure rate for specifying reliability. Reliability Engineering & System Safety Vol 64 pp 127–131.

## Discrete random variables

A random variable that may assume either a finite number of values or an infinite sequence of values such as 0, 1, 2, ... is referred to as a **discrete random variable**. For example, consider the experiment of an accountant taking the chartered accountancy (CA) examination.

The examination has four parts. We can define a random variable as  $X$  = the number of parts of the CA examination passed. It is a discrete random variable because it may assume the finite number of values 0, 1, 2, 3 or 4.

As another example of a discrete random variable, consider the experiment of cars arriving at a tollbooth. The random variable of interest is  $X$  = the number of cars arriving during a one-day period. The possible values for  $X$  come from the sequence of integers 0, 1, 2 and so on. Hence,  $X$  is a discrete random variable assuming one of the values in this infinite sequence. Although the outcomes of many experiments can naturally be described by numerical values, others cannot. For example, a survey question might ask an individual to recall the message in a recent television commercial. This experiment would have two possible outcomes: the individual cannot recall the message and the individual can recall the message.

We can still describe these experimental outcomes numerically by defining the discrete random variable  $X$  as follows: let  $X = 0$  if the individual cannot recall the message and  $X = 1$  if the individual can recall the message. The numerical values for this random variable are arbitrary (we could use 5 and 10), but they are acceptable in terms of the definition of a random variable – namely,  $X$  is a random variable because it provides a numerical description of the outcome of the experiment.

Table 5.1 provides some additional examples of discrete random variables. Note that in each example the discrete random variable assumes a finite number of values or an infinite sequence of values such as 0, 1, 2, .... These types of discrete random variables are discussed in detail in this chapter.

## Continuous random variables

A random variable that may assume any numerical value in an interval or collection of intervals is called a **continuous random variable**. Experimental outcomes based on measurement scales such as time, weight, distance and temperature can be described by continuous random variables. For example, consider an experiment of monitoring incoming telephone calls to the claims office of a major insurance company. Suppose the random variable of interest is  $X$  = the time between consecutive incoming calls in minutes. This random variable may assume any value in the interval  $X \geq 0$ . Actually, an infinite number of values are possible for  $X$ , including values such as 1.26 minutes, 2.751 minutes, 4.3333 minutes and so on. As another example, consider a 90-kilometre section of the A8 Autobahn in Germany.

**TABLE 5.1** Examples of discrete random variables

| Experiment                       | Random variable ( $X$ )                | Possible values for the random variable |
|----------------------------------|--|---|
| Contact five customers           | Number of customers who place an order | 0, 1, 2, 3, 4, 5                        |
| Inspect a shipment of 50 radios  | Number of defective radios             | 0, 1, 2, ..., 49, 50                    |
| Operate a restaurant for one day | Number of customers                    | 0, 1, 2, 3, ...                         |
| Sell a car                       | Gender of the customer                 | 0 if male; 1 if female                  |

**TABLE 5.2** Examples of continuous random variables

| Experiment                            | Random variable ( $X$ )   | Possible values for the random variable |
|---------------------------------------|---|---|
| Operate a bank                        | Time between customer arrivals  | $X \geq 0$ in minutes                   |
| Fill a soft drink can<br>(max = 350g) | Number of grams   | $0 \leq X \leq 350$                     |
| Construct a new library               | Percentage of project complete after six months                         | $0 \leq X \leq 100$                     |
| Test a new chemical process           | Temperature when the desired reaction takes place (min 65°C; max 100°C) | $65 \leq X \leq 100$                    |

For an emergency ambulance service located in Stuttgart, we might define the random variable as  $X$  = number of kilometres to the location of the next traffic accident along this section of the A8. In this case,  $X$  would be a continuous random variable assuming any value in the interval  $0 \leq X \leq 90$ . Additional examples of continuous random variables are listed in Table 5.2. Note that each example describes a random variable that may assume any value in an interval of values. Continuous random variables and their probability distributions will be the topic of Chapter 6.

## EXERCISES

### Methods

1. Consider the experiment of tossing a coin twice.
  - a. List the experimental outcomes.
  - b. Define a random variable that represents the number of heads occurring on the two tosses.
  - c. Show what value the random variable would assume for each of the experimental outcomes.
  - d. Is this random variable discrete or continuous?
2. Consider the experiment of a worker assembling a product.
  - a. Define a random variable that represents the time in minutes required to assemble the product.
  - b. What values may the random variable assume?
  - c. Is the random variable discrete or continuous?

### Applications

3. Three students have interviews scheduled for summer employment. In each case the interview results in either an offer for a position or no offer. Experimental outcomes are defined in terms of the results of the three interviews.
  - a. List the experimental outcomes.
  - b. Define a random variable that represents the number of offers made. Is the random variable continuous?
  - c. Show the value of the random variable for each of the experimental outcomes.
4. Suppose we know home mortgage rates for 12 Danish lending institutions. Assume that the random variable of interest is the number of lending institutions in this group that offers a 30-year fixed rate of 1.5 per cent or less. What values may this random variable assume?



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5. To perform a certain type of blood analysis, lab technicians must perform two procedures. The first procedure requires either one or two separate steps, and the second procedure requires either one, two or three steps.
- List the experimental outcomes associated with performing the blood analysis.
  - If the random variable of interest is the total number of steps required to do the complete analysis (both procedures), show what value the random variable will assume for each of the experimental outcomes.
6. Listed is a series of experiments and associated random variables. In each case, identify the values that the random variable can assume and state whether the random variable is discrete or continuous.



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| Experiment  | Random variable ( $X$ )                                  |
|---|--|
| a. Take a 20-question examination.                    | Number of questions answered correctly.                  |
| b. Observe cars arriving at a tollbooth for one hour. | Number of cars arriving at tollbooth.                    |
| c. Audit 50 tax returns.                              | Number of returns containing errors.                     |
| d. Observe an employee's work.                        | Number of non-productive hours in an eight-hour workday. |
| e. Weigh a shipment of goods.                         | Number of kilograms.                                     |

## 5.2 DISCRETE PROBABILITY DISTRIBUTIONS

The **probability distribution** for a random variable describes how probabilities are distributed over the values of the random variable. For a discrete random variable  $X$ , the probability distribution is defined by a **probability function**, denoted by  $p(x) = p(X = x)$  for all possible values,  $x$ . The probability function provides the probability for each value of the random variable. Consider the sales of cars at DiCarlo Motors in Sienna, Italy. Over the past 300 days of operation, sales data show 54 days with no cars sold, 117 days with one car sold, 72 days with two cars sold, 42 days with three cars sold, 12 days with four cars sold and three days with five cars sold. Suppose we consider the experiment of selecting a day of operation at DiCarlo Motors and define the random variable of interest as  $X$  = the number of cars sold during a day. From historical data, we know  $X$  is a discrete random variable that can assume the values 0, 1, 2, 3, 4 or 5. In probability function notation,  $p(0)$  provides the probability of 0 cars sold,  $p(1)$  provides the probability of one car sold and so on. Because historical data show 54 of 300 days with no cars sold, we assign the value  $54/300 = 0.18$  to  $p(0)$ , indicating that the probability of no cars being sold during a day is 0.18. Similarly, because 117 of 300 days had one car sold, we assign the value  $117/300 = 0.39$  to  $p(1)$ , indicating that the probability of exactly one car being sold during a day is 0.39. Continuing in this way for the other values of the random variable, we compute the values for  $p(2)$ ,  $p(3)$ ,  $p(4)$  and  $p(5)$  as shown in Table 5.3, the probability distribution for the number of cars sold during a day at DiCarlo Motors.

A primary advantage of defining a random variable and its probability distribution is that once the probability distribution is known, it is relatively easy to determine the probability of a variety of events that may be of interest to a decision-maker. For example, using the probability distribution for DiCarlo Motors, as shown in Table 5.3, we see that the most probable number of cars sold during a day is one with a probability of  $p(1) = 0.39$ . In addition, there is a  $p(3) + p(4) + p(5) = 0.14 + 0.04 + 0.01 = 0.19$  probability of selling three or more cars during a day. These probabilities, plus others the decision-maker may ask about, provide information that can help the decision-maker understand the process of selling cars at DiCarlo Motors.

In the development of a probability function for any discrete random variable, the following two conditions must be satisfied.

**TABLE 5.3** Probability distribution for the number of cars sold during a day at DiCarlo Motors

| <b>x</b>          | <b>p(x)</b> |
|-------------------|-------------|
| 0                 | 0.18        |
| 1                 | 0.39        |
| 2                 | 0.24        |
| 3                 | 0.14        |
| 4                 | 0.04        |
| 5                 | 0.01        |
| <b>Total 1.00</b> |             |

Required conditions for a discrete probability function are shown in equations (5.1) and (5.2).

$$p(x) \geq 0 \quad (5.1)$$

$$\sum p(x) = 1 \quad (5.2)$$

Table 5.3 shows that the probabilities for the random variable  $X$  satisfy equation (5.1);  $p(x)$  is greater than or equal to 0 for all values of  $x$ . In addition, because the probabilities sum to 1, equation (5.2) is satisfied. Thus, the DiCarlo Motors probability function is a valid discrete probability function.

We can also present probability distributions graphically. In Figure 5.1 the values of the random variable  $X$  for DiCarlo Motors are shown on the horizontal axis and the probability associated with these values is shown on the vertical axis. In addition to tables and graphs, a formula that gives the probability function,  $p(x)$ , for every value of  $X = x$  is often used to describe probability distributions. The simplest example of a discrete probability distribution given by a formula is the **discrete uniform probability distribution**. Its probability function is defined by equation (5.3).

#### Discrete uniform probability function

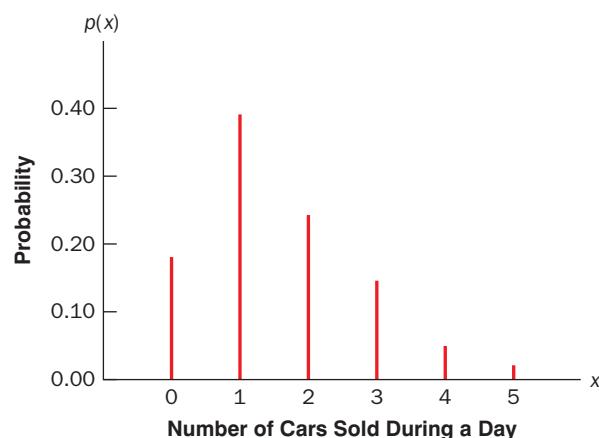
$$p(x) = 1/n \quad (5.3)$$

where

$n$  = the number of values the random variable may assume

**FIGURE 5.1**

Graphical representation of the probability distribution for the number of cars sold during a day at DiCarlo Motors



For example, suppose that for the experiment of rolling a die we define the random variable  $X$  to be the number of dots on the upward face. There are  $n = 6$  possible values for the random variable;  $X = 1, 2, 3, 4, 5, 6$ . Thus, the probability function for this discrete uniform random variable is:

$$p(x) = 1/6 \quad x = 1, 2, 3, 4, 5, 6$$

The possible values of the random variable and the associated probabilities are shown.

| $x$ | $p(x)$ |
|-----|--------|
| 1   | 1/6    |
| 2   | 1/6    |
| 3   | 1/6    |
| 4   | 1/6    |
| 5   | 1/6    |
| 6   | 1/6    |

As another example, consider the random variable  $X$  with the following discrete probability distribution.

| $x$ | $p(x)$ |
|-----|--------|
| 1   | 1/10   |
| 2   | 2/10   |
| 3   | 3/10   |
| 4   | 4/10   |

This probability distribution can be defined by the formula:

$$p(x) = \frac{x}{10} \quad \text{for } x = 1, 2, 3 \text{ or } 4$$

Evaluating  $p(x)$  for a given value of the random variable will provide the associated probability. For example, using the preceding probability function, we see that  $p(2) = 2/10$  provides the probability that the random variable assumes a value of 2. The more widely used discrete probability distributions generally are specified by formulae. Three important cases are the binomial, Poisson and hypergeometric distributions; these are discussed later in the chapter.

## EXERCISES

### Methods

7. The probability distribution for the random variable  $X$  follows.

| $x$ | $p(x)$ |
|-----|--------|
| 20  | 0.20   |
| 25  | 0.15   |
| 30  | 0.25   |
| 35  | 0.40   |

- a. Is this probability distribution valid? Explain.
- b. What is the probability that  $X = 30$ ?
- c. What is the probability that  $X$  is less than or equal to 25?
- d. What is the probability that  $X$  is greater than 30?

## Applications

8. The following data were collected by counting the number of operating rooms in use at a general hospital over a 20-day period. On three of the days only one operating room was used, on five of the days two were used, on eight of the days three were used and on four days all four of the hospital's operating rooms were used.
- Use the relative frequency approach to construct a probability distribution for the number of operating rooms in use on any given day.
  - Draw a graph of the probability distribution.
  - Show that your probability distribution satisfies the required conditions for a valid discrete probability distribution.
9. Table 5.4 summarizes the joint probability distribution for the percentage monthly return for two ordinary shares 1 and 2. In the case of share 1, the per cent return  $X$  has historically been  $-1, 0$  or  $1$ . Correspondingly, for share 2, the per cent return  $Y$  has been  $-2, 0$  or  $2$ .

**TABLE 5.4** Per cent monthly return probabilities for shares 1 and 2

|                |    | share 2 |     |        |
|----------------|----|---------|-----|--------|
|                |    | %       | -2  | Y<br>0 |
| Monthly return |    |         | -2  | 2      |
| share 1 X      | -1 |         | 0.1 | 0.1    |
|                | 0  |         | 0.1 | 0.2    |
|                | 1  |         | 0.0 | 0.1    |

- Determine  $E(Y)$ ,  $E(X)$   $\text{Var}(X)$  and  $\text{Var}(Y)$ .
  - Determine the correlation coefficient between  $X$  and  $Y$ .
  - What do you deduce from b?
10. A technician services mailing machines at companies in the Berne area. Depending on the type of malfunction, the service call can take 1, 2, 3 or 4 hours. The different types of malfunctions occur at about the same frequency.
- Develop a probability distribution for the duration of a service call.
  - Draw a graph of the probability distribution.
  - Show that your probability distribution satisfies the conditions required for a discrete probability function.
  - What is the probability a service call will take three hours?
  - A service call has just come in, but the type of malfunction is unknown. It is 3:00 p.m. and service technicians usually finish work at 5:00 p.m. What is the probability the service technician will have to work overtime to fix the machine today?
11. A college admissions tutor subjectively assessed a probability distribution for  $X$ , the number of entering students, as follows.



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| $x$  | $p(x)$ |
|------|--------|
| 1000 | 0.15   |
| 1100 | 0.20   |
| 1200 | 0.30   |
| 1300 | 0.25   |
| 1400 | 0.10   |

- a. Is this probability distribution valid? Explain.  
 b. What is the probability of 1200 or fewer entering students?
12. A psychologist determined that the number of sessions required to obtain the trust of a new patient is either 1, 2 or 3. Let  $X$  be a random variable indicating the number of sessions required to gain the patient's trust. The following probability function has been proposed.
- $$p(x) = \frac{x}{6} \quad \text{for } x = 1, 2 \text{ or } 3$$
- a. Is this probability function valid? Explain.  
 b. What is the probability that it takes exactly two sessions to gain the patient's trust?  
 c. What is the probability that it takes at least two sessions to gain the patient's trust?
13. The following table is a partial probability distribution for the MRA Company's projected profits ( $X$  = profit in €000s) for the first year of operation (the negative value denotes a loss).

| $x$  | $p(x)$ |
|------|--------|
| -100 | 0.10   |
| 0    | 0.20   |
| 50   | 0.30   |
| 100  | 0.25   |
| 150  | 0.10   |
| 200  |        |

- a. What is the proper value for  $p(200)$ ? What is your interpretation of this value?  
 b. What is the probability that MRA will be profitable?  
 c. What is the probability that MRA will make at least €100 000?

## 5.3 EXPECTED VALUE AND VARIANCE

### Expected value

The **expected value**, or mean, of a random variable is a measure of the central location for the random variable. The formula for the expected value of a discrete random variable  $X$  follows in equation (5.4).

#### Expected value of a discrete random variable

$$E(X) = \mu = \sum x p(x) \quad (5.4)$$

**TABLE 5.5** Calculation of the expected value for the number of cars sold during a day at DiCarlo Motors

| <b>x</b> | <b>p(x)</b> | <b>xp(x)</b>                  |
|----------|-------------|-------------------------------|
| 0        | 0.18        | 0 (0.18) = 0.00               |
| 1        | 0.39        | 1 (0.39) = 0.39               |
| 2        | 0.24        | 2 (0.24) = 0.48               |
| 3        | 0.14        | 3 (0.14) = 0.42               |
| 4        | 0.04        | 4 (0.04) = 0.16               |
| 5        | 0.01        | 5(0.01) = $\frac{0.05}{1.50}$ |

$E(X) = \mu = \Sigma xp(x)$

Both the notations  $E(X)$  and  $\mu$  are used to denote the expected value of a random variable. Equation (5.4) shows that to compute the expected value of a discrete random variable, we must multiply each value of the random variable by the corresponding probability  $p(x)$  and then add the resulting products. Using the DiCarlo Motors car sales example from Section 5.2, we show the calculation of the expected value for the number of cars sold during a day in Table 5.5. The sum of the entries in the  $xp(x)$  column shows that the expected value is 1.50 cars per day. We therefore know that although sales of 0, 1, 2, 3, 4 or 5 cars are possible on any one day, over time DiCarlo can anticipate selling an average of 1.50 cars per day. Assuming 30 days of operation during a month, we can use the expected value of 1.50 to forecast average monthly sales of  $30(1.50) = 45$  cars.

## Variance

Even though the expected value provides the mean value for the random variable, we often need a measure of variability, or dispersion. Just as we used the variance in Chapter 3 to summarize the variability in data, we now use **variance** to summarize the variability in the values of a random variable. The formula for the variance of a discrete random variable follows in equation (5.5).

### Variance of a discrete random variable

$$\text{Var}(X) = \sigma^2 = \Sigma(x - \mu)^2 p(x) \quad (5.5)$$

As equation (5.5) shows, an essential part of the variance formula is the deviation,  $x - \mu$ , which measures how far a particular value of the random variable is from the expected value, or mean,  $\mu$ . In computing the variance of a random variable, the deviations are squared and then weighted by the corresponding value of the probability function. The sum of these weighted squared deviations for all values of the random variable is referred to as the *variance*. The notations  $\text{Var}(X)$  and  $\sigma^2$  are both used to denote the variance of a random variable.

The calculation of the variance for the probability distribution of the number of cars sold during a day at DiCarlo Motors is summarized in Table 5.6. We see that the variance is 1.25. The **standard deviation**,  $\sigma$ , is defined as the positive square root of the variance. Thus, the standard deviation for the number of cars sold during a day is:

$$\sigma = \sqrt{1.25} = 1.118$$

**TABLE 5.6** Calculation of the variance for the number of cars sold during a day at DiCarlo Motors

| $x$ | $x - \mu$          | $(x - \mu)^2$ | $p(x)$ | $(x - \mu)^2 p(x)$                       |
|-----|--------------------|---------------|--------|--|
| 0   | $0 - 1.50 = -1.50$ | 2.25          | 0.18   | $2.25 \times 0.18 = 0.4050$              |
| 1   | $1 - 1.50 = -0.50$ | 0.25          | 0.39   | $0.25 \times 0.39 = 0.0975$              |
| 2   | $2 - 1.50 = 0.50$  | 0.25          | 0.24   | $0.25 \times 0.24 = 0.0600$              |
| 3   | $3 - 1.50 = 1.50$  | 2.25          | 0.14   | $2.25 \times 0.14 = 0.3150$              |
| 4   | $4 - 1.50 = 2.50$  | 6.25          | 0.04   | $6.25 \times 0.04 = 0.2500$              |
| 5   | $5 - 1.50 = 3.50$  | 12.25         | 0.01   | $12.25 \times 0.01 = \underline{0.1225}$ |
|     |                    |               |        | 1.2500                                   |

$\sigma^2 = \sum(x - \mu)^2 p(x)$

The standard deviation is measured in the same units as the random variable ( $\sigma = 1.118$  cars) and therefore is often preferred in describing the variability of a random variable. The variance  $\sigma^2$  is measured in squared units and is thus more difficult to interpret.

## EXERCISES

### Methods

- 14.** The following table provides a probability distribution for the random variable  $X$ .

| $x$ | $p(x)$ |
|-----|--------|
| 3   | 0.25   |
| 6   | 0.50   |
| 9   | 0.25   |

- a. Compute  $E(X)$ , the expected value of  $X$ .  
 b. Compute  $\sigma^2$ , the variance of  $X$ .  
 c. Compute  $\sigma$ , the standard deviation of  $X$ .
- 15.** The following table provides a probability distribution for the random variable  $Y$ .

| $y$ | $p(y)$ |
|-----|--------|
| 2   | 0.20   |
| 4   | 0.30   |
| 7   | 0.40   |
| 8   | 0.10   |

- a. Compute  $E(Y)$ .  
 b. Compute  $\text{Var}(Y)$  and  $\sigma$ .

### Applications

- 16.** Odds in horse race betting are defined as follows: 3/1 (three to one against) means a horse is expected to win once for every three times it loses; 3/2 means two wins out of five races; 4/5 (five to four on) means five wins for every four defeats, etc.

- Translate the above odds into ‘probabilities’ of victory.
- In the 2.45 race at L’Arc de Triomphe the odds for the five runners were:

|                  |      |
|------------------|------|
| Phillipe Bois    | 1/1  |
| Gallante Effor   | 5/2  |
| Satin Noir       | 11/2 |
| Victoire Antheme | 9/1  |
| Comme Rambleur   | 16/1 |

Calculate the ‘probabilities’ and their sum.

- How much would a bookmaker expect to profit in the long run at such odds if it is assumed each horse is backed equally? (Hint: Assume the true probabilities are proportional to the ‘probabilities’ just calculated and consider the payouts corresponding to a notional €1 wager being placed on each horse.)
  - What would the bookmaker’s expected profit have been if horses had been backed in line with the true probabilities?
- 17.** A certain machinist works an eight-hour shift. An efficiency expert wants to assess the value of this machinist where value is defined as value added minus the machinist’s labour cost. The value added for the work the machinist does is €30 per item and the machinist earns €16 per hour. From past records, the machinist’s output per shift is known to have the following probability distribution:

| Output/shift | Probability |
|--------------|-------------|
| 5            | 0.2         |
| 6            | 0.4         |
| 7            | 0.3         |
| 8            | 0.1         |

- What is the expected monetary value of the machinist to the company per shift?
  - What is the corresponding variance value?
- 18.** A company is contracted to finish a €100 000 project by 31 December. If it does not complete on time a penalty of €8000 per month (or part of a month) is incurred. The company estimates that if it continues alone there will be a 40 per cent chance of completing on time and that the project may be one, two, three or four months late with equal probability.

Subcontractors can be hired by the firm at a cost of €18 000. If the subcontractors are hired then the probability that the company completes on time is doubled. If the project is still late it will now be only one or two months late with equal probability.

- Determine the expected profit when

- subcontractors are not used
- subcontractors are used

- Which is the better option for the company?

- 19.** The following probability distributions of job satisfaction scores for a sample of information systems (IS) senior executives and IS middle managers range from a low of 1 (very dissatisfied) to a high of 5 (very satisfied).

| Job satisfaction score | Probability          |                    |
|------------------------|----------------------|--------------------|
|                        | IS senior executives | IS middle managers |
| 1                      | 0.05                 | 0.04               |
| 2                      | 0.09                 | 0.10               |
| 3                      | 0.03                 | 0.12               |
| 4                      | 0.42                 | 0.46               |
| 5                      | 0.41                 | 0.28               |

- a. What is the expected value of the job satisfaction score for senior executives?
  - b. What is the expected value of the job satisfaction score for middle managers?
  - c. Compute the variance of job satisfaction scores for executives and middle managers.
  - d. Compute the standard deviation of job satisfaction scores for both probability distributions.
  - e. Compare the overall job satisfaction of senior executives and middle managers.
- 20.** The demand for a product of Cobh Industries varies greatly from month to month. The probability distribution in the following table, based on the past two years of data, shows the company's monthly demand.

| Unit demand | Probability |
|-------------|-------------|
| 300         | 0.20        |
| 400         | 0.30        |
| 500         | 0.35        |
| 600         | 0.15        |

- a. If the company bases monthly orders on the expected value of the monthly demand, what should Cobh's monthly order quantity be for this product?
- b. Assume that each unit demanded generates €70 in revenue and that each unit ordered costs €50. How much will the company gain or lose in a month if it places an order based on your answer to part (a) and the actual demand for the item is 300 units?



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## 5.4 BINOMIAL PROBABILITY DISTRIBUTION

The binomial probability distribution is a discrete probability distribution that provides many applications. It is associated with a multiple-step experiment that we call the binomial experiment.

### A binomial experiment

A **binomial experiment** exhibits the following four properties.

#### Properties of a binomial experiment

1. The experiment consists of a sequence of  $n$  identical trials.
2. Two outcomes are possible on each trial. We refer to one outcome as a *success* and the other outcome as a *failure*.
3. The probability of a success, denoted by  $\pi$ , does not change from trial to trial. Consequently, the probability of a failure, denoted by  $1 - \pi$ , does not change from trial to trial.
4. The trials are independent.

**FIGURE 5.2**  
One possible sequence of successes and failures for an eight-day trial binomial experiment

Property 1: The experiment consists of  $n = 8$  identical trials.

Property 2: Each trial results in either success (S) or failure (F).

|          |   |   |   |   |   |   |   |   |   |
|----------|---|---|---|---|---|---|---|---|---|
| Trials   | → | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| Outcomes | → | S | F | F | S | S | F | S | S |

If properties 2, 3 and 4 are present, we say the trials are generated by a Bernoulli process. If, in addition, property 1 is present, we say we have a binomial experiment. Figure 5.2 depicts one possible sequence of successes and failures for a binomial experiment involving eight trials.

In a binomial experiment, our interest is in the *number of successes occurring in the  $n$  trials*. If we let  $X$  denote the number of successes occurring in the  $n$  trials, we see that  $X$  can assume the values of 0, 1, 2, 3, ...,  $n$ . Because the number of values is finite,  $X$  is a *discrete* random variable. The probability distribution associated with this random variable is called the **binomial probability distribution**. For example, consider the experiment of tossing a coin five times and on each toss observing whether the coin lands with a head or a tail on its upward face. Suppose we want to count the number of heads appearing over the five tosses. Does this experiment show the properties of a binomial experiment? What is the random variable of interest? Note that:

- 1 The experiment consists of five identical trials; each trial involves the tossing of one coin.
- 2 Two outcomes are possible for each trial: a head or a tail. We can designate head a success and tail a failure.
- 3 The probability of a head and the probability of a tail are the same for each trial, with  $\pi = 0.5$  and  $1 - \pi = 0.5$ .
- 4 The trials or tosses are independent because the outcome on any one trial is not affected by what happens on other trials or tosses.

Thus, the properties of a binomial experiment are satisfied. The random variable of interest is  $X =$  the number of heads appearing in the five trials. In this case,  $X$  can assume the values of 0, 1, 2, 3, 4 or 5.

As another example, consider an insurance salesperson who visits ten randomly selected families. The outcome associated with each visit is classified as a success if the family purchases an insurance policy and a failure if the family does not. From past experience, the salesperson knows the probability that a randomly selected family will purchase an insurance policy is 0.10. Checking the properties of a binomial experiment, we observe that:

- 1 The experiment consists of ten identical trials; each trial involves contacting one family.
- 2 Two outcomes are possible on each trial: the family purchases a policy (success) or the family does not purchase a policy (failure).
- 3 The probabilities of a purchase and a non-purchase are assumed to be the same for each sales call, with  $\pi = 0.10$  and  $1 - \pi = 0.90$ .
- 4 The trials are independent because the families are randomly selected.

Because the four assumptions are satisfied, this example is a binomial experiment. The random variable of interest is the number of sales obtained in contacting the ten families. In this case,  $X$  can assume the values of 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10.

Property 3 of the binomial experiment is called the *stationarity assumption* and is sometimes confused with property 4, independence of trials. To see how they differ, consider again the case of the salesperson calling on families to sell insurance policies. If, as the day wore on, the salesperson got tired and lost enthusiasm, the probability of success (selling a policy) might drop to 0.05, for example, by the tenth call.

In such a case, property 3 (stationarity) would not be satisfied, and we would not have a binomial experiment. Even if property 4 held – that is, the purchase decisions of each family were made independently – it would not be a binomial experiment if property 3 was not satisfied.

In applications involving binomial experiments, a special mathematical formula, called the *binomial probability function*, can be used to compute the probability of  $x$  successes in the  $n$  trials. We will show in the context of an illustrative problem how the formula can be developed.

### Marrine clothing store problem

Let us consider the purchase decisions of the next three customers who enter the Marrine Clothing Store. On the basis of past experience, the store manager estimates the probability that any one customer will make a purchase is 0.30. What is the probability that two of the next three customers will make a purchase?

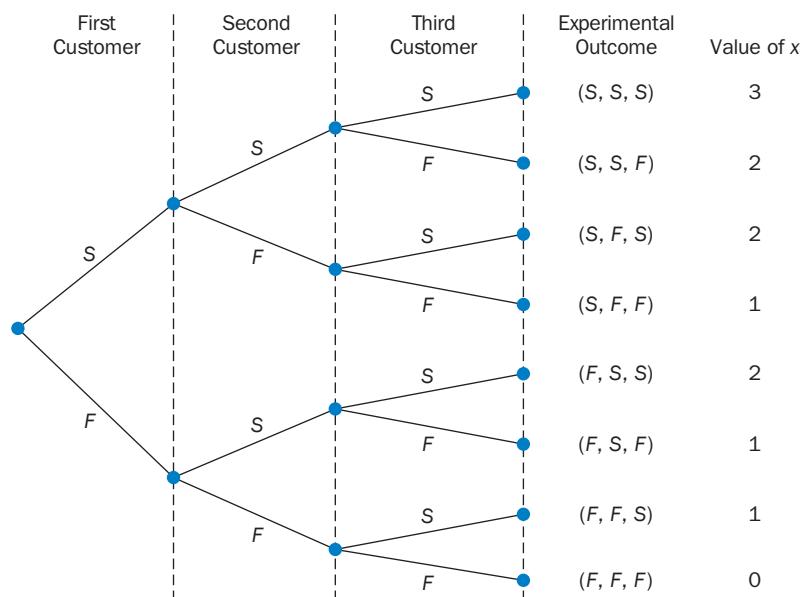
Using a tree diagram (Figure 5.3), we see that the experiment of observing the three customers each making a purchase decision has eight possible outcomes. Using  $S$  to denote success (a purchase) and  $F$  to denote failure (no purchase), we are interested in experimental outcomes involving two successes in the three trials (purchase decisions). Next, let us verify that the experiment involving the sequence of three purchase decisions can be viewed as a binomial experiment. Checking the four requirements for a binomial experiment, we note that:

- 1** The experiment can be described as a sequence of three identical trials, one trial for each of the three customers who will enter the store.
- 2** Two outcomes – the customer makes a purchase (success) or the customer does not make a purchase (failure) – are possible for each trial.
- 3** The probability that the customer will make a purchase (0.30) or will not make a purchase (0.70) is assumed to be the same for all customers.
- 4** The purchase decision of each customer is independent of the decisions of the other customers.

Hence, the properties of a binomial experiment are present.

**FIGURE 5.3**

Tree diagram for the Marrine Clothing Store problem



$S$  = Purchase

$F$  = No purchase

$x$  = Number of customers making a purchase

The number of experimental outcomes resulting in exactly  $x$  successes in  $n$  trials can be computed using the following formula.\*

**Number of experimental outcomes providing exactly  $x$  successes in  $n$  trials**

$$\binom{n}{x} = \frac{n!}{x!(n-x)!} \quad (5.6)$$

where

$$n! = n \times (n-1) \times (n-2) \times \dots \times (2) \times (1)$$

and, by definition,

$$0! = 1$$

Now let us return to the Marrine Clothing Store experiment involving three customer purchase decisions. Equation (5.6) can be used to determine the number of experimental outcomes involving two purchases; that is, the number of ways of obtaining  $X = 2$  successes in the  $n = 3$  trials. From equation (5.6) we have:

$$\binom{n}{x} = \binom{3}{2} = \frac{3!}{2! \times (3-2)!} = \frac{3 \times 2 \times 1}{(2 \times 1) \times (1)} = \frac{6}{2} = 3$$

Equation (5.6) shows that three of the experimental outcomes yield two successes. From Figure 5.3 we see these three outcomes are denoted by  $(S, S, F)$ ,  $(S, F, S)$  and  $(F, S, S)$ . Using equation (5.6) to determine how many experimental outcomes have three successes (purchases) in the three trials, we obtain:

$$\binom{n}{x} = \binom{3}{3} = \frac{3!}{3! \times (3-2)!} = \frac{3 \times 2 \times 1}{(3 \times 2 \times 1) \times (1)} = \frac{6}{6} = 1$$

From Figure 5.3 we see that the one experimental outcome with three successes is identified by  $(S, S, S)$ .

We know that equation (5.6) can be used to determine the number of experimental outcomes that result in  $X$  successes. If we are to determine the probability of  $x$  successes in  $n$  trials, however, we must also know the probability associated with each of these experimental outcomes. Because the trials of a binomial experiment are independent, we can simply multiply the probabilities associated with each trial outcome to find the probability of a particular sequence of successes and failures.

The probability of purchases by the first two customers and no purchase by the third customer, denoted  $(S, S, F)$ , is given by:

$$\pi\pi(1-\pi)$$

With a 0.30 probability of a purchase on any one trial, the probability of a purchase on the first two trials and no purchase on the third is given by:

$$0.30 \times 0.30 \times 0.70 = 0.30^2 \times 0.70 = 0.063$$

---

\*This formula, introduced in Chapter 4, determines the number of combinations of  $n$  objects selected  $x$  at a time. For the binomial experiment, this combinatorial formula provides the number of experimental outcomes (sequences of  $n$  trials) resulting in  $x$  successes.

Two other experimental outcomes also result in two successes and one failure. The probabilities for all three experimental outcomes involving two successes follow.

| Trial outcomes |              |              | Experimental outcome | Probability of experimental outcome                              |
|----------------|--------------|--------------|----------------------|--|
| 1st customer   | 2nd customer | 3rd customer |                      |  |
| Purchase       | Purchase     | No purchase  | (S, S, F)            | $\pi\pi(1 - \pi) = \pi^2(1 - \pi)$<br>$= (0.30)^2(0.70) = 0.063$ |
| Purchase       | No purchase  | Purchase     | (S, F, S)            | $\pi(1 - \pi)\pi = \pi^2(1 - \pi)$<br>$= (0.30)^2(0.70) = 0.063$ |
| No purchase    | Purchase     | Purchase     | (F, S, S)            | $(1 - \pi)\pi\pi = \pi^2(1 - \pi)$<br>$= (0.30)^2(0.70) = 0.063$ |

Observe that all three experimental outcomes with two successes have exactly the same probability. This observation holds in general. In any binomial experiment, all sequences of trial outcomes yielding  $x$  successes in  $n$  trials have the *same probability* of occurrence.

The probability of each sequence of trials yielding  $X$  successes in  $n$  trials follows in equation (5.7).

$$\text{Probability of a particular sequence of trial outcomes} = \pi^x(1 - \pi)^{(n-x)} \quad (5.7)$$

with  $X$  successes in  $n$  trials

For the Marrine Clothing Store, this formula shows that any experimental outcome with two successes has a probability of  $\pi^2(1 - \pi)^{(3-2)} = \pi^2(1 - \pi)^1 = (0.30)^2(0.70)^1 = 0.063$ . Combining equations (5.6) and (5.7) we obtain the following **binomial probability function**.

### Binomial probability function

$$p(x) = \binom{n}{x} \pi^x(1 - \pi)^{(n-x)} \quad (5.8)$$

where  $p(x)$  = the probability of  $x$  successes in  $n$  trials

$n$  = the number of trials

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

$\pi$  = the probability of a success on any one trial

$1 - \pi$  = the probability of a failure on any one trial

In the Marrine Clothing Store example, we can use this function to compute the probability that no customer makes a purchase, exactly one customer makes a purchase, exactly two customers make a purchase and all three customers make a purchase. The calculations are summarized in Table 5.7, which gives the probability distribution of the number of customers making a purchase. Figure 5.4 is a graph of this probability distribution.

The binomial probability function can be applied to *any* binomial experiment. If we are satisfied that a situation demonstrates the properties of a binomial experiment and if we know the values of  $n$  and  $\pi$ , we can use equation (5.8) to compute the probability of  $x$  successes in the  $n$  trials.

If we consider variations of the Marrine experiment, such as ten customers rather than three entering the store, the binomial probability function given by equation (5.8) is still applicable.

**TABLE 5.7** Probability distribution for the number of customers making a purchase

| $x$ | $p(x)$  |
|-----|---|
| 0   | $\frac{3!}{0!3!} (0.30)^0 (0.70)^3 = 0.343$               |
| 1   | $\frac{3!}{1!2!} (0.30)^1 (0.70)^2 = 0.441$               |
| 2   | $\frac{3!}{2!1!} (0.30)^2 (0.70)^1 = 0.189$               |
| 3   | $\frac{3!}{3!0!} (0.30)^3 (0.70)^0 = \frac{0.027}{1.000}$ |

**FIGURE 5.4**

Graphical representation of the probability distribution for the number of customers making a purchase



Suppose we have a binomial experiment with  $n = 10$ ,  $x = 4$  and  $\pi = 0.30$ . The probability of making exactly four sales to ten customers entering the store is:

$$p(4) = \frac{10!}{4!6!} (0.30)^4 (0.70)^6 = 0.2001$$

## Using tables of binomial probabilities

Tables have been developed that give the probability of  $x$  successes in  $n$  trials for a binomial experiment. The tables are generally easy to use and quicker than equation (5.8). Table 5 of Appendix B provides such a table of binomial probabilities. To use this table, we must specify the values of  $n$ ,  $\pi$  and  $x$  for the binomial experiment of interest. For example, the probability of  $x = 3$  successes in a binomial experiment with  $n = 10$  and  $\pi = 0.40$  can be seen to be 0.2150. You can use equation (5.8) to verify that you would obtain the same answer if you used the binomial probability function directly.

Now let us use the same table to verify the probability of four successes in ten trials for the Marrine Clothing Store problem. Note that the value of  $p(4) = 0.2001$  can be read directly from the table of binomial probabilities, with  $n = 10$ ,  $x = 4$  and  $\pi = 0.30$ .

Even though the tables of binomial probabilities are relatively easy to use, it is impossible to have tables that show all possible values of  $n$  and  $\pi$  that might be encountered in a binomial experiment. However, with today's calculators, using equation (5.8) to calculate the desired probability is not difficult, especially if the number of trials is not large. In the exercises, you should practice using equation (5.8) to compute the binomial probabilities unless the problem specifically requests that you use the binomial probability table.

**FIGURE 5.5**

MINITAB output showing binomial probabilities for the Marrine Clothing Store problem

| x  | P ( X = x ) |
|----|-------------|
| 0  | 0. 028248   |
| 1  | 0. 121061   |
| 2  | 0. 233474   |
| 3  | 0. 266828   |
| 4  | 0. 200121   |
| 5  | 0. 102919   |
| 6  | 0. 036757   |
| 7  | 0. 009002   |
| 8  | 0. 001447   |
| 9  | 0. 000138   |
| 10 | 0. 000006   |

Statistical software packages such as MINITAB, SPSS and spreadsheet packages such as EXCEL also provide a capability for computing binomial probabilities. Consider the Marrine Clothing Store example with  $n = 10$  and  $\pi = 0.30$ . Figure 5.5 shows the binomial probabilities generated by MINITAB for all possible values of  $x$ . Note that these values are the same as those found in the  $\pi = 0.30$  column of Table 5.5 of Appendix B. At the end of the chapter, details are given on how to generate the output in Figure 5.5 using first MINITAB, then EXCEL and finally SPSS.

## Expected value and variance for the binomial distribution

In Section 5.3 we provided formulae for computing the expected value and variance of a discrete random variable. In the special case where the random variable has a binomial distribution with a known number of trials  $n$  and a known probability of success  $\pi$ , the general formulae for the expected value and variance can be simplified. The results follow.

### Expected value and variance for the binomial distribution

$$E(X) = \mu = n\pi \quad (5.9)$$

$$\text{Var}(X) = \sigma^2 = n\pi(1 - \pi) \quad (5.10)$$

For the Marrine Clothing Store problem with three customers, we can use equation (5.9) to compute the expected number of customers who will make a purchase.

$$E(X) = n\pi = 3 \times 0.30 = 0.9$$

Suppose that for the next month the Marrine Clothing Store forecasts 1000 customers will enter the store. What is the expected number of customers who will make a purchase? The answer is  $\mu = n\pi = 1000 \times 0.3 = 300$ . Thus, to increase the expected number of purchases, Marrine must induce more customers to enter the store and/or somehow increase the probability that any individual customer will make a purchase after entering.

For the Marrine Clothing Store problem with three customers, we see that the variance and standard deviation for the number of customers who will make a purchase are:

$$\sigma^2 = n\pi(1 - \pi) = 3 \times 0.3 \times 0.7 = 0.63$$

$$\sigma = \sqrt{0.063} = 0.79$$

For the next 1000 customers entering the store, the variance and standard deviation for the number of customers who will make a purchase are:

$$\sigma^2 = n\pi(1 - \pi) = 1000 \times 0.3 \times 0.7 = 210$$

$$\sigma = \sqrt{210} = 14.49$$

## EXERCISES

### Methods

21. Consider a binomial experiment with two trials and  $\pi = 0.4$ .
- Draw a tree diagram for this experiment (see Figure 5.3).
  - Compute the probability of one success,  $p(1)$ .
  - Compute  $p(0)$ .
  - Compute  $p(2)$ .
  - Compute the probability of at least one success.
  - Compute the expected value, variance and standard deviation.
22. Consider a binomial experiment with  $n = 10$  and  $\pi = 0.10$ .
- Compute  $p(0)$ .
  - Compute  $p(2)$ .
  - Compute  $P(X \leq 2)$ .
  - Compute  $P(X \geq 1)$ .
  - Compute  $E(X)$ .
  - Compute  $\text{Var}(X)$  and  $\sigma$ .
23. Consider a binomial experiment with  $n = 20$  and  $\pi = 0.70$ .
- Compute  $p(12)$ .
  - Compute  $p(16)$ .
  - Compute  $P(X \geq 16)$ .
  - Compute  $P(X \leq 15)$ .
  - Compute  $E(X)$ .
  - Compute  $\text{Var}(X)$  and  $\sigma$ .

### Applications

24. When a new machine is functioning properly, only 3 per cent of the items produced are defective. Assume that we will randomly select two parts produced on the machine and that we are interested in the number of defective parts found.
- Describe the conditions under which this situation would be a binomial experiment.
  - Draw a tree diagram similar to Figure 5.3 showing this problem as a two-trial experiment.
  - How many experimental outcomes result in exactly one defect being found?
  - Compute the probabilities associated with finding no defects, exactly one defect and two defects.
25. It takes at least nine votes from a 12-member jury to convict a defendant. Suppose that the probability that a juror votes a guilty person innocent is 0.2 whereas the probability that the juror votes an innocent person guilty is 0.1.
- If each juror acts independently and 65 per cent of defendants are guilty, what is the probability that the jury renders a correct decision.
  - What percentage of defendants is convicted?
26. A firm bills its accounts at a 1 per cent discount for payment within ten days and the full amount is due after ten days. In the past 30 per cent of all invoices have been paid within ten days. If the firm sends out eight invoices during the first week of January, what is the probability that:
- No one receives the discount?
  - Everyone receives the discount?
  - No more than three receive the discount?
  - At least two receive the discount?

- 27.** In a game of ‘Chuck a luck’ a player bets on one of the numbers 1 to 6. Three dice are then rolled and if the number bet by the player appears  $i$  times ( $i = 1, 2, 3$ ) the player then wins  $i$  units. On the other hand if the number bet by the player does not appear on any of the dice the player loses 1 unit. If  $x$  is the player’s winnings in the game, what is the expected value of  $X$ ?

## 5.5 POISSON PROBABILITY DISTRIBUTION

In this section we consider a discrete random variable that is often useful in estimating the number of occurrences over a specified interval of time or space. For example, the random variable of interest might be the number of arrivals at a car wash in one hour, the number of repairs needed in ten kilometres of highway, or the number of leaks in 100 kilometres of pipeline.

If the following two properties are satisfied, the number of occurrences is a random variable described by the **Poisson probability distribution**.

### Properties of a Poisson experiment

1. The probability of an occurrence is the same for any two intervals of equal length.
2. The occurrence or non-occurrence in any interval is independent of the occurrence or non-occurrence in any other interval.

The **Poisson probability function** is defined by equation (5.11).

### Poisson probability function

$$p(x) = \frac{\mu^x e^{-\mu}}{x!} \quad (5.11)$$

where

$p(x)$  = the probability of  $x$  occurrences in an interval

$\mu$  = expected value or mean number of occurrences in an interval

$e$  = 2.71828

Before we consider a specific example to see how the Poisson distribution can be applied, note that the number of occurrences,  $x$ , has no upper limit. It is a discrete random variable that may assume an infinite sequence of values ( $x = 0, 1, 2, \dots$ ).

### An example involving time intervals

Suppose that we are interested in the number of arrivals at the payment kiosk of a car park during a 15-minute period on weekday mornings. If we can assume that the probability of a car arriving is the same for any two time periods of equal length and that the arrival or non-arrival of a car in any time period is independent of the arrival or non-arrival in any other time period, the Poisson probability function is applicable. Suppose these assumptions are satisfied and an analysis of historical data shows

that the average number of cars arriving in a 15-minute period of time is ten; in this case, the following probability function applies:

$$p(x) = \frac{10^x e^{-10}}{x!}$$

The random variable here is  $X$  = number of cars arriving in any 15-minute period.

If management wanted to know the probability of exactly five arrivals in 15 minutes, we would set  $X = 5$  and thus obtain:

$$\text{Probability of exactly five arrivals in 15 minutes} = p(5) = \frac{10^5 e^{-10}}{5!} = 0.0378$$

Although this probability was determined by evaluating the probability function with  $\mu = 10$  and  $x = 5$ , it is often easier to refer to a table for the Poisson distribution. The table provides probabilities for specific values of  $x$  and  $\mu$ . We include such a table as Table 7 of Appendix B. Note that to use the table of Poisson probabilities, we need know only the values of  $x$  and  $\mu$ . From this table we see that the probability of five arrivals in a 15-minute period is found by locating the value in the row of the table corresponding to  $x = 5$  and the column of the table corresponding to  $\mu = 10$ . Hence, we obtain  $p(5) = 0.0378$ .

In the preceding example, the mean of the Poisson distribution is  $\mu = 10$  arrivals per 15-minute period. A property of the Poisson distribution is that the mean of the distribution and the variance of the distribution are *equal*. Thus, the variance for the number of arrivals during 15-minute periods is  $\sigma^2 = 10$ . The standard deviation is:

$$\sigma = \sqrt{10} = 3.16$$

Our illustration involves a 15-minute period, but other time periods can be used. Suppose we want to compute the probability of one arrival in a three-minute period. Because ten is the expected number of arrivals in a 15-minute period, we see that  $10/15 = 2/3$  is the expected number of arrivals in a one-minute period and that  $2/3 \times 3 \text{ minutes} = 2$  is the expected number of arrivals in a three-minute period. Thus, the probability of  $x$  arrivals in a three-minute time period with  $\mu = 2$  is given by the following Poisson probability function.

$$p(x) = \frac{2^x e^{-2}}{x!}$$

The probability of one arrival in a three-minute period is calculated as follows:

$$\text{Probability of exactly one arrival in three minutes} = P(1) = \frac{2^1 e^{-2}}{1!} = 0.2707$$

Earlier we computed the probability of five arrivals in a 15-minute period; it was 0.0378. Note that the probability of one arrival in a three-minute period (0.2707) is not the same. When computing a Poisson probability for a different time interval, we must first convert the mean arrival rate to the time period of interest and then compute the probability.

### An example involving length or distance intervals

Consider an application not involving time intervals in which the Poisson distribution is useful. Suppose we are concerned with the occurrence of major defects in a highway, one month after resurfacing. We will assume that the probability of a defect is the same for any two highway intervals of equal length and that the occurrence or non-occurrence of a defect in any one interval is independent of the occurrence or non-occurrence of a defect in any other interval. Hence, the Poisson distribution can be applied.

Suppose that major defects one month after resurfacing occur at the average rate of two per kilometre. Let us find the probability of no major defects in a particular three-kilometre section of the highway. Because we are interested in an interval with a length of three kilometres,  $\mu = 2 \text{ defects/kilometre} \times 3 \text{ kilometres} = 6$  represents the expected number of major defects over the three-kilometre section of highway. Using equation (5.11), the probability of no major defects is  $p(0) = 6^0 e^{-6}/0! = 0.0025$ . Thus, it is unlikely that no major defects will occur in the three-kilometre section. Equivalently there is a  $1 - 0.0025 = 0.9975$  probability of at least one major defect in the three-kilometre highway section.



COMPLETE  
SOLUTIONS

## EXERCISES

### Methods

- 28.** Consider a Poisson distribution with  $\mu = 3$ .
- Write the appropriate Poisson probability function.
  - Compute  $p(2)$ .
  - Compute  $p(1)$ .
  - Compute  $P(X \geq 2)$ .
- 29.** Consider a Poisson distribution with a mean of two occurrences per time period.
- Write the appropriate Poisson probability function.
  - What is the expected number of occurrences in three time periods?
  - Write the appropriate Poisson probability function to determine the probability of  $x$  occurrences in three time periods.
  - Compute the probability of two occurrences in one time period.
  - Compute the probability of six occurrences in three time periods.
  - Compute the probability of five occurrences in two time periods.

### Applications

- 30.** A certain process produces 100m long rolls of high quality silk. In order to assess quality a 10m sample is taken from the end of each roll and inspected for blemishes. The number of blemishes in each sample is thought to follow a Poisson distribution with an average of two blemishes per 10m sample.
- What is the probability that there will be more than seven blemishes if a 30m sample is taken?
- 31.** During the period of time that a local university takes phone-in registrations, calls come in at the rate of one every two minutes.
- What is the expected number of calls in one hour?
  - What is the probability of three calls in five minutes?
  - What is the probability of no calls in a five-minute period?
- 32.** Airline passengers arrive randomly and independently at the passenger-screening facility at a major international airport. The mean arrival rate is ten passengers per minute.
- Compute the probability of no arrivals in a one-minute period.
  - Compute the probability that three or fewer passengers arrive in a one-minute period.
  - Compute the probability of no arrivals in a 15-second period.
  - Compute the probability of at least one arrival in a 15-second period.

## 5.6 HYPERGEOMETRIC PROBABILITY DISTRIBUTION

The **hypergeometric probability distribution** is closely related to the binomial distribution. The two probability distributions differ in two key ways. With the hypergeometric distribution, the trials are not independent; and the probability of success changes from trial to trial.

In the usual notation for the hypergeometric distribution,  $r$  denotes the number of elements in the population of size  $N$  labelled success, and  $N - r$  denotes the number of elements in the population labelled failure. The **hypergeometric probability function** is used to compute the probability that in a random selection of  $n$  elements, selected without replacement, we obtain  $x$  elements labelled success and  $n - x$  elements labelled failure. For this outcome to occur, we must obtain  $x$  successes from the  $r$

successes in the population and  $n - x$  failures from the  $N - r$  failures. The following hypergeometric probability function provides  $p(x)$ , the probability of obtaining  $x$  successes in a sample of size  $n$ .

### Hypergeometric probability function

$$p(x) = \frac{\binom{r}{x} \binom{N-r}{n-x}}{\binom{N}{n}} \quad (5.12)$$

where

$p(x)$  = probability of  $x$  successes in  $n$  trials

$n$  = number of trials

$N$  = number of elements in the population

$r$  = number of elements in the population labelled success

Note that  $\binom{N}{n}$  represents the number of ways a sample of size  $n$  can be selected from a population of size  $N$ ;  $\binom{r}{x}$  represents the number of ways that  $x$  successes can be selected from a total of  $r$  successes in the population; and  $\binom{N-r}{n-x}$  represents the number of ways that  $n - x$  failures can be selected from a total of  $N - r$  failures in the population.

To illustrate the computations involved in using equation (5.12), consider the following quality control application. Electric fuses produced by Warsaw Electric are packaged in boxes of 12 units each. Suppose an inspector randomly selects three of the 12 fuses in a box for testing. If the box contains exactly five defective fuses, what is the probability that the inspector will find exactly one of the three fuses defective? In this application,  $n = 3$  and  $N = 12$ . With  $r = 5$  defective fuses in the box the probability of finding  $x = 1$  defective fuse is:

$$p(1) = \frac{\binom{5}{1} \binom{7}{2}}{\binom{12}{3}} = \frac{\frac{5!}{1!4!} \frac{7!}{2!3!}}{\frac{12!}{3!9!}} = \frac{5 \times 21}{220} = 0.4733$$

Now suppose that we wanted to know the probability of finding *at least* one defective fuse. The easiest way to answer this question is to first compute the probability that the inspector does not find any defective fuses. The probability of  $x = 0$  is:

$$p(0) = \frac{\binom{5}{0} \binom{7}{3}}{\binom{12}{3}} = \frac{\frac{5!}{0!5!} \frac{7!}{3!4!}}{\frac{12!}{3!9!}} = \frac{1 \times 35}{220} = 0.1591$$

With a probability of zero defective fuses  $p(0) = 0.1591$ , we conclude that the probability of finding at least one defective fuse must be  $1 - 0.1591 = 0.8409$ . Thus, there is a reasonably high probability that the inspector will find at least one defective fuse.

The mean and variance of a hypergeometric distribution are as follows.

### Expected value for the hypergeometric distribution

$$E(x) = \mu = n\left(\frac{r}{N}\right) \quad (5.13)$$

### Variance for the hypergeometric distribution

$$\text{Var}(X) = \sigma^2 = n\left(\frac{r}{N}\right)\left(1 - \frac{r}{N}\right)\left(\frac{N-n}{N-1}\right) \quad (5.14)$$

In the preceding example  $n = 3$ ,  $r = 5$ , and  $N = 12$ . Thus, the mean and variance for the number of defective fuses is:

$$\mu = n\left(\frac{r}{N}\right) = 3\left(\frac{5}{12}\right) = 1.25$$

$$\sigma^2 = n\left(\frac{r}{N}\right)\left(1 - \frac{r}{N}\right)\left(\frac{N-n}{N-1}\right) = 3\left(\frac{5}{12}\right)\left(1 - \frac{5}{12}\right)\left(\frac{12-3}{12-1}\right) = 0.60$$

The standard deviation is:

$$\sigma = \sqrt{0.60} = 0.77$$

## EXERCISES

### Methods

- 33.** Suppose  $N = 10$  and  $r = 3$ . Compute the hypergeometric probabilities for the following values of  $n$  and  $x$ .
- $n = 4, x = 1$ .
  - $n = 2, x = 2$ .
  - $n = 2, x = 0$ .
  - $n = 4, x = 2$ .

- 34.** Suppose  $N = 15$  and  $r = 4$ . What is the probability of  $x = 3$  for  $n = 10$ ?

### Applications

- 35.** Blackjack, or Twenty-one as it is frequently called, is a popular gambling game played in Monte Carlo casinos. A player is dealt two cards. Face cards (jacks, queens and kings) and tens have a



point value of ten. Aces have a point value of one or 11. A 52-card deck contains 16 cards with a point value of ten (jacks, queens, kings and tens) and four aces.

- a. What is the probability that both cards dealt are aces or ten-point cards?
  - b. What is the probability that both of the cards are aces?
  - c. What is the probability that both of the cards have a point value of ten?
  - d. A blackjack is a ten-point card and an ace for a value of 21. Use your answers to parts (a), (b) and (c) to determine the probability that a player is dealt a blackjack. (Hint: Part (d) is not a hypergeometric problem. Develop your own logical relationship as to how the hypergeometric probabilities from parts (a), (b) and (c) can be combined to answer this question.)
- 36.** A company plans to select a team of five students from Gulf University for a business game competition from a pool of 18 undergraduates. Nine are from the second-year management course, five are third-year management and the remainder are from outside the management school. What is the probability that:
- a. All five team members are second-year management?
  - b. No students from outside the management school are selected?
- 37.** Manufactured parts are shipped in lots of 15 items. Four parts are randomly drawn from each lot and tested and the lot is considered acceptable if no defectives are among the four tested.
- a. What is the probability that the shipment will be rejected?

## ONLINE RESOURCES

For the data files, additional online summary, questions, answers and the software section for this chapter, go to the online platform.

## SUMMARY

A random variable provides a numerical description of the outcome of an experiment. The probability distribution for a random variable describes how the probabilities are distributed over the values the random variable can assume. A variety of examples are used to distinguish between discrete and continuous random variables. For any discrete random variable  $X$ , the probability distribution is defined by a probability function, denoted by  $p(x) = p(X = x)$ , which provides the probability associated with each value of the random variable. From the probability function, the expected value, variance and standard deviation for the random variable can be computed and relevant interpretations of these terms are provided.

Particular attention was devoted to the binomial distribution which can be used to determine the probability of  $x$  successes in  $n$  trials whenever the experiment has the following properties:

- 1 The experiment consists of a sequence of  $n$  identical trials.
- 2 Two outcomes are possible on each trial, one called success and the other failure.
- 3 The probability of a success  $\pi$  does not change from trial to trial. Consequently, the probability of failure,  $1 - \pi$ , does not change from trial to trial.
- 4 The trials are independent.

Formulae were also presented for the probability function, mean and variance of the binomial distribution.

The Poisson distribution can be used to determine the probability of obtaining  $x$  occurrences over an interval of time or space. The necessary assumptions for the Poisson distribution to apply in a given situation are that:

- 1 The probability of an occurrence of the event is the same for any two intervals of equal length.
- 2 The occurrence or non-occurrence of the event in any interval is independent of the occurrence or non-occurrence of the event in any other interval.

A third discrete probability distribution, the hypergeometric, was introduced in Section 5.6. Like the binomial, it is used to compute the probability of  $x$  successes in  $n$  trials. But, in contrast to the binomial, the probability of success changes from trial to trial.

## KEY TERMS

**Binomial experiment**  
**Binomial probability distribution**  
**Binomial probability function**  
**Continuous random variable**  
**Discrete random variable**  
**Discrete uniform probability distribution**  
**Expected value**  
**Hypergeometric probability distribution**

**Hypergeometric probability function**  
**Poisson probability distribution**  
**Poisson probability function**  
**Probability distribution**  
**Probability function**  
**Random variable**  
**Standard deviation**  
**Variance**

## KEY FORMULAE

### Discrete uniform probability function

$$p(x) = 1/n \quad (5.3)$$

where

$n$  = the number of values the random variable may assume

### Expected value of a discrete random variable

$$E(X) = \mu = \sum x p(x) \quad (5.4)$$

### Variance of a discrete random variable

$$\text{Var}(X) = \sigma^2 = \sum (x - \mu)^2 p(x) \quad (5.5)$$

### Number of experimental outcomes providing exactly $x$ successes in $n$ trials

$$\binom{n}{x} = \frac{n!}{x!(n-x)!} \quad (5.6)$$

**Binomial probability function**

$$p(x) = \binom{n}{x} \pi^x (1-\pi)^{n-x} \quad (5.8)$$

**Expected value for the binomial distribution**

$$E(X) = \mu = n\pi \quad (5.9)$$

**Variance for the binomial distribution**

$$\text{Var}(X)\sigma^2 = n\pi(1-\pi) \quad (5.10)$$

**Poisson probability function**

$$p(x) = \frac{\mu^x e^{-\mu}}{x!} \quad (5.11)$$

**Hypergeometric probability function**

$$p(x) = \frac{\binom{r}{x} \binom{N-r}{n-x}}{\binom{N}{n}} \quad (5.12)$$

**Expected value for the hypergeometric distribution**

$$E(x) = \mu = n \left( \frac{r}{N} \right) \quad (5.13)$$

**Variance for the hypergeometric distribution**

$$\text{Var}(X) = \sigma^2 = n \left( \frac{r}{N} \right) \left( 1 - \frac{r}{N} \right) \left( \frac{N-n}{N-1} \right) \quad (5.14)$$

**CASE PROBLEM 1*****Adapting a Bingo Game***

Gaming Machines International (GMI) is investigating the adaptation of one of its bingo machine formats to allow for a bonus game facility. With the existing setup, the player has to select seven numbers from the series 1 to 80. Fifteen numbers are then drawn randomly from the 80 available and prizes awarded, according to how many of the 15 coincide with the player's selection, as follows:

| Number of 'hits' | Payoff  |
|------------------|---------|
| 0                | 0       |
| 1                | 0       |
| 2                | 0       |
| 3                | 1       |
| 4                | 10      |
| 5                | 100     |
| 6                | 1 000   |
| 7                | 100 000 |

With the new 'two ball bonus draw' feature, players effectively have the opportunity to improve their prize by buying an extra two balls. Note, however, that the

bonus draw is only expected to be available to players who have scored 2, 3, 4 or 5 hits in the main game.

#### **Managerial report**

1. Determine the probability characteristics of GMI's original bingo game and calculate the player's expected payoff.
2. Derive corresponding probability details for the proposed bonus game. What is the probability of the player scoring:

a. 0 hits

b. 1 hit

c. 2 hits

with the extra two balls?

3. Use the results obtained from two to revise the probability distribution found for one. Hence calculate the player's expected payoff in the enhanced game. Comment on how much the player might be charged for the extra gamble.

## CASE PROBLEM 2



#### **European Airline Overbooking**

EU Regulation 261/2004 sets the minimum levels of passenger compensation for denied boarding due to overbooking, and extends its coverage to include flight cancellations and long delays. It is estimated that the annual cost to airlines over and above existing compensation will total €560 million for all EU airlines:

- Compensation for overbooking affects around 250 000 passengers (0.1 per cent of total). Higher compensation rates will add €96 million to airline costs.
- At an estimated €283 million and €176 million respectively, compensation for long delays and cancellation threaten to add most additional costs incurred by European airlines. The cost to a medium-sized European airline has been estimated at €40 million a year. That represents around 20 per cent of 2004 operating profits.

EA is a small, short-range airline headquartered in Vienna. It has a fleet of small Fokker planes with a capacity of 80 passengers each. They do not have different classes in their planes. In planning for their financial obligations, EA has requested a study of the chances of 'bumping' passengers they have to consider for their overbooking strategy. The airline reports a historical 'no shows' history of 10 per cent to 12 per cent. Compensation has been set at €250 per passenger denied boarding.

#### **Managerial report**

Write a report giving the airline some scenarios of their options. Consider scenarios according to their



policy of the number of bookings/plane: 80, 85, 89, etc.

1. What percentage of the time should they estimate that their passengers will find a seat when they show up?
2. What percentage of the time some passengers may not find a seat?
3. In each case you consider, find the average amount of loss per plane they have to take into account.

# 6

# Continuous Probability Distributions



## CHAPTER CONTENTS

Statistics in Practice Assessing the effectiveness of new medical procedures

- 6.1 Uniform probability distribution
- 6.2 Normal probability distribution
- 6.3 Normal approximation of binomial probabilities
- 6.4 Exponential probability distribution

**LEARNING OBJECTIVES** After reading this chapter and doing the exercises, you should be able to:

- 1 Understand the difference between how probabilities are computed for discrete and continuous random variables.
- 2 Compute probability values for a continuous uniform probability distribution and be able to compute the expected value and variance for such a distribution.
- 3 Compute probabilities using a normal probability distribution. Understand the role of the standard normal distribution in this process.
- 4 Use the normal distribution to approximate binomial probabilities.
- 5 Compute probabilities using an exponential probability distribution.
- 6 Understand the relationship between the Poisson and exponential probability distributions.

In this chapter we turn to the study of continuous random variables. Specifically, we discuss three continuous probability distributions: the uniform, the normal and the exponential. A fundamental difference separates discrete and continuous random variables in terms of how probabilities are computed. For a discrete random variable, the probability function  $p(x)$  provides the probability that the random variable assumes a particular value. With continuous random variables the counterpart of the probability function is the **probability density function**, denoted by  $f(x)$ . The difference is that the probability density function does not directly provide probabilities. However, the area under the graph of  $f(x)$  corresponding to a given interval does provide the probability that the continuous random variable

$X$  assumes a value in that interval. So when we compute probabilities for continuous random variables we are computing the probability that the random variable assumes any value in an interval.

One of the implications of the definition of probability for continuous random variables is that the probability of any particular value of the random variable is zero, because the area under the graph of  $f(x)$  at any particular point is zero. In Section 6.1 we demonstrate these concepts for a continuous random variable that has a uniform distribution.

Much of the chapter is devoted to describing and showing applications of the normal distribution. The main importance of normal distribution is its extensive use in statistical inference. The chapter closes with a discussion of the exponential distribution.



## STATISTICS IN PRACTICE

Assessing the effectiveness of new medical procedures

Clinical trials are a vital and commercially very important application of statistics, typically involving the random assignment of patients to two experimental groups. One group receives the treatment of interest, the second a placebo (a dummy treatment



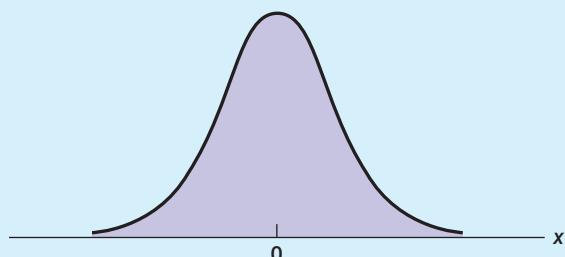
A participant takes part in a new drugs trial

that has no effect). To assess the evidence that the probability of success with the treatment will be better than that with the placebo, frequencies  $a$ ,  $b$ ,  $c$  and  $d$  can be collected for a predetermined number of trials according to the following two-way table:

|         | Treatment | Placebo |
|---------|-----------|---------|
| Success | $a$       | $b$     |
| Failure | $c$       | $d$     |

and the quantity ('log odds ratio')  $X = \log(a/c/b/d)$  calculated. Clearly the larger the value of  $X$  obtained the greater the evidence that the treatment is better than the placebo.

In the particular case that the treatment has no effect, the distribution of  $X$  can be shown to align very closely to a normal distribution with a mean of zero:



Thus, as values of  $X$  fall increasingly to the right of the zero mean this should signify stronger and stronger support for the belief in the treatment's relative effectiveness.

Intriguingly, this formulation was adapted by Copas (2005) to cast doubt on the findings of a recent study linking passive smoking to an increased risk of lung cancer.

Source: Copas, John (2005) 'The downside of publication'. *Significance* Vol. 2 Issue 4 pp. 154–157.

## 6.1 UNIFORM PROBABILITY DISTRIBUTION

Consider the random variable  $X$  representing the flight time of an aeroplane travelling from Graz to Stansted. Suppose the flight time can be any value in the interval from 120 minutes to 140 minutes. Because the random variable  $X$  can assume any value in that interval,  $X$  is a continuous rather than a discrete random variable. Let us assume that sufficient actual flight data are available to conclude that the probability of a flight time within any one-minute interval is the same as the probability of a flight time within any other one-minute interval contained in the larger interval from 120 to 140 minutes. With every one-minute interval being equally likely, the random variable  $X$  is said to have a **uniform probability distribution**.

If  $x$  is any number lying in the range that the random variable  $X$  can take then the probability density function, which defines the uniform distribution for the flight-time random variable, is:

$$f(x) = \begin{cases} 1/20 & \text{for } 120 \leq x \leq 140 \\ 0 & \text{elsewhere} \end{cases}$$

Figure 6.1 is a graph of this probability density function. In general, the uniform probability density function for a random variable  $X$  is defined by the following formula.

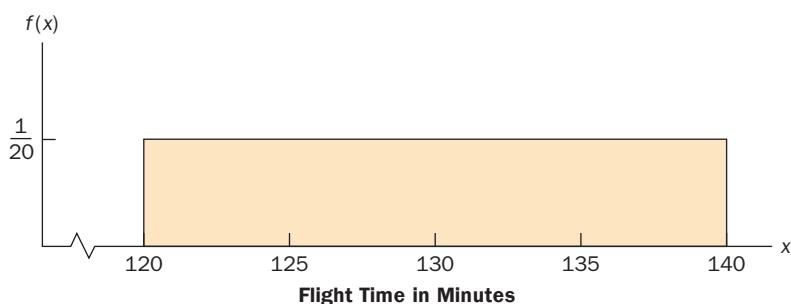
### Uniform probability density function

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b. \\ 0 & \text{elsewhere} \end{cases} \quad (6.1)$$

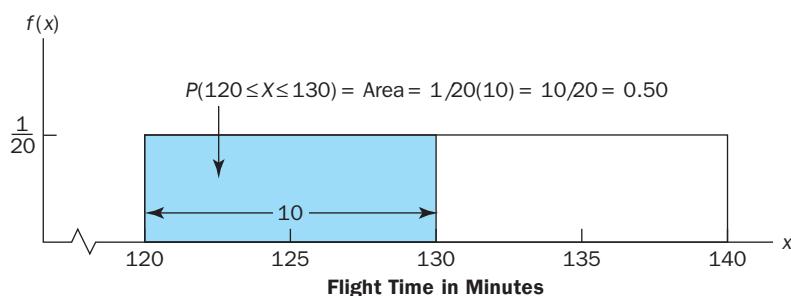
For the flight-time random variable,  $a = 120$  and  $b = 140$ .

As noted in the introduction, for a continuous random variable, we consider probability only in terms of the likelihood that a random variable assumes a value within a specified interval. In the flight time example, an acceptable probability question is: What is the probability that the flight time is between 120 and 130 minutes? That is, what is  $P(120 \leq X \leq 130)$ ? Because the flight time must be between 120 and 140 minutes and because the probability is described as being uniform over this interval, we feel comfortable saying  $P(120 \leq X \leq 130) = 0.50$ . In the following subsection we show that this probability can be computed as the area under the graph of  $f(x)$  from 120 to 130 (see Figure 6.2).

**FIGURE 6.1**  
Uniform probability density function for flight time



**FIGURE 6.2**  
Area provides probability of flight time between 120 and 130 minutes



## Area as a measure of probability

Let us make an observation about the graph in Figure 6.2. Consider the area under the graph of  $f(x)$  in the interval from 120 to 130. The area is rectangular, and the area of a rectangle is simply the width multiplied by the height. With the width of the interval equal to  $130 - 120 = 10$  and the height equal to the value of the probability density function  $f(x) = 1/20$ , we have area = width × height =  $10 \times 1/20 = 10/20 = 0.50$ .

What observation can you make about the area under the graph of  $f(x)$  and probability? They are identical! Indeed, this observation is valid for all continuous random variables. Once a probability density function  $f(x)$  is identified, the probability that  $X$  takes a value  $x$  between some lower value  $x_1$  and some higher value  $x_2$  can be found by computing the area under the graph of  $f(x)$  over the interval from  $x_1$  to  $x_2$ .

Given the uniform distribution for flight time and using the interpretation of area as probability, we can answer any number of probability questions about flight times. For example, what is the probability of a flight time between 128 and 136 minutes? The width of the interval is  $136 - 128 = 8$ . With the uniform height of  $f(x) = 1/20$ , we see that  $P(128 \leq X \leq 136) = 8 \times 1/20 = 0.40$ . Note that  $P(120 \leq X \leq 140) = 20 \times 1/20 = 1$ ; that is, the total area under the graph of  $f(x)$  is equal to 1. This property holds for all continuous probability distributions and is the analogue of the condition that the sum of the probabilities must equal 1 for a discrete probability function. For a continuous probability density function, we must also require that  $f(x) \geq 0$  for all values of  $x$ . This requirement is the analogue of the requirement that  $p(x) \geq 0$  for discrete probability functions.

Two major differences stand out between the treatment of continuous random variables and the treatment of their discrete counterparts.

- 1** We no longer talk about the probability of the random variable assuming a particular value. Instead, we talk about the probability of the random variable assuming a value within some given interval.
- 2** The probability of the random variable assuming a value within some given interval from  $x_1$  to  $x_2$  is defined to be the area under the graph of the probability density function between  $x_1$  and  $x_2$ . It implies that the probability of a continuous random variable assuming any particular value exactly is zero, because the area under the graph of  $f(x)$  at a single point is zero.

The calculation of the expected value and variance for a continuous random variable is analogous to that for a discrete random variable. However, because the computational procedure involves integral calculus, we leave the derivation of the appropriate formulae to more advanced texts.

For the uniform continuous probability distribution introduced in this section, the formulae for the expected value and variance are:

$$E(X) = \frac{a + b}{2}$$

$$\text{Var}(X) = \frac{(b - a)^2}{12}$$

In these formulae,  $a$  is the smallest value and  $b$  is the largest value that the random variable may assume.

Applying these formulae to the uniform distribution for flight times from Graz to Stansted, we obtain:

$$E(X) = \frac{(120 + 140)}{2} = 130$$

$$\text{Var}(X) = \frac{(140 - 120)^2}{12} = 33.33$$

The standard deviation of flight times can be found by taking the square root of the variance. Thus,  $\sigma = 5.77$  minutes.

## EXERCISES

### Methods

1. The random variable  $X$  is known to be uniformly distributed between 1.0 and 1.5.
  - a. Show the graph of the probability density function.
  - b. Compute  $P(X = 1.25)$ .
  - c. Compute  $P(1.0 \leq X \leq 1.25)$ .
  - d. Compute  $P(1.20 < X < 1.5)$ .
2. The random variable  $X$  is known to be uniformly distributed between 10 and 20.
  - a. Show the graph of the probability density function.
  - b. Compute  $P(X < 15)$ .
  - c. Compute  $P(12 \leq X \leq 18)$ .
  - d. Compute  $E(X)$ .
  - e. Compute  $\text{Var}(X)$ .



COMPLETE  
SOLUTIONS

### Applications

3. A continuous random variable  $X$  has probability density function:

$$f(x) = kx \quad 0 < x < 2 \\ 0 \quad \text{otherwise}$$

- a. Determine the value of  $k$ .
- b. Find  $E(X)$  and  $\text{Var}(X)$ .
- c. What is the probability that  $X$  is greater than three standard deviations above the mean?
- d. Find the distribution function  $F(X)$  and hence the median of  $X$ .
4. Most computer languages include a function that can be used to generate random numbers. In EXCEL, the RAND function can be used to generate random numbers between 0 and 1. If we let  $X$  denote a random number generated using RAND, then  $X$  is a continuous random variable with the following probability density function.

$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

- a. Graph the probability density function.
- b. What is the probability of generating a random number between 0.25 and 0.75?
- c. What is the probability of generating a random number with a value less than or equal to 0.30?
- d. What is the probability of generating a random number with a value greater than 0.60?
5. Let  $X$  denote the number of bricks a bricklayer will lay in an hour and assume that  $X$  takes values in the range 150 to 200 inclusively with equal probability (i.e. has a discrete uniform distribution). If a certain project is 170 bricks short of completion and a further project is waiting to be started as soon as this one is finished, what is the probability that:
  - a. The bricklayer will start the second project within the hour?
  - b. More than 25 bricks will have been laid on the second project at the end of the next hour?
  - c. The first project will be more than ten bricks short of completion at the end of the next hour?
  - d. The bricklayer will lay exactly 175 bricks during the next hour?

6. The label on a bottle of liquid detergent shows contents to be 12 grams per bottle. The production operation fills the bottle uniformly according to the following probability density function.

$$f(x) = \begin{cases} 8 & \text{for } 11.975 \leq x \leq 12.100 \\ 0 & \text{elsewhere} \end{cases}$$

- a. What is the probability that a bottle will be filled with between 12 and 12.05 grams?
  - b. What is the probability that a bottle will be filled with 12.02 or more grams?
  - c. Quality control accepts a bottle that is filled to within 0.02 grams of the number of grams shown on the container label. What is the probability that a bottle of this liquid detergent will fail to meet the quality control standard?
7. Suppose we are interested in bidding on a piece of land and we know there is one other bidder. The seller announced that the highest bid in excess of €10 000 will be accepted. Assume that the competitor's bid  $X$  is a random variable that is uniformly distributed between €10 000 and €15 000.
- a. Suppose you bid €12 000. What is the probability that your bid will be accepted?
  - b. Suppose you bid €14 000. What is the probability that your bid will be accepted?
  - c. What amount should you bid to maximize the probability that you get the property?
  - d. Suppose you know someone who is willing to pay you €16 000 for the property. Would you consider bidding less than the amount in part (c)? Why or why not?

## 6.2 NORMAL PROBABILITY DISTRIBUTION

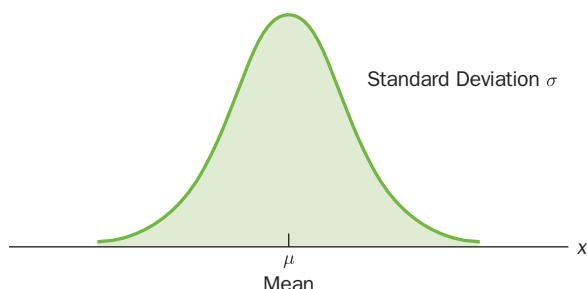
The most important probability distribution for describing a continuous random variable is the **normal probability distribution**. The normal distribution has been used in a wide variety of practical applications in which the random variables are heights and weights of people, test scores, scientific measurements, amounts of rainfall and so on. It is also widely used in statistical inference, which is the major topic of the remainder of this book. In such applications, the normal distribution provides a description of the likely results obtained through sampling.

### Normal curve

The form, or shape, of the normal distribution is illustrated by the bell-shaped normal curve in Figure 6.3. The probability density function that defines the bell-shaped curve of the normal distribution follows.

**FIGURE 6.3**

Bell-shaped curve for the normal distribution



**Normal probability density function**

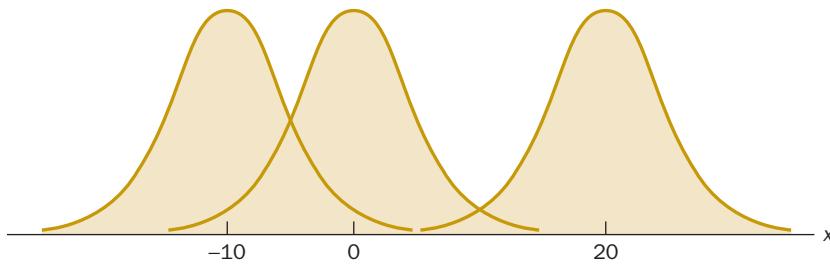
where

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \quad (6.2)$$

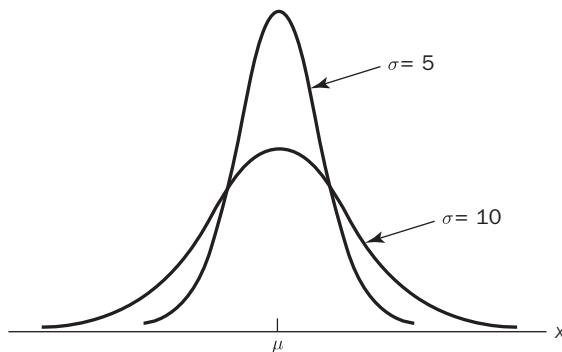
 $\mu$  = mean $\sigma$  = standard deviation $\pi = 3.14159$  $e = 2.71828$ 

We make several observations about the characteristics of the normal distribution:

- 1** The entire family of normal distributions is differentiated by its mean  $\mu$  and its standard deviation  $\sigma$ .
- 2** The highest point on the normal curve is at the mean, which is also the median and mode of the distribution.
- 3** The mean of the distribution can be any numerical value: negative, zero or positive. Three normal distributions with the same standard deviation but three different means ( $-10$ ,  $0$  and  $20$ ) are shown here.

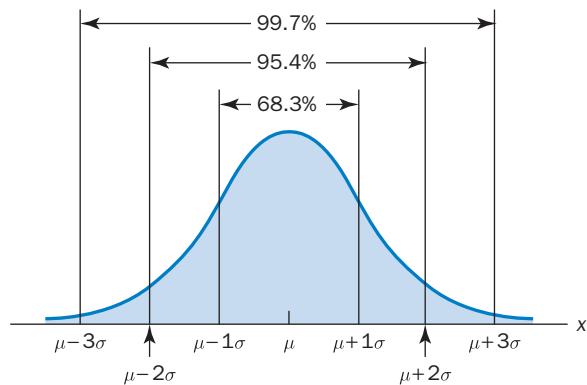


- 4** The normal distribution is symmetric, with the shape of the curve to the left of the mean a mirror image of the shape of the curve to the right of the mean. The tails of the curve extend to infinity in both directions and theoretically never touch the horizontal axis. Because it is symmetric, the normal distribution is not skewed; its skewness measure is zero.
- 5** The standard deviation determines how flat and wide the curve is. Larger values of the standard deviation result in wider, flatter curves, showing more variability in the data. Two normal distributions with the same mean but with different standard deviations are shown here.



**FIGURE 6.4**

Areas under the curve for any normal distribution



- 6** Probabilities for the normal random variable are given by areas under the curve. The total area under the curve for the normal distribution is 1. Because the distribution is symmetric, the area under the curve to the left of the mean is 0.50 and the area under the curve to the right of the mean is 0.50.
- 7** The percentage of values in some commonly used intervals are:
- 68.3 per cent of the values of a normal random variable are within plus or minus one standard deviation of its mean.
  - 95.4 per cent of the values of a normal random variable are within plus or minus two standard deviations of its mean.
  - 99.7 per cent of the values of a normal random variable are within plus or minus three standard deviations of its mean.

Figure 6.4 shows properties (a), (b) and (c) graphically.

### Standard normal probability distribution

A random variable that has a normal distribution with a mean of zero and a standard deviation of one is said to have a **standard normal probability distribution**. The letter  $Z$  is commonly used to designate this particular normal random variable. Figure 6.5 is the graph of the standard normal distribution. It has the same general appearance as other normal distributions, but with the special properties of  $\mu = 0$  and  $\sigma = 1$ .

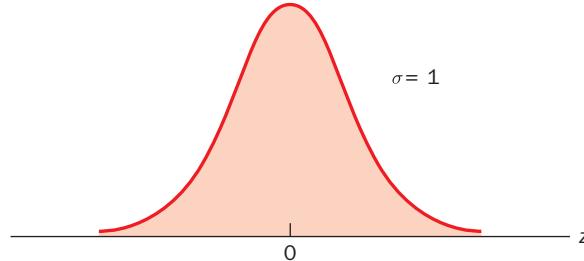
#### Standard normal density function

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

Because  $\mu = 0$  and  $\sigma = 1$ , the formula for the standard normal probability density function is a simpler version of equation (6.2).

**FIGURE 6.5**

The standard normal distribution

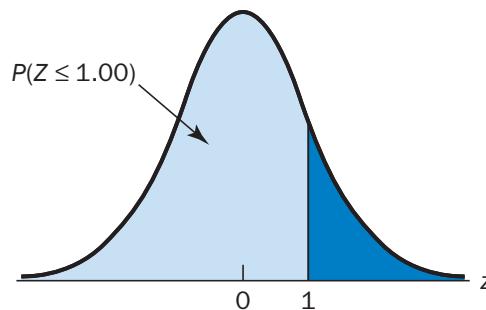


As with other continuous random variables, probability calculations with any normal distribution are made by computing areas under the graph of the probability density function. Thus, to find the probability that a normal random variable is within any specific interval, we must compute the area under the normal curve over that interval.

For the standard normal distribution, areas under the normal curve have been computed and are available in tables that can be used to compute probabilities. Such a table appears on the two pages inside the front cover of the text. The table on the left-hand page contains areas, or cumulative probabilities, for  $z$  values less than or equal to the mean of zero. The table on the right-hand page contains areas, or cumulative probabilities, for  $z$  values greater than or equal to the mean of zero.

The three types of probabilities we need to compute include (1) the probability that the standard normal random variable  $Z$  will be less than or equal to a given value; (2) the probability that  $Z$  will take a value between two given values; and (3) the probability that  $Z$  will be greater than or equal to a given value. To see how the cumulative probability table for the standard normal distribution can be used to compute these three types of probabilities, let us consider some examples.

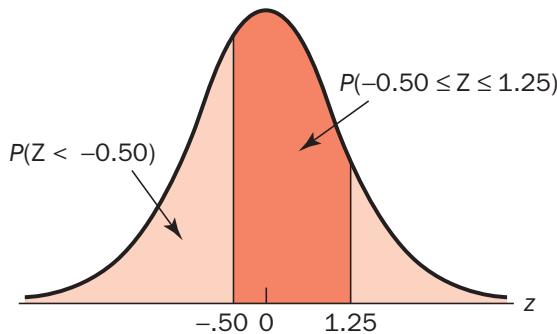
We start by showing how to compute the probability that  $Z$  is less than or equal to 1.00; that is,  $P(Z \leq 1.00)$ . This cumulative probability is the area under the normal curve to the left of  $z = 1.00$  in the following graph.



Refer to the right-hand page of the standard normal probability table inside the front cover of the text. The cumulative probability corresponding to  $z = 1.00$  is the table value located at the intersection of the row labelled 1.0 and the column labelled .00. First we find 1.0 in the left column of the table and then find .00 in the top row of the table. By looking in the body of the table, we find that the 1.0 row and the .00 column intersect at the value of 0.8413; thus,  $P(Z \leq 1.00) = 0.8413$ . The following excerpt from the probability table shows these steps.

| <b><math>z</math></b> | <b>.00</b>   | <b>.01</b>       | <b>.02</b> |
|-----------------------|--------------|------------------|------------|
| .                     | .            | .                | .          |
| .                     | .            | .                | .          |
| .9                    | .8159        | .8186            | .8212      |
| 1.0                   | <b>.8413</b> | .8438            | .8461      |
| 1.1                   | .8643        | .8665            | .8686      |
| 1.2                   | .8849        | .8869            | .8888      |
| .                     | .            | .                | .          |
| .                     | .            | .                | .          |
|                       |              | $P(Z \leq 1.00)$ |            |

To illustrate the second type of probability calculation we show how to compute the probability that  $Z$  is in the interval between  $-0.50$  and  $1.25$ ; that is,  $P(-0.50 \leq Z \leq 1.25)$ . The following graph shows this area, or probability.

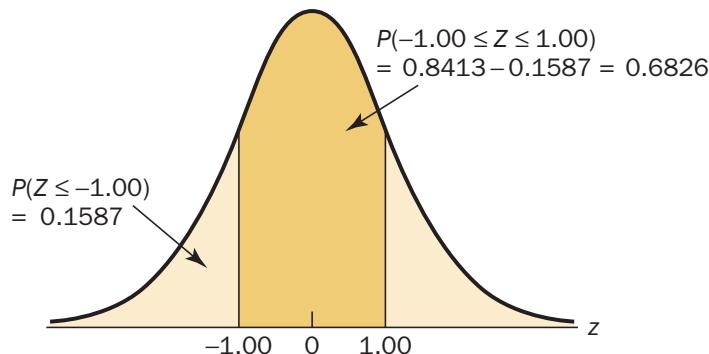


Three steps are required to compute this probability. First, we find the area under the normal curve to the left of  $z = 1.25$ . Second, we find the area under the normal curve to the left of  $z = -0.50$ . Finally, we subtract the area to the left of  $z = -0.50$  from the area to the left of  $z = 1$  to find  $P(-0.5 \leq Z \leq 1.25)$ .

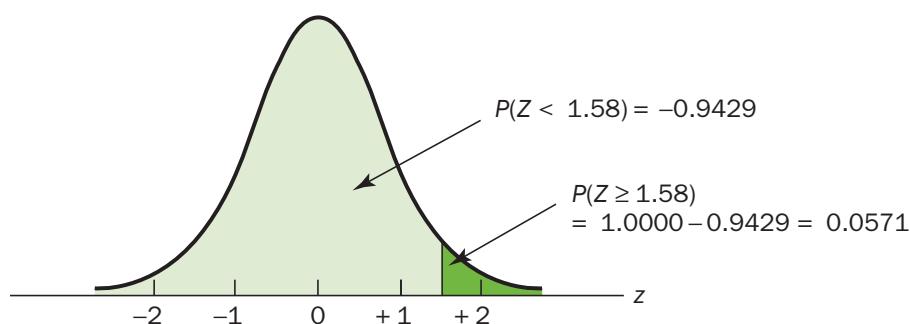
To find the area under the normal curve to the left of  $z = 1.25$ , we first locate the  $1.2$  row in the standard normal probability table and then move across to the  $.05$  column. Because the table value in the  $1.2$  row and the  $.05$  column is  $0.8944$ ,  $P(Z \leq 1.25) = 0.8944$ . Similarly, to find the area under the curve to the left of  $z = -0.50$  we use the left-hand page of the table to locate the table value in the  $-0.5$  row and the  $.00$  column; with a table value of  $0.3085$ ,  $P(Z \leq -0.50) = 0.3085$ . Thus,  $P(-0.50 \leq Z \leq 1.25) = P(Z \leq 1.25) - P(Z \leq -0.50) = 0.8944 - 0.3085 = 0.5859$ .

Let us consider another example of computing the probability that  $Z$  is in the interval between two given values. Often it is of interest to compute the probability that a normal random variable assumes a value within a certain number of standard deviations of the mean. Suppose we want to compute the probability that the standard normal random variable is within one standard deviation of the mean; that is,  $P(-1.00 \leq Z \leq 1.00)$ .

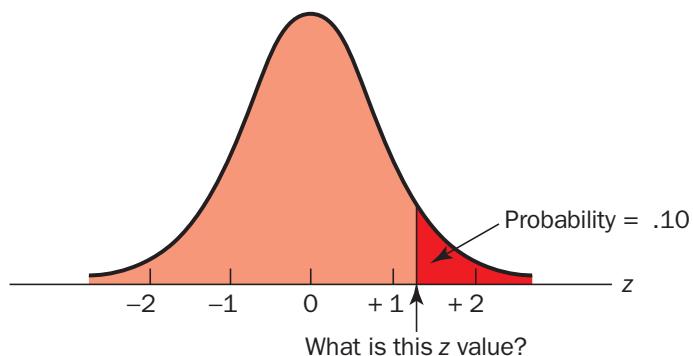
To compute this probability we must find the area under the curve between  $-1.00$  and  $1.00$ . Earlier we found that  $P(Z \leq 1.00) = 0.8413$ . Referring again to the table inside the front cover of the book, we find that the area under the curve to the left of  $z = -1.00$  is  $0.1587$ , so  $P(Z \leq -1.00) = 0.1587$ . Therefore  $P(-1.00 \leq Z \leq 1.00) = P(Z \leq 1.00) - P(Z \leq -1.00) = 0.8413 - 0.1587 = 0.6826$ . This probability is shown graphically in the following figure.



To illustrate how to make the third type of probability computation, suppose we want to compute the probability of obtaining a  $z$  value of at least  $1.58$ ; that is,  $P(Z \geq 1.58)$ . The value in the  $z = 1.5$  row and the  $.08$  column of the cumulative normal table is  $0.9429$ ; thus,  $P(Z < 1.58) = 0.9429$ . However, because the total area under the normal curve is  $1$ ,  $P(Z \geq 1.58) = 1 - 0.9429 = 0.0571$ . This probability is shown in the following figure.



In the preceding illustrations, we showed how to compute probabilities given specified  $z$  values. In some situations, we are given a probability and are interested in working backward to find the corresponding  $z$  value. Suppose we want to find a  $z$  value such that the probability of obtaining a larger  $z$  value is 0.10. The following figure shows this situation graphically.



This problem is the inverse of those in the preceding examples. Previously, we specified the  $z$  value of interest and then found the corresponding probability, or area. In this example, we are given the probability, or area, and asked to find the corresponding  $z$  value. To do so, we use the standard normal probability table somewhat differently.

| $z$  | .06   | .07   | .08   | .09   |
|--|-------|-------|-------|-------|
| .  |       |       |       |       |
| .  |       |       |       |       |
| 1.0  | .8554 | .8577 | .8599 | .8621 |
| 1.1  | .8770 | .8790 | .8810 | .8830 |
| 1.2  | .8962 | .8980 | .8997 | .9015 |
| 1.3  | .9131 | .9147 | .9162 | .9177 |
| 1.4  | .9279 | .9292 | .9306 | .9319 |
| .  |       |       |       |       |
| .  |       |       |       |       |
| Cumulative probability value closest to 0.9000 |       |       |       |       |

Recall that the standard normal probability table gives the area under the curve to the left of a particular  $z$  value. We have been given the information that the area in the upper tail of the curve is 0.10. Hence, the area under the curve to the left of the unknown  $z$  value must equal 0.9000. Scanning the body of the table, we find 0.8997 is the cumulative probability value closest to 0.9000. The section of the table providing this result is shown above. Reading the  $z$  value from the left-most column and the top row of the table, we find that the corresponding  $z$  value is 1.28. Thus, an area of approximately 0.9000 (actually

0.8997) will be to the left of  $z = 1.28$ .<sup>\*</sup> In terms of the question originally asked, the probability is approximately 0.10 that the  $z$  value will be larger than 1.28.

The examples illustrate that the table of areas for the standard normal distribution can be used to find probabilities associated with values of the standard normal random variable  $Z$ . Two types of questions can be asked. The first type of question specifies a value, or values, for  $z$  and asks us to use the table to determine the corresponding areas, or probabilities.

The second type of question provides an area, or probability, and asks us to use the table to determine the corresponding  $z$  value. Thus, we need to be flexible in using the standard normal probability table to answer the desired probability question. In most cases, sketching a graph of the standard normal distribution and shading the appropriate area or probability helps to visualize the situation and aids in determining the correct answer.

## Computing probabilities for any normal distribution

The reason for discussing the standard normal distribution so extensively is that probabilities for all normal distributions are computed by using the standard normal distribution. That is, when we have a normal distribution with any mean  $\mu$  and any standard deviation  $\sigma$ , we answer probability questions about the distribution by first converting to the standard normal distribution. Then we can use the standard normal probability table and the appropriate  $z$  values to find the desired probabilities. The formula used to convert any normal random variable  $X$  with mean  $\mu$  and standard deviation  $\sigma$  to the standard normal distribution follows as equation (6.3).

### Converting to the standard normal distribution

$$Z = \frac{X - \mu}{\sigma} \quad (6.3)$$

A value of  $X$  equal to the mean  $\mu$  results in  $z = (\mu - \mu)/\sigma = 0$ . Thus, we see that a value of  $X$  equal to the mean  $\mu$  of  $X$  corresponds to a value of  $Z$  at the mean 0 of  $Z$ . Now suppose that  $x$  is one standard deviation greater than the mean; that is,  $x = \mu + \sigma$ . Applying equation (6.3), we see that the corresponding  $z$  value  $= [(\mu + \sigma) - \mu]/\sigma = \sigma/\sigma = 1$ . Thus, a value of  $X$  that is one standard deviation above the mean  $\mu$  of  $X$  corresponds to a  $z$  value  $= 1$ . In other words, we can interpret  $Z$  as the number of standard deviations that the normal random variable  $X$  is from its mean  $\mu$ .

To see how this conversion enables us to compute probabilities for any normal distribution, suppose we have a normal distribution with  $\mu = 10$  and  $\sigma = 2$ . What is the probability that the random variable  $X$  is between 10 and 14? Using equation (6.3) we see that at  $x = 10$ ,  $z = (x - \mu)/\sigma = (10 - 10)/2 = 0$  and that at  $x = 14$ ,  $z = (14 - 10)/2 = 4/2 = 2$ . Thus, the answer to our question about the probability of  $X$  being between 10 and 14 is given by the equivalent probability that  $Z$  is between 0 and 2 for the standard normal distribution.

In other words, the probability that we are seeking is the probability that the random variable  $X$  is between its mean and two standard deviations greater than the mean. Using  $z = 2.00$  and standard normal probability table, we see that  $P(Z \leq 2) = 0.9772$ . Because  $P(Z \leq 0) = 0.5000$  we can compute  $P(0.00 \leq Z \leq 2.00) = P(Z \leq 2) - P(Z \leq 0) = 0.9772 - 0.5000 = 0.4772$ . Hence the probability that  $X$  is between 10 and 14 is 0.4772.

## Greer Tyre Company problem

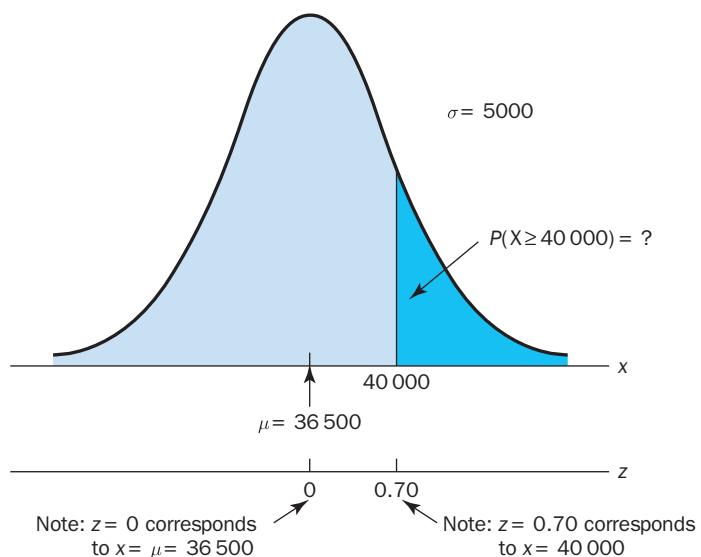
We turn now to an application of the normal distribution. Suppose the Greer Tyre Company just developed a new steel-belted radial tyre that will be sold through a national chain of discount stores. Because the tyre is a new product, Greer's managers believe that the kilometres guarantee offered with the tyre will be an

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\* We could use interpolation in the body of the table to get a better approximation of the  $z$  value that corresponds to an area of 0.9000. Doing so provides one more decimal place of accuracy and yields a  $z$  value of 1.282. However, in most practical situations, sufficient accuracy is obtained by simply using the table value closest to the desired probability.

**FIGURE 6.6**

Greer Tyre Company kilometres distribution



important factor in the acceptance of the product. Before finalizing the kilometres guarantee policy, Greer's managers want probability information about the number of kilometres the tyres will last.

From actual road tests with the tyres, Greer's engineering group estimates the mean number of kilometres the tyre will last is  $\mu = 36500$  kilometres and that the standard deviation is  $\sigma = 5000$ . In addition, the data collected indicate a normal distribution is a reasonable assumption. What percentage of the tyres can be expected to last more than 40 000 kilometres?

In other words, what is the probability that the number of kilometres the tyre lasts will exceed 40 000? This question can be answered by finding the area of the darkly shaded region in Figure 6.6. At  $x = 40000$ , we have

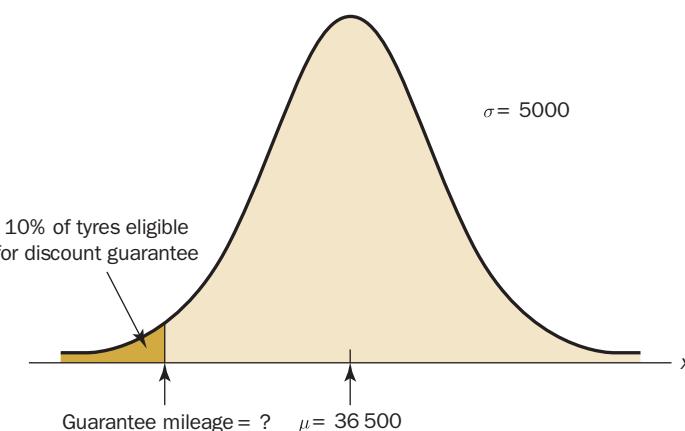
$$Z = \frac{X - \mu}{\sigma} = \frac{40000 - 36500}{5000} = \frac{3500}{5000} = 0.70$$

Refer now to the bottom of Figure 6.6. We see that a value of  $x = 40000$  on the Greer Tyre normal distribution corresponds to a value of  $z = 0.70$  on the standard normal distribution. Using the standard normal probability table, we see that the area to the left of  $z = 0.70$  is 0.7580. Referring again to Figure 6.6, we see that the area to the left of  $x = 40000$  on the Greer Tyre normal distribution is the same. Thus,  $1.000 - 0.7580 = 0.2420$  is the probability that  $X$  will exceed 40 000. We can conclude that about 24.2 per cent of the tyres will last longer than 40 000 kilometres.

Let us now assume that Greer is considering a guarantee that will provide a discount on replacement tyres if the original tyres do not exceed the number of kilometres stated in the guarantee. What should the guaranteed number of kilometres be if Greer wants no more than 10 per cent of the tyres to be eligible for the discount guarantee? This question is interpreted graphically in Figure 6.7.

**FIGURE 6.7**

Greer's discount guarantee



According to Figure 6.7, the area under the curve to the left of the unknown guaranteed number of kilometers must be 0.10. So we must find the  $z$  value that cuts off an area of 0.10 in the left tail of a standard normal distribution. Using the standard normal probability table, we see that  $z = -1.28$  cuts off an area of 0.10 in the lower tail.

Hence  $z = -1.28$  is the value of the standard normal variable corresponding to the desired number of kilometres guarantee on the Greer Tyre normal distribution. To find the value of  $X$  corresponding to  $z = -1.28$ , we have:

$$z = \frac{x - \mu}{\sigma} = -1.28$$

$$x - \mu = -1.28\sigma$$

$$x = \mu - 1.28\sigma$$

With  $\mu = 36\ 500$  and  $\sigma = 5000$ ,

$$x = 36\ 500 - 1.28 \times 5000 = 30\ 100$$

Thus, a guarantee of 30 100 kilometres will meet the requirement that approximately 10 per cent of the tyres will be eligible for the guarantee. Perhaps, with this information, the firm will set its tyre kilometres guarantee at 30 000 kilometres.

Again, we see the important role that probability distributions play in providing decision-making information. Namely, once a probability distribution is established for a particular application, it can be used quickly and easily to obtain probability information about the problem. Probability does not establish a decision recommendation directly, but it provides information that helps the decision-maker better understand the risks and uncertainties associated with the problem. Ultimately, this information may assist the decision-maker in reaching a good decision.

## EXERCISES

### Methods

8. Using Figure 6.4 as a guide, sketch a normal curve for a random variable  $X$  that has a mean of  $\mu = 100$  and a standard deviation of  $\sigma = 10$ . Label the horizontal axis with values of 70, 80, 90, 100, 110, 120 and 130.
9. A random variable is normally distributed with a mean of  $\mu = 50$  and a standard deviation of  $\sigma = 5$ .
  - a. Sketch a normal curve for the probability density function. Label the horizontal axis with values of 35, 40, 45, 50, 55, 60 and 65. Figure 6.4 shows that the normal curve almost touches the horizontal axis at three standard deviations below and at three standard deviations above the mean (in this case at 35 and 65).
  - b. What is the probability the random variable will assume a value between 45 and 55?
  - c. What is the probability the random variable will assume a value between 40 and 60?
10. Draw a graph for the standard normal distribution. Label the horizontal axis at values of  $-3, -2, -1, 0, 1, 2$  and  $3$ . Then use the table of probabilities for the standard normal distribution to compute the following probabilities.
  - a.  $P(0 \leq Z \leq 1)$ .
  - b.  $P(0 \leq Z \leq 1.5)$ .
  - c.  $P(0 < Z < 2)$ .
  - d.  $P(0 < Z < 2.5)$ .
11. Given that  $Z$  is a standard normal random variable, compute the following probabilities.
  - a.  $P(-1 \leq Z \leq 0)$ .
  - b.  $P(-1.5 \leq Z \leq 0)$ .
  - c.  $P(-2 < Z < 0)$ .
  - d.  $P(-2.5 \leq Z \leq 0)$ .
  - e.  $P(-3 \leq Z \leq 0)$ .

**12.** Given that  $Z$  is a standard normal random variable, compute the following probabilities.

- $P(0 \leq Z \leq 0.83)$ .
- $P(-1.57 \leq Z \leq 0)$ .
- $P(Z > 0.44)$ .
- $P(Z \geq -0.23)$ .
- $P(Z < 1.20)$ .
- $P(Z \leq -0.71)$ .

**13.** Given that  $Z$  is a standard normal random variable, compute the following probabilities.

- $P(-1.98 \leq Z \leq 0.49)$ .
- $P(0.52 \leq Z \leq 1.22)$ .
- $P(-1.75 \leq Z \leq -1.04)$ .

**14.** Given that  $Z$  is a standard normal random variable, find  $z$  for each situation.

- The area between 0 and  $z$  is 0.4750.
- The area between 0 and  $z$  is 0.2291.
- The area to the right of  $z$  is 0.1314.
- The area to the left of  $z$  is 0.6700.

**15.** Given that  $Z$  is a standard normal random variable, find  $z$  for each situation.

- The area to the left of  $z$  is 0.2119.
- The area between  $-z$  and  $z$  is 0.9030.
- The area between  $-z$  and  $z$  is 0.2052.
- The area to the left of  $z$  is 0.9948.
- The area to the right of  $z$  is 0.6915.

**16.** Given that  $Z$  is a standard normal random variable, find  $z$  for each situation.

- The area to the right of  $z$  is 0.01.
- The area to the right of  $z$  is 0.025.
- The area to the right of  $z$  is 0.05.
- The area to the right of  $z$  is 0.10.

## Applications

**17.** Attendance at a rock concert is normally distributed with a mean of 28 000 persons and a standard deviation of 4000 persons. What is the probability, that:

- more than 28 000 persons will attend?
- less than 14 000 persons will attend?
- between 17 000 and 25 000 persons will attend?
- Suppose the number who actually attended was  $X$  and the probability of achieving this level of attendance or higher was found to be 5 per cent. What is  $X$ ?

**18.** The holdings of clients of a successful online stockbroker are normally distributed with a mean of £20 000 and standard deviation of £1500. To increase its business, the stockbroker is looking to email special promotions to the top 20 per cent of its clientele based on the value of their holdings. What is the minimum holding of this group?

**19.** A company has been involved in developing a new pesticide. Tests show that the average proportion,  $p$ , of insects killed by administration of  $x$  units of the insecticide is given by  $p = P(X \leq x)$  where the probability  $P(X \leq x)$  relates to a normal distribution with unknown mean and standard deviation.

- Given that  $x = 10$  when  $p = 0.4$  and that  $x = 15$  when  $p = 0.9$ , determine the dose that will be lethal to 50 per cent of the insect population on average.
- If a dose of 17.5 units is administered to each of 100 insects, how many will be expected to die?



COMPLETE  
SOLUTIONS



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## 6.3 NORMAL APPROXIMATION OF BINOMIAL PROBABILITIES

In Chapter 5, Section 5.4 we presented the discrete binomial distribution. Recall that a binomial experiment consists of a sequence of  $n$  identical independent trials with each trial having two possible outcomes: a success or a failure. The probability of a success on a trial is the same for all trials and is denoted by  $\pi$  (Greek pi). The binomial random variable is the number of successes in the  $n$  trials, and probability questions pertain to the probability of  $x$  successes in the  $n$  trials. When the number of trials becomes large, evaluating the binomial probability function by hand or with a calculator is difficult. In addition, the binomial tables in Appendix B do not include values of  $n$  greater than 20. Hence, when we encounter a binomial distribution problem with a large number of trials, we may want to approximate the binomial distribution. In cases where the number of trials is greater than 20,  $n\pi \geq 5$ , and  $n(1 - \pi) \geq 5$ , the normal distribution provides an easy-to-use approximation of binomial probabilities.

When using the normal approximation to the binomial, we set  $\mu = n\pi$  and  $\sigma = \sqrt{n\pi(1 - \pi)}$  in the definition of the normal curve. Let us illustrate the normal approximation to the binomial by supposing that a particular company has a history of making errors in 10 per cent of its invoices. A sample of 100 invoices has been taken, and we want to compute the probability that 12 invoices contain errors. That is, we want to find the binomial probability of 12 successes in 100 trials.

In applying the normal approximation to the binomial, we set  $\mu = n\pi = 100 \times 0.1 = 10$  and  $\sigma = \sqrt{n\pi(1 - \pi)} = \sqrt{100 \times 0.1 \times 0.9} = 3$ . A normal distribution with  $\mu = 10$  and  $\sigma = 3$  is shown in Figure 6.8.

Recall that, with a continuous probability distribution, probabilities are computed as areas under the probability density function. As a result, the probability of any single value for the random variable is zero. Thus to approximate the binomial probability of 12 successes, we must compute the area under the corresponding normal curve between 11.5 and 12.5. The 0.5 that we add and subtract from 12 is called a **continuity correction factor**. It is introduced because a continuous distribution is being used to approximate a discrete distribution. Thus,  $P(X = 12)$  for the *discrete* binomial distribution is approximated by  $P(11.5 \leq X \leq 12.5)$  for the *continuous* normal distribution.

Converting to the standard normal distribution to compute  $P(11.5 \leq X \leq 12.5)$ , we have:

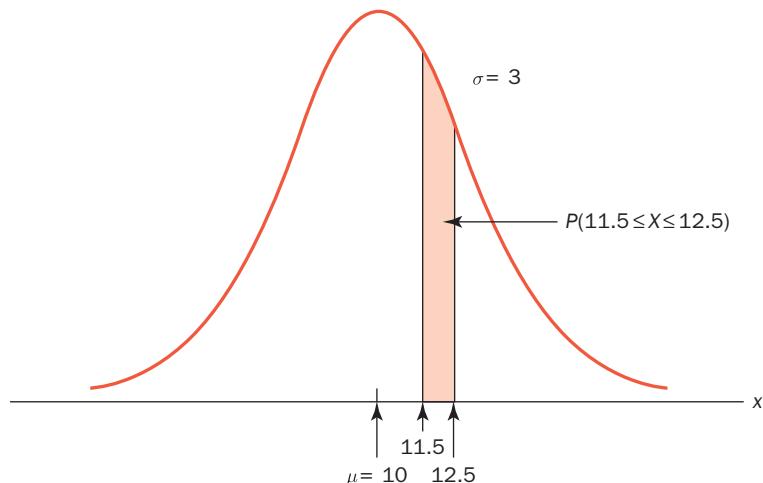
$$z = \frac{x - \mu}{\sigma} = \frac{12.5 - 10.0}{3} = 0.83 \text{ at } X = 12.5$$

And:

$$z = \frac{x - \mu}{\sigma} = \frac{11.5 - 10.0}{3} = 0.50 \text{ at } X = 11.5$$

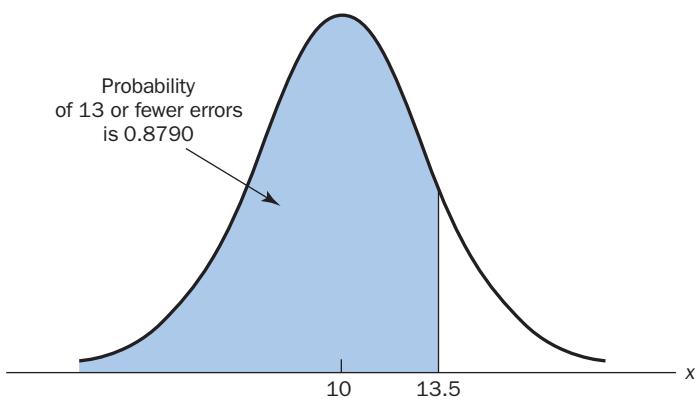
**FIGURE 6.8**

Normal approximation to a binomial probability distribution with  $n = 100$  and  $\pi = 0.10$  showing the probability of 12 errors



**FIGURE 6.9**

Normal approximation to a binomial probability distribution with  $n = 100$  and  $\pi = 0.10$  showing the probability of 13 or fewer errors



Using the standard normal probability table, we find that the area under the curve (in Figure 6.8) to the left of 12.5 is 0.7967. Similarly, the area under the curve to the left of 11.5 is 0.6915. Therefore, the area between 11.5 and 12.5 is  $0.7967 - 0.6915 = 0.1052$ . The normal approximation to the probability of 12 successes in 100 trials is 0.1052.

For another illustration, suppose we want to compute the probability of 13 or fewer errors in the sample of 100 invoices. Figure 6.9 shows the area under the normal curve that approximates this probability. Note that the use of the continuity correction factor results in the value of 13.5 being used to compute the desired probability. The  $z$  value corresponding to  $x = 13.5$  is:

$$z = \frac{13.5 - 10.0}{3} = 1.17$$

The standard normal probability table shows that the area under the standard normal curve to the left of 1.17 is 0.8790. The area under the normal curve approximating the probability of 13 or fewer errors is given by the heavily shaded portion of the graph in Figure 6.9.

## EXERCISES

### Methods

- 20.** A binomial probability distribution has  $\pi = 0.20$  and  $n = 100$ .
- What is the mean and standard deviation?
  - Is this a situation in which binomial probabilities can be approximated by the normal probability distribution? Explain.
  - What is the probability of exactly 24 successes?
  - What is the probability of 18 to 22 successes?
  - What is the probability of 15 or fewer successes?
- 21.** Assume a binomial probability distribution has  $\pi = 0.60$  and  $n = 200$ .
- What is the mean and standard deviation?
  - Is this a situation in which binomial probabilities can be approximated by the normal probability distribution? Explain.
  - What is the probability of 100 to 110 successes?
  - What is the probability of 130 or more successes?
  - What is the advantage of using the normal probability distribution to approximate the binomial probabilities? Use part (d) to explain the advantage.



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### Applications

22. A hotel in Nice has 120 rooms. In the spring months, hotel room occupancy is approximately 75 per cent.
- What is the probability that at least half of the rooms are occupied on a given day?
  - What is the probability that 100 or more rooms are occupied on a given day?
  - What is the probability that 80 or fewer rooms are occupied on a given day?

## 6.4 EXPONENTIAL PROBABILITY DISTRIBUTION

The **exponential probability distribution** may be used for random variables such as the time between arrivals at a car wash, the time required to load a truck, the distance between major defects in a highway and so on. The exponential probability density function follows.

### Exponential probability density function

$$f(x) = \frac{1}{\mu} e^{-x/\mu} \quad \text{for } x \geq 0, \mu > 0 \quad (6.4)$$

As an example of the exponential distribution, suppose that  $X$  = the time it takes to load a truck at the Schips loading dock follows such a distribution. If the mean, or average, time to load a truck is 15 minutes ( $\mu = 15$ ), the appropriate probability density function is:

$$f(x) = \frac{1}{15} e^{-x/15}$$

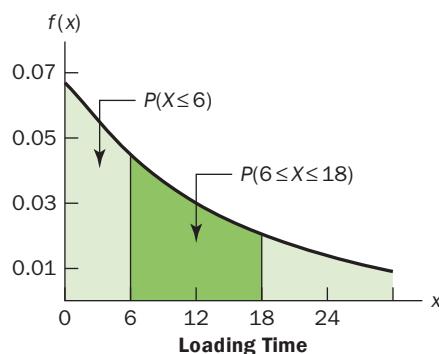
Figure 6.10 is the graph of this probability density function.

### Computing probabilities for the exponential distribution

As with any continuous probability distribution, the area under the curve corresponding to an interval provides the probability that the random variable assumes a value in that interval. In the Schips loading dock example, the probability that loading a truck will take six minutes or less ( $X \leq 6$ ) is defined to be the area under the curve in Figure 6.10 from  $x = 0$  to  $x = 6$ .

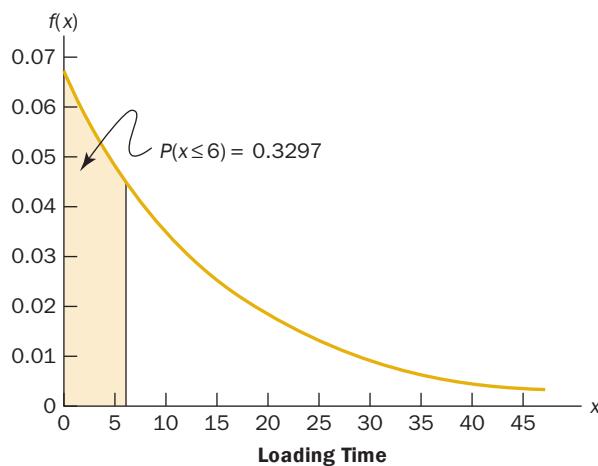
**FIGURE 6.10**

Exponential distribution for the Schips loading dock example



**FIGURE 6.11**

Probability of a loading time of six minutes or less



Similarly, the probability that loading a truck will take 18 minutes or less ( $X \leq 18$ ) is the area under the curve from  $x = 0$  to  $x = 18$ . Note also that the probability that loading a truck will take between six minutes and 18 minutes ( $6 \leq X \leq 18$ ) is given by the area under the curve from  $x = 6$  to  $x = 18$ .

To compute exponential probabilities such as those just described, we use the following formula (equation (6.5)). It provides the cumulative probability of obtaining a value for the exponential random variable of less than or equal to some specific value denoted by  $x_0$ .

#### Exponential distribution: cumulative probabilities

$$P(X \leq x_0) = 1 - e^{-x_0/\mu} \quad (6.5)$$

For the Schips loading dock example,  $X$  = loading time and  $\mu = 15$ , which gives us:

$$P(X \leq x_0) = 1 - e^{-x_0/15}$$

Hence, the probability that loading a truck will take six minutes or less is:

$$P(X \leq 6) = 1 - e^{-6/15} = 0.3297$$

Figure 6.11 shows the area or probability for a loading time of six minutes or less. Using equation (6.5), we calculate the probability of loading a truck in 18 minutes or less:

$$P(X \leq 18) = 1 - e^{-18/15} = 0.6988$$

Thus, the probability that loading a truck will take between six minutes and 18 minutes is equal to  $0.6988 - 0.3297 = 0.3691$ . Probabilities for any other interval can be computed similarly.

In the preceding example, the mean time it takes to load a truck is  $\mu = 15$  minutes. A property of the exponential distribution is that the mean of the distribution and the standard deviation of the distribution are *equal*. Thus, the standard deviation for the time it takes to load a truck is  $\sigma = 15$  minutes. The variance is  $\sigma^2 = (15)^2 = 225$ .

## Relationship between the Poisson and exponential distributions

In Chapter 5, Section 5.5 we introduced the Poisson distribution as a discrete probability distribution that is often useful in examining the number of occurrences of an event over a specified interval of time or space. Recall that the Poisson probability function is:

$$p(x) = \frac{\mu^x e^{-\mu}}{x!}$$

where:

$\mu$  = expected value or mean number of occurrences over a specified interval.

The continuous exponential probability distribution is related to the discrete Poisson distribution. If the Poisson distribution provides an appropriate description of the number of occurrences per interval, the exponential distribution provides a description of the length of the interval between occurrences.

To illustrate this relationship, suppose the number of cars that arrive at a car wash during one hour is described by a Poisson probability distribution with a mean of ten cars per hour. The Poisson probability function that gives the probability of  $X$  arrivals per hour is:

$$p(x) = \frac{10^x e^{-10}}{x!}$$

Because the average number of arrivals is ten cars per hour, the average time between cars arriving is:

$$\frac{1 \text{ hour}}{10 \text{ cars}} = 0.1 \text{ hour/car}$$

Thus, the corresponding exponential distribution that describes the time between the arrivals has a mean of  $\mu = 0.1$  hour per car; as a result, the appropriate exponential probability density function is:

$$f(x) = \frac{1}{0.1} e^{-x/0.1} = 10e^{-10x}$$

### EXERCISES

#### Methods

- 23.** Consider the following exponential probability density function.

$$f(x) = \frac{1}{8} e^{-x/2} \quad \text{for } x \geq 0$$

- a. Find  $P(X \leq 6)$ .
- b. Find  $P(X \leq 4)$ .
- c. Find  $P(X \geq 6)$ .
- d. Find  $P(4 \leq X \leq 6)$ .

- 24.** Consider the following exponential probability density function.

$$f(x) = \frac{1}{3} e^{-x/3} \quad \text{for } x \geq 0$$

- a. Write the formula for  $P(X \leq x_0)$ .
- b. Find  $P(X \leq 2)$ .
- c. Find  $P(X \geq 3)$ .
- d. Find  $P(X \leq 5)$ .
- e. Find  $P(2 \leq X \leq 5)$ .

### Applications

25. In a parts store in Mumbai, customers arrive randomly. The cashier's service time is random but it is estimated it takes an average of 30 seconds to serve each customer.
  - a. What is the probability a customer must wait more than two minutes for service?
  - b. Suppose average service time is reduced to 25 seconds. How does this affect the calculation for (a) above?
26. The time between arrivals of vehicles at a particular intersection follows an exponential probability distribution with a mean of 12 seconds.
  - a. Sketch this exponential probability distribution.
  - b. What is the probability that the arrival time between vehicles is 12 seconds or less?
  - c. What is the probability that the arrival time between vehicles is six seconds or less?
  - d. What is the probability of 30 or more seconds between vehicle arrivals?
27. According to Barron's 1998 Primary Reader Survey, the average annual number of investment transactions for a subscriber is 30 ([www.barronsmag.com](http://www.barronsmag.com), 28 July 2000). Suppose the number of transactions in a year follows the Poisson probability distribution.
  - a. Show the probability distribution for the time between investment transactions.
  - b. What is the probability of no transactions during the month of January for a particular subscriber?
  - c. What is the probability that the next transaction will occur within the next half month for a particular subscriber?



COMPLETE  
SOLUTIONS

### ONLINE RESOURCES



For the data files, additional online summary, questions, answers and the software section for this chapter, go to the online platform.

### SUMMARY

This chapter extended the discussion of probability distributions to the case of continuous random variables. The major conceptual difference between discrete and continuous probability distributions involves the method of computing probabilities. With discrete distributions, the probability function  $p(x)$  provides the probability that the random variable  $X$  assumes various values. With continuous distributions, the probability density function  $f(x)$  does not provide probability values directly. Instead, probabilities are given by areas under the curve or graph of  $f(x)$ . Three continuous probability distributions – the uniform, normal and exponential distributions were the particular focus – with detailed examples showing how probabilities could be straightforwardly computed. In addition, relationships between the binomial and normal distributions and Poisson and exponential distribution were established and related probability results, exploited.

## KEY TERMS

Continuity correction factor  
Exponential probability distribution  
Normal probability distribution

Probability density function  
Standard normal probability distribution  
Uniform probability distribution

## KEY FORMULAE

### Uniform Probability Density Function

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq X \leq b \\ 0 & \text{elsewhere} \end{cases} \quad (6.1)$$

### Normal Probability Density Function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \quad (6.2)$$

### Converting to the Standard Normal Distribution

$$Z = \frac{X - \mu}{\sigma} \quad (6.3)$$

### Exponential Probability Density Function

$$f(x) = \frac{1}{\mu} e^{-x/\mu} \quad \text{for } x \geq 0, \mu > 0 \quad (6.4)$$

### Exponential Distribution: Cumulative Probabilities

$$p(X \leq x_0) = 1 - e^{-x_0/\mu} \quad (6.5)$$

## CASE PROBLEM 1



### Prix-Fischer Toys

Prix-Fischer Toys sells a variety of new and innovative children's toys. Management learned that the pre-holiday season is the best time to introduce a new toy, because many families use this time to look for new ideas for December holiday gifts. When Prix-Fischer discovers a new toy with good market potential, it chooses an October market entry date.

In order to get toys in its stores by October, Prix-Fischer places one-time orders with its manufacturers in June or July of each year. Demand for children's toys can be highly volatile. If a new toy catches

on, a sense of shortage in the market place often increases the demand to high levels and large profits can be realized. However, new toys can also flop, leaving Prix-Fischer stuck with high levels of inventory that must be sold at reduced prices. The most important question the company faces is deciding how many units of a new toy should be purchased to meet anticipated sales demand. If too few are purchased, sales will be lost; if too many are purchased, profits will be reduced because of low prices realized in clearance sales.

For the coming season, Prix-Fischer plans to introduce a new talking bear product called Chattiest Teddy. As usual, Prix-Fischer faces the decision of how many Chattiest Teddy units to order for the coming holiday season. Members of the management

team suggested order quantities of 15 000, 18 000, 24 000 or 28 000 units. The wide range of order quantities suggested, indicate considerable disagreement concerning the market potential. The product management team asks you for an analysis of the stock-out probabilities for various order quantities, an estimate of the profit potential, and to help make an order quantity recommendation.

Prix-Fischer expects to sell Chattiest Teddy for €24 based on a cost of €16 per unit. If inventory remains after the holiday season, Prix-Fischer will sell all surplus inventory for €5 per unit. After reviewing the sales history of similar products, Prix-Fischer's

senior sales forecaster predicted an expected demand of 20 000 units with a 0.90 probability that demand would be between 10 000 units and 30 000 units.

#### Managerial report

Prepare a managerial report that addresses the following issues and recommends an order quantity for the Chattiest Teddy product.

1. Use the sales forecaster's prediction to describe a normal probability distribution that can be used to approximate the demand distribution. Sketch the distribution and show its mean and standard deviation.
2. Compute the probability of a stock-out for the order quantities suggested by members of the management team.
3. Compute the projected profit for the order quantities suggested by the management team under three scenarios: worst case in which sales = 10 000 units, most likely case in which sales = 20 000 units, and best case in which sales = 30 000 units.
4. One of Prix-Fischer's managers felt that the profit potential was so great that the order quantity should have a 70 per cent chance of meeting demand and only a 30 per cent chance of any stock-outs. What quantity would be ordered under this policy, and what is the projected profit under the three sales scenarios?
5. Provide your own recommendation for an order quantity and note the associated profit projections. Provide a rationale for your recommendation.



## CASE PROBLEM 2



### Queuing patterns in a retail furniture store

The assistant manager of one of the larger stores in a retail chain selling furniture and household appliances has recently become interested in using quan-

titative techniques in the store operation. To help resolve a longstanding queuing problem, data have been collected on the time between customer arrivals and the time that a given number of customers were in a particular store department. Relevant details are summarized in Tables 6.1 and 6.2 respectively. Corresponding data on service times per customer are tabulated in Table 6.3.

In order to arrive at an appropriate solution strategy for the department's queuing difficulties, the manager has come to you for advice on possible statistical patterns that might apply to this information.

- By plotting the arrival and service patterns shown in Tables 6.1 and 6.3, show that they can each be reasonably represented by an exponential distribution.

**TABLE 6.1** Time between arrivals (during a four-hour period)

| Time between arrivals (in minutes) | Frequency |
|------------------------------------|-----------|
| 0.0 < 0.2                          | 31        |
| 0.2 < 0.4                          | 32        |
| 0.4 < 0.6                          | 23        |
| 0.6 < 0.8                          | 21        |
| 0.8 < 1.0                          | 19        |
| 1.0 < 1.2                          | 11        |
| 1.2 < 1.4                          | 14        |
| 1.4 < 1.6                          | 8         |
| 1.6 < 1.8                          | 6         |
| 1.8 < 2.0                          | 9         |
| 2.0 < 2.2                          | 6         |
| 2.2 < 2.4                          | 4         |
| 2.4 < 2.6                          | 5         |
| 2.6 < 2.8                          | 4         |
| 2.8 < 3.0                          | 4         |
| 3.0 < 3.2                          | 3         |
| More than 3.2                      | 10        |

**TABLE 6.2** Time that  $n$  customers were in the department (during a four-hour period)

| Number of customers, $n$ | Time (in minutes) |
|--------------------------|-------------------|
| 0                        | 16.8              |
| 1                        | 35.5              |
| 2                        | 52                |
| 3                        | 49                |
| 4                        | 29.3              |
| 5                        | 18.8              |
| 6                        | 13.6              |
| 7                        | 9.6               |
| 8                        | 5.8               |
| More than 8              | 9.6               |

**TABLE 6.3** Service time (from a sample of 31)

| Service time (in minutes) | Frequency |
|---------------------------|-----------|
| Less than 1               | 5         |
| 1 < 2                     | 7         |
| 2 < 3                     | 6         |
| 3 < 4                     | 4         |
| 4 < 5                     | 2         |
| 5 < 6                     | 3         |
| 6 < 7                     | 1         |
| 7 < 8                     | 1         |
| 8 < 9                     | 1         |
| 9 < 10                    | 0         |
| More than 10              | 1         |

- If  $\lambda$  = mean arrival rate for the queuing system here and  $\mu$ , the mean service rate for each channel, estimate the mean arrival time ( $1/\lambda$ ) and mean service time ( $1/\mu$ ) respectively.
- If  $k$  = the number of service channels and the mean service time for the system (store) is greater than the mean arrival rate (i.e.  $k\mu > \lambda$ ) then the following formulae can be shown to apply to the system 'in the steady state' subject to certain additional mathematical assumptions.<sup>1</sup>

- The probability there are no customers in the system

$$p(0) = \frac{1}{\sum_{n=0}^{k-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^k}{k!} \frac{k\mu}{(k\mu - \lambda)}}$$

- The average number of customers in the queue

$$L_q = \frac{(\lambda/\mu)^k \lambda \mu}{(k-1)! (k\mu - \lambda)^2} p(0)$$

- The average number of customers in the store

$$L = L_q + \frac{\lambda}{\mu}$$

- The average time a customer spends in the queue

$$w_q = \frac{L_q}{\lambda}$$

<sup>1</sup>The queue has two or more channels; the mean service rate  $\mu$  is the same for each channel; arrivals wait in a single queue and then move to the first open channel for service; the queue discipline is first-come, first-served (FCFS).



- e. The average time a customer spends in the store

$$W = W_q + \frac{1}{\mu}$$

- f. The probability of  $n$  customers in the system

$$p(n) = \frac{(\lambda + \mu)^n}{n!} p(0) \quad \text{for } n \leq k$$

$$p(n) = \frac{(\lambda + \mu)^n}{k! k^{(n-k)}} p(0) \quad \text{for } n > k$$

According to this model, what is the smallest value that  $k$  can take? If this is the number of channels that the retailer currently operates, estimate the above operating characteristics for the store. How would these values change if the  $k$  channels were increased by one or two extra channels? Discuss what factors might influence the retailer in arriving at an appropriate value of  $k$ .



# 7 Sampling and Sampling Distributions

## CHAPTER CONTENTS

Statistics in Practice Copyright and Public Lending Right

- 7.1 The EAI sampling problem
- 7.2 Simple random sampling
- 7.3 Point estimation
- 7.4 Introduction to sampling distributions
- 7.5 Sampling distribution of  $\bar{X}$
- 7.6 Sampling distribution of  $P$

**LEARNING OBJECTIVES** After studying this chapter and doing the exercises, you should be able to:

- 1 Explain the terms simple random sample, sampling with replacement and sampling without replacement.
- 2 Select a simple random sample from a finite population using random number tables.
- 3 Explain the terms parameter, statistic, point estimator and unbiasedness.
- 4 Identify relevant point estimators for a population mean, population standard deviation and population proportion.
- 5 Explain the term sampling distribution.
- 6 Describe the form and characteristics of the sampling distribution:
  - 6.1 of the sample mean, when the sample size is large or when the population is normal.
  - 6.2 of the sample proportion when the sample size is large.

In Chapter 1, we defined the terms *element*, *population* and *sample*:

- An *element* is the entity on which data are collected.
- A *population* is the set of all the elements of interest in a study.
- A *sample* is a subset of the population.



## STATISTICS IN PRACTICE

### Copyright and Public Lending Right

**H**ow would you feel if the size of your income was determined each year by a random sampling procedure? This is the situation that often exists, for at least part of annual income, for musicians and other artists who receive copyright payments for the performance or broadcasting of their work. Even in this 21st-century world of large databases and sophisticated communication, it is not always possible, or it is too costly, to maintain 100 per cent



checks on what is being broadcast over TV, radio and the Internet, so an alternative is to sample.

In a similar vein, many book authors receive payments through a Public Lending Right (PLR) scheme. This is particularly so for authors of fiction, or of popular non-fiction, whose books are available for loan in public libraries. A PLR scheme is intended to compensate authors for potential loss of income because their books are available in public libraries, and are therefore borrowed rather than bought by readers. The website [www.plrinternational.com](http://www.plrinternational.com) listed 30 countries in mid-2012 with established PLR schemes. All except Australia, Canada and New Zealand were in Europe.

The UK PLR scheme was set up in 1979. From the outset, it was decided that it would be too costly to try and collect data from all libraries in the UK. Data on lending are collected from a sample of libraries. Similar decisions have been made in many of the other countries that operate PLR schemes. The current UK sample is reckoned to cover about 15 per cent of all library authorities in the UK (there are over 200). The PLR scheme in Finland, as another example, is estimated to cover about 10 per cent of all library loans.

The examples from copyright and PLR are cases where the sampling schemes involved can influence the income of individuals – the copyright holders or authors. The website for the UK PLR scheme acknowledges, for example, that authors of books with a ‘local interest’ – local history, say – are likely to qualify for PLR payments only if the library sample for the year contains library authorities in the relevant geographical area.

Companies and governments often make important decisions based on sample data. This chapter examines the basis and practicalities of scientific sampling.

The reason we sample is to collect data to make an inference and answer a research question about a population. Numerical characteristics of a population (e.g. population mean, population standard deviation) are called **parameters**. Numerical characteristics of a sample (e.g. sample mean, sample standard deviation) are called **sample statistics**. Primary purposes of statistical inference are to make estimates and test hypotheses about population parameters using sample statistics.

Here are two situations in which samples provide estimates of population parameters:

- 1 A European car tyre manufacturer developed a new tyre designed to provide an increase in tyre lifetime. To estimate the mean lifetime (in kilometres or miles) of the new tyre, the manufacturer

selected a sample of 120 new tyres for testing. The test results provided a sample mean of 56 000 kilometres (35 000 miles). Therefore, an estimate of the mean tyre lifetime for the population of new tyres was 56 000 kilometres.

- 2** Members of an African government were interested in estimating the proportion of registered voters likely to support a proposal for constitutional reform to be put to the electorate in a national referendum. The time and cost associated with contacting every individual in the population of registered voters were prohibitive. A sample of 5000 registered voters was therefore selected, and 2810 of the 5000 voters indicated support for the proposal. An estimate of the proportion of the population of registered voters supporting the proposal was  $2810/5000 = 0.562$ .

These two examples illustrate some of the reasons why samples are used. In the tyre lifetime example, collecting the data on tyre life involves wearing out each tyre tested. Clearly it is not feasible to test every tyre in the population. A sample is the only realistic way to obtain the tyre lifetime data. In the example involving the referendum, contacting every registered voter in the population is in principle possible, but the time and cost are prohibitive. Consequently, a sample of registered voters is preferred.

It is important to realize that sample results provide only *estimates* of the values of the population characteristics, because the sample contains only a portion of the population. A sample mean provides an estimate of a population mean, and a sample proportion provides an estimate of a population proportion. Some estimation error can be expected. This chapter provides the basis for determining how large the estimation error might be. With proper sampling methods, the sample results will provide ‘good’ estimates of the population parameters.

Let us define some of the terms used in sampling. The **sampled population** is the population from which the sample is drawn, and a **sampling frame** is a list of the elements from which the sample will be selected. In the second example above, the sampled population is all registered voters in the country, and the sampling frame is the list of all registered voters. Because the number of registered voters is finite, this is an illustration of sampling from a finite population. In Section 7.2, we consider how a simple random sample can be selected from a finite population.

The sampled population for the tyre lifetime example is more difficult to define. The sample of 120 tyres was obtained from a production process at a particular point in time. We can think of the sampled population as the conceptual population of all tyres that could be made by the production process under similar conditions to those prevailing at the time of sampling. In this context, the sampled population is considered infinite, making it impossible to construct a sampling frame. In Section 7.2, we consider how to select a random sample in such a situation.

We first show how simple random sampling can be used to select a sample from finite and from infinite populations. We then show how data from a simple random sample can be used to compute estimates of a population mean, a population standard deviation, and a population proportion. In addition, we introduce the important concept of a sampling distribution. Knowledge of the appropriate sampling distribution enables us to make statements about how close the sample estimates might be to the corresponding population parameters.

## 7.1 THE EAI SAMPLING PROBLEM

The head of personnel services for E-Applications & Informatics plc (EAI) has been given the task of constructing a profile of the company’s 2500 managers. The characteristics to be identified include the mean annual salary and the proportion of managers who have completed the company’s management training programme. The 2500 managers are the population for this study. We can find the annual salary and training programme status for each individual by referring to the firm’s personnel records. The data file containing this information for all 2500 managers in the population is on the online platform, in the file ‘EAI’.

Using the EAI data set and the formulae from Chapter 3, we calculate the population mean and the population standard deviation for the annual salary data.

$$\text{Population mean: } \mu = €51\,800$$

$$\text{Population standard deviation: } \sigma = €4000$$



EAI

The data set shows that 1500 of the 2500 managers completed the training programme. Let  $\pi$  denote the proportion of the population that completed the training programme:  $\pi = 1500/2500 = 0.60$ . The population mean annual salary ( $\mu = €51\,800$ ), the population standard deviation of annual salary ( $\sigma = €4000$ ), and the population proportion that completed the training programme ( $\pi = 0.60$ ) are parameters of the population of EAI managers.

Now, suppose the necessary information on all the EAI managers was *not* readily available in the company's database. How can the head of personnel services obtain estimates of the population parameters by using a sample of managers, rather than all 2500 managers in the population? Suppose a sample of 30 managers will be used. Clearly, the time and the cost of constructing a profile would be substantially less for 30 managers than for the entire population. If the head of personnel could be assured that a sample of 30 managers would provide adequate information about the population of 2500 managers, working with a sample would be preferable to working with the entire population. Often the cost of collecting information from a sample is substantially less than from a population, especially when personal interviews must be conducted to collect the information.

First we consider how we can identify a sample of 30 managers.

## 7.2 SIMPLE RANDOM SAMPLING

Several methods can be used to select a sample from a population. One important method is **simple random sampling**. The definition of a simple random sample and the process of selecting such a sample depend on whether the population is *finite* or *infinite*. We first consider sampling from a finite population, because the EAI sampling problem involves a finite population of 2500 managers.

### Sampling from a finite population

#### Simple random sample (finite population)

A simple random sample of size  $n$  from a finite population of size  $N$  is a sample selected such that each possible sample of size  $n$  has the same probability of being selected.

One procedure for selecting a simple random sample from a finite population is to choose the elements for the sample one at a time in such a way that, at each step, each of the elements remaining in the population has the same probability of being selected.

To select a simple random sample from the population of EAI managers, we first assign each manager a number. We can assign the managers the numbers 1 to 2500 in the order their names appear in the EAI personnel file. Next, we refer to the table of random numbers shown in Table 7.1. Using the first row of the table, each digit, 6, 3, 2, ..., is a random digit with an equal chance of occurring. The random numbers in the table are shown in groups of five for readability. Because the largest number in the population list, 2500, has four digits, we shall select random numbers from the table in groups of four digits. We may start the selection of random numbers anywhere in the table and move systematically in a direction of our choice. We shall use the first row of Table 7.1 and move from left to right. The first seven four-digit random numbers are

6327 1599 8671 7445 1102 1514 1807

These four-digit numbers are equally likely, because the numbers in the table are random. We use them to give each manager in the population an equal chance of being included in the random sample.

The first number, 6327, is greater than 2500. We discard it because it does not correspond to one of the numbered managers in the population. The second number, 1599, is between 1 and 2500.

**TABLE 7.1** Random numbers

|       |       |       |       |       |       |       |       |       |       |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 63271 | 59986 | 71744 | 51102 | 15141 | 80714 | 58683 | 93108 | 13554 | 79945 |
| 88547 | 09896 | 95436 | 79115 | 08303 | 01041 | 20030 | 63754 | 08459 | 28364 |
| 55957 | 57243 | 83865 | 09911 | 19761 | 66535 | 40102 | 26646 | 60147 | 15702 |
| 46276 | 87453 | 44790 | 67122 | 45573 | 84358 | 21625 | 16999 | 13385 | 22782 |
| 55363 | 07449 | 34835 | 15290 | 76616 | 67191 | 12777 | 21861 | 68689 | 03263 |
| 69393 | 92785 | 49902 | 58447 | 42048 | 30378 | 87618 | 26933 | 40640 | 16281 |
| 13186 | 29431 | 88190 | 04588 | 38733 | 81290 | 89541 | 70290 | 40113 | 08243 |
| 17726 | 28652 | 56836 | 78351 | 47327 | 18518 | 92222 | 55201 | 27340 | 10493 |
| 36520 | 64465 | 05550 | 30157 | 82242 | 29520 | 69753 | 72602 | 23756 | 54935 |
| 81628 | 36100 | 39254 | 56835 | 37636 | 02421 | 98063 | 89641 | 64953 | 99337 |
| 84649 | 48968 | 75215 | 75498 | 49539 | 74240 | 03466 | 49292 | 36401 | 45525 |
| 63291 | 11618 | 12613 | 75055 | 43915 | 26488 | 41116 | 64531 | 56827 | 30825 |
| 70502 | 53225 | 03655 | 05915 | 37140 | 57051 | 48393 | 91322 | 25653 | 06543 |
| 06426 | 24771 | 59935 | 49801 | 11082 | 66762 | 94477 | 02494 | 88215 | 27191 |
| 20711 | 55609 | 29430 | 70165 | 45406 | 78484 | 31639 | 52009 | 18873 | 96927 |
| 41990 | 70538 | 77191 | 25860 | 55204 | 73417 | 83920 | 69468 | 74972 | 38712 |
| 72452 | 36618 | 76298 | 26678 | 89334 | 33938 | 95567 | 29380 | 75906 | 91807 |
| 37042 | 40318 | 57099 | 10528 | 09925 | 89773 | 41335 | 96244 | 29002 | 46453 |
| 53766 | 52875 | 15987 | 46962 | 67342 | 77592 | 57651 | 95508 | 80033 | 69828 |
| 90585 | 58955 | 53122 | 16025 | 84299 | 53310 | 67380 | 84249 | 25348 | 04332 |
| 32001 | 96293 | 37203 | 64516 | 51530 | 37069 | 40261 | 61374 | 05815 | 06714 |
| 62606 | 64324 | 46354 | 72157 | 67248 | 20135 | 49804 | 09226 | 64419 | 29457 |
| 10078 | 28073 | 85389 | 50324 | 14500 | 15562 | 64165 | 06125 | 71353 | 77669 |
| 91561 | 46145 | 24177 | 15294 | 10061 | 98124 | 75732 | 00815 | 83452 | 97355 |
| 13091 | 98112 | 53959 | 79607 | 52244 | 63303 | 10413 | 63839 | 74762 | 50289 |

So the first manager selected for the random sample is number 1599 on the list of EAI managers. Continuing this process, we ignore the numbers 8671 and 7445 (greater than 2500) before identifying managers numbered 1102, 1514 and 1807 to be included in the random sample. This process continues until the simple random sample of 30 EAI managers has been obtained.

It is possible that a random number already used may appear again in the table before the sample of 30 EAI managers has been fully selected. Because we do not want to select a manager more than once, any previously used random numbers are ignored. Selecting a sample in this manner is referred to as **sampling without replacement**. If we selected a sample such that previously used random numbers are acceptable, and specific managers could be included in the sample two or more times, we would be **sampling with replacement**. Sampling with replacement is a valid way of identifying a simple random sample, but sampling without replacement is used more often. When we refer to simple random sampling, we shall assume that the sampling is without replacement.

Computer-generated random numbers can also be used to implement the random sample selection process. EXCEL, MINITAB and IBM SPSS all provide functions for generating random numbers.

The number of different simple random samples of size  $n$  that can be selected from a finite population of size  $N$  is:

$$\frac{N!}{n!(N-n)!}$$

$N!$ ,  $(N - n)!$  and  $n!$  are the factorial computations discussed in Chapter 4. For the EAI problem with  $N = 2500$  and  $n = 30$ , this expression can be used to show that approximately  $2.75 \times 10^{69}$  different simple random samples of 30 EAI managers can be selected.

## Sampling from an infinite population

In some situations, the population is either infinite, or so large that for practical purposes it must be treated as infinite. For example, suppose that a fast-food restaurant would like to obtain a profile of its customers by selecting a simple random sample of customers and asking each customer to complete a short questionnaire. The ongoing process of customer visits to the restaurant can be viewed as coming from an infinite population. In practice, a population is usually considered infinite if it involves an ongoing process that makes listing or counting every element in the population impossible. The definition of a simple random sample from an infinite population follows.

### Simple random sample (infinite population)

A simple random sample from an infinite population is a sample selected such that the following conditions are satisfied:

1. Each element selected comes from the population.
2. Each element is selected independently.

For the example of a simple random sample of customers at a fast-food restaurant, any customer who comes into the restaurant will satisfy the first requirement. The second requirement will be satisfied if a sample selection procedure is devised to select the items independently and thereby avoid any selection bias that gives higher selection probabilities to certain types of customers. Selection bias would occur if, for instance, five consecutive customers selected were all friends who arrived together. We might expect these customers to exhibit similar profiles. Selection bias can be avoided by ensuring that the selection of a particular customer does not influence the selection of any other customer. In other words, the customers must be selected independently.

Infinite populations are often associated with an ongoing process that operates continuously over time. For example, parts being manufactured on a production line, transactions occurring at a bank, telephone calls arriving at a technical support centre and customers entering stores may all be viewed as coming from an infinite population. In such cases, an effective sampling procedure will ensure that no selection bias occurs and that the sample elements are selected independently.

## EXERCISES

### Methods

1. Consider a finite population with five elements labelled A, B, C, D and E. Ten possible simple random samples of size 2 can be selected.
  - a. List the ten samples beginning with AB, AC and so on.
  - b. Using simple random sampling, what is the probability that each sample of size 2 is selected?
  - c. Assume random number 1 corresponds to A, random number 2 corresponds to B, and so on. List the simple random sample of size 2 that will be selected by using the random digits 8 0 5 7 5 3 2.



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2. Assume a finite population has 350 elements. Using the last three digits of each of the following five-digit random numbers (601, 022, 448, ...), determine the first four elements that will be selected for the simple random sample.

98601 73022 83448 02147 34229 27553 84147 93289 14209

### Applications

3. The EURO STOXX 50 share index is calculated using data for 50 blue-chip companies from 12 Eurozone countries. Assume you want to select a simple random sample of five companies from the EURO STOXX 50 list. Use the last three digits in column 9 of Table 7.1, beginning with 554. Read down the column and identify the numbers of the five companies that would be selected.
4. A student union is interested in estimating the proportion of students who favour a mandatory 'pass-fail' grading policy for optional courses. A list of names and addresses of the 645 students enrolled during the current semester is available from the registrar's office. Using three-digit random numbers in row 10 of Table 7.1 and moving across the row from left to right, identify the first ten students who would be selected using simple random sampling. The three-digit random numbers begin with 816, 283 and 610.
5. Assume that we want to identify a simple random sample of 12 of the 372 doctors practising in a particular city. The doctors' names are available from the local health authority. Use the eighth column of five-digit random numbers in Table 7.1 to identify the 12 doctors for the sample. Ignore the first two random digits in each five-digit grouping of the random numbers. This process begins with random number 108 and proceeds down the column of random numbers.
6. Indicate whether the following populations should be considered finite or infinite.
  - a. All registered voters in Ireland.
  - b. All television sets that could be produced by the Johannesburg factory of the TV-M Company.
  - c. All orders that could be processed by a mail-order firm.
  - d. All emergency telephone calls that could come into a local police station.
  - e. All components that Fibercon plc produced on the second shift on 17 February 2013.

## 7.3 POINT ESTIMATION

We return to the EAI problem. A simple random sample of 30 managers and the corresponding data on annual salary and management training programme participation are shown in Table 7.2. The notation  $x_1$ ,  $x_2$  and so on is used to denote the annual salary of the first manager in the sample, the annual salary of the second manager in the sample and so on. Completion of the management training programme is indicated by Yes in the relevant column.

To estimate the value of a population parameter, we compute a corresponding characteristic of the sample, referred to as a sample statistic. For example, to estimate the population mean  $\mu$  and the population standard deviation  $\sigma$  for the annual salary of EAI managers, we use the data in Table 7.2 to calculate the corresponding sample statistics: the sample mean and the sample standard deviation. Using the formulae from Chapter 3, the sample mean is:

$$\bar{x} = \frac{\sum x_i}{n} = \frac{1\ 554\ 420}{30} = 51\ 814 \text{ (€)}$$

**TABLE 7.2** Annual salary and training programme status for a simple random sample of 30 EAI managers

| Annual salary (€)     | Management training programme | Annual salary (€)     | Management training programme |
|-----------------------|-------------------------------|-----------------------|-------------------------------|
| $x_1 = 49\,094.30$    | Yes                           | $x_{16} = 51\,766.00$ | Yes                           |
| $x_2 = 53\,263.90$    | Yes                           | $x_{17} = 52\,541.30$ | No                            |
| $x_3 = 49\,643.50$    | Yes                           | $x_{18} = 44\,980.00$ | Yes                           |
| $x_4 = 49\,894.90$    | Yes                           | $x_{19} = 51\,932.60$ | Yes                           |
| $x_5 = 47\,621.60$    | No                            | $x_{20} = 52\,973.00$ | Yes                           |
| $x_6 = 55\,924.00$    | Yes                           | $x_{21} = 45\,120.90$ | Yes                           |
| $x_7 = 49\,092.30$    | Yes                           | $x_{22} = 51\,753.00$ | Yes                           |
| $x_8 = 51\,404.40$    | Yes                           | $x_{23} = 54\,391.80$ | No                            |
| $x_9 = 50\,957.70$    | Yes                           | $x_{24} = 50\,164.20$ | No                            |
| $x_{10} = 55\,109.70$ | Yes                           | $x_{25} = 52\,973.60$ | No                            |
| $x_{11} = 45\,922.60$ | Yes                           | $x_{26} = 50\,241.30$ | No                            |
| $x_{12} = 57\,268.40$ | No                            | $x_{27} = 52\,793.90$ | No                            |
| $x_{13} = 55\,688.80$ | Yes                           | $x_{28} = 50\,979.40$ | Yes                           |
| $x_{14} = 51\,564.70$ | No                            | $x_{29} = 55\,860.90$ | Yes                           |
| $x_{15} = 56\,188.20$ | No                            | $x_{30} = 57\,309.10$ | No                            |

and the sample standard deviation is:

$$s = \sqrt{\frac{\sum(x_i - \bar{x})^2}{n-1}} = \sqrt{\frac{325\,009\,260}{29}} = 3348 \quad (\text{€})$$

To estimate  $\pi$ , the proportion of managers in the population who completed the management training programme, we use the corresponding sample proportion. Let  $m$  denote the number of managers in the sample who completed the management training programme. The data in Table 7.2 show that  $m = 19$ . So, with a sample size of  $n = 30$ , the sample proportion is:

$$p = \frac{m}{n} = \frac{19}{30} = 0.63$$

These computations are an example of the statistical procedure called *point estimation*. We refer to the sample mean as the **point estimator** of the population mean  $\mu$ , the sample standard deviation as the point estimator of the population standard deviation  $\sigma$ , and the sample proportion as the point estimator of the population proportion  $\pi$ . The numerical value obtained for the sample mean, sample standard deviation or sample proportion is called a **point estimate**. For the simple random sample of 30 EAI managers shown in Table 7.2, €51 814 is the point estimate of  $\mu$ , €3348 is the point estimate of  $\sigma$  and 0.63 is the point estimate of  $\pi$ .

**TABLE 7.3** Summary of point estimates obtained from a simple random sample of 30 EAI managers

| Population parameter   | Parameter value    | Point estimator  | Point estimate       |
|--|--------------------|--|----------------------|
| Population mean annual salary  | $\mu = €51\,800$   | Sample mean annual salary  | $\bar{x} = €51\,814$ |
| Population standard deviation for annual salary                            | $\sigma = €4\,000$ | Sample standard deviation for annual salary                            | $s = €3\,348$        |
| Population proportion who have completed the management training programme | $\pi = 0.60$       | Sample proportion who have completed the management training programme | $p = 0.63$           |

Table 7.3 summarizes the sample results and compares the point estimates to the actual values of the population parameters.

The point estimates in Table 7.3 differ somewhat from the corresponding population parameters. This difference is to be expected because a sample, rather than a census of the entire population, is being used to obtain the point estimates. In the next chapter, we shall show how to construct an interval estimate in order to provide information about how close the point estimate is to the population parameter.

## Practical advice

The subject matter of most of the rest of the book is statistical inference. Point estimation is a form of statistical inference. We use a sample statistic to make an inference about a population parameter. When making inferences about a population based on a sample, it is important to have a close correspondence between the sampled population and the target population. The **target population** is the population we want to make inferences about, while the sampled population is the population from which the sample is actually taken. In this section, we have described the process of drawing a simple random sample from the population of EAI managers and making point estimates of characteristics of that same population. So the sampled population and the target population are identical, which is the desired situation. But in other cases, it is not as easy to obtain a close correspondence between the sampled and target populations.

Consider the case of a theme park selecting a sample of its customers to learn about characteristics such as age and time spent at the park. Suppose all the sample elements were selected on a day when park attendance was restricted to employees of a large company. Then the sampled population would be composed of employees of that company and members of their families. If the target population we wanted to make inferences about were typical park customers over a typical summer, then there might be a substantial difference between the sampled population and the target population. In such a case, we would question the validity of the point estimates being made. The park management would be in the best position to know whether a sample taken on a particular day was likely to be representative of the target population.

In summary, whenever a sample is used to make inferences about a population, we should make sure that the study is designed so that the sampled population and the target population are in close agreement. Good judgement is a necessary ingredient of sound statistical practice.

## EXERCISES

### Methods

7. The following data are from a simple random sample.

5    8    10    7    10    14

- Calculate a point estimate of the population mean.
  - Calculate a point estimate of the population standard deviation.
8. A survey question for a sample of 150 individuals yielded 75 Yes responses, 55 No responses and 20 No Opinion responses.
- Calculate a point estimate of the proportion in the population who respond Yes.
  - Calculate a point estimate of the proportion in the population who respond No.

### Applications

9. A simple random sample of five months of sales data provided the following information:

|             |    |     |    |    |    |
|-------------|----|-----|----|----|----|
| Month:      | 1  | 2   | 3  | 4  | 5  |
| Units sold: | 94 | 100 | 85 | 94 | 92 |



- a. Calculate a point estimate of the population mean number of units sold per month.  
 b. Calculate a point estimate of the population standard deviation.
- 10.** The data set Mutual Fund contains data on a sample of 40 mutual funds. These were randomly selected from 283 funds featured in *Business Week*. Use the data set to answer the following questions.
- Compute a point estimate of the proportion of the *Business Week* mutual funds that are load funds.
  - Compute a point estimate of the proportion of the funds that are classified as high risk.
  - Compute a point estimate of the proportion of the funds that have a below-average risk rating.
- 11.** In a YouGov opinion poll for the *Financial Times* in late June 2012, during the ‘Euro crisis’, a sample of 1033 German adults was asked ‘If there were a referendum tomorrow on Germany’s membership of the single currency, the euro, how would you vote?’ The responses were:

|                                |     |
|--------------------------------|-----|
| To stay in the single currency | 444 |
| To bring back the Deutschmark  | 424 |
| Would not vote                 | 72  |
| Don’t know                     | 93  |

Calculate point estimates of the following population parameters:

- The proportion of all adults who would vote to stay in the single currency.
  - The proportion of all adults who vote to bring back the Deutschmark.
  - The proportion of all adults who would not vote or don’t know.
- 12.** Many drugs used to treat cancer are expensive. *Business Week* reported on the cost per treatment of Herceptin, a drug used to treat breast cancer. Typical treatment costs (in dollars) for Herceptin are provided by a simple random sample of ten patients.

|      |      |      |      |      |
|------|------|------|------|------|
| 4376 | 5578 | 2717 | 4920 | 4495 |
| 4798 | 6446 | 4119 | 4237 | 3814 |

- Calculate a point estimate of the mean cost per treatment with Herceptin.
- Calculate a point estimate of the standard deviation of the cost per treatment with Herceptin.



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## 7.4 INTRODUCTION TO SAMPLING DISTRIBUTIONS

For the simple random sample of 30 EAI managers in Table 7.2, the point estimate of  $\mu$  is  $\bar{x} = €51\,814$  and the point estimate of  $\pi$  is  $p = 0.63$ . Suppose we select another simple random sample of 30 EAI managers and obtain the following point estimates:

Sample mean:  $\bar{x} = €52\,670$

Sample proportion:  $p = 0.70$

Note that different values of the sample mean and sample proportion were obtained. A second simple random sample of 30 EAI managers cannot be expected to provide exactly the same point estimates as the first sample.

Now, suppose we repeat the process of selecting a simple random sample of 30 EAI managers over and over again, each time computing the values of the sample mean and sample proportion. Table 7.4 contains a portion of the results obtained for 500 simple random samples, and Table 7.5 shows the

**TABLE 7.4** Values  $\bar{x}$  and  $p$  from 500 simple random samples of 30 EAI managers

| Sample number | Sample mean ( $\bar{x}$ ) | Sample proportion ( $p$ ) |
|---------------|---------------------------|---------------------------|
| 1             | 51 814                    | 0.63                      |
| 2             | 52 670                    | 0.70                      |
| 3             | 51 780                    | 0.67                      |
| 4             | 51 588                    | 0.53                      |
| .             | .                         | .                         |
| 500           | 51 752                    | 0.50                      |

**TABLE 7.5** Frequency distribution of  $\bar{X}$  values from 500 simple random samples of 30 EAI managers

| Mean annual salary (€) | Frequency | Relative frequency |
|------------------------|-----------|--------------------|
| 49 500.00–49 999.99    | 2         | 0.004              |
| 50 000.00–50 499.99    | 16        | 0.032              |
| 50 500.00–50 999.99    | 52        | 0.104              |
| 51 000.00–51 499.99    | 101       | 0.202              |
| 51 500.00–51 999.99    | 133       | 0.266              |
| 52 000.00–52 499.99    | 110       | 0.220              |
| 52 500.00–52 999.99    | 54        | 0.108              |
| 53 000.00–53 499.99    | 26        | 0.052              |
| 53 500.00–53 999.99    | 6         | 0.012              |
| Totals 500             |           | 1.000              |

frequency and relative frequency distributions for the 500 values. Figure 7.1 shows the relative frequency histogram for the values.

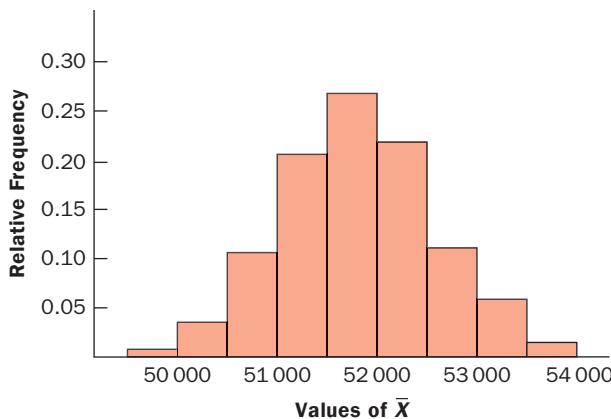
In Chapter 5 we defined a random variable as a numerical description of the outcome of an experiment. If we consider selecting a simple random sample as an experiment, the sample mean is a numerical description of the outcome of the experiment. So, the sample mean is a random variable. In accordance with the naming conventions for random variables described in Chapters 5 and Chapter 6 (i.e. use of capital letters for names of random variables), we denote this random variable  $\bar{X}$ . Just like other random variables,  $\bar{X}$  has a mean or expected value, a standard deviation and a probability distribution. Because the various possible values of  $\bar{X}$  are the result of different simple random samples, the probability distribution of  $\bar{X}$  is called the **sampling distribution** of  $\bar{X}$ . Knowledge of this sampling distribution will enable us to make probability statements about how close the sample mean is to the population mean  $\mu$ .

Let us return to Figure 7.1. We would need to enumerate every possible sample of 30 managers and compute each sample mean to completely determine the sampling distribution of  $\bar{X}$ . However, the histogram of 500  $\bar{x}$  values gives an approximation of this sampling distribution. From the approximation we observe the bell-shaped appearance of the distribution. We note that the largest concentration of the  $\bar{x}$  values and the mean of the 500  $\bar{x}$  values are near the population mean  $\mu = €51\,800$ . We shall describe the properties of the sampling distribution of  $\bar{X}$  more fully in the next section.

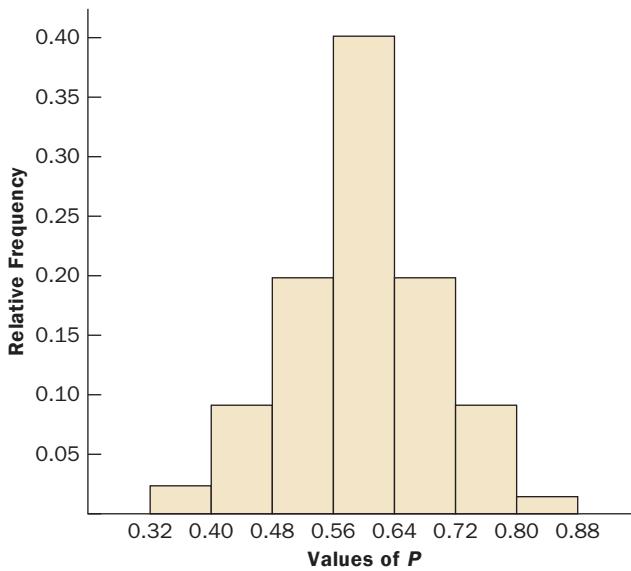
The 500 values of the sample proportion are summarized by the relative frequency histogram in Figure 7.2. As in the case of the sample mean, the sample proportion is a random variable, which we denote  $P$ . If every possible sample of size 30 were selected from the population and if a value  $p$  were computed for each sample, the resulting distribution would be the sampling distribution of  $P$ . The relative frequency histogram of the 500 sample values in Figure 7.2 provides a general idea of the appearance of the sampling distribution of  $P$ .

**FIGURE 7.1**

Relative frequency histogram of sample mean values from 500 simple random samples of size 30 each

**FIGURE 7.2**

Relative frequency histogram of sample proportion values from 500 simple random samples of size 30 each



In practice, we select only one simple random sample from the population for estimating population characteristics. We repeated the sampling process 500 times in this section simply to illustrate that many different samples are possible and that the different samples generate a variety of values  $\bar{x}$  and  $p$  for the sample statistics  $\bar{X}$  and  $P$ . The probability distribution of any particular sample statistic is called the sampling distribution of the statistic. In Section 7.5 we show the characteristics of the sampling distribution of  $\bar{X}$ . In Section 7.6 we show the characteristics of the sampling distribution of  $P$ . The ability to understand the material in subsequent chapters depends heavily on the ability to understand and use the sampling distributions presented in this chapter.

## 7.5 SAMPLING DISTRIBUTION OF $\bar{X}$

This section describes the properties of the sampling distribution of  $\bar{X}$ . Just as with other probability distributions we have studied, the sampling distribution of  $\bar{X}$  has an expected value or mean, a standard deviation and a characteristic shape or form. We begin by considering the expected value of  $\bar{X}$ .

### Sampling distribution of $\bar{X}$

The sampling distribution of  $\bar{X}$  is the probability distribution of all possible values of the sample mean.

## Expected value of $\bar{X}$

Consider the  $\bar{X}$  values generated by the various possible simple random samples. The mean of all these values is known as the expected value of  $E(\bar{X})$ . Let  $\bar{X}$  represent the expected value of  $\bar{X}$ , and  $\mu$  represent the mean of the population from which we are selecting a simple random sample. It can be shown that with simple random sampling,  $E(\bar{X})$  and  $\mu$  are equal.

### Expected value of $\bar{X}$

where

$$E(\bar{X}) = \mu \quad (7.1)$$

$E(\bar{X})$  = the expected value of  $\bar{X}$

$\mu$  = the mean of the population from which the sample is selected

In Section 7.1 we saw that the mean annual salary for the population of EAI managers is  $\mu = 51\,800$ . So according to equation (7.1), the mean of all possible sample means for the EAI study is also €51 800.

When the expected value of a point estimator equals the population parameter, we say the point estimator is an **unbiased** estimator of the population parameter.

### Unbiasedness

The sample statistic  $Q$  is an unbiased estimator of the population parameter  $\theta$  if

$$E(Q) = \theta$$

where  $E(Q)$  is the expected value of the sample statistic  $Q$ .

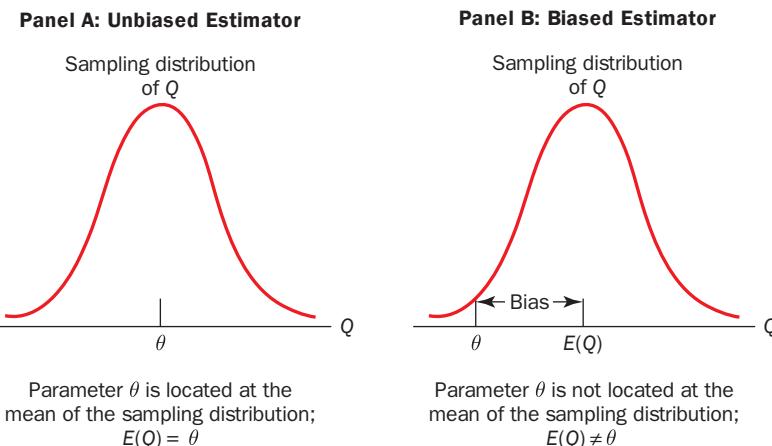
Figure 7.3 shows the cases of unbiased and biased point estimators. In the illustration showing the unbiased estimator, the mean of the sampling distribution is equal to the value of the population parameter. The estimation errors balance out in this case, because sometimes the value of the point estimator may be less than  $\theta$  and other times it may be greater than  $\theta$ .

In the case of a biased estimator, the mean of the sampling distribution is less than or greater than the value of the population parameter. In the illustration in Panel B of Figure 7.3,  $E(Q)$  is greater than  $\theta$ ; the sample statistic has a high probability of overestimating the value of the population parameter. The amount of the bias is shown in the figure.

Equation (7.1) shows that  $\bar{X}$  is an unbiased estimator of the population mean  $\mu$ .

**FIGURE 7.3**

Examples of unbiased and biased point estimators



## Standard deviation of $\bar{X}$

It can be shown that with simple random sampling, the standard deviation of  $\bar{X}$  depends on whether the population is finite or infinite. We use the following notation.

- $\sigma_{\bar{X}}$  = the standard deviation of  $\bar{X}$
- $\sigma$  = the standard deviation of the population
- $n$  = the sample size
- $N$  = the population size

### Standard deviation of $\bar{X}$

|  |  |              |
|--|--|--------------|
| Finite population  | Infinite population                          | <b>(7.2)</b> |
| $\sigma_{\bar{X}} = \sqrt{\frac{N-n}{N-1}} \left( \frac{\sigma}{\sqrt{n}} \right)$ | $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$ |              |

In comparing the two formulae in (7.2), we see that the factor  $\sqrt{(N-n)/(N-1)}$  is required for the finite population case but not for the infinite population case. This factor is commonly referred to as the **finite population correction factor**. In many practical sampling situations, we find that the population involved, although finite, is ‘large’, whereas the sample size is relatively ‘small’. In such cases the finite population correction factor is close to 1. As a result, the difference between the values of the standard deviation of  $\bar{X}$  for the finite and infinite population cases becomes negligible. Then,  $\sigma_{\bar{X}} = \sigma/\sqrt{n}$  becomes a good approximation to the standard deviation of  $\bar{X}$  even though the population is finite. This observation leads to the following general guideline, or rule of thumb, for computing the standard deviation of  $\bar{X}$ .

### Use the following expression to compute the standard deviation of $\bar{X}$

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} \quad \text{whenever:} \quad \text{1. The population is infinite; or}$$

2. The population is finite *and* the sample size is less than or equal to 5 per cent of the population size; that is,  $n / N \leq 0.05$ .

In cases where  $n/N > 0.05$ , the finite population version of formula (7.2) should be used in the computation of  $\sigma_{\bar{X}}$ . Unless otherwise noted, throughout the text we shall assume that the population size is 'large',  $n/N \leq 0.05$ , and expression (7.3) can be used to compute  $\sigma_{\bar{X}}$ .

To compute  $\sigma_{\bar{X}}$ , we need to know  $\sigma$ , the standard deviation of the population. To further emphasize the difference between  $\sigma_{\bar{X}}$  and  $\sigma$ , we refer to  $\sigma_{\bar{X}}$  as the **standard error** of the mean. The term standard error is used throughout statistical inference to refer to the standard deviation of a point estimator. Later we shall see that the value of the standard error of the mean is helpful in determining how far the sample mean may be from the population mean.

We return to the EAI example and compute the standard error of the mean associated with simple random samples of 30 EAI managers. In Section 7.1 we saw that the standard deviation of annual salary for the population of 2500 EAI managers is  $\sigma = 4000$ . In this case, the population is finite, with  $N = 2500$ . However, with a sample size of 30, we have  $n/N = 30/2500 = 0.012$ . Because the sample size is less than 5 per cent of the population size, we can ignore the finite population correction factor and use equation (7.3) to compute the standard error.

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{4000}{\sqrt{30}} = 730.3$$

## Form of the sampling distribution of $\bar{X}$

The preceding results concerning the expected value and standard deviation for the sampling distribution of  $\bar{X}$  are applicable for any population. The final step in identifying the characteristics of the sampling distribution of  $\bar{X}$  is to determine the form or shape of the sampling distribution. We shall consider two cases: (1) the population has a normal distribution; and (2) the population does not have a normal distribution.

### **Population has a normal distribution**

In many situations it is reasonable to assume that the population from which we are sampling has a normal, or nearly normal, distribution. When the population has a normal distribution, the sampling distribution of  $\bar{X}$  is normally distributed for any sample size.

### **Population does not have a normal distribution**

When the population from which we are selecting a simple random sample does not have a normal distribution, the **central limit theorem** is helpful in identifying the shape of the sampling distribution of  $\bar{X}$ .

#### **Central limit theorem**

In selecting simple random samples of size  $n$  from a population, the sampling distribution of the sample mean  $\bar{X}$  can be approximated by a *normal distribution* as the sample size becomes large.

Figure 7.4 shows how the central limit theorem works for three different populations. Each column refers to one of the populations. The top panel of the figure shows that none of the populations is normally distributed. When the samples are of size 2, we see that the sampling distribution begins to take on an appearance different from that of the population distribution. For samples of size 5, we see all three sampling distributions beginning to take on a bell-shaped appearance. Finally, the samples of size 30 show all three sampling distributions to be approximately normally distributed. For sufficiently large samples, the sampling distribution of  $\bar{X}$  can be approximated by a normal distribution. How large must the sample size be before we can assume that the central limit theorem applies? Studies of the sampling distribution of  $\bar{X}$  for a variety of populations and a variety of sample sizes have indicated that, for most applications, the sampling distribution of  $\bar{X}$  can be approximated by a normal distribution whenever the sample size is 30 or more.

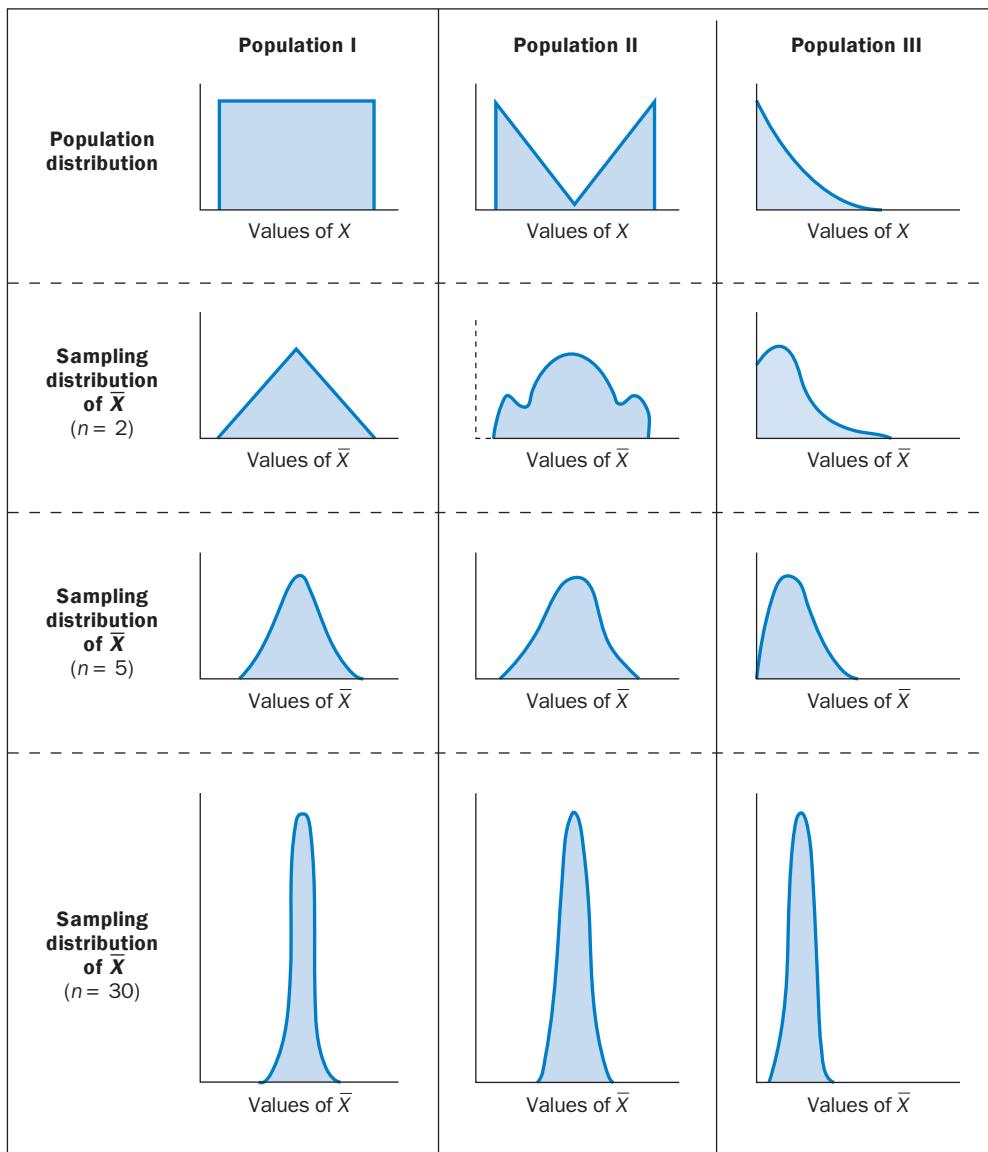
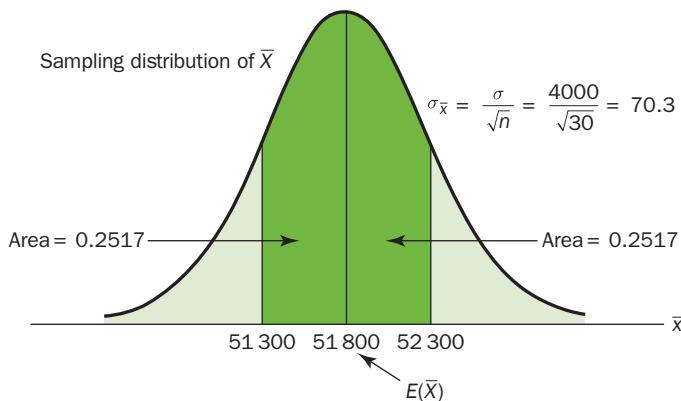
**FIGURE 7.4**

Illustration of the central limit theorem for three populations

The theoretical proof of the central limit theorem requires independent observations in the sample. This condition is met for infinite populations and for finite populations where sampling is done with replacement. Although the central limit theorem does not directly address sampling without replacement from finite populations, general statistical practice applies the findings of the central limit theorem when the population size is large.

### Sampling distribution of $\bar{X}$ for the EAI problem

For the EAI problem, we previously showed that  $E(\bar{X}) = €51\,800$  and  $\sigma_{\bar{X}} = €730.3$ . At this point, we do not have any information about the population distribution; it may or may not be normally distributed. If the population has a normal distribution, the sampling distribution of  $\bar{X}$  is normally distributed.

**FIGURE 7.5**

Sampling distribution of  $\bar{X}$  for the mean annual salary of a simple random sample of 30 EAI managers, and the probability of  $\bar{X}$  being within €500 of the population mean

If the population does not have a normal distribution, the simple random sample of 30 managers and the central limit theorem enable us to conclude that the sampling distribution can be approximated by a normal distribution. In either case, we can proceed with the conclusion that the sampling distribution can be described by the normal distribution shown in Figure 7.5.

### Practical value of the sampling distribution of $\bar{X}$

We are interested in the sampling distribution of  $\bar{X}$  because it can be used to provide probability information about the difference between the sample mean and the population mean. Suppose the head of personnel services believes the sample mean will be an acceptable estimate if it is within €500 of the population mean. It is not possible to guarantee that the sample mean will be within €500 of the population mean. Indeed, Table 7.5 and Figure 7.1 show that some of the 500 sample means differed by more than €2000 from the population mean. So we must think of the head of personnel's request in probability terms. What is the probability that the sample mean computed using a simple random sample of 30 EAI managers will be within €500 of the population mean?

We can answer this question using the sampling distribution of  $\bar{X}$ . Refer to Figure 7.5. With  $\mu = €51,800$ , the personnel manager wants to know the probability that  $\bar{X}$  is between €51,300 and €52,300. The darkly shaded area of the sampling distribution shown in Figure 7.5 gives this probability. Because the sampling distribution is normally distributed, with mean 51,800 and standard error of the mean 730.3, we can use the table of areas for the standard normal distribution to find the area or probability. At  $\bar{X} = 51,300$  we have

$$z = \frac{51300 - 51800}{730.3} = -0.68$$

Referring to the standard normal distribution table, we find the cumulative probability for  $z = -0.68$  is 0.2483. Similar calculations for  $\bar{X} = 52,300$  show a cumulative probability for  $z = +0.68$  of 0.7517. So the probability that the sample mean is between 51,300 and 52,300 is  $0.7517 - 0.2483 = 0.5034$ .

These computations show that a simple random sample of 30 EAI managers has a 0.5034 probability of providing a sample mean that is within €500 of the population mean. Hence, there is a  $1 - 0.5034 = 0.4966$  probability that the difference between  $\bar{X}$  and  $\mu$  will be more than €500. In other words, a simple random sample of 30 EAI managers has a roughly 50/50 chance of providing a sample mean within the allowable €500. Perhaps a larger sample size should be considered. We explore this possibility by considering the relationship between the sample size and the sampling distribution of  $\bar{X}$ .

## Relationship between sample size and the sampling distribution of $\bar{X}$

Suppose that in the EAI sampling problem we select a simple random sample of 100 EAI managers instead of the 30 originally considered. Intuitively, it would seem that with more sample data, the sample mean based on  $n = 100$  should provide a better estimate of the population mean than the sample mean based on  $n = 30$ . To see how much better, let us consider the relationship between the sample size and the sampling distribution of  $\bar{X}$ .

First note that  $E(\bar{X}) = \mu$ , i.e.  $\bar{X}$  is an unbiased estimator of  $\mu$ , regardless of the sample size  $n$ . However, the standard error of the mean,  $\sigma_{\bar{X}}$ , is related to the square root of the sample size. The value of  $\sigma_{\bar{X}}$  decreases when the sample size increases. With  $n = 30$ , the standard error of the mean for the EAI problem is 730.3. With the increase in the sample size to  $n = 100$ , the standard error of the mean decreases to:

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{4000}{\sqrt{100}} = 400$$

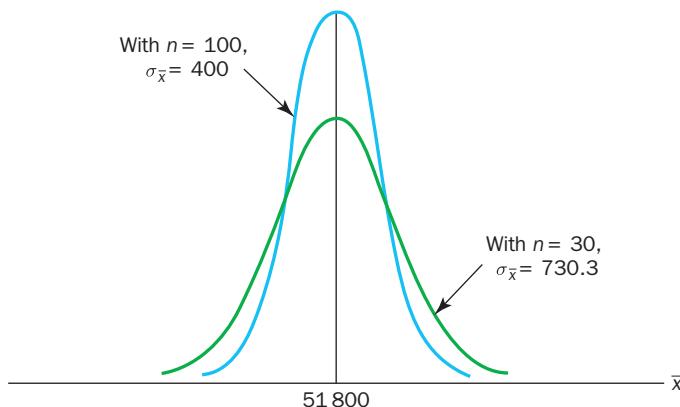
The sampling distributions of  $\bar{X}$  with  $n = 30$  and  $n = 100$  are shown in Figure 7.6. Because the sampling distribution with  $n = 100$  has a smaller standard error, the values of  $\bar{X}$  have less variation and tend to be closer to the population mean than the values of  $\bar{X}$  with  $n = 30$ .

We can use the sampling distribution of  $\bar{X}$  for  $n = 100$  to compute the probability that a simple random sample of 100 EAI managers will provide a sample mean within €500 of the population mean. Because the sampling distribution is normal, with mean 51 800 and standard error of the mean 400, we can use the standard normal distribution table to find the area or probability. At  $\bar{X} = 51\,300$  (Figure 7.7), we have:

$$z = \frac{51\,300 - 51\,800}{400} = -1.25$$

Referring to the standard normal probability distribution table, we find a cumulative probability for  $z = -1.25$  of 0.1056. With a similar calculation for  $\bar{X} = 52\,300$ , we see that the probability of the sample mean being between 51 300 and 52 300 is  $0.8944 - 0.1056 = 0.7888$ . By increasing the sample size from 30 to 100 EAI managers, we have increased the probability of obtaining a sample mean within €500 of the population mean from 0.5034 to 0.7888.

The important point in this discussion is that as the sample size is increased, the standard error of the mean decreases. As a result, the larger sample size provides a higher probability that the sample mean is within a specified distance of the population mean.

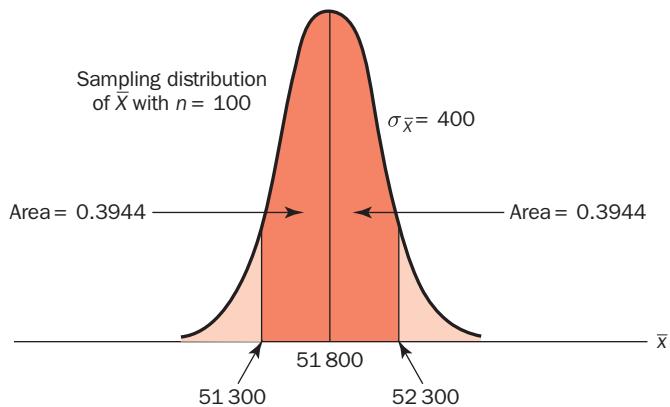


**FIGURE 7.6**

A comparison of the sampling distributions of  $\bar{X}$  for simple random samples of  $n = 30$  and  $n = 100$  EAI managers

**FIGURE 7.7**

The probability of a sample mean being within €500 of the population mean when a simple random sample of 100 EAI managers is used



In presenting the sampling distribution of  $\bar{X}$  for the EAI problem, we have taken advantage of the fact that the population mean  $\mu = 51800$  and the population standard deviation  $\sigma = 4000$  were known. However, usually the values  $\mu$  and  $\sigma$  that are needed to determine the sampling distribution of  $\bar{X}$  will be unknown. In Chapter 8 we shall show how the sample mean  $\bar{X}$  and the sample standard deviation  $S$  are used when  $\mu$  and  $\sigma$  are unknown.

## EXERCISES

### Methods

- 13.** A population has a mean of 200 and a standard deviation of 50. A simple random sample of size 100 will be taken and the sample mean will be used to estimate the population mean.
- What is the expected value of  $\bar{X}$ ?
  - What is the standard deviation of  $\bar{X}$ ?
  - Sketch the sampling distribution of  $\bar{X}$ .
  - What does the sampling distribution of  $\bar{X}$  show?
- 14.** A population has a mean of 200 and a standard deviation of 50. Suppose a simple random sample of size 100 is selected and is used to estimate  $\mu$ .
- What is the probability that the sample mean will be within  $\pm 5$  of the population mean?
  - What is the probability that the sample mean will be within  $\pm 10$  of the population mean?
- 15.** Assume the population standard deviation is  $\sigma = 25$ . Compute the standard error of the mean,  $\sigma_{\bar{X}}$ , for sample sizes of 50, 100, 150 and 200. What can you say about the size of the standard error of the mean as the sample size is increased?
- 16.** Suppose a simple random sample of size 50 is selected from a population with  $\sigma_{\bar{X}} = 25$ . Find the value of the standard error of the mean in each of the following cases (use the finite population correction factor if appropriate).
- The population size is infinite.
  - The population size is  $N = 50000$ .
  - The population size is  $N = 5000$ .
  - The population size is  $N = 500$ .



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SOLUTIONS

### Applications

17. Refer to the EAI sampling problem. Suppose a simple random sample of 60 managers is used.
- Sketch the sampling distribution of  $\bar{X}$  when simple random samples of size 60 are used.
  - What happens to the sampling distribution of  $\bar{X}$  if simple random samples of size 120 are used?
  - What general statement can you make about what happens to the sampling distribution of  $\bar{X}$  as the sample size is increased? Does this generalization seem logical? Explain.
18. In the EAI sampling problem (see Figure 7.5), we showed that for  $n = 30$ , there was a 0.5034 probability of obtaining a sample mean within  $\pm €500$  of the population mean.
- What is the probability that  $\bar{X}$  is within  $€500$  of the population mean if a sample of size 60 is used?
  - Answer part (a) for a sample of size 120.
19. The Automobile Association gave the average price of unleaded petrol in Sweden as 14.63 Swedish krona (SK) per litre in June 2012. Assume this price is the population mean, and that the population standard deviation is  $\sigma = 1$  SK.
- What is the probability that the mean price for a sample of 30 petrol stations is within 0.25 SK of the population mean?
  - What is the probability that the mean price for a sample of 50 petrol stations is within 0.25 SK of the population mean?
  - What is the probability that the mean price for a sample 100 petrol stations is within 0.25 SK of the population mean?
  - Would you recommend a sample size of 30, 50 or 100 to have at least a 0.95 probability that the sample mean is within 0.25 SK of the population mean?
20. According to *Golf Digest*, the average score for male golfers is 95 and the average score for female golfers is 106. Use these values as population means. Assume that the population standard deviation is  $\sigma = 14$  strokes for both men and women. A simple random sample of 30 male golfers and another simple random sample of 45 female golfers are taken.
- Sketch the sampling distribution of  $\bar{X}$  for male golfers.
  - What is the probability that the sample mean is within three strokes of the population mean for the sample of male golfers?
  - What is the probability that the sample mean is within three strokes of the population mean for the sample of female golfers?
  - In which case is the probability higher (b or c)? Why?
21. A researcher reports survey results by stating that the standard error of the mean is 20. The population standard deviation is 500.
- How large was the sample?
  - What is the probability that the point estimate was within  $\pm 25$  of the population mean?
22. To estimate the mean age for a population of 4000 employees in a large company in Kuwait City, a simple random sample of 40 employees is selected.
- Would you use the finite population correction factor in calculating the standard error of the mean? Explain.
  - If the population standard deviation is  $\sigma = 8.2$ , compute the standard error both with and without the finite population correction factor. What is the rationale for ignoring the finite population correction factor whenever  $n/N \leq 0.05$ ?
  - What is the probability that the sample mean age of the employees will be within  $\pm$  two years of the population mean age?



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## 7.6 SAMPLING DISTRIBUTION OF $P$

The sample proportion  $P$  is a point estimator of the population proportion  $\pi$ . The formula for computing the sample proportion is:

$$p = \frac{m}{n}$$

where:

- $m$  = the number of elements in the sample that possess the characteristic of interest
- $n$  = sample size.

The sample proportion  $P$  is a random variable and its probability distribution is called the sampling distribution of  $P$ .

### Sampling distribution of $P$

The sampling distribution of  $P$  is the probability distribution of all possible values of the sample proportion  $P$ .

To determine how close the sample proportion is to the population proportion  $\pi$ , we need to understand the properties of the sampling distribution of  $P$ : the expected value of  $P$ , the standard deviation of  $P$  and the shape of the sampling distribution of  $P$ .

### Expected value of $P$

The expected value of  $P$ , the mean of all possible values of  $P$ , is equal to the population proportion  $\pi$ .  $P$  is an unbiased estimator of  $\pi$ .

### Expected value of $P$

where:

$$E(P) = \pi \tag{7.4}$$

- $E(P)$  = the expected value of  $P$
- $\pi$  = the population proportion

In Section 7.1 we noted that  $\pi = 0.60$  for the EAI population, where  $\pi$  is the proportion of the population of managers who participated in the company's management training programme. The expected value of  $P$  for the EAI sampling problem is therefore 0.60.

### Standard deviation of $P$

Just as we found for the standard deviation of  $\bar{X}$ , the standard deviation of  $P$  depends on whether the population is finite or infinite.

### Standard deviation of $P$

|   |  |
|---|--|
| Finite population   | Infinite population                      |
| $\sigma_P = \sqrt{\frac{N-n}{N-1}} \sqrt{\frac{\pi(1-\pi)}{n}}$ | $\sigma_P = \sqrt{\frac{\pi(1-\pi)}{n}}$ |

(7.5)

Comparing the two formulae in (7.5), we see that the only difference is the use of the finite population correction factor  $\sqrt{(N-n)/(N-1)}$ .

As was the case with the sample mean, the difference between the expressions for the finite population and the infinite population becomes negligible if the size of the finite population is large in comparison to the sample size. We follow the same rule of thumb that we recommended for the sample mean. That is, if the population is finite with  $n/N \leq 0.05$ , we shall use  $\sigma_P = \sqrt{\pi(1-\pi)/n}$ .

However, if the population is finite with  $n/N > 0.05$ , the finite population correction factor should be used. Again, unless specifically noted, throughout the text we shall assume that the population size is large in relation to the sample size and so the finite population correction factor is unnecessary.

In Section 7.5 we used the term standard error of the mean to refer to the standard deviation of  $\bar{X}$ . We stated that in general the term standard error refers to the standard deviation of a point estimator. Accordingly, for proportions we use *standard error of the proportion* to refer to the standard deviation of  $P$ .

Let us now return to the EAI example and compute the standard error of the proportion associated with simple random samples of 30 EAI managers. For the EAI study we know that the population proportion of managers who participated in the management training programme is  $\pi = 0.60$ . With  $n/N = 30/2500 = 0.012$ , we can ignore the finite population correction factor when we compute the standard error of the proportion. For the simple random sample of 30 managers,  $\sigma_P$  is:

$$\sigma_P = \sqrt{\frac{\pi(1-\pi)}{n}} = \sqrt{\frac{0.60(1-0.60)}{30}} = 0.0894$$

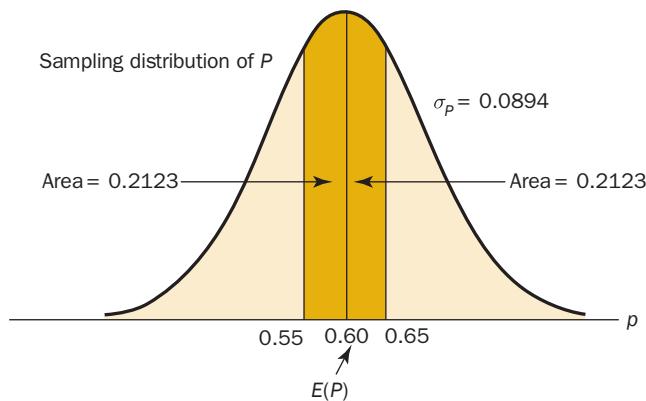
### Form of the sampling distribution of $P$

The sample proportion is  $p = m/n$ . For a simple random sample from a large population, the value of  $m$  is a binomial random variable indicating the number of elements in the sample with the characteristic of interest. Because  $n$  is a constant, the probability of each value of  $m/n$  is the same as the binomial probability of  $m$ , which means that the sampling distribution of  $P$  is also a discrete probability distribution.

In Chapter 6 we showed that a binomial distribution can be approximated by a normal distribution whenever the sample size is large enough to satisfy the following two conditions:  $n\pi \geq 5$  and  $n(1 - \pi) \geq 5$ . Assuming these two conditions are satisfied, the probability of  $m$  in the sample proportion,  $p = m/n$ , can be approximated by a normal distribution. And because  $n$  is a constant, the sampling distribution of  $P$  can also be approximated by a normal distribution. This approximation is stated as follows:

The sampling distribution of  $P$  can be approximated by a normal distribution whenever  $n\pi \geq 5$  and  $n(1 - \pi) \geq 5$ .

In practical applications, when an estimate of a population proportion is needed, we find that sample sizes are almost always large enough to permit the use of a normal approximation for the sampling distribution of  $P$ .

**FIGURE 7.8**

Sampling distribution of  $P$  for the proportion of EAI managers who participated in the management training programme

Recall that for the EAI sampling problem the population proportion of managers who participated in the training programme is  $\pi = 0.60$ . With a simple random sample of size 30, we have  $n\pi = 30(0.60) = 18$  and  $n(1 - \pi) = 30(0.40) = 12$ . Consequently, the sampling distribution of  $P$  can be approximated by the normal distribution shown in Figure 7.8.

### Practical value of the sampling distribution of $P$

The practical value of the sampling distribution of  $P$  is that it can be used to provide probability information about the difference between the sample proportion and the population proportion. For instance, suppose that in the EAI problem the head of personnel services wants to know the probability of obtaining a value of  $P$  that is within 0.05 of the population proportion of EAI managers who participated in the training programme. That is, what is the probability of obtaining a sample with a sample proportion  $P$  between 0.55 and 0.65? The darkly shaded area in Figure 7.8 shows this probability. Using the fact that the sampling distribution of  $P$  can be approximated by a normal distribution with a mean of 0.60 and a standard error of the proportion of  $\sigma_P = 0.0894$ , we find that the standard normal random variable corresponding to  $p = 0.55$  has a value of  $z = (0.55 - 0.60)/0.0894 = -0.56$ . Referring to the standard normal distribution table, we see that the cumulative probability for  $z = -0.56$  is 0.2877. Similarly, for  $p = 0.65$  we find a cumulative probability of 0.7123. Hence, the probability of selecting a sample that provides a sample proportion  $P$  within 0.05 of the population proportion  $\pi$  is  $0.7123 - 0.2877 = 0.4246$ .

If we consider increasing the sample size to  $n = 100$ , the standard error of the proportion becomes:

$$\sigma_P = \sqrt{\frac{0.60(1 - 0.60)}{100}} = 0.049$$

The probability of the sample proportion being within 0.05 of the population proportion can now be calculated, again using the standard normal distribution table to find the area or probability. At  $p = 0.55$ , we have  $z = (0.55 - 0.60)/0.049 = -1.02$ . Referring to the standard normal distribution table, we see that the cumulative probability for  $z = -1.02$  is 0.1539. Similarly, at  $p = 0.65$  the cumulative probability is 0.8461. Hence, if the sample size is increased from 30 to 100, the probability that the sample proportion is within 0.05 of the population proportion  $\pi$  will increase to  $0.8461 - 0.1539 = 0.6922$ .

## EXERCISES

### Methods

23. A simple random sample of size 100 is selected from a population with  $\pi = 0.40$ .
- What is the expected value of  $P$ ?
  - What is the standard error of  $P$ ?
  - Sketch the sampling distribution of  $P$ .
24. Assume that the population proportion is 0.55. Compute the standard error of the sample proportion,  $\sigma_P$ , for sample sizes of 100, 200, 500 and 1000. What can you say about the size of the standard error of the proportion as the sample size is increased?
25. The population proportion is 0.30. What is the probability that a sample proportion will be within  $\pm 0.04$  of the population proportion for each of the following sample sizes?
- $n = 100$ .
  - $n = 200$ .
  - $n = 500$ .
  - $n = 1000$ .
  - What is the advantage of a larger sample size?



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### Applications

26. The Chief Executive Officer of Dunkley Distributors plc believes that 30 per cent of the firm's orders come from first-time customers. A simple random sample of 100 orders will be used to estimate the proportion of first-time customers.
- Assume that the CEO is correct and  $\pi = 0.30$ . Describe the sampling distribution of the sample proportion  $P$  for this study?
  - What is the probability that the sample proportion  $P$  will be between 0.20 and 0.40?
  - What is the probability that the sample proportion  $P$  will be between 0.25 and 0.35?
27. Eurostat reported that, in 2011, 64 per cent of households in Spain had Internet access. Use a population proportion  $\pi = 0.64$  and assume that a sample of 300 households will be selected.
- Sketch the sampling distribution of  $P$ , the sample proportion of households that have Internet access.
  - What is the probability that the sample proportion  $P$  will be within  $\pm 0.03$  of the population proportion?
  - Answer part (b) for sample sizes of 600 and 1000.
28. Advertisers contract with Internet service providers and search engines to place ads on websites. They pay a fee based on the number of potential customers who click on their ads. Unfortunately, click fraud – i.e. someone clicking on an ad solely for the purpose of driving up advertising revenue – has become a problem. Forty per cent of advertisers claim they have been a victim of click fraud. Suppose a simple random sample of 380 advertisers is taken to learn about how they are affected by this practice. Assume the population proportion  $\pi = 0.40$ .
- What is the probability the sample proportion will be within  $\pm 0.04$  of the population proportion experiencing click fraud?
  - What is the probability the sample proportion will be greater than 0.45?
29. In April 2012, a Gallup poll amongst a sample of 1074 Egyptian adults reported that 58 per cent thought it would be a bad thing if the military remained involved in politics after the presidential



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election. Assume that the population proportion was  $\pi = 0.58$ , and that  $P$  is the sample proportion in a sample of  $n = 1074$ .

- Sketch the sampling distribution of  $P$ .
  - What is the probability that  $P$  will be within plus or minus 0.02 of  $\phi$ .
  - Answer part (b) for sample of 2000 adults.
- 30.** A market research firm conducts telephone surveys with a 40 per cent historical response rate. What is the probability that in a new sample of 400 telephone numbers, at least 150 individuals will cooperate and respond to the questions? In other words, what is the probability that the sample proportion will be at least  $150/400 = 0.375$ ?
- 31.** Lura Jafari is a successful sales representative for a major publisher of university textbooks. Historically, Lura secures a book adoption on 25 per cent of her sales calls. Assume that her sales calls for one month are taken as a sample of all possible sales calls, and that a statistical analysis of the data estimates the standard error of the sample proportion to be 0.0625.
- How large was the sample used in this analysis? That is, how many sales calls did Lura make during the month?
  - Let  $P$  indicate the sample proportion of book adoptions obtained during the month. Sketch the sampling distribution  $P$ .
  - Using the sampling distribution of  $P$ , compute the probability that Lura will obtain book adoptions on 30 per cent or more of her sales calls during a one-month period.



## ONLINE RESOURCES

For the data files, online summary, additional questions and answers, and software section for Chapter 7, go to the online platform.

## SUMMARY

In this chapter we presented the concepts of simple random sampling and sampling distributions.

Simple random sampling was defined for sampling without replacement and sampling with replacement. We demonstrated how a simple random sample can be selected and how the sample data can be used to calculate point estimates of population parameters.

Point estimators such as  $\bar{X}$  and  $P$  are random variables. The probability distribution of such a random variable is called a sampling distribution. In particular, we described the sampling distributions of the sample mean  $\bar{X}$  and the sample proportion  $P$ . We stated that  $E(\bar{X}) = \mu$  and  $E(P) = \pi$ , i.e. they are unbiased estimators of the respective parameters. After giving the standard deviation or standard error formulae for these estimators, we described the conditions necessary for the sampling distributions of  $\bar{X}$  and  $P$  to follow normal distributions. Finally, we gave examples of how these normal sampling distributions can be used to calculate the probability of  $\bar{X}$  or  $P$  being within any given distance of  $\mu$  or  $\pi$  respectively.

## KEY TERMS

Central limit theorem

Finite population correction factor

Parameter

Point estimate

Point estimator

Sample statistic

Sampled population

Sampling distribution

Sampling frame

Sampling with replacement

Sampling without replacement

Simple random sampling

Standard error

Target population

Unbiasedness

## KEY FORMULAE

### Expected value of $\bar{X}$

$$E(\bar{X}) = \mu \quad (7.1)$$

### Standard deviation of $\bar{X}$ (standard error)

Finite population

$$\sigma_{\bar{X}} = \sqrt{\frac{N-n}{N-1}} \left( \frac{\sigma}{\sqrt{n}} \right)$$

Infinite population

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$

(7.2)

### Expected value of $P$

$$E(P) = \pi \quad (7.4)$$

### Standard deviation of $P$ (standard error)

Finite population

$$\sigma_P = \sqrt{\frac{N-n}{N-1}} \sqrt{\frac{\pi(1-\pi)}{n}}$$

Infinite population

$$\sigma_P = \sqrt{\frac{\pi(1-\pi)}{n}}$$

(7.5)



# 8

# Interval Estimation

## CHAPTER CONTENTS

Statistics in Practice How accurate are opinion polls and market research surveys?

- 8.1 Population mean:  $\sigma$  known
- 8.2 Population mean:  $\sigma$  unknown
- 8.3 Determining the sample size
- 8.4 Population proportion

**LEARNING OBJECTIVES** After reading this chapter and doing the exercises, you should be able to:

- 1 Explain the purpose of an interval estimate of a population parameter.
- 2 Explain the terms margin of error, confidence interval, confidence level and confidence coefficient.
- 3 Construct confidence intervals for a population mean:
  - 3.1 When the population standard deviation is known, using the normal distribution.
  - 3.2 When the population standard deviation is unknown, using the  $t$  distribution.
- 4 Construct large-sample confidence intervals for a population proportion.
- 5 Calculate the sample size required to construct a confidence interval with a given margin of error for a population mean, when the population standard deviation is known.
- 6 Calculate the sample size required to construct a confidence interval with a given margin of error for a population proportion.

In Chapter 7, we stated that a point estimator is a sample statistic used to estimate a population parameter. For example, the sample mean is a point estimator of the population mean, and the sample proportion is a point estimator of the population proportion. Because a point estimator cannot be expected to provide the exact value of the population parameter, an **interval estimate** is often computed, by adding and subtracting a **margin of error**.



## STATISTICS IN PRACTICE

How accurate are opinion polls and market research surveys?

PSOS and ICM Research are two large, global market research and opinion polling companies. IPSOS has its global headquarters in Paris, ICM Research is based in London.

In July 2012, IPSOS released a report covering opinion surveys in 24 countries across the globe, entitled *IPSOS Global @dvisory: The Economic Pulse of the World*. About eighteen and a half thousand interviews were conducted over the 24 countries. Interviewees were asked to assess the economic situation in their home country: as either 'Very good', or 'Somewhat good', or 'Somewhat bad' or 'Very bad'.



In Spain, 78 per cent of respondents described the economic situation as very bad. This compared with 25 per cent of respondents in Great Britain, 15 per cent in South Africa, only 3 per cent in Germany and just 2 per cent in Sweden.

But how accurate are estimates like these based on sample evidence?

The issue of survey accuracy and margin of error features on the ICM website, [www.icmresearch.co.uk](http://www.icmresearch.co.uk). On one page, there is an interactive 'ready-reckoner' that will calculate the margin of error for any given percentage result, like those above, and for any given sample size. For example, in respect of the 78 per cent of 1012 respondents in Spain who considered the economic situation to be very bad, the ICM ready-reckoner calculates the 'accuracy at 95 per cent confidence level' to be plus or minus 2.6 percentage points. In other words, this implies we can be 95 per cent confident that the percentage of all adults in Spain who thought the economic situation was very bad was between 75.4 per cent and 80.6 per cent. By comparison, for the corresponding figure of 15 per cent in South Africa, where the sample size was 506, ICM's ready-reckoner gives the margin of error at the 95 per cent confidence level as plus or minus 3.1 per cent.

In this chapter, you will learn the basis for these margins of error, the confidence level of 95 per cent associated with them, and the calculations that underlie the ICM's ready-reckoner.

The purpose of an interval estimate is to provide information about how close the point estimate might be to the value of the population parameter. In relatively simple cases, the general form of an interval estimate is:

$$\text{Point estimate} \pm \text{Margin of error}$$

In this chapter we show how to compute interval estimates of a population mean  $\mu$  and a population proportion  $\pi$ . The interval estimates have the same general form:

$$\text{Population mean: } \bar{x} \pm \text{Margin of error}$$

$$\text{Population proportion: } p \pm \text{Margin of error}$$

The sampling distributions of  $\bar{X}$  and  $P$  play key roles in computing these interval estimates.

## 8.1 POPULATION MEAN: $\sigma$ KNOWN

To construct an interval estimate of a population mean, either the population standard deviation  $\sigma$  or the sample standard deviation  $s$  must be used to compute the margin of error. Although  $\sigma$  is rarely known exactly, historical data sometimes permit us to obtain a good estimate of the population standard deviation prior to sampling. In such cases, the population standard deviation can be considered known

for practical purposes. We refer to such cases as the  $\sigma$  known case. In this section we show how a simple random sample can be used to construct an interval estimate of a population mean for the  $\sigma$  known case.

Consider the monthly customer service survey conducted by CJW Limited, who has a website for taking customer orders and providing follow-up service. The company prides itself on providing easy online ordering, timely delivery and prompt response to customer enquiries. Good customer service is critical to the company's ongoing success.

CJW's quality assurance team uses a customer service survey to measure satisfaction with its website and online customer service. Each month, the team sends a questionnaire to a random sample of customers who placed an order or requested service during the previous month. The questionnaire asks customers to rate their satisfaction with such things as ease of placing orders, timely delivery, accurate order filling and technical advice. The team summarizes each customer's questionnaire by computing an overall satisfaction score  $x$  that ranges from 0 (worst possible score) to 100 (best possible score). A sample mean customer satisfaction score is then computed.

The sample mean satisfaction score provides a point estimate of the mean satisfaction score  $\mu$  for the population of all CJW customers. With this regular measure of customer service, CJW can promptly take corrective action if a low satisfaction score results. The company conducted this satisfaction survey for a number of months, and consistently obtained an estimate near 12 for the standard deviation of satisfaction scores. Based on these historical data, CJW now assumes a known value of  $\sigma = 12$  for the population standard deviation. The historical data also indicate that the population of satisfaction scores follows an approximately normal distribution.

During the most recent month, the quality assurance team surveyed 100 customers ( $n = 100$ ) and obtained a sample mean satisfaction score of  $\bar{x} = 72$ . This provides a point estimate of the population mean satisfaction score  $\mu$ . We show how to compute the margin of error for this estimate and construct an interval estimate of the population mean.



CJW

## Margin of error and the interval estimate

In Chapter 7 we showed that the sampling distribution of the sample mean  $\bar{X}$  can be used to compute the probability that  $\bar{X}$  will be within a given distance of  $\mu$ . In the CJW example, the historical data show that the population of satisfaction scores is normally distributed with a standard deviation of  $\sigma = 12$ . So, using what we learned in Chapter 7, we can conclude that the sampling distribution of  $\bar{X}$  follows a normal distribution with a standard error of:

$$\sigma_{\bar{X}} = \sigma/\sqrt{n} = 12/\sqrt{100} = 1.2$$

This sampling distribution is shown in Figure 8.1.\*

Using the table of cumulative probabilities for the standard normal distribution, we find that 95 per cent of the values of any normally distributed random variable are within  $\pm 1.96$  standard deviations of the mean. So, 95 per cent of the  $\bar{X}$  values must be within  $\pm 1.96\sigma_{\bar{X}}$  of the mean  $\mu$ . In the CJW example, we know that the sampling distribution of  $\bar{X}$  is normal with a standard error of  $\sigma_{\bar{X}} = 1.2$ . Because  $\pm 1.96\sigma_{\bar{X}} = \pm 1.96(1.2) = \pm 2.35$ , we conclude that 95 per cent of all  $\bar{X}$  values obtained using a sample size of  $n = 100$  will be within  $\pm 2.35$  units of the population mean  $\mu$ . See Figure 8.1.

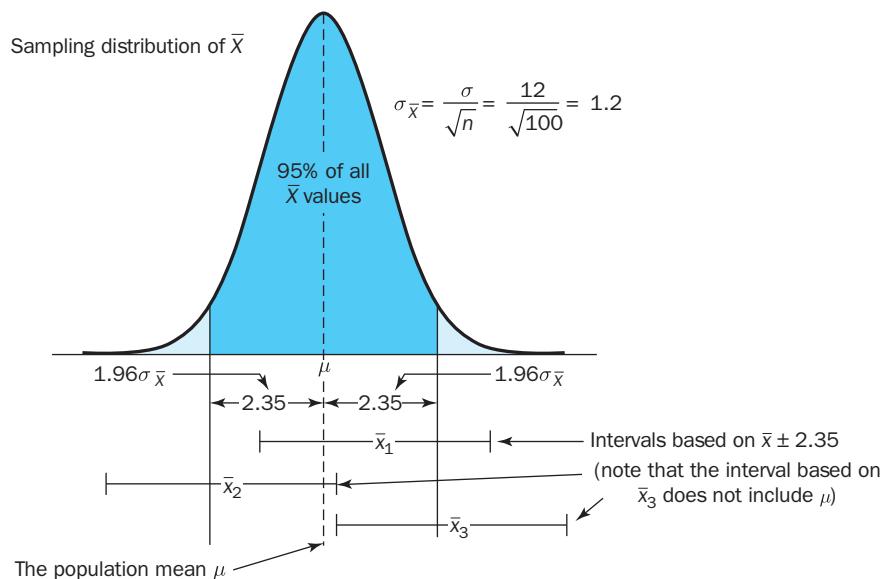
We said above that the general form of an interval estimate of the population mean  $\mu$  is  $\bar{x} \pm$  margin of error. For the CJW example, suppose we set the margin of error equal to 2.35 and compute the interval estimate of  $\mu$  using  $\bar{x} \pm 2.35$ . To provide an interpretation for this interval estimate, let us consider the values of  $\bar{x} \pm 2.35$  that could be obtained if we took three different simple random samples, each consisting of 100 CJW customers.

The first sample mean might turn out to have the value shown as  $\bar{x}_1$  in Figure 8.1. In this case, the interval formed by subtracting 2.35 from  $\bar{x}_1$  and adding 2.35 to  $\bar{x}_1$  includes the population mean  $\mu$ .

\*The population of satisfaction scores has a normal distribution, so we can conclude that the sampling distribution of  $\bar{X}$  is a normal distribution. If the population did not have a normal distribution, we could rely on the central limit theorem, and the sample size of  $n = 100$ , to conclude that the sampling distribution of  $\bar{X}$  is approximately normal. In either case, the sampling distribution would appear as shown in Figure 8.1.

**FIGURE 8.1**

Sampling distribution of the sample mean satisfaction score from simple random samples of 100 customers, also showing the location of sample means that are within  $\pm 2.35$  units of  $\mu$ , and intervals calculated from selected sample means at locations  $\bar{x}_1$ ,  $\bar{x}_2$  and  $\bar{x}_3$ .



Now consider what happens if the second sample mean turns out to have the value shown as  $\bar{x}_2$  in Figure 8.1. Although  $\bar{x}_2$  differs from  $\bar{x}_1$ , we see that the interval formed by  $\bar{x}_2 \pm 2.35$  also includes the population mean  $\mu$ . However, consider what happens if the third sample mean turns out to have the value shown as  $\bar{x}_3$  in Figure 8.1. In this case, because  $\bar{x}_3$  falls in the upper tail of the sampling distribution and is further than 2.35 units from  $\mu$ , the interval  $\bar{x}_3 \pm 2.35$  does not include the population mean  $\mu$ .

Any sample mean that is within the darkly shaded region of Figure 8.1 will provide an interval estimate that contains the population mean  $\mu$ . Because 95 per cent of all possible sample means are in the darkly shaded region, 95 per cent of all intervals formed by subtracting 2.35 from  $\bar{x}$  and adding 2.35 to  $\bar{x}$  will include the population mean  $\mu$ .

The general form of an interval estimate of a population mean for the  $\sigma$  known case is:

#### Interval estimate of a population mean: $\sigma$ known

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \quad (8.1)$$

where  $(1 - \alpha)$  is the confidence coefficient and  $z_{\alpha/2}$  is the  $z$  value providing an area  $\alpha/2$  in the upper tail of the standard normal probability distribution.

Let us use expression (8.1) to construct a 95 per cent confidence interval for the CJW problem. For a 95 per cent confidence interval, the confidence coefficient is  $(1 - \alpha) = 0.95$  and so  $\alpha = 0.05$ . As we saw above, an area of  $\alpha/2 = 0.05/2 = 0.025$  in the upper tail gives  $z_{0.025} = 1.96$ . With the CJW sample mean  $\bar{x} = 72$ ,  $\sigma = 12$  and a sample size  $n = 100$ , we obtain:

$$72 \pm 1.96 \frac{12}{\sqrt{100}} = 72 \pm 2.35$$

The specific interval estimate of  $\mu$  based on the data from the most recent month is  $72 - 2.35 = 69.65$ , to  $72 + 2.35 = 74.35$ . Because 95 per cent of all the intervals constructed using  $\bar{x} \pm 2.35$  will contain the population mean, we say that we are 95 per cent confident that the interval 69.65 to 74.35 includes the population mean  $\mu$ . We say that this interval has been established at the 95 per cent **confidence level**. The value 0.95 is referred to as the **confidence coefficient**, and the interval 69.65 to 74.35 is called the 95 per cent **confidence interval**.

**TABLE 8.1** Values of  $z_{\alpha/2}$  for the most commonly used confidence levels

| Confidence level | $\alpha$ | $\alpha/2$ | $z_{\alpha/2}$ |
|------------------|----------|------------|----------------|
| 90%              | 0.10     | 0.05       | 1.645          |
| 95%              | 0.05     | 0.025      | 1.960          |
| 99%              | 0.01     | 0.005      | 2.576          |

Although a 95 per cent confidence level is frequently used, other confidence levels such as 90 per cent and 99 per cent may be considered. Values of  $z_{\alpha/2}$  for the most commonly used confidence levels are shown in Table 8.1. Using these values and expression (8.1), the 90 per cent confidence interval for the CJW problem is:

$$72 \pm 1.645 \frac{12}{\sqrt{100}} = 72 \pm 1.97$$

At 90 per cent confidence, the margin of error is 1.97 and the confidence interval is  $72 - 1.97 = 70.03$ , to  $72 + 1.97 = 73.97$ . Similarly, the 99 per cent confidence interval is:

$$72 \pm 2.576 \frac{12}{\sqrt{100}} = 72 \pm 3.09$$

At 99 per cent confidence, the margin of error is 3.09 and the confidence interval is  $72 - 3.09 = 68.93$ , to  $72 + 3.09 = 75.09$ .

Comparing the results for the 90 per cent, 95 per cent and 99 per cent confidence levels, we see that, in order to have a higher degree of confidence, the margin of error and consequently the width of the confidence interval must be larger.

## Practical advice

If the population follows a normal distribution, the confidence interval provided by expression (8.1) is exact. Therefore, if expression (8.1) were used repeatedly to generate 95 per cent confidence intervals, 95 per cent of the intervals generated (in the long run) would contain the population mean. If the population does not follow a normal distribution, the confidence interval provided by expression (8.1) will be approximate. In this case, the quality of the approximation depends on both the distribution of the population and the sample size.

In most applications, a sample size of  $n \geq 30$  is adequate when using expression (8.1) to construct an interval estimate of a population mean. If the population is not normally distributed but is roughly symmetrical, sample sizes as small as 15 can be expected to provide good approximate confidence intervals. With smaller sample sizes, expression (8.1) should be used only if the analyst believes, or is willing to assume, that the population distribution is at least approximately normal.

## EXERCISES

### Methods

1. A simple random sample of 40 items results in a sample mean of 25. The population standard deviation is  $\sigma = 5$ .
  - a. What is the value of the standard error of the mean,  $\sigma_{\bar{X}}$ ?
  - b. At 95 per cent confidence, what is the margin of error for estimating the population mean?

2. A simple random sample of 50 items from a population with  $\sigma = 6$  results in a sample mean of 32.
  - a. Construct a 90 per cent confidence interval for the population mean.
  - b. Construct a 95 per cent confidence interval for the population mean.
  - c. Construct a 99 per cent confidence interval for the population mean.
3. A simple random sample of 60 items results in a sample mean of 80. The population standard deviation is  $\sigma = 15$ .
  - a. Compute the 95 per cent confidence interval for the population mean.
  - b. Assume that the same sample mean was obtained from a sample of 120 items. Construct a 95 per cent confidence interval for the population mean.
  - c. What is the effect of a larger sample size on the interval estimate?
4. A 95 per cent confidence interval for a population mean was reported to be 152 to 160. If  $\sigma = 15$ , what sample size was used in this study?

### Applications

5. In an effort to estimate the mean amount spent per customer for dinner at a Johannesburg restaurant, data were collected for a sample of 49 customers. Assume a population standard deviation of 40 South African rand (ZAR).
  - a. At 95 per cent confidence, what is the margin of error?
  - b. If the sample mean is ZAR186, what is the 95 per cent confidence interval for the population mean?
6. A survey of small businesses with websites found that the average amount spent on a site was €11 500 per year. Given a sample of 60 businesses and a population standard deviation of  $\sigma = €4000$ , what is the margin of error in estimating the population mean spend per year? Use 95 per cent confidence.
7. A survey of 750 university students found they were paying on average €108 per week in accommodation costs. Assume the population standard deviation for weekly accommodation costs is €22.
  - a. Construct a 90 per cent confidence interval estimate of the population mean.
  - b. Construct a 95 per cent confidence interval estimate of the population mean.
  - c. Construct a 99 per cent confidence interval estimate of the population mean.
  - d. Discuss what happens to the width of the confidence interval as the confidence level is increased. Does this result seem reasonable? Explain.



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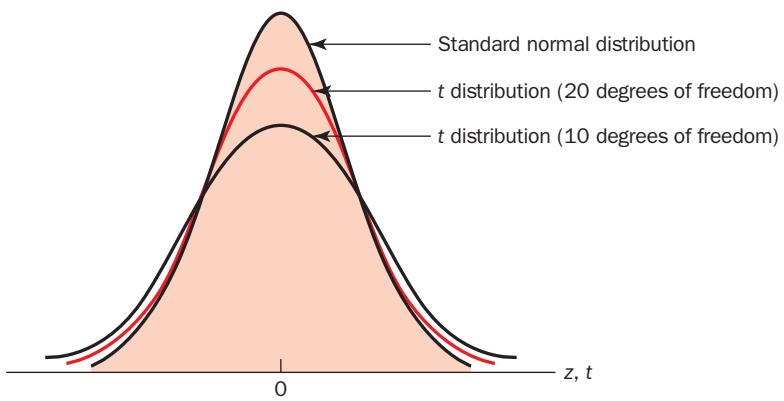
## 8.2 POPULATION MEAN: $\sigma$ UNKNOWN

If a good estimate of the population standard deviation  $\sigma$  cannot be obtained prior to sampling, we must use the sample standard deviation  $s$  to estimate  $\sigma$ . This is the  $\sigma$  unknown case. When  $s$  is used to estimate  $\sigma$ , the margin of error and the interval estimate for the population mean are based on a probability distribution known as the  **$t$  distribution**. Although the mathematical development of the  $t$  distribution is based on the assumption of a normal distribution for the population from which we are sampling, research shows that the  $t$  distribution can be successfully applied in many situations where the population deviates from normal. Later in this section we provide guidelines for using the  $t$  distribution if the population is not normally distributed.

The  $t$  distribution is a family of similar probability distributions, with a specific  $t$  distribution depending on a parameter known as the **degrees of freedom**.

**FIGURE 8.2**

Comparison of the standard normal distribution with  $t$  distributions having 10 and 20 degrees of freedom



The  $t$  distribution with one degree of freedom is unique, as is the  $t$  distribution with two degrees of freedom with three degrees of freedom and so on. As the number of degrees of freedom increases, the difference between the  $t$  distribution and the standard normal distribution becomes smaller and smaller. Figure 8.2 shows  $t$  distributions with ten and 20 degrees of freedom and their relationship to the standard normal probability distribution. Note that, the higher the degrees of freedom, the lower is the variability, and the greater the resemblance to the standard normal distribution. Note also that the mean of the  $t$  distribution is zero.

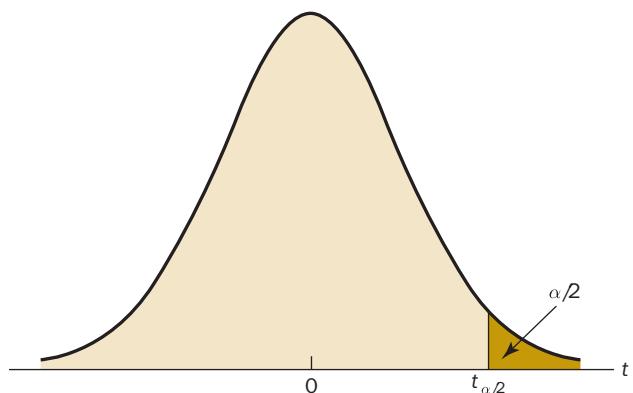
We place a subscript on  $t$  to indicate the area in the upper tail of the  $t$  distribution. For example, just as we used  $z_{0.025}$  to indicate the  $z$  value providing a 0.025 area in the upper tail of a standard normal distribution, we will use  $t_{0.025}$  to indicate a 0.025 area in the upper tail of a  $t$  distribution. So, in general, the notation  $t_{\alpha/2}$  will represent a  $t$  value with an area of  $\alpha/2$  in the upper tail of the  $t$  distribution. See Figure 8.3.

Table 2 of Appendix B is a table for the  $t$  distribution. Each row in the table corresponds to a separate  $t$  distribution with the degrees of freedom shown. For example, for a  $t$  distribution with ten degrees of freedom,  $t_{0.025} = 2.228$ . Similarly, for a  $t$  distribution with 20 degrees of freedom,  $t_{0.025} = 2.086$ . As the degrees of freedom continue to increase,  $t_{0.025}$  approaches  $z_{0.025} = 1.96$ . The standard normal distribution  $z$  values can be found in the infinite degrees of freedom row (labelled  $\infty$ ) of the  $t$  distribution table. If the degrees of freedom exceed 100, the infinite degrees of freedom row can be used to approximate the actual  $t$  value. In other words, for more than 100 degrees of freedom, the standard normal  $z$  value provides a good approximation to the  $t$  value.

William Sealy Gosset, writing under the name ‘Student’, was the originator of the  $t$  distribution. Gosset, an Oxford graduate in mathematics, worked for the Guinness Brewery in Dublin, Ireland. The distribution is sometimes referred to as ‘Student’s  $t$  distribution’.

**FIGURE 8.3**

$t$  distribution with  $\alpha/2$  area of probability in the upper tail



## Margin of error and the interval estimate

In Section 8.1 we showed that an interval estimate of a population mean for the  $\sigma$  known case is:

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

To compute an interval estimate of  $\mu$  for the  $\sigma$  unknown case, the sample standard deviation  $s$  is used to estimate  $\sigma$  and  $z_{\alpha/2}$  is replaced by the  $t$  distribution value  $t_{\alpha/2}$ . The margin of error is then  $\pm t_{\alpha/2}s/\sqrt{n}$ , and the general expression for an interval estimate of a population mean when  $\sigma$  is unknown is:

### Interval estimate of a population mean: $\sigma$ unknown

$$\bar{x} \pm t_{\alpha/2} \frac{s}{\sqrt{n}} \quad (8.2)$$

where  $s$  is the sample standard deviation,  $(1 - \alpha)$  is the confidence coefficient, and  $t_{\alpha/2}$  is the  $t$  value providing an area of  $\alpha/2$  in the upper tail of the  $t$  distribution with  $n - 1$  degrees of freedom\*.

Consider a study designed to estimate the mean credit card debt for a defined population of households. A sample of  $n = 85$  households provided the credit card balances in the file ‘Balance’ on the online platform. The first few rows of this data set are shown in the EXCEL screenshot in Figure 8.4 below. For this situation, no previous estimate of the population standard deviation  $\sigma$  is available. As a consequence, the sample data must be used to estimate both the population mean and the population standard deviation.

**FIGURE 8.4**

First few data rows and summary statistics for credit card balances

|    | A              | B    | C                    | D    |
|----|----------------|------|----------------------|------|
| 1  | <b>Balance</b> |      |                      |      |
| 2  |                | 9619 |                      |      |
| 3  |                | 5364 | mean =               | 5900 |
| 4  |                | 8348 | standard deviation = | 3058 |
| 5  |                | 7348 |                      |      |
| 6  |                | 381  |                      |      |
| 7  |                | 2998 |                      |      |
| 8  |                | 1686 |                      |      |
| 9  |                | 1962 |                      |      |
| 10 |                | 4920 |                      |      |



\*The reason the number of degrees of freedom associated with the  $t$  value in expression (8.2) is  $n - 1$  concerns the use of  $s$  as an estimate of the population standard deviation. The expression for the sample standard deviation is  $s = \sqrt{\sum(x_i - \bar{x})^2/(n-1)}$ . Degrees of freedom refers to the number of independent pieces of information that go into the computation of  $\sum(x_i - \bar{x})^2$ . The  $n$  pieces of information involved in computing  $\sum(x_i - \bar{x})^2$  are as follows:  $x_1 - \bar{x}, x_2 - \bar{x}, \dots, x_n - \bar{x}$ . In Section 3.2 we indicated that  $\sum(x_i - \bar{x}) = 0$ . Hence, only  $n - 1$  of the  $x_i - \bar{x}$  values are independent; that is, if we know  $n - 1$  of the values, the remaining value can be determined exactly by using the condition that  $\sum(x_i - \bar{x}) = 0$ . So  $n - 1$  is the number of degrees of freedom associated with  $\sum(x_i - \bar{x})^2$  and hence the number of degrees of freedom for the  $t$  distribution in expression (8.2).

Using the data in the ‘Balance’ file, we compute the sample mean  $\bar{x} = 5900$  (€) and the sample standard deviation  $s = 3058$  (€).

With 95 per cent confidence and  $n - 1 = 84$  degrees of freedom, Table 2 in Appendix B gives  $t_{0.025} = 1.989$ . We can now use expression (8.2) to compute an interval estimate of the population mean:

$$5900 \pm 1.989 \left( \frac{3058}{\sqrt{85}} \right) = 5900 \pm 660$$

The point estimate of the population mean is €5900, the margin of error is €660, and the 95 per cent confidence interval is  $5900 - 660 = €5240$  to  $5900 + 660 = €6560$ . We are 95 per cent confident that the population mean credit card balance for all households in the defined population is between €5240 and €6560.

The procedures used by MINITAB, EXCEL and IBM SPSS to construct confidence intervals for a population mean are described in the software guides on the online platform.



### Practical advice

If the population follows a normal distribution, the confidence interval provided by expression (8.2) is exact and can be used for any sample size. If the population does not follow a normal distribution, the confidence interval provided by expression (8.2) will be approximate. In this case, the quality of the approximation depends on both the distribution of the population and the sample size.

In most applications, a sample size of  $n \geq 30$  is adequate when using expression (8.2) to construct an interval estimate of a population mean. However, if the population distribution is highly skewed or contains outliers, the sample size should be 50 or more. If the population is not normally distributed but is roughly symmetrical, sample sizes as small as 15 can be expected to provide good approximate confidence intervals. With smaller sample sizes, expression (8.2) should only be used if the analyst is confident that the population distribution is at least approximately normal.

### Using a small sample

In the following example we construct an interval estimate for a population mean when the sample size is small. An understanding of the distribution of the population becomes a factor in deciding whether the interval estimation procedure provides acceptable results.

Scheer Industries is considering a new computer-assisted program to train maintenance employees to do machine repairs. To fully evaluate the program, the director of manufacturing requested an estimate of the population mean time required for maintenance employees to complete the training.

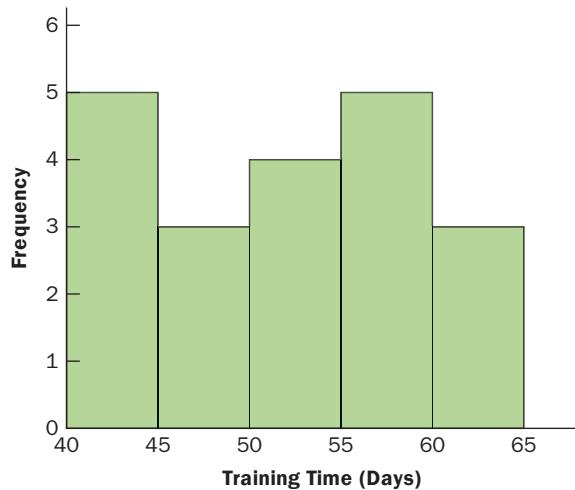
A sample of 20 employees is selected, with each employee in the sample completing the training program. Data on the training time in days for the 20 employees are shown in Table 8.2. A histogram of the sample data appears in Figure 8.5. What can we say about the distribution of the population based on this histogram? First, the sample data do not support with certainty the conclusion that the distribution of the population is normal, but we do not see any evidence of skewness or outliers. Therefore, using the guidelines in the previous subsection, we conclude that an interval estimate based on the  $t$  distribution appears acceptable for the sample of 20 employees.

**TABLE 8.2** Training time in days for a sample of 20 Scheer Industries employees

|    |    |    |    |
|----|----|----|----|
| 52 | 59 | 54 | 42 |
| 44 | 50 | 42 | 48 |
| 55 | 54 | 60 | 55 |
| 44 | 62 | 62 | 57 |
| 45 | 46 | 43 | 56 |

**FIGURE 8.5**

Histogram of training times for the Scheer Industries sample



We compute the sample mean and sample standard deviation as follows:

$$\bar{x} = \frac{\sum x_i}{n} = \frac{1030}{20} = 51.5 \text{ days}$$

$$s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}} = \sqrt{\frac{889}{20-1}} = 6.84 \text{ days}$$

For a 95 per cent confidence interval, we use Table 2 from Appendix B and  $n - 1 = 19$  degrees of freedom to obtain  $t_{0.025} = 2.093$ . Expression (8.2) provides the interval estimate of the population mean:

$$51.5 \pm 2.093 \left( \frac{6.84}{\sqrt{20}} \right) = 51.5 \pm 3.2$$

The point estimate of the population mean is 51.5 days. The margin of error is 3.2 days and the 95 per cent confidence interval is  $51.5 - 3.2 = 48.3$  days to  $51.5 + 3.2 = 54.7$  days.

Using a histogram of the sample data to learn about the distribution of a population is rarely conclusive, but in many cases it provides the only information available. The histogram, along with judgement on the part of the analyst, can often be used to decide if expression (8.2) can be used to construct the interval estimate.

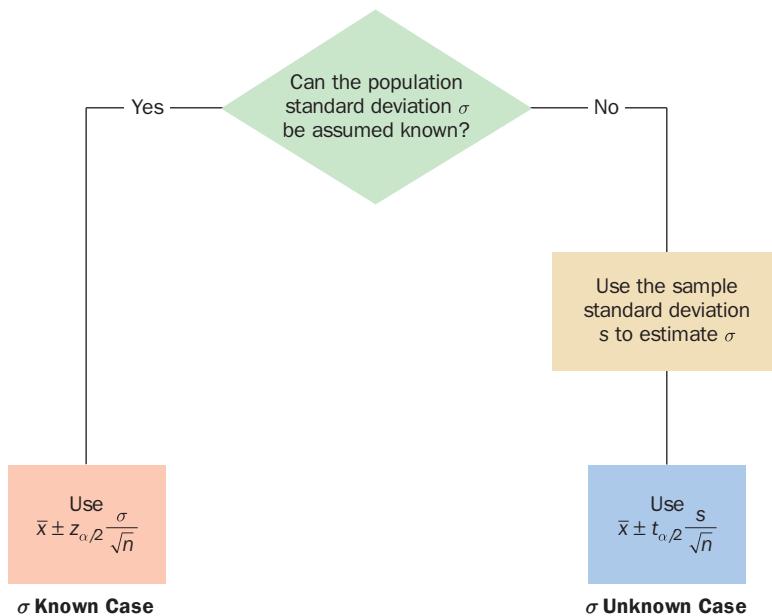
## Summary of interval estimation procedures

We provided two approaches to computing the margin of error and constructing an interval estimate of a population mean. For the  $\sigma$  known case,  $\sigma$  and the standard normal distribution are used in expression (8.1). For the  $\sigma$  unknown case, the sample standard deviation  $s$  and the  $t$  distribution are used in expression (8.2).

A summary of the interval estimation procedures for the two cases is shown in Figure 8.6. In most applications, a sample size of  $n \geq 30$  is adequate. If the population has a normal or approximately normal distribution, however, smaller sample sizes may be used. For the  $\sigma$  unknown case a sample size of  $n \geq 50$  is recommended if the population distribution is believed to be highly skewed or has outliers.

**FIGURE 8.6**

Summary of interval estimation procedures for a population mean



## EXERCISES

### Methods

8. For a  $t$  distribution with 16 degrees of freedom, find the area, or probability, in each region.
  - a. To the right of 2.120.
  - b. To the left of 1.337.
  - c. To the left of -1.746.
  - d. To the right of 2.583.
  - e. Between -2.120 and 2.120.
  - f. Between -1.746 and 1.746.
9. Find the  $t$  value(s) for each of the following cases.
  - a. Upper-tail area of 0.025 with 12 degrees of freedom.
  - b. Lower-tail area of 0.05 with 50 degrees of freedom.
  - c. Upper-tail area of 0.01 with 30 degrees of freedom.
  - d. Where 90 per cent of the area falls between these two  $t$  values with 25 degrees of freedom.
  - e. Where 95 per cent of the area falls between these two  $t$  values with 45 degrees of freedom.
10. The following sample data are from a normal population: 10, 8, 12, 15, 13, 11, 6, 5.
  - a. What is the point estimate of the population mean?
  - b. What is the point estimate of the population standard deviation?
  - c. With 95 per cent confidence, what is the margin of error for the estimation of the population mean?
  - d. What is the 95 per cent confidence interval for the population mean?



- 11.** A simple random sample with  $n = 54$  provided a sample mean of 22.5 and a sample standard deviation of 4.4.
- Construct a 90 per cent confidence interval for the population mean.
  - Construct a 95 per cent confidence interval for the population mean.
  - Construct a 99 per cent confidence interval for the population mean.
  - What happens to the margin of error and the confidence interval as the confidence level is increased?

### Applications

- 12.** Sales personnel for Emirates Distributors submit weekly reports listing the customer contacts made during the week. A sample of 65 weekly reports showed a sample mean of 19.5 customer contacts per week. The sample standard deviation was 5.2. Provide 90 per cent and 95 per cent confidence intervals for the population mean number of weekly customer contacts for the sales personnel.
- 13.** Consumption of alcoholic beverages by young women of drinking age is of concern in the UK and some other European countries. Annual consumption data (in litres) are shown below for a sample of 20 European young women.

|     |     |     |     |     |
|-----|-----|-----|-----|-----|
| 266 | 82  | 199 | 174 | 97  |
| 170 | 222 | 115 | 130 | 169 |
| 164 | 102 | 113 | 171 | 0   |
| 93  | 0   | 93  | 110 | 130 |



ALCOHOL

Assuming the population is distributed roughly symmetrically, construct a 95 per cent confidence interval for the mean annual consumption of alcoholic beverages by young European women.

- 14.** The International Air Transport Association (IATA) surveys business travellers to develop quality ratings for international airports. The maximum possible rating is 10. Suppose a simple random sample of business travellers is selected and each traveller is asked to provide a rating for Kuwait International Airport. The ratings obtained from the sample of 50 business travellers follow. Construct a 95 per cent confidence interval estimate of the population mean rating for Kuwait International.

|   |   |   |   |   |   |   |   |   |   |   |   |    |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|---|----|---|---|---|---|
| 2 | 1 | 8 | 7 | 3 | 1 | 8 | 1 | 7 | 9 | 2 | 9 | 10 | 9 | 7 | 8 | 9 |
| 1 | 0 | 3 | 0 | 1 | 6 | 2 | 3 | 1 | 6 | 8 | 7 | 7  | 7 | 7 | 7 | 1 |
| 2 | 5 | 2 | 1 | 2 | 2 | 0 | 2 | 2 | 7 | 0 | 8 | 7  | 0 | 2 | 8 |   |



IATA

- 15.** Suppose a survey of 40 first-time home buyers finds that the mean of annual household income is €40 000 and the sample standard deviation is €15 300.
- At 95 per cent confidence, what is the margin of error for estimating the population mean household income?
  - What is the 95 per cent confidence interval for the population mean annual household income for first-time home buyers?

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- 16.** A sample of 30 fast-food restaurants including McDonald's and Burger King were visited. During each visit, the customer went to the drive-through and ordered a basic meal such as a burger, fries and drink. The time between pulling up to the order kiosk and receiving the filled order was recorded. The times in minutes for the 30 visits are as follows:

|     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0.9 | 1.0 | 1.2 | 2.2 | 1.9 | 3.6 | 2.8 | 5.2 | 1.8 | 2.1 | 6.8 | 1.3 | 3.0 | 4.5 | 2.8 |
| 2.3 | 2.7 | 5.7 | 4.8 | 3.5 | 2.6 | 3.3 | 5.0 | 4.0 | 7.2 | 9.1 | 2.8 | 3.6 | 7.3 | 9.0 |



FASTFOOD



ACTTEMPS

- a. Provide a point estimate of the population mean drive-through time at fast-food restaurants.
- b. At 95 per cent confidence, what is the margin of error?
- c. What is the 95 per cent confidence interval estimate of the population mean?
- d. Discuss skewness that may be present in this population. What suggestion would you make for a repeat of this study?
- 17.** A survey by Accountemps asked a sample of 200 executives to provide data on the number of minutes per day office workers waste trying to locate mislabelled, misfiled or misplaced items. Data consistent with this survey are contained in the data set 'ActTemps'.
- a. Use 'ActTemps' to develop a point estimate of the number of minutes per day office workers waste trying to locate mislabelled, misfiled or misplaced items.
- b. What is the sample standard deviation?
- c. What is the 95 per cent confidence interval for the mean number of minutes wasted per day?

### 8.3 DETERMINING THE SAMPLE SIZE

We commented earlier on the role of the sample size in providing good approximate confidence intervals when the population is not normally distributed. In this section, we focus on another aspect of the sample size issue. We describe how to choose a sample size large enough to provide a desired margin of error. To understand how this process is done, we return to the  $\sigma$  known case presented in Section 8.1. Using expression (8.1), the interval estimate is  $\bar{x} \pm z_{\alpha/2}\sigma/\sqrt{n}$ . We see that  $z_{\alpha/2}$ , the population standard deviation  $\sigma$ , and the sample size  $n$  combine to determine the margin of error. Once we select a confidence coefficient  $1 - \alpha$ ,  $z_{\alpha/2}$  can be determined. Then, if we have a value for  $\sigma$ , we can determine the sample size  $n$  needed to provide any desired margin of error. Let  $E$  = the desired margin of error.

$$E = z_{\alpha/2} \left( \frac{\sigma}{\sqrt{n}} \right)$$

Solving for  $\sqrt{n}$ , we have:

$$\sqrt{n} = \frac{z_{\alpha/2}\sigma}{E}$$

Squaring both sides of this equation, we obtain the following expression for the sample size.

#### Sample size for an interval estimate of a population mean

$$n = \frac{(z_{\alpha/2})^2 \sigma^2}{E^2} \quad (8.3)$$

This sample size provides the desired margin of error at the chosen confidence level.

In equation (8.3),  $E$  is the acceptable margin of error, and the value of  $z_{\alpha/2}$  follows directly from the confidence level to be used. Although user preference must be considered, 95 per cent confidence is the most frequently chosen value ( $z_{0.025} = 1.96$ ). Equation (8.3) can be used to provide a good sample size recommendation. However, the analyst should use judgement in deciding whether the recommendation given by equation (8.3) needs adjustment.

Use of equation (8.3) requires a value for the population standard deviation  $\sigma$ . However, even if  $\sigma$  is unknown, we can use equation (8.3) provided we have a preliminary or *planning value* for  $\sigma$ . In practice, one of the following procedures can be chosen:

- 1** Use an estimate of the population standard deviation computed from data of previous studies as the planning value for  $\sigma$ .
- 2** Use a pilot study to select a preliminary sample. The sample standard deviation from the preliminary sample can be used as the planning value for  $\sigma$ .
- 3** Use judgement or a ‘best guess’ for the value of  $\sigma$ . For example, we might begin by estimating the largest and smallest data values in the population. The difference between the largest and smallest values provides an estimate of the range for the data. The range divided by four is often suggested as a rough approximation of the standard deviation and hence an acceptable planning value for  $\sigma$ .

Consider the following example. A travel organization would like to conduct a study to estimate the population mean daily rental cost for a family car in Ireland. The director specifies that the population mean daily rental cost be estimated with a margin of error of €2 and a 95 per cent level of confidence. A previous study some years before had found a mean cost of approximately €80 per day for renting a family car, with a standard deviation of about €10.

The director specified a desired margin of error of  $E = 2$ , and the 95 per cent level of confidence indicates  $z_{0.025} = 1.96$ . We only need a planning value for the population standard deviation  $\sigma$  to compute the required sample size. Using €10 (from the previous study) as the planning value for  $\sigma$ , we obtain:

$$n = \frac{(z_{\alpha/2})^2 \sigma^2}{E^2} = \frac{(1.96)^2 (10)^2}{(2)^2} = 96.04$$

The sample size for the new study needs to be at least 96.04 family car rentals in order to satisfy the director’s €2 margin-of-error requirement. In cases where the computed  $n$  is not an integer, we usually round up to the next integer value, in this case 97. Here, the sample size might be rounded for convenience to 100.

## EXERCISES

### Methods

- 18.** How large a sample should be selected to provide a 95 per cent confidence interval with a margin of error of 10? Assume that the population standard deviation is 40.
- 19.** The range for a set of data is estimated to be 36.
  - a.** What is the planning value for the population standard deviation?
  - b.** At 95 per cent confidence, how large a sample would provide a margin of error of 3?
  - c.** At 95 per cent confidence, how large a sample would provide a margin of error of 2?



COMPLETE  
SOLUTIONS

### Applications

- 20.** Refer to the Scheer Industries example in Section 8.2. Use 6.82 days as a planning value for the population standard deviation.
  - a.** Assuming 95 per cent confidence, what sample size would be required to obtain a margin of error of 1.5 days?
  - b.** If the precision statement was made with 90 per cent confidence, what sample size would be required to obtain a margin of error of two days?

- 21.** Suppose you are interested in estimating the average cost of staying for one night in a double room in a three-star hotel in France (outside Paris). Using €30.00 as the planning value for the population standard deviation, what sample size is recommended for each of the following cases? Use €3 as the desired margin of error.
- A 90 per cent confidence interval estimate of the population mean cost.
  - A 95 per cent confidence interval estimate of the population mean cost.
  - A 99 per cent confidence interval estimate of the population mean cost.
  - When the desired margin of error is fixed, what happens to the sample size as the confidence level is increased? Would you recommend a 99 per cent confidence level be used? Discuss.
- 22.** Suppose the price/earnings (P/E) ratios for stocks listed on a European Stock Exchange have a mean value of 35 and a standard deviation of 18. We want to estimate the population mean P/E ratio for all stocks listed on the exchange. How many stocks should be included in the sample if we want a margin of error of 3? Use 95 per cent confidence.
- 23.** Fuel consumption tests are conducted for a particular model of car. If a 98 per cent confidence interval with a margin of error of 0.2 litres per 100km is desired, how many cars should be used in the test? Assume that preliminary tests indicate the standard deviation is 0.5 litres per 100km.
- 24.** In developing patient appointment schedules, a medical centre wants to estimate the mean time that a staff member spends with each patient. How large a sample should be taken if the desired margin of error is two minutes at a 95 per cent level of confidence? How large a sample should be taken for a 99 per cent level of confidence? Use a planning value for the population standard deviation of eight minutes.



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SOLUTIONS

## 8.4 POPULATION PROPORTION

We said earlier that the general form of an interval estimate of a population proportion  $\pi$  is:  $p \pm \text{margin of error}$ . The sampling distribution of the sample proportion of plays a key role in computing the margin of error for this interval estimate.

In Chapter 7 we said that the sampling distribution of the sample proportion  $P$  can be approximated by a normal distribution whenever  $n\pi \geq 5$  and  $n(1 - \pi) \geq 5$ . Figure 8.7 shows the normal approximation of the sampling distribution of  $P$ . The mean of the sampling distribution of  $P$  is the population proportion  $\pi$ , and the standard error of  $P$  is:

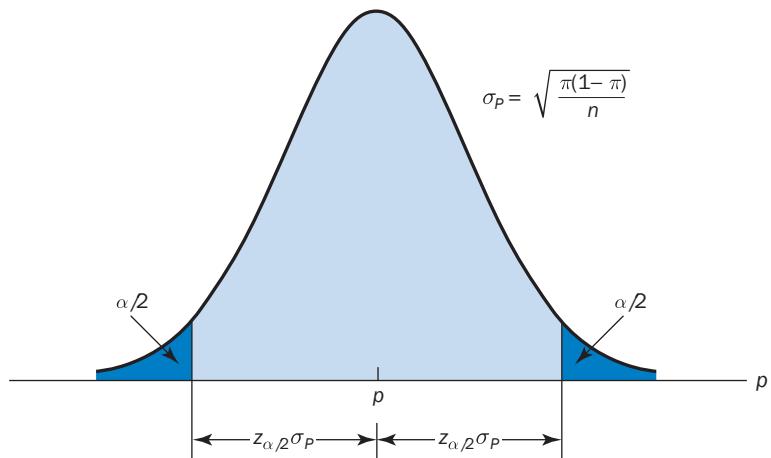
$$\sigma_P = \sqrt{\frac{\pi(1 - \pi)}{n}} \quad (8.4)$$

If we choose  $z_{\alpha/2}\sigma_P$  as the margin of error in an interval estimate of a population proportion, we know that  $100(1 - \alpha)$  per cent of the intervals generated will contain the true population proportion. But  $\sigma_P$  cannot be used directly in the computation of the margin of error because  $\pi$  will not be known;  $\pi$  is what we are trying to estimate. So,  $p$  is substituted for  $\pi$  and the margin of error for an interval estimate of a population proportion is given by:

$$\text{Margin of error} = z_{\alpha/2} \sqrt{\frac{p(1 - p)}{n}} \quad (8.5)$$

**FIGURE 8.7**

Normal approximation of the sampling distribution of  $P$



The general expression for an interval estimate of a population proportion is:

#### Interval estimate of a population proportion

$$p \pm z_{\alpha/2} \sqrt{\frac{p(1-p)}{n}} \quad (8.6)$$

where  $1 - \alpha$  is the confidence coefficient and  $z_{\alpha/2}$  is the  $z$  value providing an area of  $\alpha/2$  in the upper tail of the standard normal distribution.

Consider the following example. A national survey of 900 women golfers was conducted to learn how women golfers view their treatment at golf courses. (The data are available in the file ‘TeeTimes’ on the companion online platform.) The survey found that 396 of the women golfers were satisfied with the availability of tee times. So, the point estimate of the proportion of the population of women golfers who are satisfied is  $396/900 = 0.44$ . Using expression (8.6) and a 95 per cent confidence level,

$$p \pm z_{\alpha/2} \sqrt{\frac{p(1-p)}{n}} = 0.44 \pm 1.96 \sqrt{\frac{0.44(1-0.44)}{900}} = 0.44 \pm 0.0324$$

The margin of error is 0.0324 and the 95 per cent confidence interval estimate of the population proportion is 0.408 to 0.472. Using percentages, the survey results enable us to state with 95 per cent confidence that between 40.8 per cent and 47.2 per cent of all women golfers are satisfied with the availability of tee times.



TEETIMES

#### Determining the sample size

The rationale for the sample size determination in constructing interval estimates of  $\pi$  is similar to the rationale used in Section 8.3 to determine the sample size for estimating a population mean.

Previously in this section we said that the margin of error associated with an interval estimate of a population proportion is  $z_{\alpha/2} \sqrt{p(1-p)/n}$ . The margin of error is based on the values of  $z_{\alpha/2}$ , the sample proportion  $p$ , and the sample size  $n$ . Larger sample sizes provide a smaller margin of error and better precision. Let  $E$  denote the desired margin of error:

$$E = z_{\alpha/2} \sqrt{\frac{p(1-p)}{n}}$$

Solving this equation for  $n$  provides a formula for the sample size that will provide a margin of error of size  $E$ :

$$n = \frac{(z_{\alpha/2})^2 p(1-p)}{E^2}$$

However, we cannot use this formula to compute the sample size because  $p$  will not be known until after we select the sample. What we need, then, is a planning value for  $p$ . Using  $p^*$  to denote the planning value for  $p$ , the following formula can be used to compute the sample size.

#### Sample size for an interval estimate of a population proportion

$$n = \frac{(z_{\alpha/2})^2 p^*(1-p^*)}{E^2} \quad (8.7)$$

In practice, the planning value can be chosen by one of the following procedures:

- 1 Use the sample proportion from a previous sample of the same or similar units.
- 2 Use a pilot study to select a preliminary sample. The sample proportion from this sample can be used as the planning value.
- 3 Use judgement or a ‘best guess’ for the value of  $p^*$ .
- 4 If none of the preceding alternatives apply, use a planning value of  $p^* = 0.50$ .

Let us return to the survey of women golfers and assume that the company is interested in conducting a new survey to estimate the current proportion of the population of women golfers who are satisfied with the availability of tee times. How large should the sample be if the survey director wants to estimate the population proportion with a margin of error of 0.025 at 95 per cent confidence? With  $E = 0.025$  and  $z_{\alpha/2} = 1.96$ , we need a planning value  $p^*$  to answer the sample size question. Using the previous survey result of  $p = 0.44$  as the planning value  $p^*$ , equation (8.7) shows that:

$$n = \frac{(z_{\alpha/2})^2 p^*(1-p^*)}{E^2} = \frac{(1.96)^2(0.44)(1-0.44)}{(0.025)^2} = 1514.5$$

The sample size must be at least 1514.5 women golfers to satisfy the margin of error requirement. Rounding up to the next integer value indicates that a sample of 1515 women golfers is recommended to satisfy the margin of error requirement.

The fourth alternative suggested for selecting a planning value  $p^*$  is to use  $p^* = 0.50$ . This value of  $p^*$  is frequently used when no other information is available. To understand why, note that the numerator of equation (8.7) shows that the sample size is proportional to the quantity  $p^*(1-p^*)$ . A larger value for this quantity will result in a larger sample size. Table 8.3 gives some possible values of  $p^*(1-p^*)$ . Note that the largest value occurs when  $p^* = 0.50$ . So, in case of any uncertainty about an appropriate planning value, we know that  $p^* = 0.50$  will provide the largest sample size recommendation. If the sample proportion turns out to be different from the 0.50 planning value, the margin of error will be smaller than anticipated. In effect, we play it safe by recommending the largest possible sample size.

In the survey of women golfers example, a planning value of  $p^* = 0.50$  would have provided the sample size:

$$n = \frac{(z_{\alpha/2})^2 p^*(1-p^*)}{E^2} = \frac{(1.96)^2(0.5)(1-0.5)}{(0.025)^2} = 1536.6$$

A slightly larger sample size of 1537 women golfers would be recommended.

**TABLE 8.3** Some possible values for  $p^*(1 - p^*)$ 

| $p^*$ | $p^*(1 - p^*)$      |
|-------|---------------------|
| 0.10  | (0.10)(0.90) = 0.09 |
| 0.30  | (0.30)(0.70) = 0.21 |
| 0.40  | (0.40)(0.60) = 0.24 |
| 0.50  | (0.50)(0.50) = 0.25 |
| 0.60  | (0.60)(0.40) = 0.24 |
| 0.70  | (0.70)(0.30) = 0.21 |
| 0.90  | (0.90)(0.10) = 0.09 |

← Largest value for  $p^*(1 - p^*)$

## EXERCISES

### Methods

- 25.** A simple random sample of 400 individuals provides 100 Yes responses.
- What is the point estimate of the proportion of the population that would provide Yes responses?
  - What is your estimate of the standard error of the sample proportion?
  - Compute a 95 per cent confidence interval for the population proportion.
- 26.** A simple random sample of 800 elements generates a sample proportion  $p = 0.70$ .
- Provide a 90 per cent confidence interval for the population proportion.
  - Provide a 95 per cent confidence interval for the population proportion.
- 27.** In a survey, the planning value for the population proportion is  $p^* = 0.35$ . How large a sample should be taken to provide a 95 per cent confidence interval with a margin of error of 0.05?
- 28.** At 95 per cent confidence, how large a sample should be taken to obtain a margin of error of 0.03 for the estimation of a population proportion? Assume that past data are not available for developing a planning value for  $p$ .



**COMPLETE  
SOLUTIONS**

### Applications

- 29.** A survey of 611 office workers investigated telephone answering practices, including how often each office worker was able to answer incoming telephone calls and how often incoming telephone calls went directly to voice mail. A total of 281 office workers indicated that they never need voice mail and are able to take every telephone call.
- What is the point estimate of the proportion of the population of office workers who are able to take every telephone call?
  - At 90 per cent confidence, what is the margin of error?
  - What is the 90 per cent confidence interval for the proportion of the population of office workers who are able to take every telephone call?
- 30.** The French market research and polling company CSA carried out surveys to investigate job satisfaction among professionally qualified employees of private companies. A total of 629

professionals were involved in the surveys, of whom 195 said that they were dissatisfied with their employer's recognition of their professional experience.

- a. What is the point estimate of the proportion of the population of employees who were dissatisfied with their employer's recognition of their professional experience?
  - b. At 95 per cent confidence, what is the margin of error?
  - c. What is the 95 per cent confidence interval for the proportion of the population of employees who were dissatisfied with their employer's recognition of their professional experience?
- 31.** In a sample of 162 companies, 104 reported profits that beat prior estimates, 29 matched estimates and 29 fell short of prior estimates.
- a. What is the point estimate of the proportion that fell short of estimates?
  - b. Determine the margin of error and provide a 95 per cent confidence interval for the proportion that fell short of estimates.
  - c. How large a sample is needed if the desired margin of error is 0.05?
- 32.** In early December 2008, the Palestinian Center for Policy and Survey Research carried out an opinion poll among adults in the West Bank and Gaza Strip. Respondents were asked their opinion about the chance of an independent Palestinian state being established alongside Israel in the next five years. Among the 1270 respondents, 34.6 per cent felt there was no chance of this happening.
- a. Provide a 95 per cent confidence interval for the population proportion of adults who thought there was no chance of an independent Palestinian state being established alongside Israel in the next five years.
  - b. Provide a 99 per cent confidence interval for the population proportion of adults who thought there was no chance of an independent Palestinian state being established alongside Israel in the next five years.
  - c. What happens to the margin of error as the confidence is increased from 95 per cent to 99 per cent?
- 33.** In a survey conducted by ICM Research in the UK, 710 out of 1000 adults interviewed said that, if there were to be a referendum, they would vote for the UK not to join the European currency (the euro). What is the margin of error and what is the interval estimate of the population proportion of British adults who would vote for the UK not to join the European currency? Use 95 per cent confidence.
- 34.** A well-known bank credit card firm wishes to estimate the proportion of credit card holders who carry a non-zero balance at the end of the month and incur an interest charge. Assume that the desired margin of error is 0.03 at 98 per cent confidence.
- a. How large a sample should be selected if it is anticipated that roughly 70 per cent of the firm's cardholders carry a non-zero balance at the end of the month?
  - b. How large a sample should be selected if no planning value for the proportion could be specified?



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SOLUTIONS



### ONLINE RESOURCES

For the data files, additional online summary, questions, answers and software section go to the accompanying online platform.

## SUMMARY

In this chapter we introduced the idea of an interval estimate of a population parameter. A point estimator may or may not provide a good estimate of a population parameter. The use of an interval estimate provides a measure of the precision of an estimate. A common form of interval estimate is a confidence interval.

We presented methods for computing confidence intervals of a population mean and a population proportion. Both are of the form: point estimate  $\pm$  margin of error. The confidence interval has a confidence coefficient associated with it.

We presented interval estimates for a population mean for two cases. In the  $\sigma$  known case, historical data or other information is used to make an estimate of  $\sigma$  prior to taking a sample. Analysis of new sample data then proceeds based on the assumption that  $\sigma$  is known. In the  $\sigma$  unknown case, the sample data are used to estimate both the population mean and the population standard deviation. In the  $\sigma$  known case, the interval estimation procedure is based on the assumed value of  $\sigma$  and the use of the standard normal distribution. In the  $\sigma$  unknown case, the interval estimation procedure uses the sample standard deviation  $s$  and the  $t$  distribution.

In both cases the quality of the interval estimates obtained depends on the distribution of the population and the sample size. Practical advice about the sample size necessary to obtain good approximations was included in Sections 8.1 and 8.2.

The general form of the interval estimate for a population proportion is  $p \pm$  margin of error. In practice, the sample sizes used for interval estimates of a population proportion are generally large. Consequently, the interval estimation procedure is based on the standard normal distribution.

We explained how the expression for margin of error can be used to calculate the sample size required to achieve a desired margin of error at a given level of confidence. We did this for two cases: estimating a population mean when the population standard deviation is known, and estimating a population proportion.

## KEY TERMS

**Confidence coefficient**  
**Confidence interval**  
**Confidence level**  
**Degrees of freedom**  
**Interval estimate**

**Margin of error**  
 **$\sigma$  known**  
 **$\sigma$  unknown**  
 **$t$  distribution**

## KEY FORMULAE

### Interval estimate of a population mean: $\sigma$ known

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \quad (8.1)$$

### Interval estimate of a population mean: $\sigma$ unknown

$$\bar{x} \pm t_{\alpha/2} \frac{s}{\sqrt{n}} \quad (8.2)$$

### Sample size for an interval estimate of a population mean

$$n = \frac{(z_{\alpha/2})^2 \sigma^2}{E^2} \quad (8.3)$$

### Interval estimate of a population proportion

$$p \pm z_{\alpha/2} \sqrt{\frac{p(1-p)}{n}} \quad (8.6)$$

### Sample size for an interval estimate of a population proportion

$$n = \frac{(z_{\alpha/2})^2 p^*(1-p^*)}{E^2} \quad (8.7)$$

## CASE PROBLEM 1



### International bank

The manager of a city-centre branch of a well-known international bank commissioned a customer satisfaction survey. The survey investigated three areas of customer satisfaction: their experience waiting for service at a till, their experience being served at the till and their experience of self-service facilities at the branch. Within each of these categories, respondents to the survey were asked to give ratings on a number of aspects of the bank's service. These ratings were then summed to give an overall satisfaction rating in each of the three areas of service. The summed ratings are scaled such that they lie between 0 and 100, with 0 representing extreme dissatisfaction and 100 representing extreme satisfaction. The data file for this case study ('IntnlBank' on the online platform) contains the 0–100 ratings for the three areas of service, together with particulars of respondents' gender and whether they would recommend the bank to other people (a simple Yes/No response was required to this question). A table containing the first few rows of the data file is shown.

| <i>Waiting</i> | <i>Service</i> | <i>Self-service</i> | <i>Gender</i> | <i>Recommend</i> |
|----------------|----------------|---------------------|---------------|------------------|
| 55             | 65             | 50                  | male          | no               |
| 50             | 80             | 88                  | male          | no               |
| 30             | 40             | 44                  | male          | no               |
| 65             | 60             | 69                  | male          | yes              |
| 55             | 65             | 63                  | male          | no               |
| 40             | 60             | 56                  | male          | no               |
| 15             | 65             | 38                  | male          | yes              |
| 45             | 60             | 56                  | male          | no               |
| 55             | 65             | 75                  | male          | no               |
| 50             | 50             | 69                  | male          | yes              |

### Managerial report

1. Use descriptive statistics to summarize each of the five variables in the data file (the three service ratings, customer gender and customer recommendation).
2. Calculate a 95 per cent confidence interval estimate of the mean service rating for the population of customers of the branch, for each of the three service areas. Provide a managerial interpretation of each interval estimate.
3. Calculate a 95 per cent confidence interval estimate of the proportion of the branch's customers who would recommend the bank, and a 95 per cent confidence interval estimate of



the proportion of the branch's customers who are female. Provide a managerial interpretation of each interval estimate.

4. Suppose the branch manager required an estimate of the percentage of branch customers who would recommend the branch within a margin of error of 3 percentage points. Using 95 per cent confidence, how large should the sample size be?
5. Suppose the branch manager required an estimate of the percentage of branch customers who are female within a margin of error of 5 percentage points. Using 95 per cent confidence, how large should the sample size be?



## CASE PROBLEM 2



### Consumer Knowhow

Consumer Knowhow is a consumer research organization that conducts surveys designed to evaluate a wide variety of products and services available to consumers. In one particular study, Consumer Knowhow looked at consumer satisfaction with the performance of cars produced by a major European manufacturer. A questionnaire sent to owners of one of the manufacturer's family cars revealed several complaints about early transmission problems.



To learn more about the transmission failures, Consumer Knowhow used a sample of transmission repairs provided by a transmission repair firm located near the manufacturing plant. The data in the file 'Repairs' are the kilometres driven for 50 cars at the time of transmission failure.

### Managerial report

1. Use appropriate descriptive statistics to summarize the transmission failure data.
2. Construct a 95 per cent confidence interval for the mean number of kilometres driven until transmission failure for the population of cars with transmission failure. Provide a managerial interpretation of the interval estimate.
3. Discuss the implication of your statistical findings in relation to the proposition that some owners of the cars experienced early transmission failures.
4. How many repair records should be sampled if the research company wants the population mean number of kilometres driven until transmission failure to be estimated with a margin of error of 5000 kilometres? Use 95 per cent confidence.
5. What other information would you like to gather to evaluate the transmission failure problem more fully?



REPAIRS



# 9

# Hypothesis Tests

## CHAPTER CONTENTS

Statistics in Practice Hypothesis testing in business research

- 9.1 Developing null and alternative hypotheses
- 9.2 Type I and Type II errors
- 9.3 Population mean:  $\sigma$  known
- 9.4 Population mean:  $\sigma$  unknown
- 9.5 Population proportion
- 9.6 Hypothesis testing and decision-making
- 9.7 Calculating the probability of Type II errors
- 9.8 Determining the sample size for hypothesis tests about a population mean

**LEARNING OBJECTIVES** After studying this chapter and doing the exercises, you should be able to:

- 1 Set up appropriate null and alternative hypotheses for testing research hypotheses, and for testing the validity of a claim.
- 2 Give an account of the logical steps involved in a statistical hypothesis test.
- 3 Explain the meaning of the terms null hypothesis, alternative hypothesis, Type I error, Type II error, level of significance,  $p$ -value and critical value in statistical hypothesis testing.
- 4 Construct and interpret hypothesis tests for a population mean:
  - 4.1 When the population standard deviation is known.
  - 4.2 When the population standard deviation is unknown.
- 5 Construct and interpret hypothesis tests for a population proportion.
- 6 Explain the relationship between the construction of hypothesis tests and confidence intervals.
- 7 Calculate the probability of a Type II error for a hypothesis test of a population mean when the population standard deviation is known.
- 8 Estimate the sample size required for a hypothesis test of a population mean when the population standard deviation is known.

In Chapters 7 and 8 we showed how a sample could be used to construct point and interval estimates of population parameters. In this chapter we continue the discussion of statistical inference by showing how hypothesis testing can be used to determine whether a statement about the value of a population parameter should or should not be rejected.

In hypothesis testing we begin by making a tentative assumption about a population parameter. This tentative assumption is called the **null hypothesis** and is denoted by  $H_0$ . We then define another hypothesis, called the **alternative hypothesis**, which is the opposite of what is stated in the null hypothesis. We denote the alternative hypothesis by  $H_1$ . The hypothesis testing procedure uses data from a sample to assess the two competing statements indicated by  $H_0$  and  $H_1$ .

This chapter shows how hypothesis tests can be conducted about a population mean and a population proportion. We begin by providing examples of approaches to formulating null and alternative hypotheses.



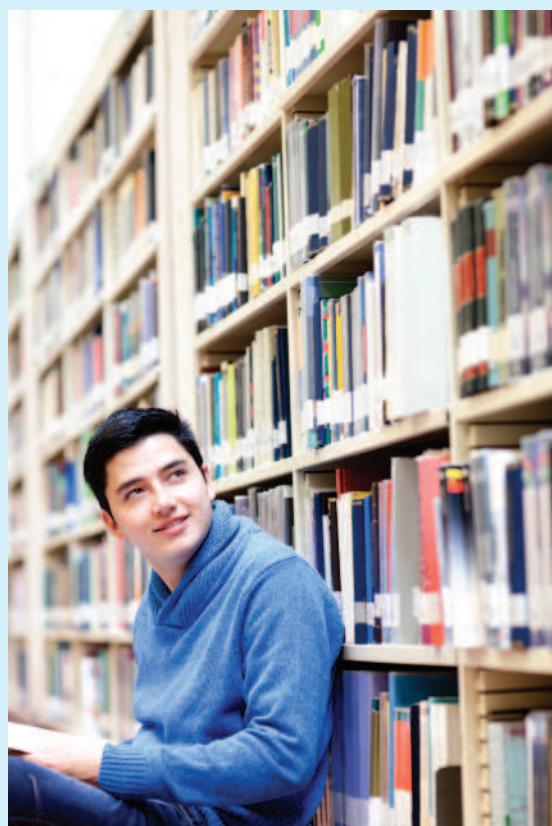
## STATISTICS IN PRACTICE

Hypothesis testing in business research

The *British Journal of Management (BJM)* is one of the most highly rated academic journals globally in the field of management. It is published quarterly, and contains articles giving accounts of the latest research in the field. Any particular issue typically shows an authorship with wide geographic spread. For example, the June 2011 issue contained nine articles written by researchers from Germany, Switzerland, Italy, UK, Lebanon, Australia and Canada. The research topics addressed included links between work/home culture and employee well-being, attitudes towards corporate social responsibility, age discrimination in recruitment and assessment of research quality in UK universities.

Of the nine articles in the June 2011 *BJM* issue, seven reported research based on quantitative methodology. The other two featured qualitative research. All of the seven quantitative papers featured both descriptive statistics and extensive use of inferential statistics. The main tool in regard to the inferential results reported was the *statistical hypothesis test*. Between them, the seven articles reported a total of over 400 statistical hypothesis tests. In other words, most of these articles involved over 50 hypothesis tests per article. The *BJM* is not unusual in this respect. Similar results would be found if other academic journals in business were examined, as indeed academic journals in economics, finance, psychology and many other fields.

Many of the hypothesis tests in the *BJM* articles were those described in Chapters 10 to 18 of this book. In the present chapter, we set the scene by setting out the logic of statistical hypothesis testing, and illustrating the logic by describing several simple hypothesis tests.



## 9.1 DEVELOPING NULL AND ALTERNATIVE HYPOTHESES

It is not always obvious how the null and alternative hypotheses should be formulated. Care must be taken to structure the hypotheses appropriately so that the hypothesis testing conclusion provides the information the researcher or decision maker wants. The context of the situation is very important in determining how the hypotheses should be stated. All hypothesis testing applications involve collecting a sample and using the sample results to provide evidence for drawing a conclusion. Good questions to consider when formulating the null and alternative hypotheses are: What is the purpose of collecting the sample? What conclusions are we hoping to make?

In the chapter introduction, we stated that the null hypothesis  $H_0$  is a tentative assumption about a population parameter such as a population mean or a population proportion. The alternative hypothesis  $H_1$  states the opposite (or complement) of the null hypothesis. In some situations it is easier to identify the alternative hypothesis first and then develop the null hypothesis. In other situations it is easier to identify the null hypothesis first and then develop the alternative hypothesis. We shall illustrate these situations in the following examples.

### The alternative hypothesis as a research hypothesis

Many applications of hypothesis testing involve an attempt to gather evidence in support of a research hypothesis. In these situations, it is often best to begin with the alternative hypothesis and make it the conclusion that the researcher hopes to support. Consider a particular model of car that currently attains an average fuel consumption of seven litres of fuel per 100 kilometres of driving. A product research group develops a new fuel injection system specifically designed to decrease the fuel consumption. To evaluate the new system, several will be manufactured, installed in cars and subjected to research-controlled driving tests. Here the product research group is looking for evidence to conclude that the new system *decreases* the mean fuel consumption. In this case, the research hypothesis is that the new fuel injection system will provide a mean litres-per-100 km rating below 7; that is,  $\mu < 7$ . As a general guideline, a research hypothesis should be stated as the *alternative hypothesis*. Hence, the appropriate null and alternative hypotheses for the study are:

$$H_0: \mu \geq 7$$

$$H_1: \mu < 7$$

If the sample results lead to the conclusion to reject  $H_0$ , the inference can be made that  $H_1: \mu < 7$  is true. The researchers have the statistical support to state that the new fuel injection system decreases the mean litres of fuel consumed per 100km. The production of cars with the new fuel injection system should be considered. However, if the sample results lead to the conclusion that  $H_0$  cannot be rejected, the researchers cannot conclude that the new fuel injection system is better than the current system. Production of cars with the new fuel injection system on the basis of improved fuel consumption cannot be justified. Perhaps more research and further testing can be conducted.

*The conclusion that the research hypothesis is true is made if the sample data provide sufficient evidence to show that the null hypothesis can be rejected.*

Successful companies stay competitive by developing new products, new methods, new systems and the like, that are better than those currently available. Before adopting something new, it is desirable to do research to determine if there is statistical support for the conclusion that the new approach is indeed better. In such cases, the research hypothesis is stated as the alternative hypothesis. For example, a new teaching method is developed that is believed to be better than the current method. The alternative hypothesis is that the new method is better. The null hypothesis is that the new method is no better than the old method. A new sales force bonus plan is developed in an attempt to increase sales. The alternative hypothesis is that the new bonus plan increases sales. The null hypothesis is that the new bonus plan does not increase sales. A new drug is developed with the goal of lowering blood pressure more than an existing drug. The alternative hypothesis is that the new drug lowers blood pressure more than the existing drug. The null hypothesis is that the new drug does not provide lower blood pressure than the existing drug. In each case, rejection of the null hypothesis  $H_0$  provides statistical support for the

research hypothesis. We will see many examples of hypothesis tests in research situations such as these throughout this chapter and in the remainder of the text.

## The null hypothesis as an assumption to be challenged

Of course, not all hypothesis tests involve research hypotheses. In the following discussion we consider applications of hypothesis testing where we begin with a belief or an assumption that a statement about the value of a population parameter is true. We will then use a hypothesis test to challenge the assumption and determine if there is statistical evidence to conclude that the assumption is incorrect. In these situations, it is helpful to develop the null hypothesis first. The null hypothesis  $H_0$  expresses the belief or assumption about the value of the population parameter. The alternative hypothesis  $H_1$  is that the belief or assumption is incorrect.

As an example, consider the situation of a soft drinks manufacturer. The label on the bottle states that it contains 1.5 litres. We consider the label correct provided the population mean filling volume for the bottles is *at least* 1.5 litres. Without any reason to believe otherwise, we would give the manufacturer the benefit of the doubt and assume that the statement on the label is correct. So, in a hypothesis test about the population mean volume per bottle, we would begin with the assumption that the label is correct and state the null hypothesis as  $\mu \geq 1.5$ . The challenge to this assumption would imply that the label is incorrect and the bottles are being underfilled. This challenge would be stated as the alternative hypothesis  $\mu < 1.5$ . The null and alternative hypotheses are:

$$H_0: \mu \geq 1.5$$

$$H_1: \mu < 1.5$$

A trading standards office (TSO) with the responsibility for validating manufacturing labels could select a sample of soft drinks bottles, compute the sample mean filling weight and use the sample results to test the preceding hypotheses. If the sample results lead to the conclusion to reject  $H_0$ , the inference that  $H_1: \mu < 1.5$  is true can be made. With this statistical support, the TSO is justified in concluding that the label is incorrect and underfilling of the bottles is occurring. Appropriate action to force the manufacturer to comply with labelling standards would be considered. However, if the sample results indicate  $H_0$  cannot be rejected, the assumption that the manufacturer's labelling is correct cannot be rejected. With this conclusion, no action would be taken.

*A manufacturer's product information is usually assumed to be true and stated as the null hypothesis. The conclusion that the information is incorrect can be made if the null hypothesis is rejected.*

Let us now consider a variation of the soft drink bottle filling example by viewing the same situation from the manufacturer's point of view. The bottle-filling operation has been designed to fill soft drink bottles with 1.5 litres as stated on the label. The company does not want to underfill the containers because that could result in an underfilling complaint from customers or, perhaps, a TSO. However, the company does not want to overfill containers either because putting more soft drink than necessary into the containers would be an unnecessary cost. The company's goal would be to adjust the bottle-filling operation so that the population mean filling weight per bottle is 1.5 litres as specified on the label.

Although this is the company's goal, from time to time any production process can get out of adjustment. If this occurs in our example, underfilling or overfilling of the soft drink bottles will occur. In either case, the company would like to know about it in order to correct the situation by re-adjusting the bottle-filling operation to the designed 1.5 litres. In a hypothesis testing application, we would again begin with the assumption that the production process is operating correctly and state the null hypothesis as  $\mu = 1.5$  litres. The alternative hypothesis that challenges this assumption is that  $\mu \neq 1.5$ , which indicates either overfilling or underfilling is occurring. The null and alternative hypotheses for the manufacturer's hypothesis test are:

$$H_0: \mu = 1.5$$

$$H_1: \mu \neq 1.5$$

Suppose that the soft drink manufacturer uses a quality control procedure to periodically select a sample of bottles from the filling operation and computes the sample mean filling volume per bottle. If the sample results lead to the conclusion to reject  $H_0$ , the inference is made that  $H_1: \mu \neq 1.5$  is true. We conclude that the bottles are not being filled properly and the production process should be adjusted to restore the population mean to 1.5 litres per bottle. However, if the sample results indicate  $H_0$  cannot be rejected, the assumption that the manufacturer's bottle filling operation is functioning properly cannot be rejected. In this case, no further action would be taken and the production operation would continue to run.

The two preceding forms of the soft drink manufacturing hypothesis test show that the null and alternative hypotheses may vary depending upon the point of view of the researcher or decision maker. To correctly formulate hypotheses it is important to understand the context of the situation and structure the hypotheses to provide the information the researcher or decision maker wants.

### Summary of forms for null and alternative hypotheses

The hypothesis tests in this chapter involve one of two population parameters: the population mean and the population proportion. Depending on the situation, hypothesis tests about a population parameter may take one of three forms: two include inequalities in the null hypothesis, the third uses only an equality in the null hypothesis. For hypothesis tests involving a population mean, we let  $\mu_0$  denote the hypothesized value and choose one of the following three forms for the hypothesis test.

$$\begin{array}{lll} H_0: \mu \geq \mu_0 & H_0: \mu \leq \mu_0 & H_0: \mu = \mu_0 \\ H_1: \mu < \mu_0 & H_1: \mu > \mu_0 & H_1: \mu \neq \mu_0 \end{array}$$

For reasons that will be clear later, the first two forms are called one-tailed tests. The third form is called a two-tailed test.

In many situations, the choice of  $H_0$  and  $H_1$  is not obvious and judgement is necessary to select the proper form. However, as the preceding forms show, the equality part of the expression (either  $\geq$ ,  $\leq$  or  $=$ ) *always* appears in the null hypothesis. In selecting the proper form of  $H_0$  and  $H_1$ , keep in mind that the alternative hypothesis is often what the test is attempting to establish. Hence, asking whether the user is looking for evidence to support  $\mu < \mu_0$ ,  $\mu > \mu_0$  or  $\mu \neq \mu_0$  will help determine  $H_1$ . The following exercises are designed to provide practice in choosing the proper form for a hypothesis test involving a population mean.

### EXERCISES

- 1.** The manager of the Costa Resort Hotel stated that the mean weekend guest bill is €600 or less. A member of the hotel's accounting staff noticed that the total charges for guest bills have been increasing in recent months. The accountant will use a sample of weekend guest bills to test the manager's claim.

- a. Which form of the hypotheses should be used to test the manager's claim? Explain.

$$\begin{array}{lll} H_0: \mu \geq 600 & H_0: \mu \leq 600 & H_0: \mu = 600 \\ H_1: \mu < 600 & H_1: \mu > 600 & H_1: \mu \neq 600 \end{array}$$

- b. What conclusion is appropriate when  $H_0$  cannot be rejected?  
c. What conclusion is appropriate when  $H_0$  can be rejected?

- 2.** The manager of a car dealership is considering a new bonus plan designed to increase sales volume. Currently, the mean sales volume is 14 cars per month. The manager wants to conduct a research study to see whether the new bonus plan increases sales volume. To collect data on the plan, a sample of sales personnel will be allowed to sell under the new bonus plan for a one-month period.

- a. Formulate the null and alternative hypotheses most appropriate for this research situation.  
b. Comment on the conclusion when  $H_0$  cannot be rejected.  
c. Comment on the conclusion when  $H_0$  can be rejected.



3. A production line operation is designed to fill cartons with laundry detergent to a mean weight of 0.75kg. A sample of cartons is periodically selected and weighed to determine whether underfilling or overfilling is occurring. If the sample data lead to a conclusion of underfilling or overfilling, the production line will be shut down and adjusted to obtain proper filling.
- Formulate the null and alternative hypotheses that will help in deciding whether to shut down and adjust the production line.
  - Comment on the conclusion and the decision when  $H_0$  cannot be rejected.
  - Comment on the conclusion and the decision when  $H_0$  can be rejected.
4. Because of high production-changeover time and costs, a director of manufacturing must convince management that a proposed manufacturing method reduces costs before the new method can be implemented. The current production method operates with a mean cost of €320 per hour. A research study will measure the cost of the new method over a sample production period.
- Formulate the null and alternative hypotheses most appropriate for this study.
  - Comment on the conclusion when  $H_0$  cannot be rejected.
  - Comment on the conclusion when  $H_0$  can be rejected.

## 9.2 TYPE I AND TYPE II ERRORS

The null and alternative hypotheses are competing statements about the population. Either the null hypothesis  $H_0$  is true or the alternative hypothesis  $H_1$  is true, but not both. Ideally the hypothesis testing procedure should lead to the acceptance of  $H_0$  when  $H_0$  is true and the rejection of  $H_0$  when  $H_1$  is true. Unfortunately, the correct conclusions are not always possible. Because hypothesis tests are based on sample information, we must allow for the possibility of errors. Table 9.1 illustrates the two kinds of errors that can be made in hypothesis testing.

The first row of Table 9.1 shows what can happen if the conclusion is to accept  $H_0$ . If  $H_0$  is true, this conclusion is correct. However, if  $H_1$  is true, we make a **Type II error**; that is, we accept  $H_0$  when it is false. The second row of Table 9.1 shows what can happen if the conclusion is to reject  $H_0$ . If  $H_0$  is true, we make a **Type I error**; that is, we reject  $H_0$  when it is true. However, if  $H_1$  is true, rejecting  $H_0$  is correct.

Recall the hypothesis testing illustration discussed in Section 9.1 in which a product research group developed a new fuel injection system designed to decrease the fuel consumption of a particular car. With the current model achieving an average of seven litres of fuel per 100km, the hypothesis test was formulated as follows.

$$\begin{aligned} H_0: \mu &\geq 7 \\ H_1: \mu &< 7 \end{aligned}$$

The alternative hypothesis,  $H_1: \mu < 7$ , indicates that the researchers are looking for sample evidence to support the conclusion that the population mean fuel consumption with the new fuel injection system is less than 7.

In this application, the Type I error of rejecting  $H_0$  when it is true corresponds to the researchers claiming that the new system reduces fuel consumption ( $\mu < 7$ ) when in fact the new system is no better than the current system.

**TABLE 9.1** Errors and correct conclusions in hypothesis testing

|            | Population condition |                    |
|------------|----------------------|--------------------|
|            | $H_0$ true           | $H_1$ true         |
| Conclusion | Accept $H_0$         | Correct conclusion |
|            | Reject $H_0$         | Type I error       |
|            |                      | Type II error      |
|            |                      | Correct conclusion |

In contrast, the Type II error of accepting  $H_0$  when it is false corresponds to the researchers concluding that the new system is no better than the current system ( $\mu \geq 7$ ) when in fact the new system reduces fuel consumption.

For the fuel consumption hypothesis test, the null hypothesis is  $H_0: \mu \geq 7$ . Suppose the null hypothesis is true as an equality; that is,  $\mu = 7$ . The probability of making a Type I error when the null hypothesis is true as an equality is called the **level of significance**. This is an important concept. For the fuel efficiency hypothesis test, the level of significance is the probability of rejecting  $H_0: \mu \geq 7$  when  $\mu = 7$ .

### Level of significance

The level of significance is the probability of making a Type I error when the null hypothesis is true as an equality.

The Greek symbol  $\alpha$  (alpha) is used to denote the level of significance. In practice, the person conducting the hypothesis test specifies the level of significance. By selecting  $\alpha$ , that person is controlling the probability of making a Type I error. If the cost of making a Type I error is high, small values of  $\alpha$  are preferred. If the cost of making a Type I error is not too high, larger values of  $\alpha$  are typically used. Common choices for  $\alpha$  are 0.05 and 0.01. Applications of hypothesis testing that only control for the Type I error are often called *significance tests*. Most applications of hypothesis testing are of this type.

Although most applications of hypothesis testing control for the probability of making a Type I error, they do not always control for the probability of making a Type II error. Hence, if we decide to accept  $H_0$ , we cannot determine how confident we can be with that decision. Because of the uncertainty associated with making a Type II error, statisticians often recommend that we use the statement ‘do not reject  $H_0$ ’ instead of ‘accept  $H_0$ ’. Using the statement ‘do not reject  $H_0$ ’ carries the recommendation to withhold both judgement and action. In effect, by not directly accepting  $H_0$ , the statistician avoids the risk of making a Type II error. Whenever the probability of making a Type II error has not been determined and controlled, we will not make the statement ‘accept  $H_0$ ’. In such cases, the two conclusions possible are: *do not reject  $H_0$*  or *reject  $H_0$* .

Although controlling for a Type II error in hypothesis testing is not common, it can be done. In Sections 9.7 and 9.8 we shall illustrate procedures for determining and controlling the probability of making a Type II error. If proper controls have been established for this error, action based on the ‘accept  $H_0$ ’ conclusion can be appropriate.

## EXERCISES

5. The label on a container of yoghurt claims that the yoghurt contains an average of one gram of fat or less. Answer the following questions for a hypothesis test that could be used to test the claim on the label.
  - a. Formulate the appropriate null and alternative hypotheses.
  - b. What is the Type I error in this situation? What are the consequences of making this error?
  - c. What is the Type II error in this situation? What are the consequences of making this error?
6. Carpetland salespersons average €5000 per week in sales. The company’s chief executive officer (CEO) proposes a remuneration plan with new selling incentives. The CEO hopes that the results of a trial selling period will enable them to conclude that the remuneration plan increases the average sales per salesperson.
  - a. Formulate the appropriate null and alternative hypotheses.
  - b. What is the Type I error in this situation? What are the consequences of making this error?
  - c. What is the Type II error in this situation? What are the consequences of making this error?



7. Suppose a new production method will be implemented if a hypothesis test supports the conclusion that the new method reduces the mean operating cost per hour.
- State the appropriate null and alternative hypotheses if the mean cost for the current production method is €320 per hour.
  - What is the Type I error in this situation? What are the consequences of making this error?
  - What is the Type II error in this situation? What are the consequences of making this error?

## 9.3 POPULATION MEAN: $\sigma$ KNOWN

In this section we show how to conduct a hypothesis test about a population mean for the  $\sigma$  known case, i.e. where historical data and/or other information are available that enable us to obtain a good estimate of the population standard deviation prior to sampling. The methods presented in this section are exact if the sample is selected from a population that is normally distributed. In cases where it is not reasonable to assume the population is normally distributed, these methods are still applicable if the sample size is large enough. We provide some practical advice concerning the population distribution and the sample size at the end of this section.

### One-tailed test

**One-tailed tests** about a population mean take one of the following two forms.

| <i>Lower-tail test</i> | <i>Upper-tail test</i> |
|------------------------|------------------------|
| $H_0: \mu \geq \mu_0$  | $H_0: \mu \leq \mu_0$  |
| $H_1: \mu < \mu_0$     | $H_1: \mu > \mu_0$     |

Consider an example. Trading Standards Offices (TSOs) periodically conduct statistical studies to test the claims that manufacturers make about their products. For example, suppose the label on a large bottle of Cola states that the bottle contains three litres of Cola. European legislation acknowledges that the bottling process cannot guarantee exactly three litres of Cola in each bottle, even if the mean filling volume for the population of all bottles filled is three litres per bottle. However, if the population mean filling volume is at least three litres per bottle, the rights of consumers will be protected. The legislation interprets the label information on a large bottle of Cola as a claim that the population mean filling weight is at least three litres per bottle. We shall show how a TSO can check the claim by conducting a lower-tail hypothesis test.

The first step is to formulate the null and alternative hypotheses for the test. If the population mean filling volume is at least three litres per bottle, the manufacturer's claim is correct. This establishes the null hypothesis for the test. However, if the population mean weight is less than three litres per bottle, the manufacturer's claim is incorrect. This establishes the alternative hypothesis. With  $\mu$  denoting the population mean filling volume, the null and alternative hypotheses are as follows:

$$H_0: \mu \geq 3$$

$$H_1: \mu < 3$$

Note that the hypothesized value of the population mean is  $\mu_0 \geq 3$ . If the sample data indicate that  $H_0$  cannot be rejected, the statistical evidence does not support the conclusion that a labelling violation has occurred. Hence, no action should be taken against the manufacturer. However, if the sample data indicate  $H_0$  can be rejected, we shall conclude that the alternative hypothesis,  $H_1: \mu < 3$ , is true. In this case a conclusion of underfilling and a charge of a labelling violation against the manufacturer would be justified.

Suppose a sample of 36 bottles is selected and the sample mean is computed as an estimate of the population mean  $\mu$ . If the value of the sample mean is less than three litres, the sample results will cast doubt on the null hypothesis. What we want to know is how much less than three litres the sample mean must be before we would be willing to declare the difference significant and risk making a Type I error by falsely accusing the manufacturer of a labelling violation. A key factor in addressing this issue is the value the decision-maker selects for the level of significance.

As noted in the preceding section, the level of significance, denoted by  $\alpha$ , is the probability of making a Type I error by rejecting  $H_0$  when the null hypothesis is true as an equality. The decision-maker must specify the level of significance. If the cost of making a Type I error is high, a small value should be chosen for the level of significance. If the cost is not high, a larger value is more appropriate. Suppose that in the Cola bottling study, the TSO made the following statement: 'If the manufacturer is meeting its weight specifications at  $\mu = 3$ , I would like a 99 per cent chance of not taking any action against the manufacturer. Although I do not want to accuse the manufacturer wrongly of underfilling, I am willing to risk a 1 per cent chance of making such an error.' From the TSO's statement, we set the level of significance for the hypothesis test at  $\alpha = 0.01$ . Hence, we must design the hypothesis test so that the probability of making a Type I error when  $\mu = 3$  is 0.01.

For the Cola bottling study, by developing the null and alternative hypotheses and specifying the level of significance for the test, we carry out the first two steps required in conducting every hypothesis test. We are now ready to perform the third step of hypothesis testing: collect the sample data and compute the value of an appropriate test statistic.

### Test statistic

For the Cola bottling study, previous Trading Standards tests show that the population standard deviation can be assumed known with a value of  $\sigma = 0.18$ . In addition, these tests also show that the population of filling weights can be assumed to have a normal distribution. From the study of sampling distributions in Chapter 7 we know that if the population from which we are sampling is normally distributed, the sampling distribution of the sample mean will also be normal in shape. Hence, for the Cola bottling study, the sampling distribution of  $\bar{X}$  is normal. With a known value of  $\sigma = 0.18$  and a sample size of  $n = 36$ , Figure 9.1 shows the sampling distribution of  $\bar{X}$  when the null hypothesis is true as an equality; that is, when  $\mu = \mu_0 = 3$ . In constructing sampling distributions for hypothesis tests, it is assumed that  $H_0$  is satisfied as an equality. Note that the standard error of is given by:

$$\sigma_{\bar{X}} = \sigma/\sqrt{n} = 0.18/\sqrt{36} = 0.03$$

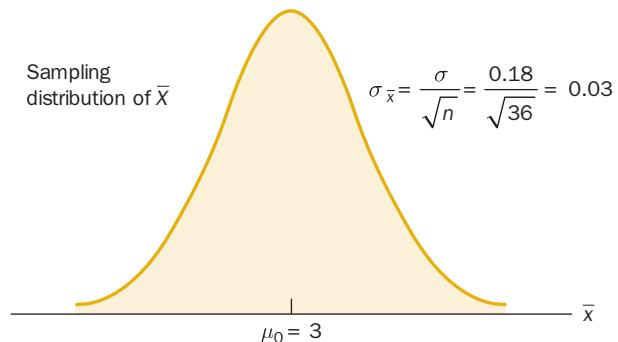
Because  $\bar{X}$  has a normal sampling distribution, the sampling distribution of:

$$Z = \frac{\bar{X} - \mu_0}{\sigma_{\bar{X}}} = \frac{\bar{X} - 3}{0.03}$$

is a standard normal distribution. A value  $z = -1$  means that  $\bar{x}$  is one standard error below the mean, a value  $z = -2$  means that  $\bar{x}$  is two standard errors below the mean, and so on. We can use the standard normal distribution table to find the lower-tail probability corresponding to any  $z$  value. For instance, the standard normal table shows that the cumulative probability for  $z = -3.00$  is 0.0014.

**FIGURE 9.1**

Sampling distribution of  $\bar{X}$  for the Cola bottling study when the null hypothesis is true as an equality ( $\mu_0 = 3$ )



This is the probability of obtaining a value that is three or more standard errors below the mean. As a result, the probability of obtaining a value  $\bar{x}$  that is 3 or more standard errors below the hypothesized population mean  $\mu_0 = 3$  is also 0.0014. Such a result is unlikely if the null hypothesis is true.

For hypothesis tests about a population mean for the  $\sigma$  known case, we use the standard normal random variable  $Z$  as a **test statistic** to determine whether  $\bar{x}$  deviates from the hypothesized value  $\mu_0$  enough to justify rejecting the null hypothesis. The test statistic used in the  $\sigma$  known case is as follows (note that  $\sigma_{\bar{x}} = \sigma/\sqrt{n}$ ).

### Test statistic for hypothesis tests about a population mean: $\sigma$ known

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \quad (9.1)$$

The key question for a lower-tail test is: How small must the test statistic  $z$  be before we choose to reject the null hypothesis? Two approaches can be used to answer this question.

The first approach uses the value  $z$  from expression (9.1) to compute a probability called a ***p*-value**. The *p*-value measures the support provided by the sample for the null hypothesis, and is the basis for determining whether the null hypothesis should be rejected given the level of significance. The second approach requires that we first determine a value for the test statistic called the **critical value**. For a lower-tail test, the critical value serves as a benchmark for determining whether the value of the test statistic is small enough to reject the null hypothesis. We begin with the *p*-value approach.

### ***p*-value approach**

The *p*-value approach has become the preferred method of determining whether the null hypothesis can be rejected, especially when using computer software packages such as MINITAB, IBM SPSS and EXCEL. We begin with a formal definition for a *p*-value.

#### ***p*-value**

The *p*-value is a probability, computed using the test statistic, that measures the degree to which the sample supports the null hypothesis.

Because a *p*-value is a probability, it ranges from 0 to 1. A small *p*-value indicates a sample result that is unusual given the assumption that  $H_0$  is true. Small *p*-values lead to rejection of  $H_0$ , whereas large *p*-values indicate the null hypothesis should not be rejected.

First, we use the value of the test statistic to compute the *p*-value. The method used to compute a *p*-value depends on whether the test is lower-tail, upper-tail, or a two-tailed test. For a lower tail test, the *p*-value is the probability of obtaining a value for the test statistic at least as small as that provided by the sample. To compute the *p*-value for the lower tail test in the  $\sigma$  known case, we find the area under the standard normal curve to the left of the test statistic. After computing the *p*-value, we then decide whether it is small enough to reject the null hypothesis. As we will show, this involves comparing it to the level of significance.

We now illustrate the *p*-value approach by computing the *p*-value for the Cola bottling lower-tail test. Suppose the sample of 36 Cola bottles provides a sample mean of  $\bar{x} = 2.92$  litres. Is  $\bar{x} = 2.92$  small enough to cause us to reject  $H_0$ ? Because this test is a lower-tail test, the *p*-value is the area under the standard normal curve to the left of the test statistic. Using  $\bar{x} = 2.92$ ,  $\sigma = 0.18$  and  $n = 36$ , we compute the value  $z$  of the test statistic:

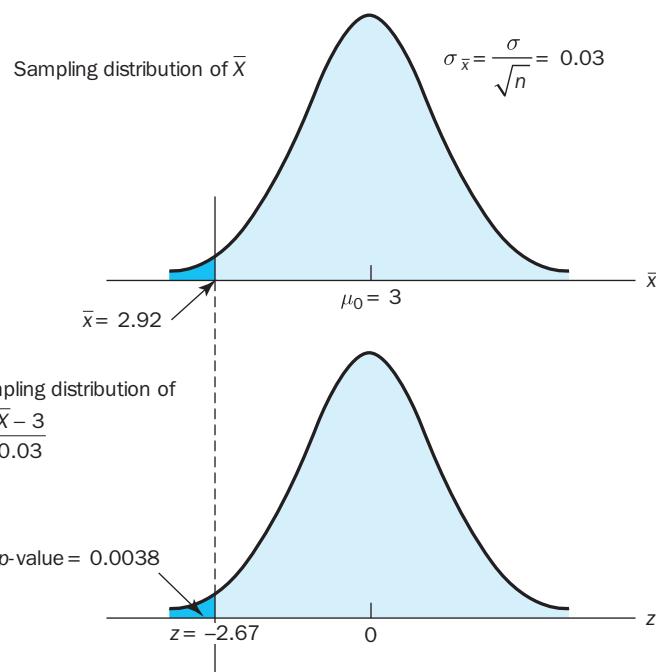
$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{2.92 - 3}{0.18/\sqrt{36}} = -2.67$$



COLA

**FIGURE 9.2**

*p*-value for the Cola bottling study when  $\bar{x} = 2.92$  and  $z = -2.67$



The *p*-value is the probability that the test statistic  $Z$  is less than or equal to  $-2.67$  (the area under the standard normal curve to the left of  $z = -2.67$ ).

Using the standard normal distribution table, we find that the cumulative probability for  $z = -2.67$  is 0.00382. Figure 9.2 shows that  $\bar{x} = 2.92$  corresponds to  $z = -2.67$  and a  $p$ -value = 0.0038. This  $p$ -value indicates a small probability of obtaining a sample mean of  $\bar{x} = 2.92$  or smaller when sampling from a population with  $\mu = 3$ . This  $p$ -value does not provide much support for the null hypothesis, but is it small enough to cause us to reject  $H_0$ ? The answer depends upon the level of significance for the test.

As noted previously, the TSO selected a value of 0.01 for the level of significance. The selection of  $\alpha = 0.01$  means that the TSO is willing to accept a probability of 0.01 of rejecting the null hypothesis when it is true as an equality ( $\mu_0 = 3$ ). The sample of 36 bottles in the Cola bottling study resulted in a  $p$ -value = 0.0038, which means that the probability of obtaining a value of  $\bar{x} = 2.92$  or less when the null hypothesis is true as an equality is 0.0038. Because 0.0038 is less than  $\alpha = 0.01$  we reject  $H_0$ . Therefore, we find sufficient statistical evidence to reject the null hypothesis at the 0.01 level of significance.

We can now state the general rule for determining whether the null hypothesis can be rejected when using the  $p$ -value approach. For a level of significance  $\alpha$ , the rejection rule using the  $p$ -value approach is as follows:

#### Rejection rule using $p$ -value

Reject  $H_0$  if  $p$ -value  $\leq \alpha$

In the Cola bottling test, the  $p$ -value of 0.0038 resulted in the rejection of the null hypothesis. The basis for rejecting  $H_0$  is a comparison of the  $p$ -value to the level of significance ( $\alpha = 0.01$ ) specified by the TSO. However, the observed  $p$ -value of 0.0038 means that we would reject  $H_0$  for any value  $\alpha \geq 0.0038$ . For this reason, the  $p$ -value is also called the *observed level of significance* or the *attained level of significance*.

Different decision-makers may express different opinions concerning the cost of making a Type I error and may choose a different level of significance. By providing the  $p$ -value as part of the hypothesis testing results, another decision-maker can compare the reported  $p$ -value to their own level of significance and possibly make a different decision with respect to rejecting  $H_0$ . The smaller the  $p$ -value, the greater the evidence against  $H_0$ , and the more the evidence in favour of  $H_1$ . Here are some guidelines statisticians suggest for interpreting small  $p$ -values:

- Less than 0.01 – Very strong evidence to conclude  $H_1$  is true.
- Between 0.01 and 0.05 – Moderately strong evidence to conclude  $H_1$  is true.
- Between 0.05 and 0.10 – Weak evidence to conclude  $H_1$  is true.
- Greater than 0.10 – Insufficient evidence to conclude  $H_1$  is true.

### Critical value approach

For a lower-tail test, the critical value is the value of the test statistic that corresponds to an area of  $\alpha$  (the level of significance) in the lower tail of the sampling distribution of the test statistic. In other words, the critical value is the largest value of the test statistic that will result in the rejection of the null hypothesis. Let us return to the Cola bottling example and see how this approach works.

In the  $\sigma$  known case, the sampling distribution for the test statistic  $Z$  is a standard normal distribution. Therefore, the critical value is the value of the test statistic that corresponds to an area of  $\alpha = 0.01$  in the lower tail of a standard normal distribution. Using the standard normal distribution table, we find that  $z = -2.33$  gives an area of 0.01 in the lower tail (see Figure 9.3). So if the sample results in a value of the test statistic that is less than or equal to  $-2.33$ , the corresponding  $p$ -value will be less than or equal to 0.01; in this case, we should reject the null hypothesis. Hence, for the Cola bottling study the critical value rejection rule for a level of significance of 0.01 is:

$$\text{Reject } H_0 \text{ if } z \leq -2.33$$

In the Cola bottling example,  $\bar{x} = 2.92$  and the test statistic is  $z = -2.67$ . Because  $z = -2.67 < -2.33$ , we can reject  $H_0$  and conclude that the Cola manufacturer is under-filling bottles.

We can generalize the rejection rule for the critical value approach to handle any level of significance. The rejection rule for a lower-tail test follows.

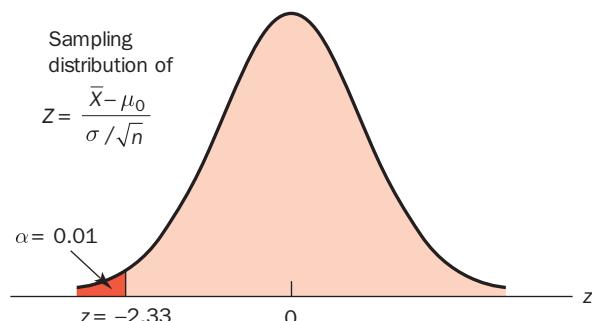
#### Rejection rule for a lower-tail test: critical value approach

$$\text{Reject } H_0 \text{ if } z \leq -z_\alpha$$

where  $-z_\alpha$  is the critical value; that is, the  $z$  value that provides an area of  $\alpha$  in the lower tail of the standard normal distribution.

**FIGURE 9.3**

Critical value for the Cola bottling hypothesis test



The  $p$ -value approach and the critical value approach will always lead to the same rejection decision. That is, whenever the  $p$ -value is less than or equal to  $\alpha$ , the value of the test statistic will be less than or equal to the critical value. The advantage of the  $p$ -value approach is that the  $p$ -value tells us *how* statistically significant the results are (the observed level of significance). If we use the critical value approach, we only know that the results are significant at the stated level of significance  $\alpha$ .

Computer procedures for hypothesis testing provide the  $p$ -value, so it is rapidly becoming the preferred method of doing hypothesis tests. If you do not have access to a computer, you may prefer to use the critical value approach. For some probability distributions it is easier to use statistical tables to find a critical value than to use the tables to compute the  $p$ -value. This topic is discussed further in the next section.

At the beginning of this section, we said that one-tailed tests about a population mean take one of the following two forms:

| <i>Lower-tail test</i> | <i>Upper-tail test</i> |
|------------------------|------------------------|
| $H_0: \mu \geq \mu_0$  | $H_0: \mu \leq \mu_0$  |
| $H_1: \mu < \mu_0$     | $H_1: \mu > \mu_0$     |

We used the Cola bottling study to illustrate how to conduct a lower-tail test. We can use the same general approach to conduct an upper-tail test. The test statistic is still computed using equation (9.1). But, for an upper-tail test, the  $p$ -value is the probability of obtaining a value for the test statistic at least as large as that provided by the sample. To compute the  $p$ -value for the upper-tail test in the  $\sigma$  known case, we must find the area under the standard normal curve to the right of the test statistic. Using the critical value approach causes us to reject the null hypothesis if the value of the test statistic is greater than or equal to the critical value  $z_\alpha$ . In other words, we reject  $H_0$  if  $z \geq z_\alpha$ .

## Two-tailed test

In hypothesis testing, the general form for a **two-tailed test** about a population mean is as follows:

$$H_0: \mu = \mu_0$$

$$H_1: \mu \neq \mu_0$$

In this subsection we show how to conduct a two-tailed test about a population mean for the  $\sigma$  known case. As an illustration, we consider the hypothesis testing situation facing MaxFlight, a manufacturer of golf equipment who use a high technology manufacturing process to produce golf balls with an average driving distance of 295 metres. Sometimes the process gets out of adjustment and produces golf balls with average distances different from 295 metres. When the average distance falls below 295 metres, the company worries about losing sales because the golf balls do not provide as much distance as advertised. However, some of the national golfing associations impose equipment standards for professional competition and when the average driving distance exceeds 295 metres, MaxFlight's golf balls may be rejected for exceeding the overall distance standard concerning carry and roll.

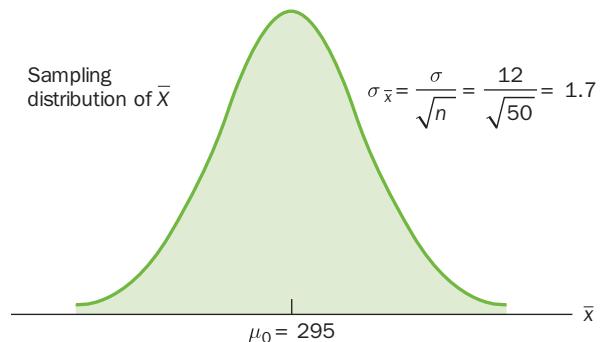
MaxFlight's quality control programme involves taking periodic samples of 50 golf balls to monitor the manufacturing process. For each sample, a hypothesis test is done to determine whether the process has fallen out of adjustment. Let us formulate the null and alternative hypotheses. We begin by assuming that the process is functioning correctly; that is, the golf balls being produced have a mean driving distance of 295 metres. This assumption establishes the null hypothesis. The alternative hypothesis is that the mean driving distance is not equal to 295 yards. The null and alternative hypotheses for the MaxFlight hypothesis test are as follows:

$$H_0: \mu = 295$$

$$H_1: \mu \neq 295$$

**FIGURE 9.4**

Sampling distribution of  $\bar{X}$  for the MaxFlight hypothesis test



If the sample mean is significantly less than 295 metres or significantly greater than 295 metres, we will reject  $H_0$ . In this case, corrective action will be taken to adjust the manufacturing process. On the other hand, if  $\bar{X}$  does not deviate from the hypothesized mean  $\mu_0 = 295$  by a significant amount,  $H_0$  will not be rejected and no action will be taken to adjust the manufacturing process.

The quality control team selected  $\alpha = 0.05$  as the level of significance for the test. Data from previous tests conducted when the process was known to be in adjustment show that the population standard deviation can be assumed known with a value of  $\sigma = 12$ . With a sample size of  $n = 50$ , the standard error of the sample mean is:

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{12}{\sqrt{50}} = 1.7$$

Because the sample size is large, the central limit theorem (see Chapter 7) allows us to conclude that the sampling distribution of  $\bar{X}$  can be approximated by a normal distribution. Figure 9.4 shows the sampling distribution of  $\bar{X}$  for the MaxFlight hypothesis test with a hypothesized population mean of  $\mu_0 = 295$ .

Suppose that a sample of 50 golf balls is selected and that the sample mean is 297.6 metres. This sample mean suggests that the population mean may be larger than 295 metres. Is this value  $\bar{x} = 297.6$  sufficiently larger than 295 to cause us to reject  $H_0$  at the 0.05 level of significance? In the previous section we described two approaches that can be used to answer this question: the *p*-value approach and the critical value approach.



### **p-value approach**

The *p*-value is a probability that measures the degree of support provided by the sample for the null hypothesis. For a two-tailed test, values of the test statistic in *either* tail show a lack of support for the null hypothesis. For a two-tailed test, the *p*-value is the probability of obtaining a value for the test statistic *at least as unlikely* as that provided by the sample. Let us compute the *p*-value for the MaxFlight hypothesis test.

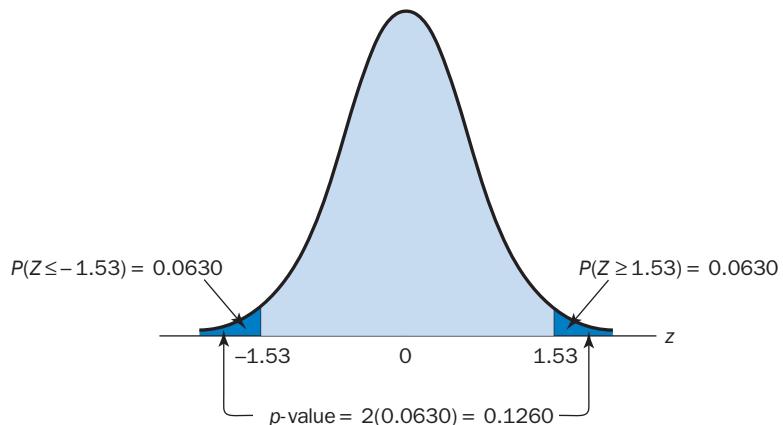
First, we compute the value of the test statistic. For the  $\sigma$  known case, the test statistic  $Z$  is a standard normal random variable. Using equation (9.1) with  $\bar{x} = 297.6$ , the value of the test statistic is:

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{297.6 - 295}{12/\sqrt{50}} = 1.53$$

Now we find the probability of obtaining a value for the test statistic *at least as unlikely* as  $z = 1.53$ . Clearly values  $\geq 1.53$  are *at least as unlikely*. But, because this is a two-tailed test, values  $\leq -1.53$  are also *at least as unlikely* as the value of the test statistic provided by the sample. Referring to Figure 9.5, we see that the two-tailed *p*-value in this case is given by  $P(Z \leq -1.53) + P(Z \geq 1.53)$ . Because the normal curve is symmetrical, we can compute this probability by finding the area under the standard normal curve to the left of  $z = -1.53$  and doubling it. The table of cumulative probabilities for the standard normal distribution shows that the area to the left of  $z = -1.53$  is 0.0630. Doubling this, we find the *p*-value for the MaxFlight two-tailed hypothesis test is  $2(0.0630) = 0.126$ .

**FIGURE 9.5**

*p*-value for the MaxFlight hypothesis test



Next we compare the *p*-value to the level of significance  $\alpha$ . With  $\alpha = 0.05$ , we do not reject  $H_0$  because the *p*-value  $= 0.126 > 0.05$ . Because the null hypothesis is not rejected, no action will be taken to adjust the MaxFlight manufacturing process.

The computation of the *p*-value for a two-tailed test may seem a bit confusing as compared to the computation of the *p*-value for a one-tailed test. But it can be simplified by following these three steps:

- 1 Compute the value of the test statistic  $z$ .
- 2 If the value of the test statistic is in the upper tail ( $z > 0$ ), find the area under the standard normal curve to the right of  $z$ . If the value of the test statistic is in the lower tail, find the area under the standard normal curve to the left of  $z$ .
- 3 Double the tail area, or probability, obtained in step 2 to obtain the *p*-value.

In practice, the computation of the *p*-value is done automatically when using computer software such as MINITAB, IBM SPSS and EXCEL.

### Critical value approach

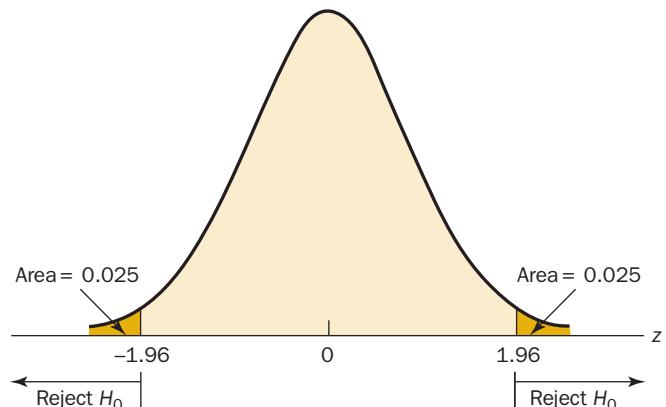
Now let us see how the test statistic can be compared to a critical value to make the hypothesis testing decision for a two-tailed test. Figure 9.6 shows that the critical values for the test will occur in both the lower and upper tails of the standard normal distribution. With a level of significance of  $\alpha = 0.05$ , the area in each tail beyond the critical values is  $\alpha/2 = 0.05/2 = 0.025$ . Using the table of probabilities for the standard normal distribution, we find the critical values for the test statistic are  $-z_{0.025} = -1.96$  and  $z_{0.025} = 1.96$ . Using the critical value approach, the two-tailed rejection rule is:

$$\text{Reject } H_0 \text{ if } z \leq -1.96 \text{ or if } z \geq 1.96$$

Because the value of the test statistic for the MaxFlight study is  $z = 1.53$ , the statistical evidence will not permit us to reject the null hypothesis at the 0.05 level of significance.

**FIGURE 9.6**

Critical values for the MaxFlight hypothesis test



**TABLE 9.2** Summary of hypothesis tests about a population mean:  $\sigma$  known case

|  | Lower-tail test                               | Upper-tail test                               | Two-tailed test   |
|--|---|---|---|
| <b>Hypotheses</b>                                  | $H_0: \mu \geq \mu_0$<br>$H_1: \mu < \mu_0$   | $H_0: \mu \leq \mu_0$<br>$H_1: \mu > \mu_0$   | $H_0: \mu = \mu_0$<br>$H_1: \mu \neq \mu_0$                           |
| <b>Test statistic</b>                              | $z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$ | $z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$ | $z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$                         |
| <b>Rejection rule:</b>                             | Reject $H_0$ if                               | Reject $H_0$ if                               | Reject $H_0$ if   |
| <b>p-value approach</b>                            | $p\text{-value} \leq \alpha$                  | $p\text{-value} \leq \alpha$                  | $p\text{-value} \leq \alpha$  |
| <b>Rejection rule:<br/>critical value approach</b> | Reject $H_0$ if<br>$z \leq -z_\alpha$         | Reject $H_0$ if<br>$z \geq z_\alpha$          | Reject $H_0$ if<br>$z \leq -z_{\alpha/2}$ or if $z \geq z_{\alpha/2}$ |

## Summary and practical advice

We presented examples of a lower-tail test and a two-tailed test about a population mean. Based upon these examples, we can now summarize the hypothesis testing procedures about a population mean for the  $\sigma$  known case as shown in Table 9.2. Note that  $\mu_0$  is the hypothesized value of the population mean. The hypothesis testing steps followed in the two examples presented in this section are common to every hypothesis test.

## Steps of hypothesis testing

- Step 1** Formulate the null and alternative hypotheses.
- Step 2** Specify the level of significance  $\alpha$ .
- Step 3** Collect the sample data and compute the value of the test statistic.

### p-value approach

- Step 4** Use the value of the test statistic to compute the  $p$ -value.
- Step 5** Reject  $H_0$  if the  $p$ -value  $\leq \alpha$ .

### Critical value approach

- Step 4** Use the level of significance  $\alpha$  to determine the critical value and the rejection rule.
- Step 5** Use the value of the test statistic and the rejection rule to determine whether to reject  $H_0$ .

Practical advice about the sample size for hypothesis tests is similar to the advice we provided about the sample size for interval estimation in Chapter 8. In most applications, a sample size of  $n \geq 30$  is adequate when using the hypothesis testing procedure described in this section. In cases where the sample size is less than 30, the distribution of the population from which we are sampling becomes an important consideration. If the population is normally distributed, the hypothesis testing procedure that we described is exact and can be used for any sample size. If the population is not normally distributed but is at least roughly symmetrical, sample sizes as small as 15 can be expected to provide acceptable results. With smaller sample sizes, the hypothesis testing procedure presented in this section should only be used if the analyst believes, or is willing to assume, that the population is at least approximately normally distributed.

## Relationship between interval estimation and hypothesis testing

We close this section by discussing the relationship between interval estimation and hypothesis testing. In Chapter 8 we showed how to construct a confidence interval estimate of a population mean. For the  $\sigma$

known case, the confidence interval estimate of a population mean corresponding to a  $1 - \alpha$  confidence coefficient is given by:

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \quad (9.2)$$

Doing a hypothesis test requires us first to formulate the hypotheses about the value of a population parameter. In the case of the population mean, the two-tailed test takes the form:

$$\begin{aligned} H_0: \mu &= \mu_0 \\ H_1: \mu &\neq \mu_0 \end{aligned}$$

where  $\mu_0$  is the hypothesized value for the population mean. Using the two-tailed critical value approach, we do not reject  $H_0$  for values of the sample mean that are within  $-z_{\alpha/2}$  and  $+z_{\alpha/2}$  standard errors of  $\mu_0$ . Hence, the do-not-reject region for the sample mean in a two-tailed hypothesis test with a level of significance of  $\alpha$  is given by:

$$\mu_0 \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \quad (9.3)$$

A close look at expression (9.2) and expression (9.3) provides insight about the relationship between the estimation and hypothesis testing approaches to statistical inference. Both procedures require the computation of the values  $z_{\alpha/2}$  and  $\sigma/\sqrt{n}$ . Focusing on  $\alpha$ , we see that a confidence coefficient of  $(1 - \alpha)$  for interval estimation corresponds to a level of significance of  $\alpha$  in hypothesis testing. For example, a 95 per cent confidence interval corresponds to a 0.05 level of significance for hypothesis testing. Furthermore, expressions (9.2) and (9.3) show that, because  $z_{\alpha/2}\sigma/\sqrt{n}$  is the plus or minus value for both expressions, if  $\bar{x}$  is in the do-not-reject region defined by (9.3), the hypothesized value  $\mu_0$  will be in the confidence interval defined by (9.2). Conversely, if the hypothesized value  $\mu_0$  is in the confidence interval defined by (9.2), the sample mean will be in the do-not-reject region for the hypothesis  $H_0: \mu = \mu_0$  as defined by (9.3). These observations lead to the following procedure for using a confidence interval to conduct a two-tailed hypothesis test.

#### A confidence interval approach to testing a hypothesis of the form

$$\begin{aligned} H_0: \mu &= \mu_0 \\ H_1: \mu &\neq \mu_0 \end{aligned}$$

- Select a simple random sample from the population and use the value of the sample mean to construct the confidence interval for the population mean  $\mu$ .

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

- If the confidence interval contains the hypothesized value  $\mu_0$ , do not reject  $H_0$ . Otherwise, reject  $H_0$ .

We return to the MaxFlight hypothesis test, which resulted in the following two-tailed test:

$$\begin{aligned} H_0: \mu &= 295 \\ H_1: \mu &\neq 295 \end{aligned}$$

To test this hypothesis with a level of significance of  $\alpha = 0.05$ , we sampled 50 golf balls and found a sample mean distance of  $\bar{x} = 297.6$  yards. The population standard deviation is  $\sigma = 12$ . Using these results with  $z_{0.025} = 1.96$ , we find that the 95 per cent confidence interval estimate of the population mean is:

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = 297.6 \pm 1.96 \frac{12}{\sqrt{50}} = 297.6 \pm 3.3$$

This finding enables the quality control manager to conclude with 95 per cent confidence that the mean distance for the population of golf balls is between 294.3 and 300.9 metres. Because the hypothesized value for the population mean,  $\mu_0 = 295$ , is in this interval, the conclusion from the hypothesis test is that the null hypothesis,  $H_0: \mu = 295$ , cannot be rejected.

Note that this discussion and example pertain to two-tailed hypothesis tests about a population mean. The same confidence interval and two-tailed hypothesis testing relationship exists for other population parameters. The relationship can also be extended to one-tailed tests about population parameters. Doing so, however, requires the construction of one-sided confidence intervals.

## EXERCISES

*Note to student:* Some of the exercises ask you to use the *p*-value approach and others ask you to use the critical value approach. Both methods will provide the same hypothesis testing conclusion. We provide exercises with both methods to give you practice using both. In later sections and in following chapters, we will generally emphasize the *p*-value approach as the preferred method, but you may select either based on personal preference.

### Methods

8. Consider the following hypothesis test:

$$H_0: \mu \geq 20$$

$$H_1: \mu < 20$$

A sample of 50 gave a sample mean of 19.4. The population standard deviation is 2.

- Compute the value of the test statistic.
- What is the *p*-value?
- Using  $\alpha = 0.05$ , what is your conclusion?
- What is the rejection rule using the critical value? What is your conclusion?

9. Consider the following hypothesis test:

$$H_0: \mu = 15$$

$$H_1: \mu \neq 15$$

A sample of 50 provided a sample mean of 14.15. The population standard deviation is 3.

- Compute the value of the test statistic.
- What is the *p*-value?
- At  $\alpha = 0.05$ , what is your conclusion?
- What is the rejection rule using the critical value? What is your conclusion?

10. Consider the following hypothesis test:

$$H_0: \mu \leq 50$$

$$H_1: \mu > 50$$

A sample of 60 is used and the population standard deviation is 8. Use the critical value approach to state your conclusion for each of the following sample results. Use  $\alpha = 0.05$ .

- $\bar{x} = 52.5$ .
- $\bar{x} = 51.0$ .
- $\bar{x} = 51.8$ .



COMPLETE  
SOLUTIONS

**Applications**

- 11.** Suppose that the mean length of the working week for a population of workers has been previously reported as 39.2 hours. We would like to take a current sample of workers to see whether the mean length of a working week has changed from the previously reported 39.2 hours.
- State the hypotheses that will help us determine whether a change occurred in the mean length of a working week.
  - Suppose a current sample of 112 workers provided a sample mean of 38.5 hours. Use a population standard deviation  $\sigma = 4.8$  hours. What is the  $p$ -value?
  - At  $\alpha = 0.05$ , can the null hypothesis be rejected? What is your conclusion?
  - Repeat the preceding hypothesis test using the critical value approach.
- 12.** Suppose the national mean sales price for new two-bedroom houses is £181 900. A sample of 40 new two-bedroom house sales in the north-east of England showed a sample mean of £166 400. Use a population standard deviation of £33 500.
- Formulate the null and alternative hypotheses that can be used to determine whether the sample data support the conclusion that the population mean sales price for new two-bedroom houses in the north-east is less than the national mean of £181 900.
  - What is the value of the test statistic?
  - What is the  $p$ -value?
  - At  $\alpha = 0.01$ , what is your conclusion?
- 13.** Fowler Marketing Research bases charges to a client on the assumption that telephone surveys can be completed in a mean time of 15 minutes or less per interview. If a longer mean interview time is necessary, a premium rate is charged. Suppose a sample of 35 interviews shows a sample mean of 17 minutes. Use  $\sigma = 4$  minutes. Is the premium rate justified?
- Formulate the null and alternative hypotheses for this application.
  - Compute the value of the test statistic.
  - What is the  $p$ -value?
  - At  $\alpha = 0.01$ , what is your conclusion?
- 14.** CCN and ActMedia provided a television channel targeted to individuals waiting in supermarket checkout lines. The channel showed news, short features and advertisements. The length of the programme was based on the assumption that the population mean time a shopper stands in a supermarket checkout line is eight minutes. A sample of actual waiting times will be used to test this assumption and determine whether actual mean waiting time differs from this standard.
- Formulate the hypotheses for this application.
  - A sample of 120 shoppers showed a sample mean waiting time of eight and a half minutes. Assume a population standard deviation  $\sigma = 3.2$  minutes. What is the  $p$ -value?
  - At  $\alpha = 0.05$ , what is your conclusion?
  - Compute a 95 per cent confidence interval for the population mean. Does it support your conclusion?
- 15.** During the global economic upheavals in late 2008, research companies affiliated to the Worldwide Independent Network of Market Research carried out polls in 17 countries to assess people's views on the economic outlook. One of the questions asked respondents to rate their trust in their government's management of the financial situation, on a 0 to 10 scale (10 being maximum trust). Suppose the worldwide population mean on this trust question was 5.2, and we are interested in the question of whether the population mean in Germany was different from this worldwide mean.



**COMPLETE  
SOLUTIONS**

- a. State the hypotheses that could be used to address this question.
- b. In the Germany survey, respondents gave the government a mean trust score of 4.0. Suppose the sample size in Germany was 1050, and the population standard deviation score was  $\sigma = 2.9$ . What is the 95 per cent confidence interval estimate of the population mean trust score for Germany?
- c. Use the confidence interval to conduct a hypothesis test. Using  $\alpha = 0.05$ , what is your conclusion?
- 16.** A production line operates with a mean filling weight of 500 grams per container. Overfilling or underfilling presents a serious problem and when detected requires the operator to shut down the production line to readjust the filling mechanism. From past data, a population standard deviation  $\sigma = 25$  grams is assumed. A quality control inspector selects a sample of 30 items every hour and at that time makes the decision of whether to shut down the line for readjustment. The level of significance is  $\alpha = 0.05$ .
- a. State the hypotheses in the hypothesis test for this quality control application.
- b. If a sample mean of 510 grams were found, what is the  $p$ -value? What action would you recommend?
- c. If a sample mean of 495 grams were found, what is the  $p$ -value? What action would you recommend?
- d. Use the critical value approach. What is the rejection rule for the preceding hypothesis testing procedure? Repeat parts (b) and (c). Do you reach the same conclusion?

## 9.4 POPULATION MEAN: $\sigma$ UNKNOWN

In this section we describe how to do hypothesis tests about a population mean for the  $\sigma$  unknown case. In this case, the sample must be used to compute estimates of both  $\mu$  (estimated by  $\bar{x}$ ) and  $\sigma$  (estimated by  $s$ ). The steps of the hypothesis testing procedure are the same as those for the  $\sigma$  known case described in Section 9.3. But, with  $\sigma$  unknown, the computation of the test statistic and  $p$ -value are a little different. For the  $\sigma$  known case, the sampling distribution of the test statistic has a standard normal distribution. For the  $\sigma$  unknown case, the sampling distribution of the test statistic has slightly more variability because the sample is used to compute estimates of both  $\mu$  and  $\sigma$ .

In Chapter 8, Section 8.2 we showed that an interval estimate of a population mean for the  $\sigma$  unknown case is based on a probability distribution known as the  $t$  distribution. Hypothesis tests about a population mean for the  $\sigma$  unknown case are also based on the  $t$  distribution. The test statistic has a  $t$  distribution with  $n - 1$  degrees of freedom.

### Test statistic for hypothesis tests about a population mean: $\sigma$ unknown

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \quad (9.4)$$

In Chapter 8 we said that the  $t$  distribution is based on an assumption that the population from which we are sampling has a normal distribution. However, research shows that this assumption can be relaxed considerably when the sample size is large enough. We provide some practical advice concerning the population distribution and sample size at the end of the section.

## One-tailed test

Consider an example of a one-tailed test about a population mean for the  $\sigma$  unknown case. A travel magazine wants to classify international airports according to the mean rating given by business travellers. A rating scale from 0 to 10 will be used, and airports with a population mean rating greater than seven will be designated as superior service airports. The magazine staff surveyed a sample of 60 business travellers at each airport. Suppose the sample for Abu Dhabi International Airport provided a sample mean rating of  $\bar{x} = 7.25$  and a sample standard deviation of  $s = 1.052$ . Do the data indicate that Abu Dhabi should be designated as a superior service airport?

We want to construct a hypothesis test for which the decision to reject  $H_0$  will lead to the conclusion that the population mean rating for Abu Dhabi International Airport is *greater* than seven. Accordingly, an upper-tail test with  $H_1: \mu > 7$  is required. The null and alternative hypotheses for this upper-tail test are as follows:



AIRRATING

$$H_0: \mu \leq 7$$

$$H_1: \mu > 7$$

We will use  $\alpha = 0.05$  as the level of significance for the test.

Using expression (9.4) with  $\bar{x} = 7.25$ ,  $s = 1.052$  and  $n = 60$ , the value of the test statistic is:

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{7.25 - 7}{1.052/\sqrt{60}} = 1.84$$

The sampling distribution of  $t$  has  $n - 1 = 60 - 1 = 59$  degrees of freedom. Because the test is an upper-tail test, the  $p$ -value is the area under the curve of the  $t$  distribution to the right of  $t = 1.84$ .

The  $t$  distribution table in most textbooks will not contain sufficient detail to determine the exact  $p$ -value, such as the  $p$ -value corresponding to  $t = 1.84$ . For instance, using Table 2 in Appendix B, the  $t$  distribution with 59 degrees of freedom provides the following information.

| Area in upper tail | 0.20  | 0.10  | 0.05  | 0.025 | 0.01  | 0.005 |
|--------------------|-------|-------|-------|-------|-------|-------|
| $t$ value (59 df)  | 0.848 | 1.296 | 1.671 | 2.001 | 2.391 | 2.662 |

$t = 1.84$

We see that  $t = 1.84$  is between 1.671 and 2.001. Although the table does not provide the exact  $p$ -value, the values in the 'Area in upper tail' row show that the  $p$ -value must be less than 0.05 and greater than 0.025. With a level of significance of  $\alpha = 0.05$ , this placement is all we need to know to make the decision to reject the null hypothesis and conclude that Abu Dhabi should be classified as a superior service airport. Computer packages such as MINITAB, IBM SPSS and EXCEL can easily determine the exact  $p$ -value associated with the test statistic  $t = 1.84$ . Each of these packages will show that the  $p$ -value is 0.035 for this example. A  $p$ -value  $= 0.035 < 0.05$  leads to the rejection of the null hypothesis and to the conclusion Abu Dhabi should be classified as a superior service airport.

The critical value approach can also be used to make the rejection decision. With  $\alpha = 0.05$  and the  $t$  distribution with 59 degrees of freedom,  $t_{0.05} = 1.671$  is the critical value for the test. The rejection rule is therefore:

$$\text{Reject } H_0 \text{ if } t \geq 1.671$$

With the test statistic  $t = 1.84 > 1.671$ ,  $H_0$  is rejected and we can conclude that Abu Dhabi can be classified as a superior service airport.

## Two-tailed test

To illustrate a two-tailed test about a population mean for the  $\sigma$  unknown case, consider the hypothesis testing situation facing Mega Toys. The company manufactures and distributes its products through more than 1000 retail outlets. In planning production levels for the coming winter season, Mega Toys must decide how many units of each product to produce prior to knowing the actual demand at the retail level. For this year's most important new toy, Mega Toys' marketing director is expecting demand to average 40 units per retail outlet. Prior to making the final production decision based on this estimate, Mega Toys decided to survey a sample of 25 retailers to gather more information about the demand for the new product. Each retailer was provided with information about the features of the new toy along with the cost and the suggested selling price. Then each retailer was asked to specify an anticipated order quantity.

With  $\mu$  denoting the population mean order quantity per retail outlet, the sample data will be used to conduct the following two-tailed hypothesis test:

$$H_0: \mu = 40$$

$$H_1: \mu \neq 40$$

If  $H_0$  cannot be rejected, Mega Toys will continue its production planning based on the marketing director's estimate that the population mean order quantity per retail outlet will be  $\mu = 40$  units. However, if  $H_0$  is rejected, Mega Toys will immediately re-evaluate its production plan for the product. A two-tailed hypothesis test is used because Mega Toys wants to re-evaluate the production plan if the population mean quantity per retail outlet is less than anticipated or greater than anticipated. Because no historical data are available (it is a new product), the population mean and the population standard deviation must both be estimated using  $\bar{x}$  and  $s$  from the sample data.

The sample of 25 retailers provided a mean of  $\bar{x} = 37.4$  and a standard deviation of  $s = 11.79$  units. Before going ahead with the use of the  $t$  distribution, the analyst constructed a histogram of the sample data in order to check on the form of the population distribution. The histogram of the sample data showed no evidence of skewness or any extreme outliers, so the analyst concluded that the use of the  $t$  distribution with  $n - 1 = 24$  degrees of freedom was appropriate. Using equation (9.4) with  $\bar{x} = 37.4$ ,  $\mu_0 = 40$ ,  $s = 11.79$ , and  $n = 25$ , the value of the test statistic is:

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{37.4 - 40}{11.79/\sqrt{25}} = -1.10$$

Because this is a two-tailed test, the  $p$ -value is two times the area under the curve for the  $t$  distribution to the left of  $t = -1.10$ . Using Table 2 in Appendix B, the  $t$  distribution table for 24 degrees of freedom provides the following information.

| Area in upper tail | 0.20  | 0.10  | 0.05  | 0.025 | 0.01  | 0.005 |
|--------------------|-------|-------|-------|-------|-------|-------|
| $t$ value (24 df)  | 0.858 | 1.318 | 1.711 | 2.064 | 2.492 | 2.797 |

$t = 1.10$

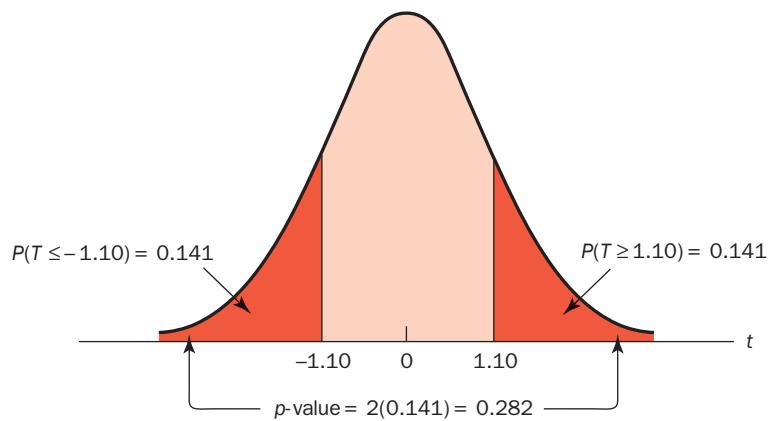
The  $t$  distribution table only contains positive  $t$  values. Because the  $t$  distribution is symmetrical, however, we can find the area under the curve to the right of  $t = 1.10$  and double it to find the  $p$ -value. We see that  $t = 1.10$  is between 0.858 and 1.318. From the 'Area in upper tail' row, we see that the area in the tail to the right of  $t = 1.10$  is between 0.20 and 0.10. Doubling these amounts, we see that the  $p$ -value must be between 0.40 and 0.20. With a level of significance of  $\alpha = 0.05$ , we now know that the  $p$ -value is greater than  $\alpha$ . Therefore,  $H_0$  cannot be rejected. There is insufficient evidence to conclude that Mega Toys should change its production plan for the coming season. Using MINITAB, IBM SPSS or EXCEL, we find that the exact  $p$ -value is 0.282. Figure 9.7 shows the two areas under the curve of the  $t$  distribution corresponding to the exact  $p$ -value.



ORDERS

**FIGURE 9.7**

Area under the curve in both tails provides the *p*-value



The test statistic can also be compared to the critical value to make the two-tailed hypothesis testing decision. With  $\alpha = 0.05$  and the *t* distribution with 24 degrees of freedom,  $-t_{0.025} = -2.064$  and  $t_{0.025} = 2.064$  are the critical values for the two-tailed test. The rejection rule using the test statistic is:

$$\text{Reject } H_0 \text{ if } t \leq -2.064 \text{ or if } t \geq 2.064$$

Based on the test statistic  $t = -1.10$ ,  $H_0$  cannot be rejected. This result indicates that Mega Toys should continue its production planning for the coming season based on the expectation that  $\mu = 40$  or do further investigation amongst its retailers.

## Summary and practical advice

Table 9.3 provides a summary of the hypothesis testing procedures about a population mean for the  $\sigma$  unknown case. The key difference between these procedures and the ones for the  $\sigma$  known case are that  $s$  is used, instead of  $\sigma$ , in the computation of the test statistic. For this reason, the test statistic follows the *t* distribution.

The applicability of the hypothesis testing procedures of this section is dependent on the distribution of the population being sampled and the sample size. When the population is normally distributed, the hypothesis tests described in this section provide exact results for any sample size. When the population is not normally distributed, the procedures are approximations. Nonetheless, we find that sample sizes greater than 50 will provide good results in almost all cases. If the population is approximately normal, small sample sizes (e.g.  $n < 15$ ) can provide acceptable results. In situations where the population cannot be approximated by a normal distribution, sample sizes of  $n \geq 15$  will provide acceptable results as long as the population is not significantly skewed and does not contain outliers. If the population is significantly skewed or contains outliers, samples sizes approaching 50 are a good idea.

**TABLE 9.3** Summary of hypothesis tests about a population mean:  $\sigma$  unknown case

|  | Lower-tail test                                 | Upper-tail test                                 | Two-tailed test   |
|--|---|---|---|
| <b>Hypotheses</b>  | $H_0: \mu \geq \mu_0$<br>$H_1: \mu < \mu_0$     | $H_0: \mu \leq \mu_0$<br>$H_1: \mu > \mu_0$     | $H_0: \mu = \mu_0$<br>$H_1: \mu \neq \mu_0$                           |
| <b>Test statistic</b>                                    | $t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$        | $t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$        | $t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$                              |
| <b>Rejection rule:</b><br><b><i>p</i>-value approach</b> | Reject $H_0$ if<br>$p\text{-value} \leq \alpha$ | Reject $H_0$ if<br>$p\text{-value} \leq \alpha$ | Reject $H_0$ if<br>$p\text{-value} \leq \alpha$                       |
| <b>Rejection rule:</b><br><b>critical value approach</b> | Reject $H_0$ if<br>$t \leq -t_\alpha$           | Reject $H_0$ if<br>$t \geq t_\alpha$            | Reject $H_0$ if<br>$t \leq -t_{\alpha/2}$ or if $t \geq t_{\alpha/2}$ |

## EXERCISES

### Methods

- 17.** Consider the following hypothesis test:

$$H_0: \mu \leq 12$$

$$H_1: \mu > 12$$

A sample of 25 provided a sample mean  $\bar{x} = 14$  and a sample standard deviation  $s = 4.32$ .

- a. Compute the value of the test statistic.
- b. What does the  $t$  distribution table (Table 2 in Appendix B) tell you about the  $p$ -value?
- c. At  $\alpha = 0.05$ , what is your conclusion?
- d. What is the rejection rule using the critical value? What is your conclusion?

- 18.** Consider the following hypothesis test:

$$H_0: \mu = 18$$

$$H_1: \mu \neq 18$$

A sample of 48 provided a sample mean  $\bar{x} = 17$  and a sample standard deviation  $s = 4.5$ .

- a. Compute the value of the test statistic.
- b. What does the  $t$  distribution table (Table 2 in Appendix B) tell you about the  $p$ -value?
- c. At  $\alpha = 0.05$ , what is your conclusion?
- d. What is the rejection rule using the critical value? What is your conclusion?

- 19.** Consider the following hypothesis test:

$$H_0: \mu \geq 45$$

$$H_1: \mu < 45$$

A sample of size 36 is used. Using  $\alpha = 0.01$ , identify the  $p$ -value and state your conclusion for each of the following sample results.

- a.  $\bar{x} = 44$  and  $s = 5.2$ .
- b.  $\bar{x} = 43$  and  $s = 4.6$ .
- c.  $\bar{x} = 46$  and  $s = 5.0$ .

### Applications

- 20.** Grolsch lager, like some of its competitors, can be bought in handy 300ml bottles. If a bottle such as Grolsch is marked as containing 300ml, legislation requires that the production batch from which the bottle came must have a mean fill volume of at least 300ml.

- a. Formulate hypotheses that could be used to determine whether the mean fill volume for a production batch satisfies the legal requirement of being at least 300ml.
- b. Suppose you take a random sample of 30 bottles from a lager-bottling production line and find that the mean fill for the sample of 30 bottles is 299.5ml, with a sample standard deviation of 1.9ml. What is the  $p$ -value?
- c. At  $\alpha = 0.01$ , what is your conclusion?

- 21.** Consider a daily TV programme – like the 10 o'clock news – that over the last calendar year had a mean daily audience of 4.0 million viewers. Assume that for a sample of 40 days during the current year, the daily audience was 4.15 million viewers with a sample standard deviation of 0.45 million viewers.



COMPLETE  
SOLUTIONS



COMPLETE  
SOLUTIONS

- a. If the TV management company would like to test for a change in mean viewing audience, what statistical hypotheses should be set up?
- b. What is the  $p$ -value?
- c. Select your own level of significance. What is your conclusion?
- 22.** A popular pastime amongst football fans is participation in ‘fantasy football’ competitions. Participants choose a squad of players and a manager, with the objective of increasing the valuation of the squad over the season. Suppose that at the start of the competition, the mean valuation of all available strikers was £4.7 million per player.
- a. Formulate the null and alternative hypotheses that could be used by a football pundit to determine whether mid-fielders have a higher mean valuation than strikers.
- b. Suppose a random sample of 30 mid-fielders from the available list had a mean valuation at the start of the competition of £5.80 million with a sample standard deviation of £2.46 million. On average, by how much did the valuation of mid-fielders exceed that of strikers?
- c. At  $\alpha = 0.05$ , what is your conclusion?
- 23.** Most new models of car sold in the European Union have to undergo an official test for fuel consumption. The test is in two parts: an urban cycle and an extra-urban cycle. The urban cycle is carried out under laboratory conditions, over a total distance of 4km at an average speed of 19km per hour. Consider a new car model for which the official fuel consumption figure for the urban cycle is published as 11.8 litres of fuel per 100km. A consumer affairs organization is interested in examining whether this published figure is truly indicative of urban driving.
- a. State the hypotheses that would enable the consumer affairs organization to conclude that the model’s fuel consumption is more than the published 11.8 litres per 100km.
- b. A sample of 50 mileage tests with the new model of car showed a sample mean of 12.10 litres per 100km and a sample standard deviation of 0.92 litre per 100km. What is the  $p$ -value?
- c. What conclusion should be drawn from the sample results? Use  $\alpha = 0.01$ .
- d. Repeat the preceding hypothesis test using the critical value approach.
- 24.** SuperScapes specializes in custom-designed landscaping for residential areas. The estimated labour cost associated with a particular landscaping proposal is based on the number of plantings of trees, shrubs and so on to be used for the project. For cost-estimating purposes, managers use two hours of labour time for the planting of a medium-sized tree. Actual times from a sample of ten plantings during the past month follow (times in hours).

1.7    1.5    2.6    2.2    2.4    2.3    2.6    3.0    1.4    2.3

With a 0.05 level of significance, test to see whether the mean tree-planting time differs from two hours.

- a. State the null and alternative hypotheses.
- b. Compute the sample mean.
- c. Compute the sample standard deviation.
- d. What is the  $p$ -value?
- e. What is your conclusion?

## 9.5 POPULATION PROPORTION

In this section we show how to do a hypothesis test about a population proportion  $\pi$ . Using  $\pi_0$  to denote the hypothesized value for  $\pi$ , the three forms for a hypothesis test are as follows.

$$\begin{array}{lll} H_0: \pi \geq \pi_0 & H_0: \pi \leq \pi_0 & H_0: \pi = \pi_0 \\ H_1: \pi < \pi_0 & H_1: \pi > \pi_0 & H_1: \pi \neq \pi_0 \end{array}$$

The first form is a lower-tail test, the second form is an upper-tail test, and the third form is a two-tailed test.

Hypothesis tests about a population proportion are based on the difference between the sample proportion  $p$  and the hypothesized population proportion  $\pi_0$ . The methods used to do the hypothesis test are similar to those used for hypothesis tests about a population mean. The only difference is that we use the sample proportion and its standard error to compute the test statistic. The  $p$ -value approach or the critical value approach is then used to determine whether the null hypothesis should be rejected.

Consider an example involving a situation faced by Aspire gymnasium. Over the past year, 20 per cent of the users of Aspire were women. In an effort to increase the proportion of women users, Aspire implemented a special promotion designed to attract women. One month afterwards, the gym manager requested a statistical study to determine whether the proportion of women users at Aspire had increased. An upper-tail test with  $H_1: \pi > 0.20$  is appropriate, because the objective of the study is to determine whether the proportion of women users increased. The null and alternative hypotheses for the Aspire hypothesis test are as follows:

$$H_0: \pi \leq 0.20$$

$$H_1: \pi > 0.20$$

If  $H_0$  can be rejected, the test results will give statistical support for the conclusion that the proportion of women users increased and the promotion was beneficial. The gym manager specified that a level of significance of  $\alpha = 0.05$  be used in carrying out this hypothesis test.

The next step of the hypothesis testing procedure is to select a sample and compute the value of an appropriate test statistic. We begin with a general discussion of how to compute the value of the test statistic for any form of a hypothesis test about a population proportion. The sampling distribution of  $P$ , the point estimator of the population parameter  $\pi$ , is the basis for constructing the test statistic.

When the null hypothesis is true as an equality, the expected value of  $P$  equals the hypothesized value  $\pi_0$ ; that is,  $E(P) = \pi_0$ . The standard error of  $P$  is given by:

$$\sigma_P = \sqrt{\frac{\pi_0(1 - \pi_0)}{n}}$$

In Chapter 7 we said that if  $n\pi \geq 5$  and  $n(1 - \pi) \geq 5$ , the sampling distribution of  $P$  can be approximated by a normal distribution.\* Under these conditions, which usually apply in practice, the quantity:

$$Z = \frac{P - \pi_0}{\sigma_P} \tag{9.5}$$

has a standard normal probability distribution, with  $\sigma_P = \sqrt{\pi_0(1 - \pi_0)/n}$ . Expression (9.5) gives the test statistic used to do hypothesis tests about a population proportion.

#### Test statistic for hypothesis tests about a population proportion

$$z = \frac{P - \pi_0}{\sqrt{\frac{\pi_0(1 - \pi_0)}{n}}} \tag{9.6}$$

We can now compute the test statistic for the Aspire hypothesis test. Suppose a random sample of 400 gym users was selected and that 100 of the users were women.

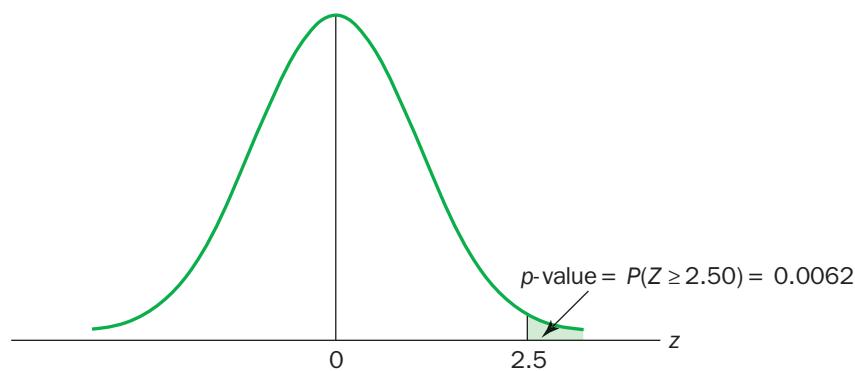
\*In most applications involving hypothesis tests of a population proportion, sample sizes are large enough to use the normal approximation. The exact sampling distribution of  $P$  is discrete with the probability for each value of  $P$  given by the binomial distribution. So hypothesis testing is more complicated for small samples when the normal approximation cannot be used.



WOMENGYM

**FIGURE 9.8**

Calculation of the  $p$ -value for the Aspire hypothesis



The proportion of women users in the sample is  $p = 100/400 = 0.25$ . Using expression (9.6), the value of the test statistic is:

$$z = \frac{p - \pi_0}{\sqrt{\frac{\pi_0(1 - \pi_0)}{n}}} = \frac{0.25 - 0.20}{\sqrt{\frac{0.20(1 - 0.20)}{400}}} = \frac{0.05}{\sqrt{\frac{0.02}{400}}} = 2.50$$

Because the Aspire hypothesis test is an upper-tail test, the  $p$ -value is the probability that  $Z$  is greater than or equal to  $z = 2.50$ . That is, it is the area under the standard normal curve to the right of  $z = 2.50$ . Using the table of cumulative probabilities for the standard normal distribution, we find that the  $p$ -value for the Aspire test is therefore  $(1 - 0.9938) = 0.0062$ . Figure 9.8 shows this  $p$ -value calculation.

Recall that the gym manager specified a level of significance of  $\alpha = 0.05$ . A  $p$ -value  $= 0.0062 < 0.05$  gives sufficient statistical evidence to reject  $H_0$  at the 0.05 level of significance. The test provides statistical support for the conclusion that the special promotion increased the proportion of women users at the Aspire gymnasium.

The decision whether to reject the null hypothesis can also be made using the critical value approach. The critical value corresponding to an area of 0.05 in the upper tail of a standard normal distribution is  $z_{0.05} = 1.645$ . Hence, the rejection rule using the critical value approach is to reject  $H_0$  if  $z \geq 1.645$ . Because  $z = 2.50 > 1.645$ ,  $H_0$  is rejected.

Again, we see that the  $p$ -value approach and the critical value approach lead to the same hypothesis testing conclusion, but the  $p$ -value approach provides more information. With a  $p$ -value  $= 0.0062$ , the null hypothesis would be rejected for any level of significance greater than or equal to 0.0062.

## Summary of hypothesis tests about a population proportion

The procedure used to conduct a hypothesis test about a population proportion is similar to the procedure used to conduct a hypothesis test about a population mean. Although we only illustrated how to conduct a hypothesis test about a population proportion for an upper-tail test, similar procedures can be used for lower-tail and two-tailed tests. Table 9.4 provides a summary of the hypothesis tests about a population proportion.

**TABLE 9.4** Summary of hypothesis tests about a population proportion

|   | <b>Lower-tail test</b>                                    | <b>Upper-tail test</b>                                    | <b>Two-tailed test</b>   |
|---|---|---|--|
| <b>Hypotheses</b>   | $H_0: \pi \geq \pi_0$<br>$H_1: \pi < \pi_0$               | $H_0: \pi \leq \pi_0$<br>$H_1: \pi > \pi_0$               | $H_0: \pi = \pi_0$<br>$H_1: \pi \neq \pi_0$                              |
| <b>Test statistic</b>   | $z = \frac{p - \pi_0}{\sqrt{\frac{\pi_0(1 - \pi_0)}{n}}}$ | $z = \frac{p - \pi_0}{\sqrt{\frac{\pi_0(1 - \pi_0)}{n}}}$ | $z = \frac{p - \pi_0}{\sqrt{\frac{\pi_0(1 - \pi_0)}{n}}}$                |
| <b>Rejection rule:</b><br><b><i>p</i>-value approach</b>            | Reject $H_0$ if<br>$p\text{-value} \leq \alpha$           | Reject $H_0$ if<br>$p\text{-value} \leq \alpha$           | Reject $H_0$ if<br>$p\text{-value} \leq \alpha$                          |
| <b>Rejection rule:</b><br><b><i>critical value</i><br/>approach</b> | Reject $H_0$<br>if $z \leq -z_\alpha$                     | Reject $H_0$<br>if $z \geq z_\alpha$                      | Reject $H_0$ if<br>$z \leq -z_{\alpha/2}$<br>or if $z \geq z_{\alpha/2}$ |

## EXERCISES

### Methods

- 25.** Consider the following hypothesis test:

$$H_0: \pi = 0.20$$

$$H_1: \pi \neq 0.20$$

A sample of 400 provided a sample proportion  $p = 0.175$ .

- a. Compute the value of the test statistic.
- b. What is the  $p$ -value?
- c. At  $\alpha = 0.05$ , what is your conclusion?
- d. What is the rejection rule using the critical value? What is your conclusion?

- 26.** Consider the following hypothesis test:

$$H_0: \pi \geq 0.75$$

$$H_1: \pi < 0.75$$

A sample of 300 items was selected. At  $\alpha = 0.05$ , compute the  $p$ -value and state your conclusion for each of the following sample results.

- a.  $p = 0.68$ .
- b.  $p = 0.72$ .
- c.  $p = 0.70$ .
- d.  $p = 0.77$ .

### Applications

- 27.** An airline promotion to business travellers is based on the assumption that at least two-thirds of business travellers use a laptop computer on overnight business trips.

- a. State the hypotheses that can be used to test the assumption.
- b. What is the sample proportion from an American Express sponsored survey that found 355 of 546 business travellers use a laptop computer on overnight business trips?
- c. What is the  $p$ -value?
- d. Use  $\alpha = 0.05$ . What is your conclusion?

- 28.** Eagle Outfitters is a chain of stores specializing in outdoor clothing and camping gear. It is considering a promotion that involves sending discount coupons to all their credit card customers by direct mail. This promotion will be considered a success if more than 10 per cent of those receiving the coupons use them. Before going nationwide with the promotion, coupons were sent to a sample of 100 credit card customers.

- a. Formulate hypotheses that can be used to test whether the population proportion of those who will use the coupons is sufficient to go national.
- b. The file ‘Eagle’ contains the sample data. Compute a point estimate of the population proportion.
- c. Use  $\alpha = 0.05$  to conduct your hypothesis test. Should Eagle go national with the promotion?

- 29.** In an IPSOS South Africa opinion poll in May 2012, a sample of adult South Africans were asked their opinions about the performance of the president, Jacob Zuma. One of the response options was the view that the president was performing ‘well’.

- a. Formulate the hypotheses that can be used to help determine whether more than 50 per cent of the adult population believe the president was performing well.
- b. Suppose that, of the 3565 respondents to the poll, 2140 expressed the view that the president was performing well. What is the sample proportion? What is the  $p$ -value?
- c. At  $\alpha = 0.01$ , what is your conclusion?



EAGLE



- 30.** A study by *Consumer Reports* showed that 64 per cent of supermarket shoppers believe supermarket brands to be as good as national name brands. To investigate whether this result applies to its own product, the manufacturer of a national name-brand ketchup asked a sample of shoppers whether they believed that supermarket ketchup was as good as the national brand ketchup.
- Formulate the hypotheses that could be used to determine whether the percentage of supermarket shoppers who believe that the supermarket ketchup was as good as the national brand ketchup differed from 64 per cent.
  - If a sample of 100 shoppers showed 52 stating that the supermarket brand was as good as the national brand, what is the  $p$ -value?
  - At  $\alpha = 0.05$ , what is your conclusion?
  - Should the national brand ketchup manufacturer be pleased with this conclusion? Explain.
- 31.** Microsoft Outlook is the most widely used email manager. A Microsoft executive claims that Microsoft Outlook is used by at least 75 per cent of Internet users. A sample of Internet users will be used to test this claim.
- Formulate the hypotheses that can be used to test the claim.
  - A Merrill Lynch study reported that Microsoft Outlook is used by 72 per cent of Internet users. Assume that the report was based on a sample size of 300 Internet users. What is the  $p$ -value?
  - At  $\alpha = 0.05$ , should the executive's claim of at least 75 per cent be rejected?
- 32.** In the elections in Greece in mid-June 2012, the centre-right New Democracy party polled 29.66 per cent of the vote. About a month before the election, a Public Issue opinion poll had estimated the proportion of support for each party. Did New Democracy's support change during the last month of the election campaign?
- Formulate the null and alternative hypotheses.
  - Suppose the Public Issue opinion poll in May had a random sample of 1200 potential voters, and that 26.0 per cent expressed support for New Democracy. What is the  $p$ -value?
  - Using  $\alpha = 0.05$ , what is your conclusion?

## 9.6 HYPOTHESIS TESTING AND DECISION-MAKING

In the previous sections of this chapter we have illustrated hypothesis testing applications that are considered significance tests. After formulating the null and alternative hypotheses, we selected a sample and computed the value of a test statistic and the associated  $p$ -value. We then compared the  $p$ -value to a controlled probability of Type I error,  $\alpha$ , which is called the level of significance for the test. If  $p$ -value  $\leq \alpha$ , we concluded 'reject  $H_0$ ' and declared the results significant; otherwise, we made the conclusion 'do not reject  $H_0$ '. With a significance test, we control the probability of making the Type I error, but not the Type II error. Consequently, we recommended the conclusion 'do not reject  $H_0$ ' rather than 'accept  $H_0$ ' because the latter puts us at risk of making the Type II error of accepting  $H_0$  when it is false. With the conclusion 'do not reject  $H_0$ ', the statistical evidence is considered inconclusive and is usually an indication to postpone a decision or action until further research and testing can be undertaken.

However, if the purpose of a hypothesis test is to make a decision when  $H_0$  is true and a different decision when  $H_1$  is true, the decision-maker may want to, and in some cases be forced to, take action with both the conclusion *do not reject  $H_0$*  and the conclusion *reject  $H_0$* . If this situation occurs, statisticians generally recommend controlling the probability of making a Type II error. With the probabilities of both the Type I and Type II error controlled, the conclusion from the hypothesis test is either to *accept  $H_0$*  or *reject  $H_0$* . In the first case,  $H_0$  is concluded to be true, while in the second case,  $H_1$  is concluded true. Thus, a decision and appropriate action can be taken when either conclusion is reached.

A good illustration of this situation is lot-acceptance sampling, a topic we will discuss in more depth in Chapter 20 (on the online platform). For example, a quality control manager must decide to accept a

shipment of batteries from a supplier or to return the shipment because of poor quality. Assume that design specifications require batteries from the supplier to have a mean useful life of at least 120 hours. To evaluate the quality of an incoming shipment, a sample of 36 batteries will be selected and tested. On the basis of the sample, a decision must be made to accept the shipment of batteries or to return it to the supplier because of poor quality. Let  $\mu$  denote the mean number of hours of useful life for batteries in the shipment. The null and alternative hypotheses about the population mean follow.

$$H_0: \mu \geq 120$$

$$H_1: \mu < 120$$

If  $H_0$  is rejected, the alternative hypothesis is concluded to be true. This conclusion indicates that the appropriate action is to return the shipment to the supplier. However, if  $H_0$  is not rejected, the decision-maker must still determine what action should be taken. Therefore, without directly concluding that  $H_0$  is true, but merely by not rejecting it, the decision-maker will have made the decision to accept the shipment as being of satisfactory quality.

In such decision-making situations, it is recommended that the hypothesis testing procedure be extended to control the probability of making a Type II error. Knowledge of the probability of making a Type II error will be helpful because a decision will be made and action taken when we do not reject  $H_0$ . In Sections 9.7 and 9.8 we explain how to compute the probability of making a Type II error and how the sample size can be adjusted to help control the probability of making a Type II error.

## 9.7 CALCULATING THE PROBABILITY OF TYPE II ERRORS

In this section we show how to calculate the probability of making a Type II error for a hypothesis test about a population mean. We illustrate the procedure by using the lot-acceptance example described in Section 9.6. The null and alternative hypotheses about the mean number of hours of useful life for a shipment of batteries are  $H_0: \mu \geq 120$  and  $H_1: \mu < 120$ . If  $H_0$  is rejected, the decision will be to return the shipment to the supplier because the mean hours of useful life are less than the specified 120 hours. If  $H_0$  is not rejected, the decision will be to accept the shipment.

Suppose a level of significance of  $\alpha = 0.05$  is used to conduct the hypothesis test. The test statistic in the  $\sigma$  known case is:

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{\bar{x} - 120}{\sigma/\sqrt{n}}$$

Based on the critical value approach and  $z_{0.05} = 1.645$ , the rejection rule for the lower-tail test is to reject  $H_0$  if  $z \leq -1.645$ . Suppose a sample of 36 batteries will be selected and based upon previous testing the population standard deviation can be assumed known with a value of  $\sigma = 12$  hours. The rejection rule indicates that we will reject  $H_0$  if:

$$z = \frac{\bar{x} - 120}{12/\sqrt{36}} \leq -1.645$$

Solving for  $\bar{x}$  in the preceding expression indicates that we will reject  $H_0$  if:

$$\bar{x} \leq 120 - 1.645 \left( \frac{12}{\sqrt{36}} \right) = 116.71$$

Rejecting  $H_0$  when  $\bar{x} \leq 116.71$  means we will accept the shipment whenever  $\bar{x} > 116.71$ . We are now ready to compute probabilities associated with making a Type II error. We make a Type II error whenever the true shipment mean is less than 120 hours and we decide to accept  $H_0: \mu \geq 120$ . To compute the probability of making a Type II error, we must therefore select a value of  $\mu$  less than 120 hours. For example, suppose the shipment is considered to be of poor quality if the batteries have a mean life of  $\mu = 112$  hours. If  $\mu = 112$ , what is the probability of accepting  $H_0: \mu \geq 120$  and hence committing a Type II error? This probability is the probability that the sample mean  $\bar{x}$  is greater than 116.71 when  $\mu = 112$ .

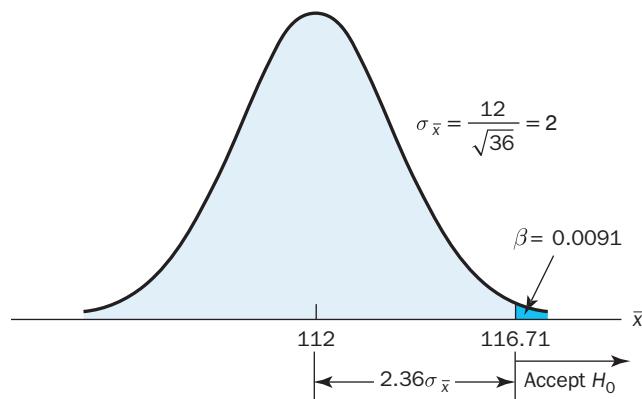
**FIGURE 9.9**Probability of a Type II error when  $\mu = 112$ 

Figure 9.9 shows the sampling distribution of the sample mean when the mean is  $\mu = 112$ . The shaded area in the upper tail gives the probability of obtaining  $\bar{x} > 116.71$ . Using the standard normal distribution, we see that at  $\bar{x} = 116.71$ :

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{116.71 - 112}{12/\sqrt{36}} = 2.36$$

The standard normal distribution table shows that with  $z = 2.36$ , the area in the upper tail is  $1 - 0.0909 = 0.0091$ . Denoting the probability of making a Type II error as  $\beta$ , we see if  $\mu = 112$ ,  $\beta = 0.0091$ . If the mean of the population is 112 hours, the probability of making a Type II error is only 0.0091.

We can repeat these calculations for other values of  $\mu$  less than 120. Doing so will show a different probability of making a Type II error for each value of  $\mu$ . For example, suppose the shipment of batteries has a mean useful life of  $\mu = 115$  hours. Because we will accept  $H_0$  whenever  $\bar{x} > 116.71$  the  $z$  value for  $\mu = 115$  is given by:

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{116.71 - 115}{12/\sqrt{36}} = 0.86$$

From the standard normal distribution table, we find that the area in the upper tail of the standard normal distribution for  $z = 0.86$  is  $1 - 0.8051 = 0.1949$ . The probability of making a Type II error is  $\beta = 0.1949$  when the true mean is  $\mu = 115$ .

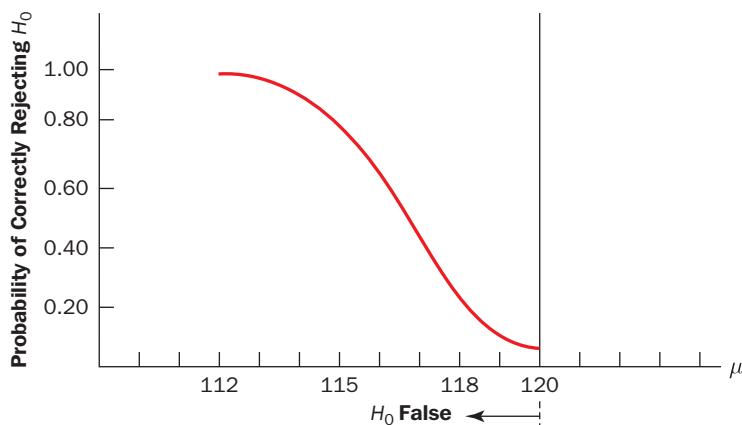
In Table 9.5 we show the probability of making a Type II error for a number of values of  $\mu$  less than 120. Note that as  $\mu$  increases towards 120, the probability of making a Type II error increases towards an upper bound of 0.95. However, as  $\mu$  decreases to values further below 120, the probability of making a Type II error diminishes. This pattern is what we should expect. When the true population mean  $\mu$  is close to the null hypothesis value of  $\mu = 120$ , the probability is high that we will make a Type II error. However, when the true population mean  $\mu$  is far below the null hypothesis value of 120, the probability is low that we will make a Type II error.

**TABLE 9.5** Probability of making a Type II error for the lot-acceptance hypothesis test

| Value of $\mu$ | $z = \frac{116.71 - \mu}{12/\sqrt{36}}$ | Probability of a Type II error ( $\beta$ ) | Power ( $1 - \beta$ ) |
|----------------|---|--|-----------------------|
| 112            | 2.36                                    | 0.0091                                     | 0.9909                |
| 114            | 1.36                                    | 0.0869                                     | 0.9131                |
| 115            | 0.86                                    | 0.1949                                     | 0.8051                |
| 116.71         | 0.00                                    | 0.5000                                     | 0.5000                |
| 117            | -0.15                                   | 0.5596                                     | 0.4404                |
| 118            | -0.65                                   | 0.7422                                     | 0.2578                |
| 119.999        | -1.645                                  | 0.9500                                     | 0.0500                |

**FIGURE 9.10**

Power curve for the lot-acceptance hypothesis test



The probability of correctly rejecting  $H_0$  when it is false is called the **power** of the test. For any particular value of  $\mu$ , the power is  $1 - \beta$ ; that is, the probability of correctly rejecting the null hypothesis is 1 minus the probability of making a Type II error. Values of power are also listed in Table 9.5. On the basis of these values, the power associated with each value of  $\mu$  is shown graphically in Figure 9.10. Such a graph is called a **power curve**. Note that the power curve extends over the values of  $\mu$  for which the null hypothesis is false. The height of the power curve at any value of  $\mu$  indicates the probability of correctly rejecting  $H_0$  when  $H_0$  is false. Another graph, called the *operating characteristic curve*, is sometimes used to provide information about the probability of making a Type II error. The operating characteristic curve shows the probability of accepting  $H_0$  and thus provides  $\beta$  for the values of  $\mu$  where the null hypothesis is false. The probability of making a Type II error can be read directly from this graph.

In summary, the following step-by-step procedure can be used to compute the probability of making a Type II error in hypothesis tests about a population mean.

- 1** Formulate the null and alternative hypotheses.
- 2** Use the level of significance  $\alpha$  and the critical value approach to determine the critical value and the rejection rule for the test.
- 3** Use the rejection rule to solve for the value of the sample mean corresponding to the critical value of the test statistic.
- 4** Use the results from step 3 to state the values of the sample mean that lead to the acceptance of  $H_0$ . These values define the acceptance region for the test.
- 5** Use the sampling distribution of  $\bar{X}$  for a value of  $\mu$  satisfying the alternative hypothesis, and the acceptance region from step 4, to compute the probability that the sample mean will be in the acceptance region. This probability is the probability of making a Type II error at the chosen value of  $\mu$ .

## EXERCISES

### Methods

- 33.** Consider the following hypothesis test.

$$\begin{aligned} H_0: \mu &\geq 10 \\ H_1: \mu &< 10 \end{aligned}$$

The sample size is 120 and the population standard deviation is assumed known,  $\sigma = 5$ . Use  $\alpha = 0.05$ .

- a. If the population mean is 9, what is the probability that the sample mean leads to the conclusion *do not reject  $H_0$* ?



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- b. What type of error would be made if the actual population mean is 9 and we conclude that  $H_0: \mu \geq 10$  is true?
- c. What is the probability of making a Type II error if the actual population mean is 8?

- 34.** Consider the following hypothesis test.

$$H_0: \mu = 20$$

$$H_1: \mu \neq 20$$

A sample of 200 items will be taken and the population standard deviation is  $\sigma = 10$ . Use  $\alpha = 0.05$ . Compute the probability of making a Type II error if the population mean is:

- a.  $\mu = 18.0$ .
- b.  $\mu = 22.5$ .
- c.  $\mu = 21.0$ .

### Applications

- 35.** Fowler Marketing Research bases charges to a client on the assumption that telephone survey interviews can be completed within 15 minutes or less. If more time is required, a premium rate is charged. With a sample of 35 interviews, a population standard deviation of four minutes, and a level of significance of 0.01, the sample mean will be used to test the null hypothesis  $H_0: \mu \leq 15$ .
- a. What is your interpretation of the Type II error for this problem? What is its impact on the firm?
  - b. What is the probability of making a Type II error when the actual mean time is  $\mu = 17$  minutes?
  - c. What is the probability of making a Type II error when the actual mean time is  $\mu = 18$  minutes?
  - d. Sketch the general shape of the power curve for this test.
- 36.** Refer to Exercise 35. Assume the firm selects a sample of 50 interviews and repeat parts (b) and (c). What observation can you make about how increasing the sample size affects the probability of making a Type II error?
- 37.** *Young Adult* magazine states the following hypotheses about the mean age of its subscribers.

$$H_0: \mu = 28$$

$$H_1: \mu \neq 28$$

- a. What would it mean to make a Type II error in this situation?
  - b. The population standard deviation is assumed known at  $\sigma = 6$  years and the sample size is 100. With  $\alpha = 0.05$ , what is the probability of accepting  $H_0$  for  $\mu$  equal to 26, 27, 29 and 30?
  - c. What is the power at  $\mu = 26$ ? What does this result tell you?
- 38.** Sparr Investments specializes in tax-deferred investment opportunities for its clients. Recently Sparr offered a payroll deduction investment scheme for the employees of a particular company. Sparr estimates that the employees are currently averaging €100 or less per month in tax-deferred investments. A sample of 40 employees will be used to test Sparr's hypothesis about the current level of investment activity among the population of employees. Assume the employee monthly tax-deferred investment amounts have a standard deviation of €75 and that a 0.05 level of significance will be used in the hypothesis test.
- a. What would it mean to make a Type II error in this situation?
  - b. What is the probability of the Type II error if the actual mean employee monthly investment is €120?
  - c. What is the probability of the Type II error if the actual mean employee monthly investment is €130?
  - d. Assume a sample size of 80 employees is used and repeat parts (b) and (c).

## 9.8 DETERMINING THE SAMPLE SIZE FOR HYPOTHESIS TESTS ABOUT A POPULATION MEAN

Consider a hypothesis test about the value of a population mean. The level of significance specified by the user determines the probability of making a Type I error for the test. By controlling the sample size, the user can also control the probability of making a Type II error. Let us show how a sample size can be determined for the following lower-tail test about a population mean.

$$H_0: \mu \geq \mu_0$$

$$H_1: \mu < \mu_0$$

The upper panel of Figure 9.11 is the sampling distribution of  $\bar{x}$  when  $H_0$  is true with  $\mu = \mu_0$ . For a lower-tail test, the critical value of the test statistic is denoted  $-z_\alpha$ . In the upper panel of the figure the vertical line, labelled  $c$ , is the corresponding value of  $\bar{x}$ . Note that, if we reject  $H_0$  when  $\bar{x} = c$  the probability of a Type I error will be  $\alpha$ . With  $z_\alpha$  representing the  $z$  value corresponding to an area of  $\alpha$  in the upper tail of the standard normal distribution, we compute  $c$  using the following formula:

$$c = \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} \quad (9.7)$$

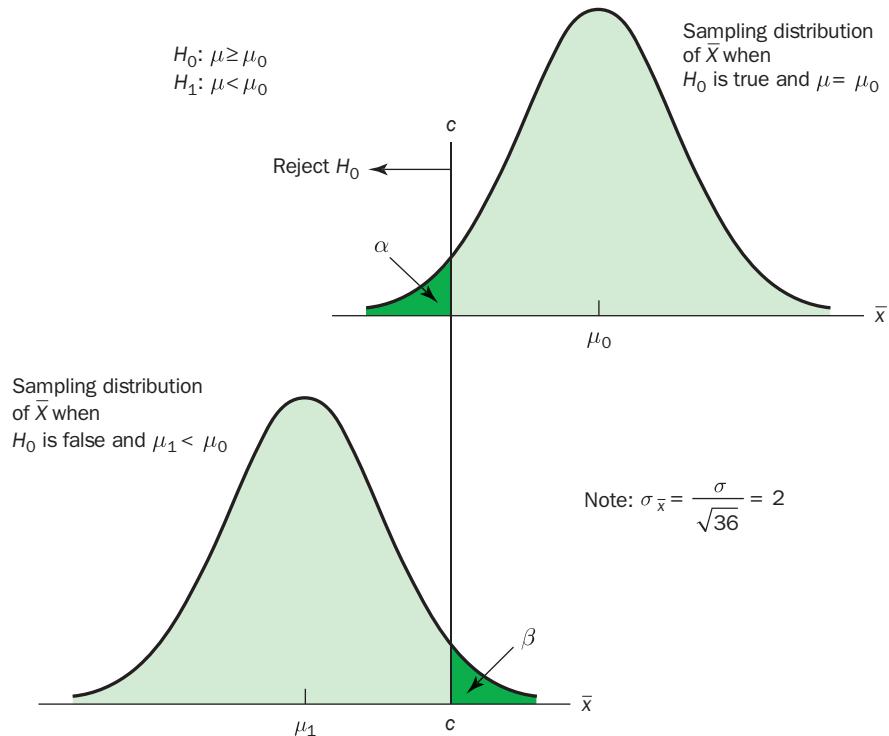
The lower panel of Figure 9.11 is the sampling distribution of  $\bar{X}$  when the alternative hypothesis is true with  $\mu = \mu_1 < \mu_0$ . The shaded region shows  $\beta$ , the probability of a Type II error if the null hypothesis is accepted when  $\bar{x} > c$ . With  $z_\beta$  representing the  $z$  value corresponding to an area of  $\beta$  in the upper tail of the standard normal distribution, we compute  $c$  using the following formula:

$$c = \mu_1 + z_\beta \frac{\sigma}{\sqrt{n}} \quad (9.8)$$

We wish to select a value for  $c$  so that when we reject or do not reject  $H_0$ , the probability of a Type I error is equal to the chosen value of  $\alpha$  and the probability of a Type II error is equal to the chosen value of  $\beta$ .

**FIGURE 9.11**

Determining the sample size for specified levels of the Type I ( $\alpha$ ) and Type II ( $\beta$ ) errors



Therefore, both equations (9.7) and (9.8) must provide the same value for  $c$ . Hence, the following equation must be true.

$$\mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} = \mu_1 + z_\beta \frac{\sigma}{\sqrt{n}}$$

To determine the required sample size, we first solve for  $\sqrt{n}$  as follows.

$$\mu_0 - \mu_1 = z_\alpha \frac{\sigma}{\sqrt{n}} + z_\beta \frac{\sigma}{\sqrt{n}} = \frac{(z_\alpha + z_\beta)\sigma}{\sqrt{n}}$$

and:

$$\sqrt{n} = \frac{(z_\alpha + z_\beta)\sigma}{(\mu_0 - \mu_1)}$$

Squaring both sides of the expression provides the following sample size formula for a one-tailed hypothesis test about a population mean.

**Sample size for a one-tailed hypothesis test about a population mean**

$$n = \frac{(z_\alpha + z_\beta)^2 \sigma^2}{(\mu_0 - \mu_1)^2} \quad (9.9)$$

$z_\alpha$  =  $z$  value giving an area of  $\alpha$  in the upper tail of a standard normal distribution.

$z_\beta$  =  $z$  value giving an area of  $\beta$  in the upper tail of a standard normal distribution.

$\sigma$  = the population standard deviation.

$\mu_0$  = the value of the population mean in the null hypothesis.

$\mu_1$  = the value of the population mean used for the Type II error.

Although the logic of equation (9.9) was developed for the hypothesis test shown in Figure 9.11, it holds for any one-tailed test about a population mean. Note that in a two-tailed hypothesis test about a population mean,  $z_{\alpha/2}$  is used instead of  $z_\alpha$  in equation (9.9).

Let us return to the lot-acceptance example from Sections 9.6 and 9.7. The design specification for the shipment of batteries indicated a mean useful life of at least 120 hours for the batteries. Shipments were rejected if  $H_0: \mu \geq 120$  was rejected. Let us assume that the quality control manager makes the following statements about the allowable probabilities for the Type I and Type II errors:

Type I error statement: If the mean life of the batteries in the shipment is  $\mu = 120$ , I am willing to risk an  $\alpha = 0.05$  probability of rejecting the shipment.

Type II error statement: If the mean life of the batteries in the shipment is five hours under the specification (i.e.  $\mu = 115$ ), I am willing to risk a  $\beta = 0.10$  probability of accepting the shipment.

Statements about the allowable probabilities of both errors must be made before the sample size can be determined. These statements are based on the judgement of the manager. Someone else might specify different restrictions on the probabilities.

In the example,  $\alpha = 0.05$  and  $\beta = 0.10$ . Using the standard normal probability distribution, we have  $z_{0.05} = 1.645$  and  $z_{0.10} = 1.28$ . From the statements about the error probabilities, we note that  $\mu_0 = 120$  and  $\mu_1 = 115$ . The population standard deviation was assumed known at  $\sigma = 12$ . By using equation (9.9), we find that the recommended sample size for the lot-acceptance example is:

$$n = \frac{(1.645 + 1.28)^2 (12)^2}{(120 - 115)^2} = 49.3$$

Rounding up, we recommend a sample size of 50.

Because both the Type I and Type II error probabilities have been controlled at allowable levels with  $n = 50$ , the quality control manager is now justified in using the *accept  $H_0$*  and *reject  $H_0$*  statements for the hypothesis test. The accompanying inferences are made with allowable probabilities of making Type I and Type II errors.

We can make three observations about the relationship among  $\alpha$ ,  $\beta$  and the sample size  $n$ .

- 1** Once two of the three values are known, the other can be computed.
- 2** For a given level of significance  $\alpha$ , increasing the sample size will reduce  $\beta$ .
- 3** For a given sample size, decreasing  $\alpha$  will increase  $\beta$ , whereas increasing  $\alpha$  will decrease  $\beta$ .

The third observation should be kept in mind when the probability of a Type II error is not being controlled. It suggests that one should not choose unnecessarily small values for the level of significance  $\alpha$ . For a given sample size, choosing a smaller level of significance means more exposure to a Type II error. Inexperienced users of hypothesis testing often think that smaller values of  $\alpha$  are always better. They are better if we are concerned only about making a Type I error. However, smaller values of  $\alpha$  have the disadvantage of increasing the probability of making a Type II error.

## EXERCISES

### Methods

- 39.** Consider the following hypothesis test.

$$H_0: \mu \geq 10$$

$$H_1: \mu < 10$$

The sample size is 120 and the population standard deviation is 5. Use  $\alpha = 0.05$ . If the actual population mean is 9, the probability of a Type II error is 0.2912. Suppose the researcher wants to reduce the probability of a Type II error to 0.10 when the actual population mean is 9. What sample size is recommended?

- 40.** Consider the following hypothesis test.

$$H_0: \mu = 20$$

$$H_1: \mu \neq 20$$

The population standard deviation is 10. Use  $\alpha = 0.05$ . How large a sample should be taken if the researcher is willing to accept a 0.05 probability of making a Type II error when the actual population mean is 22?

### Applications

- 41.** A special industrial battery must have a mean life of at least 400 hours. A hypothesis test is to be conducted with a 0.02 level of significance. If the batteries from a particular production run have an actual mean use life of 385 hours, the production manager wants a sampling procedure that only 10 per cent of the time would show erroneously that the batch is acceptable. What sample size is recommended for the hypothesis test? Use 30 hours as an estimate of the population standard deviation.

- 42.** *Young Adult* magazine states the following hypotheses about the mean age of its subscribers.

$$H_0: \mu = 28$$

$$H_1: \mu \neq 28$$

If the manager conducting the test will permit a 0.15 probability of making a Type II error when the true mean age is 29, what sample size should be selected? Assume  $\sigma = 6$  and a 0.05 level of significance.



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SOLUTIONS

43.  $H_0: \mu = 120$  and  $H_1: \mu \neq 120$  are used to test whether a bath soap production process is meeting the standard output of 120 bars per batch. Use a 0.05 level of significance for the test and a planning value of 5 for the standard deviation.

- a. If the mean output drops to 117 bars per batch, the firm wants to have a 98 per cent chance of concluding that the standard production output is not being met. How large a sample should be selected?
- b. With your sample size from part (a), what is the probability of concluding that the process is operating satisfactorily for each of the following actual mean outputs: 117, 118, 119, 121, 122 and 123 bars per batch? That is, what is the probability of a Type II error in each case?



## ONLINE RESOURCES

For the data files, online summary, additional questions and answers, and software section for Chapter 9, go to the online platform.

## SUMMARY

Hypothesis testing uses sample data to determine whether a statement about the value of a population parameter should or should not be rejected. The hypotheses are two competing statements about a population parameter. One is called the null hypothesis ( $H_0$ ), and the other is called the alternative hypothesis ( $H_1$ ). In Section 9.1 we provided guidelines for formulating hypotheses for situations frequently encountered in practice.

In all hypothesis tests, a relevant test statistic is calculated using sample data. The test statistic can be used to compute a  $p$ -value for the test. A  $p$ -value is a probability that measures the degree of support provided by the sample for the null hypothesis. If the  $p$ -value is less than or equal to the level of significance  $\alpha$ , the null hypothesis can be rejected.

Conclusions can also be drawn by comparing the value of the test statistic to a critical value. For lower-tail tests, the null hypothesis is rejected if the value of the test statistic is less than or equal to the critical value. For upper-tail tests, the null hypothesis is rejected if the value of the test statistic is greater than or equal to the critical value. Two-tailed tests consist of two critical values: one in the lower tail of the sampling distribution and one in the upper tail. In this case, the null hypothesis is rejected if the value of the test statistic is less than or equal to the critical value in the lower tail or greater than or equal to the critical value in the upper tail.

We illustrated the relationship between hypothesis testing and interval construction in Section 9.3.

When historical data or other information provides a basis for assuming that the population standard deviation is known, the hypothesis testing procedure is based on the standard normal distribution. When  $\sigma$  is unknown, the sample standard deviation  $s$  is used to estimate  $\sigma$  and the hypothesis testing procedure is based on the  $t$  distribution.

In the case of hypothesis tests about a population proportion, the hypothesis testing procedure uses a test statistic based on the standard normal distribution.

Extensions of hypothesis testing procedures to include an analysis of the Type II error were also presented. In Section 9.7 we showed how to compute the probability of making a Type II error. In Section 9.8 we showed how to determine a sample size that will control for both the probability of making a Type I error and a Type II error.

## KEY TERMS

Alternative hypothesis  
 Critical value  
 Level of significance  
 Null hypothesis  
 One-tailed test  
 $p$ -value

Power  
 Power curve  
 Test statistic  
 Two-tailed test  
 Type I error  
 Type II error

## KEY FORMULAE

**Test statistic for hypothesis tests about a population mean:  $\sigma$  known**

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \quad (9.1)$$

**Test statistic for hypothesis tests about a population mean:  $\sigma$  unknown**

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \quad (9.4)$$

**Test statistic for hypothesis tests about a population proportion**

$$z = \frac{p - \pi_0}{\sqrt{\frac{\pi_0(1 - \pi_0)}{n}}} \quad (9.6)$$

**Sample size for a one-tailed hypothesis test about a population mean**

$$n = \frac{(z_\alpha + z_\beta)^2 \sigma^2}{(\mu_0 - \mu_1)^2} \quad (9.9)$$

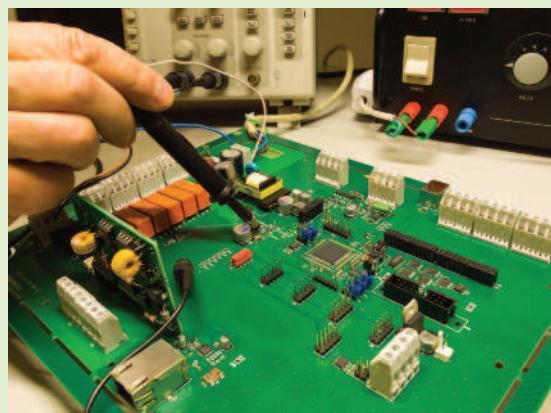
In a two-tailed test, replace  $z_\alpha$  with  $z_{\alpha/2}$ .

### CASE PROBLEM 1



#### Quality Associates

Quality Associates, a consulting firm, advises its clients about sampling and statistical procedures that can be used to control their manufacturing processes. In one particular application, a client gave Quality Associates a sample of 800 observations taken during a time in which that client's process was operating satisfactorily. The sample standard deviation for these data was 0.21; hence, with so



The components of an electronic product are tested

much data, the population standard deviation was assumed to be 0.21. Quality Associates then suggested that random samples of size 30 be taken periodically to monitor the process on an ongoing basis. By analyzing the new samples, the client could quickly learn whether the process was operating satisfactorily. When the process was not operating satisfactorily, corrective action could be taken to eliminate the problem. The design specification indicated the mean for the process should be 12. The hypothesis test suggested by Quality Associates follows.

$$H_0: \mu = 12$$

$$H_1: \mu \neq 12$$

Corrective action will be taken any time  $H_0$  is rejected.

The data set 'Quality' on the online platform contains data from four samples, each of size 30, collected at hourly intervals during the first day of operation of the new statistical control procedure.



QUALITY

### Managerial report

1. Do a hypothesis test for each sample at the 0.01 level of significance and determine what action, if any, should be taken. Provide the test statistic and  $p$ -value for each test.
2. Compute the standard deviation for each of the four samples. Does the assumption of 0.21 for the population standard deviation appear reasonable?
3. Compute limits for the sample mean  $X$  around  $\mu = 12$  such that, as long as a new sample mean is within those limits, the process will be considered to be operating satisfactorily. If  $X$  exceeds the upper limit or is below the lower limit, corrective action will be taken. These limits are referred to as upper and lower control limits for quality control purposes.
4. Discuss the implications of changing the level of significance to a larger value. What mistake or error could increase if the level of significance is increased?

## CASE PROBLEM 2



### Ethical behaviour of business students at the World Academy

During the global recession of 2008 and 2009, there were many accusations of unethical behaviour by bank directors, financial managers and other corporate officers. At that time, an article appeared that suggested part of the reason for such unethical business behaviour may stem from the fact that cheating has become more prevalent among business students (*Chronicle of Higher Education*, February 10, 2009). The article reported that 56 per cent of business students admitted to cheating at some time during their academic career as compared to 47 per cent of non-business students.

Cheating has been a concern of the dean of the Faculty of Business at the World Academy for several years. Some faculty members believe that cheating is more widespread at the World Academy than at other universities, while other faculty members think that cheating is not a

major problem in the Academy. To resolve some of these issues, the dean commissioned a study to assess the current ethical behaviour of business students at the World Academy. As part of this study, an anonymous exit survey was administered to a sample of 90 business students from this year's graduating class. Responses to the following questions were used to obtain data regarding three types of cheating.

During your time at the World Academy, did you ever present work copied off the Internet as your own?

Yes \_\_\_\_\_ No \_\_\_\_\_

During your time at the World Academy, did you ever copy answers off another student's exam?

Yes \_\_\_\_\_ No \_\_\_\_\_

During your time at the World Academy, did you ever collaborate with other students on projects that were supposed to be completed individually?

Yes \_\_\_\_\_ No \_\_\_\_\_

Any student who answered Yes to one or more of these questions was considered to have been involved in some type of cheating. A portion of

the data collected follows. The complete data set is in the file named ‘World Academy’ on the accompanying online platform.

| Student | Copied from Internet | Copied on exam | Collaborated on Individual project | Gender |
|---------|----------------------|----------------|------------------------------------|--------|
| 1       | No                   | No             | No                                 | Female |
| 2       | No                   | No             | No                                 | Male   |
| 3       | Yes                  | No             | Yes                                | Male   |
| 4       | Yes                  | Yes            | No                                 | Male   |
| 5       | No                   | No             | Yes                                | Male   |
| 6       | Yes                  | No             | No                                 | Female |
| .       | .                    | .              | .                                  | .      |
| .       | .                    | .              | .                                  | .      |
| 88      | No                   | No             | No                                 | Male   |
| 89      | No                   | Yes            | Yes                                | Male   |
| 90      | No                   | No             | No                                 | Female |

#### Managerial report

Prepare a report for the dean of the faculty that summarizes your assessment of the nature of cheating by business students at the World Acad-

emy. Be sure to include the following items in your report.

1. Use descriptive statistics to summarize the data and comment on your findings.
2. Develop 95 per cent confidence intervals for the proportion of all students, the proportion of male students and the proportion of female students who were involved in some type of cheating.
3. Conduct a hypothesis test to determine if the proportion of business students at the World Academy who were involved in some type of cheating is less than that of business students at other institutions as reported by the *Chronicle of Higher Education*.
4. Conduct a hypothesis test to determine if the proportion of business students at the World Academy who were involved in some form of cheating is less than that of non-business students at other institutions as reported by the *Chronicle of Higher Education*.
5. What advice would you give to the dean based up-on your analysis of the data?



WORLD  
ACADEMY





# 10

# Statistical

# Inference about

# Means and

# Proportions with

# Two Populations

## CHAPTER CONTENTS

Statistics in Practice How your name affects your buying behaviour

- 10.1 Inferences about the difference between two population means:  $\sigma_1$  and  $\sigma_2$  known
- 10.2 Inferences about the difference between two population means:  $\sigma_1$  and  $\sigma_2$  unknown
- 10.3 Inferences about the difference between two population means: matched samples
- 10.4 Inferences about the difference between two population proportions

**LEARNING OBJECTIVES** After studying this chapter and doing the exercises, you should be able to:

- 1 Construct and interpret confidence intervals and hypothesis tests for the difference between two population means, given independent samples from the two populations:
  - 1.1 When the standard deviations of the two populations are known.
  - 1.2 When the standard deviations of the two populations are unknown.
- 2 Construct and interpret confidence intervals and hypothesis tests for the difference between two population means, given matched samples from the two populations.
- 3 Construct and interpret confidence intervals and hypothesis tests for the difference between two population proportions, given independent samples from the two populations.

In Chapters 8 and 9 we showed how to construct interval estimates and do hypothesis tests for situations involving a single population mean or a single population proportion. In this chapter we extend our discussion by showing how interval estimates and hypothesis tests can be constructed when the difference between two population means or two population proportions is of prime importance. For example, we may want to construct an interval estimate of the difference between the mean starting salary for a population of men and the mean starting salary for a population of women. Or we may want to conduct a hypothesis test to determine whether there is any difference between the proportion of defective parts in a population of parts produced by supplier A and the proportion of defective parts in a population of parts produced by supplier B.



## STATISTICS IN PRACTICE

How your name affects your buying behaviour

In an article in the *Journal of Consumer Research* in 2011, two researchers reported results of studies on a phenomenon they called the ‘last name effect’. In the consumer behaviour field, *acquisition timing* refers to the speed with which consumers respond to opportunities to acquire goods or services – for instance, opportunities to get discounts or free offers, to acquire new technology or to replace consumer goods with new models. The researchers hypothesized that people with family names starting with a letter near the end of the alphabet would react



more quickly to such opportunities than people with names beginning with a letter near the beginning of the alphabet.

Their reasoning was that, during childhood, people with names near the beginning of the alphabet tend to develop a relatively laid-back approach to ‘queuing’ opportunities, because their name often gives them an advantage in situations where queuing is arranged on an alphabetical basis. On the other hand, people with names near the end of the alphabet tend to be more proactive, to counteract the disadvantage they experience in alphabetically queued situations.

One of the studies reported in the *Journal of Consumer Research* measured the acquisition timing, or reaction time, of a sample of MBA students to an email offer of free tickets to a basketball game. The mean reaction time of respondents with a family name beginning with one of the last nine letters of the alphabet was 19.38 minutes, compared to 25.08 minutes for respondents whose name began with one of the first nine letters of the alphabet. This difference was found to be statistically significant, using a statistical hypothesis test known as an independent-samples *t* test. This result offered support for the researchers’ hypothesis.

In this chapter, you will learn how to construct interval estimates and do hypothesis tests about mean and proportions with two populations. The independent-samples *t* test used in the consumer behaviour research is an example of such a test.

We begin by showing how to construct interval estimates and do hypothesis tests for the difference between two population means when the population standard deviations are assumed known.

## 10.1 INFERENCES ABOUT THE DIFFERENCE BETWEEN TWO POPULATION MEANS: $\sigma_1$ AND $\sigma_2$ KNOWN

Let  $\mu_1$  denote the mean of population 1 and  $\mu_2$  denote the mean of population 2. We focus on inferences about the difference between the means:  $\mu_1 - \mu_2$ . We select a simple random sample of  $n_1$  units from population 1 and a second simple random sample of  $n_2$  units from population 2. The two samples, taken separately and independently, are referred to as independent simple random samples. In this section, we assume the two population standard deviations,  $\sigma_1$  and  $\sigma_2$ , are known prior to collecting the samples. We refer to this situation as the  $\sigma_1$  and  $\sigma_2$  known case. In the following example we show how to compute a margin of error and construct an interval estimate of the difference between the two population means when  $\sigma_1$  and  $\sigma_2$  are known.

## Interval estimation of $\mu_1 - \mu_2$

Suppose a retailer such as Currys (selling TVs, DVD players, computers, photographic equipment and so on) operates two stores in Dublin, Ireland. One store is in the inner city and the other is in an out-of-town shopping centre. The regional manager noticed that products selling well in one store do not always sell well in the other. The manager believes this may be attributable to differences in customer demographics at the two locations. Customers may differ in age, education, income and so on. Suppose the manager asks us to investigate the difference between the mean ages of the customers who shop at the two stores.

Let us define population 1 as all customers who shop at the inner-city store and population 2 as all customers who shop at the out-of-town store.

$$\mu_1 = \text{mean age of population 1}$$

$$\mu_2 = \text{mean age of population 2}$$

The difference between the two population means is  $\mu_1 - \mu_2$ . To estimate  $\mu_1 - \mu_2$ , we shall select a simple random sample of  $n_1$  customers from population 1 and a simple random sample of  $n_2$  customers from population 2. We then compute the two sample means.

$$\bar{x}_1 = \text{sample mean age for the simple random sample of } n_1 \text{ inner-city customers}$$

$$\bar{x}_2 = \text{sample mean age for the simple random sample of } n_2 \text{ out-of-town customers}$$

The point estimator of the difference between the two populations is the difference between the sample means.

### Point estimator of the difference between two population means

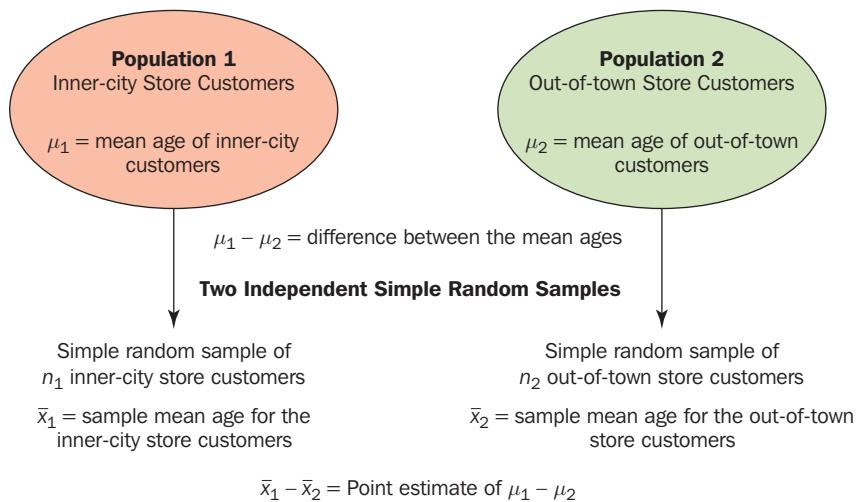
$$\bar{X}_1 - \bar{X}_2$$

(10.1)

Figure 10.1 provides an overview of the process used to estimate the difference between two population means based on two independent simple random samples.

**FIGURE 10.1**

Estimating the difference between two population means



As with other point estimators, the point estimator  $\bar{X}_1 - \bar{X}_2$  has a standard error that describes the variation in the sampling distribution of the estimator. With two independent simple random samples, the standard error of  $\bar{X}_1 - \bar{X}_2$  is as follows:

**Standard error of  $\bar{X}_1 - \bar{X}_2$** 

$$\sigma_{\bar{X}_1 - \bar{X}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \quad (10.2)$$

If both populations have a normal distribution, or if the sample sizes are large enough to use a normal approximation for the sampling distributions of  $\bar{X}_1$  and  $\bar{X}_2$ , the sampling distribution of  $\bar{X}_1 - \bar{X}_2$  will be normal, with a mean of  $\mu_1 - \mu_2$ .

As we showed in Chapter 8, an interval estimate is given by a point estimate  $\pm$  a margin of error. In the case of estimation of the difference between two population means, an interval estimate will take the form  $(\bar{x}_1 - \bar{x}_2) \pm$  margin of error. When the sampling distribution of  $\bar{X}_1 - \bar{X}_2$  is a normal distribution, we can write the margin of error as follows:

$$\text{Margin of error} = z_{\alpha/2} \sigma_{\bar{X}_1 - \bar{X}_2} = z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \quad (10.3)$$

Therefore the interval estimate of the difference between two population means is as follows:

**Interval estimate of the difference between two population means:  $\sigma_1$  and  $\sigma_2$  known**

$$(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \quad (10.4)$$

where  $1 - \alpha$  is the confidence coefficient.

We return to the example of the Dublin retailer. Based on data from previous customer demographic studies, the two population standard deviations are known,  $\sigma_1 = 9$  years and  $\sigma_2 = 10$  years. The data collected from the two independent simple random samples of the retailer's customers provided the following results:

|             | <i>Inner-city store</i> | <i>Out-of-town store</i> |
|-------------|-------------------------|--------------------------|
| Sample size | $n_1 = 36$              | $n_2 = 49$               |
| Sample mean | $\bar{x}_1 = 40$ years  | $\bar{x}_2 = 35$ years   |

Using expression (10.1), we find that the point estimate of the difference between the mean ages of the two populations is  $\bar{x}_1 - \bar{x}_2 = 40 - 35 = 5$  years. We estimate that the customers at the inner-city store have a mean age five years greater than the mean age of the out-of-town customers. We can now use expression (10.4) to compute the margin of error and provide the interval estimate of  $\mu_1 - \mu_2$ . Using 95 per cent confidence and  $z_{\alpha/2} = z_{0.025} = 1.96$ , we have:

$$\bar{x}_1 - \bar{x}_2 \pm z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = (40 - 35) \pm 1.96 \sqrt{\frac{(9)^2}{36} + \frac{(10)^2}{49}} = 5 \pm 4.1$$

The margin of error is 4.1 years and the 95 per cent confidence interval estimate of the difference between the two population means is  $5 - 4.1 = 0.9$  years to  $5 + 4.1 = 9.1$  years.

## Hypothesis tests about $\mu_1 - \mu_2$

Let us consider hypothesis tests about the difference between two population means. Using  $D_0$  to denote the hypothesized difference between  $\mu_1$  and  $\mu_2$ , the three forms for a hypothesis test are as follows:

$$\begin{array}{lll} H_0: \mu_1 - \mu_2 \geq D_0 & H_0: \mu_1 - \mu_2 \leq D_0 & H_0: \mu_1 - \mu_2 = D_0 \\ H_1: \mu_1 - \mu_2 < D_0 & H_1: \mu_1 - \mu_2 > D_0 & H_1: \mu_1 - \mu_2 \neq D_0 \end{array}$$

In most applications,  $D_0 = 0$ . Using the two-tailed test as an example, when  $D_0 = 0$  the null hypothesis is  $H_0: \mu_1 - \mu_2 = 0$ , i.e. the null hypothesis is that  $\mu_1$  and  $\mu_2$  are equal. Rejection of  $H_0$  leads to the conclusion that  $H_1: \mu_1 - \mu_2 \neq 0$  is true, i.e.  $\mu_1$  and  $\mu_2$  are not equal.

The steps for doing hypothesis tests presented in Chapter 9 are applicable here. We must choose a level of significance, compute the value of the test statistic and find the  $p$ -value to determine whether the null hypothesis should be rejected. With two independent simple random samples, we showed that the point estimator  $\bar{X}_1 - \bar{X}_2$  has a standard error  $\sigma_{\bar{X}_1 - \bar{X}_2}$  given by expression (10.2), and the distribution of  $\bar{X}_1 - \bar{X}_2$  can be described by a normal distribution. In this case, the test statistic for the difference between two population means when  $\sigma_1$  and  $\sigma_2$  are known is as follows.

### Test statistic for hypothesis tests about $\mu_1 - \mu_2$ : $\sigma_1$ and $\sigma_2$ known

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \quad (10.5)$$

Here is an example. As part of a study to evaluate differences in education quality between two training centres, a standardized examination is given to individuals trained at the centres. The difference between the mean examination scores is used to assess quality differences between the centres. The population means for the two centres are as follows:

- $\mu_1$  = the mean examination score for the population of individuals trained at centre A
- $\mu_2$  = the mean examination score for the population of individuals trained at centre B

We begin with the tentative assumption that no difference exists between the average training quality provided at the two centres. Hence, in terms of the mean examination scores, the null hypothesis is that  $\mu_1 - \mu_2 = 0$ . If sample evidence leads to the rejection of this hypothesis, we shall conclude that the mean examination scores differ for the two populations. This conclusion indicates a quality differential between the two centres and suggests that a follow-up study investigating the reason for the differential may be warranted. The null and alternative hypotheses for this two-tailed test are written as follows:

$$H_0: \mu_1 - \mu_2 = 0$$

$$H_1: \mu_1 - \mu_2 \neq 0$$



EXAMS SCORES

The standardized examination given previously in a variety of settings always resulted in an examination score standard deviation near ten points. We shall use this information to assume that the population standard deviations are known with  $\sigma_1 = 10$  and  $\sigma_2 = 10$ . An  $\alpha = 0.05$  level of significance is specified for the study.

Independent simple random samples of  $n_1 = 30$  individuals from training centre A and  $n_2 = 40$  individuals from training centre B are taken. The respective sample means are  $\bar{x}_1 = 82$  and  $\bar{x}_2 = 78$ . Do these data suggest a difference between the population means at the two training centres? To help answer this question, we compute the test statistic using equation (10.5):

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{(82 - 78) - 0}{\sqrt{\frac{(10)^2}{30} + \frac{(10)^2}{40}}} = 1.66$$

Next we compute the  $p$ -value for this two-tailed test. Because the test statistic  $z$  is in the upper tail, we first compute the area under the curve to the right of  $z = 1.66$ . Using the standard normal distribution table, the cumulative probability for  $z = 1.66$  is 0.9515, so the area in the upper tail of the distribution is  $1 - 0.9515 = 0.0485$ . Because this test is a two-tailed test, we must double the tail area:  $p\text{-value} = 2(0.0485) = 0.0970$ . Following the usual rule to reject  $H_0$  if  $p\text{-value} \leq \alpha$ , we see that the  $p$ -value of 0.0970 does not allow us to reject  $H_0$  at the 0.05 level of significance. The sample results do not provide sufficient evidence to conclude that the training centres differ in quality.

In this chapter we shall use the  $p$ -value approach to hypothesis testing as described in Chapter 9. However, if you prefer, the test statistic and the critical value rejection rule may be used. With  $\alpha = 0.05$  and  $z_{\alpha/2} = z_{0.025} = 1.96$ , the rejection rule using the critical value approach would be to reject  $H_0$  if  $z \leq -1.96$  or if  $z \geq 1.96$ . With  $z = 1.66$ , we reach the same ‘do not reject  $H_0$ ’ conclusion.

In the preceding example, we demonstrated a two-tailed hypothesis test about the difference between two population means. Lower-tail and upper-tail tests can also be considered. These tests use the same test statistic as given in equation (10.5). The procedure for computing the  $p$ -value and the rejection rules for these one-tailed tests are the same as those presented in Chapter 9.

## Practical advice

In most applications of the interval estimation and hypothesis testing procedures presented in this section, random samples with  $n_1 \geq 30$  and  $n_2 \geq 30$  are adequate. In cases where either or both sample sizes are less than 30, the distributions of the populations become important considerations. In general, with smaller sample sizes, it is more important for the analyst to be satisfied that the distributions of the two populations are at least approximately normal.

## EXERCISES

### Methods

- Consider the following results for two independent random samples taken from two populations.

| Sample 1           | Sample 2           |
|--------------------|--------------------|
| $n_1 = 50$         | $n_2 = 35$         |
| $\bar{x}_1 = 13.6$ | $\bar{x}_2 = 11.6$ |
| $\sigma_1 = 2.2$   | $\sigma_2 = 3.0$   |

- What is the point estimate of the difference between the two population means?
  - Construct a 90 per cent confidence interval for the difference between the two population means.
  - Construct a 95 per cent confidence interval for the difference between the two population means.
- Consider the following hypothesis test.

$$H_0: \mu_1 - \mu_2 \leq 0$$

$$H_1: \mu_1 - \mu_2 > 0$$



COMPLETE  
SOLUTIONS

The following results are for two independent samples taken from the two populations.

| Sample 1           | Sample 2           |
|--------------------|--------------------|
| $n_1 = 40$         | $n_2 = 50$         |
| $\bar{x}_1 = 25.2$ | $\bar{x}_2 = 22.8$ |
| $\sigma_1 = 5.2$   | $\sigma_2 = 6.0$   |

- a. What is the value of the test statistic?
  - b. What is the  $p$ -value?
  - c. With  $\alpha = 0.05$ , what is your hypothesis testing conclusion?
3. Consider the following hypothesis test.

$$H_0: \mu_1 - \mu_2 = 0$$

$$H_1: \mu_1 - \mu_2 \neq 0$$

The following results are for two independent samples taken from the two populations.

| Sample 1          | Sample 2          |
|-------------------|-------------------|
| $n_1 = 80$        | $n_2 = 70$        |
| $\bar{x}_1 = 104$ | $\bar{x}_2 = 106$ |
| $\sigma_1 = 8.4$  | $\sigma_2 = 7.6$  |

- a. What is the value of the test statistic?
- b. What is the  $p$ -value?
- c. With  $\alpha = 0.05$ , what is your hypothesis testing conclusion?

### Applications

4. A study of wage differentials between men and women reported that one of the reasons wages for men are higher than wages for women is that men tend to have more years of work experience than women. Assume that the following sample summaries show the years of experience for each group.

| Men                      | Women                    |
|--------------------------|--------------------------|
| $n_1 = 100$              | $n_2 = 85$               |
| $\bar{x}_1 = 14.9$ years | $\bar{x}_2 = 10.3$ years |
| $\sigma_1 = 5.2$ years   | $\sigma_2 = 3.8$ years   |

- a. What is the point estimate of the difference between the two population means?
  - b. At 95 per cent confidence, what is the margin of error?
  - c. What is the 95 per cent confidence interval estimate of the difference between the two population means?
5. The Dublin retailer age study (used as an example above) provided the following data on the ages of customers from independent random samples taken at the two store locations.

| Inner-city store       | Out-of-town store      |
|------------------------|------------------------|
| $n_1 = 36$             | $n_2 = 49$             |
| $\bar{x}_1 = 40$ years | $\bar{x}_2 = 35$ years |
| $\sigma_1 = 9$ years   | $\sigma_2 = 10$ years  |

- a. State the hypotheses that could be used to detect a difference between the population mean ages at the two stores.
  - b. What is the value of the test statistic?
  - c. What is the  $p$ -value?
  - d. At  $\alpha = 0.05$ , what is your conclusion?
6. Consider the following results from a survey looking at how much people spend on gifts on Valentine's Day (14 February). The average expenditure of 40 males was €135.67, and the average expenditure of 30 females was €68.64. Based on past surveys, the standard deviation for males is assumed to be €35, and the standard deviation for females is assumed to be €20. Do male and female consumers differ in the average amounts they spend?
- a. What is the point estimate of the difference between the population mean expenditure for males and the population mean expenditure for females?
  - b. At 99 per cent confidence, what is the margin of error?
  - c. Construct a 99 per cent confidence interval for the difference between the two population means.

## 10.2 INFERENCES ABOUT THE DIFFERENCE BETWEEN TWO POPULATION MEANS: $\sigma_1$ AND $\sigma_2$ UNKNOWN

In this section we extend the discussion of inferences about  $\mu_1 - \mu_2$  to the case when the two population standard deviations,  $\sigma_1$  and  $\sigma_2$ , are unknown. In this case, we use the sample standard deviations,  $s_1$  and  $s_2$ , to estimate the unknown  $\sigma_1$  and  $\sigma_2$ . The interval estimation and hypothesis testing procedures are based on the  $t$  distribution rather than the standard normal distribution.

### Interval estimation of $\mu_1 - \mu_2$

The Union Bank is conducting a study designed to identify differences between cheque account practices by customers at two of its branches. A simple random sample of 28 cheque accounts is selected from the Northern Branch and an independent simple random sample of 22 cheque accounts is selected from the Eastern Branch. The current cheque account balance is recorded for each of the accounts. A summary of the account balances follows:

|                           | <i>Northern</i>     | <i>Eastern</i>     |
|---------------------------|---------------------|--------------------|
| Sample size               | $n_1 = 28$          | $n_2 = 22$         |
| Sample mean               | $\bar{x}_1 = €1025$ | $\bar{x}_2 = €910$ |
| Sample standard deviation | $s_1 = €150$        | $s_2 = €125$       |



CHEQACCT

The Union Bank would like to estimate the difference between the mean cheque account balances maintained by the population of Northern customers and the population of Eastern customers. In Section 10.1, we provided the following interval estimate for the case when the population standard deviations,  $\sigma_1$  and  $\sigma_2$ , are known:

$$(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

With  $\sigma_1$  and  $\sigma_2$  unknown, we shall use the sample standard deviations  $s_1$  and  $s_2$  to estimate  $\sigma_1$  and  $\sigma_2$  and replace  $z_{\alpha/2}$  with  $t_{\alpha/2}$ . As a result, the interval estimate of the difference between two population means is given by the following expression:

**Interval estimate of the difference between two population means:  $\sigma_1$  and  $\sigma_2$  unknown**

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \quad (10.6)$$

where  $1 - \alpha$  is the confidence coefficient.

In this expression, the use of the  $t$  distribution is an approximation, but it provides excellent results and is relatively easy to use. The only difficulty in using expression (10.6) is determining the appropriate degrees of freedom for  $t_{\alpha/2}$ . Statistical software packages compute the appropriate degrees of freedom automatically. The formula used is as follows:

**Degrees of freedom for the  $t$  distribution using two independent random samples**

$$df = \frac{\left( \frac{s_1^2}{n_1} + \frac{s_2^2}{n_2} \right)^2}{\left( \frac{1}{n_1-1} \right) \left( \frac{s_1^2}{n_1} \right)^2 + \left( \frac{1}{n_2-1} \right) \left( \frac{s_2^2}{n_2} \right)^2} \quad (10.7)$$

We return to the Union Bank example. The sample data show  $n_1 = 28$ ,  $\bar{x}_1 = €1025$  and  $s_1 = €150$  for the Northern Branch, and  $n_2 = 22$ ,  $\bar{x}_2 = €910$  and  $s_2 = €125$  for the Eastern Branch. The calculation for degrees of freedom for  $t_{\alpha/2}$  is as follows:

$$df = \frac{\left( \frac{s_1^2}{n_1} + \frac{s_2^2}{n_2} \right)^2}{\left( \frac{1}{n_1-1} \right) \left( \frac{s_1^2}{n_1} \right)^2 + \left( \frac{1}{n_2-1} \right) \left( \frac{s_2^2}{n_2} \right)^2} = \frac{\left( \frac{(150)^2}{28} + \frac{(125)^2}{22} \right)^2}{\left( \frac{1}{28-1} \right) \left( \frac{(150)^2}{28} \right)^2 + \left( \frac{1}{22-1} \right) \left( \frac{(125)^2}{22} \right)^2} = 47.8$$

We round the non-integer degrees of freedom *down* to 47 to provide a larger  $t$ -value and a more conservative interval estimate. Using the  $t$  distribution table with 47 degrees of freedom, we find  $t_{0.025} = 2.012$ . Using expression (10.6), we construct the 95 per cent confidence interval estimate of the difference between the two population means as follows:

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = (1025 - 910) \pm 2.012 \sqrt{\frac{(150)^2}{28} + \frac{(125)^2}{22}} = 115 \pm 78$$

The point estimate of the difference between the population mean cheque account balances at the two branches is €115. The margin of error is €78, and the 95 per cent confidence interval estimate of the difference between the two population means is  $115 - 78 = €37$  to  $115 + 78 = €193$ .

The computation of the degrees of freedom (equation (10.7)) is cumbersome if you are doing the calculation by hand, but it is easily implemented with a computer software package. Note that the terms  $s_1^2/n_1$  and  $s_2^2/n_2$  appear in both expression (10.6) and in (10.7). These need to be computed only once in order to evaluate both (10.6) and (10.7).

## Hypothesis tests about $\mu_1 - \mu_2$

Let us now consider hypothesis tests for  $\mu_1 - \mu_2$  when the population standard deviations  $\sigma_1$  and  $\sigma_2$  are unknown. Letting  $D_0$  denote the hypothesized value for  $\mu_1 - \mu_2$ , Section 10.1 showed that the test statistic used for the case where  $\sigma_1$  and  $\sigma_2$  are known is as follows. The test statistic,  $z$ , follows the standard normal distribution:

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

When  $\sigma_1$  and  $\sigma_2$  are unknown, we use  $s_1$  as an estimator of  $\sigma_1$  and  $s_2$  as an estimator of  $\sigma_2$ . Substituting these sample standard deviations for  $\sigma_1$  and  $\sigma_2$  gives the following test statistic when  $\sigma_1$  and  $\sigma_2$  are unknown.

### Test statistic for hypothesis tests about $\mu_1 - \mu_2$ : $\sigma_1$ and $\sigma_2$ unknown

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \quad (10.8)$$

The degrees of freedom for  $t$  are given by equation (10.7).

Consider an example involving a new computer software package developed to help systems analysts reduce the time required to design, develop and implement an information system. To evaluate the benefits of the new software package, a random sample of 24 systems analysts is selected. Each analyst is given specifications for a hypothetical information system. Then 12 of the analysts are instructed to produce the information system by using current technology. The other 12 analysts are trained in the use of the new software package and then instructed to use it to produce the information system.

This study involves two populations: a population of systems analysts using the current technology and a population of systems analysts using the new software package. In terms of the time required to complete the information system design project, the population means are as follows:

- $\mu_1$  = the mean project completion time for systems analysts using the current technology
- $\mu_2$  = the mean project completion time for systems analysts using the new software package

The researcher in charge of the new software evaluation project hopes to show that the new software package will provide a shorter mean project completion time, i.e. the researcher is looking for evidence to conclude that  $\mu_2$  is less than  $\mu_1$ . In this case,  $\mu_1 - \mu_2$  will be greater than zero. The research hypothesis  $\mu_1 - \mu_2 > 0$  is stated as the alternative hypothesis. The hypothesis test becomes:

$$H_0: \mu_1 - \mu_2 \leq 0$$

$$H_1: \mu_1 - \mu_2 > 0$$

We shall use  $\alpha = 0.05$  as the level of significance. Suppose that the 24 analysts complete the study with the results shown in Table 10.1.



Using the test statistic in equation (10.8), we have:

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{(325 - 286) - 0}{\sqrt{\frac{(40)^2}{12} + \frac{(44)^2}{12}}} = 2.27$$

**TABLE 10.1** Completion time data and summary statistics for the software testing study

|                           | Current technology      | New software            |
|---------------------------|-------------------------|-------------------------|
|                           | 300                     | 274                     |
|                           | 280                     | 220                     |
|                           | 344                     | 308                     |
|                           | 385                     | 336                     |
|                           | 372                     | 198                     |
|                           | 360                     | 300                     |
|                           | 288                     | 315                     |
|                           | 321                     | 258                     |
|                           | 376                     | 318                     |
|                           | 290                     | 310                     |
|                           | 301                     | 332                     |
|                           | 283                     | 263                     |
| <b>Summary statistics</b> |                         |                         |
| Sample size               | $n_1 = 12$              | $n_2 = 12$              |
| Sample mean               | $\bar{x}_1 = 325$ hours | $\bar{x}_2 = 286$ hours |
| Sample standard deviation | $s_1 = 40$              | $s_2 = 44$              |

Computing the degrees of freedom using equation (10.7), we have:

$$df = \frac{\left( \frac{s_1^2}{n_1} + \frac{s_2^2}{n_2} \right)^2}{\left( \frac{1}{n_1 - 1} \right) \left( \frac{s_1^2}{n_1} \right)^2 + \left( \frac{1}{n_2 - 1} \right) \left( \frac{s_2^2}{n_2} \right)^2} = \frac{\left( \frac{(40)^2}{12} + \frac{(44)^2}{12} \right)^2}{\left( \frac{1}{12 - 1} \right) \left( \frac{(40)^2}{12} \right)^2 + \left( \frac{1}{12 - 1} \right) \left( \frac{(44)^2}{12} \right)^2} = 21.8$$

Rounding down, we shall use a  $t$  distribution with 21 degrees of freedom. This row of the  $t$  distribution table is as follows:

| Area in upper tail | 0.20  | 0.10  | 0.05  | 0.025 | 0.01  | 0.005 |
|--------------------|-------|-------|-------|-------|-------|-------|
| $t$ value (21 df)  | 0.859 | 1.323 | 1.721 | 2.080 | 2.518 | 2.831 |

$t = 2.27$

With an upper-tail test, the  $p$ -value is the area in the upper tail to the right of  $t = 2.27$ . From the above results, we see that the  $p$ -value is between 0.025 and 0.01. Hence, the  $p$ -value is less than  $\alpha = 0.05$  and  $H_0$  is rejected. The sample results enable the researcher to conclude that  $\mu_1 - \mu_2 > 0$  or  $\mu_1 > \mu_2$ . The research study supports the conclusion that the new software package provides a smaller population mean completion time.

### Practical advice

The interval estimation and hypothesis testing procedures presented in this section are robust and can be used with relatively small sample sizes. In most applications, equal or nearly equal sample sizes such that the total sample size  $n_1 + n_2$  is at least 20 can be expected to provide very good results even if the populations are not normal. Larger sample sizes are recommended if the distributions of the populations are highly skewed or contain outliers. Smaller sample sizes should only be used if the analyst is satisfied that the distributions of the populations are at least approximately normal.

Another approach sometimes used to make inferences about the difference between two population means when  $\sigma_1$  and  $\sigma_2$  are unknown is based on the assumption that the two population standard

deviations are equal. You will find this approach as an option in MINITAB, IBM SPSS and EXCEL. Under the assumption of equal population variances, the two sample standard deviations are combined to provide the following 'pooled' sample variance  $s^2$ :

$$s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

The  $t$  test statistic becomes:

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{s \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

and has  $n_1 + n_2 - 2$  degrees of freedom. At this point, the computation of the  $p$ -value and the interpretation of the sample results are identical to the procedures discussed earlier in this section. A difficulty with this procedure is that the assumption of equal population standard deviations is usually difficult to verify. Unequal population standard deviations are frequently encountered. Using the pooled procedure may not provide satisfactory results especially if the sample sizes  $n_1$  and  $n_2$  are quite different. The  $t$  procedure that we presented in this section does not require the assumption of equal population standard deviations and can be applied whether the population standard deviations are equal or not. It is a more general procedure and is recommended for most applications.

## EXERCISES

### Methods

7. Consider the following results for independent random samples taken from two populations.

| Sample 1           | Sample 2           |
|--------------------|--------------------|
| $n_1 = 20$         | $n_2 = 30$         |
| $\bar{x}_1 = 22.5$ | $\bar{x}_2 = 20.1$ |
| $s_1 = 2.5$        | $s_2 = 4.8$        |

- a. What is the point estimate of the difference between the two population means?
  - b. What are the degrees of freedom for the  $t$  distribution?
  - c. At 95 per cent confidence, what is the margin of error?
  - d. What is the 95 per cent confidence interval for the difference between the two population means?
8. Consider the following hypothesis test.

$$H_0: \mu_1 - \mu_2 = 0$$

$$H_1: \mu_1 - \mu_2 \neq 0$$

The following results are from independent samples taken from two populations.

| Sample 1           | Sample 2           |
|--------------------|--------------------|
| $n_1 = 35$         | $n_2 = 40$         |
| $\bar{x}_1 = 13.6$ | $\bar{x}_2 = 10.1$ |
| $s_1 = 5.2$        | $s_2 = 8.5$        |



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- a. What is the value of the test statistic?  
 b. What are the degrees of freedom for the  $t$  distribution?  
 c. What is the  $p$ -value?  
 d. At  $\alpha = 0.05$ , what is your conclusion?
9. Consider the following data for two independent random samples taken from two normal populations.

|          |    |   |    |   |   |   |
|----------|----|---|----|---|---|---|
| Sample 1 | 10 | 7 | 13 | 7 | 9 | 8 |
| Sample 2 | 8  | 7 | 8  | 4 | 6 | 9 |

- a. Compute the two sample means.  
 b. Compute the two sample standard deviations.  
 c. What is the point estimate of the difference between the two population means?  
 d. What is the 90 per cent confidence interval estimate of the difference between the two population means?

### Applications

10. The International Air Transport Association surveyed business travellers to determine ratings of various international airports. The maximum possible score was ten. Suppose 50 business travellers were asked to rate airport L and 50 other business travellers were asked to rate airport M. The rating scores follow.



AIRPORTS

Airport L  
 10 9 6 7 8 7 9 8 10 7 6 5 7 3 5 6 8 7 10 8 4 7 8 6 9  
 9 5 3 1 8 9 6 8 5 4 6 10 9 8 3 2 7 9 5 3 10 3 5 10 8

Airport M  
 6 4 6 8 7 7 6 3 3 8 10 4 8 7 8 7 5 9 5 8 4 3 8 5 5  
 4 4 4 8 4 5 6 2 5 9 9 8 4 8 9 9 5 9 7 8 3 10 8 9 6

Construct a 95 per cent confidence interval estimate of the difference between the mean ratings of the airports L and M.

11. Suppose independent random samples of 15 unionized women and 20 non-unionized women in a skilled manufacturing job provide the following hourly wage rates (€).



UNION

Union workers  
 22.40 18.90 16.70 14.05 16.20 20.00 16.10 16.30 19.10 16.50  
 18.50 19.80 17.00 14.30 17.20

Non-union workers  
 17.60 14.40 16.60 15.00 17.65 15.00 17.55 13.30 11.20 15.90  
 19.20 11.85 16.65 15.20 15.30 17.00 15.10 14.30 13.90 14.50

- a. What is the point estimate of the difference between mean hourly wages for the two populations?  
 b. Develop a 95 per cent confidence interval estimate of the difference between the two population means.  
 c. Does there appear to be any difference in the mean wage rate for these two groups? Explain.
12. The Scholastic Aptitude Test (SAT) is a commonly used entrance qualification for university. Consider the research hypothesis that students whose parents had attained a higher level of

education would on average score higher on the SAT. SAT verbal scores for independent samples of students follow. The first sample shows the SAT verbal test scores for students whose parents are college graduates with a bachelor's degree. The second sample shows the SAT verbal test scores for students whose parents are high school graduates but do not have a college degree.

| Students' parents |                   |     |     |
|-------------------|-------------------|-----|-----|
| College grads     | High school grads |     |     |
| 485               | 487               | 442 | 492 |
| 534               | 533               | 580 | 478 |
| 650               | 526               | 479 | 425 |
| 554               | 410               | 486 | 485 |
| 550               | 515               | 528 | 390 |
| 572               | 578               | 524 | 535 |
| 497               | 448               |     |     |
| 592               | 469               |     |     |

- a. Formulate the hypotheses that can be used to determine whether the sample data support the hypothesis that students show a higher population mean verbal score on the SAT if their parents attained a higher level of education.
  - b. What is the point estimate of the difference between the means for the two populations?
  - c. Compute the  $p$ -value for the hypothesis test.
  - d. At  $\alpha = 0.05$ , what is your conclusion?
13. Periodically, Merrill Lynch customers are asked to evaluate Merrill Lynch financial consultants and services. Higher ratings on the client satisfaction survey indicate better service, with 7 the maximum service rating. Independent samples of service ratings for two financial consultants in the Dubai office are summarized here. Consultant A has ten years of experience while consultant B has one year of experience. Use  $\alpha = 0.05$  and test to see whether the consultant with more experience has the higher population mean service rating.

| Consultant A       | Consultant B       |
|--------------------|--------------------|
| $n_1 = 16$         | $n_2 = 10$         |
| $\bar{x}_1 = 6.82$ | $\bar{x}_2 = 6.25$ |
| $s_1 = 0.64$       | $s_2 = 0.75$       |

- a. State the null and alternative hypotheses.
  - b. Compute the value of the test statistic.
  - c. What is the  $p$ -value?
  - d. What is your conclusion?
14. Safegate Foods is redesigning the checkouts in its supermarkets throughout the country and is considering two designs. Tests on customer checkout times conducted at two stores where the two new systems have been installed result in the following summary of the data.

| System A                  | System B                  |
|---------------------------|---------------------------|
| $n_1 = 120$               | $n_2 = 100$               |
| $\bar{x}_1 = 4.1$ minutes | $\bar{x}_2 = 3.4$ minutes |
| $s_1 = 2.2$ minutes       | $s_2 = 1.5$ minutes       |



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Test at the 0.05 level of significance to determine whether the population mean checkout times of the two systems differ. Which system is preferred?

- 15.** Samples of final examination scores for two statistics classes with different instructors provided the following results.

| Instructor A     | Instructor B     |
|------------------|------------------|
| $n_1 = 12$       | $n_2 = 15$       |
| $\bar{x}_1 = 72$ | $\bar{x}_2 = 76$ |
| $s_1 = 8$        | $s_2 = 10$       |

With  $\alpha = 0.05$ , test whether these data are sufficient to conclude that the population mean grades for the two classes differ.

- 16.** Educational testing companies provide tutoring, classroom learning and practice tests in an effort to help students perform better on tests such as the Scholastic Aptitude Test (SAT). The test preparation companies claim that their courses will improve SAT score performances by an average of 120 points. A researcher is uncertain of this claim and believes that 120 points may be an overstatement in an effort to encourage students to take the test preparation course. In an evaluation study of one test preparation service, the researcher collects SAT score data for 35 students who took the test preparation course and 48 students who did not take the course.

|                           | Course | No course |
|---------------------------|--------|-----------|
| Sample mean               | 1058   | 983       |
| Sample standard deviation | 90     | 105       |

- a. Formulate the hypotheses that can be used to test the researcher's belief that the improvement in SAT scores may be less than the stated average of 120 points.
- b. Use  $\alpha = 0.05$  and the data above. What is your conclusion?
- c. What is the point estimate of the improvement in the average SAT scores provided by the test preparation course? Provide a 95 per cent confidence interval estimate of the improvement.
- d. What advice would you have for the researcher after seeing the confidence interval?

## 10.3 INFERENCES ABOUT THE DIFFERENCE BETWEEN TWO POPULATION MEANS: MATCHED SAMPLES

Suppose employees at a manufacturing company can use two different methods to perform a production task. To maximize production output, the company wants to identify the method with the smaller population mean completion time. Let  $\mu_1$  denote the population mean completion time for production method 1 and  $\mu_2$  denote the population mean completion time for production method 2. With no preliminary indication of the preferred production method, we begin by tentatively assuming that the two production methods have the same population mean completion time. The null hypothesis is  $H_0: \mu_1 - \mu_2 = 0$ . If this hypothesis is rejected, we can conclude that the population mean completion

times differ. In this case, the method providing the smaller mean completion time would be recommended. The null and alternative hypotheses are written as follows:

$$H_0: \mu_1 - \mu_2 = 0$$

$$H_1: \mu_1 - \mu_2 \neq 0$$

In choosing the sampling procedure that will be used to collect production time data and test the hypotheses, we consider two alternative designs. One is based on **independent samples** and the other is based on **matched samples**.

- 1 Independent sample design:** A simple random sample of workers is selected and each worker in the sample uses method 1. A second independent simple random sample of workers is selected and each worker in this sample uses method 2. The test of the difference between population means is based on the procedures in Section 10.2.
- 2 Matched sample design:** One simple random sample of workers is selected. Each worker first uses one method and then uses the other method. The order of the two methods is assigned randomly to the workers, with some workers performing method 1 first and others performing method 2 first. Each worker provides a pair of data values, one value for method 1 and another value for method 2.

In the matched sample design the two production methods are tested under similar conditions (i.e. with the same workers). Hence this design often leads to a smaller sampling error than the independent sample design. The primary reason is that in a matched sample design, variation between workers is eliminated because the same workers are used for both production methods.

Let us demonstrate the analysis of a matched sample design by assuming it is the method used to test the difference between population means for the two production methods. A random sample of six workers is used. The data on completion times for the six workers are given in Table 10.2. Note that each worker provides a pair of data values, one for each production method. Also note that the last column contains the difference in completion times  $d_i$  for each worker in the sample.

The key to the analysis of the matched sample design is to realize that we consider only the column of differences. Therefore, we have six data values (0.6, -0.2, 0.5, 0.3, 0.0, 0.6) that will be used to analyze the difference between population means of the two production methods.

Let  $\mu_d$  = the mean of the *difference* values for the population of workers. With this notation, the null and alternative hypotheses are rewritten as follows:

$$H_0: \mu_d = 0$$

$$H_1: \mu_d \neq 0$$

If  $H_0$  is rejected, we can conclude that the population mean completion times differ. The  $d$  notation is a reminder that the matched sample provides *difference* data. The sample mean and sample standard deviation for the six difference values in Table 10.2 follow:



MATCHED

**TABLE 10.2** Task completion times for a matched sample design

| Worker | Completion time for Method 1 (minutes) | Completion time for Method 2 (minutes) | Difference in completion times ( $d_i$ ) |
|--------|--|--|--|
| 1      | 6.0                                    | 5.4                                    | 0.6                                      |
| 2      | 5.0                                    | 5.2                                    | -0.2                                     |
| 3      | 7.0                                    | 6.5                                    | 0.5                                      |
| 4      | 6.2                                    | 5.9                                    | 0.3                                      |
| 5      | 6.0                                    | 6.0                                    | 0.0                                      |
| 6      | 6.4                                    | 5.8                                    | 0.6                                      |

Other than the use of the  $d$  notation, the formulae for the sample mean and sample standard deviation are the same ones used previously in the text.

$$\bar{d} = \frac{\sum d_i}{n} = \frac{1.8}{8} = 0.30$$

$$s_d = \sqrt{\frac{\sum (d_i - \bar{d})^2}{n-1}} = \sqrt{\frac{0.56}{5}} = 0.335$$

With the small sample of  $n = 6$  workers, we need to make the assumption that the population of differences has a normal distribution. This assumption is necessary so that we may use the  $t$  distribution for hypothesis testing and interval estimation procedures. Sample size guidelines for using the  $t$  distribution were presented in Chapters 8 and 9. Based on this assumption, the following test statistic has a  $t$  distribution with  $n - 1$  degrees of freedom.

#### Test statistic for hypothesis test involving matched samples

$$t = \frac{\bar{d} - \mu_d}{s_d/\sqrt{n}} \quad (10.9)$$

Let us use equation (10.9) to test the hypotheses  $H_0: \mu_d = 0$  and  $H_1: \mu_d \neq 0$ , using  $\alpha = 0.05$ . Substituting the sample results  $\bar{d} = 0.30$ ,  $s_d = 0.335$  and  $n = 6$  into equation (10.9), we compute the value of the test statistic.

$$t = \frac{\bar{d} - \mu_d}{s_d/\sqrt{n}} = \frac{0.30 - 0}{0.335/\sqrt{6}} = 2.20$$

Now let us compute the  $p$ -value for this two-tailed test. Because  $t = 2.20 > 0$ , the test statistic is in the upper tail of the  $t$  distribution. With  $t = 2.20$ , the area in the upper tail to the right of the test statistic can be found by using the  $t$  distribution table with degrees of freedom =  $n - 1 = 6 - 1 = 5$ . Information from the five degrees of freedom row of the  $t$  distribution table is as follows:

| Area in upper tail | 0.20  | 0.10  | 0.05  | 0.025 | 0.01  | 0.005 |
|--------------------|-------|-------|-------|-------|-------|-------|
| $t$ value (5 df)   | 0.920 | 1.476 | 2.015 | 2.571 | 3.365 | 4.032 |

$t = 2.20$

We see that the area in the upper tail is between 0.05 and 0.025. Because this test is a two-tailed test, we double these values to conclude that the  $p$ -value is between 0.10 and 0.05. This  $p$ -value is greater than  $\alpha = 0.05$ , so the null hypothesis  $H_0: \mu_d = 0$  is not rejected. MINITAB, EXCEL and IBM SPSS show the  $p$ -value as 0.080.

In addition we can obtain an interval estimate of the difference between the two population means by using the single population methodology of Chapter 8. At 95 per cent confidence, the calculation follows:

$$\bar{d} \pm t_{0.025} \frac{s_d}{\sqrt{n}} = 0.30 \pm 2.527 \left( \frac{0.335}{\sqrt{6}} \right) = 0.30 \pm 0.35$$

The margin of error is 0.35 and the 95 per cent confidence interval for the difference between the population means of the two production methods is  $-0.05$  minutes to  $0.65$  minutes.

In the example presented in this section, workers performed the production task with first one method and then the other method. This example illustrates a matched sample design in which each sampled element (worker) provides a pair of data values. It is also possible to use different but 'similar' elements to

provide the pair of data values. For example, a worker at one location could be matched with a similar worker at another location (similarity based on age, education, gender, experience, etc.). The pairs of workers would provide the difference data that could be used in the matched sample analysis. A matched sample procedure for inferences about two population means generally provides better precision than the independent samples approach, therefore it is the recommended design. However, in some applications matching is not feasible, or perhaps the time and cost associated with matching are excessive. In such cases, the independent samples design should be used.

## EXERCISES

### Methods

- 17.** Consider the following hypothesis test.

$$H_0: \mu_d \leq 0$$

$$H_1: \mu_d > 0$$

The following data are from matched samples taken from two populations.

| Element | Population |    |
|---------|------------|----|
|         | 1          | 2  |
| 1       | 21         | 20 |
| 2       | 28         | 26 |
| 3       | 18         | 18 |
| 4       | 20         | 20 |
| 5       | 26         | 24 |

- a. Compute the difference value for each element.
- b. Compute  $\bar{d}$ .
- c. Compute the standard deviation  $s_d$ .
- d. Conduct a hypothesis test using  $\alpha = 0.05$ . What is your conclusion?

- 18.** The following data are from matched samples taken from two populations.

| Element | Population |    |
|---------|------------|----|
|         | 1          | 2  |
| 1       | 11         | 8  |
| 2       | 7          | 8  |
| 3       | 9          | 6  |
| 4       | 12         | 7  |
| 5       | 13         | 10 |
| 6       | 15         | 15 |
| 7       | 15         | 14 |

- a. Compute the difference value for each element.
- b. Compute  $\bar{d}$ .
- c. Compute the standard deviation  $s_d$ .
- d. What is the point estimate of the difference between the two population means?
- e. Provide a 95 per cent confidence interval for the difference between the two population means.



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TVRADIO

### Applications

- 19.** In recent years, a growing array of entertainment options has been competing for consumer time. Researchers used a sample of 15 individuals and collected data on the hours per week spent watching cable television and hours per week spent listening to the radio.

| Individual | Television | Radio | Individual | Television | Radio |
|------------|------------|-------|------------|------------|-------|
| 1          | 22         | 25    | 9          | 21         | 21    |
| 2          | 8          | 10    | 10         | 23         | 23    |
| 3          | 25         | 29    | 11         | 14         | 15    |
| 4          | 22         | 19    | 12         | 14         | 18    |
| 5          | 12         | 13    | 13         | 14         | 17    |
| 6          | 26         | 28    | 14         | 16         | 15    |
| 7          | 22         | 23    | 15         | 24         | 23    |
| 8          | 19         | 21    |            |            |       |

- a. What is the sample mean number of hours per week spent watching cable television? What is the sample mean number of hours per week spent listening to radio? Which medium has the greater usage?
- b. Use a 0.05 level of significance and test for a difference between the population mean usage for cable television and radio. What is the  $p$ -value?
- 20.** A market research firm used a sample of individuals to rate the purchase potential of a particular product before and after the individuals saw a new television commercial about the product. The purchase potential ratings were based on a 0 to 10 scale, with higher values indicating a higher purchase potential. The null hypothesis stated that the mean rating 'after' would be less than or equal to the mean rating 'before'. Rejection of this hypothesis would show that the commercial improved the mean purchase potential rating. Use  $\alpha = 0.05$  and the following data to test the hypothesis and comment on the value of the commercial.

| Purchase rating |       |        | Purchase rating |       |        |
|-----------------|-------|--------|-----------------|-------|--------|
| Individual      | After | Before | Individual      | After | Before |
| 1               | 6     | 5      | 5               | 3     | 5      |
| 2               | 6     | 4      | 6               | 9     | 8      |
| 3               | 7     | 7      | 7               | 7     | 5      |
| 4               | 4     | 3      | 8               | 6     | 6      |

- 21.** Figures on profit margins (%) for 2010 and 2011 are given below for a sample of large French companies. Use the data to comment on differences between profit margins in the two years.

| Company       | Profit margin (%) |       |
|---------------|-------------------|-------|
|               | 2010              | 2011  |
| BNP Paribus   | 29.74             | 23.43 |
| Carrefour     | 1.29              | -1.50 |
| Danone        | 14.64             | 12.59 |
| Lafarge       | 8.83              | 4.42  |
| L'Oréal       | 16.17             | 17.03 |
| Michelin      | 8.69              | 9.63  |
| Pernod-Ricard | 16.43             | 17.94 |
| Renault       | 8.61              | 6.17  |
| Thales        | -2.90             | 4.61  |
| Vinci         | 8.04              | 7.87  |

- a. Use  $\alpha = 0.05$  and test for any difference between the population mean profit margins in 2010 and 2011. What is the  $p$ -value? What is your conclusion?
- b. What is the point estimate of the difference between the two mean profit margins?
- c. At 95 per cent confidence, what is the margin of error for the estimate in part (b)?
22. A survey was made of Book-of-the-Month Club members to ascertain whether members spend more time watching television than they do reading. Assume a sample of 15 respondents provided the following data on weekly hours of television watching and weekly hours of reading. Using a 0.05 level of significance, can you conclude that Book-of-the-Month Club members spend more hours per week watching television than reading?



PROFITS



TVREAD

| Respondent | Television | Reading | Respondent | Television | Reading |
|------------|------------|---------|------------|------------|---------|
| 1          | 10         | 6       | 9          | 4          | 7       |
| 2          | 14         | 16      | 10         | 8          | 8       |
| 3          | 16         | 8       | 11         | 16         | 5       |
| 4          | 18         | 10      | 12         | 5          | 10      |
| 5          | 15         | 10      | 13         | 8          | 3       |
| 6          | 14         | 8       | 14         | 19         | 10      |
| 7          | 10         | 14      | 15         | 11         | 6       |
| 8          | 12         | 14      |            |            |         |

## 10.4 INFERENCES ABOUT THE DIFFERENCE BETWEEN TWO POPULATION PROPORTIONS

Let  $\pi_1$  denote the proportion for population 1 and  $\pi_2$  denote the proportion for population 2. We next consider inferences about the difference between the two population proportions:  $\pi_1 - \pi_2$ . We shall select two independent random samples consisting of  $n_1$  units from population 1 and  $n_2$  units from population 2.

### Interval estimation of $\pi_1 - \pi_2$

An accountancy firm specializing in the preparation of income tax returns is interested in comparing the quality of work at two of its regional offices. The firm will be able to estimate the proportion of erroneous returns by randomly selecting samples of tax returns prepared at each office and verifying their accuracy. The difference between these proportions is of particular interest:

$\pi_1$  = proportion of erroneous returns for population 1 (office 1)

$\pi_2$  = proportion of erroneous returns for population 2 (office 2)

$P_1$  = sample proportion for a simple random sample from population 1

$P_2$  = sample proportion for a simple random sample from population 2

The difference between the two population proportions is given by  $\pi_1 - \pi_2$ . The point estimator of  $\pi_1 - \pi_2$  is as follows:

#### Point estimator of the difference between two population proportions

$$P_1 - P_2$$

(10.10)

The point estimator of the difference between two population proportions is the difference between the sample proportions of two independent simple random samples.

As with other point estimators, the point estimator  $P_1 - P_2$  has a sampling distribution that reflects the possible values of  $P_1 - P_2$  if we repeatedly took two independent random samples. The mean of this sampling distribution is  $\pi_1 - \pi_2$  and the standard error of  $P_1 - P_2$  is as follows:

#### Standard error of $P_1 - P_2$

$$\sigma_{P_1 - P_2} = \sqrt{\frac{\pi_1(1 - \pi_1)}{n_1} + \frac{\pi_2(1 - \pi_2)}{n_2}} \quad (10.11)$$

If the sample sizes are large enough that  $n_1\pi_1$ ,  $n_1(1 - \pi_1)$ ,  $n_2\pi_2$  and  $n_2(1 - \pi_2)$  are all greater than or equal to five, the sampling distribution of  $P_1 - P_2$  can be approximated by a normal distribution.

As we showed previously, an interval estimate is given by a point estimate  $\pm$  a margin of error. In the estimation of the difference between two population proportions, an interval estimate will take the form  $P_1 - P_2 \pm$  margin of error. With the sampling distribution of  $P_1 - P_2$  approximated by a normal distribution, we would like to use  $z_{\alpha/2}\sigma_{P_1 - P_2}$  as the margin of error. However,  $\sigma_{P_1 - P_2}$  given by equation (10.11) cannot be used directly because the two population proportions,  $\pi_1$  and  $\pi_2$ , are unknown. Using the sample proportion  $p_1$  to estimate  $\pi_1$  and the sample proportion  $p_2$  to estimate  $\pi_2$ , the margin of error is as follows:

$$\text{Margin of error} = z_{\alpha/2} \sqrt{\frac{p_1(1 - p_1)}{n_1} + \frac{p_2(1 - p_2)}{n_2}} \quad (10.12)$$

The general form of an interval estimate of the difference between two population proportions is as follows:

#### Interval estimate of the difference between two population proportions

$$(p_1 - p_2) \pm z_{\alpha/2} \sqrt{\frac{p_1(1 - p_1)}{n_1} + \frac{p_2(1 - p_2)}{n_2}} \quad (10.13)$$

where  $1 - \alpha$  is the confidence coefficient.

Returning to the tax returns example, we find that independent simple random samples from the two offices provide the following information:

| <i>Office 1</i>                    | <i>Office 2</i>                    |
|------------------------------------|------------------------------------|
| $n_1 = 250$                        | $n_1 = 300$                        |
| Number of returns with errors = 35 | Number of returns with errors = 27 |

The sample proportions for the two offices are:

$$p_1 = \frac{35}{250} = 0.14 \quad p_2 = \frac{27}{300} = 0.09$$

The point estimate of the difference between the proportions of erroneous tax returns for the two populations is  $p_1 - p_2 = 0.14 - 0.09 = 0.05$ . We estimate that Office 1 has a 0.05, or 5 percentage points, greater error rate than Office 2.



Expression (10.13) can now be used to provide a margin of error and interval estimate of the difference between the two population proportions. Using a 90 per cent confidence interval with  $z_{\alpha/2} = z_{0.05} = 1.645$ , we have:

$$\begin{aligned}(p_1 - p_2) &\pm z_{\alpha/2} \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}} \\ &= (0.14 - 0.09) \pm 1.645 \sqrt{\frac{0.14(1-0.14)}{250} + \frac{0.09(1-0.09)}{300}} = 0.05 \pm 0.045\end{aligned}$$

The margin of error is 0.045, and the 90 per cent confidence interval is 0.005 to 0.095.

## Hypothesis tests about $\pi_1 - \pi_2$

Let us now consider hypothesis tests about the difference between the proportions of two populations. The three forms for a hypothesis test are as follows:

$$\begin{array}{lll}H_0: \pi_1 - \pi_2 \geq 0 & H_0: \pi_1 - \pi_2 \leq 0 & H_0: \pi_1 - \pi_2 = 0 \\ H_1: \pi_1 - \pi_2 < 0 & H_1: \pi_1 - \pi_2 > 0 & H_1: \pi_1 - \pi_2 \neq 0\end{array}$$

When we assume  $H_0$  is true as an equality, we have  $\pi_1 - \pi_2 = 0$ , which is the same as saying that the population proportions are equal,  $\pi_1 = \pi_2$ . The test statistic is based on the sampling distribution of the point estimator  $P_1 - P_2$ .

In expression (10.11), we showed that the standard error of  $P_1 - P_2$  is given by:

$$\sigma_{P_1 - P_2} = \sqrt{\frac{\pi_1(1-\pi_1)}{n_1} + \frac{\pi_2(1-\pi_2)}{n_2}}$$

Under the assumption that  $H_0$  is true as an equality, the population proportions are equal and  $\pi_1 = \pi_2 = \pi$ . In this case,  $\sigma_{P_1 - P_2}$  becomes:

### Standard error of $P_1 - P_2$ when $\pi_1 = \pi_2 = \pi$

$$\sigma_{P_1 - P_2} = \sqrt{\frac{\pi(1-\pi)}{n_1} + \frac{\pi(1-\pi)}{n_2}} = \sqrt{\pi(1-\pi)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} \quad (10.14)$$

With  $\pi$  unknown, we pool, or combine, the point estimates from the two samples ( $p_1$  and  $p_2$ ) to obtain a single point estimate of  $\pi$  as follows:

### Pooled estimate of $\pi$ when $\pi_1 = \pi_2 = \pi$

$$p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} \quad (10.15)$$

This **pooled estimate of  $\pi$**  is a weighted average of  $p_1$  and  $p_2$ .

Substituting  $p$  for  $\pi$  in equation (10.14), we obtain an estimate of  $\sigma_{P_1 - P_2}$ , which is used in the test statistic. The general form of the test statistic for hypothesis tests about the difference between two population proportions is the point estimator divided by the estimate of  $\sigma_{P_1 - P_2}$ .

**Test statistic for hypothesis tests about  $\pi_1 - \pi_2$** 

$$z = \frac{(p_1 - p_2)}{\sqrt{p(1-p)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \quad (10.16)$$

This test statistic applies to large sample situations where  $n_1\pi_1$ ,  $n_1(1 - \pi_1)$ ,  $n_2\pi_2$  and  $n_2(1 - \pi_2)$  are all greater than or equal to five.

Let us return to the tax returns example and assume that the firm wants to use a hypothesis test to determine whether the error proportions differ between the two offices. A two-tailed test is required. The null and alternative hypotheses are as follows:

$$H_0: \pi_1 - \pi_2 = 0$$

$$H_1: \pi_1 - \pi_2 \neq 0$$

If  $H_0$  is rejected, the firm can conclude that the error rates at the two offices differ. We shall use  $\alpha = 0.10$  as the level of significance.

The sample data previously collected showed  $p_1 = 0.14$  for the  $n_1 = 250$  returns sampled at Office 1 and  $p_2 = 0.09$  for the  $n_2 = 300$  returns sampled at Office 2. The pooled estimate of  $\pi$  is:

$$p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{250(0.14) + 300(0.09)}{20 + 300} = 0.1127$$

Using this pooled estimate and the difference between the sample proportions, the value of the test statistic is as follows:

$$z = \frac{(p_1 - p_2)}{\sqrt{p(1-p)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{(0.14 - 0.09)}{\sqrt{0.1127(1-0.1127)\left(\frac{1}{250} + \frac{1}{300}\right)}} = 1.85$$

To compute the  $p$ -value for this two-tailed test, we first note that  $z = 1.85$  is in the upper tail of the standard normal distribution. Using the standard normal distribution table, we find the area in the upper tail for  $z = 1.85$  is  $1 - 0.9678 = 0.0322$ . Doubling this area for a two-tailed test, we find the  $p$ -value =  $2(0.0322) = 0.0644$ . With the  $p$ -value less than  $\alpha = 0.10$ ,  $H_0$  is rejected at the 0.10 level of significance. The firm can conclude that the error rates differ between the two offices. This hypothesis test conclusion is consistent with the earlier interval estimation results that showed the interval estimate of the difference between the population error rates at the two offices to be 0.005 to 0.095, with Office 1 having the higher error rate.

## EXERCISES

### Methods

- 23.** Consider the following results for independent samples taken from two populations.

| <i>Sample 1</i> | <i>Sample 2</i> |
|-----------------|-----------------|
| $n_1 = 400$     | $n_2 = 300$     |
| $p_1 = 0.48$    | $p_2 = 0.36$    |

- What is the point estimate of the difference between the two population proportions?
- Construct a 90 per cent confidence interval for the difference between the two population proportions.
- Construct a 95 per cent confidence interval for the difference between the two population proportions.

**24.** Consider the hypothesis test

$$H_0: \pi_1 - \pi_2 \leq 0$$

$$H_1: \pi_1 - \pi_2 > 0$$

The following results are for independent samples taken from the two populations.



COMPLETE  
SOLUTIONS

| Sample 1     | Sample 2     |
|--------------|--------------|
| $n_1 = 200$  | $n_2 = 300$  |
| $p_1 = 0.22$ | $p_2 = 0.10$ |

- What is the *p*-value?
- With  $\alpha = 0.05$ , what is your hypothesis testing conclusion?

### Applications

- In November and December 2008, research companies affiliated to the Worldwide Independent Network of Market Research carried out polls in 17 countries to assess people's views on the economic outlook. In the Canadian survey, conducted by Léger Marketing, 61 per cent of the sample of 1511 people thought the economic situation would worsen over the next three months. In the UK survey, conducted by ICM Research, 78 per cent of the sample of 1050 felt that economic conditions would worsen over that period. Provide a 95 per cent confidence interval estimate for the difference between the population proportions in the two countries. What is your interpretation of the interval estimate?
- In the results of the NUS 2011/12 Student Experience Research, it was reported that 34.3 per cent of students studying Business ( $n = 2171$ ) said a main reason for choosing their course was that the course was well-regarded by potential employers. The corresponding figure amongst students studying Maths and Computer Science ( $n = 1180$ ) was 28.1 per cent. Construct a 95 per cent confidence interval for the difference between the proportion of Business students who gave this as main reason and the proportion of Maths and Computer Science students who did likewise.
- In a test of the quality of two television commercials, each commercial was shown in a separate test area six times over a one-week period. The following week a telephone survey was conducted to identify individuals who had seen the commercials. Those individuals were asked to state the primary message in the commercials. The following results were recorded.

|                             | Commercial A | Commercial B |
|-----------------------------|--------------|--------------|
| Number who saw commercial   | 150          | 200          |
| Number who recalled message | 63           | 60           |

- Use  $\alpha = 0.05$  and test the hypothesis that there is no difference in the recall proportions for the two commercials.
- Compute a 95 per cent confidence interval for the difference between the recall proportions for the two populations.



**COMPLETE  
SOLUTIONS**

- 28.** In the UNITE 2007 *Student Experience Report*, it was reported that 49 per cent of 1600 student respondents in UK universities considered the academic reputation of the university an important factor in their choice of university. In the 2012 *Student Experience Report*, 343 out of 488 respondents considered academic reputation to be important. Test the hypothesis  $\pi_1 - \pi_2 = 0$  with  $\alpha = 0.05$ . What is the *p*-value. What is your conclusion?
- 29.** A large car insurance company selected samples of single and married male policyholders and recorded the number who made an insurance claim over the preceding three-year period.

| <i>Single policyholders</i> | <i>Married policyholders</i> |
|-----------------------------|------------------------------|
| $n_1 = 400$                 | $n_2 = 900$                  |
| Number making claims = 76   | Number making claims = 90    |

- a. Use  $\alpha = 0.05$  and test to determine whether the claim rates differ between single and married male policyholders.
- b. Provide a 95 per cent confidence interval for the difference between the proportions for the two populations.



## ONLINE RESOURCES

For the data files, online summary, additional questions and answers, and software section for Chapter 10, go to the online platform.

## SUMMARY

In this chapter we discussed procedures for constructing interval estimates and doing hypothesis tests involving two populations. First, we showed how to make inferences about the difference between two population means when independent simple random samples are selected. We considered the case where the population standard deviations,  $\sigma_1$  and  $\sigma_2$ , could be assumed known. The standard normal distribution  $z$  was used to develop the interval estimate and served as the test statistic for hypothesis tests. We then considered the case where the population standard deviations were unknown and estimated by the sample standard deviations  $s_1$  and  $s_2$ . In this case, the  $t$  distribution was used to develop the interval estimate and served as the test statistic for hypothesis tests.

Inferences about the difference between two population means were then discussed for the matched sample design. In the matched sample design each element provides a pair of data values, one from each population. The difference between the paired data values is then used in the statistical analysis. The matched sample design is generally preferred to the independent sample design, when it is feasible, because the matched-samples procedure often improves the precision of the estimate.

Finally, interval estimation and hypothesis testing about the difference between two population proportions were discussed. Statistical procedures for analyzing the difference between two population proportions are similar to the procedures for analyzing the difference between two population means.

## KEY TERMS

Independent samples  
Matched samples

Pooled estimator of  $\pi$

## KEY FORMULAE

**Point estimator of the difference between two population means**

$$\bar{X}_1 - \bar{X}_2 \quad (10.1)$$

**Standard error of  $\bar{X}_1 - \bar{X}_2$**

$$\sigma_{\bar{X}_1 - \bar{X}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \quad (10.2)$$

**Interval estimate of the difference between two population means:  $\sigma_1$  and  $\sigma_2$  known**

$$(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \quad (10.4)$$

**Test statistic for hypothesis tests about  $\mu_1 - \mu_2$ :  $\sigma_1$  and  $\sigma_2$  known**

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \quad (10.5)$$

**Interval estimate of the difference between two population means:  $\sigma_1$  and  $\sigma_2$  unknown**

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \quad (10.6)$$

**Degrees of freedom for the  $t$  distribution using two independent random samples**

$$df = \frac{\left( \frac{s_1^2}{n_1} + \frac{s_2^2}{n_2} \right)^2}{\left( \frac{1}{n_1 - 1} \right) \left( \frac{s_1^2}{n_1} \right)^2 + \left( \frac{1}{n_2 - 1} \right) \left( \frac{s_2^2}{n_2} \right)^2} \quad (10.7)$$

**Test statistic for hypothesis tests about  $\mu_1 - \mu_2$ :  $\sigma_1$  and  $\sigma_2$  unknown**

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \quad (10.8)$$

### Test statistic for hypothesis test involving matched samples

$$t = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}} \quad (10.9)$$

### Point estimator of the difference between two population proportions

$$P_1 - P_2 \quad (10.10)$$

### Standard error of $P_1 - P_2$

$$\sigma_{P_1 - P_2} = \sqrt{\frac{\pi_1(1 - \pi_1)}{n_1} + \frac{\pi_2(1 - \pi_2)}{n_2}} \quad (10.11)$$

### Interval estimate of the difference between two population proportions

$$(p_1 - p_2) \pm z_{\alpha/2} \sqrt{\frac{p_1(1 - p_1)}{n_1} + \frac{p_2(1 - p_2)}{n_2}} \quad (10.13)$$

### Standard error of $P_1 - P_2$ when $\pi_1 = \pi_2 = \pi$

$$\sigma_{P_1 - P_2} = \sqrt{\frac{\pi(1 - \pi)}{n_1} + \frac{\pi(1 - \pi)}{n_2}} = \sqrt{\pi(1 - \pi) \left( \frac{1}{n_1} + \frac{1}{n_2} \right)} \quad (10.14)$$

### Pooled estimate of $\pi$ when $\pi_1 = \pi_2 = \pi$

$$p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} \quad (10.15)$$

### Test statistic for hypothesis tests about $\pi_1 - \pi_2$

$$z = \frac{(p_1 - p_2)}{\sqrt{p(1 - p) \left( \frac{1}{n_1} + \frac{1}{n_2} \right)}} \quad (10.16)$$

### CASE PROBLEM



#### Par Products

Par Products is a major manufacturer of golf equipment. Management believes that Par's market share could be increased with the introduction of a cut-resistant,

longer-lasting golf ball. Therefore, the research group at Par has been investigating a new golf ball coating designed to resist cuts and provide a more durable ball. The tests with the coating have been promising.

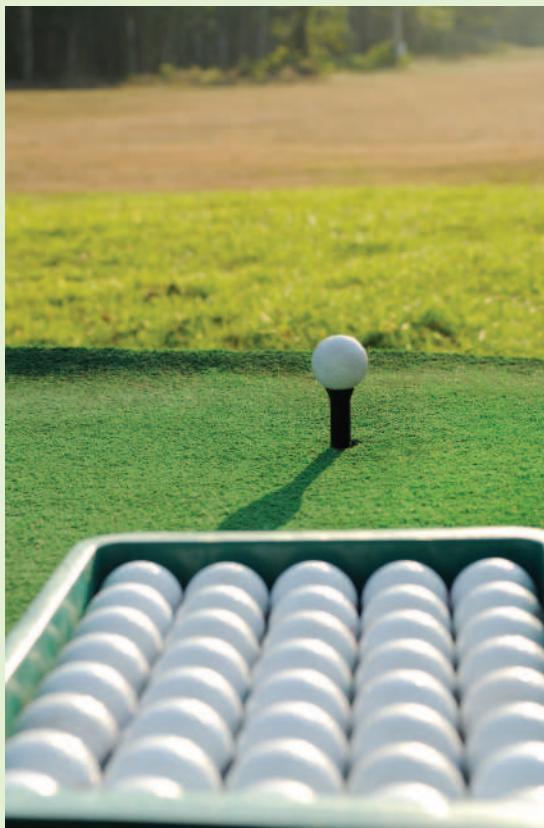
One of the researchers voiced concern about the effect of the new coating on driving distances. Par would like the new cut-resistant ball to offer driving distances comparable to those of the current-model

golf ball. To compare the driving distances for the two balls, 40 balls of both the new and current models were subjected to distance tests. The testing was performed with a mechanical hitting machine so that any difference between the mean

distances for the two models could be attributed to a difference in the two models. The results of the tests, with distances measured to the nearest metre, are available on the online platform, in the file 'Golf'.



| <i>Model</i>   | <i>Model</i> | <i>Model</i>   | <i>Model</i> |                |            |                |            |
|----------------|--------------|----------------|--------------|----------------|------------|----------------|------------|
| <i>Current</i> | <i>New</i>   | <i>Current</i> | <i>New</i>   | <i>Current</i> | <i>New</i> | <i>Current</i> | <i>New</i> |
| 264            | 277          | 270            | 272          | 263            | 274        | 281            | 283        |
| 261            | 269          | 287            | 259          | 264            | 266        | 274            | 250        |
| 267            | 263          | 289            | 264          | 284            | 262        | 273            | 253        |
| 272            | 266          | 280            | 280          | 263            | 271        | 263            | 260        |
| 258            | 262          | 272            | 274          | 260            | 260        | 275            | 270        |
| 283            | 251          | 275            | 281          | 283            | 281        | 267            | 263        |
| 258            | 262          | 265            | 276          | 255            | 250        | 279            | 261        |
| 266            | 289          | 260            | 269          | 272            | 263        | 274            | 255        |
| 259            | 286          | 278            | 268          | 266            | 278        | 276            | 263        |
| 270            | 264          | 275            | 262          | 268            | 264        | 262            | 279        |



### Managerial report

1. Formulate and present the rationale for a hypothesis test that Par could use to compare the driving distances of the current and new golf balls.
2. Analyze the data to provide the hypothesis test conclusion. What is the  $p$ -value for your test? What is your recommendation for Par Products?
3. Provide descriptive statistical summaries of the data for each model.
4. What is the 95 per cent confidence interval for the population mean of each model, and what is the 95 per cent confidence interval for the difference between the means of the two populations?
5. Do you see a need for larger sample sizes and more testing with the golf balls? Discuss.



# 11

# Inferences about Population Variances

## CHAPTER CONTENTS

Statistics in Practice The behaviour of financial markets: do we like Mondays?

- 11.1 Inferences about a population variance
- 11.2 Inferences about two population variances

**LEARNING OBJECTIVES** After studying this chapter and doing the exercises, you should be able to:

- 1 Construct confidence intervals for a population standard deviation or population variance, using the chi-squared distribution.
- 2 Conduct and interpret the results of hypothesis tests for a population standard deviation or population variance, using the  $F$  distribution.
- 3 Construct confidence intervals for a population variance, using the chi-squared distribution.

In the preceding four chapters we examined methods of statistical inference involving population means and population proportions. In this chapter we extend the discussion to inferences about population variances.

In many manufacturing processes, controlling the process variance is extremely important for maintaining quality. Consider the production process of filling containers with a liquid detergent product, for example. The filling mechanism is adjusted so the mean filling weight is 500 grams per container. In addition, the variance of the filling weights is critical. Even with the filling mechanism properly adjusted for the mean of 500 grams, we cannot expect every container to contain exactly 500 grams. By selecting a sample of containers, we can compute a sample variance for the number of grams placed in a container. This value will serve as an estimate of the variance for the population of containers being filled by the production process. If the sample variance is modest, the production process will be continued. However, if the sample variance is excessive, overfilling and underfilling may be occurring, even though the mean is correct at 500 grams. In this case, the filling mechanism will be re-adjusted in an attempt to reduce the filling variance for the containers.



## STATISTICS IN PRACTICE

The behaviour of financial markets:  
do we like Mondays?

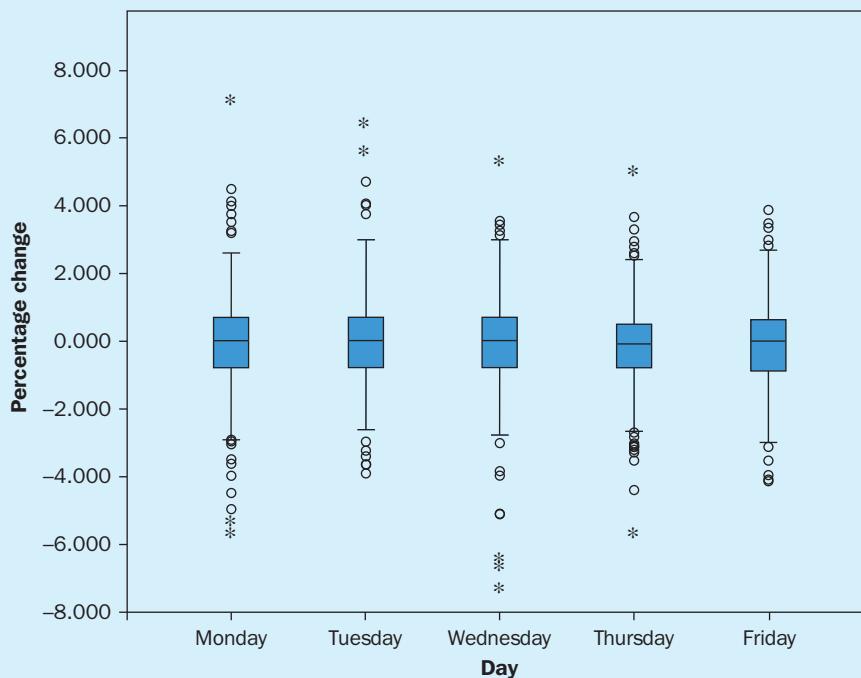
In recent years, in the wake of financial and banking crises in several countries, there has been severe questioning in the media about the behaviour of financial markets. Academic interest in this area is more long-standing. Over several decades there have been many published research studies examining whether various markets are ‘perfect’ (in an economic sense), and probing the existence of possible anomalies in the markets. Part of the motivation for the interest is the possibility that the existence of anomalies provides opportunities for investors.



One of the anomalies or effects that has been extensively examined is the so-called ‘day of the week’ effect – do markets behave differently on Mondays, for example, compared to Fridays? If a trader is trying to make profits by investing on the basis of daily movements in selected markets, are some days a better bet than others? A simple Google search will quickly reveal at least a dozen pieces of academic and professional research on this question over the last few years, published in a range of economics and finance journals, and covering markets in a variety of locations: Greece, Turkey, several central European countries, South Africa, Nigeria, Kuwait, India, Thailand, Muscat and Australia. Some of these studies find evidence for a day of the week effect, others do not.

Most of the studies looked for evidence of differences in *average* performance on different days of the week, as well as for differences in volatility (i.e. *variability*) on different days of the week. As an illustration of the possibilities, the IBM SPSS boxplot shown here charts the daily percentage changes (from opening level to closing level) in the CAC 40 share index (Paris Stock Exchange), for about five and a half years from 2007 to 2012. The plot gives an impression of higher variability on Mondays and Wednesdays, and shows Tuesdays as having marginally the highest average (median).

The previous chapter in this text looked at methods for examining differences between two mean values. The present chapter turns to estimation and testing of standard deviations and variances.



In the first section we consider inferences about the variance of a single population. Subsequently, we shall discuss procedures that can be used to make inferences comparing the variances of two populations.

## 11.1 INFERENCES ABOUT A POPULATION VARIANCE

Recall that sample variance is calculated as follows:

$$s^2 = \frac{\sum(x_i - \bar{x})^2}{n-1} \quad (11.1)$$

The sample variance ( $S^2$ ) is a point estimator of the population variance  $\sigma^2$ . To make inferences about  $\sigma^2$ , the sampling distribution of the quantity  $(n-1)S^2/\sigma^2$  can be used, under appropriate circumstances.

### Sampling distribution of $(n-1)S^2/\sigma^2$

When a simple random sample of size  $n$  is selected from a normal population, the sampling distribution of

$$\frac{(n-1)S^2}{\sigma^2} \quad (11.2)$$

has a chi-squared distribution with  $n-1$  degrees of freedom.

Figure 11.1 shows some possible forms of the sampling distribution of  $(n-1)S^2/\sigma^2$ . Because the sampling distribution is a chi-squared distribution, under the conditions described above, we can use this distribution to construct interval estimates and do hypothesis tests about a population variance. Tables of areas or probabilities are readily available for the chi-squared distribution.

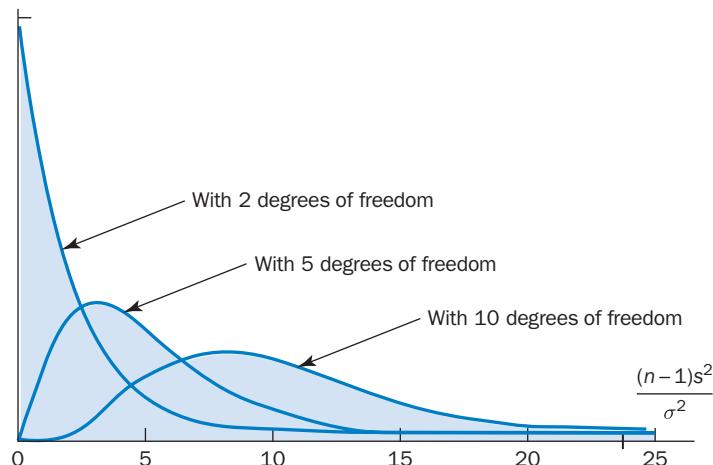
### Interval estimation

Suppose we are interested in estimating the population variance for the production filling process described above. A sample of 20 containers is taken and the sample variance for the filling quantities is found to be  $s^2 = 2.50$  (in appropriate units). However, we cannot expect the variance of a sample of 20 containers to provide the exact value of the variance for the population of containers filled by the production process. Our interest is in constructing an interval estimate for the population variance.

The Greek letter chi is  $\chi$ , so chi-squared is often denoted  $\chi^2$ . We shall use the notation  $\chi_\alpha^2$  to denote the value for the chi-squared distribution that gives an area or probability of  $\alpha$  to the right of the  $\chi_\alpha^2$  value. For example, in Figure 11.2 the chi-squared distribution with 19 degrees of freedom is shown, with  $\chi_{0.025}^2 = 32.852$  indicating that 2.5 per cent of the chi-squared values are to the right of 32.852, and  $\chi_{0.975}^2 = 8.907$  indicating that 97.5 per cent of the chi-squared values are to the right of 8.907.

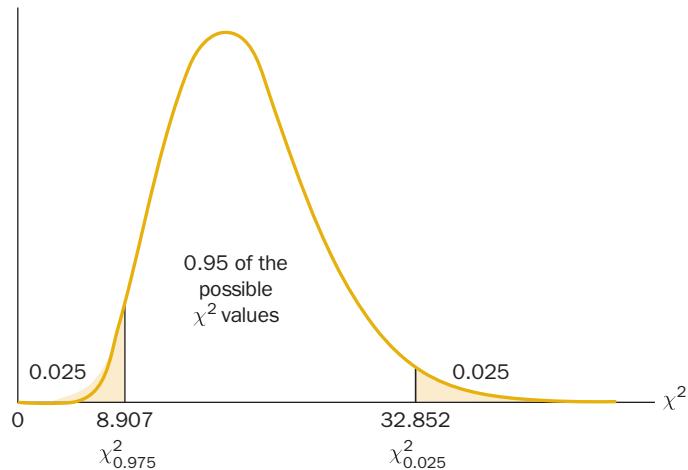
**FIGURE 11.1**

Examples of the sampling distribution of  $(n-1)S^2/\sigma^2$  (chi-squared distribution)



**FIGURE 11.2**

A chi-squared distribution with 19 degrees of freedom



Refer to Table 3 of Appendix B and verify that these chi-squared values with 19 degrees of freedom are correct (19th row of the table).

From Figure 11.2 we see that 0.95, or 95 per cent, of the chi-squared values are between  $\chi^2_{0.975}$  and  $\chi^2_{0.025}$ . That is, there is a 0.95 probability of obtaining a  $\chi^2$  value such that:

$$\chi^2_{0.975} \leq \chi^2 \leq \chi^2_{0.025}$$

We stated in expression (11.2) that the random variable  $(n - 1)s^2/\sigma^2$  follows a chi-squared distribution, therefore we can substitute  $(n - 1)s^2/\sigma^2$  for  $\chi^2$  and write:

$$\chi^2_{0.975} \leq \frac{(n - 1)s^2}{\sigma^2} \leq \chi^2_{0.025} \quad (11.3)$$

Expression (11.3) provides the basis for an interval estimate because 95 per cent of all possible values for  $(n - 1)s^2/\sigma^2$  will be in the interval  $\chi^2_{0.975}$  to  $\chi^2_{0.025}$ . We now need to do some algebraic manipulations with expression (11.3) to construct an interval estimate for the population variance  $\sigma^2$ . Using the leftmost inequality in expression (11.3), we have:

$$\chi^2_{0.975} \leq \frac{(n - 1)s^2}{\sigma^2}$$

So:

$$\chi^2_{0.975}\sigma^2 \leq (n - 1)s^2$$

or:

$$\sigma^2 \leq \frac{(n - 1)s^2}{\chi^2_{0.975}} \quad (11.4)$$

Doing similar algebraic manipulations with the rightmost inequality in expression (11.3) gives:

$$\frac{(n - 1)s^2}{\chi^2_{0.025}} \leq \sigma^2 \quad (11.5)$$

Expressions (11.4) and (11.5) can be combined to provide:

$$\frac{(n - 1)s^2}{\chi^2_{0.025}} \leq \sigma^2 \leq \frac{(n - 1)s^2}{\chi^2_{0.975}} \quad (11.6)$$

Because expression (11.3) is true for 95 per cent of the  $(n - 1)s^2/\sigma^2$  values, expression (11.6) provides a 95 per cent confidence interval estimate for the population variance  $\sigma^2$ .

We return to the problem of providing an interval estimate for the population variance of filling quantities. The sample of 20 containers provided a sample variance of  $s^2 = 2.50$ . With a sample size of 20,

we have 19 degrees of freedom. As shown in Figure 11.2, we have already determined that  $\chi^2_{0.975} = 8.907$  and  $\chi^2_{0.025} = 32.852$ . Using these values in expression (11.6) provides the following interval estimate for the population variance.

$$\frac{(19)(2.50)}{32.852} \leq \sigma^2 \leq \frac{(19)(2.50)}{8.907}$$

or

$$1.45 \leq \sigma^2 \leq 5.33$$

Taking the square root of these values provides the following 95 per cent confidence interval for the population standard deviation.

$$1.20 \leq \sigma \leq 2.31$$

Because  $\chi^2_{0.975} = 8.907$  and  $\chi^2_{0.025} = 32.852$  were used, the interval estimate has a 0.95 confidence coefficient. Extending expression (11.6) to the general case of any confidence coefficient, we have the following interval estimate of a population variance.

#### Interval estimate of a population variance

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2}} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}} \quad (11.7)$$

where the  $\chi^2$  values are based on a chi-squared distribution with  $n - 1$  degrees of freedom and where  $1 - \alpha$  is the confidence coefficient.

## Hypothesis testing

Using  $\sigma_0^2$  to denote the hypothesized value for the population variance, the three forms for a hypothesis test about a population variance are as follows:

$$\begin{array}{lll} H_0: \sigma^2 \geq \sigma_0^2 & H_0: \sigma^2 \leq \sigma_0^2 & H_0: \sigma^2 = \sigma_0^2 \\ H_1: \sigma^2 < \sigma_0^2 & H_1: \sigma^2 > \sigma_0^2 & H_1: \sigma^2 \neq \sigma_0^2 \end{array}$$

These three forms are similar to the three forms we used to do one-tailed and two-tailed hypothesis tests about population means and proportions in Chapters 9 and 10.

Hypothesis tests about a population variance use the hypothesized value for the population variance and the sample variance  $s^2$  to compute the value of a  $\chi^2$  test statistic. Assuming that the population has a normal distribution, the test statistic is:

#### Test statistic for hypothesis tests about a population variance

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} \quad (11.8)$$

where  $\chi^2$  has a chi-squared distribution with  $n - 1$  degrees of freedom.

After computing the value of the  $\chi^2$  test statistic, either the  $p$ -value approach or the critical value approach may be used to determine whether the null hypothesis can be rejected.

Here is an example. The EuroBus Company wants to promote an image of reliability by encouraging its drivers to maintain consistent schedules. The company would like arrival times at bus stops to have

low variability. The company standard specifies an arrival time variance of four or less when arrival times are measured in minutes.

The following hypothesis test is formulated to help the company determine whether the arrival time population variance is excessive.

$$\begin{aligned} H_0: \sigma^2 &\leq 4 \\ H_1: \sigma^2 &> 4 \end{aligned}$$

In tentatively assuming  $H_0$  is true, we are assuming the population variance of arrival times is within the company guideline. We reject  $H_0$  if the sample evidence indicates that the population variance exceeds the guideline. In this case, follow-up steps should be taken to reduce the population variance. We conduct the hypothesis test using a level of significance of  $\alpha = 0.05$ .

Suppose a random sample of 24 bus arrivals taken at a city-centre bus stop provides a sample variance of  $s^2 = 4.9$ . Assuming the population distribution of arrival times is approximately normal, the value of the test statistic is as follows:

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(24-1)4.9}{4} = 28.18$$

The chi-squared distribution with  $n - 1 = 24 - 1 = 23$  degrees of freedom is shown in Figure 11.3. Because this is an upper-tail test, the area under the curve to the right of the test statistic  $\chi^2 = 28.18$  is the  $p$ -value for the test.

Like the  $t$  distribution table, the chi-squared distribution table does not contain sufficient detail to enable us to determine the  $p$ -value exactly. However, we can use the chi-squared distribution table to obtain a range for the  $p$ -value. For example, using Table 3 of Appendix B, we find the following information for a chi-squared distribution with 23 degrees of freedom.

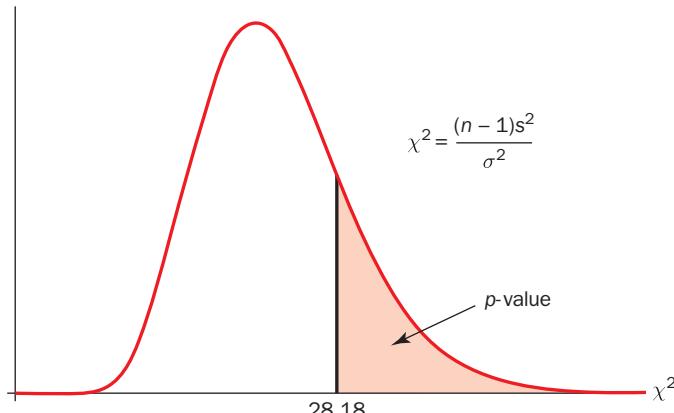
| Area in upper tail     | 0.10   | 0.05   | 0.025  | 0.01   |
|------------------------|--------|--------|--------|--------|
| $\chi^2$ value (23 df) | 32.007 | 35.172 | 38.076 | 41.638 |

$\chi^2 = 28.18$

Because  $\chi^2 = 28.18$  is less than 32.007, the area in the upper tail (the  $p$ -value) is greater than 0.10. With the  $p$ -value  $> \alpha = 0.05$ , we cannot reject the null hypothesis. The sample does not support the conclusion that the population variance of the arrival times is excessive.

Because of the difficulty of determining the exact  $p$ -value directly from the chi-squared distribution table, a computer software package such as IBM SPSS, MINITAB or EXCEL is helpful.

The guides on the online platform describe the procedures showing that with 23 degrees of freedom,  $\chi^2 = 28.18$  provides a  $p$ -value = 0.2091.



**FIGURE 11.3**

Chi-squared distribution for the EuroBus example

As with other hypothesis testing procedures, the critical value approach can also be used to draw the conclusion. With  $\alpha = 0.05$ ,  $\chi^2_{0.05}$  provides the critical value for the upper-tail hypothesis test. Using Table 3 of Appendix B and 23 degrees of freedom,  $\chi^2_{0.05} = 35.172$ . Consequently, the rejection rule for the bus arrival time example is as follows:

$$\text{Reject } H_0 \text{ if } \chi^2 \geq 35.172$$

Because the value of the test statistic is  $\chi^2 = 28.18$ , we cannot reject the null hypothesis.

In practice, upper-tail tests as presented here are the most frequently encountered tests about a population variance. In situations involving arrival times, production times, filling weights, part dimensions and so on, low variances are desirable, whereas large variances are unacceptable. With a statement about the maximum allowable population variance, we can test the null hypothesis that the population variance is less than or equal to the maximum allowable value against the alternative hypothesis that the population variance is greater than the maximum allowable value. With this test structure, corrective action will be taken whenever rejection of the null hypothesis indicates the presence of an excessive population variance.

As we saw with population means and proportions, other forms of hypothesis test can be done. We demonstrate a two-tailed test about a population variance by considering a situation faced by a car driver licensing authority. Historically, the variance in test scores for individuals applying for driving licences has been  $\sigma^2 = 100$ . A new examination with a new style of test questions has been developed. Administrators of the licensing authority would like the variance in the test scores for the new examination to remain at the historical level. To evaluate the variance in the new examination test scores, the following two-tailed hypothesis test has been proposed:

$$\begin{aligned} H_0: \sigma^2 &= 100 \\ H_1: \sigma^2 &\neq 100 \end{aligned}$$

Rejection of  $H_0$  will indicate that a change in the variance has occurred and suggest that some questions in the new examination may need revision to make the variance of the new test scores similar to the variance of the old test scores.

A sample of 30 applicants for driving licences is given the new version of the examination. The sample provides a sample variance  $s^2 = 162$ . We shall use a level of significance  $\alpha = 0.05$  to do the hypothesis test. The value of the chi-squared test statistic is as follows:

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(30-1)162}{100} = 46.98$$

Now, let us compute the  $p$ -value. Using Table 3 of Appendix B and  $n - 1 = 30 - 1 = 29$  degrees of freedom, we find the following:

| Area in upper tail     | 0.10   | 0.05   | 0.025  | 0.01   |
|------------------------|--------|--------|--------|--------|
| $\chi^2$ value (29 df) | 39.087 | 42.557 | 45.722 | 49.588 |

$\chi^2 = 46.98$

The value of the test statistic  $\chi^2 = 46.98$  gives an area between 0.025 and 0.01 in the upper tail of the chi-squared distribution. Doubling these values shows that the two-tailed  $p$ -value is between 0.05 and 0.02. IBM SPSS, EXCEL or MINITAB can be used to show the exact  $p$ -value = 0.0374. With  $p$ -value  $< \alpha = 0.05$ , we reject  $H_0$  and conclude that the new examination test scores have a population variance different from the historical variance of  $\sigma^2 = 100$ .

A summary of the hypothesis testing procedures for a population variance is shown in Table 11.1.

**TABLE 11.1** Summary of hypothesis tests about a population variance

|  | Lower-tail test   | Upper-tail test   | Two-tailed test  |
|--|---|---|--|
| <b>Hypotheses</b>                              | $H_0 : \sigma^2 \geq \sigma_0^2$<br>$H_1 : \sigma^2 > \sigma_0^2$ | $H_0 : \sigma^2 \leq \sigma_0^2$<br>$H_1 : \sigma^2 > \sigma_0^2$ | $H_0 : \sigma^2 = \sigma_0^2$<br>$H_1 : \sigma^2 \neq \sigma_0^2$                                |
| <b>Test statistic</b>                          | $\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$                            | $\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$                            | $\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$   |
| <b>Rejection rule: <i>p</i>-value approach</b> | Reject $H_0$ if <i>p</i> -value $\leq \alpha$                     | Reject $H_0$ if <i>p</i> -value $\leq \alpha$                     | Reject $H_0$ if <i>p</i> -value $\leq \alpha$  |
| <b>Rejection rule: critical value approach</b> | Reject $H_0$ if $\chi^2 \leq \chi_{1-\alpha}^2$                   | Reject $H_0$ if $\chi^2 \geq \chi_\alpha^2$                       | Reject $H_0$ if<br>$\chi^2 \leq \chi_{1-\alpha/2}^2$<br>or if<br>$\chi^2 \geq \chi_{\alpha/2}^2$ |

## EXERCISES

### Methods

- Find the following chi-squared distribution values from Table 3 of Appendix B.
  - $\chi_{0.05}^2$  with df = 5.
  - $\chi_{0.025}^2$  with df = 15.
  - $\chi_{0.975}^2$  with df = 20.
  - $\chi_{0.01}^2$  with df = 10.
  - $\chi_{0.95}^2$  with df = 18.
- A sample of 20 items provides a sample standard deviation of five.
  - Compute a 90 per cent confidence interval estimate of the population variance.
  - Compute a 95 per cent confidence interval estimate of the population variance.
  - Compute a 95 per cent confidence interval estimate of the population standard deviation.
- A sample of 16 items provides a sample standard deviation of 9.5. Test the following hypotheses using  $\alpha = 0.05$ . What is your conclusion? Use both the *p*-value approach and the critical value approach.
 
$$H_0: \sigma^2 \leq 50$$

$$H_1: \sigma^2 > 50$$

### Applications

- The variance in drug weights is critical in the pharmaceutical industry. For a specific drug, with weights measured in grams, a sample of 18 units provided a sample variance of  $s^2 = 0.36$ .
  - Construct a 90 per cent confidence interval estimate of the population variance for the weight of this drug.
  - Construct a 90 per cent confidence interval estimate of the population standard deviation.



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SOLUTIONS

5. The table below shows estimated P/E ratios for December 2012, for a sample of eight companies listed on the Tel Aviv stock exchange (Source: Bloomberg, July 2012).

| Company                        | P/E ratio |
|--------------------------------|-----------|
| Avner Oil Exploration          | 37.69     |
| Bank Hapoalim BM               | 6.59      |
| Cellcom Israel Ltd             | 5.30      |
| Delek Group Ltd                | 14.53     |
| Nice Systems Ltd               | 14.46     |
| Partner Communications Co. Ltd | 5.09      |
| Paz Oil Co. Ltd                | 16.13     |
| Teva Pharmaceutical            | 7.29      |

- a. Compute the sample variance and sample standard deviation for these data.  
 b. What is the 95 per cent confidence interval for the population variance?  
 c. What is the 95 per cent confidence interval for the population standard deviation?
6. Because of staffing decisions, managers of the Worldview Hotel are interested in the variability in the number of rooms occupied per day during a particular season of the year. A sample of 20 days of operation shows a sample mean of 290 rooms occupied per day and a sample standard deviation of 30 rooms.
- a. What is the point estimate of the population variance?  
 b. Provide a 90 per cent confidence interval estimate of the population variance.  
 c. Provide a 90 per cent confidence interval estimate of the population standard deviation.
7. The CAC 40 is a share index based on the price movements of shares quoted on the Paris stock exchange. The figures below are the quarterly percentage returns for a tracker fund linked to the CAC 40, over the period January 2007 to June 2012.

|      | 1st quarter | 2nd quarter | 3rd quarter | 4th quarter |
|------|-------------|-------------|-------------|-------------|
| 2007 | 6.27        | -3.51       | 1.68        | -16.73      |
| 2008 | 2.60        | -12.09      | -20.61      | -14.72      |
| 2009 | 6.25        | 8.43        | 5.29        | 3.65        |
| 2010 | 2.07        | -4.55       | 5.23        | 4.49        |
| 2011 | 2.53        | -10.57      | -11.71      | 1.72        |
| 2012 | -2.60       | -1.12       |             |             |

- a. Compute the mean, variance and standard deviation for the quarterly returns.  
 b. Financial analysts often use standard deviation of percentage returns as a measure of risk for stocks and mutual funds. Construct a 95 per cent confidence interval for the population standard deviation of quarterly returns for the CAC 40 tracker fund.
8. In the file 'Travel' on the online platform, there are estimated daily living costs (in euros) for a businessman travelling to 20 major cities. The estimates include a single room at a four-star hotel, beverages, breakfast, taxi fares and incidental costs.
- a. Compute the sample mean.  
 b. Compute the sample standard deviation.  
 c. Compute a 95 per cent confidence interval for the population standard deviation.



CAC40

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TRAVEL

| <i>City</i>  | <i>Daily living cost</i> | <i>City</i>    | <i>Daily living cost</i> |
|--------------|--------------------------|----------------|--------------------------|
| Bangkok      | 242.87                   | Madrid         | 283.56                   |
| Bogota       | 260.93                   | Mexico City    | 212.00                   |
| Bombay       | 139.16                   | Milan          | 284.08                   |
| Cairo        | 194.19                   | Paris          | 436.72                   |
| Dublin       | 260.76                   | Rio de Janeiro | 240.87                   |
| Frankfurt    | 355.36                   | Seoul          | 310.41                   |
| Hong Kong    | 346.32                   | Tel Aviv       | 223.73                   |
| Johannesburg | 165.37                   | Toronto        | 181.25                   |
| Lima         | 250.08                   | Warsaw         | 238.20                   |
| London       | 326.76                   | Washington, DC | 250.61                   |

9. Gold Fields Ltd is a South African mining company quoted on several stock exchanges, including NASDAQ Dubai. To analyze the risk, or volatility, associated with investing in Gold Fields Ltd shares, a sample of the monthly percentage return for 12 months was taken using the NASDAQ prices. The returns for the last six months of 2011 and the first six months of 2012 are shown here.

| <i>Month (2012)</i> | <i>Return (%)</i> | <i>Month (2011)</i> | <i>Return (%)</i> |
|---------------------|-------------------|---------------------|-------------------|
| January             | -5.26             | July                | 10.25             |
| February            | -6.45             | August              | 6.24              |
| March               | -9.66             | September           | 0.72              |
| April               | -9.66             | October             | -8.15             |
| May                 | -7.06             | November            | 12.13             |
| June                | -9.03             | December            | -0.58             |

- a. Compute the sample variance and sample standard deviation monthly return for Gold Fields, as measures of volatility.
- b. Construct a 95 per cent confidence interval for the population variance.
- c. Construct a 95 per cent confidence interval for the population standard deviation.
10. Part variability is critical in the manufacturing of ball bearings. Large variances in the size of the ball bearings cause bearing failure and rapid wear. Production standards call for a maximum variance of 0.0025 when the bearing sizes are measured in millimetres. A sample of 15 bearings shows a sample standard deviation of 0.066 mm.
- a. Use  $\alpha = 0.10$  to determine whether the sample indicates that the maximum acceptable variance is being exceeded.
- b. Compute a 90 per cent confidence interval estimate for the variance of the ball bearings in the population.
11. Suppose that any investment with an annualized standard deviation of percentage returns greater than 20 per cent is classified as 'high-risk'. The annualized standard deviation of percentage returns for the MSCI Emerging Markets index, based on a sample of size 36, is 25.2 per cent. Construct a hypothesis test that can be used to determine whether an investment based on the movements in the MSCI index would be classified as 'high-risk'. With a 0.05 level of significance, what is your conclusion?
12. A sample standard deviation for the number of passengers taking a particular airline flight is 8. A 95 per cent confidence interval estimate of the population standard deviation is 5.86 passengers to 12.62 passengers.
- a. Was a sample size of 10 or 15 used in the statistical analysis?
- b. Suppose the sample standard deviation of  $s = 8$  was based on a sample of 25 flights. What change would you expect in the confidence interval for the population standard deviation? Compute a 95 per cent confidence interval estimate of  $\sigma$  with a sample size of 25.

## 11.2 INFERENCES ABOUT TWO POPULATION VARIANCES

In some statistical applications we may want to compare the variances in product quality resulting from two different production processes, the variances in assembly times for two assembly methods or the variances in temperatures for two heating devices. In making comparisons about the two population variances, we shall be using data collected from two independent random samples, one from population 1 and another from population 2. The two sample variances  $s_1^2$  and  $s_2^2$  will be the basis for making inferences about the two population variances  $\sigma_1^2$  and  $\sigma_2^2$ . Whenever the variances of two normal populations are equal ( $\sigma_1^2 = \sigma_2^2$ ), the sampling distribution of the ratio of the two sample variances is as follows.

### Sampling distribution of $S_1^2/S_2^2$ when $\sigma_1^2 = \sigma_2^2$

When independent simple random samples of sizes  $n_1$  and  $n_2$  are selected from two normal populations with equal variances, the sampling distribution of:

$$\frac{S_1^2}{S_2^2} \quad (11.9)$$

has an  $F$  distribution with  $n_1 - 1$  degrees of freedom for the numerator and  $n_2 - 1$  degrees of freedom for the denominator.  $S_1^2$  is the sample variance for the random sample of  $n_1$  items from population 1, and  $S_2^2$  is the sample variance for the random sample of  $n_2$  items from population 2.

Figure 11.4 is a graph of the  $F$  distribution with 20 degrees of freedom for both the numerator and denominator. As can be seen from this graph,  $F$  values can never be negative, and the  $F$  distribution is not symmetrical. The shape of any particular  $F$  distribution depends on its numerator and denominator degrees of freedom.

We shall use  $F_\alpha$  to denote the value of  $F$  that gives an area or probability of  $\alpha$  in the upper tail of the distribution. For example, as noted in Figure 11.4,  $F_{0.05}$  identifies the upper tail area of 0.05 for an  $F$  distribution with 20 degrees of freedom for both the numerator and for the denominator. The specific value of  $F_{0.05}$  can be found by referring to the  $F$  distribution table, Table 4 of Appendix B. Using 20 degrees of freedom for the numerator, 20 degrees of freedom for the denominator and the row corresponding to an area of 0.05 in the upper tail, we find  $F_{0.05} = 2.12$ . Note that the table can be used to find  $F$  values for upper tail areas of 0.10, 0.05, 0.025 and 0.01.

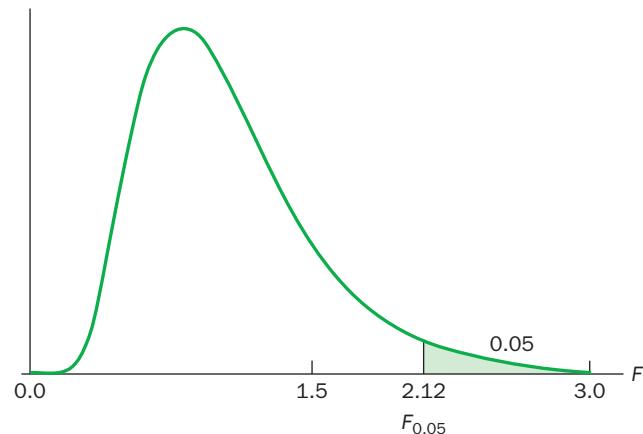
We now show how the  $F$  distribution can be used to do a hypothesis test about the equality of two population variances. The hypotheses are stated as follows:

$$H_0: \sigma_1^2 = \sigma_2^2$$

$$H_1: \sigma_1^2 \neq \sigma_2^2$$

**FIGURE 11.4**

$F$  distribution with 20 degrees of freedom for the numerator and 20 degrees of freedom for the denominator



We make the tentative assumption that the population variances are equal. If  $H_0$  is rejected, we will draw the conclusion that the population variances are not equal.

The hypothesis test requires two independent random samples, one from each population. The two sample variances are then computed. We refer to the population providing the *larger* sample variance as population 1. A sample size of  $n_1$  and a sample variance of  $s_1^2$  correspond to population 1, and a sample size of  $n_2$  and a sample variance of  $s_2^2$  correspond to population 2. Based on the assumption that both populations have a normal distribution, the ratio of sample variances provides the following  $F$  test statistic.

**Test statistic for hypothesis tests about population variances with  $\sigma_1^2 = \sigma_2^2$**

$$F = \frac{s_1^2}{s_2^2} \quad (11.10)$$

Denoting the population with the larger sample variance as population 1, the test statistic has an  $F$  distribution with  $n_1 - 1$  degrees of freedom for the numerator and  $n_2 - 1$  degrees of freedom for the denominator.

Because the  $F$  test statistic is constructed with the larger sample variance in the numerator, the value of the test statistic will be in the upper tail of the  $F$  distribution. Therefore, the  $F$  distribution table (Table 4 of Appendix B) need only provide upper-tail areas or probabilities.

We now consider an example. New Century Schools is renewing its school bus service contract for the coming year and must select one of two bus companies, the Red Bus Company or the Route One Company. We shall assume that the two companies have similar performance for average punctuality (i.e. mean arrival time) and use the variance of the arrival times as a primary measure of the quality of the bus service. Low variance values indicate the more consistent and higher quality service. If the variances of arrival times associated with the two services are equal, New Century Schools' managers will select the company offering the better financial terms. However, if the sample data on bus arrival times for the two companies indicate a significant difference between the variances, the administrators may want to give special consideration to the company with the better or lower variance service. The appropriate hypotheses follow.

$$H_0: \sigma_1^2 = \sigma_2^2$$

$$H_1: \sigma_1^2 \neq \sigma_2^2$$



If  $H_0$  can be rejected, the conclusion of unequal service quality is appropriate. We shall use a level of significance of  $\alpha = 0.10$  to do the hypothesis test. A sample of 26 arrival times for the Red Bus service provides a sample variance of 48 and a sample of 16 arrival times for the Route One service provides a sample variance of 20. Because the Red Bus sample provided the larger sample variance, we shall denote Red Bus as population 1. Using equation (11.10), the value of the test statistic is:

$$F = \frac{s_1^2}{s_2^2} = \frac{48}{20} = 2.40$$

The corresponding  $F$  distribution has  $n_1 - 1 = 26 - 1 = 25$  numerator degrees of freedom and  $n_2 - 1 = 16 - 1 = 15$  denominator degrees of freedom. As with other hypothesis testing procedures, we can use the  $p$ -value approach or the critical value approach to reach a conclusion. Table 4 of Appendix B shows the following areas in the upper tail and corresponding  $F$  values for an  $F$  distribution with 25 numerator degrees of freedom and 15 denominator degrees of freedom.

| Area in upper tail                      | 0.10 | 0.05       | 0.025 | 0.01 |
|---|------|------------|-------|------|
| $F$ value ( $df_1 = 25$ , $df_2 = 15$ ) | 1.89 | 2.28       | 2.69  | 3.28 |
|   |      | ↑          |       |      |
|   |      | $F = 2.40$ |       |      |

Because  $F = 2.40$  is between 2.28 and 2.69, the area in the upper tail of the distribution is between 0.05 and 0.025. Since this is a two-tailed test, we double the upper-tail area, which results in a  $p$ -value between 0.10 and 0.05. For this test, we selected  $\alpha = 0.10$  as the level of significance, which gives us a  $p$ -value  $< \alpha = 0.10$ . Hence, the null hypothesis is rejected. This finding leads to the conclusion that the two bus services differ in terms of arrival time variances. The recommendation is that the New Century Schools' managers give special consideration to the better or lower variance service offered by the Route One Company.

We can use EXCEL, MINITAB or IBM SPSS to show that the test statistic  $F = 2.40$  provides a two-tailed  $p$ -value = 0.0811. With  $0.0811 < \alpha = 0.10$ , the null hypothesis of equal population variances is rejected.

To use the critical value approach to do the two-tailed hypothesis test at the  $\alpha = 0.10$  level of significance, we select critical values with an area of  $\alpha/2 = 0.10/2 = 0.05$  in each tail of the distribution. Because the value of the test statistic computed using equation (11.10) will always be in the upper tail, we only need to determine the upper-tail critical value. From Table 4 of Appendix B, we see that  $F_{0.05} = 2.28$ . So, even though we use a two-tailed test, the rejection rule is stated as follows:

$$\text{Reject } H_0 \text{ if } F \geq 2.28$$

Because the test statistic  $F = 2.40$  is greater than 2.28, we reject  $H_0$  and conclude that the two bus services differ in terms of arrival time variances.

One-tailed tests involving two population variances are also possible. In this case, we use the  $F$  distribution to determine whether one population variance is significantly greater than the other. If we are using tables of the  $F$  distribution to compute the  $p$ -value or determine the critical value, a one-tailed hypothesis test about two population variances will always be formulated as an *upper-tail test*:

$$\begin{aligned} H_0: \sigma_1^2 &\leq \sigma_2^2 \\ H_1: \sigma_1^2 &> \sigma_2^2 \end{aligned}$$

This form of the hypothesis test always places the  $p$ -value and the critical value in the upper tail of the  $F$  distribution. As a result, only upper-tail  $F$  values will be needed, simplifying both the computations and the table for the  $F$  distribution.

As an example of a one-tailed test, consider a public opinion survey. Samples of 31 men and 41 women were used to study attitudes about current political issues. The researcher conducting the study wants to test to see if women show a greater variation in attitude on political issues than men. In the form of the one-tailed hypothesis test given previously, women will be denoted as population 1 and men will be denoted as population 2. The hypothesis test will be stated as follows:

$$\begin{aligned} H_0: \sigma_{\text{women}}^2 &\leq \sigma_{\text{men}}^2 \\ H_1: \sigma_{\text{women}}^2 &> \sigma_{\text{men}}^2 \end{aligned}$$

**TABLE 11.2** Summary of hypothesis tests about two population variances

|  | Upper-tail test   | Two-tailed test   |
|--|---|---|
| <b>Hypotheses</b>  | $H_0: \sigma_1^2 \leq \sigma_2^2$<br>$H_1: \sigma_1^2 > \sigma_2^2$ | $H_0: \sigma_1^2 = \sigma_2^2$<br>$H_1: \sigma_1^2 \neq \sigma_2^2$ |
|  | Note: Population 1 has the larger sample variance                   |   |
| <b>Test statistic</b>                                    | $F = \frac{s_1^2}{s_2^2}$   | $F = \frac{s_1^2}{s_2^2}$   |
| <b>Rejection rule:</b><br><b><i>p</i>-value approach</b> | Reject $H_0$ if<br>$p\text{-value} \leq \alpha$                     | Reject $H_0$ if<br>$p\text{-value} \leq \alpha$                     |
| <b>Rejection rule:</b><br><b>critical value approach</b> | Reject $H_0$ if<br>$F \geq F_\alpha$                                | Reject $H_0$ if<br>$F \geq F_{\alpha/2}$                            |

Rejection of  $H_0$  will give the researcher the statistical support necessary to conclude that women show a greater variation in attitude on political issues.

With the sample variance for women in the numerator and the sample variance for men in the denominator, the  $F$  distribution will have  $n_1 - 1 = 41 - 1 = 40$  numerator degrees of freedom and  $n_2 - 1 = 31 - 1 = 30$  denominator degrees of freedom. We shall use a level of significance  $\alpha = 0.05$  for the hypothesis test. The survey results provide a sample variance of  $s_1^2 = 120$  for women and a sample variance of  $s_2^2 = 80$  for men. The test statistic is as follows:

$$F = \frac{s_1^2}{s_2^2} = \frac{120}{80} = 1.50$$

Referring to Table 4 in Appendix B, we find that an  $F$  distribution with 40 numerator degrees of freedom and 30 denominator degrees of freedom has  $F_{0.10} = 1.57$ . Because the test statistic  $F = 1.50$  is less than 1.57, the area in the upper tail must be greater than 0.10. Hence, we can conclude that the  $p$ -value is greater than 0.10. Using MINITAB, IBM SPSS or EXCEL provides a  $p$ -value = 0.1256. Because the  $p$ -value  $> \alpha = 0.05$ ,  $H_0$  cannot be rejected. Hence, the sample results do not support the conclusion that women show greater variation in attitude on political issues than men.

Table 11.2 provides a summary of hypothesis tests about two population variances. Research confirms that the  $F$  distribution is sensitive to the assumption of normal populations. The  $F$  distribution should not be used unless it is reasonable to assume that both populations are at least approximately normally distributed.

## EXERCISES

### Methods

- 13.** Find the following  $F$  distribution values from Table 4 of Appendix B.

- a.  $F_{0.05}$  with degrees of freedom 5 and 10.
- b.  $F_{0.025}$  with degrees of freedom 20 and 15.
- c.  $F_{0.01}$  with degrees of freedom 8 and 12.
- d.  $F_{0.10}$  with degrees of freedom 10 and 20.

- 14.** A sample of 16 items from population 1 has a sample variance  $s_1^2 = 5.8$  and a sample of 21 items from population 2 has a sample variance  $s_2^2 = 2.4$ . Test the following hypotheses at the 0.05 level of significance.

$$\begin{aligned} H_0: \sigma_1^2 &\leq \sigma_2^2 \\ H_1: \sigma_1^2 &> \sigma_2^2 \end{aligned}$$

- a. What is your conclusion using the  $p$ -value approach?
- b. Repeat the test using the critical value approach.

- 15.** Consider the following hypothesis test.

$$\begin{aligned} H_0: \sigma_1^2 &= \sigma_2^2 \\ H_1: \sigma_1^2 &\neq \sigma_2^2 \end{aligned}$$

- a. What is your conclusion if  $n_1 = 21$ ,  $s_1^2 = 8.2$ ,  $n_2 = 26$ ,  $s_2^2 = 4.0$ ? Use  $\alpha = 0.05$  and the  $p$ -value approach.
- b. Repeat the test using the critical value approach.



COMPLETE  
SOLUTIONS

### Applications

- 16.** Most individuals are aware of the fact that the average annual repair cost for a car depends on its age. A researcher is interested in finding out whether the variance of the annual repair costs also increases with the age of the car. A sample of 26 cars that were eight years old showed a sample standard deviation for annual repair costs of £170 and a sample of 25 cars that were four years old showed a sample standard deviation for annual repair costs of £100.
- Suppose the research hypothesis is that the variance in annual repair costs is larger for the older cars. State the null and alternative hypotheses for an appropriate hypothesis test.
  - At a 0.01 level of significance, what is your conclusion? What is the  $p$ -value? Discuss the reasonableness of your findings.
- 17.** On the basis of data provided by a salary survey, the variance in annual salaries for seniors in accounting firms is approximately 2.1 and the variance in annual salaries for managers in accounting firms is approximately 11.1. The salary data were provided in thousands of euros. Assuming that the salary data were based on samples of 25 seniors and 26 managers, test the hypothesis that the population variances in the salaries are equal. At a 0.05 level of significance, what is your conclusion?
- 18.** For a sample of 100 days in 2012, the euro to US dollars and the British pound to euro exchange rates were recorded. The sample means were 1.2852 US\$/€ and 1.2294 €/£. The respective sample standard deviations were 0.03565 US\$/€ and 0.02290 €/£. Do a hypothesis test to determine whether there is a difference in variability between the two exchange rates. Use  $\alpha = 0.05$  as the level of significance. Discuss briefly whether the comparison you have made is a 'fair' one.
- 19.** Two new assembly methods are tested and the variances in assembly times are reported. Use  $\alpha = 0.10$  and test for equality of the two population variances.

|                  | Method A     | Method B     |
|------------------|--------------|--------------|
| Sample size      | $n_1 = 31$   | $n_2 = 25$   |
| Sample variation | $s_1^2 = 25$ | $s_2^2 = 12$ |



COMPLETE  
SOLUTIONS

- 20.** A research hypothesis is that the variance of stopping distances of cars on wet roads is greater than the variance of stopping distances of cars on dry roads. In the research study, 16 cars travelling at the same speeds are tested for stopping distances on wet roads and 16 cars are tested for stopping distances on dry roads. On wet roads, the standard deviation of stopping distances is ten metres. On dry roads, the standard deviation is five metres.
- At a 0.05 level of significance, do the sample data justify the conclusion that the variance in stopping distances on wet roads is greater than the variance in stopping distances on dry roads? What is the  $p$ -value?
  - What are the implications of your statistical conclusions in terms of driving safety recommendations?
- 21.** The grade point averages of 352 students who completed a college course in financial accounting have a standard deviation of 0.940. The grade point averages of 73 students who dropped out of the same course have a standard deviation of 0.797. Do the data indicate a difference between the variances of grade point averages for students who completed a financial accounting course and students who dropped out? Use a 0.05 level of significance.

Note:  $F_{0.025}$  with 351 and 72 degrees of freedom is 1.466.

- 22.** The variance in a production process is an important measure of the quality of the process. A large variance often signals an opportunity for improvement in the process by finding ways to reduce the process variance. The file ‘Bags’ on the online platform contains data for two machines that fill bags with powder. The file has 25 bag weights for Machine 1 and 22 bag weights for Machine 2. Conduct a statistical test to determine whether there is a significant difference between the variances in the bag weights for the two machines. Use a 0.05 level of significance. What is your conclusion? Which machine, if either, provides the greater opportunity for quality improvements?



## ONLINE RESOURCES

For the data files, online summary, additional questions and answers, and the software section for Chapter 11, visit the online platform.

## SUMMARY

In this chapter we presented statistical procedures that can be used to make inferences about population variances. In the process we introduced two new probability distributions: the chi-squared distribution and the  $F$  distribution. The chi-squared distribution can be used as the basis for interval estimation and hypothesis tests about the variance of a normal population.

We illustrated the use of the  $F$  distribution in hypothesis tests about the variances of two normal populations. With independent simple random samples of sizes  $n_1$  and  $n_2$  selected from two normal populations with equal variances, the sampling distribution of the ratio of the two sample variances has an  $F$  distribution with  $n_1 - 1$  degrees of freedom for the numerator and  $n_2 - 1$  degrees of freedom for the denominator.

## KEY FORMULAE

### Interval estimate of a population variance

$$\frac{(n-1)s^2}{\chi_{\alpha/2}^2} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi_{1-\alpha/2}^2} \quad (11.7)$$

### Test statistic for hypothesis tests about a population variance

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} \quad (11.8)$$

### Test statistic for hypothesis tests about population variances with $\sigma_1^2 = \sigma_2^2$

$$F = \frac{S_1^2}{S_2^2} \quad (11.10)$$

## CASE PROBLEM



### *Recovery from the global economic problems of 2008–09*

In 2008, particularly in the latter part of the year, there were global economic problems, including banking crises in a number of countries, clear indications of economic recession and increased stock market volatility. Since then, governments across Europe have been trying to lift their economies out of recession.

One method of measuring volatility in stock markets is to calculate the standard deviation of percentage changes in stock market prices or share index levels (e.g. daily percentage changes or weekly percentage changes). Although this is a relatively unsophisticated method, it is the first operational definition offered for the concept of ‘volatility’ in many texts on finance.

The data in the file ‘Share indices 2008–2012’ (on the online platform) are samples of daily percentage changes in four well-known stock market indices for two 12-month periods: one from mid-2008 to mid-2009, and the other from mid-2011 to mid-2012. The four indices are the FTSE 100 (London Stock Exchange, UK), the DAX 40 (Frankfurt Stock Exchange, Germany), the Athens Com-

posite Index (Athens Stock Exchange, Greece) and the TA 100 (Tel Aviv Stock Exchange, Israel).

Have stock markets become less volatile than during the problems of 2008–09? The report you are asked to prepare below should be focused particularly on the question of whether the stock markets showed greater volatility in 2008–09 than in 2011–12.

#### **Analyst's report**

1. Use appropriate descriptive statistics to summarize the daily percentage change data for each index in 2008–09 and 2011–12. What similarities or differences do you observe from the sample data?
2. Use the methods of Chapter 10 to comment on any difference between the population mean daily percentage change in each index for 2008–09 versus 2011–12. Discuss your findings.
3. Compute the standard deviation of the daily percentage changes for each share index, for 2008–09 and for 2011–12. For each share index, do a hypothesis test to examine the equality of population variances in 2008–09 and 2011–12. Discuss your findings.
4. What conclusions can you reach about any differences between 2008–09 and 2011–12?



SHARE  
INDICES

2008-2012

# 12

# Tests of Goodness of Fit and Independence



## CHAPTER CONTENTS

Statistics in Practice Pan-European and National lotteries

- 12.1 Goodness of fit test: a multinomial population
- 12.2 Test of independence
- 12.3 Goodness of fit test: Poisson and normal distributions

**LEARNING OBJECTIVES** After studying this chapter and doing the exercises, you should be able to construct and interpret the results of goodness of fit tests, using the chi-squared distribution, for several situations:

- 1 A multinomial population with given probabilities.
- 2 A test of independence in a two-way contingency table.
- 3 A Poisson distribution.
- 4 A normal distribution.

In Chapter 11 we showed how the chi-squared distribution could be used in estimation and in hypothesis tests about a population variance. In the present chapter, we introduce two additional hypothesis testing procedures, both based on the use of the chi-squared distribution. Like other hypothesis testing procedures, these tests compare sample results with those expected when the null hypothesis is true.

In the following section we introduce a goodness of fit test for a multinomial population. Later we discuss the test for independence using contingency tables and then show goodness of fit tests for the Poisson and normal distributions.

## 12.1 GOODNESS OF FIT TEST: A MULTINOMIAL POPULATION

Suppose each element of a population is assigned to one, and only one, of several classes or categories. Such a population is a **multinomial population**. The multinomial distribution can be thought of as an extension of the binomial distribution to three or more categories of outcomes. On each trial of a multinomial experiment, one and only one of the outcomes occurs. Each trial of the experiment is assumed to be independent of all others, and the probabilities of the outcomes remain the same at each trial.



## STATISTICS IN PRACTICE

### Pan-European and National lotteries

**E**very week, hundreds of millions of people across Europe pay to take a small gamble, in the hope of becoming an instant millionaire, by buying one or more tickets in a national lottery or a pan-European lottery. Since its inception in 2004, average sales in the EuroMillions lottery have topped 60 million tickets per draw (draws are held twice each week). The competitor EuroJackpot lottery, which started in 2012, expected sales of over 50 million tickets per draw. The European Lotteries association reported the 2011 revenues of its members as over €80 billion.



The precise details of the game, or gamble, differ from lottery to lottery, but the general principle is that each ticket buyer chooses several numbers from a prescribed set. The jackpot winner (or winners) is the ticket holder whose chosen numbers exactly match those picked out from the full set by a 'randomizing device' on the day the lottery is decided. For example, in the UK Lotto game, and in several others around Europe, six numbers are chosen from the set 1 to 49. The randomizing device is usually a sophisticated (and TV-friendly) piece of machinery that thoroughly mixes a set of numbered balls and picks out balls one by one. The objective is to give each ball an equal probability of being picked, so that every possible combination of numbers has equal probability.

Checks are periodically made to provide assurance on this principle of fairness. The checks are usually made by an independent body. For example, the Centre for the Study of Gambling at the University of Salford, UK reported to the National Lotteries Commission in January 2010 on the randomness of the EuroMillions draws. In the report, comparisons were made between the actual frequencies with which individual balls have been drawn and the frequencies expected assuming fairness or randomness. In statistical parlance, these are known as goodness of fit tests, more specifically as chi-squared tests.

In this chapter you will learn how chi-squared tests like those in the EuroMillions report are done.

As an example, consider a market share study being conducted by Scott Market Research. Over the past year market shares stabilized at 30 per cent for company A, 50 per cent for company B and 20 per cent for company C. Recently company C developed a 'new and improved' product to replace its current offering in the market. Company C retained Scott Market Research to assess whether the new product will alter market shares.

In this case, the population of interest is a multinomial population. Each customer is classified as buying from company A, company B or company C. So we have a multinomial population with three possible outcomes. We use the following notation:

$$\pi_A = \text{market share for company A}$$

$$\pi_B = \text{market share for company B}$$

$$\pi_C = \text{market share for company C}$$

Scott Market Research will conduct a sample survey and find the sample proportion preferring each company's product. A hypothesis test will then be done to assess whether the new product will lead to a change in market shares. The null and alternative hypotheses are:

$$H_0: \pi_A = 0.30, \pi_B = 0.50 \text{ and } \pi_C = 0.20$$

$$H_1: \text{The population proportions are not } \pi_A = 0.30, \pi_B = 0.50 \text{ and } \pi_C = 0.20$$

If the sample results lead to the rejection of  $H_0$ , Scott Market Research will have evidence that the introduction of the new product may affect market shares.

The market research firm has used a consumer panel of 200 customers for the study, in which each individual is asked to specify a purchase preference for one of three alternatives: company A's product, company B's product and company C's new product. This is equivalent to a multinomial experiment with 200 trials. The 200 responses are summarized here.

| Observed frequency  |                     |                         |
|---------------------|---------------------|-------------------------|
| Company A's product | Company B's product | Company C's new product |
| 48                  | 98                  | 54                      |

We now do a **goodness of fit test** to assess whether the sample of 200 customer purchase preferences is consistent with the null hypothesis. The goodness of fit test is based on a comparison of the sample of *observed* results with the *expected* results under the assumption that the null hypothesis is true. The next step is therefore to compute expected purchase preferences for the 200 customers under the assumption that  $\pi_A = 0.30$ ,  $\pi_B = 0.50$  and  $\pi_C = 0.20$ . The expected frequency for each category is found by multiplying the sample size of 200 by the hypothesized proportion for the category.

| Expected frequency  |                     |                         |
|---------------------|---------------------|-------------------------|
| Company A's product | Company B's product | Company C's new product |
| $200(0.30) = 60$    | $200(0.50) = 100$   | $200(0.20) = 40$        |

The goodness of fit test now focuses on the differences between the observed frequencies and the expected frequencies. Large differences between observed and expected frequencies cast doubt on the assumption that the hypothesized proportions or market shares are correct. Whether the differences between the observed and expected frequencies are 'large' or 'small' is a question answered with the aid of the following test statistic.

#### Test statistic for goodness of fit

$$\chi^2 = \sum_{i=1}^k \frac{(f_i - e_i)^2}{e_i} \quad (12.1)$$

where

$f_i$  = observed frequency for category  $i$

$e_i$  = expected frequency for category  $i$

$k$  = the number of categories

*Note:* The test statistic has a chi-squared distribution with  $k - 1$  degrees of freedom provided the expected frequencies are five or more for all categories.

In the Scott Market Research example we use the sample data to test the hypothesis that the multinomial population has the proportions  $\pi_A = 0.30$ ,  $\pi_B = 0.50$  and  $\pi_C = 0.20$ . We shall use level of significance  $\alpha = 0.05$ . The computation of the chi-squared test statistic is shown in Table 12.1, giving  $\chi^2 = 7.34$ .

We shall reject the null hypothesis if the differences between the observed and expected frequencies are large, which in turn will result in a large value for the test statistic. Hence the goodness of fit test will always be an upper-tail test. With  $k - 1 = 3 - 1 = 2$  degrees of freedom, the chi-squared table (Table 3 of Appendix B) provides the following (an introduction to the chi-squared distribution and the use of the chi-squared table were presented in Section 11.1).

**TABLE 12.1** Computation of the chi-squared test statistic for the Scott Market Research market share study

|                  | Hypothesized proportion | Observed frequency ( $f_i$ ) | Expected frequency ( $e_i$ ) | Difference ( $f_i - e_i$ ) | Squared difference ( $f_i - e_i$ ) <sup>2</sup> | Squared difference divided by expected frequency ( $(f_i - e_i)^2/e_i$ ) |
|------------------|-------------------------|------------------------------|------------------------------|----------------------------|---|--|
| <b>Company A</b> | 0.30                    | 48                           | 60                           | -12                        | 144   | 2.40   |
| <b>Company B</b> | 0.50                    | 98                           | 100                          | -2                         | 4   | 0.04   |
| <b>Company C</b> | 0.20                    | 54                           | 40                           | 14                         | 196   | 4.90   |
| <b>Total</b>     |                         | 200                          |                              |                            |   | $\chi^2 = 7.34$  |

| Area in upper tail    | 0.10  | 0.05  | 0.025 | 0.01  |
|-----------------------|-------|-------|-------|-------|
| $\chi^2$ value (2 df) | 4.605 | 5.991 | 7.378 | 9.210 |

$\chi^2 = 7.34$

The test statistic  $\chi^2 = 7.34$  is between 5.991 and 7.378 (very close to 7.378), so the corresponding upper-tail area or  $p$ -value must be between 0.05 and 0.025 (very close to 0.025). With  $p$ -value  $\alpha = 0.05$ , we reject  $H_0$  and conclude that the introduction of the new product by company C may alter the current market share structure. MINITAB, IBM SPSS or EXCEL can be used to show that  $\chi^2 = 7.34$  gives a  $p$ -value = 0.0255 (see the software guides on the online platform).

Instead of using the  $p$ -value, we could use the critical value approach to draw the same conclusion. With  $\alpha = 0.05$  and 2 degrees of freedom, the critical value for the test statistic is  $\chi^2 = 5.991$ . The upper tail rejection rule becomes:

$$\text{Reject } H_0 \text{ if } \chi^2 \geq 5.991$$

With  $\chi^2 = 7.34 > 5.991$ , we reject  $H_0$ . The  $p$ -value approach and critical value approach provide the same conclusion.

Although the test itself does not directly tell us about *how* market shares may change, we can compare the observed and expected frequencies descriptively to get an idea of the change in market structure. We see that the observed frequency of 54 for company C is larger than the expected frequency of 40. Because the latter was based on current market shares, the larger observed frequency suggests that the new product will have a positive effect on company C's market share. Similar comparisons for the other two companies suggest that company C's gain in market share will hurt company A more than company B.

Here are the steps for doing a goodness of fit test for a hypothesized multinomial population distribution.

#### Multinomial distribution goodness of fit test: a summary

- State the null and alternative hypotheses.

$H_0$ : The population follows a multinomial distribution with specified probabilities for each of the  $k$  categories

$H_1$ : The population does not follow a multinomial distribution with the specified probabilities for each of the  $k$  categories

2. Select a random sample and record the observed frequencies  $f_i$  for each category.
3. Assume the null hypothesis is true and determine the expected frequency  $e_i$  in each category by multiplying the category probability by the sample size.
4. Compute the value of the test statistic.
5. Rejection rule:

$$\begin{array}{ll} p\text{-value approach:} & \text{Reject } H_0 \text{ if } p\text{-value} \leq \alpha \\ \text{Critical value approach:} & \text{Reject } H_0 \text{ if } \chi^2 \geq \chi_{\alpha}^2 \end{array}$$

where  $\alpha$  is the level of significance for the test and there are  $k - 1$  degrees of freedom.

## EXERCISES

### Methods

1. Test the following hypotheses using the  $\chi^2$  goodness of fit test.

$$H_0: \pi_A = 0.40, \pi_B = 0.40, \pi_C = 0.20$$

$H_1$ : The population proportions are not  $\pi_A = 0.40, \pi_B = 0.40, \pi_C = 0.20$

A sample of size 200 yielded 60 in category A, 120 in category B and 20 in category C. Use  $\alpha = 0.01$  and test to see whether the proportions are as stated in  $H_0$ .

- a. Use the  $p$ -value approach.
- b. Repeat the test using the critical value approach.

2. Suppose we have a multinomial population with four categories: A, B, C and D. The null hypothesis is that the proportion of items is the same in every category, i.e.

$$H_0: \pi_A = \pi_B = \pi_C = \pi_D = 0.25$$

A sample of size 300 yielded the following results.

$$\text{A: 85 B: 95 C: 50 D: 70}$$

Use  $\alpha = 0.05$  to determine whether  $H_0$  should be rejected. What is the  $p$ -value?



COMPLETE  
SOLUTIONS

### Applications

3. One of the questions on *Business Week's* Subscriber Study was, ‘When making investment purchases, do you use full service or discount brokerage firms?’ Survey results showed that 264 respondents use full service brokerage firms only, 255 use discount brokerage firms only and 229 use both full service and discount firms. Use  $\alpha = 0.10$  to determine whether there are any differences in preference among the three service choices.
4. How well do airline companies serve their customers? A study by *Business Week* showed the following customer ratings: 3 per cent excellent, 28 per cent good, 45 per cent fair and 24 per cent poor. In a follow-up study of service by telephone companies, assume that a sample of 400 adults found the following customer ratings: 24 excellent, 124 good, 172 fair and 80 poor. Taking the figures from the *Business Week* study as ‘population’ values, is the distribution of the customer ratings for telephone companies different from the distribution of customer ratings for airline companies? Test with  $\alpha = 0.01$ . What is your conclusion?
5. In setting sales quotas, the marketing manager of a multinational company makes the assumption that order potentials are the same for each of four sales territories in the Middle East. A sample of 200 sales follows. Should the manager’s assumption be rejected? Use  $\alpha = 0.05$ .

| Sales territories |    |    |    |
|-------------------|----|----|----|
| 1                 | 2  | 3  | 4  |
| 60                | 45 | 59 | 36 |

6. A community park will open soon in a large European city. A sample of 210 individuals are asked to state their preference for when they would most like to visit the park. The sample results follow.

| Monday | Tuesday | Wednesday | Thursday | Friday | Saturday | Sunday |
|--------|---------|-----------|----------|--------|----------|--------|
| 20     | 30      | 30        | 25       | 35     | 20       | 50     |

In developing a staffing plan, should the park manager plan on the same number of individuals visiting the park each day? Support your conclusion with a statistical test. Use  $\alpha = 0.05$ .

7. The results of *ComputerWorld's* Annual Job Satisfaction Survey showed that 28 per cent of information systems (IS) managers are very satisfied with their job, 46 per cent are somewhat satisfied, 12 per cent are neither satisfied or dissatisfied, 10 per cent are somewhat dissatisfied and 4 per cent are very dissatisfied. Suppose that a sample of 500 computer programmers yielded the following results.

| Category              | Number of respondents |
|-----------------------|-----------------------|
| Very satisfied        | 105                   |
| Somewhat satisfied    | 235                   |
| Neither               | 55                    |
| Somewhat dissatisfied | 90                    |
| Very dissatisfied     | 15                    |

Taking the *ComputerWorld* figures as ‘population’ values, use  $\alpha = 0.05$  and test to determine whether the job satisfaction for computer programmers is different from the job satisfaction for IS managers.

## 12.2 TEST OF INDEPENDENCE

Another important application of the chi-squared distribution involves testing for the independence of two qualitative (categorical) variables. Consider a study conducted by the Millennium Brewery, which manufactures and distributes three types of beer: pilsner, export and dark beer. In an analysis of the market segments for the three beers, the firm’s market research group raised the question of whether preferences for the three beers differ between male and female beer drinkers. If beer preference is independent of gender, a single advertising campaign will be initiated for all of the Millennium beers. However, if beer preference depends on the gender of the beer drinker, the firm will tailor its promotions to different target markets.

A test of independence addresses the question of whether the beer preference (pilsner, export or dark) is independent of the gender of the beer drinker (male, female). The hypotheses for this test are:

$$H_0: \text{Beer preference is independent of the gender of the beer drinker}$$

$$H_1: \text{Beer preference is not independent of the gender of the beer drinker}$$

Table 12.2 can be used to describe the situation. The population under study is all male and female beer drinkers. A sample can be selected from this population and each individual asked to state his or her preference among the three Millennium beers. Every individual in the sample will be classified in one of the six cells in the table. For example, an individual may be a male preferring export (cell (1,2)), a female preferring pilsner (cell (2,1)), a female preferring dark beer (cell (2,3)) and so on.

**TABLE 12.2** Contingency table for beer preference and gender of beer drinker

| Gender | Beer preference |           |           |
|--------|-----------------|-----------|-----------|
|        | Pilsner         | Export    | Dark      |
| Male   | cell(1,1)       | cell(1,2) | cell(1,3) |
| Female | cell(2,1)       | cell(2,2) | cell(2,3) |

**TABLE 12.3** Sample results for beer preferences of male and female beer drinkers (observed frequencies)

| Gender | Beer preference |        |      | Total |
|--------|-----------------|--------|------|-------|
|        | Pilsner         | Export | Dark |       |
| Male   | 20              | 40     | 20   | 80    |
| Female | 30              | 30     | 10   | 70    |
| Total  | 50              | 70     | 30   | 150   |

Because we have listed all possible combinations of beer preference and gender – in other words, listed all possible contingencies – Table 12.2 is called a **contingency table**. The test of independence is sometimes referred to as a *contingency table test*.

Suppose a simple random sample of 150 beer drinkers is selected. After tasting each beer, the individuals in the sample are asked to state their first-choice preference. The cross-tabulation in Table 12.3 summarizes the responses. The data for the test of independence are collected in terms of counts or frequencies for each cell or category. Of the 150 individuals in the sample, 20 were men favouring pilsner, 40 were men favouring export, 20 were men favouring dark beer and so on. The data in Table 12.3 are the observed frequencies for the six classes or categories.

If we can determine the expected frequencies under the assumption of independence between beer preference and gender of the beer drinker, we can use the chi-squared distribution to determine whether there is a significant difference between observed and expected frequencies.

Expected frequencies for the cells of the contingency table are based on the following rationale. We assume the null hypothesis of independence between beer preference and gender of the beer drinker is true. Then we note that in the entire sample of 150 beer drinkers, a total of 50 prefer pilsner, 70 prefer export and 30 prefer dark beer. In terms of fractions,  $50/150$  of the beer drinkers prefer pilsner,  $70/150$  prefer export and  $30/150$  prefer dark beer. If the *independence* assumption is valid, these fractions must be applicable to both male and female beer drinkers. So we would expect the sample of 80 male beer drinkers to contain  $(50/150)80 = 26.67$  who prefer pilsner,  $(70/150)80 = 37.33$  who prefer export, and  $(30/150)80 = 16$  who prefer dark beer. Application of the same fractions to the 70 female beer drinkers provides the expected frequencies shown in Table 12.4.

**TABLE 12.4** Expected frequencies if beer preference is independent of the gender of the beer drinker

| Gender | Beer preference |        |       | Total |
|--------|-----------------|--------|-------|-------|
|        | Pilsner         | Export | Dark  |       |
| Male   | 26.67           | 37.33  | 16.00 | 80    |
| Female | 23.33           | 32.67  | 14.00 | 70    |
| Total  | 50.00           | 70.00  | 30.00 | 150   |

Let  $e_{ij}$  denote the expected frequency for the contingency table category in row  $i$  and column  $j$ . With this notation, consider the expected frequency calculation for males (row  $i = 1$ ) who prefer lager (column  $j = 2$ ): that is, expected frequency  $e_{12}$ . The argument above showed that:

$$e_{12} = \left( \frac{70}{150} \right) 80 = 37.33$$

This expression can be written slightly differently as:

$$e_{12} = \left( \frac{70}{150} \right) 80 = \frac{(80)(70)}{150} = 37.33$$

Note that the 80 in the expression is the total number of males (row 1 total), 70 is the total number of individuals who prefer export (column 2 total) and 150 is the total sample size. Hence, we see that:

$$e_{12} = \frac{(\text{Row 1 Total})(\text{Column 2 Total})}{\text{Sample Size}}$$

Generalization of this expression shows that the following formula provides the expected frequencies for a contingency table in the test of independence.

#### Expected frequencies for contingency tables under the assumption of independence

$$e_{ij} = \frac{(\text{Row } i \text{ Total})(\text{Column } j \text{ Total})}{\text{Sample Size}} \quad (12.2)$$

Using this formula for male beer drinkers who prefer dark beer, we find an expected frequency of  $e_{13} = (80)(30)/(150) = 16.00$ , as shown in Table 12.4. Use equation (12.2) to verify the other expected frequencies shown in Table 12.4.

The test procedure for comparing the observed frequencies of Table 12.3 with the expected frequencies of Table 12.4 is similar to the goodness of fit calculations made in Section 12.1. Specifically, the  $\chi^2$  value based on the observed and expected frequencies is computed as follows.

#### Test statistic for independence

$$\chi^2 = \sum_i \sum_j \frac{(f_{ij} - e_{ij})^2}{e_{ij}} \quad (12.3)$$

where

- $f_{ij}$  = observed frequency for contingency table category in row  $i$  and column  $j$
- $e_{ij}$  = expected frequency for contingency table category in row  $i$  and column  $j$  on the assumption of independence

Note: With  $n$  rows and  $m$  columns in the contingency table, the test statistic has a chi-squared distribution with  $(n - 1)(m - 1)$  degrees of freedom provided that the expected frequencies are five or more for all categories.

The double summation in equation (12.3) is used to indicate that the calculation must be made for all the cells in the contingency table.

**TABLE 12.5** Computation of the chi-squared test statistic for determining whether beer preference is independent of the gender of the beer drinker

| Gender | Beer preference | Observed frequency ( $f_{ij}$ ) | Expected frequency ( $e_{ij}$ ) | Difference ( $f_{ij} - e_{ij}$ ) | Squared difference ( $f_{ij} - e_{ij}$ ) <sup>2</sup> | Squared difference divided by expected frequency ( $(f_{ij} - e_{ij})^2/e_{ij}$ ) |
|--------|-----------------|---------------------------------|---------------------------------|----------------------------------|---|---|
| Male   | Pilsner         | 20                              | 26.67                           | -6.67                            | 44.44   | 1.67  |
| Male   | Export          | 40                              | 37.33                           | 2.67                             | 7.11  | 0.19  |
| Male   | Dark            | 20                              | 16.00                           | 4.00                             | 16.00   | 1.00  |
| Female | Pilsner         | 30                              | 23.33                           | 6.67                             | 44.44   | 1.90  |
| Female | Export          | 30                              | 32.67                           | -2.67                            | 7.11  | 0.22  |
| Female | Dark            | 10                              | 14.00                           | -4.00                            | 16.00   | 1.14  |
| Total  |                 | 150                             |                                 |                                  |   | $\chi^2 = 6.12$   |

The expected frequencies are five or more for each category. We therefore proceed with the computation of the chi-squared test statistic, as shown in Table 12.5. We see that the value of the test statistic is  $\chi^2 = 6.12$ .

The number of degrees of freedom for the appropriate chi-squared distribution is computed by multiplying the number of rows minus one by the number of columns minus one. With two rows and three columns, we have  $(2 - 1)(3 - 1) = 3$  degrees of freedom. Just like the test for goodness of fit, the test for independence rejects  $H_0$  if the differences between observed and expected frequencies provide a large value for the test statistic. So the test for independence is also an upper-tail test. Using the chi-squared table (Table 3 of Appendix B), we find that the upper-tail area or  $p$ -value at  $\chi^2 = 6.12$  is between 0.025 and 0.05. At the 0.05 level of significance,  $p$ -value  $\alpha = 0.05$ . We reject the null hypothesis of independence and conclude that beer preference is not independent of the gender of the beer drinker.

Computer software packages such as IBM SPSS, MINITAB and EXCEL can simplify the computations for a test of independence and provide the  $p$ -value for the test (see the software guides on the online platform). In the Millennium Brewery example, EXCEL, MINITAB or IBM SPSS shows  $p$ -value = 0.0468.

The test itself does not tell us directly about the nature of the dependence between beer preference and gender, but we can compare the observed and expected frequencies descriptively to get an idea. Refer to Tables 12.3 and 12.4. Male beer drinkers have higher observed than expected frequencies for both export and dark beer, whereas female beer drinkers have a higher observed than expected frequency only for pilsner. These observations give us insight about the beer preference differences between male and female beer drinkers.

Here are the steps in a contingency table test of independence.



### Test of independence: a summary

- State the null and alternative hypotheses.

$H_0$ : the column variable is independent of the row variable

$H_1$ : the column variable is not independent of the row variable

- Select a random sample and record the observed frequencies for each cell of the contingency table.
- Use equation (12.2) to compute the expected frequency for each cell.
- Use equation (12.3) to compute the value of the test statistic.
- Rejection rule:

$p$ -value approach: Reject  $H_0$  if  $p$ -value  $\leq \alpha$

Critical value approach: Reject  $H_0$  if  $\chi^2 \geq \chi^2_\alpha$

where  $\alpha$  is the level of significance for the test, with  $n$  rows and  $m$  columns providing  $(n - 1) \times (m - 1)$  degrees of freedom.

Note: The test statistic for the chi-squared tests in this chapter requires an expected frequency of five or more for each category. When a category has fewer than five, it is often appropriate to combine two adjacent rows or columns to obtain an expected frequency of five or more in each category.

## EXERCISES

### Methods



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8. The following  $2 \times 3$  contingency table contains observed frequencies for a sample of 200. Test for independence of the row and column variables using the  $\chi^2$  test with  $\alpha = 0.05$ .

|              |   | Column variable |    |    |
|--------------|---|-----------------|----|----|
|              |   | A               | B  | C  |
| Row variable | P | 20              | 44 | 50 |
|              | Q | 30              | 26 | 30 |

9. The following  $3 \times 3$  contingency table contains observed frequencies for a sample of 240. Test for independence of the row and column variables using the  $\chi^2$  test with  $\alpha = 0.05$ .

|              |   | Column variable |    |    |
|--------------|---|-----------------|----|----|
|              |   | A               | B  | C  |
| Row variable | P | 20              | 30 | 20 |
|              | Q | 30              | 60 | 25 |
|              | R | 10              | 15 | 30 |

### Applications

10. One of the questions on the *Business Week* Subscriber Study was, ‘In the past 12 months, when travelling for business, what type of airline ticket did you purchase most often?’ The data obtained are shown in the following contingency table.

|                |                  | Type of flight        |  |
|----------------|------------------|-----------------------|--|
| Type of ticket | Domestic flights | International flights |  |
| First class    | 29               | 22                    |  |
| Business class | 95               | 121                   |  |
| Economy class  | 518              | 135                   |  |

Using  $\alpha = 0.05$ , test for the independence of type of flight and type of ticket. What is your conclusion?

- 11.** First-destination jobs for Business and Engineering graduates are classified by industry as shown in the following table.

| <i>Degree major</i> | <i>Industry</i> |                 |                   |                 |
|---------------------|-----------------|-----------------|-------------------|-----------------|
|                     | <i>Oil</i>      | <i>Chemical</i> | <i>Electrical</i> | <i>Computer</i> |
| <i>Business</i>     | 30              | 15              | 15                | 40              |
| <i>Engineering</i>  | 30              | 30              | 20                | 20              |

Test for independence of degree major and industry type, using  $\alpha = 0.01$ .

- 12.** Businesses are increasingly placing orders online. The Performance Measurement Group collected data on the rates of correctly filled electronic orders by industry. Assume a sample of 700 electronic orders provided the following results.



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| <i>Order</i>     | <i>Industry</i>       |                 |                  |                           |
|------------------|-----------------------|-----------------|------------------|---------------------------|
|                  | <i>Pharmaceutical</i> | <i>Consumer</i> | <i>Computers</i> | <i>Telecommunications</i> |
| <i>Correct</i>   | 207                   | 136             | 151              | 178                       |
| <i>Incorrect</i> | 3                     | 4               | 9                | 12                        |

- a.** Test whether order fulfillment is independent of industry. Use  $\alpha = 0.05$ . What is your conclusion?  
**b.** Which industry has the highest percentage of correctly filled orders?
- 13.** Three suppliers provide the following data on defective parts.

| <i>Supplier</i> | <i>Part quality</i> |                     |                     |
|-----------------|---------------------|---------------------|---------------------|
|                 | <i>Good</i>         | <i>Minor defect</i> | <i>Major defect</i> |
| <i>A</i>        | 90                  | 3                   | 7                   |
| <i>B</i>        | 170                 | 18                  | 7                   |
| <i>C</i>        | 135                 | 6                   | 9                   |

Using  $\alpha = 0.05$ , test for independence between supplier and part quality. What does the result of your analysis tell the purchasing department?

- 14.** A sample of parts taken in a machine shop in Karachi provided the following contingency table data on part quality by production shift.

| <i>Shift</i>  | <i>Number good</i> | <i>Number defective</i> |
|---------------|--------------------|-------------------------|
| <i>First</i>  | 368                | 32                      |
| <i>Second</i> | 285                | 15                      |
| <i>Third</i>  | 176                | 24                      |

Test the hypothesis that part quality is independent of the production shift, using  $\alpha = 0.05$ . What is your conclusion?

- 15.** Visa studied how frequently consumers of various age groups use plastic cards (debit and credit cards) when making purchases. Sample data for 300 customers show the use of plastic cards by four age groups.

| Payment        | Age group |       |       |             |
|----------------|-----------|-------|-------|-------------|
|                | 18–24     | 25–34 | 35–44 | 45 and over |
| Plastic        | 21        | 27    | 27    | 36          |
| Cash or Cheque | 21        | 36    | 42    | 90          |

- a. Test for the independence between method of payment and age group. What is the  $p$ -value? Using  $\alpha = 0.05$ , what is your conclusion?
- b. If method of payment and age group are not independent, what observation can you make about how different age groups use plastic to make purchases?
- c. What implications does this study have for companies such as Visa and MasterCard?
16. The following cross-tabulation shows industry type and P/E ratio for 100 companies in the consumer products and banking industries.

| Industry | P/E ratio |       |       |       |       | Total |
|----------|-----------|-------|-------|-------|-------|-------|
|          | 5–9       | 10–14 | 15–19 | 20–24 | 25–29 |       |
| Consumer | 4         | 10    | 18    | 10    | 8     | 50    |
| Banking  | 14        | 14    | 12    | 6     | 4     | 50    |
| Total    | 18        | 24    | 30    | 16    | 12    | 100   |

Does there appear to be a relationship between industry type and P/E ratio? Support your conclusion with a statistical test using  $\alpha = 0.05$ .

## 12.3 GOODNESS OF FIT TEST: POISSON AND NORMAL DISTRIBUTIONS

In general, the chi-squared goodness of fit test can be used with any hypothesized probability distribution. In this section we illustrate for cases in which the population is hypothesized to have a Poisson or a normal distribution. The goodness of fit test follows the same general procedure as in Section 12.1.

### Poisson distribution

Consider the arrival of customers at the Mediterranean Food Market. Because of recent staffing problems, the Mediterranean's managers asked a local consultancy to assist with the scheduling of checkout assistants. After reviewing the checkout operation, the consultancy will make a recommendation for a scheduling procedure. The procedure, based on a mathematical analysis of waiting times, is applicable only if the number of customers arriving during a specified time period follows the Poisson distribution. Therefore, before the scheduling process is implemented, data on customer arrivals must be collected and a statistical test done to see whether an assumption of a Poisson distribution for arrivals is reasonable.

We define the arrivals at the store in terms of the *number of customers* entering the store during five-minute intervals. The following null and alternative hypotheses are appropriate:

$H_0$ : The number of customers entering the store during five-minute intervals has a Poisson probability distribution

$H_1$ : The number of customers entering the store during five-minute intervals does not have a Poisson distribution

**TABLE 12.6** Observed frequency of the Mediterranean's customer arrivals for a sample of 128 five-minute time periods

| Number of customers arriving | Observed frequency |
|------------------------------|--------------------|
| 0                            | 2                  |
| 1                            | 8                  |
| 2                            | 10                 |
| 3                            | 12                 |
| 4                            | 18                 |
| 5                            | 22                 |
| 6                            | 22                 |
| 7                            | 16                 |
| 8                            | 12                 |
| 9                            | 6                  |
| Total                        | 128                |

If a sample of customer arrivals provides insufficient evidence to reject  $H_0$ , the Mediterranean will proceed with the implementation of the consultancy's scheduling procedure. However, if the sample leads to the rejection of  $H_0$ , the assumption of the Poisson distribution for the arrivals cannot be made and other scheduling procedures will be considered.

To test the assumption of a Poisson distribution for the number of arrivals during weekday morning hours, a store assistant randomly selects a sample,  $n = 128$ , of five-minute intervals during weekday mornings over a three-week period. For each five-minute interval in the sample, the store assistant records the number of customer arrivals. The store assistant then summarizes the data by counting the number of five-minute intervals with no arrivals, the number of five-minute intervals with one arrival and so on. These data are summarized in Table 12.6.

To do the goodness of fit test, we need to consider the expected frequency for each of the ten categories, under the assumption that the Poisson distribution of arrivals is true. The Poisson probability function, first introduced in Chapter 5, is:

$$p(X = x) = \frac{\mu^x e^{-\mu}}{x!} \quad (12.4)$$

In this function,  $\mu$  represents the mean or expected number of customers arriving per five-minute period,  $X$  is a random variable indicating the number of customers arriving during a five-minute period and  $p(X = x)$  is the probability that exactly  $x$  customers will arrive in a five-minute interval.

To use (12.4), we must obtain an estimate of  $\mu$ , the mean number of customer arrivals during a five-minute time period. The sample mean for the data in Table 12.6 provides this estimate. With no customers arriving in two five-minute time periods, one customer arriving in eight five-minute time periods and so on, the total number of customers who arrived during the sample of 128 five-minute time periods is given by  $0(2) + 1(8) + 2(10) + \dots + 9(6) = 640$ . The 640 customer arrivals over the sample of 128 periods provide an estimated mean arrival rate of  $640/128 = 5$  customers per five-minute period. With this value for the mean of the distribution, an estimate of the Poisson probability function for the Mediterranean Food Market is:

$$p(X = x) = \frac{5^x e^{-5}}{x!} \quad (12.5)$$

This probability function can be evaluated for different values  $x$  to determine the probability associated with each category of arrivals. These probabilities, which can also be found in Table 7 of Appendix B, are given in Table 12.7. For example, the probability of zero customers arriving during a five-minute interval is  $p(0) = 0.0067$ , the probability of one customer arriving during a five-minute interval is  $p(1) = 0.0337$  and so on. As we saw in Section 12.1, the expected frequencies for the categories are found by multiplying the probabilities by the sample size.

**TABLE 12.7** Expected frequency of Mediterranean's customer arrivals, assuming a Poisson distribution with  $\mu = 5$

| Number of customers arriving ( $x$ ) | Poisson probability $p(x)$ | Expected number of five-minute time periods with $x$ arrivals, $128p(x)$ |
|--------------------------------------|----------------------------|--|
| 0                                    | 0.0067                     | 0.86   |
| 1                                    | 0.0337                     | 4.31   |
| 2                                    | 0.0842                     | 10.78  |
| 3                                    | 0.1404                     | 17.97  |
| 4                                    | 0.1755                     | 22.46  |
| 5                                    | 0.1755                     | 22.46  |
| 6                                    | 0.1462                     | 18.71  |
| 7                                    | 0.1044                     | 13.36  |
| 8                                    | 0.0653                     | 8.36   |
| 9                                    | 0.0363                     | 4.65   |
| 10 or more                           | 0.0318                     | 4.07   |
| Total                                |                            | 128.00   |

For example, the expected number of periods with zero arrivals is given by  $(0.0067)(128) = 0.86$ , the expected number of periods with one arrival is given by  $(0.0337)(128) = 4.31$  and so on.

In Table 12.7, four of the categories have an expected frequency less than five. This condition violates the requirements for use of the chi-squared distribution. However, adjacent categories can be combined to satisfy the 'at least five' expected frequency requirement. In particular, we shall combine 0 and 1 into a single category, and then combine 9 with '10 or more' into another single category. Table 12.8 shows the observed and expected frequencies after combining categories.

As in Section 12.1, the goodness of fit test focuses on the differences between observed and expected frequencies,  $f_i - e_i$ . The calculations are shown in Table 12.8. The value of the test statistic is  $\chi^2 = 10.96$ .

In general, the chi-squared distribution for a goodness of fit test has  $k - p - 1$  degrees of freedom, where  $k$  is the number of categories and  $p$  is the number of population parameters estimated from the sample data.

**TABLE 12.8** Observed and expected frequencies for the Mediterranean's customer arrivals after combining categories, and computation of the chi-squared test statistic

| Number of customers arriving ( $x$ ) | Observed frequency ( $f_i$ ) | Expected frequency ( $e_i$ ) | Difference ( $f_i - e_i$ ) | Squared difference $(f_i - e_i)^2$ | Squared difference divided by expected frequency $(f_i - e_i)^2/e_i$ |
|--------------------------------------|------------------------------|------------------------------|----------------------------|------------------------------------|--|
| 0 or 1                               | 10                           | 5.17                         | 4.83                       | 23.28                              | 4.50   |
| 2                                    | 10                           | 10.78                        | -0.78                      | 0.61                               | 0.06   |
| 3                                    | 12                           | 17.97                        | -5.97                      | 35.62                              | 1.98   |
| 4                                    | 18                           | 22.46                        | -4.46                      | 19.89                              | 0.89   |
| 5                                    | 22                           | 22.46                        | -0.46                      | 0.21                               | 0.01   |
| 6                                    | 22                           | 18.72                        | 3.28                       | 10.78                              | 0.58   |
| 7                                    | 16                           | 13.37                        | 2.63                       | 6.92                               | 0.52   |
| 8                                    | 12                           | 8.36                         | 3.64                       | 13.28                              | 1.59   |
| 9 or more                            | 6                            | 8.72                         | -2.72                      | 7.38                               | 0.85   |
| Total                                | 128                          | 128.00                       |                            |                                    | $\chi^2 = 10.96$   |

Table 12.8 shows  $k = 9$  categories. Because the sample data were used to estimate the mean of the Poisson distribution,  $p = 1$ . Hence, there are  $k - p - 1 = 9 - 1 - 1 = 7$  degrees of freedom.

Suppose we test the null hypothesis with a 0.05 level of significance. We need to determine the  $p$ -value for the test statistic  $\chi^2 = 10.96$  by finding the area in the upper tail of a chi-squared distribution with seven degrees of freedom. Using Table 3 of Appendix B, we find that  $\chi^2 = 10.96$  provides an area in the upper tail greater than 0.10. So we know that the  $p$ -value is greater than 0.10. MINITAB, IBM SPSS or EXCEL shows  $p$ -value = 0.1403. With  $p$ -value  $> \alpha = 0.10$ , we cannot reject  $H_0$ . The assumption of a Poisson probability distribution for weekday morning customer arrivals cannot be rejected. As a result, the Mediterranean's management may proceed with the consulting firm's scheduling procedure for weekday mornings.

### Poisson distribution goodness of fit test: a summary

1. State the null and alternative hypotheses.

$H_0$ : The population has a Poisson distribution

$H_1$ : The population does not have a Poisson distribution

2. Select a random sample and

- a. Record the observed frequency  $f_i$  for each value of the Poisson random variable.
- b. Compute the mean number of occurrences.

3. Compute the expected frequency of occurrences  $e_i$  for each value of the Poisson random variable.

Multiply the sample size by the Poisson probability of occurrence for each value of the Poisson random variable. If there are fewer than five expected occurrences for some values, combine adjacent values and reduce the number of categories as necessary.

4. Compute the value of the test statistic.

$$\chi^2 = \sum_{i=1}^k \frac{(f_j - e_j)^2}{e_j}$$

5. Rejection rule:

$p$ -value approach: Reject  $H_0$  if  $p$ -value  $\leq \alpha$

Critical value approach: Reject  $H_0$  if  $\chi^2 \geq \chi^2_\alpha$

where  $\alpha$  is the level of significance for the test, and there are  $k - 2$  degrees of freedom.

## Normal distribution

A goodness of fit test for a normal distribution can also be based on the use of the chi-squared distribution. It is similar to the procedure for the Poisson distribution. In particular, observed frequencies for several categories of sample data are compared to expected frequencies under the assumption that the population has a normal distribution. Because the normal distribution is continuous, we must modify the way the categories are defined and how the expected frequencies are computed.

Consider the job applicant test data for Pharmaco plc, listed in Table 12.9. Pharmaco hires approximately 400 new employees annually for its four plants located in Europe and the Middle East. The personnel director asks whether a normal distribution applies for the population of test scores. If such a distribution can be used, the distribution would be helpful in evaluating specific test scores; that is, scores in the upper 20 per cent, lower 40 per cent and so on, could be identified quickly. Hence, we want to test the null hypothesis that the population of test scores has a normal distribution.

**TABLE 12.9** Pharmaco employee aptitude test scores for 50 randomly chosen job applicants

|    |    |    |    |    |    |    |    |    |    |
|----|----|----|----|----|----|----|----|----|----|
| 71 | 65 | 54 | 93 | 60 | 86 | 70 | 70 | 73 | 73 |
| 55 | 63 | 56 | 62 | 76 | 54 | 82 | 79 | 76 | 68 |
| 53 | 58 | 85 | 80 | 56 | 61 | 64 | 65 | 62 | 90 |
| 69 | 76 | 79 | 77 | 54 | 64 | 74 | 65 | 65 | 61 |
| 56 | 63 | 80 | 56 | 71 | 79 | 84 | 66 | 61 | 61 |

We first use the data in Table 12.9 to calculate estimates of the mean and standard deviation of the normal distribution that will be considered in the null hypothesis. We use the sample mean and the sample standard deviation as point estimators of the mean and standard deviation of the normal distribution. The calculations follow.

$$\bar{x} = \frac{\sum x_i}{n} = \frac{3421}{50} = 68.42$$

$$s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}} = \sqrt{\frac{5310.04}{49}} = 10.41$$

Using these values, we state the following hypotheses about the distribution of the job applicant test scores.

$H_0$ : The population of test scores has a normal distribution with mean 68.42 and standard deviation 10.41.

$H_1$ : The population of test scores does not have a normal distribution with mean 68.42 and standard deviation 10.41.

Now we look at how to define the categories for a goodness of fit test involving a normal distribution. For the discrete probability distribution in the Poisson distribution test, the categories were readily defined in terms of the number of customers arriving, such as 0, 1, 2 and so on. However, with the continuous normal probability distribution, we must use a different procedure for defining the categories. We need to define the categories in terms of *intervals* of test scores.

Recall the rule of thumb for an expected frequency of at least five in each interval or category. We define the categories of test scores such that the expected frequencies will be at least five for each category. With a sample size of 50, one way of establishing categories is to divide the normal distribution into ten equal-probability intervals (see Figure 12.1). With a sample size of 50, we would expect five outcomes in each interval or category and the rule of thumb for expected frequencies would be satisfied.

When the normal probability distribution is assumed, the standard normal distribution tables can be used to determine the category boundaries. First consider the test score cutting off the lowest 10 per cent of the test scores. From Table 1 of Appendix B we find that the  $z$  value for this test score is  $-1.28$ . Therefore, the test score  $x = 68.42 - 1.28(10.41) = 55.10$  provides this cut-off value for the lowest 10 per cent of the scores. For the lowest 20 per cent, we find  $z = -0.84$  and so  $x = 68.42 - 0.84(10.41) = 59.68$ . Working through the normal distribution in that way provides the following test score values.

$$\text{Lower 10\%: } 68.42 - 1.28(10.41) = 55.10$$

$$\text{Lower 20\%: } 68.42 - 0.84(10.41) = 59.68$$

$$\text{Lower 30\%: } 68.42 - 0.52(10.41) = 63.01$$

$$\text{Lower 40\%: } 68.42 - 0.25(10.41) = 65.82$$

$$\text{Mid-score: } 68.42 - 0(10.41) = 68.42$$

$$\text{Upper 40\%: } 68.42 + 0.25(10.41) = 71.02$$

$$\text{Upper 30\%: } 68.42 + 0.52(10.41) = 73.83$$

$$\text{Upper 20\%: } 68.42 + 0.84(10.41) = 77.16$$

$$\text{Upper 10\%: } 68.42 + 1.28(10.41) = 81.74$$

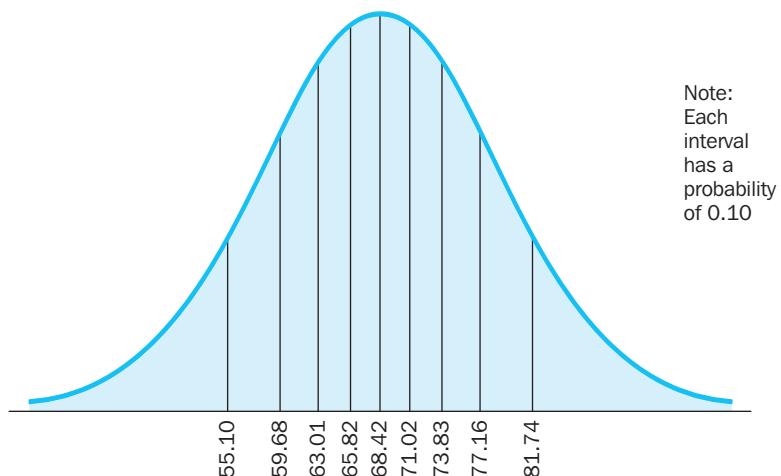
These cutoff or interval boundary points are identified on the graph in Figure 12.1.



PHARMACO

**FIGURE 12.1**

Normal distribution for the Pharmaco example with ten equal-probability intervals



**TABLE 12.10** Observed and expected frequencies for Pharmaco job applicant test scores, and computation of the chi-squared test statistic

| Test score interval | Observed frequency ( $f_i$ ) | Expected frequency ( $e_i$ ) | Difference ( $f_i - e_i$ ) | Squared difference $(f_i - e_i)^2$ | Squared difference divided by expected frequency $(f_i - e_i)^2/e_i$ |
|---------------------|------------------------------|------------------------------|----------------------------|------------------------------------|--|
| Less than 55.10     | 5                            | 5                            | 0                          | 0                                  | 0.0  |
| 55.10 to 59.67      | 5                            | 5                            | 0                          | 0                                  | 0.0  |
| 59.68 to 63.00      | 9                            | 5                            | 4                          | 16                                 | 3.2  |
| 63.01 to 65.81      | 6                            | 5                            | 1                          | 1                                  | 0.2  |
| 65.82 to 68.41      | 2                            | 5                            | 3                          | 9                                  | 1.8  |
| 68.42 to 71.01      | 5                            | 5                            | 0                          | 0                                  | 0.0  |
| 71.02 to 73.82      | 2                            | 5                            | 3                          | 9                                  | 1.8  |
| 73.83 to 77.15      | 5                            | 5                            | 0                          | 0                                  | 0.0  |
| 77.16 to 81.73      | 5                            | 5                            | 0                          | 0                                  | 0.0  |
| 81.74 and over      | 6                            | 5                            | 1                          | 1                                  | 0.2  |
| Total               | 50                           | 50                           |                            |                                    | $\chi^2 = 7.2$   |

We can now return to the sample data of Table 12.9 and determine the observed frequencies for the categories. The results are in Table 12.10. The goodness of fit calculations now proceed exactly as before. Namely, we compare the observed and expected results by computing a  $\chi^2$  value. The computations are also shown in Table 12.10. We see that the value of the test statistic is  $\chi^2 = 7.2$ .

To determine whether the computed  $\chi^2$  value of 7.2 is large enough to reject  $H_0$ , we need to refer to the appropriate chi-squared distribution tables. Using the rule for computing the number of degrees of freedom for the goodness of fit test, we have  $k - p - 1 = 10 - 2 - 1 = 7$  degrees of freedom based on  $k = 10$  categories and  $p = 2$  parameters (mean and standard deviation) estimated from the sample data.

Suppose we do the test with a 0.10 level of significance. To test this hypothesis, we need to determine the  $p$ -value for the test statistic  $\chi^2 = 7.2$  by finding the area in the upper tail of a chi-squared distribution with 7 degrees of freedom. Using Table 3 of Appendix B, we find that  $\chi^2 = 7.2$  provides an area in the upper tail greater than 0.10. So we know that the  $p$ -value is greater than 0.10. EXCEL, IBM SPSS or MINITAB shows  $p$ -value = 0.4084.

With  $p$ -value  $> \alpha = 0.10$ , the hypothesis that the probability distribution for the Pharmaco job applicant test scores is a normal distribution cannot be rejected. The normal distribution may be applied to assist in the interpretation of test scores.

A summary of the goodness fit test for a normal distribution follows.

### Normal distribution goodness of fit test: a summary

1. State the null and alternative hypotheses.

$H_0$ : The population has a normal distribution

$H_1$ : The population does not have a normal distribution

2. Select a random sample and

- a. Compute the sample mean and sample standard deviation.
- b. Define intervals of values so that the expected frequency is at least five for each interval. Using equal probability intervals is a good approach.
- c. Record the observed frequency of data values  $f_i$  in each interval defined.

3. Compute the expected number of occurrences  $e_i$  for each interval of values defined in step 2(b). Multiply the sample size by the probability of a normal random variable being in the interval.

4. Compute the value of the test statistic.

$$\chi^2 = \sum_{i=1}^k \frac{(f_j - e_j)^2}{e_j}$$

5. Rejection rule:

$p$ -value approach: Reject  $H_0$  if  $p$ -value  $\leq \alpha$

Critical value approach: Reject  $H_0$  if  $\chi^2 \geq \chi^2_\alpha$

where  $\alpha$  is the level of significance for the test, and there are  $k - 3$  degrees of freedom.

## EXERCISES

### Methods

17. The following data are believed to have come from a normal distribution. Use a goodness of fit test with  $\alpha = 0.05$  to test this claim.

17    23    22    24    19    23    18    22    20    13    11    21    18    20    21  
21    18    15    24    23    23    43    29    27    26    30    28    33    23    29

18. Data on the number of occurrences per time period and observed frequencies follow. Use a goodness of fit test with  $\alpha = 0.05$  to see whether the data fit a Poisson distribution.


**COMPLETE  
SOLUTIONS**

| <i>Number of occurrences</i> | <i>Observed frequency</i> |
|------------------------------|---------------------------|
| 0                            | 39                        |
| 1                            | 30                        |
| 2                            | 30                        |
| 3                            | 18                        |
| 4                            | 3                         |

### Applications

19. The number of incoming phone calls to a small call centre in Mumbai, during one-minute intervals, is believed to have a Poisson distribution. Use  $\alpha = 0.10$  and the following data to test the assumption that the incoming phone calls follow a Poisson distribution.

| <i>Number of incoming phone calls<br/>during a one-minute interval</i> | <i>Observed frequency</i> |
|--|---------------------------|
| 0  | 15                        |
| 1  | 31                        |
| 2  | 20                        |
| 3  | 15                        |
| 4  | 13                        |
| 5  | 4                         |
| 6  | 2                         |
| Total  | 100                       |

20. The weekly demand for a particular product in a white-goods store is thought to be normally distributed. Use a goodness of fit test and the following data to test this assumption. Use  $\alpha = 0.10$ . The sample mean is 24.5 and the sample standard deviation is 3.0.

|    |    |    |    |    |    |    |    |    |    |
|----|----|----|----|----|----|----|----|----|----|
| 18 | 20 | 22 | 27 | 22 | 25 | 22 | 27 | 25 | 24 |
| 26 | 23 | 20 | 24 | 26 | 27 | 25 | 19 | 21 | 25 |
| 26 | 25 | 31 | 29 | 25 | 25 | 28 | 26 | 28 | 24 |


**COMPLETE  
SOLUTIONS**

21. A random sample of final examination grades for a college course in Middle-East studies follows.

|    |    |    |    |    |    |    |    |    |    |
|----|----|----|----|----|----|----|----|----|----|
| 55 | 85 | 72 | 99 | 48 | 71 | 88 | 70 | 59 | 98 |
| 80 | 74 | 93 | 85 | 74 | 82 | 90 | 71 | 83 | 60 |
| 95 | 77 | 84 | 73 | 63 | 72 | 95 | 79 | 51 | 85 |
| 76 | 81 | 78 | 65 | 75 | 87 | 86 | 70 | 80 | 64 |

Using  $\alpha = 0.05$ , determine whether a normal distribution should be rejected as being representative of the population's distribution of grades.

22. The number of car accidents per day in a particular city is believed to have a Poisson distribution. A sample of 80 days during the past year gives the following data. Do these data support the belief that the number of accidents per day has a Poisson distribution? Use  $\alpha = 0.05$ .

| <i>Number of accidents</i> | <i>Observed frequency (days)</i> |
|----------------------------|----------------------------------|
| 0                          | 34                               |
| 1                          | 25                               |
| 2                          | 11                               |
| 3                          | 7                                |
| 4                          | 3                                |



## ONLINE RESOURCES

For the data files, online summary, additional questions and answers, and software section, go to the online platform.

## SUMMARY

The purpose of a goodness of fit test is to determine whether a hypothesized probability distribution can be used as a model for a particular population of interest. The computations for the goodness of fit test involve comparing observed frequencies from a sample with expected frequencies when the hypothesized probability distribution is assumed true. A chi-squared distribution is used to determine whether the differences between observed and expected frequencies are large enough to reject the hypothesized probability distribution.

In this chapter we introduced the goodness of fit test for a multinomial distribution. A test of independence for two variables is an extension of the methodology used in the goodness of fit test for a multinomial population. A contingency table is used to set out the observed and expected frequencies. Then a chi-squared value is computed.

We also illustrated the goodness of fit test for Poisson and normal distributions.

## KEY TERMS

**Contingency table**

**Multinomial population**

**Goodness of fit test**

## KEY FORMULAE

### Test statistic for goodness of fit

$$\chi^2 = \sum_{i=1}^k \frac{(f_i - e_i)^2}{e_i} \quad (12.1)$$

### Expected frequencies for contingency tables under the assumption of independence

$$e_{ij} = \frac{(\text{Row } i \text{ Total})(\text{Column } j \text{ Total})}{\text{Sample Size}} \quad (12.2)$$

### Test statistic for independence

$$\chi^2 = \sum_i \sum_j \frac{(f_{ij} - e_{ij})^2}{e_{ij}} \quad (12.3)$$

## CASE PROBLEM 1



### Evaluation of Management School website pages

A group of MSc students at an international university conducted a survey to assess the students' views regarding the web pages of the university's Management School. Among the questions in the survey were items that asked respondents to express agreement or disagreement with the following statements.

1. The Management School web pages are attractive for prospective students.
2. I find it easy to navigate the Management School web pages.
3. There is up-to-date information about courses on the Management School web pages.
4. If I were to recommend the university to someone else, I would suggest that they go to the Management School web pages.

Responses were originally given on a five-point scale, but in the data file on the online platform ('Web Pages'), the responses have been re-coded as binary variables. For each questionnaire item, those who agreed or agreed strongly with the statement have been grouped into one category (Agree). Those who disagreed, disagreed strongly, were indifferent or opted for a 'Don't know' response, have been grouped into a second category (Don't Agree). The data file also contains particulars of respondent gender and level of study (undergraduate or postgraduate). The first few rows of the data file are shown below.



### Managerial report

1. Use descriptive statistics to summarize the data from this study. What are your preliminary conclusions about the independence of the response (Agree or Don't Agree) and gender for each of the four items? What are your preliminary conclusions about the independence of the response (Agree or Don't Agree) and level of study for each of the four items?
2. With regard to each of the four items, test for the independence of the response (Agree or Don't Agree) and gender. Use  $\alpha = 0.05$ .
3. With regard to each of the four items, test for the independence of the response (Agree or Don't Agree) and level of study. Use  $\alpha = 0.05$ .
4. Does it appear that views regarding the web pages are consistent for students of both genders and both levels of study? Explain.



WEB  
PAGES

| Gender | Study level   | Attractiveness | Navigation  | Up-to-date  | Referrals   |
|--------|---------------|----------------|-------------|-------------|-------------|
| Female | Undergraduate | Don't Agree    | Agree       | Agree       | Agree       |
| Female | Undergraduate | Agree          | Agree       | Agree       | Agree       |
| Male   | Undergraduate | Don't Agree    | Don't Agree | Don't Agree | Don't Agree |
| Male   | Undergraduate | Agree          | Agree       | Agree       | Agree       |
| Male   | Undergraduate | Agree          | Agree       | Agree       | Agree       |
| Female | Undergraduate | Don't Agree    | Don't Agree | Agree       | Agree       |
| Male   | Undergraduate | Don't Agree    | Agree       | Agree       | Agree       |
| Male   | Undergraduate | Agree          | Agree       | Agree       | Agree       |
| Male   | Undergraduate | Don't Agree    | Agree       | Agree       | Agree       |
| Male   | Undergraduate | Don't Agree    | Don't Agree | Don't Agree | Agree       |

## CASE PROBLEM 2



LOTTO

### Checking for randomness in Lotto draws

In the main Lotto game of the UK National Lottery, six balls are randomly selected from a set of balls numbered 1, 2, ..., 49. The file ‘Lotto’ on the online platform contains details of the numbers drawn in the main Lotto game from January 2007 up to early July 2012 (Wednesdays and Saturdays each week).

The first few rows of the data file are shown below. In addition to showing the six numbers drawn in the game each time, and the order in which they were drawn, the file also gives details of the day on which the draw took place, the machine that was used to do the draw, and the set of balls that was used. A number of similar machines are used for the draws: Sapphire, Amethyst, etc., and eight sets of balls are used.

### Analyst's report

1. Use an appropriate hypothesis test to assess whether there is any evidence of non-randomness in the first ball drawn. Similarly, test for non-randomness in the second ball drawn, third ball drawn, ..., sixth ball drawn.
2. Use an appropriate hypothesis test to assess whether there is any evidence of non-randomness overall in the drawing of the 49 numbers (regardless of the order of selection).
3. Use an appropriate hypothesis test to assess whether there is evidence of any dependence between the numbers drawn and the day on which the draw is made.
4. Use an appropriate hypothesis test to assess whether there is evidence of any dependence between the numbers drawn and the machine on which the draw is made.
5. Use an appropriate hypothesis test to assess whether there is evidence of any dependence between the numbers drawn and the set of balls that is used.



| No.  | Day | DD | MMM | YYYY | N1 | N2 | N3 | N4 | N5 | N6 | Machine   | Set |
|------|-----|----|-----|------|----|----|----|----|----|----|-----------|-----|
| 1732 | Sat | 28 | Jul | 2012 | 7  | 10 | 22 | 29 | 43 | 44 | Guinevere | 1   |
| 1731 | Wed | 25 | Jul | 2012 | 8  | 14 | 15 | 22 | 41 | 48 | Lancelot  | 3   |
| 1730 | Sat | 21 | Jul | 2012 | 5  | 14 | 20 | 40 | 41 | 42 | Lancelot  | 2   |
| 1729 | Wed | 18 | Jul | 2012 | 11 | 26 | 34 | 38 | 40 | 46 | Guinevere | 4   |
| 1728 | Sat | 14 | Jul | 2012 | 13 | 27 | 29 | 42 | 43 | 46 | Guinevere | 4   |
| 1727 | Wed | 11 | Jul | 2012 | 12 | 19 | 28 | 29 | 38 | 49 | Guinevere | 1   |
| 1726 | Sat | 7  | Jul | 2012 | 5  | 22 | 23 | 30 | 33 | 45 | Guinevere | 4   |
| 1725 | Wed | 4  | Jul | 2012 | 3  | 9  | 14 | 19 | 34 | 38 | Guinevere | 3   |
| 1724 | Sat | 30 | Jun | 2012 | 1  | 18 | 19 | 24 | 30 | 38 | Guinevere | 6   |
| 1723 | Wed | 27 | Jun | 2012 | 10 | 22 | 29 | 39 | 46 | 47 | Guinevere | 4   |

# 13

# Experimental Design and Analysis of Variance



## CHAPTER CONTENTS

Statistics in Practice Product customization and manufacturing trade-offs

- 13.1 An introduction to experimental design and analysis of variance
- 13.2 Analysis of variance and the completely randomized design
- 13.3 Multiple comparison procedures
- 13.4 Randomized block design
- 13.5 Factorial experiments

**LEARNING OBJECTIVES** After reading this chapter and doing the exercises, you should be able to:

- 1 Understand the basics of experimental design and how the analysis of variance procedure can be used to determine if the means of more than two populations are equal.
- 2 Know the assumptions necessary to use the analysis of variance procedure.
- 3 Understand the use of the  $F$  distribution in performing the analysis of variance procedure.
- 4 Know how to set up an ANOVA table and interpret the entries in the table.
- 5 Use output from computer software packages to solve analysis of variance problems.
- 6 Know how to use Fisher's least significant difference (LSD) procedure and Fisher's LSD with the Bonferroni adjustment to conduct statistical comparisons between pairs of population means.
- 7 Understand the difference between a completely randomized design, a randomized block design and factorial experiments.
- 8 Know the definition of the following terms: comparisonwise Type I error rate; experimentwise Type I error rate; factor; level; treatment; partitioning; blocking; main effect; interaction; replication.

In Chapter 1 we stated that statistical studies can be classified as either experimental or observational. In an experimental statistical study, an experiment is conducted to generate the data. An experiment begins with identifying a variable of interest. Then one or more other variables, thought to be related, are identified and controlled, and data are collected about how those variables influence the variable of interest.



## STATISTICS IN PRACTICE

Product customization and manufacturing trade-offs

The analysis of variance technique was used recently in a study to investigate trade-offs between product customization and other manufacturing priorities. A total of 102 UK manufacturers from eight industrial sectors were involved in the research. Three levels of customization were considered: full customization where customer input was incorporated at the product design or fabrication stages; partial customization with customer input incorporated into product assembly or delivery stages and standard products which did not incorporate any customer input at all.

The impact of customization was considered against four competitive imperatives – cost, quality, delivery and volume flexibility.

It was found that customization had a significant effect on delivery (both in terms of speed and lead times); also on manufacturers' costs – although not design, component, delivery and servicing costs.



The findings suggest that customization is not cost-free and that the advent of mass customization is unlikely to see the end of trade-offs with other key priorities.

Source: Squire, B., Brown, S., Readman, J. and Bessant, J. (2005) 'The impact of mass customization on manufacturing trade-offs'. *Production and Operations Management Journal* 15(1): 10–21

In an observational study, data are usually obtained through sample surveys and not a controlled experiment. Good design principles are still employed, but the rigorous controls associated with an experimental statistical study are often not possible.

For instance, in a study of the relationship between smoking and lung cancer the researcher cannot assign a smoking habit to subjects. The researcher is restricted to simply observing the effects of smoking on people who already smoke and the effects of not smoking on people who do not already smoke.

In this chapter we introduce three types of experimental designs: a completely randomized design, a randomized block design and a factorial experiment. For each design we show how a statistical procedure called analysis of variance (ANOVA) can be used to analyze the data available. ANOVA can also be used to analyze the data obtained through an observation study. For instance, we will see that the ANOVA procedure used for a completely randomized experimental design also works for testing the equality of three or more population means when data are obtained through an observational study. In the following chapters we will see that ANOVA plays a key role in analyzing the results of regression studies involving both experimental and observational data.

In the first section, we introduce the basic principles of an experimental study and show how they are employed in a completely randomized design. In the second section, we then show how ANOVA can be used to analyze the data from a completely randomized experimental design. In later sections we discuss multiple comparison procedures and two other widely used experimental designs: the randomized block design and the factorial experiment.

## 13.1 AN INTRODUCTION TO EXPERIMENTAL DESIGN AND ANALYSIS OF VARIANCE

As an example of an experimental statistical study, let us consider the problem facing the Chemitech company. Chemitech developed a new filtration system for municipal water supplies.

The components for the new filtration system will be purchased from several suppliers, and Chemitech will assemble the components at its plant in North Saxony. The industrial engineering group is responsible for determining the best assembly method for the new filtration system. After considering a variety of possible approaches, the group narrows the alternatives to three: method A, method B and method C. These methods differ in the sequence of steps used to assemble the system. Managers at Chemitech want to determine which assembly method can produce the greatest number of filtration systems per week.

In the Chemitech experiment, assembly method is the independent variable or **factor**. Because three assembly methods correspond to this factor, we say that three treatments are associated with this experiment; each **treatment** corresponds to one of the three assembly methods. The Chemitech problem is an example of a **single-factor experiment**; it involves one qualitative factor (method of assembly). More complex experiments may consist of multiple factors; some factors may be qualitative and others may be quantitative.

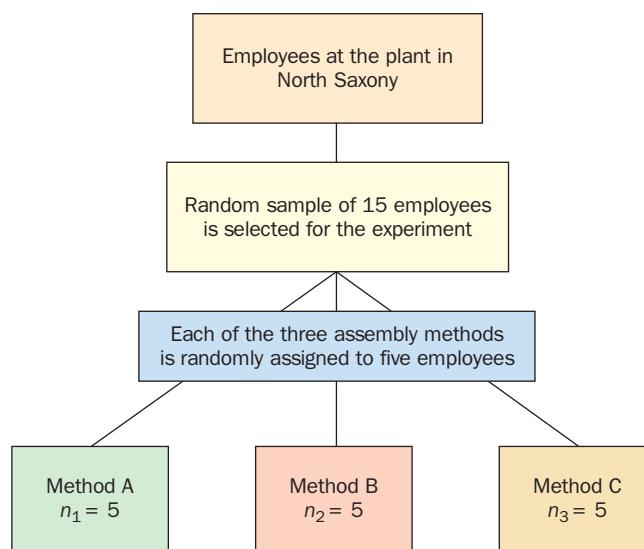
The three assembly methods or treatments define the three populations of interest for the Chemitech experiment. One population is all Chemitech employees who use assembly method A, another is those who use method B and the third is those who use method C. Note that for each population the dependent or **response variable** is the number of filtration systems assembled per week, and the primary statistical objective of the experiment is to determine whether the mean number of units produced per week is the same for all three populations (methods).

Suppose a random sample of three employees is selected from all assembly workers at the Chemitech production facility. In experimental design terminology, the three randomly selected workers are the **experimental units**. The experimental design that we will use for the Chemitech problem is called a **completely randomized design**. This type of design requires that each of the three assembly methods or treatments be assigned randomly to one of the experimental units or workers. For example, method A might be randomly assigned to the second worker, method B to the first worker and method C to the third worker. The concept of *randomization*, as illustrated in this example, is an important principle of all experimental designs.

Note that this experiment would result in only one measurement or number of units assembled for each treatment. To obtain additional data for each assembly method, we must repeat or replicate the basic experimental process. Suppose, for example, that instead of selecting just three workers at random we selected 15 workers and then randomly assigned each of the three treatments to five of the workers. Because each method of assembly is assigned to five workers, we say that five replicates have been obtained. The process of *replication* is another important principle of experimental design. Figure 13.1 shows the completely randomized design for the Chemitech experiment.

**FIGURE 13.1**

Completely randomized design for evaluating the Chemitech assembly method experiment



## Data collection

Once we are satisfied with the experimental design, we proceed by collecting and analyzing the data. In the Chemitech case, the employees would be instructed in how to perform the assembly method assigned to them and then would begin assembling the new filtration systems using that method. After this assignment and training, the number of units assembled by each employee during one week is as shown in Table 13.1. The sample means, sample variances and sample standard deviations for each assembly method are also provided. Therefore, the sample mean number of units produced using method A is 62; the sample mean using method B is 66; and the sample mean using method C is 52. From these data, method B appears to result in higher production rates than either of the other methods.

The real issue is whether the three sample means observed are different enough for us to conclude that the means of the populations corresponding to the three methods of assembly are different. To write this question in statistical terms, we introduce the following notation.

$\mu_1$  = mean number of units produced per week using method A

$\mu_2$  = mean number of units produced per week using method B

$\mu_3$  = mean number of units produced per week using method C

Although we will never know the actual values of  $\mu_1$ ,  $\mu_2$  and  $\mu_3$ , we want to use the sample means to test the following hypotheses.



CHEMITECH

$$H_0: \mu_1 = \mu_2 = \mu_3$$

$H_1$ : Not all population means are equal

As we will demonstrate shortly, analysis of variance (ANOVA) is the statistical procedure used to determine whether the observed differences in the three sample means are large enough to reject  $H_0$ .

## Assumptions for analysis of variance

Three assumptions are required to use analysis of variance.

- 1 For each population, the response variable is normally distributed.** Implication: In the Chemitech experiment the number of units produced per week (response variable) must be normally distributed for each assembly method.
- 2 The variance of the response variable, denoted  $\sigma^2$ , is the same for all of the populations.** Implication: In the Chemitech experiment, the variance of the number of units produced per week must be the same for each assembly method.
- 3 The observations must be independent.** Implication: In the Chemitech experiment, the number of units produced per week for each employee must be independent of the number of units produced per week for any other employee.

**TABLE 13.1** Number of units produced by 15 workers

|                           | Method |       |       |
|---------------------------|--------|-------|-------|
|                           | A      | B     | C     |
| 58                        | 58     | 58    | 48    |
| 64                        | 69     | 69    | 57    |
| 55                        | 71     | 71    | 59    |
| 66                        | 64     | 64    | 47    |
| 67                        | 68     | 68    | 49    |
| Sample mean               | 62     | 66    | 52    |
| Sample variance           | 27.5   | 26.5  | 31.0  |
| Sample standard deviation | 5.244  | 5.148 | 5.568 |

## Analysis of variance: a conceptual overview

If the means for the three populations are equal, we would expect the three sample means to be close together. In fact, the closer the three sample means are to one another, the more evidence we have for the conclusion that the population means are equal. Alternatively, the more the sample means differ, the more evidence we have for the conclusion that the population means are not equal. In other words, if the variability among the sample means is ‘small’ it supports  $H_0$ ; if the variability among the sample means is ‘large’ it supports  $H_1$ .

If the null hypothesis,  $H_0: \mu_1 = \mu_2 = \mu_3$ , is true, we can use the variability among the sample means to develop an estimate of  $\sigma^2$ . First, note that if the assumptions for analysis of variance are satisfied, each sample will have come from the same normal distribution with mean  $\mu$  and variance  $\sigma^2$ . Recall from Chapter 7 that the sampling distribution of the sample mean  $\bar{x}$  for a simple random sample of size  $n$  from a normal population will be normally distributed with mean  $\mu$  and variance  $\sigma^2/n$ . Figure 13.2 illustrates such a sampling distribution.

Therefore, if the null hypothesis is true, we can think of each of the three sample means,  $\bar{x}_1 = 62$ ,  $\bar{x}_2 = 66$  and  $\bar{x}_3 = 52$  from Table 13.1, as values drawn at random from the sampling distribution shown in Figure 13.2. In this case, the mean and variance of the three  $\bar{x}$  values can be used to estimate the mean and variance of the sampling distribution. When the sample sizes are equal, as in the Chemitech experiment, the best estimate of the mean of the sampling distribution of  $\bar{x}$  is the mean or average of the sample means. Thus, in the Chemitech experiment, an estimate of the mean of the sampling distribution of  $\bar{x}$  is  $(62 + 66 + 52)/3 = 60$ . We refer to this estimate as the *overall sample mean*. An estimate of the variance of the sampling distribution of  $\bar{x}$ ,  $s_{\bar{x}}^2$ , is provided by the variance of the three sample means:

$$s_{\bar{x}}^2 = \frac{(62 - 60)^2 + (66 - 60)^2 + (52 - 60)^2}{3 - 1} = \frac{104}{2} = 52$$

Because  $\sigma_{\bar{x}}^2 = \sigma^2/n$ , solving for  $\sigma^2$  gives:

$$\sigma^2 = n\sigma_{\bar{x}}^2$$

Hence,

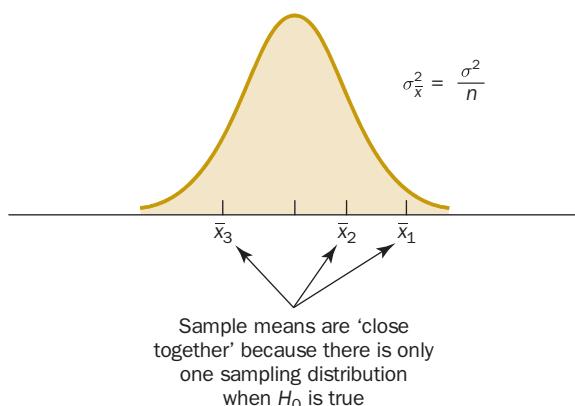
$$\text{Estimate of } \sigma^2 = n \text{ (Estimate of } \sigma_{\bar{x}}^2) = ns_{\bar{x}}^2 = 5(52) = 260$$

The result,  $ns_{\bar{x}}^2 = 260$ , is referred to as the *between-treatments estimate of  $\sigma^2$* .

The between-treatments estimate of  $\sigma^2$  is based on the assumption that the null hypothesis is true. In this case, each sample comes from the same population, and there is only one sampling distribution of  $\bar{X}$ . To illustrate what happens when  $H_0$  is false, suppose the population means all differ. Note that because the three samples are from normal populations with different means, they will result in three different sampling distributions. Figure 13.3 shows that, in this case, the sample means are not as close together as they were when  $H_0$  was true. Therefore,  $s_{\bar{x}}^2$  will be larger, causing the between-treatments estimate of  $\sigma^2$  to be larger.

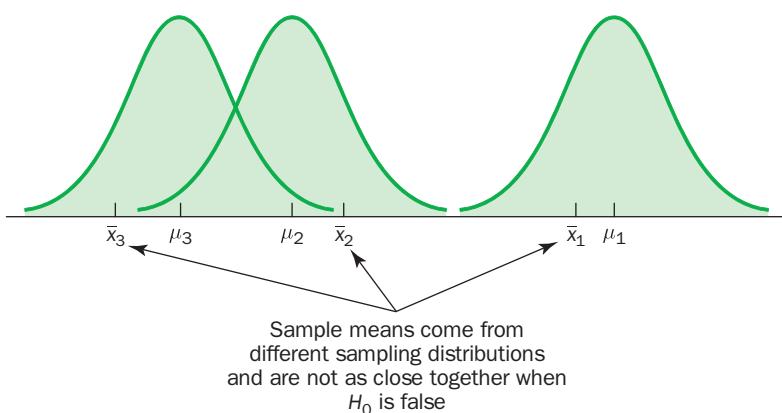
**FIGURE 13.2**

Sampling distribution of  $\bar{X}$  given  $H_0$  is true



**FIGURE 13.3**

Sampling distribution of  $\bar{X}$   
given  $H_0$  is false



In general, when the population means are not equal, the between-treatments estimate will overestimate the population variance  $\sigma^2$ .

The variation within each of the samples also has an effect on the conclusion we reach in analysis of variance. When a simple random sample is selected from each population, each of the sample variances provides an unbiased estimate of  $\sigma^2$ . Hence, we can combine or pool the individual estimates of  $\sigma^2$  into one overall estimate. The estimate of  $\sigma^2$  obtained in this way is called the *pooled* or *within-treatments* estimate of  $\sigma^2$ . Because each sample variance provides an estimate of  $\sigma^2$  based only on the variation within each sample, the within-treatments estimate of  $\sigma^2$  is not affected by whether the population means are equal.

When the sample sizes are equal, the within-treatments estimate of  $\sigma^2$  can be obtained by computing the average of the individual sample variances. For the Chemitech experiment we obtain:

$$\text{Within-treatments estimate of } \sigma^2 = \frac{27.5 + 26.5 + 31.0}{3} = \frac{85}{3} = 28.33$$

In the Chemitech experiment, the between-treatments estimate of  $\sigma^2$  (260) is much larger than the within-treatments estimate of  $\sigma^2$  (28.33). In fact, the ratio of these two estimates is  $260/28.33 = 9.18$ . Recall, however, that the between-treatments approach provides a good estimate of  $\sigma^2$  only if the null hypothesis is true; if the null hypothesis is false, the between-treatments approach overestimates  $\sigma^2$ . The within-treatments approach provides a good estimate of  $\sigma^2$  in either case. Therefore, if the null hypothesis is true, the two estimates will be similar and their ratio will be close to 1. If the null hypothesis is false, the between-treatments estimate will be larger than the within-treatments estimate, and their ratio will be large. In the next section we will show how large this ratio must be to reject  $H_0$ .

In summary, the logic behind ANOVA is based on the development of two independent estimates of the common population variance  $\sigma^2$ . One estimate of  $\sigma^2$  is based on the variability among the sample means themselves, and the other estimate of  $\sigma^2$  is based on the variability of the data within each sample. By comparing these two estimates of  $\sigma^2$ , we will be able to determine whether the population means are equal.

## 13.2 ANALYSIS OF VARIANCE AND THE COMPLETELY RANDOMIZED DESIGN

In this section we show how analysis of variance can be used to test for the equality of  $k$  population means for a completely randomized design. The general form of the hypotheses tested is:

$$H_0: \mu_1 = \mu_2 = \dots = \mu_k$$

$$H_1: \text{Not all population means are equal}$$

Where:

$\mu_j$  = mean of the  $j$ th population

We assume that a simple random sample of size  $n_j$  has been selected from each of the  $k$  populations or treatments. For the resulting sample data, let:

- $x_{ij}$  = value of observation  $i$  for treatment  $j$
- $n_j$  = number of observations for treatment  $j$
- $\bar{x}_j$  = sample mean for treatment  $j$
- $s_j^2$  = sample variance for treatment  $j$
- $s_j$  = sample standard deviation for treatment  $j$

The formulae for the sample mean and sample variance for treatment  $j$  are as follows:

#### Testing for the Equality of $k$ Population means sample mean for Treatment $j$

$$\bar{x}_j = \frac{\sum_{i=1}^{n_j} x_{ij}}{n_j} \quad (13.1)$$

#### Sample Variance for Treatment $j$

$$s_j^2 = \frac{\sum_{i=1}^{n_j} (x_{ij} - \bar{x}_j)^2}{n_j - 1} \quad (13.2)$$

The overall sample mean, denoted  $\bar{\bar{x}}$ , is the sum of all the observations divided by the total number of observations. That is,

#### Overall Sample Mean

$$\bar{\bar{x}} = \frac{\sum_{j=1}^k \sum_{i=1}^{n_j} x_{ij}}{n_T} \quad (13.3)$$

where:

$$n_T = n_1 + n_2 + \dots + n_k \quad (13.4)$$

If the size of each sample is  $n$ ,  $n_T = kn$ ; in this case equation (13.3) reduces to:

$$\bar{\bar{x}} = \frac{\sum_{j=1}^k \sum_{i=1}^{n_j} x_{ij}}{kn} = \frac{\sum_{j=1}^k \sum_{i=1}^{n_j} x_{ij}/n}{k} = \frac{\sum_{j=1}^k \bar{x}_j}{k} \quad (13.5)$$

In other words, whenever the sample sizes are the same, the overall sample mean is just the average of the  $k$  sample means.

Because each sample in the Chemitech experiment consists of  $n = 5$  observations, the overall sample mean can be computed by using equation (13.5). For the data in Table 13.1 we obtained the following result.

$$\bar{\bar{x}} = \frac{62 + 66 + 52}{3} = 60$$

If the null hypothesis is true ( $\mu_1 = \mu_2 = \mu_3 = \mu$ ), the overall sample mean of 60 is the best estimate of the population mean  $\mu$ .

## Between-treatments estimate of population variance

In the preceding section, we introduced the concept of a between-treatments estimate of  $\sigma^2$  and showed how to compute it when the sample sizes were equal. This estimate of  $\sigma^2$  is called the *mean square due to treatments* and is denoted MSTR. The general formula for computing MSTR is:

$$\text{MSTR} = \frac{\sum_{j=1}^k n_j(\bar{x}_j - \bar{x})^2}{k-1} \quad (13.6)$$

The numerator in equation (13.6) is called the *sum of squares due to treatments* and is denoted SSTR. The denominator,  $k - 1$ , represents the degrees of freedom associated with SSTR. Hence, the mean square due to treatments can be computed using the following formula.

### Mean square due to treatments

$$\text{MSTR} = \frac{\text{SSTR}}{k-1} \quad (13.7)$$

where:

$$\text{SSTR} = \sum_{j=1}^k n_j(\bar{x}_j - \bar{x})^2 \quad (13.8)$$

If  $H_0$  is true, MSTR provides an unbiased estimate of  $\sigma^2$ . However, if the means of the  $k$  populations are not equal, MSTR is not an unbiased estimate of  $\sigma^2$ ; in fact, in that case, MSTR should overestimate  $\sigma^2$ .

For the Chemitech data in Table 13.1, we obtain the following results.

$$\text{SSTR} = \sum_{j=1}^k n_j(\bar{x}_j - \bar{x})^2 = 5(62 - 60)^2 + 5(66 - 60)^2 + 5(52 - 60)^2 = 520$$

$$\text{MSTR} = \frac{\text{SSTR}}{k-1} = \frac{520}{2} = 260$$

## Within-treatments estimate of population variance

Earlier we introduced the concept of a within-treatments estimate of  $\sigma^2$  and showed how to compute it when the sample sizes were equal. This estimate of  $\sigma^2$  is called the *mean square due to error* and is denoted MSE. The general formula for computing MSE is:

$$\text{MSE} = \frac{\sum_{j=1}^k (n_j - 1)s_j^2}{n_T - k} \quad (13.9)$$

The numerator in equation (13.9) is called the *sum of squares due to error* and is denoted SSE. The denominator of MSE is referred to as the degrees of freedom associated with SSE. Hence, the formula for MSE can also be stated as follows.

### Mean square due to error

$$\text{MSE} = \frac{\text{SSE}}{n_T - k} \quad (13.10)$$

where:

$$\text{SSE} = \sum_{j=1}^k (n_j - 1)s_j^2 \quad (13.11)$$

Note that MSE is based on the variation within each of the treatments; it is not influenced by whether the null hypothesis is true. Therefore, MSE always provides an unbiased estimate of  $\sigma^2$ .

For the Chemitech data in Table 13.1 we obtain the following results.

$$\begin{aligned}\text{SSE} &= \sum_{j=1}^k (n_j - 1)s_j^2 = (5 - 1)27.5 + (5 - 1)26.5 + (5 - 1)31 = 340 \\ \text{MSE} &= \frac{\text{SSE}}{n_T - k} = \frac{340}{15 - 3} = \frac{340}{12} = 28.33\end{aligned}$$

## Comparing the variance estimates: the *F* test

If the null hypothesis is true, MSTR and MSE provide two independent, unbiased estimates of  $\sigma^2$ . Based on the material covered in Chapter 11 we know that, for normal populations, the sampling distribution of the ratio of two independent estimates of  $\sigma^2$  follows an *F* distribution. Hence, if the null hypothesis is true and the ANOVA assumptions are valid, the sampling distribution of MSTR/MSE is an *F* distribution with numerator degrees of freedom equal to  $k - 1$  and denominator degrees of freedom equal to  $n_T - k$ . In other words, if the null hypothesis is true, the value of MSTR/MSE should appear to have been selected from this *F* distribution.

However, if the null hypothesis is false, the value of MSTR/MSE will be inflated because MSTR overestimates  $\sigma^2$ . Hence, we will reject  $H_0$  if the resulting value of MSTR/MSE appears to be too large to have been selected from an *F* distribution with  $k - 1$  numerator degrees of freedom and  $n_T - k$  denominator degrees of freedom. Because the decision to reject  $H_0$  is based on the value of MSTR/MSE, the test statistic used to test for the equality of  $k$  population means is as follows.

### Test statistic for the equality of $k$ population means

$$F = \frac{\text{MSTR}}{\text{MSE}} \quad (13.12)$$

The test statistic follows an *F* distribution with  $k - 1$  degrees of freedom in the numerator and  $n_T - k$  degrees of freedom in the denominator.

Let us return to the Chemitech experiment and use a level of significance  $\alpha = 0.05$  to conduct the hypothesis test. The value of the test statistic is:

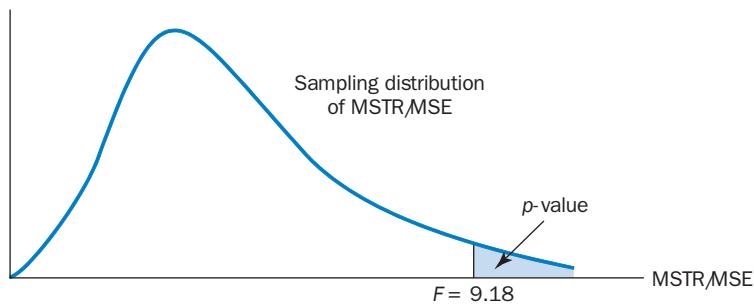
$$F = \frac{\text{MSTR}}{\text{MSE}} = \frac{260}{28.33} = 9.18$$

The numerator degrees of freedom is  $k - 1 = 3 - 1 = 2$  and the denominator degrees of freedom is  $n_T - k = 15 - 3 = 12$ . Because we will only reject the null hypothesis for large values of the test statistic, the *p*-value is the upper tail area of the *F* distribution to the right of the test statistic  $F = 9.18$ . Figure 13.4 shows the sampling distribution of  $F = \text{MSTR}/\text{MSE}$ , the value of the test statistic, and the upper tail area that is the *p*-value for the hypothesis test.

From Table 4 of Appendix B we find the following areas in the upper tail of an *F* distribution with two numerator degrees of freedom and 12 denominator degrees of freedom.

**FIGURE 13.4**

Computation of  $p$ -value using the sampling distribution of MSTR/MSE



| Area in Upper Tail          | .10  | .05  | .025 | .01  |
|-----------------------------|------|------|------|------|
| F Value (df1 = 2, df2 = 12) | 2.81 | 3.89 | 5.10 | 6.93 |
| $F = 9.18$                  |      |      |      |      |

Because  $F = 9.18$  is greater than 6.93, the area in the upper tail at  $F = 9.18$  is less than .01. Therefore, the  $p$ -value is less than .01. MINITAB, EXCEL or SPSS can be used to show that the exact  $p$ -value is .004. With  $p$ -value  $\leq \alpha = 0.05$ ,  $H_0$  is rejected. The test provides sufficient evidence to conclude that the means of the three populations are not equal. In other words, analysis of variance supports the conclusion that the population mean number of units produced per week for the three assembly methods are not equal.

As with other hypothesis testing procedures, the critical value approach may also be used. With  $\alpha = 0.05$ , the critical  $F$  value occurs with an area of 0.05 in the upper tail of an  $F$  distribution with two and 12 degrees of freedom. From the  $F$  distribution table, we find  $F_{.05} = 3.89$ . Hence, the appropriate upper tail rejection rule for the Chemitech experiment is:

$$\text{Reject } H_0 \text{ if } F \geq 3.89$$

With  $F = 9.18$ , we reject  $H_0$  and conclude that the means of the three populations are not equal. A summary of the overall procedure for testing for the equality of  $k$  population means follows.

### Test for the equality of $k$ population means

$$H_0: \mu_1 = \mu_2 = \dots = \mu_k$$

$$H_1: \text{Not all population means are equal}$$

#### Test statistic

$$F = \frac{\text{MSTR}}{\text{MSE}}$$

#### Rejection rule

*p*-value approach: Reject  $H_0$  if  $p$ -value  $\leq \alpha$

Critical value approach: Reject  $H_0$  if  $F \geq F_\alpha$

where the value of  $F_\alpha$  is based on an  $F$  distribution with  $k - 1$  numerator degrees of freedom and  $n_T - k$  denominator degrees of freedom.

## ANOVA table

The results of the preceding calculations can be displayed conveniently in a table referred to as the analysis of variance or **ANOVA table**. The general form of the ANOVA table for a completely randomized design is shown in Table 13.2; Table 13.3 is the corresponding ANOVA table for the Chemitech experiment. The sum of squares associated with the source of variation referred to as 'Total' is called the total sum of squares (SST). Note that the results for the Chemitech experiment suggest that  $SST = SSTR + SSE$ , and that the degrees of freedom associated with this total sum of squares is the sum of the degrees of freedom associated with the sum of squares due to treatments and the sum of squares due to error.

We point out that SST divided by its degrees of freedom  $n_T - 1$  is nothing more than the overall sample variance that would be obtained if we treated the entire set of 15 observations as one data set. With the entire data set as one sample, the formula for computing the total sum of squares, SST, is:

### Total sum of squares

$$SST = \sum_{j=1}^k \sum_{i=1}^{n_j} (x_{ij} - \bar{x})^2 \quad (13.13)$$

It can be shown that the results we observed for the analysis of variance table for the Chemitech experiment also apply to other problems. That is,

### Partitioning of sum of squares

$$SST = SSTR + SSE \quad (13.14)$$

In other words, SST can be partitioned into two sums of squares: the sum of squares due to treatments and the sum of squares due to error.

**TABLE 13.2** ANOVA table for a completely randomized design

| Source of variation | Degrees of freedom | Sum of squares | Mean square               | F                  | p-value |
|---------------------|--------------------|----------------|---------------------------|--------------------|---------|
| Treatments          | $k - 1$            | SSTR           | $MSTR = \frac{SSTR}{k-1}$ | $\frac{MSTR}{MSE}$ |         |
| Error               | $n_T - k$          | SSE            | $MSE = \frac{SSE}{n_T-k}$ |                    |         |
| Total               | $n_T - 1$          | SST            |                           |                    |         |

**TABLE 13.3** Analysis of variance table for the Chemitech experiment

| Source of variation | Degrees of freedom | Sum of squares | Mean square | F    | p-value |
|---------------------|--------------------|----------------|-------------|------|---------|
| Treatments          | 2                  | 520            | 260.00      | 9.18 | .004    |
| Error               | 12                 | 340            | 28.33       |      |         |
| Total               | 14                 | 860            |             |      |         |

Note also that the degrees of freedom corresponding to SST,  $n_T - 1$ , can be partitioned into the degrees of freedom corresponding to SSTR,  $k - 1$ , and the degrees of freedom corresponding to SSE,  $n_T - k$ . The analysis of variance can be viewed as the process of **partitioning** the total sum of squares and the degrees of freedom into their corresponding sources: treatments and error. Dividing the sum of squares by the appropriate degrees of freedom provides the variance estimates, the  $F$  value, and the  $p$ -value used to test the hypothesis of equal population means.

## Computer results for analysis of variance

Using statistical computer packages, analysis of variance computations with large sample sizes or a large number of populations can be performed easily. Appendices 13.1–13.3 show the steps required to use MINITAB, EXCEL and SPSS to perform the analysis of variance computations. In Figure 13.5 we show output for the Chemitech experiment obtained using MINITAB. The first part of the computer output contains the familiar ANOVA table format.

Note that following the ANOVA table the computer output contains the respective sample sizes, the sample means and the standard deviations. In addition, MINITAB provides a figure that shows individual 95 per cent confidence interval estimates of each population mean. In developing these confidence interval estimates, MINITAB uses MSE as the estimate of  $\sigma^2$ . Therefore, the square root of MSE provides the best estimate of the population standard deviation  $\sigma$ . This estimate of  $\sigma$  on the computer output is Pooled StDev; it is equal to 5.323. To provide an illustration of how these interval estimates are developed, we will compute a 95 per cent confidence interval estimate of the population mean for method A.

From our study of interval estimation in Chapter 8, we know that the general form of an interval estimate of a population mean is:

$$\bar{x} \pm t_{\alpha/2} \frac{s}{\sqrt{n}} \quad (13.15)$$

where  $s$  is the estimate of the population standard deviation  $\sigma$ . Because the best estimate of  $\sigma$  is provided by the Pooled StDev, we use a value of 5.323 for  $s$  in expression (13.15). The degrees of freedom for the  $t$  value is 12, the degrees of freedom associated with the error sum of squares. Hence, with  $t_{.025} = 2.179$  we obtain:

$$62 \pm 2.179 \frac{5.323}{\sqrt{5}} = 62 \pm 5.19$$

Therefore, the individual 95 per cent confidence interval for method A goes from  $62 - 5.19 = 56.81$  to  $62 + 5.19 = 67.19$ . Because the sample sizes are equal for the Chemitech experiment, the individual confidence intervals for methods B and C are also constructed by adding and subtracting 5.19 from each sample mean.

| Source               | DF            | SS                 | MS   | F                             | P     |
|----------------------|---------------|--------------------|--|-------------------------------|-------|
| Factor               | 2             | 520.0              | 260.0  | 9.18                          | 0.004 |
| Error                | 12            | 340.0              | 28.3   |                               |       |
| Total                | 14            | 860.0              |  |                               |       |
| S = 5.323            | R-Sq = 60.47% | R-Sq(adj) = 53.88% |  |                               |       |
|                      |               |                    | Individual 95% CIs For Mean Based on<br>Pooled StDev |                               |       |
| Level                | N             | Mean               | StDev  | -----+-----+-----+-----+----- |       |
| A                    | 5             | 62.000             | 5.244  | (-----*-----)                 |       |
| B                    | 5             | 66.000             | 4.148  | (-----*-----)                 |       |
| C                    | 5             | 52.000             | 5.568  | (-----*-----)                 |       |
| Pooled StDev = 5.323 |               |                    | -----+-----+-----+-----+-----                        |                               |       |
|                      |               |                    | 49.0   | 56.0                          | 63.0  |
|                      |               |                    |  |                               | 70.0  |

**FIGURE 13.5**

MINITAB output for the Chemitech experiment analysis of variance

Therefore, in the figure provided by MINITAB we see that the widths of the confidence intervals are the same.

## Testing for the equality of $k$ population means: an observational study

National Computer Products (NCP) manufactures printers and fax machines at plants located in Ayr, Dusseldorf and Stockholm. To measure how much employees at these plants know about quality management, a random sample of six employees was selected from each plant and the employees selected were given a quality awareness examination. The examination scores for these 18 employees are shown in Table 13.4. The sample means, sample variances and sample standard deviations for each group are also provided. Managers want to use these data to test the hypothesis that the mean examination score is the same for all three plants.

We define population 1 as all employees at the Ayr plant, population 2 as all employees at the Dusseldorf plant and population 3 as all employees at the Stockholm plant. Let

$$\mu_1 = \text{mean examination score for population 1}$$

$$\mu_2 = \text{mean examination score for population 2}$$

$$\mu_3 = \text{mean examination score for population 3}$$

Although we will never know the actual values of  $\mu_1$ ,  $\mu_2$  and  $\mu_3$ , we want to use the sample results to test the following hypotheses.

$$H_0: \mu_1 = \mu_2 = \mu_3$$

$$H_1: \text{Not all population means are equal}$$

Note that the hypothesis test for the NCP observational study is exactly the same as the hypothesis test for the Chemitech experiment. Indeed, the same analysis of variance methodology we used to analyze the Chemitech experiment can also be used to analyze the data from the NCP observational study.

Even though the same ANOVA methodology is used for the analysis, it is worth noting how the NCP observational statistical study differs from the Chemitech experimental statistical study. The individuals who conducted the NCP study had no control over how the plants were assigned to individual employees. That is, the plants were already in operation and a particular employee worked at one of the three plants. All that NCP could do was to select a random sample of six employees from each plant and administer the quality awareness examination. To be classified as an experimental study, NCP would have had to be able to randomly select 18 employees and then assign the plants to each employee in a random fashion.

**TABLE 13.4** Examination scores for 18 employees

|                           | Plant 1 Ayr | Plant 2 Dusseldorf | Plant 3 Stockholm |
|---------------------------|-------------|--------------------|-------------------|
|                           | 85          | 71                 | 59                |
|                           | 75          | 75                 | 64                |
|                           | 82          | 73                 | 62                |
|                           | 76          | 74                 | 69                |
|                           | 71          | 69                 | 75                |
|                           | 85          | 82                 | 67                |
| Sample mean               | 79          | 74                 | 66                |
| Sample variance           | 34          | 20                 | 32                |
| Sample standard deviation | 5.83        | 4.47               | 5.66              |



## EXERCISES

### Methods

1. The following data are from a completely randomized design.

|                 | Treatment |       |       |
|-----------------|-----------|-------|-------|
|                 | A         | B     | C     |
|                 | 162       | 142   | 126   |
|                 | 142       | 156   | 122   |
|                 | 165       | 124   | 138   |
|                 | 145       | 142   | 140   |
|                 | 148       | 136   | 150   |
|                 | 174       | 152   | 128   |
| Sample mean     | 156       | 142   | 134   |
| Sample variance | 164.4     | 131.2 | 110.4 |

- a. Compute the sum of squares between treatments.  
 b. Compute the mean square between treatments.  
 c. Compute the sum of squares due to error.  
 d. Compute the mean square due to error.  
 e. Set up the ANOVA table for this problem.  
 f. At the  $\alpha = 0.05$  level of significance, test whether the means for the three treatments are equal.
2. In a completely randomized design, seven experimental units were used for each of the five levels of the factor. Complete the following ANOVA table.

| Source of variation | Sum of squares | Degrees of freedom | Mean square | F | p-value |
|---------------------|----------------|--------------------|-------------|---|---------|
| Treatments          | 300            |                    |             |   |         |
| Error               |                |                    |             |   |         |
| Total               | 460            |                    |             |   |         |

3. Refer to Exercise 2.
- a. What hypotheses are implied in this problem?  
 b. At the  $\alpha = 0.05$  level of significance, can we reject the null hypothesis in part (a)? Explain.
4. In an experiment designed to test the output levels of three different treatments, the following results were obtained:  $SST = 400$ ,  $SSTR = 150$ ,  $n_T = 19$ . Set up the ANOVA table and test for any significant difference between the mean output levels of the three treatments. Use  $\alpha = .05$ .
5. In a completely randomized design, 12 experimental units were used for the first treatment, 15 for the second treatment and 20 for the third treatment. Complete the following analysis of variance. At a 0.05 level of significance, is there a significant difference between the treatments?

| Source of variation | Sum of squares | Degrees of freedom | Mean square | F | p-value |
|---------------------|----------------|--------------------|-------------|---|---------|
| Treatments          | 1200           |                    |             |   |         |
| Error               |                |                    |             |   |         |
| Total               | 1800           |                    |             |   |         |

6. Develop the analysis of variance computations for the following completely randomized design. At  $\alpha = 0.05$ , is there a significant difference between the treatment means?

| <i>Treatment</i> |          |          |
|------------------|----------|----------|
| <i>A</i>         | <i>B</i> | <i>C</i> |
| 136              | 107      | 92       |
| 120              | 114      | 82       |
| 113              | 125      | 85       |
| 107              | 104      | 101      |
| 131              | 107      | 89       |
| 114              | 109      | 117      |
| 129              | 97       | 110      |
| 102              | 114      | 120      |
|                  | 104      | 98       |
|                  | 89       | 106      |
| $\bar{x}_j$      | 119      | 100      |
| $s_j^2$          | 146.86   | 96.44    |
|                  |          | 173.78   |

### Applications

7. To test whether the mean time needed to mix a batch of material is the same for machines produced by three manufacturers, the Jacobs Chemical Company obtained the following data on the time (in minutes) needed to mix the material. Use these data to test whether the population mean times for mixing a batch of material differ for the three manufacturers. Use  $\alpha = 0.05$ .

| <i>Manufacturer</i> |    |    |
|---------------------|----|----|
| 1                   | 2  | 3  |
| 20                  | 28 | 20 |
| 26                  | 26 | 19 |
| 24                  | 31 | 23 |
| 22                  | 27 | 22 |

8. Managers at all levels of an organization need adequate information to perform their respective tasks. One study investigated the effect the source has on the dissemination of information. In this particular study the sources of information were a superior, a peer and a subordinate. In each case, a measure of dissemination was obtained, with higher values indicating greater dissemination of information. Use  $\alpha = 0.05$  and the following data to test whether the source of information significantly affects dissemination. What is your conclusion, and what does it suggest about the use and dissemination of information?

| <i>Superior</i> | <i>Peer</i> | <i>Subordinate</i> |
|-----------------|-------------|--------------------|
| 8               | 6           | 6                  |
| 5               | 6           | 5                  |
| 4               | 7           | 7                  |
| 6               | 5           | 4                  |
| 6               | 3           | 3                  |
| 7               | 4           | 5                  |
| 5               | 7           | 7                  |
| 5               | 6           | 5                  |



EXER6



**COMPLETE  
SOLUTIONS**



**COMPLETE  
SOLUTIONS**



**PAINT**

9. A study investigated the perception of corporate ethical values among individuals specializing in marketing. Use  $\alpha = 0.05$  and the following data (higher scores indicate higher ethical values) to test for significant differences in perception among the three groups.

|   | <i>Marketing managers</i> | <i>Marketing research</i> | <i>Advertising</i> |
|---|---------------------------|---------------------------|--------------------|
| 6 |                           | 5                         | 6                  |
| 5 |                           | 5                         | 7                  |
| 4 |                           | 4                         | 6                  |
| 5 |                           | 4                         | 5                  |
| 6 |                           | 5                         | 6                  |
| 4 |                           | 4                         | 6                  |

10. A study reported in the *Journal of Small Business Management* concluded that self-employed individuals experience higher job stress than individuals who are not self-employed. In this study job stress was assessed with a 15-item scale designed to measure various aspects of ambiguity and role conflict. Ratings for each of the 15 items were made using a scale with 1–5 response options ranging from strong agreement to strong disagreement. The sum of the ratings for the 15 items for each individual surveyed is between 15 and 75, with higher values indicating a higher degree of job stress. Suppose that a similar approach, using a 20-item scale with 1–5 response options, was used to measure the job stress of individuals for 15 randomly selected property agents, 15 architects and 15 stockbrokers. The results obtained follow.

|    | <i>Property agent</i> | <i>Architect</i> | <i>Stockbroker</i> |
|----|-----------------------|------------------|--------------------|
| 81 |                       | 43               | 65                 |
| 48 |                       | 63               | 48                 |
| 68 |                       | 60               | 57                 |
| 69 |                       | 52               | 91                 |
| 54 |                       | 54               | 70                 |
| 62 |                       | 77               | 67                 |
| 76 |                       | 68               | 83                 |
| 56 |                       | 57               | 75                 |
| 61 |                       | 61               | 53                 |
| 65 |                       | 80               | 71                 |
| 64 |                       | 50               | 54                 |
| 69 |                       | 37               | 72                 |
| 83 |                       | 73               | 65                 |
| 85 |                       | 84               | 58                 |
| 75 |                       | 58               | 58                 |

Use  $\alpha = 0.05$  to test for any significant difference in job stress among the three professions.

11. Four different paints are advertised as having the same drying time. To check the manufacturer's claims, five samples were tested for each of the paints. The time in minutes until the paint was dry enough for a second coat to be applied was recorded. The following data were obtained.

|     | <i>Paint 1</i> | <i>Paint 2</i> | <i>Paint 3</i> | <i>Paint 4</i> |
|-----|----------------|----------------|----------------|----------------|
| 128 | 144            | 133            | 150            |                |
| 137 | 133            | 143            | 142            |                |
| 135 | 142            | 137            | 135            |                |
| 124 | 146            | 136            | 140            |                |
| 141 | 130            | 131            | 153            |                |

At the  $\alpha = 0.05$  level of significance, test to see whether the mean drying time is the same for each type of paint.

- 12.** The *Consumer Reports*' Restaurant Customer Satisfaction Survey is based upon 148 599 visits to full-service restaurant chains (Consumer Reports website). One of the variables in the study is meal price, the average amount paid per person for dinner and drinks, minus the tip. Suppose a reporter for the *Sun Coast Times* thought that it would be of interest to their readers to conduct a similar study for restaurants located on the Grand Strand section in Myrtle Beach, South Carolina. The reporter selected a sample of eight seafood restaurants, eight Italian restaurants and eight steakhouses. The following data show the meal prices (\$) obtained for the 24 restaurants sampled. Use  $\alpha = 0.05$  to test whether there is a significant difference among the mean meal price for the three types of restaurants.

| <i>Italian</i> | <i>Seafood</i> | <i>Steakhouse</i> |
|----------------|----------------|-------------------|
| \$12           | \$16           | \$24              |
| 13             | 18             | 19                |
| 15             | 17             | 23                |
| 17             | 26             | 25                |
| 18             | 23             | 21                |
| 20             | 15             | 22                |
| 17             | 19             | 27                |
| 24             | 18             | 31                |



GRANDSTRAND

### 13.3 MULTIPLE COMPARISON PROCEDURES

When we use analysis of variance to test whether the means of  $k$  populations are equal, rejection of the null hypothesis allows us to conclude only that the population means are *not all equal*. In some cases we will want to go a step further and determine where the differences among means occur. The purpose of this section is to show how **multiple comparison procedures** can be used to conduct statistical comparisons between pairs of population means.

#### Fisher's LSD

Suppose that analysis of variance provides statistical evidence to reject the null hypothesis of equal population means. In this case, Fisher's least significant difference (LSD) procedure can be used to determine where the differences occur. To illustrate the use of Fisher's LSD procedure in making pairwise comparisons of population means, recall the Chemitech experiment introduced in Section 13.1. Using analysis of variance, we concluded that the mean number of units produced per week are not the same for the three assembly methods. In this case, the follow-up question is: We believe the assembly methods differ, but where do the differences occur? That is, do the means of populations 1 and 2 differ? Or those of populations 1 and 3? Or those of populations 2 and 3?

In Chapter 10 we presented a statistical procedure for testing the hypothesis that the means of two populations are equal. With a slight modification in how we estimate the population variance, Fisher's LSD procedure is based on the  $t$  test statistic presented for the two-population case. The following details summarize Fisher's LSD procedure.

**Fisher's LSD procedure**

$$H_0: \mu_i = \mu_j$$

$$H_1: \mu_i \neq \mu_j$$

**Test statistic**

$$t = \frac{\bar{x}_i - \bar{x}_j}{\sqrt{MSE \left( \frac{1}{n_i} + \frac{1}{n_j} \right)}} \quad (13.16)$$

**Rejection rule**

*p*-value approach: Reject  $H_0$  if  $p$ -value  $\leq \alpha$

Critical value: Reject  $H_0$  if  $t \leq -t_{\alpha/2}$  or  $t \geq t_{\alpha/2}$

where the value of  $t_{\alpha/2}$  is based on a  $t$  distribution with  $n_T - k$  degrees of freedom.

Let us now apply this procedure to determine whether there is a significant difference between the means of population 1 (method A) and population 2 (method B) at the  $\alpha = 0.05$  level of significance. Table 13.1 showed that the sample mean is 62 for method A and 66 for method B. Table 13.3 showed that the value of MSE is 28.33; it is the estimate of  $\sigma^2$  and is based on 12 degrees of freedom. For the Chemitech data the value of the test statistic is:

$$t = \frac{62 - 66}{\sqrt{28.33 \left( \frac{1}{5} + \frac{1}{5} \right)}} = -1.19$$

Because we have a two-tailed test, the *p*-value is two times the area under the curve for the  $t$  distribution to the left of  $t = -1.19$ . Using Table 2 in Appendix B, the  $t$  distribution table for 12 degrees of freedom provides the following information.

|                    |      |       |       |       |       |       |
|--------------------|------|-------|-------|-------|-------|-------|
| Area in Upper Tail | .20  | .10   | .05   | .025  | .01   | .005  |
| $t$ Value (12 df)  | .873 | 1.356 | 1.782 | 2.179 | 2.681 | 3.055 |

$$t = 1.19$$

The  $t$  distribution table only contains positive  $t$  values. Because the  $t$  distribution is symmetric, however, we can find the area under the curve to the right of  $t = 1.19$  and double it to find the *p*-value corresponding to  $t = -1.19$ . We see that  $t = 1.19$  is between .20 and .10. Doubling these amounts, we see that the *p*-value must be between .40 and .20. EXCEL or MINITAB can be used to show that the exact *p*-value is .2571. Because the *p*-value is greater than  $\alpha = 0.05$ , we cannot reject the null hypothesis. Hence, we cannot conclude that the population mean number of units produced per week for method A is different from the population mean for method B.

Many practitioners find it easier to determine how large the difference between the sample means must be to reject  $H_0$ . In this case the test statistic is  $\bar{x}_i - \bar{x}_j$  and the test is conducted by the following procedure.

**Fisher's LSD procedure based on the test statistic  $\bar{x}_i - \bar{x}_j$** 

$$H_0: \mu_i = \mu_j$$

$$H_1: \mu_i \neq \mu_j$$

**Test statistic**

$$\bar{x}_i - \bar{x}_j$$

**Rejection rule at a level of significance  $\alpha$** 

Reject  $H_0$  if  $|\bar{x}_i - \bar{x}_j| \geq \text{LSD}$

where

$$\text{LSD} = t_{\alpha/2} \sqrt{\text{MSE} \left( \frac{1}{n_i} + \frac{1}{n_j} \right)} \quad (13.17)$$

For the Chemitech experiment the value of LSD is:

$$\text{LSD} = 2.179 \sqrt{28.33 \left( \frac{1}{5} + \frac{1}{5} \right)} = 7.34$$

Note that when the sample sizes are equal, only one value for LSD is computed. In such cases we can simply compare the magnitude of the difference between any two sample means with the value of LSD. For example, the difference between the sample means for population 1 (method A) and population 3 (method C) is  $62 - 52 = 10$ . This difference is greater than  $\text{LSD} = 7.34$ , which means we can reject the null hypothesis that the population mean number of units produced per week for method A is equal to the population mean for method C. Similarly, with the difference between the sample means for populations 2 and 3 of  $66 - 52 = 14 > 7.34$ , we can also reject the hypothesis that the population mean for method B is equal to the population mean for method C. In effect, our conclusion is that methods A and B both differ from method C.

Fisher's LSD can also be used to develop a confidence interval estimate of the difference between the means of two populations. The general procedure follows.

**Confidence interval estimate of the difference between two population means using Fisher's LSD procedure**

$$\bar{x}_i - \bar{x}_j \pm \text{LSD} \quad (13.18)$$

where:

$$\text{LSD} = t_{\alpha/2} \sqrt{\text{MSE} \left( \frac{1}{n_i} + \frac{1}{n_j} \right)} \quad (13.19)$$

and  $t_{\alpha/2}$  is based on a  $t$  distribution with  $n_T - k$  degrees of freedom.

If the confidence interval in expression (13.18) includes the value zero, we cannot reject the hypothesis that the two population means are equal. However, if the confidence interval does not include the value zero, we conclude that there is a difference between the population means. For the Chemitech experiment, recall that  $\text{LSD} = 7.34$  (corresponding to  $t_{0.025} = 2.179$ ). Therefore, a 95 per cent confidence interval estimate of the difference between the means of populations 1 and 2 is  $62 - 66 \pm 7.34 = -4 \pm 7.34 = -11.34$  to  $3.34$ ; because this interval includes zero, we cannot reject the hypothesis that the two population means are equal.

**Type I error rates**

We began the discussion of Fisher's LSD procedure with the premise that analysis of variance gave us statistical evidence to reject the null hypothesis of equal population means. We showed how Fisher's LSD procedure can be used in such cases to determine where the differences occur. Technically, it is referred to

as a *protected* or *restricted* LSD test because it is employed only if we first find a significant  $F$  value by using analysis of variance. To see why this distinction is important in multiple comparison tests, we need to explain the difference between a *comparisonwise* Type I error rate and an *experimentwise* Type I error rate.

In the Chemitech experiment we used Fisher's LSD procedure to make three pairwise comparisons.

| Test 1                  | Test 2                  | Test 3                  |
|-------------------------|-------------------------|-------------------------|
| $H_0: \mu_1 = \mu_2$    | $H_0: \mu_1 = \mu_3$    | $H_0: \mu_2 = \mu_3$    |
| $H_1: \mu_1 \neq \mu_2$ | $H_1: \mu_1 \neq \mu_3$ | $H_1: \mu_2 \neq \mu_3$ |

In each case, we used a level of significance of  $\alpha = 0.05$ . Therefore, for each test, if the null hypothesis is true, the probability that we will make a Type I error is  $\alpha = 0.05$ ; hence, the probability that we will not make a Type I error on each test is  $1 - 0.05 = 0.95$ . In discussing multiple comparison procedures we refer to this probability of a Type I error ( $\alpha = 0.05$ ) as the **comparisonwise Type I error rate**; comparisonwise Type I error rates indicate the level of significance associated with a single pairwise comparison.

Let us now consider a slightly different question. What is the probability that in making three pairwise comparisons, we will commit a Type I error on at least one of the three tests? To answer this question, note that the probability that we will not make a Type I error on any of the three tests is  $(0.95)(0.95)(0.95) = 0.8574$ .<sup>\*</sup> Therefore, the probability of making at least one Type I error is  $1 - 0.8574 = .1426$ . When we use Fisher's LSD procedure to make all three pairwise comparisons, the Type I error rate associated with this approach is not 0.05, but actually 0.1426; we refer to this error rate as the *overall* or **experimentwise Type I error rate**. To avoid confusion, we denote the experimentwise Type I error rate as  $\alpha_{EW}$ .

The experimentwise Type I error rate gets larger for problems with more populations. For example, a problem with five populations has ten possible pairwise comparisons. If we tested all possible pairwise comparisons by using Fisher's LSD with a comparisonwise error rate of  $\alpha = 0.05$ , the experimentwise Type I error rate would be  $1 - (1 - 0.05)^{10} = .40$ . In such cases, practitioners look to alternatives that provide better control over the experimentwise error rate.

One alternative for controlling the overall experimentwise error rate, referred to as the Bonferroni adjustment, involves using a smaller comparisonwise error rate for each test. For example, if we want to test  $C$  pairwise comparisons and want the maximum probability of making a Type I error for the overall experiment to be  $\alpha_{EW}$ , we simply use a comparisonwise error rate equal to  $\alpha_{EW}/C$ . In the Chemitech experiment, if we want to use Fisher's LSD procedure to test all three pairwise comparisons with a maximum experimentwise error rate of  $\alpha_{EW} = 0.05$ , we set the comparisonwise error rate to be  $\alpha = .05/3 = .017$ . For a problem with five populations and ten possible pairwise comparisons, the Bonferroni adjustment would suggest a comparisonwise error rate of  $0.05/10 = 0.005$ . Recall from our discussion of hypothesis testing in Chapter 9 that for a fixed sample size, any decrease in the probability of making a Type I error will result in an increase in the probability of making a Type II error, which corresponds to accepting the hypothesis that the two population means are equal when in fact they are not equal. As a result, many practitioners are reluctant to perform individual tests with a low comparisonwise Type I error rate because of the increased risk of making a Type II error.

Several other procedures, such as Tukey's procedure and Duncan's multiple range test, have been developed to help in such situations. However, there is considerable controversy in the statistical community as to which procedure is 'best.' The truth is that no one procedure is best for all types of problems.

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\* The assumption is that the three tests are independent, and hence the joint probability of the three events can be obtained by simply multiplying the individual probabilities. In fact, the three tests are not independent because MSE is used in each test; therefore, the error involved is even greater than that shown.

## EXERCISES

### Methods

- 13.** The following data are from a completely randomized design.

|                 | Treatment A | Treatment B | Treatment C |
|-----------------|-------------|-------------|-------------|
|                 | 32          | 44          | 33          |
|                 | 30          | 43          | 36          |
|                 | 30          | 44          | 35          |
|                 | 26          | 46          | 36          |
|                 | 32          | 48          | 40          |
| Sample mean     | 30          | 45          | 36          |
| Sample variance | 6.00        | 4.00        | 6.50        |

- a. At the  $\alpha = 0.05$  level of significance, can we reject the null hypothesis that the means of the three treatments are equal?
  - b. Use Fisher's LSD procedure to test whether there is a significant difference between the means for treatments A and B, treatments A and C and treatments B and C. Use  $\alpha = .05$ .
  - c. Use Fisher's LSD procedure to develop a 95 per cent confidence interval estimate of the difference between the means of treatments A and B.
- 14.** The following data are from a completely randomized design. In the following calculations, use  $\alpha = .05$ .

|             | Treatment 1 | Treatment 2 | Treatment 3 |
|-------------|-------------|-------------|-------------|
|             | 63          | 82          | 69          |
|             | 47          | 72          | 54          |
|             | 54          | 88          | 61          |
|             | 40          | 66          | 48          |
| $\bar{x}_j$ | 51          | 77          | 58          |
| $s_j^2$     | 96.67       | 97.34       | 81.99       |

- a. Use analysis of variance to test for a significant difference among the means of the three treatments.
- b. Use Fisher's LSD procedure to determine which means are different.

### Applications

- 15.** To test whether the mean time needed to mix a batch of material is the same for machines produced by three manufacturers, the Jacobs Chemical Company obtained the following data on the time (in minutes) needed to mix the material.

| Manufacturer |    |    |
|--------------|----|----|
| 1            | 2  | 3  |
| 20           | 28 | 20 |
| 26           | 26 | 19 |
| 24           | 31 | 23 |
| 22           | 27 | 22 |



**COMPLETE  
SOLUTIONS**



**COMPLETE  
SOLUTIONS**

- a. Use these data to test whether the population mean times for mixing a batch of material differ for the three manufacturers. Use  $\alpha = .05$ .
  - b. At the  $\alpha = 0.05$  level of significance, use Fisher's LSD procedure to test for the equality of the means for manufacturers 1 and 3. What conclusion can you draw after carrying out this test?
- 16.** Refer to Exercise 15. Use Fisher's LSD procedure to develop a 95 per cent confidence interval estimate of the difference between the means for manufacturer 1 and manufacturer 2.
- 17.** The following data are from an experiment designed to investigate the perception of corporate ethical values among individuals specializing in marketing (higher scores indicate higher ethical values).

|   | <i>Marketing managers</i> | <i>Marketing research</i> | <i>Advertising</i> |
|---|---------------------------|---------------------------|--------------------|
| 6 |                           | 5                         | 6                  |
| 5 |                           | 5                         | 7                  |
| 4 |                           | 4                         | 6                  |
| 5 |                           | 4                         | 5                  |
| 6 |                           | 5                         | 6                  |
| 4 |                           | 4                         | 6                  |

- a. Use  $\alpha = 0.05$  to test for significant differences in perception among the three groups.
  - b. At the  $\alpha = 0.05$  level of significance, we can conclude that there are differences in the perceptions for marketing managers, marketing research specialists and advertising specialists. Use the procedures in this section to determine where the differences occur. Use  $\alpha = .05$ .
- 18.** To test for any significant difference in the number of hours between breakdowns for four machines, the following data were obtained.

|     | <i>Machine 1</i> | <i>Machine 2</i> | <i>Machine 3</i> | <i>Machine 4</i> |
|-----|------------------|------------------|------------------|------------------|
| 6.4 | 8.7              | 11.1             | 9.9              |                  |
| 7.8 | 7.4              | 10.3             | 12.8             |                  |
| 5.3 | 9.4              | 9.7              | 12.1             |                  |
| 7.4 | 10.1             | 10.3             | 10.8             |                  |
| 8.4 | 9.2              | 9.2              | 11.3             |                  |
| 7.3 | 9.8              | 8.8              | 11.5             |                  |

- a. At the  $\alpha = 0.05$  level of significance, what is the difference, if any, in the population mean times among the four machines?
  - b. Use Fisher's LSD procedure to test for the equality of the means for machines 2 and 4. Use a 0.05 level of significance.
- 19.** Refer to Exercise 18. Use the Bonferroni adjustment to test for a significant difference between all pairs of means. Assume that a maximum overall experiment wise error rate of 0.05 is desired.

## 13.4 RANDOMIZED BLOCK DESIGN

This far we have considered the completely randomized experimental design. Recall that to test for a difference among treatment means, we computed an  $F$  value by using the ratio:

**F test statistic**

$$F = \frac{\text{MSTR}}{\text{MSE}} \quad (13.20)$$

A problem can arise whenever differences due to extraneous factors (ones not considered in the experiment) cause the MSE term in this ratio to become large. In such cases, the  $F$  value in equation (13.20) can become small, signalling no difference among treatment means when in fact such a difference exists.

In this section we present an experimental design known as a **randomized block design**. Its purpose is to control some of the extraneous sources of variation by removing such variation from the MSE term. This design tends to provide a better estimate of the true error variance and leads to a more powerful hypothesis test in terms of the ability to detect differences among treatment means. To illustrate, let us consider a stress study for air traffic controllers.

### Air traffic controller stress test

A study measuring the fatigue and stress of air traffic controllers resulted in proposals for modification and redesign of the controller's work station. After consideration of several designs for the work station, three specific alternatives are selected as having the best potential for reducing controller stress. The key question is: To what extent do the three alternatives differ in terms of their effect on controller stress? To answer this question, we need to design an experiment that will provide measurements of air traffic controller stress under each alternative.

In a completely randomized design, a random sample of controllers would be assigned to each work station alternative. However, controllers are believed to differ substantially in their ability to handle stressful situations. What is high stress to one controller might be only moderate or even low stress to another. Hence, when considering the within-group source of variation (MSE), we must realize that this variation includes both random error and error due to individual controller differences. In fact, managers expected controller variability to be a major contributor to the MSE term.

One way to separate the effect of the individual differences is to use a randomized block design. Such a design will identify the variability stemming from individual controller differences and remove it from the MSE term. The randomized block design calls for a single sample of controllers. Each controller in the sample is tested with each of the three work station alternatives. In experimental design terminology, the work station is the *factor of interest* and the controllers are the *blocks*. The three treatments or populations associated with the work station factor correspond to the three work station alternatives. For simplicity, we refer to the work station alternatives as system A, system B and system C.

The *randomized* aspect of the randomized block design is the random order in which the treatments (systems) are assigned to the controllers. If every controller were to test the three systems in the same order, any observed difference in systems might be due to the order of the test rather than to true differences in the systems.

To provide the necessary data, the three work station alternatives were installed at the Berlin Control Centre. Six controllers were selected at random and assigned to operate each of the systems. A follow-up interview and a medical examination of each controller participating in the study provided a measure of the stress for each controller on each system. The data are reported in Table 13.5.

Table 13.6 is a summary of the stress data collected. In this table we include column totals (treatments) and row totals (blocks) as well as some sample means that will be helpful in making the sum of squares computations for the ANOVA procedure. Because lower stress values are viewed as better, the sample data seem to favour system B with its mean stress rating of 13. However, the usual question remains: Do the sample results justify the conclusion that the population mean stress levels for the three systems differ? That is, are the differences statistically significant? An analysis of variance computation similar to the one performed for the completely randomized design can be used to answer this statistical question.



**TABLE 13.5** A randomized block design for the air traffic controller stress test

|        |              | Treatments |          |          |
|--------|--------------|------------|----------|----------|
|        |              | System A   | System B | System C |
| Blocks | Controller 1 | 15         | 15       | 18       |
|        | Controller 2 | 14         | 14       | 14       |
|        | Controller 3 | 10         | 11       | 15       |
|        | Controller 4 | 13         | 12       | 17       |
|        | Controller 5 | 16         | 13       | 16       |
|        | Controller 6 | 13         | 13       | 13       |

**TABLE 13.6** Summary of stress data for the air traffic controller stress test

|                               |              | Treatments                               |  |  |                        |                                   |
|-------------------------------|--------------|--|--|--|------------------------|-----------------------------------|
|                               |              | System A                                 | System B                                 | System C                                 | Row or<br>block totals | Block means                       |
| Blocks                        | Controller 1 | 15                                       | 15                                       | 18                                       | 48                     | $x_{1\cdot} = 48/3 = 16.0$        |
|                               | Controller 2 | 14                                       | 14                                       | 14                                       | 42                     | $x_{2\cdot} = 42/3 = 14.0$        |
|                               | Controller 3 | 10                                       | 11                                       | 15                                       | 36                     | $x_{3\cdot} = 36/3 = 12.0$        |
|                               | Controller 4 | 13                                       | 12                                       | 17                                       | 42                     | $x_{4\cdot} = 42/3 = 14.0$        |
|                               | Controller 5 | 16                                       | 13                                       | 16                                       | 45                     | $x_{5\cdot} = 45/3 = 15.0$        |
|                               | Controller 6 | 13                                       | 13                                       | 13                                       | 39                     | $x_{6\cdot} = 39/3 = 13.0$        |
| Column or<br>Treatment totals |              | 81                                       | 78                                       | 93                                       | 252                    | $\bar{x} = \frac{252}{18} = 14.0$ |
| Treatment means               |              | $\bar{x}_{1\cdot} = \frac{81}{6} = 13.5$ | $\bar{x}_{2\cdot} = \frac{78}{6} = 13.0$ | $\bar{x}_{3\cdot} = \frac{93}{6} = 15.5$ |                        |                                   |

## ANOVA procedure

The ANOVA procedure for the randomized block design requires us to partition the sum of squares total (SST) into three groups: sum of squares due to treatments (SSTR), sum of squares due to blocks (SSBL) and sum of squares due to error (SSE). The formula for this partitioning follows.

$$\text{SST} = \text{SSTR} + \text{SSBL} + \text{SSE} \quad (13.21)$$

This sum of squares partition is summarized in the ANOVA table for the randomized block design as shown in Table 13.7. The notation used in the table is:

- $k$  = the number of treatments
- $b$  = the number of blocks
- $n_T$  = the total sample size ( $n_T = kb$ )

Note that the ANOVA table also shows how the  $n_T - 1$  total degrees of freedom are partitioned such that  $k - 1$  degrees of freedom go to treatments,  $b - 1$  go to blocks and  $(k - 1)(b - 1)$  go to the error term. The mean square column shows the sum of squares divided by the degrees of freedom, and  $F = \text{MSTR}/\text{MSE}$  is the  $F$  ratio used to test for a significant difference among the treatment means. The primary contribution of the randomized block design is that, by including blocks, we remove the individual controller differences from the MSE term and obtain a more powerful test for the stress differences in the three work station alternatives.

**TABLE 13.7** ANOVA table for the randomized block design with  $k$  treatments and  $b$  blocks

| Source of variation | Degrees of freedom | Sum of squares | Mean square                    | $F$                |
|---------------------|--------------------|----------------|--------------------------------|--------------------|
| Treatments          | $k - 1$            | SSTR           | $MSTR = \frac{SSTR}{k-1}$      | $\frac{MSTR}{MSE}$ |
| Blocks              | $b - 1$            | SSBL           | $MSBL = \frac{SSBL}{b-1}$      |                    |
| Error               | $(k - 1)(b - 1)$   | SSE            | $MSE = \frac{SSE}{(k-1)(b-1)}$ |                    |
| Total               | $n_T - 1$          | SST            |                                |                    |

## Computations and conclusions

To compute the  $F$  statistic needed to test for a difference among treatment means with a randomized block design, we need to compute MSTR and MSE. To calculate these two mean squares, we must first compute SSTR and SSE; in doing so, we will also compute SSBL and SST. To simplify the presentation, we perform the calculations in four steps. In addition to  $k$ ,  $b$  and  $n_T$  as previously defined, the following notation is used.

$\bar{x}_{ij}$  = value of the observation corresponding to treatment  $j$  in block  $i$

$\bar{x}_j$  = sample mean of the  $j$ th treatment

$\bar{x}_{\cdot i}$  = sample mean for the  $i$ th block

$\bar{\bar{x}}$  = overall sample mean

**Step 1.** Compute the total sum of squares (SST).

$$SST = \sum_{i=1}^b \sum_{j=1}^k (\bar{x}_{ij} - \bar{\bar{x}})^2 \quad (13.22)$$

**Step 2.** Compute the sum of squares due to treatments (SSTR).

$$SSTR = b \sum_{j=1}^k (\bar{x}_j - \bar{\bar{x}})^2 \quad (13.23)$$

**Step 3.** Compute the sum of squares due to blocks (SSBL).

$$SSBL = k \sum_{i=1}^b (\bar{x}_{\cdot i} - \bar{\bar{x}})^2 \quad (13.24)$$

**Step 4.** Compute the sum of squares due to error (SSE).

$$SSE = SST - SSTR - SSBL \quad (13.25)$$

For the air traffic controller data in Table 13.6, these steps lead to the following sums of squares.

**Step 1.**  $SST = (15 - 14)^2 + (15 - 14)^2 + (18 - 14)^2 + \dots + (13 - 14)^2 = 70$

**Step 2.**  $SSTR = [(13.5 - 14)^2 + (13.0 - 14)^2 + (15.5 - 14)^2] = 21$

**Step 3.**  $SSBL = 3[(16 - 14)^2 + (14 - 14)^2 + (12 - 14)^2 + (14 - 14)^2 + (15 - 14)^2 + (13 - 14)^2] = 30$

**Step 4.**  $SSE = 70 - 21 - 30 = 19$

**TABLE 13.8** ANOVA table for the air traffic controller stress test

| Source of variation | Sum of squares | Degrees of freedom | Mean square | F               | p-value |
|---------------------|----------------|--------------------|-------------|-----------------|---------|
| Treatments          | 2              | 21                 | 10.5        | 10.5/1.9 = 5.53 | .024    |
| Blocks              | 5              | 30                 | 6.0         |                 |         |
| Error               | 10             | 19                 | 1.9         |                 |         |
| Total               | 17             | 70                 |             |                 |         |

These sums of squares divided by their degrees of freedom provide the corresponding mean square values shown in Table 13.8.

Let us use a level of significance  $\alpha = 0.05$  to conduct the hypothesis test. The value of the test statistic is:

$$F = \frac{\text{MSTR}}{\text{MSE}} = \frac{10.5}{1.9} = 5.53$$

The numerator degrees of freedom is  $k - 1 = 3 - 1 = 2$  and the denominator degrees of freedom is  $(k - 1)(b - 1) = (3 - 1)(6 - 1) = 10$ . Because we will only reject the null hypothesis for large values of the test statistic, the *p*-value is the area under the *F* distribution to the right of  $F = 5.53$ . From Table 4 of Appendix B we find that with the degrees of freedom 2 and 10,  $F = 5.53$  is between  $F_{0.025} = 5.46$  and  $F_{0.01} = 7.56$ . As a result, the area in the upper tail, or the *p*-value, is between .01 and .025. Alternatively, we can use or MINITAB, EXCEL or SPSS to show that the exact *p*-value for  $F = 5.53$  is 0.024. With *p*-value  $\leq \alpha = 0.05$ , we reject the null hypothesis  $H_0: \mu_1 = \mu_2 = \mu_3$  and conclude that the population mean stress levels differ for the three work station alternatives.

Some general comments can be made about the randomized block design. The experimental design described in this section is a *complete* block design; the word ‘complete’ indicates that each block is subjected to all  $k$  treatments. That is, all controllers (blocks) were tested with all three systems (treatments). Experimental designs in which some but not all treatments are applied to each block are referred to as *incomplete* block designs. A discussion of incomplete block designs is beyond the scope of this text.

Because each controller in the air traffic controller stress test was required to use all three systems, this approach guarantees a complete block design. In some cases, however, **blocking** is carried out with ‘similar’ experimental units in each block. For example, assume that in a pretest of air traffic controllers, the population of controllers was divided into groups ranging from extremely high-stress individuals to extremely low-stress individuals.

The blocking could still be accomplished by having three controllers from each of the stress classifications participate in the study. Each block would then consist of three controllers in the same stress group. The randomized aspect of the block design would be the random assignment of the three controllers in each block to the three systems.

Finally, note that the ANOVA table shown in Table 13.7 provides an *F* value to test for treatment effects but *not* for blocks. The reason is that the experiment was designed to test a single factor – work station design. The blocking based on individual stress differences was conducted to remove such variation from the MSE term. However, the study was not designed to test specifically for individual differences in stress.

Some analysts compute  $F = \text{MSB}/\text{MSE}$  and use that statistic to test for significance of the blocks. Then they use the result as a guide to whether the same type of blocking would be desired in future experiments. However, if individual stress difference is to be a factor in the study, a different experimental design should be used. A test of significance on blocks should not be performed as a basis for a conclusion about a second factor.

## EXERCISES

### Methods

- 20.** Consider the experimental results for the following randomized block design. Make the calculations necessary to set up the analysis of variance table.

|        |   | <i>Treatments</i> |          |          |
|--------|---|-------------------|----------|----------|
|        |   | <i>A</i>          | <i>B</i> | <i>C</i> |
| Blocks | 1 | 10                | 9        | 8        |
|        | 2 | 12                | 6        | 5        |
|        | 3 | 18                | 15       | 14       |
|        | 4 | 20                | 18       | 18       |
|        | 5 | 8                 | 7        | 8        |

Use  $\alpha = 0.05$  to test for any significant differences.

- 21.** The following data were obtained for a randomized block design involving five treatments and three blocks: SST = 430, SSTR = 310, SSBL = 85. Set up the ANOVA table and test for any significant differences. Use  $\alpha = .05$
- 22.** An experiment has been conducted for four treatments with eight blocks. Complete the following analysis of variance table.

| <i>Source of variation</i> | <i>Degrees of freedom</i> | <i>Sum of squares</i> | <i>Mean square</i> | <i>F</i> |
|----------------------------|---------------------------|-----------------------|--------------------|----------|
| Treatments                 |                           | 900                   |                    |          |
| Blocks                     |                           | 400                   |                    |          |
| Error                      |                           |                       |                    |          |
| Total                      |                           | 1800                  |                    |          |

Use  $\alpha = 0.05$  to test for any significant differences.

### Applications

- 23.** A car dealer, AfricaDrive, conducted a test to determine if the time in minutes needed to complete a minor engine tune-up depends on whether a computerized engine analyzer or an electronic analyzer is used. Because tune-up time varies among compact, intermediate and full-sized cars, the three types of cars were used as blocks in the experiment. The data obtained follow.

|              |                     | <i>Analyzer</i>   |  |
|--------------|---------------------|-------------------|--|
| <i>Car</i>   | <i>Computerized</i> | <i>Electronic</i> |  |
| Compact      | 50                  | 42                |  |
| Intermediate | 55                  | 44                |  |
| Full-sized   | 63                  | 46                |  |

Use  $\alpha = 0.05$  to test for any significant differences.

- 24.** A textile mill produces a silicone proofed fabric for making into rainwear. The chemist in charge thinks that a silicone solution of about 12 per cent strength should yield a fabric with maximum



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waterproofing index. He also suspected there may be some batch to batch variation because of slight differences in the cloth. To allow for this possibility five different strengths of solution were used on each of the three different batches of fabric. The following values of waterproofing index were obtained:

|        |   | <i>[Strength of silicone solution (%)]</i> |      |      |      |      |
|--------|---|--|------|------|------|------|
|        |   | 6  | 9    | 12   | 15   | 18   |
| Fabric | 1 | 20.8                                       | 20.6 | 22.0 | 22.6 | 20.9 |
|        | 2 | 19.4                                       | 21.2 | 21.8 | 23.9 | 22.4 |
|        | 3 | 19.9                                       | 21.1 | 22.7 | 22.7 | 22.1 |

Using  $\alpha = 0.05$ , carry out an appropriate test of these data and comment on the chemist's original beliefs.

25. An important factor in selecting software for word-processing and database management systems is the time required to learn how to use the system. To evaluate three file management systems, a firm designed a test involving five word-processing operators. Because operator variability was believed to be a significant factor, each of the five operators was trained on each of the three file management systems. The data obtained follow.

| Operator | <i>System</i> |    |    |
|----------|---------------|----|----|
|          | A             | B  | C  |
| 1        | 6             | 16 | 24 |
| 2        | 9             | 17 | 22 |
| 3        | 4             | 13 | 19 |
| 4        | 3             | 12 | 18 |
| 5        | 8             | 17 | 22 |

Use  $\alpha = 0.05$  to test for any difference in the mean training time (in hours) for the three systems.

## 13.5 FACTORIAL EXPERIMENT

The experimental designs we have considered thus far enable us to draw statistical conclusions about one factor. However, in some experiments we want to draw conclusions about more than one variable or factor. A **factorial experiment** is an experimental design that allows simultaneous conclusions about two or more factors. The term *factorial* is used because the experimental conditions include all possible combinations of the factors. For example, for  $a$  levels of factor A and  $b$  levels of factor B, the experiment will involve collecting data on  $ab$  treatment combinations. In this section we will show the analysis for a two-factor factorial experiment. The basic approach can be extended to experiments involving more than two factors.

As an illustration of a two-factor factorial experiment, we will consider a study involving the Graduate Management Admissions Test (GMAT), a standardized test used by graduate schools of business to evaluate an applicant's ability to pursue a graduate programme in that field. Scores on the GMAT range from 200 to 800, with higher scores implying higher aptitude.

In an attempt to improve students' performance on the GMAT, a major Spanish university is considering offering the following three GMAT preparation programmes.

- 1 A three-hour review session covering the types of questions generally asked on the GMAT.
- 2 A one-day programme covering relevant exam material, along with the taking and grading of a sample exam.
- 3 An intensive ten-week course involving the identification of each student's weaknesses and the setting up of individualized programmes for improvement.

Hence, one factor in this study is the GMAT preparation programme, which has three treatments: three-hour review, one-day programme and ten-week course. Before selecting the preparation programme to adopt, further study will be conducted to determine how the proposed programmes affect GMAT scores.

The GMAT is usually taken by students from three colleges: the College of Business, the College of Engineering and the College of Arts and Sciences. Therefore, a second factor of interest in the experiment is whether a student's undergraduate college affects the GMAT score. This second factor, undergraduate college, also has three treatments: business, engineering and arts and sciences. The factorial design for this experiment with three treatments corresponding to factor A, the preparation programme, and three treatments corresponding to factor B, the undergraduate college, will have a total of  $3 \times 3 = 9$  treatment combinations. These treatment combinations or experimental conditions are summarized in Table 13.9.

Assume that a sample of two students will be selected corresponding to each of the nine treatment combinations shown in Table 13.9: two business students will take the three-hour review, two will take the one-day programme and two will take the ten-week course. In addition, two engineering students and two arts and sciences students will take each of the three preparation programmes. In experimental design terminology, the sample size of two for each treatment combination indicates that we have two **replications**. Additional replications and a larger sample size could easily be used, but we elect to minimize the computational aspects for this illustration.

This experimental design requires that six students who plan to attend graduate school be randomly selected from *each* of the three undergraduate colleges. Then two students from each college should be assigned randomly to each preparation programme, resulting in a total of 18 students being used in the study.

Assume that the randomly selected students participated in the preparation programmes and then took the GMAT. The scores obtained are reported in Table 13.10.

**TABLE 13.9** Nine treatment combinations for the two-factor GMAT experiment

|                                       |                   | Factor B: College |             |                   |
|---------------------------------------|-------------------|-------------------|-------------|-------------------|
|                                       |                   | Business          | Engineering | Arts and Sciences |
| Factor A:<br>Preparation<br>Programme | Three-hour review | 1                 | 2           | 3                 |
|                                       | One-day programme | 4                 | 5           | 6                 |
|                                       | ten-week course   | 7                 | 8           | 9                 |

**TABLE 13.10** GMAT scores for the two-factor experiment

|                                       |                   | Factor B: College |             |                   |
|---------------------------------------|-------------------|-------------------|-------------|-------------------|
|                                       |                   | Business          | Engineering | Arts and Sciences |
| Factor A:<br>Preparation<br>Programme | Three-hour review | 500               | 540         | 480               |
|                                       | Three-hour review | 580               | 460         | 400               |
|                                       | One-day programme | 460               | 560         | 420               |
|                                       | One-day programme | 540               | 620         | 480               |
|                                       | ten-week course   | 560               | 600         | 480               |
|                                       | ten-week course   | 600               | 580         | 410               |

The analysis of variance computations with the data in Table 13.10 will provide answers to the following questions:

- **Main effect (factor A):** Do the preparation programmes differ in terms of effect on GMAT scores?
- **Main effect (factor B):** Do the undergraduate colleges differ in terms of effect on GMAT scores?
- **Interaction effect (factors A and B):** Do students in some colleges do better on one type of preparation programme whereas others do better on a different type of preparation programme?

The term **interaction** refers to a new effect that we can now study because we used a factorial experiment. If the interaction effect has a significant impact on the GMAT scores, we can conclude that the effect of the type of preparation programme depends on the undergraduate college.

## ANOVA procedure

The ANOVA procedure for the two-factor factorial experiment requires us to partition the sum of squares total (SST) into four groups: sum of squares for factor A (SSA), sum of squares for factor B (SSB), sum of squares for interaction (SSAB) and sum of squares due to error (SSE). The formula for this partitioning follows.

$$\text{SST} = \text{SSA} + \text{SSB} + \text{SSAB} + \text{SSE} \quad (13.26)$$

The partitioning of the sum of squares and degrees of freedom is summarized in Table 13.11. The following notation is used.

$a$  = number of levels of factor A

$b$  = number of levels of factor B

$r$  = number of replications

$n_T$  = total number of observations taken in the experiment;  $n_T = abr$

## Computations and conclusions

To compute the  $F$  statistics needed to test for the significance of factor A, factor B and interaction, we need to compute MSA, MSB, MSAB and MSE. To calculate these four mean squares, we must first compute SSA, SSB, SSAB and SSE; in doing so we will also compute SST. To simplify the presentation, we perform the calculations in five steps. In addition to  $a$ ,  $b$ ,  $r$  and  $n_T$  as previously defined, the following notation is used.

**TABLE 13.11** ANOVA table for the two-factor factorial experiment with  $r$  replications

| Source of variation | Degrees of freedom | Sum of squares | Mean square                                    | $F$                              |
|---------------------|--------------------|----------------|--|----------------------------------|
| Factor A            | $a - 1$            | SSA            | $\text{MSA} = \frac{\text{SSA}}{a-1}$          | $\frac{\text{MSA}}{\text{MSE}}$  |
| Factor B            | $b - 1$            | SSB            | $\text{MSB} = \frac{\text{SSB}}{b-1}$          | $\frac{\text{MSB}}{\text{MSE}}$  |
| Interaction         | $(a - 1)(b - 1)$   | SSAB           | $\text{MSAB} = \frac{\text{SSAB}}{(a-1)(b-1)}$ | $\frac{\text{MSAB}}{\text{MSE}}$ |
| Error               | $ab(r - 1)$        | SSE            | $\text{MSE} = \frac{\text{SSE}}{ab(r-1)}$      |                                  |
| Total               | $n_T - 1$          | SST            |  |                                  |

- $x_{ijk}$  = observation corresponding to the  $k$ th replicate taken from treatment  $i$  of factor A and treatment  $j$  of factor B  
 $\bar{x}_{i\cdot}$  = sample mean for the observations in treatment  $i$  (factor A)  
 $\bar{x}_{j\cdot}$  = sample mean for the observations in treatment  $j$  (factor B)  
 $\bar{x}_{ij\cdot}$  = sample mean for the observations corresponding to the combination of treatment  $i$  (factor A) and treatment  $j$  (factor B)  
 $\bar{\bar{x}}$  = overall sample mean of all  $n_T$  observations

**Step 1** Compute the total sum of squares.

$$\text{SST} = \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^r (x_{ijk} - \bar{\bar{x}})^2 \quad (13.27)$$

**Step 2** Compute the sum of squares for factor A.

$$\text{SSA} = br \sum_{i=1}^a (\bar{x}_{i\cdot} - \bar{\bar{x}})^2 \quad (13.28)$$

**Step 3** Compute the sum of squares for factor B.

$$\text{SSB} = ar \sum_{j=1}^b (\bar{x}_{j\cdot} - \bar{\bar{x}})^2 \quad (13.29)$$

**Step 4** Compute the sum of squares for interaction.

$$\text{SSAB} = r \sum_{i=1}^a \sum_{j=1}^b (\bar{x}_{ij\cdot} - \bar{x}_{i\cdot} - \bar{x}_{j\cdot} + \bar{\bar{x}})^2 \quad (13.30)$$

**Step 5** Compute the sum of squares due to error.

$$\text{SSE} = \text{SST} - \text{SSA} - \text{SSB} - \text{SSAB} \quad (13.31)$$

Table 13.12 reports the data collected in the experiment and the various sums that will help us with the sum of squares computations. Using equations (13.27) through (13.31), we calculate the following sums of squares for the GMAT two-factor factorial experiment.

**Step 1**  $\text{SST} = (500 - 515)^2 + (580 - 515)^2 + (540 - 515)^2 + \dots + (410 - 515)^2 = 82\,450$

**Step 2**  $\text{SSA} = (3)(2)[(493.33 - 515)^2 + (513.33 - 515)^2 + (538.33 - 515)^2] = 6100$

**Step 3**  $\text{SSB} = (3)(2)[(540 - 515)^2 + (560 - 515)^2 + (445 - 515)^2] = 45\,300$

**Step 4**  $\text{SSAB} = 2[(540 - 493.33 - 540 + 515)^2 + (500 - 493.33 - 560 + 515)^2 + \dots + (445 - 538.33 - 445 + 515)^2] = 11\,200$

**Step 5**  $\text{SSE} = 82\,450 - 6100 - 45\,300 - 11\,200 = 19\,850$

These sums of squares divided by their corresponding degrees of freedom provide the appropriate mean square values for testing the two main effects (preparation programme and undergraduate college) and the interaction effect.

Because of the computational effort involved in any modest- to large-size factorial experiment, the computer usually plays an important role in performing the analysis of variance computations shown above and in the calculation of the  $p$ -values used to make the hypothesis testing decisions. Figure 13.6 shows the MINITAB output for the analysis of variance for the GMAT two-factor factorial experiment. Let us use the MINITAB output and a level of significance  $\alpha = 0.05$  to conduct the hypothesis tests for the two-factor GMAT study. The  $p$ -value used to test for significant differences among the three preparation programmes (factor A) is 0.299.

**TABLE 13.12** GMAT summary data for the two-factor experiment

|                                    |                   | Factor B: College                     |                                       |                                      | Row totals                        | Factor A means                           |  |
|------------------------------------|-------------------|---------------------------------------|---------------------------------------|--------------------------------------|-----------------------------------|--|--|
| Treatment combination totals       |                   | Business                              | Engineering                           | Arts and Sciences                    |                                   |  |  |
| Factor A:<br>Preparation programme | Three-hour review | 500                                   | 540                                   | 480                                  |                                   |  |  |
|                                    |                   | 580                                   | 460                                   | 400                                  |                                   |  |  |
|                                    |                   | 1080                                  | 1000                                  | 880                                  | s                                 | 2960                                     |  |
|                                    | One-day programme |                                       |                                       |                                      |                                   | $\bar{x}_{1.} = \frac{2960}{6} = 493.33$ |  |
|                                    |                   | $\bar{x}_{11} = \frac{1080}{2} = 540$ | $\bar{x}_{12} = \frac{1000}{2} = 500$ | $\bar{x}_{13} = \frac{880}{2} = 440$ |                                   |  |  |
|                                    |                   | 460                                   | 560                                   | 420                                  |                                   |  |  |
| 10-week course                     |                   |                                       |                                       |                                      |                                   |  |  |
|                                    |                   | $\frac{540}{1000}$                    | $\frac{620}{1180}$                    | $\frac{480}{900}$                    |                                   |  |  |
|                                    |                   |                                       |                                       |                                      | 3080                              | $\bar{x}_{2.} = \frac{3080}{6} = 513.33$ |  |
|                                    |                   | $\bar{x}_{21} = \frac{1000}{2} = 500$ | $\bar{x}_{22} = \frac{1180}{2} = 590$ | $\bar{x}_{23} = \frac{900}{2} = 450$ |                                   |  |  |
|                                    |                   | 560                                   | 600                                   | 480                                  |                                   |  |  |
|                                    |                   | $\frac{600}{1160}$                    | $\frac{580}{1180}$                    | $\frac{410}{890}$                    |                                   |  |  |
|                                    | Column totals     |                                       |                                       |                                      | 3230                              | $\bar{x}_{3.} = \frac{3230}{6} = 538.33$ |  |
|                                    |                   |                                       |                                       |                                      |                                   |  |  |
|                                    |                   | $\bar{x}_{31} = \frac{1160}{2} = 580$ | $\bar{x}_{32} = \frac{1180}{2} = 590$ | $\bar{x}_{33} = \frac{890}{2} = 445$ |                                   |  |  |
| Factor B means                     |                   | 3240                                  | 3360                                  | 2670                                 | 9270                              | Overall total                            |  |
|                                    |                   | $\bar{x}_1 = \frac{3240}{6} = 540$    | $\bar{x}_2 = \frac{3360}{6} = 560$    | $\bar{x}_3 = \frac{2670}{6} = 445$   | $\bar{x} = \frac{9270}{18} = 515$ |  |  |

**FIGURE 13.6**

MINITAB output for the GMAT two-factor design

### Two-way ANOVA: Score versus Factor A, Factor B

| Source      | DF | SS    | MS      | F     | P     |
|-------------|----|-------|---------|-------|-------|
| Factor A    | 2  | 6100  | 3050.0  | 1.38  | 0.299 |
| Factor B    | 2  | 45300 | 22650.0 | 10.27 | 0.005 |
| Interaction | 4  | 11200 | 2800.0  | 1.27  | 0.350 |
| Error       | 9  | 19850 | 2206.0  |       |       |
| Total       | 17 | 82450 |         |       |       |

**S = 46.96    R-Sq = 75.92%    R-Sq(adj) = 54.52%**

Because the  $p$ -value = 0.299 is greater than  $\alpha = 0.05$ , there is no significant difference in the mean GMAT test scores for the three preparation programmes. However, for the undergraduate college effect, the  $p$ -value = 0.005 is less than  $\alpha = 0.05$ ; thus, there is a significant difference in the mean GMAT test scores among the three undergraduate colleges. Finally, because the  $p$ -value of 0.350 for the interaction effect is greater than  $\alpha = 0.05$ , there is no significant interaction effect. Therefore, the study provides no reason to believe that the three preparation programmes differ in their ability to prepare students from the different colleges for the GMAT.

Undergraduate college was found to be a significant factor. Checking the calculations in Table 13.12, we see that the sample means are: business students  $\bar{x}_{1\cdot} = 540 = 540$ , engineering students  $\bar{x}_{2\cdot} = 560 = 560$ , and arts and sciences students  $\bar{x}_{3\cdot} = 445 = 445$ . Tests on individual treatment means can be conducted; yet after reviewing the three sample means, we would anticipate no difference in preparation for business and engineering graduates. However, the arts and sciences students appear to be significantly less prepared for the GMAT than students in the other colleges. Perhaps this observation will lead the university to consider other options for assisting these students in preparing for the Graduate Management Admission Test.

## EXERCISES

### Methods

26. A factorial experiment involving two levels of factor A and three levels of factor B resulted in the following data.

|          |         | Factor B |         |         |
|----------|---------|----------|---------|---------|
|          |         | Level 1  | Level 2 | Level 3 |
| Factor A | Level 1 | 135      | 90      | 75      |
|          | Level 2 | 165      | 66      | 93      |
|          | Level 2 | 125      | 127     | 120     |
|          | Level 2 | 95       | 105     | 136     |

Test for any significant main effects and any interaction. Use  $\alpha = .05$ .

27. The calculations for a factorial experiment involving four levels of factor A, three levels of factor B, and three replications resulted in the following data: SST = 280, SSA = 26, SSB = 23, SSAB = 175. Set up the ANOVA table and test for any significant main effects and any interaction effect. Use  $\alpha = .05$ .



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### Applications

- 28.** A mail-order catalogue firm designed a factorial experiment to test the effect of the size of a magazine advertisement and the advertisement design on the number of catalogue requests received (data in thousands). Three advertising designs and two different size advertisements were considered. The data obtained follow. Use the ANOVA procedure for factorial designs to test for any significant effects due to type of design, size of advertisement or interaction. Use  $\alpha = .05$ .

|        |   | Size of advertisement |       |
|--------|---|-----------------------|-------|
|        |   | Small                 | Large |
|        |   | A                     | 12    |
| Design | A | 8                     | 12    |
|        |   | 12                    | 8     |
|        | B | 22                    | 26    |
|        |   | 14                    | 30    |
|        | C | 10                    | 18    |
|        |   | 18                    | 14    |

- 29.** An amusement park studied methods for decreasing the waiting time (minutes) for rides by loading and unloading riders more efficiently. Two alternative loading/unloading methods have been proposed. To account for potential differences due to the type of ride and the possible interaction between the method of loading and unloading and the type of ride, a factorial experiment was designed. Use the following data to test for any significant effect due to the loading and unloading method, the type of ride and interaction. Use  $\alpha = .05$ .

|          |    | Type of ride   |                 |           |
|----------|----|----------------|-----------------|-----------|
|          |    | Roller-coaster | Screaming Demon | Log Flume |
| Method 1 | 41 | 52             | 50              |           |
|          | 43 | 44             | 46              |           |
| Method 2 | 49 | 50             | 48              |           |
|          | 51 | 46             | 44              |           |

- 30.** As part of a study designed to compare hybrid and similarly equipped conventional vehicles, *Consumer Reports* tested a variety of classes of hybrid and all-gas model cars and sport utility vehicles (SUVs). The following data show the miles-per-gallon rating *Consumer Reports* obtained for two hybrid small cars, two hybrid mid-size cars, two hybrid small SUVs and two hybrid mid-sized SUVs; also shown are the miles per gallon obtained for eight similarly equipped conventional models (*Consumer Reports*, October 2008).

| Make/Model       | Class        | Type         | MPG |
|------------------|--------------|--------------|-----|
| Honda Civic      | Small car    | Hybrid       | 37  |
| Honda Civic      | Small car    | Conventional | 28  |
| Toyota Prius     | Small car    | Hybrid       | 44  |
| Toyota Corolla   | Small car    | Conventional | 32  |
| Chevrolet Malibu | Mid-size car | Hybrid       | 27  |
| Chevrolet Malibu | Mid-size car | Conventional | 23  |
| Nissan Altima    | Mid-size car | Hybrid       | 32  |
| Nissan Altima    | Mid-size car | Conventional | 25  |

|                   |              |              |    |
|-------------------|--------------|--------------|----|
| Ford Escape       | Small SUV    | Hybrid       | 27 |
| Ford Escape       | Small SUV    | Conventional | 21 |
| Saturn Vue        | Small SUV    | Hybrid       | 28 |
| Saturn Vue        | Small SUV    | Conventional | 22 |
| Lexus RX          | Mid-size SUV | Hybrid       | 23 |
| Lexus RX          | Mid-size SUV | Conventional | 19 |
| Toyota Highlander | Mid-size SUV | Hybrid       | 24 |
| Toyota Highlander | Mid-size SUV | Conventional | 18 |

At the  $\alpha = 0.05$ . level of significance, test for significant effects due to class, type and interaction.



HYBRIDTEST

## ONLINE RESOURCES

For the data files, additional online summary, questions, answers and software section go to the online platform.



## SUMMARY

In this chapter we showed how analysis of variance can be used to test for differences among means of several populations or treatments. We introduced the completely randomized design, the randomized block design and the two-factor factorial experiment. The completely randomized design and the randomized block design are used to draw conclusions about differences in the means of a single factor. The primary purpose of blocking in the randomized block design is to remove extraneous sources of variation from the error term. Such blocking provides a better estimate of the true error variance and a better test to determine whether the population or treatment means of the factor differ significantly.

We showed that the basis for the statistical tests used in analysis of variance and experimental design is the development of two independent estimates of the population variance  $\sigma^2$ . In the single-factor case, one estimator is based on the variation between the treatments; this estimator provides an unbiased estimate of  $\sigma^2$  only if the means  $\mu_1, \mu_2, \dots, \mu_k$  are all equal. A second estimator of  $\sigma^2$  is based on the variation of the observations within each sample; this estimator will always provide an unbiased estimate of  $\sigma^2$ . By computing the ratio of these two estimators (the  $F$  statistic) we developed a rejection rule for determining whether to reject the null hypothesis that the population or treatment means are equal. In all the experimental designs considered, the partitioning of the sum of squares and degrees of freedom into their various sources enabled us to compute the appropriate values for the analysis of variance calculations and tests. We also showed how Fisher's LSD procedure and the Bonferroni adjustment can be used to perform pairwise comparisons to determine which means are different.

## KEY TERMS

|                                  |                                |
|----------------------------------|--------------------------------|
| ANOVA table                      | Interaction                    |
| Blocking                         | Multiple comparison procedures |
| Comparisonwise Type I error rate | Partitioning                   |
| Completely randomized design     | Randomized block design        |
| Experimental units               | Replications                   |
| Experimentwise Type I error rate | Response variable              |
| Factor                           | Single-factor experiment       |
| Factorial experiments            | Treatment                      |

## KEY FORMULAE

**Completely randomized design Sample mean for treatment  $j$**

$$\bar{x}_j = \frac{\sum_{i=1}^{n_j} x_{ij}}{n_j} \quad (13.1)$$

**Sample variance for treatment  $j$**

$$s_j^2 = \frac{\sum_{i=1}^{n_j} (x_{ij} - \bar{x}_j)^2}{n_j - 1} \quad (13.2)$$

**Overall sample mean**

$$\bar{\bar{x}} = \frac{\sum_{j=1}^k \sum_{i=1}^{n_j} x_{ij}}{n_T} \quad (13.3)$$

$$n_T = n_1 + n_2 + \dots + n_k \quad (13.4)$$

**Mean square due to treatments**

$$MSTR = \frac{SSTR}{k-1} \quad (13.7)$$

**Sum of squares due to treatments**

$$SSTR = \sum_{j=1}^k n_j (\bar{x}_j - \bar{\bar{x}})^2 \quad (13.8)$$

**Mean square due to error**

$$MSE = \frac{SSE}{n_T - k} \quad (13.10)$$

**Sum of squares due to error**

$$\text{SSE} = \sum_{j=1}^k (n_j - 1) s_j^2 \quad (13.11)$$

**Test statistic for the equality of  $k$  population means**

$$F = \frac{\text{MSTR}}{\text{MSE}} \quad (13.12)$$

**Total sum of squares**

$$\text{SST} = \sum_{j=1}^k \sum_{i=1}^{n_j} (x_{ij} - \bar{x})^2 \quad (13.13)$$

**Partitioning of sum of squares**

$$\text{SST} = \text{SSTR} + \text{SSE} \quad (13.14)$$

**Multiple comparison procedures Test statistic for Fisher's LSD procedure**

$$t = \frac{\bar{x}_i - \bar{x}_j}{\sqrt{\text{MSE} \left( \frac{1}{n_i} + \frac{1}{n_j} \right)}} \quad (13.16)$$

**Fisher's LSD**

$$\text{LSD} = t_{a/2} \sqrt{\text{MSE} \left( \frac{1}{n_i} + \frac{1}{n_j} \right)} \quad (13.17)$$

**Randomized block design Total sum of squares**

$$\text{SST} = \sum_{i=1}^b \sum_{j=1}^k (x_{ij} - \bar{x})^2 \quad (13.22)$$

**Sum of squares due to treatments**

$$\text{SSTR} = b \sum_{j=1}^k (x_j - \bar{x})^2 \quad (13.23)$$

**Sum of squares due to blocks**

$$\text{SSBL} = k \sum_{i=1}^b (\bar{x}_{i\cdot} - \bar{x})^2 \quad (13.24)$$

**Sum of squares due to error**

$$\text{SSE} = \text{SST} - \text{SSTR} - \text{SSBL} \quad (13.25)$$

**Factorial experiments Total sum of squares**

$$\text{SST} = \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^r (x_{ijk} - \bar{x})^2 \quad (13.27)$$

### Sum of squares for factor A

$$SSA = br \sum_{i=1}^a (\bar{x}_{i\cdot} - \bar{\bar{x}})^2 \quad (13.28)$$

### Sum of squares for factor B

$$SSB = ar \sum_{j=1}^b (\bar{x}_{\cdot j} - \bar{\bar{x}})^2 \quad (13.29)$$

### Sum of squares for interaction

$$SSAB = r \sum_{i=1}^a \sum_{j=1}^b (\bar{x}_{ij} - \bar{x}_{i\cdot} - \bar{x}_{\cdot j} + \bar{\bar{x}})^2 \quad (13.30)$$

### Sum of squares for error

$$SSE = SST - SSA - SSB - SSAB \quad (13.31)$$

### CASE PROBLEM



#### *Product design testing*

An engineering manager has been designated the task of evaluating a commercial device subject to marked variations in temperature. Three different types of component are being considered for the device. When the device is manufactured and is shipped to the field, the manager has no control over the temperature extremes that the device will encounter, but knows from experience that temperature is an important factor in relation to the component's life. Notwithstanding this, temperature can be controlled in the laboratory for the purposes of the test.

The engineering manager arranges for all three components to be tested at the temperature levels:  $-10^\circ\text{C}$ ,  $20^\circ\text{C}$  and  $50^\circ\text{C}$  – as these temperature levels are consistent with the product end-use environment. Four components are tested for each combination of type and temperature, and all 36 tests are run in random order. The resulting observed component life data are presented in Table 1.



DEVICE



A product component is tested for its capability of enduring extreme heat

**TABLE 1** Component lifetimes (000s of hours)

| Type | Temperature (°C) |      |      |      | 50   |
|------|------------------|------|------|------|------|
|      | -10              | 20   | 20   | 50   |      |
| 1    | 3.12             | 3.70 | 0.82 | 0.96 | 0.48 |
|      | 1.80             | 4.32 | 1.92 | 1.80 | 1.97 |
| 2    | 3.60             | 4.51 | 3.02 | 2.93 | 0.60 |
|      | 3.82             | 3.02 | 2.54 | 2.76 | 1.39 |
| 3    | 3.31             | 2.64 | 4.18 | 2.88 | 2.30 |
|      | 4.03             | 3.84 | 3.60 | 3.34 | 1.44 |

***Managerial report***

1. What are the effects of the chosen factors on the life of the component?
2. Do any components have a consistently long life regardless of temperature?
3. What recommendation would you make to the engineering manager?



# 14

# Simple Linear Regression

## CHAPTER CONTENTS

Statistics in Practice Foreign direct investment (FDI) in China

- 14.1 Simple linear regression model
- 14.2 Least squares method
- 14.3 Coefficient of determination
- 14.4 Model assumptions
- 14.5 Testing for significance
- 14.6 Using the estimated regression equation for estimation and prediction
- 14.7 Computer solution
- 14.8 Residual analysis: validating model assumptions
- 14.9 Residual analysis: autocorrelation
- 14.10 Residual analysis: outliers and influential observations

**LEARNING OBJECTIVES** After reading this chapter and doing the exercises, you should be able to:

- 1 Understand how regression analysis can be used to develop an equation that estimates mathematically how two variables are related.
- 2 Understand the differences between the regression model, the regression equation and the estimated regression equation.
- 3 Know how to fit an estimated regression equation to a set of sample data based upon the least squares method.
- 4 Determine how good a fit is provided by the estimated regression equation and compute the sample correlation coefficient from the regression analysis output.
- 5 Understand the assumptions necessary for statistical inference and be able to test for a significant relationship.
- 6 Know how to develop confidence interval estimates of the mean value of  $Y$  and an individual value of  $Y$  for a given value of  $X$ .
- 7 Learn how to use a residual plot to make a judgement as to the validity of the regression assumptions, recognize outliers and identify influential observations.
- 8 Use the Durbin–Watson test to test for autocorrelation.
- 9 Know the definition of the following terms: independent and dependent variable; simple linear regression; regression model; regression equation and estimated regression equation; scatter diagram; coefficient of determination; standard error of the estimate; confidence interval; prediction interval; residual plot; standardized residual plot; outlier; influential observation; leverage.

**M**anagerial decisions are often based on the relationship between two or more variables. For example, after considering the relationship between advertising expenditures and sales, a marketing manager might attempt to predict sales for a given level of advertising expenditure. In another case, a public utility might use the relationship between the daily high temperature and the demand for electricity to predict electricity usage on the basis of next month's anticipated daily high temperatures. Sometimes a manager will rely on intuition to judge how two variables are related. However, if data can be obtained, a statistical procedure called *regression analysis* can be used to develop an equation showing how the variables are related.



### STATISTICS IN PRACTICE

#### Foreign direct investment (FDI) in China

In a recent study by Kingston Business School, regression modelling was used to investigate patterns of FDI in China as well as to assess the particular potential of the autonomous region of Guangxi in south-west China as an FDI attractor. A variety of simple models were developed based on positive correlations between gross domestic product (GDP) and FDI in provinces using data collected from official statistical sources.

Estimated regression equations obtained were as follows:

$$\hat{y} = 1.1m + 21.7x \quad 1990-1993$$

$$\hat{y} = 2.1m + 8.9x \quad 1995-1998$$

$$\hat{y} = 3.3m + 14.6x \quad 2000-2003$$

where:  $\hat{y}$  = estimated GDP

$$x = FDI$$

across all provinces.

In terms of FDI *per capita*, Guangxi has been ranked around 27 of 31 over the last ten years or so. FDI is a key driver of economic growth in modern China. But clearly Guangxi needs to improve its ranking if it is to be able to compete effectively with the more successful eastern coastal provinces and great municipalities.

Source: Foster, M. J. (2002) 'On evaluation of FDI's: Principles, actualities and possibilities'. *International Journal of Management and Decision-Making* 3(1): 67-82



In regression terminology, the variable being predicted is called the **dependent variable**. The variable or variables being used to predict the value of the dependent variable are called the **independent variables**. For example, in analyzing the effect of advertising expenditures on sales, a marketing manager's desire to predict sales would suggest making sales the dependent variable. Advertising expenditure would be the independent variable used to help predict sales. In statistical notation,  $Y$  denotes the dependent variable and  $X$  denotes the independent variable.

In this chapter we consider the simplest type of regression analysis involving one independent variable and one dependent variable in which the relationship between the variables is approximated by a straight line. It is called **simple linear regression**. Regression analysis involving two or more independent variables is called *multiple regression analysis*; multiple regression and cases involving curvilinear relationships are covered in Chapters 15 and 16.

## 14.1 SIMPLE LINEAR REGRESSION MODEL

Armand's Pizza Parlours is a chain of Italian food restaurants located in northern Italy. Armand's most successful locations are near college campuses. The managers believe that quarterly sales for these restaurants (denoted by  $Y$ ) are related positively to the size of the student population (denoted by  $X$ ); that is, restaurants near campuses with a large student population tend to generate more sales than those located near campuses with a small student population. Using regression analysis, we can develop an equation showing how the dependent variable  $Y$  is related to the independent variable  $X$ .

### Regression model and regression equation

In the Armand's Pizza Parlours example, the population consists of all the Armand's restaurants.

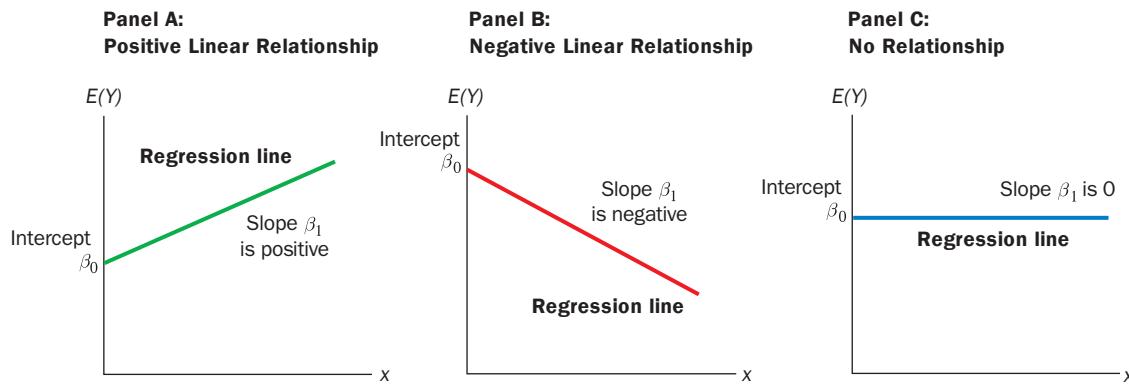
For every restaurant in the population, there is a value  $x$  of  $X$  (student population) and a corresponding value  $y$  of  $Y$  (quarterly sales). The equation that describes how  $Y$  is related to  $x$  and an error term is called the **regression model**. The regression model used in simple linear regression follows.

#### Simple linear regression model

$$Y = \beta_0 + \beta_1 x + \epsilon \quad (14.1)$$

$\beta_0$  and  $\beta_1$  are referred to as the parameters of the model, and  $\epsilon$  (the Greek letter epsilon) is a random variable referred to as the *error term*. The error term  $\epsilon$  accounts for the variability in  $Y$  that cannot be explained by the linear relationship between  $X$  and  $Y$ .

The population of all Armand's restaurants can also be viewed as a collection of subpopulations, one for each distinct value of  $X$ . For example, one subpopulation consists of all Armand's restaurants located near college campuses with 8000 students; another subpopulation consists of all Armand's restaurants located near college campuses with 9000 students and so on. Each subpopulation has a corresponding distribution of  $Y$  values. Thus, a distribution of  $Y$  values is associated with restaurants located near campuses with 8000 students a distribution of  $Y$  values is associated with restaurants located near campuses with 9000 students and so on. Each distribution of  $Y$  values has its own mean or expected value. The equation that describes how the expected value of  $Y$  – denoted by  $E(Y)$  or equivalently  $E(Y | X = x)$  – is related to  $x$  is called the **regression equation**. The regression equation for simple linear regression follows.

**FIGURE 14.1**

Possible regression lines in simple linear regression

### Simple linear regression equation

$$E(Y) = \beta_0 + \beta_1 x \quad (14.2)$$

The graph of the simple linear regression equation is a straight line;  $\beta_0$  is the  $y$ -intercept of the regression line;  $\beta_1$  is the slope and  $E(Y)$  is the mean or expected value of  $Y$  for a given value of  $X$ .

Examples of possible regression lines are shown in Figure 14.1. The regression line in Panel A shows that the mean value of  $Y$  is related positively to  $X$ , with larger values of  $E(Y)$  associated with larger values of  $X$ . The regression line in Panel B shows the mean value of  $Y$  is related negatively to  $X$ , with smaller values of  $E(Y)$  associated with larger values of  $X$ . The regression line in Panel C shows the case in which the mean value of  $Y$  is not related to  $X$ ; that is, the mean value of  $Y$  is the same for every value of  $X$ .

### Estimated regression equation

If the values of the population parameters  $\beta_0$  and  $\beta_1$  were known, we could use equation (14.2) to compute the mean value of  $Y$  for a given value of  $X$ . In practice, the parameter values are not known, and must be estimated using sample data. Sample statistics (denoted  $b_0$  and  $b_1$ ) are computed as estimates of the population parameters  $\beta_0$  and  $\beta_1$ . Substituting the values of the sample statistics  $b_0$  and  $b_1$  for  $\beta_0$  and  $\beta_1$  in the regression equation, we obtain the **estimated regression equation**. The estimated regression equation for simple linear regression follows.

### Estimated simple linear regression equation

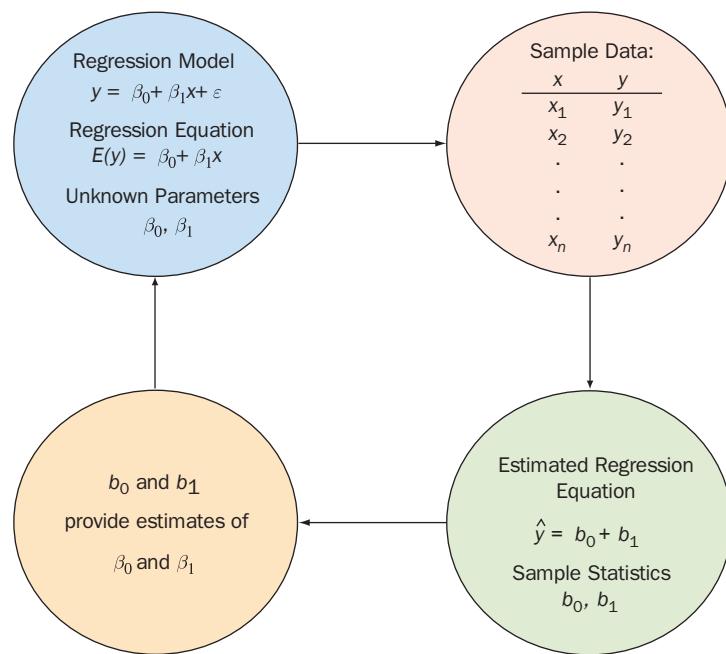
$$\hat{y} = b_0 + b_1 x \quad (14.3)$$

The graph of the estimated simple linear regression equation is called the *estimated regression line*;  $b_0$  is the  $y$  intercept and  $b_1$  is the slope. In the next section, we show how the least squares method can be used to compute the values of  $b_0$  and  $b_1$  in the estimated regression equation.

In general,  $\hat{y}$  is the point estimator of  $E(Y)$ , the mean value of  $Y$  for a given value of  $X$ . Thus, to estimate the mean or expected value of quarterly sales for all restaurants located near campuses with 10 000 students, Armand's would substitute the value of 10 000 for  $X$  in equation (14.3).

**FIGURE 14.2**

The estimation process in simple linear regression



In some cases, however, Armand's may be more interested in predicting sales for one particular restaurant. For example, suppose Armand's would like to predict quarterly sales for the restaurant located near Cabot College, a school with 10 000 students.

As it turns out, the best estimate of  $Y$  for a given value of  $X$  is also provided by  $\hat{y}$ . Thus, to predict quarterly sales for the restaurant located near Cabot College, Armand's would also substitute the value of 10 000 for  $X$  in equation (14.3). Because the value of  $\hat{y}$  provides both a point estimate of  $E(Y)$  and an individual value of  $Y$  for a given value of  $X$ , we will refer to  $\hat{y}$  simply as the *estimated value of  $y$* .

Figure 14.2 provides a summary of the estimation process for simple linear regression.

## 14.2 LEAST SQUARES METHOD

The **least squares method** is a procedure for using sample data to find the estimated regression equation. To illustrate the least squares method, suppose data were collected from a sample of ten Armand's Pizza Parlour restaurants located near college campuses. For the  $i$ th observation or restaurant in the sample,  $x_i$  is the size of the student population (in thousands) and  $y_i$  is the quarterly sales (in thousands of euros). The values of  $x_i$  and  $y_i$  for the ten restaurants in the sample are summarized in Table 14.1. We see that restaurant 1, with  $x_1 = 2$  and  $y_1 = 58$ , is near a campus with 2000 students and has quarterly sales of €58 000. Restaurant 2, with  $x_2 = 6$  and  $y_2 = 105$ , is near a campus with 6000 students and has quarterly sales of €105 000. The largest sales value is for restaurant 10, which is near a campus with 26 000 students and has quarterly sales of €202 000.

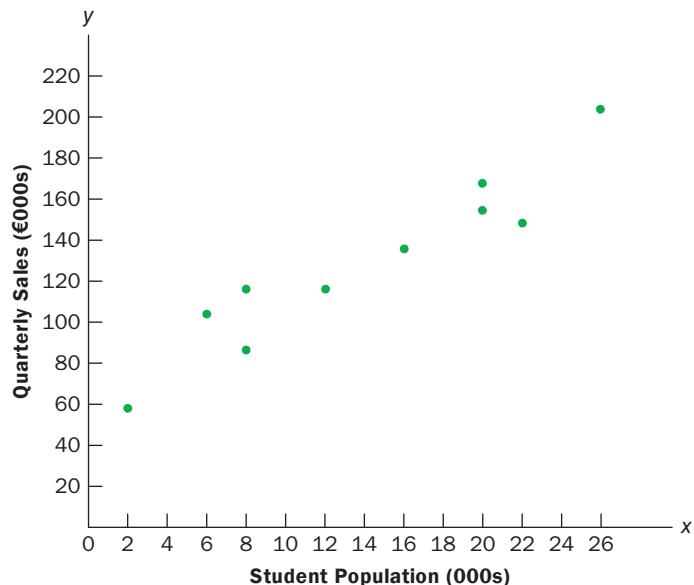
Figure 14.3 is a scatter diagram of the data in Table 14.1. Student population is shown on the horizontal axis and quarterly sales are shown on the vertical axis. **Scatter diagrams** for regression analysis are constructed with the independent variable  $X$  on the horizontal axis and the dependent variable  $Y$  on the vertical axis. The scatter diagram enables us to observe the data graphically and to draw preliminary conclusions about the possible relationship between the variables.

What preliminary conclusions can be drawn from Figure 14.3? Quarterly sales appear to be higher at campuses with larger student populations. In addition, for these data the relationship between the size of the student population and quarterly sales appears to be approximated by a straight line; indeed, a positive linear relationship is indicated between  $X$  and  $Y$ .



**FIGURE 14.3**

Scatter diagram of student population and quarterly sales for Armand's Pizza Parlours

**TABLE 14.1** Student population and quarterly sales data for ten Armand's Pizza Parlours

| Restaurant <i>i</i> | Student population (000s) <i>x<sub>i</sub></i> | Quarterly sales (€ <i>y<sub>i</sub></i> ) |
|---------------------|--|---|
| 1                   | 2  | 58  |
| 2                   | 6  | 105                                       |
| 3                   | 8  | 88  |
| 4                   | 8  | 118                                       |
| 5                   | 12   | 117                                       |
| 6                   | 16   | 137                                       |
| 7                   | 20   | 157                                       |
| 8                   | 20   | 169                                       |
| 9                   | 22   | 149                                       |
| 10                  | 26   | 202                                       |

We therefore choose the simple linear regression model to represent the relationship between quarterly sales and student population. Given that choice, our next task is to use the sample data in Table 14.1 to determine the values of  $b_0$  and  $b_1$  in the estimated simple linear regression equation. For the *i*th restaurant, the estimated regression equation provides:

$$\hat{y}_i = b_0 + b_1 x_i \quad (14.4)$$

where:

$\hat{y}_i$  = estimated value of quarterly sales (€000s) for the *i*th restaurant

$b_0$  = of the estimated regression line

$b_1$  = the slope of the estimated regression line

$x_i$  = size of the student population (000s) for the *i*th restaurant

Every restaurant in the sample will have an observed value of sales  $y_i$  and an estimated value of sales  $\hat{y}_i$ . For the estimated regression line to provide a good fit to the data, we want the differences between the observed sales values and the estimated sales values to be small.

The least squares method uses the sample data to provide the values of  $b_0$  and  $b_1$  that minimize the *sum of the squares of the deviations* between the observed values of the dependent variable  $y_i$  and the estimated values of the dependent variable. The criterion for the least squares method is given by expression (14.5).

**Least squares criterion**

where:

$$\text{Min } \sum(y_i - \hat{y}_i)^2$$
(14.5)

$y_i$  = observed value of the dependent variable for the  $i$ th observation  
 $\hat{y}_i$  = estimated value of the dependent variable for the  $i$ th observation

Differential calculus can be used to show that the values of  $b_0$  and  $b_1$  that minimize expression (14.5) can be found by using equations (14.6) and (14.7).

**Slope and y-intercept for the estimated regression equation\***

$$b_1 = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2}$$
(14.6)

$$b_0 = \bar{y} - b_1 \bar{x}$$
(14.7)

where:

$x_i$  = value of the independent variable for the  $i$ th observation  
 $y_i$  = value of the dependent variable for the  $i$ th observation  
 $\bar{x}$  = mean value for the independent variable  
 $\bar{y}$  = total number of observations  
 $n$  = total number of observations

Some of the calculations necessary to develop the least squares estimated regression equation for Armand's Pizza Parlours are shown in Table 14.2. With the sample of ten restaurants, we have  $n = 10$  observations. Because equations (14.6) and (14.7) require  $\bar{x}$  and  $\bar{y}$  we begin the calculations by computing  $\bar{x}$  and  $\bar{y}$ .

$$\bar{x} = \frac{\sum x_i}{n} = \frac{140}{10} = 14$$

$$\bar{y} = \frac{\sum y_i}{n} = \frac{1300}{10} = 130$$

Using equations (14.6) and (14.7) and the information in Table 14.2, we can compute the slope and intercept of the estimated regression equation for Armand's Pizza Parlours. The calculation of the slope ( $b_1$ ) proceeds as follows.

$$b_1 = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2}$$

$$= \frac{2840}{568} = 5$$

\*An alternative formula for  $b_1$  is:

$$b_1 = \frac{\sum x_i y_i - (\sum x_i \sum y_i)/n}{\sum x_i^2 - (\sum x_i)^2/n}$$

This form of equation (14.6) is often recommended when using a calculator to compute  $b_1$ .

**TABLE 14.2** Calculations for the least squares estimated regression equation for Armand's Pizza Parlours

| Restaurant $i$ | $x_i$      | $y_i$      | $x_i - \bar{x}$ | $y_i - \bar{y}$ | $(x_i - \bar{x})(y_i - \bar{y})$      | $(x_i - \bar{x})^2$      |
|----------------|------------|------------|-----------------|-----------------|---------------------------------------|--------------------------|
| 1              | 2          | 58         | -12             | -72             | 864                                   | 144                      |
| 2              | 6          | 105        | -8              | -25             | 200                                   | 64                       |
| 3              | 8          | 88         | -6              | -42             | 252                                   | 36                       |
| 4              | 8          | 118        | -6              | -12             | 72                                    | 36                       |
| 5              | 12         | 117        | -2              | -13             | 26                                    | 4                        |
| 6              | 16         | 137        | 2               | 7               | 14                                    | 4                        |
| 7              | 20         | 157        | 6               | 27              | 162                                   | 36                       |
| 8              | 20         | 169        | 6               | 39              | 234                                   | 36                       |
| 9              | 22         | 149        | 8               | 19              | 152                                   | 64                       |
| 10             | 26         | 202        | 12              | 72              | 864                                   | 144                      |
| Totals         | 140        | 1300       |                 |                 | 2840                                  | 568                      |
|                | $\sum x_i$ | $\sum y_i$ |                 |                 | $\sum (x_i - \bar{x})(y_i - \bar{y})$ | $\sum (x_i - \bar{x})^2$ |

The calculation of the  $y$  intercept ( $b_0$ ) follows.

$$\begin{aligned} b_0 &= y - b_1 x \\ &= 130 - 5(14) \\ &= 60 \end{aligned}$$

Thus, the estimated regression equation is:

$$\hat{y} = 60 + 5x$$

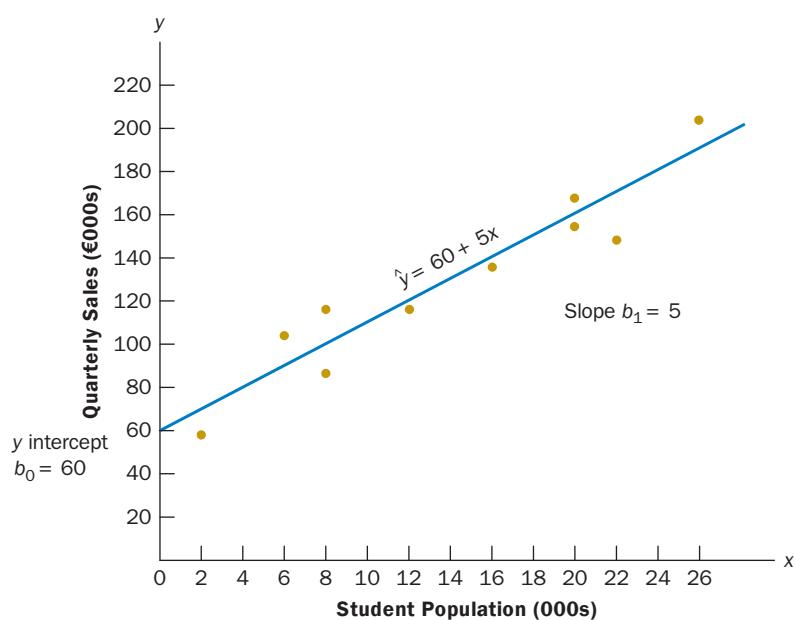
Figure 14.4 shows the graph of this equation on the scatter diagram.

The slope of the estimated regression equation ( $b_1 = 5$ ) is positive, implying that as student population increases, sales increase. In fact, we can conclude (based on sales measured in €000s and student population in 000s) that an increase in the student population of 1000 is associated with an increase of €5000 in expected sales; that is, quarterly sales are expected to increase by €5 per student.

If we believe the least squares estimated regression equation adequately describes the relationship between  $X$  and  $Y$ , it would seem reasonable to use the estimated regression equation to predict the value of  $Y$  for a given value of  $X$ .

**FIGURE 14.4**

Graph of the estimated regression equation for Armand's Pizza Parlours  
 $\hat{y} = 60 + 5x$



For example, if we wanted to predict quarterly sales for a restaurant to be located near a campus with 16 000 students, we would compute:

$$\hat{y} = 60 + 5(16) = 140$$

Therefore, we would predict quarterly sales of €140 000 for this restaurant. In the following sections we will discuss methods for assessing the appropriateness of using the estimated regression equation for estimation and prediction.

## EXERCISES

### Methods

- 1.** Given are five observations for two variables,  $X$  and  $Y$ .

|       |   |   |   |    |    |
|-------|---|---|---|----|----|
| $x_i$ | 1 | 2 | 3 | 4  | 5  |
| $y_i$ | 3 | 7 | 5 | 11 | 14 |

- a. Develop a scatter diagram for these data.
- b. What does the scatter diagram developed in part (a) indicate about the relationship between the two variables?
- c. Try to approximate the relationship between  $X$  and  $Y$  by drawing a straight line through the data.
- d. Develop the estimated regression equation by computing the values of  $b_0$  and  $b_1$  using equations (14.6) and (14.7).
- e. Use the estimated regression equation to predict the value of  $Y$  when  $X = 4$ .

- 2.** Given are five observations for two variables,  $X$  and  $Y$ .

|       |    |    |    |    |
|-------|----|----|----|----|
| $x_i$ | 2  | 3  | 5  | 8  |
| $y_i$ | 25 | 25 | 20 | 16 |

- a. Develop a scatter diagram for these data.
- b. What does the scatter diagram developed in part (a) indicate about the relationship between the two variables?
- c. Try to approximate the relationship between  $X$  and  $Y$  by drawing a straight line through the data.
- d. Develop the estimated regression equation by computing the values of  $b_0$  and  $b_1$  using equations (14.6) and (14.7).
- e. Use the estimated regression equation to predict the value of  $Y$  when  $X = 6$ .

- 3.** Given are five observations collected in a regression study on two variables.

|       |   |   |   |   |   |
|-------|---|---|---|---|---|
| $x_i$ | 2 | 4 | 5 | 7 | 8 |
| $y_i$ | 2 | 3 | 2 | 6 | 4 |

- a. Develop a scatter diagram for these data.
- b. Develop the estimated regression equation for these data.
- c. Use the estimated regression equation to predict the value of  $Y$  when  $X = 4$ .



COMPLETE  
SOLUTIONS



DOWS & P

### Applications

- 4.** The following data were collected on the height (cm) and weight (kg) of women swimmers.

|        |     |     |     |     |     |
|--------|-----|-----|-----|-----|-----|
| Height | 173 | 163 | 157 | 165 | 168 |
| Weight | 60  | 49  | 46  | 52  | 58  |

- a. Develop a scatter diagram for these data with height as the independent variable.
- b. What does the scatter diagram developed in part (a) indicate about the relationship between the two variables?

- c. Try to approximate the relationship between height and weight by drawing a straight line through the data.
- d. Develop the estimated regression equation by computing the values of  $b_0$  and  $b_1$ .
- e. If a swimmer's height is 160cm, what would you estimate their weight to be?
5. The Dow Jones Industrial Average (DJIA) and the Standard & Poor's 500 (S&P) indexes are both used as measures of overall movement in the stock market. The DJIA is based on the price movements of 30 large companies; the S&P 500 is an index composed of 500 stocks. Some say the S&P 500 is a better measure of stock market performance because it is broader based. The closing prices for the DJIA and the S&P 500 for ten weeks, beginning with 11 February 2009, follow (<http://uk.finance.yahoo.com>, 21 April 2009).
- | Date      | DJIA    | S&P    |
|-----------|---------|--------|
| 11 Feb 09 | 7939.53 | 833.74 |
| 18 Feb 09 | 7555.63 | 788.42 |
| 25 Feb 09 | 7270.89 | 764.90 |
| 03 Mar 09 | 6726.02 | 696.33 |
| 10 Mar 09 | 6926.49 | 719.60 |
| 17 Mar 09 | 7395.70 | 778.12 |
| 24 Mar 09 | 7660.21 | 806.12 |
| 31 Mar 09 | 7608.92 | 797.87 |
| 07 Apr 09 | 7789.56 | 815.55 |
| 14 Apr 09 | 7920.18 | 841.50 |
- a. Develop a scatter diagram for these data with DJIA as the independent variable.
- b. Develop the least squares estimated regression equation.
- c. Suppose the closing price for the DJIA is 8000. Estimate the closing price for the S&P 500.
6. The following table shows the observations of transportation time and distance for a sample of ten rail shipments made by a motor parts supplier.

| Delivery time (days) | Distance (kilometres) |
|----------------------|-----------------------|
| 5                    | 210                   |
| 7                    | 290                   |
| 6                    | 350                   |
| 11                   | 480                   |
| 8                    | 490                   |
| 11                   | 730                   |
| 12                   | 780                   |
| 8                    | 850                   |
| 15                   | 920                   |
| 12                   | 1010                  |

- a. Develop a scatter diagram for these data with distance as the independent variable.
- b. Develop an estimated regression equation that can be used to predict delivery time given the distance.
- c. Use the estimated regression equation to predict delivery time for a customer situated 600 miles from the company.



## 14.3 COEFFICIENT OF DETERMINATION

For the Armand's Pizza Parlours example, we developed the estimated regression equation  $\hat{y} = 60 + 5x$  to approximate the linear relationship between the size of student population  $X$  and quarterly sales  $Y$ . A question now is: How well does estimated regression equation fit the data? In this section, we show that **coefficient of determination** provides a measure of the goodness of fit for the estimated regression equation.

For the  $i$ th observation, the difference between the observed value of the dependent variable,  $y_i$ , and the estimated value of the dependent variable,  $\hat{y}_i$ , is called the  **$i$ th residual**. The  $i$ th residual represents the error in using  $y_i$  to estimate  $\hat{y}_i$ . Thus, for the  $i$ th observation, the residual is  $y_i - \hat{y}_i$ . The sum of squares of these residuals or errors is the quantity that is minimized by the least squares method. This quantity, also known as the *sum of squares due to error*, is denoted by SSE.

### Sum of squares due to error

$$\text{SSE} = \sum (y_i - \hat{y}_i)^2 \quad (14.8)$$

The value of SSE is a measure of the error in using the least squares regression equation to estimate the values of the dependent variable in the sample.

In Table 14.3 we show the calculations required to compute the sum of squares due to error for the Armand's Pizza Parlours example. For instance, for restaurant 1 the values of the independent and dependent variables are  $x_1 = 2$  and  $y_1 = 58$ . Using the estimated regression equation, we find that the estimated value of quarterly sales for restaurant 1 is  $\hat{y}_1 = 60 + 5(2) = 70$ . Thus, the error in using  $y_1 - \hat{y}_1 = 58 - 70 = -12$ . The squared error,  $(-12)^2 = 144$ , is shown in the last column of Table 14.3. After computing and squaring the residuals for each restaurant in the sample, we sum them to obtain  $\text{SSE} = 1530$ . Thus, SSE = 1530 measures the error in using the estimated regression equation  $\hat{y}_1 = 60 + 5x$  to predict sales.

Now suppose we are asked to develop an estimate of quarterly sales without knowledge of the size of the student population. Without knowledge of any related variables, we would use the sample mean as an estimate of quarterly sales at any given restaurant. Table 14.2 shows that for the sales data,  $\sum y_i = 1300$ . Hence, the mean value of quarterly sales for the sample of ten Armand's restaurants is  $\bar{y} = \sum y/n = 1300/10 = 130$ .

**TABLE 14.3** Calculation of SSE for Armand's Pizza Parlours

| Restaurant<br><i>i</i> | $x_i = \text{Student population (000s)}$ | $y_i = \text{Quarterly sales (\$000s)}$ | Predicted sales<br>$\hat{y}_i = 60 + 5x_i$ | Error<br>$y_i - \hat{y}_i$ | Squared error<br>$(y_i - \hat{y}_i)^2$ |
|------------------------|--|---|--|----------------------------|--|
| 1                      | 2  | 58                                      | 70   | -12                        | 144                                    |
| 2                      | 6  | 105                                     | 90   | 15                         | 225                                    |
| 3                      | 8  | 88                                      | 100  | -12                        | 144                                    |
| 4                      | 8  | 118                                     | 100  | 18                         | 324                                    |
| 5                      | 12                                       | 117                                     | 120  | -3                         | 9                                      |
| 6                      | 16                                       | 137                                     | 140  | -3                         | 9                                      |
| 7                      | 20                                       | 157                                     | 160  | -3                         | 9                                      |
| 8                      | 20                                       | 169                                     | 160  | 9                          | 81                                     |
| 9                      | 22                                       | 149                                     | 170  | -21                        | 441                                    |
| 10                     | 26                                       | 202                                     | 190  | 12                         | 144                                    |
|                        |  |   |  |                            | SSE = 1530                             |

**TABLE 14.4** Computation of the total sum of squares for Armand's Pizza Parlours

| Restaurant $i$ | $x_i = \text{Student population (000s)}$ | $y_i = \text{Quarterly sales (\text{€}000s)}$ | Deviation<br>$y_i - \bar{y}$ | Squared deviation<br>$(y_i - \bar{y})^2$ |
|----------------|--|---|------------------------------|--|
| 1              | 2  | 58  | -72                          | 5184                                     |
| 2              | 6  | 105   | -25                          | 625                                      |
| 3              | 8  | 88  | -42                          | 1764                                     |
| 4              | 8  | 118   | -12                          | 144                                      |
| 5              | 12                                       | 117   | -13                          | 169                                      |
| 6              | 16                                       | 137   | 7                            | 49                                       |
| 7              | 20                                       | 157   | 27                           | 729                                      |
| 8              | 20                                       | 169   | 39                           | 1521                                     |
| 9              | 22                                       | 149   | 19                           | 361                                      |
| 10             | 26                                       | 202   | 72                           | 5184                                     |
|                |  |   |                              | SST = 15 730                             |

In Table 14.4 we show the sum of squared deviations obtained by using the sample mean  $\bar{y} = 130$  to estimate the value of quarterly sales for each restaurant in the sample. For the  $i$ th restaurant in the sample, the difference  $y_i - \bar{y}$  provides a measure of the error involved in using  $\bar{y}$  to estimate sales. The corresponding sum of squares, called the *total sum of squares*, is denoted SST.

### Total sum of squares

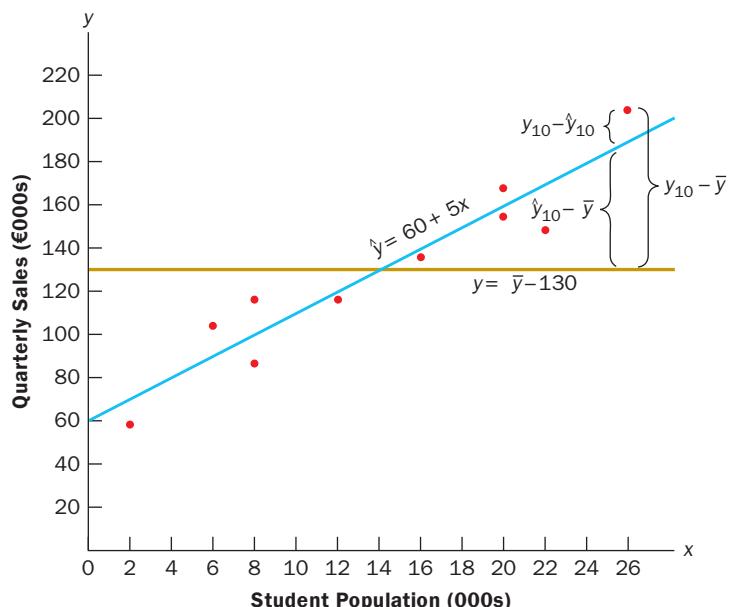
$$\text{SST} = \sum(y_i - \hat{y})^2 \quad (14.9)$$

The sum at the bottom of the last column in Table 14.4 is the total sum of squares for Armand's Pizza Parlours; it is SST = 15 730.

In Figure 14.5 we show the estimated regression line  $\hat{y}_i = 60 + 5x$  and the line corresponding to  $\bar{y} = 130$ . Note that the points cluster more closely around the estimated regression line than they do about the line  $\bar{y} = 130$ .

**FIGURE 14.5**

Deviations about the estimated regression line and the line  $y = \bar{y}$  for Armand's Pizza Parlours



For example, for the tenth restaurant in the sample we see that the error is much larger when  $\bar{y} = 130$  is used as an estimate of  $y_{10}$  than when  $\hat{y}_i = 60 + 5(26) = 190$  is used. We can think of SST as a measure of how well the observations cluster about the  $y$  line and SSE as a measure of how well the observations cluster about the  $\hat{y}$  line.

To measure how much the  $\hat{y}$  values on the estimated regression line deviate from  $\bar{y}$ , another sum of squares is computed. This sum of squares, called the *sum of squares due to regression*, is denoted SSR.

### Sum of squares due to regression

$$\text{SSR} = \sum(\hat{y}_i - \bar{y})^2 \quad (14.10)$$

From the preceding discussion, we should expect that SST, SSR and SSE are related. Indeed, the relationship among these three sums of squares provides one of the most important results in statistics.

### Relationship among SST, SSR and SSE

where:

$$\text{SST} = \text{SSR} + \text{SSE} \quad (14.11)$$

SST = total sum of squares

SSR = sum of squares due to regression

SSE = sum of squares due to error

Equation (14.11) shows that the total sum of squares can be partitioned into two components, the regression sum of squares and the sum of squares due to error. Hence, if the values of any two of these sum of squares are known, the third sum of squares can be computed easily. For instance, in the Armand's Pizza Parlours example, we already know that SSE = 1530 and SST = 15 730; therefore, solving for SSR in equation (14.11), we find that the sum of squares due to regression is

$$\text{SSR} = \text{SST} - \text{SSE} = 15\,730 - 1530 = 14\,200$$

Now let us see how the three sums of squares, SST, SSR and SSE, can be used to provide a measure of the goodness of fit for the estimated regression equation. The estimated regression equation would provide a perfect fit if every value of the dependent variable  $y_i$  happened to lie on the estimated regression line. In this case,  $y_i - \hat{y}_i$  would be zero for each observation, resulting in SSE = 0. Because SST = SSR + SSE, we see that for a perfect fit SSR must equal SST and the ratio (SSR/SST) must equal one. Poorer fits will result in larger values for SSE. Solving for SSE in equation (14.11), we see that SSE = SST - SSR. Hence, the largest value for SSE (and hence the poorest fit) occurs when SSR = 0 and SSE = SST. The ratio SSR/SST, which will take values between zero and one, is used to evaluate the goodness of fit for the estimated regression equation. This ratio is called the *coefficient of determination* and is denoted by  $r^2$ .

### Coefficient of determination

$$r^2 = \frac{\text{SSR}}{\text{SST}} \quad (14.12)$$

For the Armand's Pizza Parlours example, the value of the coefficient of determination is

$$r^2 = \frac{\text{SSR}}{\text{SST}} = \frac{14\,200}{15\,730} = 0.9027$$

When we express the coefficient of determination as a percentage,  $r^2$  can be interpreted as the percentage of the total sum of squares that can be explained by using the estimated regression equation. For Armand's Pizza Parlours, we can conclude that 90.27 per cent of the total sum of squares can be explained by using the estimated regression equation  $\hat{y} = 60 + 5x$  to predict quarterly sales. In other words, 90.27 per cent of the variability in sales can be explained by the linear relationship between the size of the student population and sales. We should be pleased to find such a good fit for the estimated regression equation.

## Correlation coefficient

In Chapter 3 we introduced the **correlation coefficient** as a descriptive measure of the strength of linear association between two variables,  $X$  and  $Y$ . Values of the correlation coefficient are always between  $-1$  and  $+1$ . A value of  $+1$  indicates that the two variables  $X$  and  $Y$  are perfectly related in a positive linear sense. That is, all data points are on a straight line that has a positive slope. A value of  $-1$  indicates that  $X$  and  $Y$  are perfectly related in a negative linear sense, with all data points on a straight line that has a negative slope. Values of the correlation coefficient close to zero indicate that  $X$  and  $Y$  are not linearly related.

In Section 3.5 we presented the equation for computing the sample correlation coefficient. If a regression analysis has already been performed and the coefficient of determination  $r^2$  computed, the sample correlation coefficient can be computed as follows.

### Sample correlation coefficient

$$\begin{aligned} r_{XY} &= (\text{sign of } b_1) \sqrt{\text{Coefficient of determination}} \\ &= (\text{sign of } b_1) \sqrt{r^2} \end{aligned} \tag{14.13}$$

where:

$b_1$  = the slope of the estimated regression equation  $\hat{y} = b_0 + b_1x$

The sign for the sample correlation coefficient is positive if the estimated regression equation has a positive slope ( $b_1 > 0$ ) and negative if the estimated regression equation has a negative slope ( $b_1 < 0$ ).

For the Armand's Pizza Parlour example, the value of the coefficient of determination corresponding to the estimated regression equation  $\hat{y} = 60 + 5x$  is 0.9027. Because the slope of the estimated regression equation is positive, equation (14.13) shows that the sample correlation coefficient is  $=\sqrt{0.9027} + 0.9501$ .

With a sample correlation coefficient of  $r_{XY} = 0.9501$ , we would conclude that a strong positive linear association exists between  $X$  and  $Y$ .

In the case of a linear relationship between two variables, both the coefficient of determination and the sample correlation coefficient provide measures of the strength of the relationship. The coefficient of determination provides a measure between zero and one whereas the sample correlation coefficient provides a measure between  $-1$  and  $+1$ . Although the sample correlation coefficient is restricted to a linear relationship between two variables, the coefficient of determination can be used for nonlinear relationships and for relationships that have two or more independent variables. Thus, the coefficient of determination provides a wider range of applicability.

## EXERCISES

### Methods

7. The data from Exercise 1 follow.

|       |   |   |   |    |    |
|-------|---|---|---|----|----|
| $x_i$ | 1 | 2 | 3 | 4  | 5  |
| $y_i$ | 3 | 7 | 5 | 11 | 14 |

The estimated regression equation for these data is  $\hat{y} = 0.20 + 2.60x$ .

- a. Compute SSE, SST and SSR using equations (14.8), (14.9) and (14.10).
- b. Compute the coefficient of determination  $r^2$ . Comment on the goodness of fit.
- c. Compute the sample correlation coefficient.

8. The data from Exercise 2 follow.

|       |    |    |    |    |    |
|-------|----|----|----|----|----|
| $x_i$ | 2  | 3  | 5  | 1  | 8  |
| $y_i$ | 25 | 25 | 20 | 30 | 16 |

The estimated regression equation for these data is  $\hat{y} = 30.33 - 1.88x$ .

- a. Compute SSE, SST and SSR.
- b. Compute the coefficient of determination  $r^2$ . Comment on the goodness of fit.
- c. Compute the sample correlation coefficient.

9. The data from Exercise 3 follow.

|       |   |   |   |   |   |
|-------|---|---|---|---|---|
| $x_i$ | 2 | 4 | 5 | 7 | 8 |
| $y_i$ | 2 | 3 | 2 | 6 | 4 |

The estimated regression equation for these data is  $\hat{y} = 0.75 + 0.51x$ . What percentage of the total sum of squares can be accounted for by the estimated regression equation? What is the value of the sample correlation coefficient?

### Applications

10. The estimated regression equation for the data in Exercise 5 can be shown to be  $\hat{y} = -75.586 + 0.115x$ . What percentage of the total sum of squares can be accounted for by the estimated regression equation?

Comment on the goodness of fit. What is the sample correlation coefficient?

11. An investment manager studying haulage companies calculates for a random sample of six such firms, the percentage capital investment in vehicles and the profit before tax as a percentage of turnover with the following results:

|                                |    |    |    |    |    |    |
|--------------------------------|----|----|----|----|----|----|
| % Capital investment, vehicles | 37 | 47 | 10 | 22 | 41 | 25 |
| % Profit                       | 14 | 21 | -5 | 16 | 19 | 8  |

- a. Calculate the coefficient of determination. What percentage of the variation in total cost can be explained by production volume?
- b. Carry out a linear regression analysis for the data.
- c. Hence estimate the percentage profit when the percentage capital investment, vehicles is
  - (i) 30%.
  - (ii) 90%.



COMPLETE  
SOLUTIONS



DOWS & P

- 12.** *PC World* provided details for ten of the most economical laser printers (*PC World*, April 2009). The following data show the maximum printing speed in pages per minute (ppm) and the price (in euros including 15 per cent value added tax) for each printer.

| Name                | Speed (ppm) | Price (€) |
|---------------------|-------------|-----------|
| Brother HL 2035     | 18          | 61.35     |
| HP Laserjet P1005   | 15          | 70.13     |
| Samsung ML-1640     | 16          | 77.39     |
| HP Laserjet P1006   | 17          | 82.93     |
| Brother HL-2140     | 22          | 92.34     |
| Brother DCP7030     | 22          | 96.04     |
| HP Laserjet P1009   | 16          | 99.52     |
| HP Laserjet P1505   | 24          | 119.10    |
| Samsung 4300        | 18          | 121.64    |
| Epson EPL-6200 Mono | 20          | 133.53    |

- a. Develop the estimated regression equation with speed as the independent variable.
- b. Compute  $r^2$ . What percentage of the variation in cost can be explained by the printing speed?
- c. What is the sample correlation coefficient between speed and price? Does it reflect a strong or weak relationship between printing speed and cost?

## 14.4 MODEL ASSUMPTIONS

We saw in the previous section that the value of the coefficient of determination ( $r^2$ ) is a measure of the goodness of fit of the estimated regression equation. However, even with a large value of  $r^2$ , the estimated regression equation should not be used until further analysis of the appropriateness of the assumed model has been conducted. An important step in determining whether the assumed model is appropriate involves testing for the significance of the relationship. The tests of significance in regression analysis are based on the following assumptions about the error term  $\epsilon$ .

### Assumptions about the error term $\epsilon$ in the regression model

$$Y = \beta_0 + \beta_1 x + \epsilon$$

- 1.** The error term  $E(\epsilon) = 0$  is a random variable with a mean or expected value of zero; that is,  $E(\epsilon) = 0$ .

*Implication:*  $\beta_0$  and  $\beta_1$  are constants, therefore  $E(\beta_0) = \beta_0$  and  $E(\beta_1) = \beta_1$ ; thus, for a given value  $x$  of  $X$ , the expected value of  $Y$  is:

$$E(Y) = \beta_0 + \beta_1 x \quad (14.14)$$

As we indicated previously, equation (14.14) is referred to as the regression equation.

- 2.** The variance of  $\epsilon$ , denoted by  $\sigma^2$ , is the same for all values of  $X$

*Implication:* The variance of  $Y$  about the regression line equals  $\sigma^2$  and is the same for all values of  $X$ .

- 3.** The values of  $\epsilon$  are independent.

*Implication:* The value of  $\epsilon$  for a particular value of  $X$  is not related to the value of  $\epsilon$  for any other value of  $X$ ; thus, the value of  $Y$  for a particular value of  $X$  is not related to the value of for any other value of  $X$ .

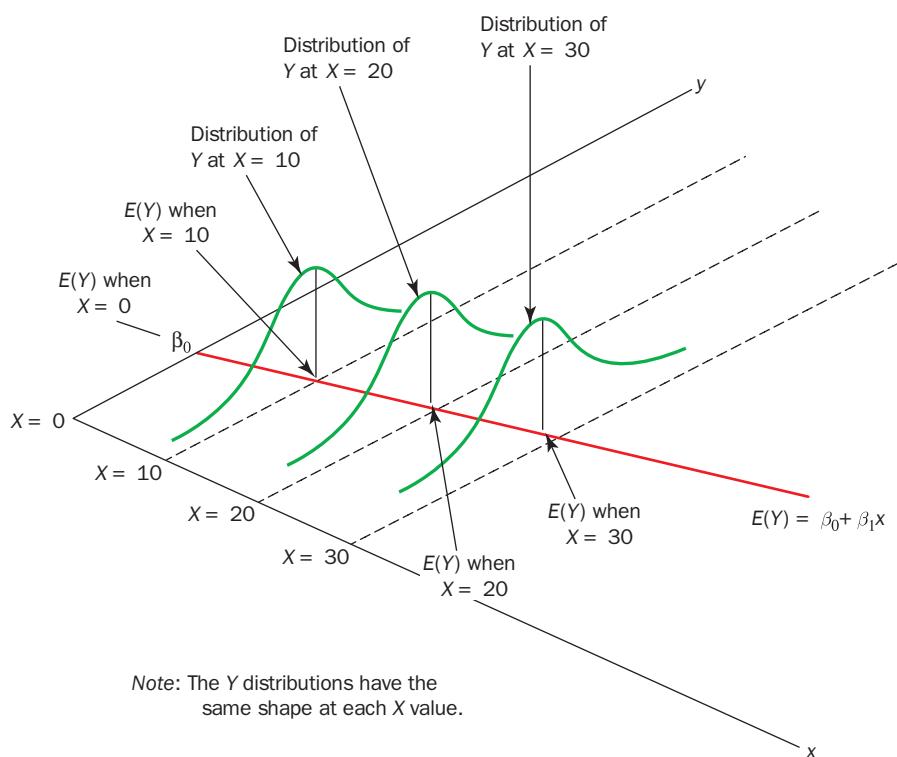
4. The error term  $\epsilon$  is a normally distributed random variable.

*Implication:* Because  $Y$  is a linear function of  $\epsilon$ ,  $Y$  is also a normally distributed random variable.

Figure 14.6 illustrates the model assumptions and their implications; note that in this graphical interpretation, the value of  $E(Y)$  changes according to the specific value of  $X$  considered. However, regardless of the  $X$  value, the probability distribution of  $\epsilon$  and hence the probability distributions of  $Y$  are normally distributed, each with the same variance. The specific value of the error  $\epsilon$  at any particular point depends on whether the actual value of  $Y$  is greater than or less than  $E(Y)$ .

**FIGURE 14.6**

Assumptions for the regression model



At this point, we must keep in mind that we are also making an assumption or hypothesis about the form of the relationship between  $X$  and  $Y$ . That is, we assume that a straight line represented by  $\beta_0 + \beta_1 x$  is the basis for the relationship between the variables. We must not lose sight of the fact that some other model, for instance  $Y = \beta_0 + \beta_1 x^2 + \epsilon$  may turn out to be a better model for the underlying relationship.

## 14.5 TESTING FOR SIGNIFICANCE

In a simple linear regression equation, the mean or expected value of  $E(Y) = \beta_0 + \beta_1 x$ . If the value of  $E(Y) = \beta_0 + (0)x = \beta_0$ . In this case, the mean value of  $Y$  does not depend on the value of  $X$  and hence we would conclude that  $X$  and  $Y$  are not linearly related. Alternatively, if the value of  $\beta_1$  is not equal to zero, we would conclude that the two variables are related. Thus, to test for a significant regression relationship, we must conduct a hypothesis test to determine whether the value of  $\beta_1$  is zero. Two tests are commonly used. Both require an estimate of  $\sigma^2$ , the variance of  $\epsilon$  in the regression model.

## Estimate of $\sigma^2$

From the regression model and its assumptions we can conclude that  $\sigma^2$  also represents the variance of the  $Y$  values about the regression line. Recall that the deviations of the  $Y$  values about the estimated regression line are called residuals. Thus, SSE, the sum of squared residuals, is a measure of the variability of the actual observations about the estimated regression line. The **mean square error (MSE)** provides the estimate of  $\sigma^2$ ; it is SSE divided by its degrees of freedom.

With  $\hat{y}_i = b_0 + b_1x_i$ , SSE can be written as:

$$\text{SSE} = \sum(y_1 - \hat{y})^2 = \sum(y_1 - b_0 - b_1x_i)^2$$

Every sum of squares is associated with a number called its degrees of freedom. Statisticians have shown that SSE has  $n - 2$  degrees of freedom because two parameters ( $\beta_0$  and  $\beta_1$ ) must be estimated to compute SSE. Thus, the mean square is computed by dividing SSE by  $n - 2$ . MSE provides an unbiased estimator of  $\sigma^2$ . Because the value of MSE provides an estimate of  $\sigma^2$ , the notation  $s^2$  is also used.

### Mean square error (estimate of $\sigma^2$ )

$$s^2 = \text{MSE} = \frac{\text{SSE}}{n-2} \quad (14.15)$$

In Section 14.3 we showed that for the Armand's Pizza Parlours example,  $\text{SSE} = 1530$ ; hence,

$$s^2 = \text{MSE} = \frac{1530}{8} = 191.25$$

provides an unbiased estimate of  $\sigma^2$ .

To estimate  $\sigma$  we take the square root of  $s^2$ . The resulting value,  $s$ , is referred to as the **standard error of the estimate**.

### Standard error of estimate

$$s = \sqrt{\text{MSE}} = \sqrt{\frac{\text{SSE}}{n-2}} \quad (14.16)$$

For the Armand's Pizza Parlours example,  $s = \sqrt{\text{MSE}} = \sqrt{191.25} = 13.829$ . In the following discussion, we use the standard error of the estimate in the tests for a significant relationship between  $X$  and  $Y$ .

## **t test**

The simple linear regression model is  $Y = \beta_0 + \beta_1x + \epsilon$ . If  $X$  and  $Y$  are linearly related, we must have  $\beta_1 \neq 0$ . The purpose of the *t* test is to see whether we can conclude that  $\beta_1 \neq 0$ .

We will use the sample data to test the following hypotheses about the parameter  $\beta_1$ .

$$\begin{aligned} H_0: \beta_1 &= 0 \\ H_1: \beta_1 &\neq 0 \end{aligned}$$

If  $H_0$  is rejected, we will conclude that  $\beta_1 \neq 0$  and that a statistically significant relationship exists between the two variables. However, if  $H_0$  cannot be rejected, we will have insufficient evidence to

conclude that a significant relationship exists. The properties of the sampling distribution of  $b_1$ , the least squares estimator of  $\beta_1$ , provide the basis for the hypothesis test.

First, let us consider what would happen if we used a different random sample for the same regression study. For example, suppose that Armand's Pizza Parlours used the sales records of a different sample of ten restaurants. A regression analysis of this new sample might result in an estimated regression equation similar to our previous estimated regression equation  $\hat{y} = 60 + 5x$ . However, it is doubtful that we would obtain exactly the same equation (with an intercept of exactly 60 and a slope of exactly 5). Indeed,  $b_0$  and  $b_1$ , the least squares estimators, are sample statistics with their own sampling distributions. The properties of the sampling distribution of  $b_1$  follow.

### Sampling distribution of $b_1$

Expected value  $E(b_1) = \beta_1$

Standard deviation

$$\sigma_{b_1} = \frac{\sigma}{\sqrt{\sum(x_i - \bar{x})^2}} \quad (14.17)$$

Distribution form

Normal

Note that the expected value of  $b_1$  is equal to  $\beta_1$ , so  $b_1$  is an unbiased estimator of  $\beta_1$ . As we do not know the value of  $\sigma$ , so we estimate  $\sigma_{b_1}$  by  $s_{b_1}$  where  $s_{b_1}$  is derived by substituting  $s$  for  $\sigma$  in equation (14.17):

### Estimated standard deviation of $b_1$

$$s_{b_1} = \frac{s}{\sqrt{\sum(x_i - \bar{x})^2}} \quad (14.18)$$

For Armand's Pizza Parlours,  $s = 13.829$ . Hence, using  $\sum(x_i - \bar{x})^2 = 568$  as shown in Table 14.2, we have:

$$s_{b_1} = \frac{13.829}{\sqrt{568}} = 0.5803$$

as the estimated standard deviation of  $b_1$ .

The  $t$  test for a significant relationship is based on the fact that the test statistic:

$$\frac{b_1 - \beta_1}{s_{b_1}}$$

follows a  $t$  distribution with  $n - 2$  degrees of freedom. If the null hypothesis is true, then  $\beta_1 = 0$  and  $t = b_1/s_{b_1}$ .

Let us conduct this test of significance for Armand's Pizza Parlours at the  $\alpha = 0.01$  level of significance. The test statistic is:

$$t = \frac{b_1}{s_{b_1}} = \frac{5}{0.5803} = 8.62$$

The  $t$  distribution table shows that with  $n - 2 = 10 - 2 = 8$  degrees of freedom,  $t = 3.355$  provides an area of 0.005 in the upper tail. Thus, the area in the upper tail of the  $t$  distribution corresponding to the test statistic  $t = 8.62$  must be less than 0.005. Because this test is a two-tailed test, we double this value to

conclude that the  $p$ -value associated with  $t = 8.62$  must be less than  $2(0.005) = 0.01$ . MINITAB, SPSS or EXCEL show the  $p$ -value = 0.000. Because the  $p$ -value is less than  $\alpha = 0.01$ , we reject  $H_0$  and conclude that  $\beta_1$  is not equal to zero. This evidence is sufficient to conclude that a significant relationship exists between student population and quarterly sales. A summary of the  $t$  test for significance in simple linear regression follows.

### **t test for significance in simple linear regression**

$$\begin{aligned} H_0: \beta_1 &= 0 \\ H_1: \beta_1 &\neq 0 \end{aligned}$$

#### **Test statistic**

$$t = \frac{b_1}{s_{b_1}} \quad (14.19)$$

#### **Rejection rule**

$p$ -value approach: Reject  $H_0$  if  $p$ -value  $\leq \alpha$

Critical value approach: Reject  $H_0$  if  $t \leq -t_{\alpha/2}$  or if  $t \geq t_{\alpha/2}$

where  $t_{\alpha/2}$  is based on a  $t$  distribution with  $n - 2$  degrees of freedom.

## **Confidence interval for $\beta_1$**

The form of a confidence interval for  $\beta_1$  is as follows:

$$b_1 \pm t_{\alpha/2} s_{b_1}$$

The point estimator is  $b_1$  and the margin of error is  $t_{\alpha/2} s_{b_1}$ . The confidence coefficient associated with this interval is  $1 - \alpha$ , and  $t_{\alpha/2}$  is the  $t$  value providing an area of  $\alpha/2$  in the upper tail of a  $t$  distribution with  $n - 2$  degrees of freedom. For example, suppose that we wanted to develop a 99 per cent confidence interval estimate of  $\beta_1$  for Armand's Pizza Parlours. From Table 2 of Appendix B we find that the  $t$  value corresponding to  $\alpha = 0.01$  and  $n - 2 = 10 - 2 = 8$  degrees of freedom is  $t_{0.005} = 3.355$ . Thus, the 99 per cent confidence interval estimate of  $\beta_1$  is:

$$b_1 \pm t_{\alpha/2} s_{b_1} = 5 \pm 3.355(0.5803) = 5 \pm 1.95$$

or 3.05 to 6.95.

In using the  $t$  test for significance, the hypotheses tested were:

$$\begin{aligned} H_0: \beta_1 &= 0 \\ H_1: \beta_1 &\neq 0 \end{aligned}$$

At the  $\alpha = 0.01$  level of significance, we can use the 99 per cent confidence interval as an alternative for drawing the hypothesis testing conclusion for the Armand's data. Because 0, the hypothesized value of  $\beta_1$ , is not included in the confidence interval (3.05 to 6.95), we can reject  $H_0$  and conclude that a significant statistical relationship exists between the size of the student population and quarterly sales. In general, a confidence interval can be used to test any two-sided hypothesis about  $\beta_1$ . If the hypothesized value of  $\beta_1$  is contained in the confidence interval, do not reject  $H_0$ . Otherwise, reject  $H_0$ .

## **F test**

An  $F$  test, based on the  $F$  probability distribution, can also be used to test for significance in regression. With only one independent variable, the  $F$  test will provide the same conclusion as the  $t$  test; that is, if the  $t$  test indicates  $\beta_1 \neq 0$  and hence a significant relationship, the  $F$  test will also indicate a significant

relationship.\* But with more than one independent variable, only the  $F$  test can be used to test for an overall significant relationship.

The logic behind the use of the  $F$  test for determining whether the regression relationship is statistically significant is based on the development of two independent estimates of  $\sigma^2$ . We explained how MSE provides an estimate of  $\sigma^2$ . If the null hypothesis  $H_0: \beta_1 = 0$  is true, the sum of squares due to regression, SSR, divided by its degrees of freedom provides another independent estimate of  $\sigma^2$ . This estimate is called the *mean square due to regression*, or simply the *mean square regression*, and is denoted MSR. In general,

$$\text{MSR} = \frac{\text{SSR}}{\text{Regression degrees of freedom}}$$

For the models we consider in this text, the regression degrees of freedom is always equal to the number of independent variables in the model:

### Mean square regression

$$\text{MSR} = \frac{\text{SSR}}{\text{Number of independent variables}} \quad (14.20)$$

Because we consider only regression models with one independent variable in this chapter, we have  $\text{MSR} = \text{SSR}/1 = \text{SSR}$ . Hence, for Armand's Pizza Parlours,  $\text{MSR} = \text{SSR} = 14\,200$ .

If the null hypothesis ( $H_0: \beta_1 = 0$ ) is true, MSR and MSE are two independent estimates of  $\sigma^2$  and the sampling distribution of MSR/MSE follows an  $F$  distribution with numerator degrees of freedom equal to one and denominator degrees of freedom equal to  $n - 2$ . Therefore, when  $\beta_1 = 0$ , the value of MSR/MSE should be close to one. However, if the null hypothesis is false ( $\beta_1 \neq 0$ ), MSR will overestimate  $\sigma^2$  and the value of MSR/MSE will be inflated; thus, large values of MSR/MSE lead to the rejection of  $H_0$  and the conclusion that the relationship between  $X$  and  $Y$  is statistically significant.

Let us conduct the  $F$  test for the Armand's Pizza Parlours example. The test statistic is:

$$F = \frac{\text{MSR}}{\text{MSE}} = \frac{14\,200}{191.25} = 74.25$$

The  $F$  distribution table (Table 4 of Appendix B) shows that with one degree of freedom in the denominator and  $n - 2 = 10 - 2 = 8$  degrees of freedom in the numerator,  $F = 11.26$  provides an area of 0.01 in the upper tail. Thus, the area in the upper tail of the  $F$  distribution corresponding to the test statistic  $F = 74.25$  must be less than 0.01. Thus, we conclude that the  $p$ -value must be less than 0.01. MINITAB, SPSS or EXCEL show the  $p$ -value = 0.000. Because the  $p$ -value is less than  $\alpha = 0.01$ , we reject  $H_0$  and conclude that a significant relationship exists between the size of the student population and quarterly sales. A summary of the  $F$  test for significance in simple linear regression follows.

### **F test for significance in simple linear regression**

$$\begin{aligned} H_0: \beta_1 &= 0 \\ H_1: \beta_1 &\neq 0 \end{aligned}$$

\*In fact  $F = t^2$  for a simple regression model.

**Test statistic**

$$F = \frac{\text{MSR}}{\text{MSE}} \quad (14.21)$$

**Rejection rule**

*p*-value approach: Reject  $H_0$  if  $p\text{-value} \leq \alpha$

Critical value approach: Reject  $H_0$  if  $F \geq F_\alpha$

where  $F_\alpha$  is based on a  $F$  distribution with one degree of freedom in the numerator and  $n - 2$  degrees of freedom in the denominator.

In Chapter 13 we covered analysis of variance (ANOVA) and showed how an **ANOVA table** could be used to provide a convenient summary of the computational aspects of analysis of variance. A similar ANOVA table can be used to summarize the results of the  $F$  test for significance in regression. Table 14.5 is the general form of the ANOVA table for simple linear regression. Table 14.6 is the ANOVA table with the  $F$  test computations performed for Armand's Pizza Parlours. Regression, Error and Total are the labels for the three sources of variation, with SSR, SSE and SST appearing as the corresponding sum of squares in column 3. The degrees of freedom, 1 for SSR,  $n - 2$  for SSE and  $n - 1$  for SST, are shown in column 2. Column 4 contains the values of MSR and MSE and column 5 contains the value of  $F = \text{MSR}/\text{MSE}$ . Almost all computer printouts of regression analysis include an ANOVA table summary and the  $F$  test for significance.

### Some cautions about the interpretation of significance tests

Rejecting the null hypothesis  $H_0: \beta_1 = 0$  and concluding that the relationship between  $X$  and  $Y$  is significant does not enable us to conclude that a cause-and-effect relationship is present between  $X$  and  $Y$ . Concluding a cause-and-effect relationship is warranted only if the analyst can provide some type of theoretical justification that the relationship is in fact causal. In the Armand's Pizza Parlours example, we can conclude that there is a significant relationship between the size of the student population  $X$  and quarterly sales  $Y$ ; moreover, the estimated regression equation  $\hat{y} = 60 + 5x$  provides the least squares estimate of the relationship. We cannot, however, conclude that changes in student population  $X$  cause changes in quarterly sales  $Y$  just because we identified a statistically significant relationship. The appropriateness of such a cause-and-effect conclusion is left to supporting theoretical justification and to good judgement on the part of the analyst. Armand's managers felt that increases in the student population were a likely cause of increased quarterly sales.

**TABLE 14.5** General form of the ANOVA table for simple linear regression

| Source of variation | Degrees of freedom | Sum of squares | Mean square                             | $F$                       |
|---------------------|--------------------|----------------|---|---------------------------|
| Regression          | 1                  | SSR            | $\text{MSR} = \frac{\text{SSR}}{1}$     | $\text{MSR} / \text{MSE}$ |
| Error               | $n - 2$            | SSE            | $\text{MSE} = \frac{\text{SSE}}{n - 2}$ |                           |
| Total               | $n - 1$            | SST            |   |                           |

**TABLE 14.6** ANOVA table for the Armand's Pizza Parlours problem

| Source of variation | Degrees of freedom | Sum of squares | Mean square                   | F                               |
|---------------------|--------------------|----------------|-------------------------------|---------------------------------|
| Regression          | 1                  | 14 200         | $\frac{14\ 200}{1} = 14\ 200$ | $\frac{14\ 200}{191.25} = 74.2$ |
| Error               | 8                  | 1 530          | $\frac{1\ 530}{8} = 191.25$   |                                 |
| Total               | 9                  | 15 730         |                               |                                 |

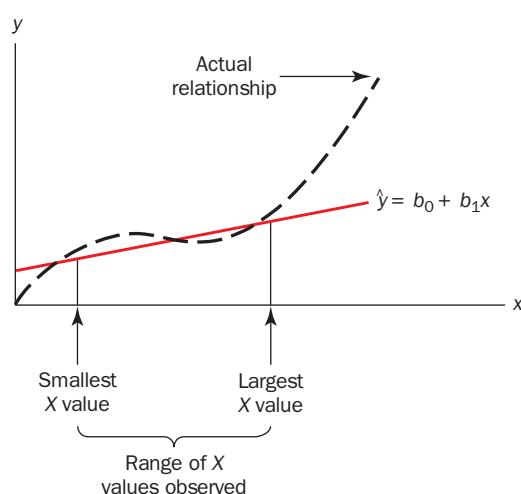
Thus, the result of the significance test enabled them to conclude that a cause-and-effect relationship was present.

In addition, just because we are able to reject  $H_0: \beta_1 = 0$  and demonstrate statistical significance does not enable us to conclude that the relationship between  $X$  and  $Y$  is linear. We can state only that  $X$  and  $Y$  are related and that a linear relationship explains a significant portion of the variability in  $Y$  over the range of values for  $X$  observed in the sample. Figure 14.7 illustrates this situation. The test for significance calls for the rejection of the null hypothesis  $H_0: \beta_1 = 0$  and leads to the conclusion that  $X$  and  $Y$  are significantly related, but the figure shows that the actual relationship between  $X$  and  $Y$  is not linear. Although the linear approximation provided by  $\hat{y} = b_0 + b_1x$  is good over the range of  $X$  values observed in the sample, it becomes poor for  $X$  values outside that range.

Given a significant relationship, we should feel confident in using the estimated regression equation for predictions corresponding to  $X$  values within the range of the  $X$  values observed in the sample. For Armand's Pizza Parlours, this range corresponds to values of  $X$  between 2 and 26. Unless other reasons indicate that the model is valid beyond this range, predictions outside the range of the independent variable should be made with caution. For Armand's Pizza Parlours, because the regression relationship has been found significant at the 0.01 level, we should feel confident using it to predict sales for restaurants where the associated student population is between 2000 and 26 000.

**FIGURE 14.7**

Example of a linear approximation of a nonlinear relationship



## EXERCISES

### Methods

**13.** The data from Exercise 1 follow.

|       |   |   |   |    |    |
|-------|---|---|---|----|----|
| $x_i$ | 1 | 2 | 3 | 4  | 5  |
| $y_i$ | 3 | 7 | 5 | 11 | 14 |

- a. Compute the mean square error using equation (14.15).
- b. Compute the standard error of the estimate using equation (14.16).
- c. Compute the estimated standard deviation of  $b_1$  using equation (14.18).
- d. Use the  $t$  test to test the following hypotheses ( $\alpha = 0.05$ ):

$$H_0: \beta_1 = 0$$

$$H_1: \beta_1 \neq 0$$

- e. Use the  $F$  test to test the hypotheses in part (d) at a 0.05 level of significance. Present the results in the analysis of variance table format.

**14.** The data from Exercise 2 follow.

|       |    |    |    |    |    |
|-------|----|----|----|----|----|
| $x_i$ | 2  | 3  | 5  | 1  | 8  |
| $y_i$ | 25 | 25 | 20 | 30 | 16 |

- a. Compute the mean square error using equation (14.15).
- b. Compute the standard error of the estimate using equation (14.16).
- c. Compute the estimated standard deviation of  $b_1$  using equation (14.18).
- d. Use the  $t$  test to test the following hypotheses ( $\beta = 0.05$ ):

$$H_0: \beta_1 = 0$$

$$H_1: \beta_1 \neq 0$$

- e. Use the  $F$  test to test the hypotheses in part (d) at a 0.05 level of significance. Present the results in the analysis of variance table format.

**15.** The data from Exercise 3 follow.

|       |   |   |   |   |   |
|-------|---|---|---|---|---|
| $x_i$ | 2 | 4 | 5 | 7 | 8 |
| $y_i$ | 2 | 3 | 2 | 6 | 4 |

- a. What is the value of the standard error of the estimate?
- b. Test for a significant relationship by using the  $t$  test. Use  $\alpha = 0.05$ .
- c. Use the  $F$  test to test for a significant relationship. Use  $\alpha = 0.05$ . What is your conclusion?

### Applications

**16.** The Supplies Office of a local authority is reviewing its policy for the replacement of photocopiers. For the ten photocopiers in use within the local authority, the number of breakdowns during the past year has been recorded.

| Photocopier       | A  | B | C  | D  | E  | F  | G  | H | I  | J  |
|-------------------|----|---|----|----|----|----|----|---|----|----|
| No. of breakdowns | 11 | 9 | 13 | 10 | 18 | 13 | 15 | 8 | 16 | 10 |
| Age (years)       | 6  | 4 | 6  | 2  | 9  | 4  | 8  | 1 | 7  | 3  |

The Supplies Offices wishes to determine how the number of breakdowns depends upon the age of the photocopier.

Use  $\alpha = 0.05$  to test whether number of breakdowns is significantly related to the age. Show the ANOVA table. What is your conclusion?



PRINTERS  
2009

- 17.** Refer to Exercise 12 where the data were used to determine whether the price of a printer is related to the speed for plain text printing (*PC World*, April 2009). Does the evidence indicate a significant relationship between printing speed and price? Conduct the appropriate statistical test and state your conclusion. Use  $\alpha = 0.05$ .

## 14.6 USING THE ESTIMATED REGRESSION EQUATION FOR ESTIMATION AND PREDICTION

When using the simple linear regression model we are making an assumption about the relationship between  $X$  and  $Y$ . We then use the least squares method to obtain the estimated simple linear regression equation. If a significant relationship exists between  $X$  and  $Y$ , and the coefficient of determination shows that the fit is good, the estimated regression equation should be useful for estimation and prediction.

### Point estimation

In the Armand's Pizza Parlours example, the estimated regression equation  $\hat{y} = 60 + 5x$  provides an estimate of the relationship between the size of the student population  $X$  and quarterly sales  $Y$ . We can use the estimated regression equation to develop a point estimate of either the mean value of  $Y$  or an individual value of  $Y$  corresponding to a given value of  $X$ . For instance, suppose Armand's managers want a point estimate of the mean quarterly sales for all restaurants located near college campuses with 10 000 students. Using the estimated regression equation  $\hat{y} = 60 + 5x$ , we see that for  $X = 10$  (or 10 000 students),  $\hat{y} = 60 + 5(10) = 110$ . Thus, a point estimate of the mean quarterly sales for all restaurants located near campuses with 10 000 students is €110 000.

Now suppose Armand's managers want to predict sales for an individual restaurant located near Cabot College, a school with 10 000 students. Then, as the point estimate for an individual value of  $Y$  is the same as the point estimate for the mean value of  $Y$  we would predict quarterly sales of  $\hat{y} = 60 + 5(10) = 110$  or €110 000 for this one restaurant.

### Interval estimation

Point estimates do not provide any information about the precision associated with an estimate. For that we must develop interval estimates much like those in Chapters 10 and 11. The first type of interval estimate, a **confidence interval**, is an interval estimate of the *mean value of  $Y$*  for a given value of  $X$ . The second type of interval estimate, a **prediction interval**, is used whenever we want an interval estimate of an *individual value of  $Y$*  for a given value of  $X$ . The point estimate of the mean value of  $Y$  is the same as the point estimate of an individual value of  $Y$ . But the interval estimates we obtain for the two cases are different. The margin of error is larger for a prediction interval.

### Confidence interval for the mean value of $Y$

The estimated regression equation provides a point estimate of the mean value of  $Y$  for a given value of  $X$ . In developing the confidence interval, we will use the following notation.

$x_p$  = the particular or given value of the independent variable  $X$

$Y_p$  = the dependent variable  $Y$  corresponding to the given  $x_p$

$E(Y_p)$  = the mean or expected value of the dependent variable  $Y_p$  corresponding to the given  $x_p$

$\hat{y}_p = b_0 + b_1 x_p$  = the point estimate of  $E(Y_p)$  when  $X = x_p$

Using this notation to estimate the mean sales for all Armand's restaurants located near a campus with 10 000 students, we have  $x_p = 10$ , and  $E(Y_p)$  denotes the unknown mean value of sales for all restaurants where  $x_p = 10$ . The point estimate of  $E(Y_p)$  is given by  $\hat{y}_p = 60 + 5(10) = 110$ .

In general, we cannot expect  $\hat{y}_p$  to equal  $E(Y_p)$  exactly. If we want to make an inference about how close  $\hat{y}_p$  is to the true mean value  $E(Y_p)$ , we will have to estimate the variance of  $\hat{y}_p$ . The formula for estimating the variance of  $\hat{y}_p$  given  $x_p$ , denoted by  $s_{\hat{y}_p}^2$  is:

$$s_{\hat{y}_p}^2 = s^2 \left[ \frac{1}{n} + \frac{(x_p - \bar{x})^2}{\sum(x_i - \bar{x})^2} \right]$$

The general expression for a confidence interval follows.

#### Confidence Interval for $E(Y_p)$

$$\hat{y}_p \pm t_{\alpha/2}s \sqrt{\frac{1}{n} + \frac{(x_p - \bar{x})^2}{\sum(x_i - \bar{x})^2}} \quad (14.22)$$

where the confidence coefficient is  $1 - \alpha$  and  $t_{\alpha/2}$  is based on a  $t$  distribution with  $n - 2$  degrees of freedom.

Using expression (14.22) to develop a 95 per cent confidence interval of the mean quarterly sales for all Armand's restaurants located near campuses with 10 000 students, we need the value of  $t$  for  $\alpha/2 = 0.025$  and  $n - 2 = 10 - 2 = 8$  degrees of freedom. Using Table 2 of Appendix B, we have  $t_{0.025} = 2.306$ . Thus, with  $\hat{y}_p = 110$ , the 95 per cent confidence interval estimate is:

$$\begin{aligned} \hat{y}_p &\pm t_{\alpha/2}s \sqrt{\frac{1}{n} + \frac{(x_p - \bar{x})^2}{\sum(x_i - \bar{x})^2}} \\ 110 &\pm 2.306 \times 13.829 \sqrt{\frac{1}{10} + \frac{(10 - 14)^2}{568}} \\ &= 110 \pm 11.415 \end{aligned}$$

In euros, the 95 per cent confidence interval for the mean quarterly sales of all restaurants near campuses with 10 000 students is €110 000  $\pm$  €11 415. Therefore, the 95 per cent confidence interval for the mean quarterly sales when the student population is 10 000 is €98 585 to €121 415.

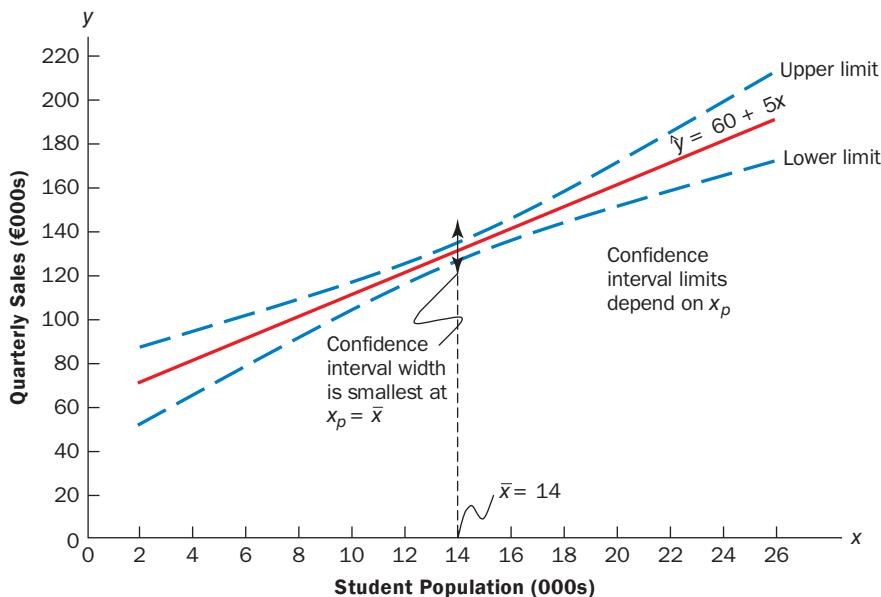
Note that the estimated standard deviation of  $\hat{y}_p$  is smallest when  $x_p = \bar{x}$  so that the quantity  $x_p - \bar{x} = 0$ . In this case, the estimated standard deviation of  $\hat{y}_p$  becomes:

$$s \sqrt{\frac{1}{n} + \frac{(x_p - \bar{x})^2}{\sum(x_i - \bar{x})^2}} = s \sqrt{\frac{1}{n}}$$

This result implies that the best or most precise estimate of the mean value of  $Y$  occurs when  $x_p = \bar{x}$ . But, the further  $x_p$  is from  $x_p = \bar{x}$  the larger  $x_p - \bar{x}$  becomes and thus the wider confidence intervals will be for the mean value of  $Y$ . This pattern is shown graphically in Figure 14.8.

#### Prediction interval for an individual value of $Y$

Suppose that instead of estimating the mean value of sales for all Armand's restaurants located near campuses with 10 000 students, we want to estimate the sales for an individual restaurant located near Cabot College, a school with 10 000 students.

**FIGURE 14.8**

Confidence intervals for the mean sales  $Y$  at given values of student population  $x$

As noted previously, the point estimate of  $y_p$ , the value of  $Y$  corresponding to the given  $x_p$ , is provided by the estimated regression equation  $\hat{y}_p = b_0 + b_1 x_p$ . For the restaurant at Cabot College, we have  $x_p = 10$  and a corresponding predicted quarterly sales of  $\hat{y}_p = 60 + 5(10) = 110$  or €110 000.

Note that this value is the same as the point estimate of the mean sales for all restaurants located near campuses with 10 000 students.

To develop a prediction interval, we must first determine the variance associated with using  $\hat{y}_p$  as an estimate of an individual value of  $Y$  when  $X = x_p$ . This variance is made up of the sum of the following two components.

- 1 The variance of individual  $Y$  values about the mean  $E(Y_p)$ , an estimate of which is given by  $s^2$ .
- 2 The variance associated with using  $\hat{y}_p$  to estimate  $E(Y_p)$ , an estimate of which is given by:

$$s_{\hat{y}_p}^2 = s^2 \left[ \frac{1}{n} + \frac{(x_p - \bar{x})^2}{\sum(x_i - \bar{x})^2} \right]$$

Thus the formula for estimating the variance of an individual value of  $Y_p$ , is:

$$s^2 + s_{\hat{y}_p}^2 = s^2 + s^2 \left[ \frac{1}{n} + \frac{(x_p - \bar{x})^2}{\sum(x_i - \bar{x})^2} \right] = s^2 \left[ 1 + \frac{1}{n} + \frac{(x_p - \bar{x})^2}{\sum(x_i - \bar{x})^2} \right]$$

The general expression for a prediction interval follows.

### Prediction interval for $y_p$

$$\hat{y}_p \pm t_{\alpha/2} s \sqrt{1 + \frac{1}{n} + \frac{(x_p - \bar{x})^2}{\sum(x_i - \bar{x})^2}} \quad (14.23)$$

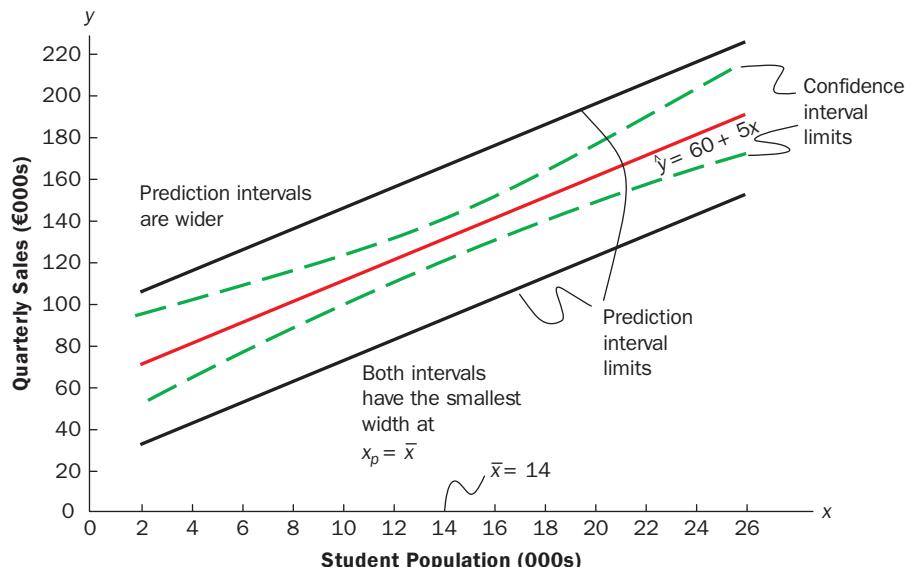
where the confidence coefficient is  $1 - \alpha$  and  $t_{\alpha/2}$  is based on a  $t$  distribution with  $n - 2$  degrees of freedom.

Thus the 95 per cent prediction interval of sales for one specific restaurant located near a campus with 10 000 students is:

$$\begin{aligned}\hat{y}_p &\pm t_{z/2} s \sqrt{1 + \frac{1}{n} + \frac{(x_p - \bar{x})^2}{\sum(x_i - \bar{x})^2}} \\ &= 110 \pm 2.306 \times 13.829 \sqrt{1 + \frac{1}{10} + \frac{(10 - 14)^2}{568}} \\ &= 110 \pm 33.875\end{aligned}$$

In euros, this prediction interval is €110 000 ± €33 875 or €76 125 to €143 875. Note that the prediction interval for an individual restaurant located near a campus with 10 000 students is wider than the confidence interval for the mean sales of all restaurants located near campuses with 10 000 students. The difference reflects the fact that we are able to estimate the mean value of  $Y$  more precisely than we can an individual value of  $Y$ .

Both confidence interval estimates and prediction interval estimates are most precise when the value of the independent variable is  $x_p = \bar{x}$ . The general shapes of confidence intervals and the wider prediction intervals are shown together in Figure 14.9.



**FIGURE 14.9**

Confidence and prediction intervals for sales  $Y$  at given values of student population  $X$

## EXERCISES

### Methods

- 18.** The data from Exercise 1 follow.

|       |   |   |   |    |    |
|-------|---|---|---|----|----|
| $x_i$ | 1 | 2 | 3 | 4  | 5  |
| $y_i$ | 3 | 7 | 5 | 11 | 14 |

- a. Use expression (14.22) to develop a 95 per cent confidence interval for the expected value of  $Y$  when  $X = 4$ .  
 b. Use expression (14.23) to develop a 95 per cent prediction interval for  $Y$  when  $X = 4$ .



COMPLETE  
SOLUTIONS

**19.** The data from Exercise 2 follow.

|       |    |    |    |    |    |
|-------|----|----|----|----|----|
| $x_i$ | 2  | 3  | 5  | 1  | 8  |
| $y_i$ | 25 | 25 | 20 | 30 | 16 |

- Estimate the standard deviation of  $\hat{y}_p$  when  $X = 3$ .
- Develop a 95 per cent confidence interval for the expected value of  $Y$  when  $X = 3$ .
- Estimate the standard deviation of an individual value of  $Y$  when  $X = 3$ .
- Develop a 95 per cent prediction interval for  $Y$  when  $X = 3$ .

**20.** The data from Exercise 3 follow.

|       |   |   |   |   |   |
|-------|---|---|---|---|---|
| $x_i$ | 2 | 4 | 5 | 7 | 8 |
| $y_i$ | 2 | 3 | 2 | 6 | 4 |

Develop the 95 per cent confidence and prediction intervals when  $X = 3$ . Explain why these two intervals are different.

### Applications

**21.** A company that manufactures ballpoint pens has a cost function of the form:

$$T = T_0 + kx^2$$

where  $T_0$  is a constant value linked to the production method used and  $x$  is the quantity of pens (in thousands) manufactured. During the last year, the total costs of the company were recorded as follows (where pens were recorded in thousands and costs are recorded in €000s)

| Month | # of pens ( $x$ ) | Total cost ( $T$ ) |
|-------|-------------------|--------------------|
| Jan.  | 5.5               | 80.1               |
| Feb.  | 4.2               | 80.4               |
| Mar.  | 6.4               | 58.0               |
| Apr.  | 3.3               | 90.1               |
| May   | 7.2               | 47.2               |
| Jun.  | 8.6               | 27.0               |
| Jul.  | 9.2               | 17.4               |
| Aug.  | 3.9               | 82.8               |
| Sep.  | 6.8               | 53.8               |
| Oct.  | 8.3               | 33.1               |
| Nov.  | 5.9               | 63.2               |
| Dec.  | 8.2               | 32.8               |

- Derive least squares estimates of  $T_0$  and  $k$ .
- Hence determine a 95 per cent interval estimate of Total Cost when 6000 pens are manufactured.

## 14.7 COMPUTER SOLUTION

Performing the regression analysis computations without the help of a computer can be quite time consuming. In this section we discuss how the computational burden can be minimized by using a computer software package such as MINITAB.

We entered Armand's student population and sales data into a MINITAB worksheet. The independent variable was named Pop and the dependent variable was named Sales to assist with interpretation of the

computer output. Using MINITAB, we obtained the printout for Armand's Pizza Parlours shown in Figure 14.10.\* The interpretation of this printout follows.

- 1 MINITAB prints the estimated regression equation as Sales = 60.0 + 5.00Pop.
- 2 A table is printed that shows the values of the coefficients  $b_0$  and  $b_1$ , the standard deviation of each coefficient, the  $t$  value obtained by dividing each coefficient value by its standard deviation, and the  $p$ -value associated with the  $t$  test. Because the  $p$ -value is zero (to three decimal places), the sample results indicate that the null hypothesis ( $H_0: \beta_1 = 0$ ) should be rejected. Alternatively, we could compare 8.62 (located in the  $t$ -ratio column) to the appropriate critical value. This procedure for the  $t$  test was described in Section 14.5.
- 3 MINITAB prints the standard error of the estimate,  $s = 13.83$ , as well as information about the goodness of fit. Note that ' $R$ -sq = 90.3 per cent' is the coefficient of determination expressed as a percentage. The value ' $R$ -sq(adj) = 89.1 per cent' is discussed in Chapter 15.
- 4 The ANOVA table is printed below the heading Analysis of Variance. MINITAB uses the label Residual Error for the error source of variation. Note that DF is an abbreviation for degrees of freedom and that MSR is given as 14 200 and MSE as 191.

The ratio of these two values provides the  $F$  value of 74.25 and the corresponding  $p$ -value of 0.000. Because the  $p$ -value is zero (to three decimal places), the relationship between Sales and Pop is judged statistically significant.

- 5 The 95 per cent confidence interval estimate of the expected sales and the 95 per cent prediction interval estimate of sales for an individual restaurant located near a campus with 10 000 students are printed below the ANOVA table. The confidence interval is (98.58, 121.42) and the prediction interval is (76.12, 143.88) as we showed in Section 14.6.

## EXERCISES

### Applications

22. The commercial division of the Supreme real estate firm in Cyprus is conducting a regression analysis of the relationship between  $X$ , annual gross rents (in thousands of euros), and  $Y$ , selling price (in thousands of euros) for apartment buildings. Data were collected on several properties recently sold and the following computer selective output was obtained.

| The regression equation is<br>$Y = 20.0 + 7.21 X$ |        |         |      |
|---|--------|---------|------|
| Predictor   | Coef   | SE Coef | T    |
| Constant  | 20.000 | 3.2213  | 6.21 |
| X   | 7.210  | 1.3626  | 5.29 |
| Analysis of Variance                              |        |         |      |
| SOURCE  | DF     | SS      |      |
| Regression  | 1      | 41587.3 |      |
| Residual Error                                    | 7      |         |      |
| Total   | 8      | 51984.1 |      |

- a. How many apartment buildings were in the sample?
- b. Write the estimated regression equation.
- c. What is the value of  $s_{b0}$ ?
- d. Use the  $F$  statistic to test the significance of the relationship at a 0.05 level of significance.
- e. Estimate the selling price of an apartment building with gross annual rents of €50 000.



COMPLETE  
SOLUTIONS



\*The MINITAB steps necessary to generate the output are given in the software section on the online platform.

- 23.** Following is a portion of the computer output for a regression analysis relating  $Y$  = maintenance expense (euros per month) to  $X$  = usage (hours per week) of a particular brand of computer terminal.

|  |
|--|
| The regression equation is<br>$Y = 6.1092 + .8951 X$ |
| Predictor      Coef      SE Coef                     |
| Constant      6.1092      0.9361                     |
| X                0.8951      0.1490                  |
| Analysis of Variance                                 |
| SOURCE      DF      SS      MS                       |
| Regression      1      1575.76      1575.76          |
| Residual Error      8      349.14      43.64         |
| Total      9      1924.90                            |

- a. Write the estimated regression equation.
  - b. Use a  $t$  test to determine whether monthly maintenance expense is related to usage at the 0.05 level of significance.
  - c. Use the estimated regression equation to predict mean monthly maintenance expense for any terminal that is used 25 hours per week.
- 24.** A regression model relating  $X$ , number of salespersons at a branch office, to  $Y$ , annual sales at the office (in thousands of euros) provided the following computer output from a regression analysis of the data.

|  |
|--|
| The regression equation is<br>$Y = 80.0 + 50.00 X$ |
| Predictor      Coef      SE Coef      T            |
| Constant      80.0      11.333      7.06           |
| X                50.0      5.482      9.12         |
| Analysis of Variance                               |
| SOURCE      DF      SS      MS                     |
| Regression      1      6828.6      6828.6          |
| Residual Error      28      2298.8      82.1       |
| Total      29      9127.4                          |

- a. Write the estimated regression equation.
- b. How many branch offices were involved in the study?
- c. Compute the  $F$  statistic and test the significance of the relationship at a 0.05 level of significance.
- d. Predict the annual sales at the Marseilles branch office. This branch employs 12 salespersons.

## 14.8 RESIDUAL ANALYSIS: VALIDATING MODEL ASSUMPTIONS

As we noted previously, the *residual* for observation  $i$  is the difference between the observed value of the dependent variable ( $y_i$ ) and the estimated value of the dependent variable ( $\hat{y}_i$ ).

### Residual for observation $i$

where:

$$y_i - \hat{y}_i \quad (14.24)$$

$y_i$  is the observed value of the dependent variable  
 $\hat{y}_i$  is the estimated value of the dependent variable

**TABLE 14.7** Residuals for Armand's Pizza Parlour

| Student population $x_i$ | Sales $y_i$ | Estimated sales $\hat{y}_i = 60 - 5x_i$ | Residuals $y_i - \hat{y}_i$ |
|--------------------------|-------------|---|-----------------------------|
| 2                        | 58          | 70                                      | -12                         |
| 6                        | 105         | 90                                      | 15                          |
| 8                        | 88          | 100                                     | -12                         |
| 8                        | 118         | 100                                     | 18                          |
| 12                       | 117         | 120                                     | -3                          |
| 16                       | 137         | 140                                     | -3                          |
| 20                       | 157         | 160                                     | -3                          |
| 20                       | 169         | 160                                     | 9                           |
| 22                       | 149         | 170                                     | -21                         |
| 26                       | 202         | 190                                     | 12                          |

In other words, the  $i$ th residual is the error resulting from using the estimated regression equation to predict the value of the dependent variable. The residuals for the Armand's Pizza Parlours example are computed in Table 14.7. The observed values of the dependent variable are in the second column and the estimated values of the dependent variable, obtained using the estimated regression equation  $\hat{y} = 60 + 5x$ , are in the third column. An analysis of the corresponding residuals in the fourth column will help determine whether the assumptions made about the regression model are appropriate.

Recall that for the Armand's Pizza Parlours example it was assumed the simple linear regression model took the form:

$$Y = \beta_0 + \beta_1 x + \epsilon \quad (14.25)$$

In other words we assumed quarterly sales ( $Y$ ) to be a linear function of the size of the student population ( $X$ ) plus an error term  $\epsilon$ . In Section 14.4 we made the following assumptions about the error term  $\epsilon$ .

- 1  $E(\epsilon) = 0$ .
- 2 The variance of  $\epsilon$ , denoted by  $\sigma^2$ , is the same for all values of  $X$ .
- 3 The values of  $\epsilon$  are independent.
- 4 The error term  $\epsilon$  has a normal distribution.

These assumptions provide the theoretical basis for the  $t$  test and the  $F$  test used to determine whether the relationship between  $X$  and  $Y$  is significant, and for the confidence and prediction interval estimates presented in Section 14.6. If the assumptions about the error term  $\epsilon$  appear questionable, the hypothesis tests about the significance of the regression relationship and the interval estimation results may not be valid.

The residuals provide the best information about  $\epsilon$ ; hence an analysis of the residuals is an important step in determining whether the assumptions for  $\epsilon$  are appropriate. Much of **residual analysis** is based on an examination of graphical plots. In this section, we discuss the following residual plots.

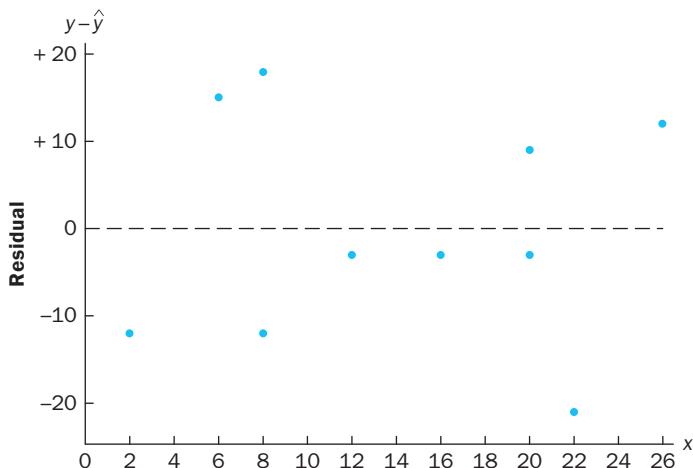
- 1 A plot of the residuals against values of the independent variable  $X$ .
- 2 A plot of residuals against the predicted values  $\hat{y}$  of the dependent variable.
- 3 A standardized residual plot.
- 4 A normal probability plot.

## Residual plot against $X$

A **residual plot** against the independent variable  $X$  is a graph in which the values of the independent variable are represented by the horizontal axis and the corresponding residual values are represented by the vertical axis. A point is plotted for each residual.

**FIGURE 14.11**

Plot of the residuals against the independent variable for Armand's Pizza Parlours



The first coordinate for each point is given by the value of  $x_i$  and the second coordinate is given by the corresponding value of the residual  $y_i - \hat{y}_i$ . For a residual plot against  $X$  with the Armand's Pizza Parlours data from Table 14.7, the coordinates of the first point are  $(2, -12)$ , corresponding to  $x_1 = 2$  and  $y_1 - \hat{y}_1 = -12$ ; the coordinates of the second point are  $(6, 15)$ , corresponding to  $x_2 = 6$  and  $y_2 - \hat{y}_2 = 15$  and so on. Figure 14.11 shows the resulting residual plot.

Before interpreting the results for this residual plot, let us consider some general patterns that might be observed in any residual plot. Three examples appear in Figure 14.12.

If the assumption that the variance of  $\varepsilon$  is the same for all values of  $X$  and the assumed regression model is an adequate representation of the relationship between the variables, the residual plot should give an overall impression of a horizontal band of points such as the one in Panel A of Figure 14.12. However, if the variance of  $\varepsilon$  is not the same for all values of  $X$  – for example, if variability about the regression line is greater for larger values of  $X$  – a pattern such as the one in Panel B of Figure 14.12 could be observed. In this case, the assumption of a constant variance of  $\varepsilon$  is violated. Another possible residual plot is shown in Panel C. In this case, we would conclude that the assumed regression model is not an adequate representation of the relationship between the variables. A curvilinear regression model or multiple regression model should be considered.

Now let us return to the residual plot for Armand's Pizza Parlours shown in Figure 14.11. The residuals appear to approximate the horizontal pattern in Panel A of Figure 14.12. Hence, we conclude that the residual plot does not provide evidence that the assumptions made for Armand's regression model should be challenged. At this point, we are confident in the conclusion that Armand's simple linear regression model is valid.

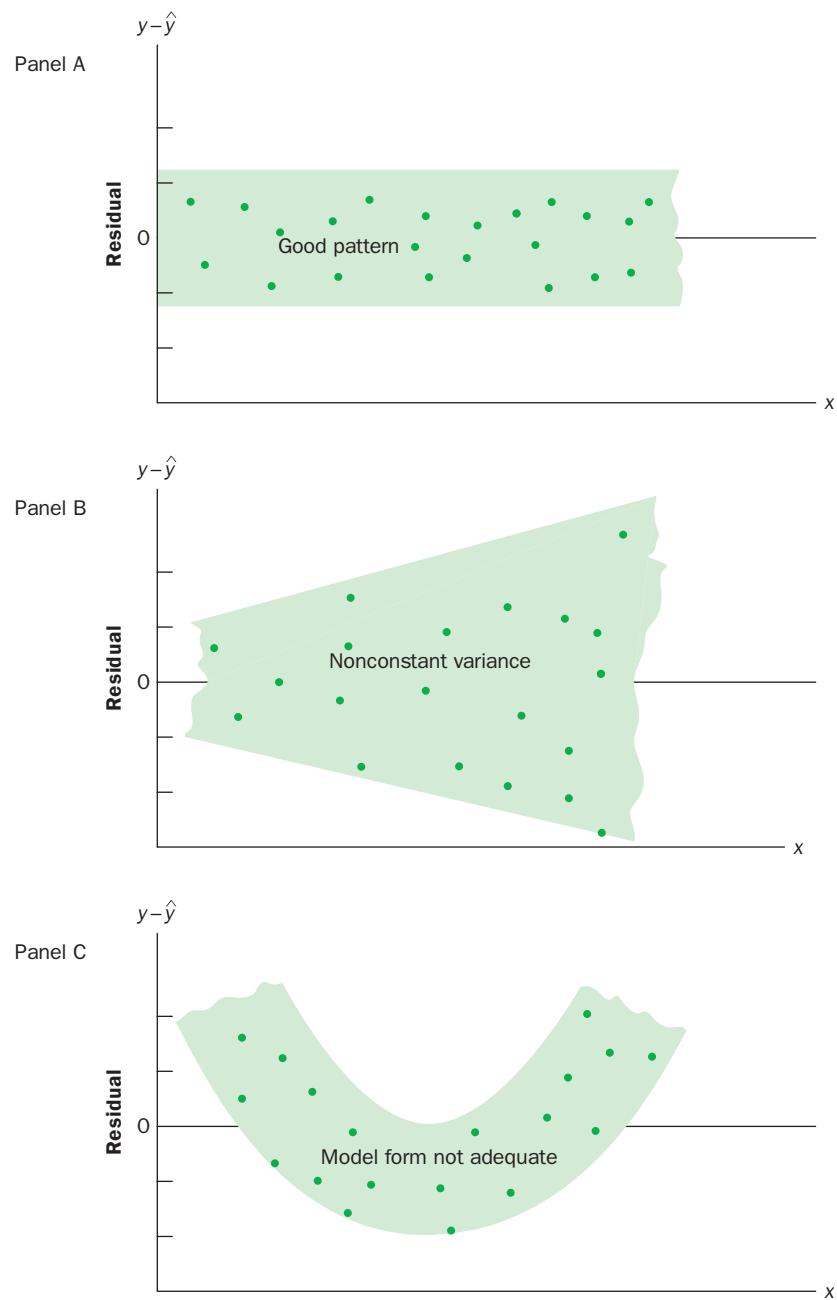
Experience and good judgement are always factors in the effective interpretation of residual plots. Seldom does a residual plot conform precisely to one of the patterns in Figure 14.12. Yet analysts who frequently conduct regression studies and frequently review residual plots become adept at understanding the differences between patterns that are reasonable and patterns that indicate the assumptions of the model should be questioned. A residual plot provides one technique to assess the validity of the assumptions for a regression model.

### Residual plot against $\hat{y}$

Another residual plot represents the predicted value of the dependent variable  $\hat{y}$  on the horizontal axis and the residual values on the vertical axis. A point is plotted for each residual. The first coordinate for each point is given by  $\hat{y}_i$  and the second coordinate is given by the corresponding value of the  $i$ th residual  $y_i - \hat{y}_i$ . With the Armand's data from Table 14.7, the coordinates of the first point are  $(70, -12)$ , corresponding to  $\hat{y}_1 = 70$  and  $y_1 - \hat{y}_1 = -12$ ; the coordinates of the second point are  $(90, 15)$  and so on. Figure 14.13 provides the residual plot. Note that the pattern of this residual plot is the same as the pattern of the residual plot against the independent variable  $X$ .

**FIGURE 14.12**

Residual plots from three regression studies



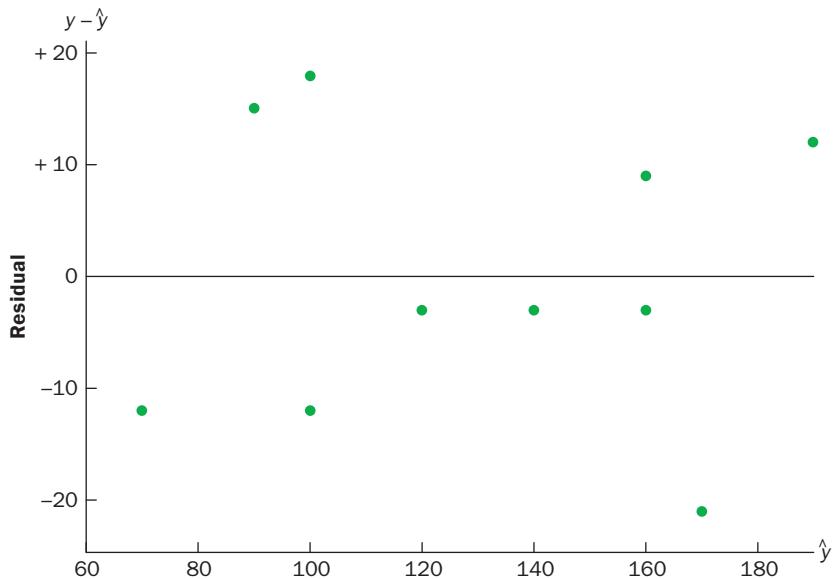
It is not a pattern that would lead us to question the model assumptions. For simple linear regression, both the residual plot against  $X$  and the residual plot against  $\hat{y}$  provide the same pattern. For multiple regression analysis, the residual plot against  $\hat{y}$  is more widely used because of the presence of more than one independent variable.

## Standardized residuals

Many of the residual plots provided by computer software packages use a standardized version of the residuals. As demonstrated in preceding chapters, a random variable is standardized by subtracting its mean and dividing the result by its standard deviation. With the least squares method, the mean of the residuals is zero. Thus, simply dividing each residual by its standard deviation provides the **standardized residual**.

**FIGURE 14.13**

Plot of the residuals against the predicted values  $\hat{y}$  for Armand's Pizza Parlours



It can be shown that the standard deviation of residual  $i$  depends on the standard error of the estimate  $s$  and the corresponding value of the independent variable  $x_i$ .

Note that equation (14.26) shows that the standard deviation of the  $i$ th residual depends on  $x_i$  because of the presence of  $h_i$  in the formula.<sup>†</sup> Once the standard deviation of each residual is calculated, we can compute the standardized residual by dividing each residual by its corresponding standard deviation.

#### Standard deviation of the $i$ th residual\*

$$\text{where: } s_{y_i - \hat{y}_i} = s\sqrt{1 - h_i} \quad (14.26)$$

$s_{y_i - \hat{y}_i}$  = the standard deviation of residual  $i$

$s$  = the standard error of the estimate

$$h_i = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum(x_i - \bar{x})^2} \quad (14.27)$$

#### Standardized residual for observation $i$

$$\frac{y_i - \hat{y}_i}{s_{y_i - \hat{y}_i}} \quad (14.28)$$

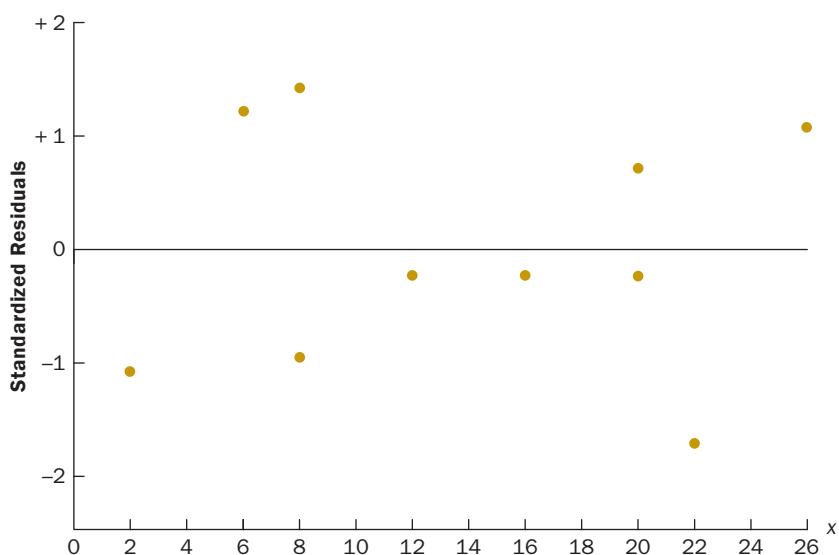
Table 14.8 shows the calculation of the standardized residuals for Armand's Pizza Parlours. Recall that previous calculations showed  $s = 13.829$ . Figure 14.14 is the plot of the standardized residuals against the independent variable  $X$ .

<sup>†</sup> $h_i$  is referred to as the *leverage of observation  $i$* . Leverage will be discussed further when we consider influential observations in Section 14.9.

\*This equation actually provides an estimate of the standard deviation of the  $i$ th residual, because  $s$  is used instead of  $\sigma$ .

**FIGURE 14.14**

Plot of the standardized residuals against the independent variable X for Armand's Pizza Parlours



The standardized residual plot can provide insight about the assumption that the error term  $\varepsilon$  has a normal distribution. If this assumption is satisfied, the distribution of the standardized residuals should appear to come from a standard normal probability distribution.\*

Thus, when looking at a standardized residual plot, we should expect to see approximately 95 per cent of the standardized residuals between  $-2$  and  $+2$ . We see in Figure 14.14 that for the Armand's example all standardized residuals are between  $-2$  and  $+2$ . Therefore, on the basis of the standardized residuals, this plot gives us no reason to question the assumption that  $\varepsilon$  has a normal distribution.

Because of the effort required to compute the estimated values  $\hat{y}$ , the residuals, and the standardized residuals, most statistical packages provide these values as optional regression output. Hence, residual plots can be easily obtained. For large problems computer packages are the only practical means for developing the residual plots discussed in this section.

## Normal probability plot

Another approach for determining the validity of the assumption that the error term has a normal distribution is the **normal probability plot**. To show how a normal probability plot is developed, we introduce the concept of *normal scores*.

Suppose ten values are selected randomly from a normal probability distribution with a mean of zero and a standard deviation of one, and that the sampling process is repeated over and over with the values in each sample of ten ordered from smallest to largest. For now, let us consider only the smallest value in each sample. The random variable representing the smallest value obtained in repeated sampling is called the first-order statistic.

Statisticians show that for samples of size ten from a standard normal probability distribution, the expected value of the first-order statistic is  $-1.55$ . This expected value is called a normal score. For the case with a sample of size  $n = 10$ , there are ten order statistics and ten normal scores (see Table 14.9). In general, a data set consisting of  $n$  observations will have  $n$  order statistics and hence  $n$  normal scores.

Let us now show how the ten normal scores can be used to determine whether the standardized residuals for Armand's Pizza Parlours appear to come from a standard normal probability distribution.

---

\*Because  $s$  is used instead of  $\sigma$  in equation (14.26), the probability distribution of the standardized residuals is not technically normal. However, in most regression studies, the sample size is large enough that a normal approximation is very good.

**TABLE 14.8** Computation of standardized residuals for Armand's Pizza Parlours

| i     | $x_i$ | $x_i - \bar{x}$ | $(x_i - \bar{x})^2$ | $\frac{(x_i - \bar{x})^2}{\sum(x_i - \bar{x})^2}$ | $h_i$  | $s_{y_i - \bar{y}_i}$ | $y_i - \hat{y}_i$ | Standardized residual |
|-------|-------|-----------------|---------------------|---|--------|-----------------------|-------------------|-----------------------|
| 1     | 2     | -12             | 144                 | 0.2535  | 0.3535 | 11.1193               | -12               | -1.0792               |
| 2     | 6     | -8              | 64                  | 0.1127  | 0.2127 | 12.2709               | 15                | 1.2224                |
| 3     | 8     | -6              | 36                  | 0.0634  | 0.1634 | 12.6493               | -12               | -0.9487               |
| 4     | 8     | -6              | 36                  | 0.0634  | 0.1634 | 12.6493               | 18                | 1.4230                |
| 5     | 12    | -2              | 4                   | 0.0070  | 0.1070 | 13.0682               | -3                | -0.2296               |
| 6     | 16    | 2               | 4                   | 0.0070  | 0.1070 | 13.0682               | -3                | -0.2296               |
| 7     | 20    | 6               | 36                  | 0.0634  | 0.1634 | 12.6493               | -3                | -0.2372               |
| 8     | 20    | 6               | 36                  | 0.0634  | 0.1634 | 12.6493               | 9                 | 0.7115                |
| 9     | 22    | 8               | 64                  | 0.1127  | 0.2127 | 12.2709               | -21               | -1.7114               |
| 10    | 26    | 12              | 144                 | 0.2535  | 0.3535 | 11.1193               | 12                | 1.0792                |
| Total |       | 568             |                     |   |        |                       |                   |                       |

Note: The values of the residuals were computed in Table 14.7.

**TABLE 14.9** Normal scores for  $n = 10$ 

| Order statistic | Normal score |
|-----------------|--------------|
| 1               | -1.55        |
| 2               | -1.00        |
| 3               | -0.65        |
| 4               | -0.37        |
| 5               | -0.12        |
| 6               | 0.12         |
| 7               | 0.37         |
| 8               | 0.65         |
| 9               | 1.00         |
| 10              | 1.55         |

We begin by ordering the ten standardized residuals from Table 14.8. The ten normal scores and the ordered standardized residuals are shown together in Table 14.10. If the normality assumption is satisfied, the smallest standardized residual should be close to the smallest normal score, the next smallest standardized residual should be close to the next smallest normal score and so on. If we were to develop a plot with the normal scores on the horizontal axis and the corresponding standardized residuals on the vertical axis, the plotted points should cluster closely around a 45-degree line passing through the origin if the standardized residuals are approximately normally distributed. Such a plot is referred to as a *normal probability plot*.

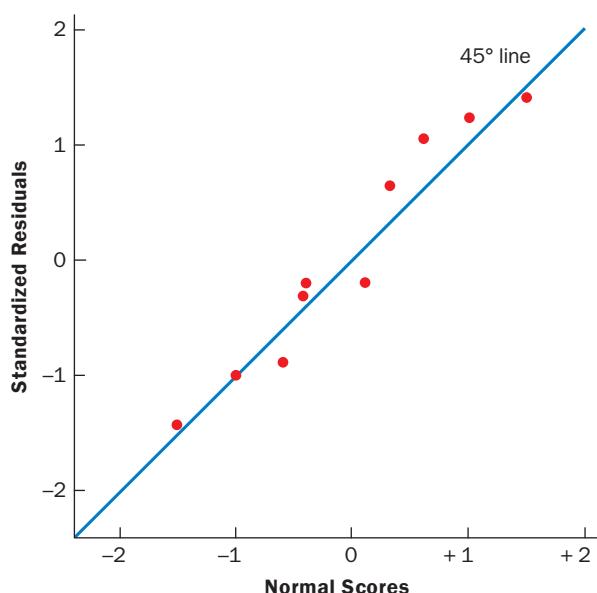
Figure 14.15 is the normal probability plot for the Armand's Pizza Parlours example. Judgement is used to determine whether the pattern observed deviates from the line enough to conclude that the standardized residuals are not from a standard normal probability distribution. In Figure 14.15, we see that the points are grouped closely about the line. We therefore conclude that the assumption of the error term having a normal probability distribution is reasonable. In general, the more closely the points are clustered about the 45-degree line, the stronger the evidence supporting the normality assumption. Any substantial curvature in the normal probability plot is evidence that the residuals have not come from a normal distribution. Normal scores and the associated normal probability plot can be obtained easily from statistical packages such as MINITAB.

**TABLE 14.10** Normal scores and ordered standardized residuals for Armand's Pizza Parlours

| Ordered normal scores | Standardized residuals |
|-----------------------|------------------------|
| -1.55                 | -1.7114                |
| -1.00                 | -1.0792                |
| -0.65                 | -0.9487                |
| -0.37                 | -0.2372                |
| -0.12                 | -0.2296                |
| 0.12                  | -0.2296                |
| 0.37                  | 0.7115                 |
| 0.65                  | 1.0792                 |
| 1.00                  | 1.2224                 |
| 1.55                  | 1.4230                 |

**FIGURE 14.15**

Normal probability plot for Armand's Pizza Parlours



## 14.9 RESIDUAL ANALYSIS: AUTOCORRELATION

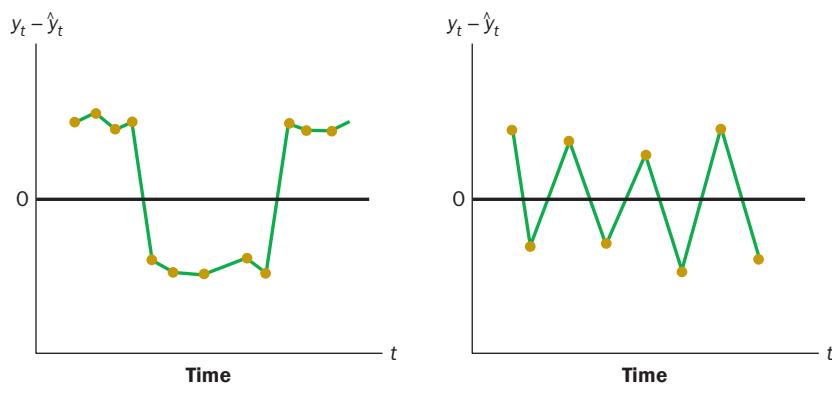
In the last section we showed how residual plots can be used to detect violations of assumptions about the error term  $\varepsilon$  in the regression model. In many regression studies, particularly involving data collected over time, a special type of correlation among the error terms can cause problems; it is called **serial correlation** or **autocorrelation**. In this section we show how the **Durbin–Watson test** can be used to detect significant autocorrelation.

### Autocorrelation and the Durbin–Watson test

Often, the data used for regression studies in business and economics are collected over time. It is not uncommon for the value of  $Y$  at time  $t$ , denoted by  $y_t$ , to be related to the value of  $Y$  at previous time periods. In such cases, we say autocorrelation (also called serial correlation) is present in the data. If the value of  $Y$  in time period  $t$  is related to its value in time period  $t - 1$ , first-order autocorrelation is present. If the value of  $Y$  in time period  $t$  is related to the value of  $Y$  in time period  $t - 2$ , second-order autocorrelation is present and so on.

**FIGURE 14.16**

Two data sets with  
first-order  
autocorrelation

**Panel A. Positive Autocorrelation****Panel B. Negative Autocorrelation**

When autocorrelation is present, one of the assumptions of the regression model is violated: the error terms are not independent. In the case of first-order autocorrelation, the error at time  $t$ , denoted  $\epsilon_t$ , will be related to the error at time period  $t - 1$ , denoted  $\epsilon_{t-1}$ . Two cases of first-order autocorrelation are illustrated in Figure 14.16. Panel A is the case of positive autocorrelation; panel B is the case of negative autocorrelation. With positive autocorrelation we expect a positive residual in one period to be followed by a positive residual in the next period, a negative residual in one period to be followed by a negative residual in the next period and so on. With negative autocorrelation, we expect a positive residual in one period to be followed by a negative residual in the next period, then a positive residual and so on. When autocorrelation is present, serious errors can be made in performing tests of statistical significance based upon the assumed regression model. It is therefore important to be able to detect autocorrelation and take corrective action. We will show how the Durbin–Watson statistic can be used to detect first-order autocorrelation.

Suppose the values of  $\epsilon$  are not independent but are related in the following manner:

#### First-order autocorrelation

$$\epsilon_t = \rho\epsilon_{t-1} + z_t \quad (14.29)$$

where  $\rho$  is a parameter with an absolute value less than one and  $z_t$  is a normally and independently distributed random variable with a mean of zero and a variance of  $\sigma^2$ . From equation (14.29) we see that if  $\rho = 0$ , the error terms are not related, and each has a mean of zero and a variance of  $\sigma^2$ . In this case, there is no autocorrelation and the regression assumptions are satisfied. If  $\rho > 0$ , we have positive autocorrelation; if  $\rho < 0$ , we have negative autocorrelation. In either of these cases, the regression assumptions about the error term are violated.

The Durbin–Watson test for autocorrelation uses the residuals to determine whether  $\rho = 0$ . To simplify the notation for the Durbin–Watson statistic, we denote the  $i$ th residual by  $e_t = y_t - \hat{y}_t$ . The Durbin–Watson test statistic is computed as follows.

#### Durbin–Watson test statistic

$$d = \frac{\sum_{t=2}^n (e_t - e_{t-1})^2}{\sum_{t=1}^n e_t^2} \quad (14.30)$$

If successive values of the residuals are close together (positive autocorrelation), the value of the Durbin–Watson test statistic will be small. If successive values of the residuals are far apart (negative autocorrelation), the value of the Durbin–Watson statistic will be large.

The Durbin–Watson test statistic ranges in value from zero to four, with a value of two indicating no autocorrelation is present. Durbin and Watson developed tables that can be used to determine when their test statistic indicates the presence of autocorrelation. Table 14 in Appendix B shows lower and upper bounds ( $d_L$  and  $d_U$ ) for hypothesis tests using  $\alpha = 0.05$ ,  $\alpha = 0.025$  and  $\alpha = 0.01$ ;  $n$  denotes the number of observations.

The null hypothesis to be tested is always that there is no autocorrelation.

$$H_0: \rho = 0$$

The alternative hypothesis to test for positive autocorrelation is:

$$H_1: \rho > 0$$

The alternative hypothesis to test for negative autocorrelation is:

$$H_1: \rho < 0$$

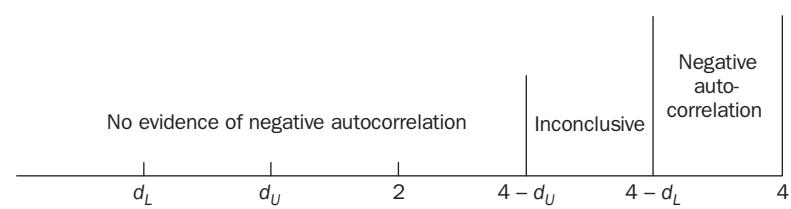
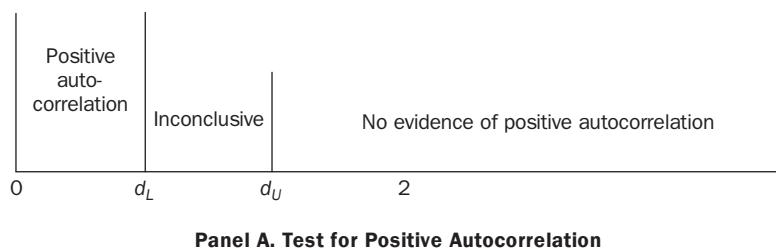
A two-sided test is also possible. In this case the alternative hypothesis is:

$$H_1: \rho \neq 0$$

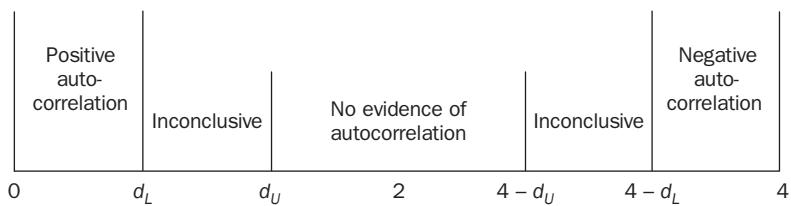
Figure 14.17 shows how the values of  $d_L$  and  $d_U$  in Table 7.0 in Appendix B are used to test for autocorrelation.

**Panel A** illustrates the test for positive autocorrelation. If  $d < d_L$ , we conclude that positive autocorrelation is present. If  $d_L \leq d \leq d_U$ , we say the test is inconclusive. If  $d > d_U$ , we conclude that there is no evidence of positive autocorrelation.

**FIGURE 14.17**  
Hypothesis test for autocorrelation using the Durbin–Watson test



Panel B. Test for Negative Autocorrelation



Panel C. Two-sided Test for Autocorrelation

**Panel B** illustrates the test for negative autocorrelation. If  $d > 4 - d_L$ , we conclude that negative autocorrelation is present. If  $4 - d_U \leq d \leq 4 - d_L$ , we say the test is inconclusive. If  $d < 4 - d_U$ , we conclude that there is no evidence of negative autocorrelation.

Note: Entries in Table 7.0 in Appendix B are the critical values for a one-tailed Durbin–Watson test for autocorrelation. For a two-tailed test, the level of significance is doubled.

**Panel C** illustrates the two-sided test. If  $d < d_L$  or  $d > 4 - d_L$ , we reject  $H_0$  and conclude that autocorrelation is present. If  $d_L \leq d \leq d_U$  or  $4 - d_U \leq d \leq 4 - d_L$ , we say the test is inconclusive. If  $d_U \leq d \leq 4 - d_U$ , we conclude that there is no evidence of autocorrelation.

If significant autocorrelation is identified, we should investigate whether we omitted one or more key independent variables that have time-ordered effects on the dependent variable. If no such variables can be identified, including an independent variable that measures the time of the observation (for instance, the value of this variable could be one for the first observation, two for the second observation and so on) will sometimes eliminate or reduce the autocorrelation. When these attempts to reduce or remove autocorrelation do not work, transformations on the dependent or independent variables can prove helpful; a discussion of such transformations can be found in more advanced texts on regression analysis.

Note that the Durbin–Watson tables list the smallest sample size as 15. The reason is that the test is generally inconclusive for smaller sample sizes; in fact, many statisticians believe the sample size should be at least 50 for the test to produce worthwhile results.

## EXERCISES

### Methods

- 25.** Given are data for two variables,  $X$  and  $Y$ .

|       |   |    |    |    |    |
|-------|---|----|----|----|----|
| $x_i$ | 6 | 11 | 15 | 18 | 20 |
| $y_i$ | 6 | 8  | 12 | 20 | 30 |

- Develop an estimated regression equation for these data.
- Compute the residuals.
- Develop a plot of the residuals against the independent variable  $X$ . Do the assumptions about the error terms seem to be satisfied?
- Compute the standardized residuals.
- Develop a plot of the standardized residuals against  $\hat{y}$ . What conclusions can you draw from this plot?

- 26.** The following data were used in a regression study.

| Observation | $x_i$ | $y_i$ | Observation | $x_i$ | $y_i$ |
|-------------|-------|-------|-------------|-------|-------|
| 1           | 2     | 4     | 6           | 7     | 6     |
| 2           | 3     | 5     | 7           | 7     | 9     |
| 3           | 4     | 4     | 8           | 8     | 5     |
| 4           | 5     | 6     | 9           | 9     | 11    |
| 5           | 7     | 4     |             |       |       |

- Develop an estimated regression equation for these data.
- Construct a plot of the residuals. Do the assumptions about the error term seem to be satisfied?



COMPLETE  
SOLUTIONS

### Applications

27. A doctor has access to historical data as follows:

|                | Vehicles per 100 population | Road death per 100 000 population |
|----------------|-----------------------------|-----------------------------------|
| Great Britain  | 31                          | 14                                |
| Belgium        | 32                          | 29                                |
| Denmark        | 30                          | 22                                |
| France         | 47                          | 32                                |
| Germany        | 30                          | 25                                |
| Irish Republic | 19                          | 20                                |
| Italy          | 36                          | 21                                |
| Netherlands    | 40                          | 22                                |
| Canada         | 47                          | 30                                |
| USA            | 58                          | 35                                |

- a. First identifying the  $X$  and  $Y$  variables appropriately, use the method of least squares to develop a straight line approximation of the relationship between the two variables.
  - b. Test whether vehicles and road deaths are related at a 0.05 level of significance.
  - c. Prepare a residual plot of  $y - \hat{y}$  versus  $\hat{y}$ . Use the result from part (a) to obtain the values of  $\hat{y}$ .
  - d. What conclusions can you draw from residual analysis? Should this model be used, or should we look for a better one?
28. Refer to Exercise 6, where an estimated regression equation relating years of experience and annual sales was developed.
- a. Compute the residuals and construct a residual plot for this problem.
  - b. Do the assumptions about the error terms seem reasonable in light of the residual plot?

## 14.10 RESIDUAL ANALYSIS: OUTLIERS AND INFLUENTIAL OBSERVATIONS

In Section 14.8 we showed how residual analysis could be used to determine when violations of assumptions about the regression model occur. In this section, we discuss how residual analysis can be used to identify observations that can be classified as outliers or as being especially influential in determining the estimated regression equation. Some steps that should be taken when such observations occur are discussed.

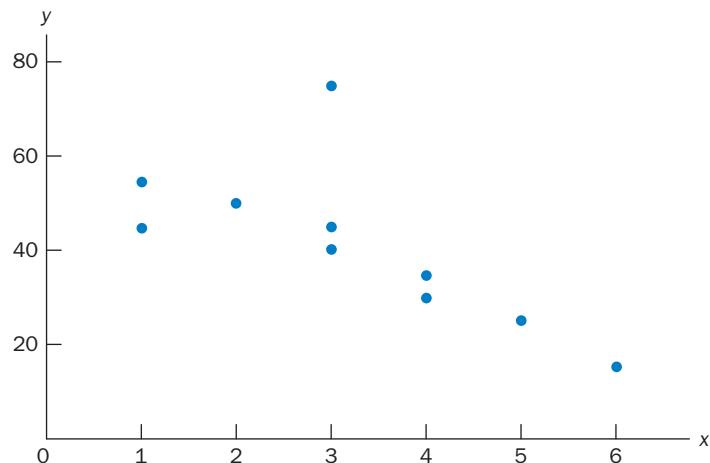
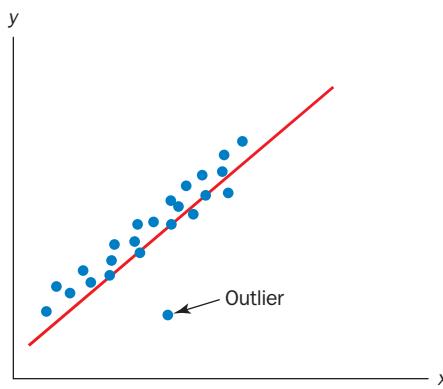
### Detecting outliers

Figure 14.18 is a scatter diagram for a data set that contains an **outlier**, a data point (observation) that does not fit the trend shown by the remaining data. Outliers represent observations that are suspect and warrant careful examination. They may represent erroneous data; if so, the data should be corrected. They may signal a violation of model assumptions; if so, another model should be considered. Finally, they may simply be unusual values that occurred by chance. In this case, they should be retained.

To illustrate the process of detecting outliers, consider the data set in Table 14.11; Figure 14.19 is a scatter diagram. Except for observation 4 ( $x_4 = 3, y_4 = 75$ ), a pattern suggesting a negative linear relationship is apparent. Indeed, given the pattern of the rest of the data, we would expect  $y_4$  to be much smaller and hence would identify the corresponding observation as an outlier. For the case of simple linear regression, one can often detect outliers by simply examining the scatter diagram.

**FIGURE 14.18**

A data set with an outlier

**FIGURE 14.19**

Scatter diagram for outlier data set

**TABLE 14.11** Data set illustrating the effect of an outlier

| $x_i$ | $y_i$ |
|-------|-------|
| 1     | 45    |
| 1     | 55    |
| 2     | 50    |
| 3     | 75    |
| 3     | 40    |
| 3     | 45    |
| 4     | 30    |
| 4     | 35    |
| 5     | 25    |
| 6     | 15    |

The standardized residuals can also be used to identify outliers. If an observation deviates greatly from the pattern of the rest of the data (e.g. the outlier in Figure 14.18), the corresponding standardized residual will be large in absolute value. Many computer packages automatically identify observations with standardized residuals that are large in absolute value.

**FIGURE 14.20**

MINITAB output for regression analysis of the outlier data set

### Regression Analysis: y versus x

The regression equation is  
 $y = 65.0 - 7.33 x$

| Predictor | Coef   | SE Coef | T     | P     |
|-----------|--------|---------|-------|-------|
| Constant  | 64.958 | 9.258   | 7.02  | 0.000 |
| x         | -7.331 | 2.608   | -2.81 | 0.023 |

S = 12.6704 R-Sq = 49.7% R-Sq(adj) = 43.4%

#### Analysis of Variance

| Source         | DF | SS     | MS     | F    | P     |
|----------------|----|--------|--------|------|-------|
| Regression     | 1  | 1268.2 | 1268.2 | 7.90 | 0.023 |
| Residual Error | 8  | 1284.3 | 160.5  |      |       |
| Total          | 9  | 2552.5 |        |      |       |

#### Unusual Observations

| Obs | x    | y     | Fit   | SE Fit | Residual | St Resid |
|-----|------|-------|-------|--------|----------|----------|
| 4   | 3.00 | 75.00 | 42.97 | 4.04   | 32.03    | 2.67R    |

R denotes an observation with a large standardized residual.

In Figure 14.20 we show the MINITAB output from a regression analysis of the data in Table 14.11. The next to last line of the output shows that the standardized residual for observation 4 is 2.67. MINITAB identifies any observation with a standardized residual of less than  $-2$  or greater than  $+2$  as an unusual observation; in such cases, the observation is printed on a separate line with an R next to the standardized residual, as shown in Figure 14.20. With normally distributed errors, standardized residuals should be outside these limits approximately 5 per cent of the time.

In deciding how to handle an outlier, we should first check to see whether it is a valid observation. Perhaps an error was made in initially recording the data or in entering the data into the computer file. For example, suppose that in checking the data for the outlier in Table 14.11, we find an error; the correct value for observation 4 is  $x_4 = 3$ ,  $y_4 = 30$ . Figure 14.21 is the MINITAB output obtained after correction of the value of  $y_4$ . We see that using the incorrect data value substantially affected the goodness of fit. With the correct data, the value of  $R$ -sq increased from 49.7 per cent to 83.8 per cent and the value of  $b_0$  decreased from 64.958 to 59.237. The slope of the line changed from  $-7.331$  to  $-6.949$ . The identification of the outlier enabled us to correct the data error and improve the regression results.

**FIGURE 14.21**

MINITAB output for the revised outlier data set

### Regression Analysis: y versus x

The regression equation is  
 $y = 59.2 - 6.95 x$

| Predictor | Coef   | SE Coef | T     | P     |
|-----------|--------|---------|-------|-------|
| Constant  | 59.237 | 3.835   | 15.45 | 0.000 |
| x         | -6.949 | 1.080   | -6.43 | 0.000 |

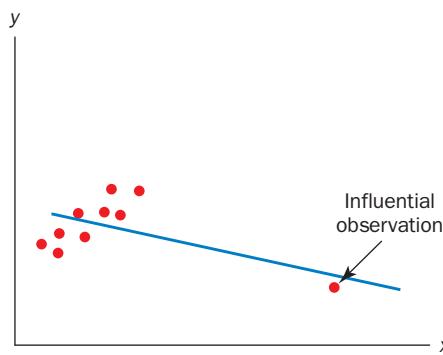
S = 5.24808 R-Sq = 83.8% R-Sq(adj) = 81.8%

#### Analysis of Variance

| Source         | DF | SS     | MS     | F     | P     |
|----------------|----|--------|--------|-------|-------|
| Regression     | 1  | 1139.7 | 1139.7 | 41.38 | 0.000 |
| Residual Error | 8  | 220.3  | 27.5   |       |       |
| Total          | 9  | 1360.0 |        |       |       |

**FIGURE 14.22**

A data set with an influential observation



## Detecting influential observations

Sometimes one or more observations exert a strong influence on the results obtained. Figure 14.22 shows an example of an **influential observation** in simple linear regression. The estimated regression line has a negative slope. However, if the influential observation were dropped from the data set, the slope of the estimated regression line would change from negative to positive and the  $y$ -intercept would be smaller. Clearly, this one observation is much more influential in determining the estimated regression line than any of the others; dropping one of the other observations from the data set would have little effect on the estimated regression equation.

Influential observations can be identified from a scatter diagram when only one independent variable is present. An influential observation may be an outlier (an observation with a  $Y$  value that deviates substantially from the trend), it may correspond to an  $X$  value far away from its mean (e.g. see Figure 14.22), or it may be caused by a combination of the two (a somewhat off-trend  $Y$  value and a somewhat extreme  $X$  value).

Because influential observations may have such a dramatic effect on the estimated regression equation, they must be examined carefully. We should first check to make sure that no error was made in collecting or recording the data. If an error occurred, it can be corrected and a new estimated regression equation can be developed. If the observation is valid, we might consider ourselves fortunate to have it. Such a point, if valid, can contribute to a better understanding of the appropriate model and can lead to a better estimated regression equation. The presence of the influential observation in Figure 14.22, if valid, would suggest trying to obtain data on intermediate values of  $X$  to understand better the relationship between  $X$  and  $Y$ .

Observations with extreme values for the independent variables are called **high leverage points**. The influential observation in Figure 14.22 is a point with high leverage. The leverage of an observation is determined by how far the values of the independent variables are from their mean values. For the single-independent-variable case, the leverage of the  $i$ th observation, denoted  $h_i$ , can be computed by using equation (14.31).

### Leverage of observation $i$

$$h_i = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum(x_i - \bar{x})^2} \quad (14.31)$$

From the formula, it is clear that the farther  $x_i$  is from its mean  $\bar{x}$ , the higher the leverage of observation  $i$ .

**TABLE 14.12** Data set with a high leverage observation

| $x_i$ | $y_i$ |
|-------|-------|
| 10    | 125   |
| 10    | 130   |
| 15    | 120   |
| 20    | 115   |
| 20    | 120   |
| 25    | 110   |
| 70    | 100   |

Many statistical packages automatically identify observations with high leverage as part of the standard regression output. As an illustration of how the MINITAB statistical package identifies points with high leverage, let us consider the data set in Table 14.12.

From Figure 14.23, a scatter diagram for the data set in Table 14.12, it is clear that observation 7 ( $X = 70, Y = 100$ ) is an observation with an extreme value of  $X$ . Hence, we would expect it to be identified as a point with high leverage. For this observation, the leverage is computed by using equation (14.31) as follows.

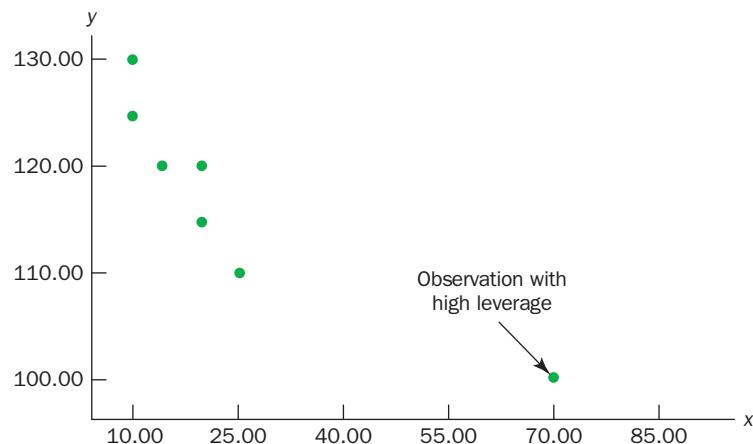
$$h_7 = \frac{1}{n} + \frac{(x_7 - \bar{x})^2}{\sum(x_i - \bar{x})^2} = \frac{1}{7} + \frac{(70 - 24.286)^2}{2621.43} = 0.94$$

For the case of simple linear regression, MINITAB identifies observations as having high leverage if  $h_i > 6/n$ ; for the data set in Table 14.12,  $6/n = 6/7 = 0.86$ . Because  $h_7 = 0.94 > 0.86$ , MINITAB will identify observation 7 as an observation whose  $X$  value gives it large influence. Figure 14.24 shows the MINITAB output for a regression analysis of this data set. Observation 7 ( $X=70, Y=100$ ) is identified as having large influence; it is printed on a separate line at the bottom, with an X in the right margin.

Influential observations that are caused by an interaction of large residuals and high leverage can be difficult to detect. Diagnostic procedures are available that take both into account in determining when an observation is influential. One such measure, called Cook's  $D$  statistic, will be discussed in Chapter 15.

**FIGURE 14.23**

Scatter diagram for the data set with a high leverage observation



**FIGURE 14.24**

MINITAB output for the data set with a high leverage observation

**Regression Analysis: y versus x**

The regression equation is  
 $y = 127 - 0.425x$

| Predictor | Coef     | SE Coef | T     | P     |
|-----------|----------|---------|-------|-------|
| Constant  | 127.466  | 2.961   | 43.04 | 0.000 |
| x         | -0.42507 | 0.09537 | -4.46 | 0.007 |

$$S = 4.88282 \quad R-Sq = 79.9\% \quad R-Sq(\text{adj}) = 75.9\%$$

**Analysis of Variance**

| Source         | DF | SS     | MS     | F     | P     |
|----------------|----|--------|--------|-------|-------|
| Regression     | 1  | 473.65 | 473.65 | 19.87 | 0.007 |
| Residual Error | 5  | 119.21 | 23.84  |       |       |
| Total          | 6  | 592.86 |        |       |       |

**Unusual Observations**

| Obs | x    | y      | Fit   | SE Fit | Residual | St Resid |
|-----|------|--------|-------|--------|----------|----------|
| 7   | 70.0 | 100.00 | 97.71 | 4.73   | 2.29     | 1.91 X   |

X denotes an observation whose X value gives it large leverage.

**EXERCISES****Methods**

- 29.** Consider the following data for two variables, X and Y.

|       |     |     |     |     |     |     |     |
|-------|-----|-----|-----|-----|-----|-----|-----|
| $x_i$ | 135 | 110 | 130 | 145 | 175 | 160 | 120 |
| $y_i$ | 145 | 100 | 120 | 120 | 130 | 130 | 110 |

- a. Compute the standardized residuals for these data. Do there appear to be any outliers in the data? Explain.
  - b. Plot the standardized residuals against  $\hat{y}$ . Does this plot reveal any outliers?
  - c. Develop a scatter diagram for these data. Does the scatter diagram indicate any outliers in the data? In general, what implications does this finding have for simple linear regression?
- 30.** Consider the following data for two variables, X and Y.

|       |    |    |    |    |    |    |    |    |
|-------|----|----|----|----|----|----|----|----|
| $x_i$ | 4  | 5  | 7  | 8  | 10 | 12 | 12 | 22 |
| $y_i$ | 12 | 14 | 16 | 15 | 18 | 20 | 24 | 19 |

- a. Compute the standardized residuals for these data. Do there appear to be any outliers in the data? Explain.
- b. Compute the leverage values for these data. Do there appear to be any influential observations in these data? Explain.
- c. Develop a scatter diagram for these data. Does the scatter diagram indicate any influential observations? Explain.



## ONLINE RESOURCES



For data files, additional online summary, questions, answers and software section, please go to the online platform.

## SUMMARY

In this chapter we showed how regression analysis can be used to determine how a dependent variable  $Y$  is related to an independent variable  $X$ . In simple linear regression, the regression model is  $Y = \beta_0 + \beta_1x + \epsilon$ . The simple linear regression equation  $E(\hat{y}) = \beta_0 + \beta_1x$  describes how the mean or expected value of  $Y$  is related to  $X$ . We used sample data and the least squares method to develop the estimated regression equation  $\hat{y} = b_0 + b_1x$  for a given value  $x$  of  $X$ . In effect,  $b_0$  and  $b_1$  are the sample statistics used to estimate the unknown model parameters  $\beta_0$  and  $\beta_1$ .

The coefficient of determination was presented as a measure of the goodness of fit for the estimated regression equation; it can be interpreted as the proportion of the variation in the dependent variable  $Y$  that can be explained by the estimated regression equation. We reviewed correlation as a descriptive measure of the strength of a linear relationship between two variables.

The assumptions about the regression model and its associated error term  $\hat{y}$  were discussed, and  $t$  and  $F$  tests, based on those assumptions, were presented as a means for determining whether the relationship between two variables is statistically significant. We showed how to use the estimated regression equation to develop confidence interval estimates of the mean value of  $Y$  and prediction interval estimates of individual values of  $Y$ .

The chapter concluded with a section on the computer solution of regression problems and two sections on the use of residual analysis to validate the model assumptions and to identify outliers and influential observations.

## KEY TERMS

- ANOVA table
- Autocorrelation
- Coefficient of determination
- Confidence interval
- Correlation coefficient
- Dependent variable
- Durbin–Watson test
- Estimated regression equation
- High leverage points
- Independent variable
- Influential observation
- $i$ th residual
- Least squares method

- Mean square error (MSE)
- Normal probability plot
- Outlier
- Prediction interval
- Regression equation
- Regression model
- Residual analysis
- Residual plot
- Scatter diagram
- Serial correlation
- Simple linear regression
- Standard error of the estimate
- Standardized residual

## KEY FORMULAE

**Simple linear regression model**

$$Y = \beta_0 + \beta_1 x + \epsilon \quad (\mathbf{14.1})$$

**Simple linear regression equation**

$$E(Y) = \beta_0 + \beta_1 x \quad (\mathbf{14.2})$$

**Estimated simple linear regression equation**

$$\hat{y} = b_0 + b_1 x \quad (\mathbf{14.3})$$

**Least squares criterion**

$$\text{Min } \Sigma(y_i - \hat{y}_i)^2 \quad (\mathbf{14.5})$$

**Slope and  $y$ -intercept for the estimated regression equation**

$$b_1 = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x - \bar{x})^2} \quad (\mathbf{14.6})$$

$$b_0 = \bar{y} - b_1 \bar{x} \quad (\mathbf{14.7})$$

**Sum of squares due to error**

$$\text{SSE} = \Sigma(y_i - \hat{y}_i)^2 \quad (\mathbf{14.8})$$

**Total sum of squares**

$$\text{SST} = \Sigma(y_i - \bar{y})^2 \quad (\mathbf{14.9})$$

**Sum of squares due to regression**

$$\text{SSR} = \Sigma(\hat{y}_i - \bar{y})^2 \quad (\mathbf{14.10})$$

**Relationship among SST SSR and SSE**

$$\text{SST} = \text{SSR} + \text{SSE} \quad (\mathbf{14.11})$$

**Coefficient of determination**

$$r^2 = \frac{\text{SSR}}{\text{SST}} \quad (\mathbf{14.12})$$

**Sample correlation coefficient**

$$\begin{aligned} r_{XY} &= (\text{sign of } b_i) \sqrt{\text{Coefficient of determination}} \\ &= (\text{sign of } b_i) \sqrt{r^2} \end{aligned} \quad (14.13)$$

**Mean square error (estimate of  $s^2$ )**

$$s^2 = \text{MSE} = \frac{\text{SSE}}{n-2} \quad (14.15)$$

**Standard error of the estimate**

$$s = \sqrt{\text{MSE}} = \sqrt{\frac{\text{SSE}}{n-2}} \quad (14.16)$$

**Standard deviation of  $b_1$** 

$$\sigma_{b_1} = \frac{\sigma}{\sqrt{\sum(X_i - \bar{X})^2}} \quad (14.17)$$

**Estimated standard deviation of  $b_1$** 

$$s_{b_1} = \frac{s}{\sqrt{\sum(x_i - \bar{x})^2}} \quad (14.18)$$

 **$t$  test statistic**

$$t = \frac{b_1}{s_{b_1}} \quad (14.19)$$

**Mean square regression**

$$\text{MSR} = \frac{\text{SSR}}{\text{Number of independent variables}} \quad (14.20)$$

 **$F$  test statistic**

$$F = \frac{\text{MSR}}{\text{MSE}} \quad (14.21)$$

**Confidence interval for  $E(Y_p)$** 

$$\hat{y}_p \pm t_{\alpha/2}s \sqrt{\frac{1}{n} + \frac{(x_p - \bar{x})^2}{\sum(x_i - \bar{x})^2}} \quad (14.22)$$

**Prediction interval for  $Y_p$** 

$$\hat{y}_p \pm t_{\alpha/2}s \sqrt{\frac{1}{n} + \frac{(x_p - \bar{x})^2}{\sum(x_i - \bar{x})^2}} \quad (14.23)$$

**Residual for observation *i***

$$y_i - \hat{y}_i \quad (14.24)$$

**Standard deviation of the *i*th residual**

$$s_{y_i - \hat{y}_i} = s\sqrt{l - h_i} \quad (14.26)$$

**Standardized residual for observation *i***

$$\frac{y_i - \hat{y}_i}{s_y - \hat{y}} \quad (14.28)$$

**First-order autocorrelation**

$$\epsilon_t = \rho\epsilon_{t-1} + z_t \quad (14.29)$$

**Durbin–Watson test statistic**

$$d = \frac{\sum_{t=2}^n (e_t - e_{t-1})^2}{\sum_{t=1}^n e_t^2} \quad (14.30)$$

**Leverage of observation *i***

$$h_i = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum(x_i - \bar{x})^2} \quad (14.31)$$

**CASE PROBLEM 1*****Investigating the relationship between weight loss and triglyceride level reduction<sup>†</sup>***

Epidemiological studies have shown that there is a relationship between raised blood levels of triglyceride and coronary heart disease but it is not certain how important a risk factor triglycerides are. It is

believed that exercise and lower consumption of fatty acids can help to reduce triglyceride levels.\*

In 1998 Knoll Pharmaceuticals received authorization to market sibutramine for the treatment of obesity in the US. One of their suite of studies involved 35 obese patients who followed a treatment regime comprising a combination of diet, exercise and drug treatment.

Each patient's weight and triglyceride level were recorded at the start (known as *baseline*) and at week eight. The information recorded for each patient was:

<sup>†</sup>Source: STARS ([www.stars.ac.uk](http://www.stars.ac.uk)).

\*Triglycerides are lipids (fats) which are formed from glycerol and fatty acids. They can be absorbed into the body from food intake, particularly from fatty food, or produced in the body itself when the uptake of energy (food) exceeds the expenditure (exercise). Triglycerides provide the principal energy store for the body. Compared with carbohydrates or proteins, triglycerides produce a substantially higher number of calories per gram.



| Patient ID | Weight at baseline | Weight at week 8 | Triglyceride level at baseline | Triglyceride level at week 8 |
|------------|--------------------|------------------|--------------------------------|------------------------------|
| 201        | 84.0               | 82.4             | 90                             | 131                          |
| 202        | 88.8               | 87.0             | 137                            | 82                           |
| 203        | 87.0               | 81.8             | 182                            | 152                          |
| 204        | 84.5               | 80.4             | 72                             | 72                           |
| 205        | 69.4               | 69.0             | 143                            | 126                          |
| 206        | 104.7              | 102.0            | 96                             | 157                          |
| 207        | 90.0               | 87.6             | 115                            | 88                           |
| 208        | 89.4               | 86.8             | 124                            | 123                          |
| 209        | 95.2               | 92.8             | 188                            | 255                          |
| 210        | 108.1              | 100.9            | 167                            | 87                           |
| 211        | 93.9               | 90.2             | 143                            | 213                          |
| 212        | 83.4               | 75.0             | 143                            | 102                          |
| 213        | 104.4              | 102.9            | 276                            | 313                          |
| 214        | 103.7              | 95.7             | 84                             | 84                           |
| 215        | 99.2               | 99.2             | 142                            | 135                          |
| 216        | 95.6               | 88.5             | 64                             | 114                          |
| 217        | 126.0              | 123.2            | 226                            | 152                          |
| 218        | 103.7              | 95.5             | 199                            | 120                          |
| 219        | 133.1              | 130.8            | 212                            | 156                          |
| 220        | 85.0               | 80.0             | 268                            | 250                          |
| 221        | 83.8               | 77.9             | 111                            | 107                          |
| 222        | 104.5              | 98.3             | 132                            | 117                          |
| 223        | 76.8               | 73.2             | 165                            | 96                           |
| 224        | 90.5               | 88.9             | 57                             | 63                           |
| 225        | 106.9              | 103.7            | 163                            | 131                          |
| 226        | 81.5               | 78.9             | 111                            | 54                           |
| 227        | 96.5               | 94.9             | 300                            | 241                          |
| 228        | 103.0              | 97.2             | 192                            | 124                          |
| 229        | 127.5              | 124.7            | 176                            | 215                          |
| 230        | 103.2              | 102.0            | 146                            | 138                          |
| 231        | 113.5              | 115.0            | 446                            | 795                          |
| 232        | 107.0              | 99.2             | 232                            | 63                           |
| 233        | 106.0              | 103.5            | 255                            | 204                          |
| 234        | 114.9              | 105.3            | 187                            | 144                          |
| 235        | 103.4              | 96.0             | 154                            | 96                           |



TRIGLYCERID

- Patient ID.
- Weight at baseline (kg).
- Weight at week 8 (kg).
- Triglyceride level at baseline (mg/dl).
- Triglyceride level at week 8 (mg/dl).

### Triglyceride

The results are shown below.

### Managerial report

1. Are weight loss and triglyceride level reduction (linearly) correlated?
2. Is there a linear relationship between weight loss and triglyceride level reduction?
3. How can a more detailed regression analysis be undertaken?

**CASE PROBLEM 2****Measuring stock market risk**

One measure of the risk or volatility of an individual stock is the standard deviation of the total return (capital appreciation plus dividends) over several periods of time. Although the standard deviation is easy to compute, it does not take into account the extent to which the price of a given stock varies as a function of a standard market index, such as the S&P 500. As a result, many financial analysts prefer to use another measure of risk referred to as *beta*.

Betas for individual stocks are determined by simple linear regression. The dependent variable is the total return for the stock and the independent variable is the total return for the stock market.\* For this Case Problem we will use the S&P 500 index as the measure of the total return for the stock market, and an estimated regression equation will be developed using monthly data. The beta for the stock is the slope of the estimated regression equation ( $b_1$ ). The data contained in the file named 'Beta' provides the total return (capital appreciation plus dividends) over 36 months for eight widely traded common stocks and the S&P 500.



BETA

**Beta**

The value of beta for the stock market will always be 1; thus, stocks that tend to rise and fall with the stock market will also have a beta close to 1. Betas greater than 1 indicate that the stock is more volatile than the market, and betas less than 1 indicate that the stock is less volatile than the market. For instance, if a stock has a beta of 1.4, it is 40 per cent *more* volatile than the market, and if a stock has a beta of .4, it is 60 per cent *less* volatile than the market.



The Frankfurt Stock Exchange

**Managerial report**

You have been assigned to analyze the risk characteristics of these stocks. Prepare a report that includes but is not limited to the following items.

- a. Compute descriptive statistics for each stock and the S&P 500. Comment on your results. Which stocks are the most volatile?
- b. Compute the value of beta for each stock. Which of these stocks would you expect to perform best in an up market? Which would you expect to hold their value best in a down market?
- c. Comment on how much of the return for the individual stocks is explained by the market.

## CASE PROBLEM 3



### Can we detect dyslexia?

Data were collected on 34 pre-school children and then in follow-up tests (on the same children) three years later when they were seven years old.

Scores were obtained from a variety of tests on all the children at age four when they were at nursery school. The tests were:

- Knowledge of vocabulary, measured by the British Picture Vocabulary Test (BPVT) in three versions – as raw scores, standardized scores and percentile norms.
- Another vocabulary test – non-word repetition.
- Motor skills, where the children were scored on the time in seconds to complete five different peg board tests.
- Knowledge of prepositions, scored as the number correct out of ten.
- Three tests on the use of rhyming, scored as the number correct out of ten.

Three years later the same children were given a reading test, from which a reading deficiency was calculated as Reading Age – Chronological Age (in months), this being known as Reading Age Deficiency (RAD). The children were then classified into ‘poor’ or ‘normal’ readers, depending on their RAD scores. Poor reading ability is taken as an indication of potential dyslexia.

One purpose of this study is to identify which of the tests at age four might be used as predictors of poor reading ability, which in turn is a possible indication of dyslexia.

### Data

The data set ‘Dyslexia’ contains 18 variables:

- Child Code an identification number for each child (1–34)
- Sex m for male, f for female

The BPVT scores:

- |             |                               |
|-------------|-------------------------------|
| BPVT raw    | the raw score                 |
| BPVT std    | the standardized score        |
| BPVT % norm | cumulative percentage scores  |
| Non-wd repn | score for non-word repetition |

Scores in motor skills:

- |                                  |   |
|----------------------------------|---|
| ● Pegboard set1 to Pegboard set5 | the time taken to complete each test    |
| ● Mean                           | child’s average over the pegboard tests |
| ● Preps score                    | knowledge of prepositions (6–10)        |

Scores in rhyming tests (2–10):

- |               |   |
|---------------|---|
| ● Rhyme set1  |   |
| ● Rhyme set2  |   |
| ● Rhyme set3  |   |
| ● RAD         |   |
| ● Poor/Normal | RAD scores, categorized as 1 = normal, 2 = poor |

Details for ten records from the dataset are shown below.

| Child code | Sex | BPVT raw | BPVT std | BPVT % norm | Non-wd repn | Pegboard set1 | Pegboard set2 | Pegboard set3 | Pegboard set4 | Pegboard set5 |
|------------|-----|----------|----------|-------------|-------------|---------------|---------------|---------------|---------------|---------------|
| 1          | m   | 29       | 88       | 22          | 15          | 20.21         | 28.78         | 28.04         | 20.00         | 24.37         |
| 2          | m   | 21       | 77       | 6           | 11          | 26.34         | 26.20         | 20.35         | 28.25         | 20.87         |
| 3          | m   | 50       | 107      | 68          | 17          | 21.13         | 19.88         | 17.63         | 16.25         | 19.76         |
| 4          | m   | 23       | 80       | 9           | 5           | 16.46         | 16.47         | 16.63         | 14.16         | 17.25         |
| 5          | f   | 35       | 91       | 28          | 13          | 17.88         | 15.13         | 17.81         | 18.41         | 15.99         |
| 6          | m   | 36       | 97       | 42          | 16          | 20.41         | 18.64         | 17.03         | 16.69         | 14.47         |
| 7          | f   | 47       | 109      | 72          | 25          | 21.31         | 18.06         | 28.00         | 21.88         | 18.03         |
| 8          | m   | 32       | 92       | 30          | 12          | 14.57         | 14.22         | 13.47         | 12.29         | 18.38         |
| 9          | f   | 38       | 101      | 52          | 14          | 22.07         | 22.69         | 21.19         | 22.72         | 20.62         |
| 10         | f   | 44       | 105      | 63          | 15          | 16.40         | 14.48         | 13.83         | 17.59         | 34.68         |



DYSLEXIA

| Child code | Mean | Preps score | Rhyme set1 | Rhyme set2 | Rhyme set3 | RAD    | Poor/normal |
|------------|------|-------------|------------|------------|------------|--------|-------------|
| 1          | 24.3 | 6           | 5          | 5          | 5          | -6.50  | P           |
| 2          | 24.4 | 9           | 3          | 3          | 4          | -7.33  | P           |
| 3          | 18.9 | 10          | 9          | 8          | *          | 49.33  | N           |
| 4          | 16.2 | 7           | 4          | 6          | 4          | -11.00 | P           |
| 5          | 17.0 | 10          | 10         | 6          | 6          | -2.67  | N           |
| 6          | 17.5 | 10          | 6          | 5          | 5          | -8.33  | P           |
| 7          | 21.5 | 8           | 9          | 10         | 10         | 26.33  | N           |
| 8          | 14.6 | 10          | 8          | 6          | 3          | 9.00   | N           |
| 9          | 21.9 | 9           | 10         | 10         | 7          | 2.67   | N           |
| 10         | 19.4 | 10          | 7          | 8          | 4          | 9.67   | N           |



### Managerial report

1. Is there a (linear) relationship between scores in tests at ages four and seven?
2. Can we predict RAD from scores at age four?

# 15

# Multiple

# Regression



## CHAPTER CONTENTS

Statistics in Practice Jura

- 15.1 Multiple regression model
- 15.2 Least squares method
- 15.3 Multiple coefficient of determination
- 15.4 Model assumptions
- 15.5 Testing for significance
- 15.6 Using the estimated regression equation for estimation and prediction
- 15.7 Qualitative independent variables
- 15.8 Residual analysis
- 15.9 Logistic regression

**LEARNING OBJECTIVES** After reading this chapter and doing the exercises you should be able to:

- 1 Understand how multiple regression analysis can be used to develop relationships involving one dependent variable and several independent variables.
- 2 Interpret the coefficients in a multiple regression analysis.
- 3 Appreciate the background assumptions necessary to conduct statistical tests involving the hypothesized regression model.
- 4 Understand the role of computer packages in performing multiple regression analysis.
- 5 Interpret and use computer output to develop the estimated regression equation.
- 6 Determine how good a fit is provided by the estimated regression equation.
- 7 Test the significance of the regression equation.
- 8 Understand how multicollinearity affects multiple regression analysis.
- 9 Understand how residual analysis can be used to make a judgement as to the appropriateness of the model, identify outliers and determine which observations are influential.
- 10 Understand how logistic regression is used for regression analyses involving a binary dependent variable.

In Chapter 14 we presented simple linear regression and demonstrated its use in developing an estimated regression equation that describes the relationship between two variables. Recall that the variable being predicted or explained is called the dependent variable and the variable being used to predict or explain the dependent variable is called the independent variable. In this chapter we continue our study of regression analysis by considering situations involving two or more independent variables. This subject area, called **multiple regression analysis**, enables us to consider more than one potential predictor and thus obtain better estimates than are possible with simple linear regression.



## STATISTICS IN PRACTICE

Jura

Jura is a large island (380 sq km) off the South West of Scotland, famous for its malt whisky and the large deer population that wander the quartz mountains ('the Paps') that dominate the landscape. With a population of a mere 461 it has one of the lowest population densities of any place in the UK. Currently Jura is only accessible via the adjoining island, Islay, which has three ferry services a day – crossings taking about two hours. However, because Jura is only four miles from the mainland it has been suggested that a direct car ferry taking less than half an hour would be preferable and more economical than existing provisions.

In exploring the case for an alternative service, Riddington (1996) arrives at a number of alternative mathematical formulations that essentially reduce to multiple regression analysis. In particular, using historical data that also encompasses other inner Hebridean islands of Arran, Bute, Mull and Skye, he obtains the estimated binary logistic regression model:

The ferry to Jura

$$\begin{aligned} \text{Log}_e \frac{Q_{1it}}{Q_{2it}} = & 6.48 - 0.89 \frac{P_{1it}}{P_{2it}} \\ & + 0.129 \frac{F_{1it}}{F_{2it}} - 6.18 \frac{J_{1it}}{J_{2it}} \end{aligned}$$

where:

$Q_{1it}/Q_{2it}$  is the number of cars travelling by route 1 relative to the number travelling by route 2 to island  $i$  in year  $t$

$P_{1it}/P_{2it}$  is the relative price between route 1 and route 2 to  $i$  in year  $t$



$F_{1it}/F_{2it}$  is the relative frequency between route 1 and route 2 to  $i$  in year  $t$   
 $J_{1it}/J_{2it}$  is the relative journey time between route 1 and route 2 to  $i$  in year  $t$

Based on appropriate economic assumptions he estimates from this that some 132 000 passengers and 38 000 cars would use the new service each year rising over time. Initially this would yield a revenue of £426 000. Allowing for annual running costs of £322 000, the resultant gross profit would therefore be of the order of £100 000.

Source: Riddington, Geoff (1996) How many for the ferry boat? *OR Insight* Vol. 9:2: 26–32

## 15.1 MULTIPLE REGRESSION MODEL

Multiple regression analysis is the study of how a dependent variable  $Y$  is related to two or more independent variables. In the general case, we will use  $p$  to denote the number of independent variables.

### Regression model and regression equation

The concepts of a regression model and a regression equation introduced in the preceding chapter are applicable in the multiple regression case. The equation that describes how the dependent variable  $Y$  is related to the independent variables  $X_1, X_2, \dots, X_p$  and an error term is called the **multiple regression model**. We begin with the assumption that the multiple regression model takes the following form.

#### Multiple regression model

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p + \varepsilon \quad (15.1)$$

where  $X_1 = x_1, X_2 = x_2, \dots, X_p = x_p$

In the multiple regression model,  $\beta_0, \beta_1, \dots, \beta_p$ , are the parameters and  $\varepsilon$  (the Greek letter epsilon) is a random variable. A close examination of this model reveals that  $Y$  is a linear function of  $x_1, x_2, \dots, x_p$  (the  $\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p$  part) plus an error term  $\varepsilon$ . The error term accounts for the variability in  $Y$  that cannot be explained by the linear effect of the  $p$  independent variables.

In Section 15.4 we will discuss the assumptions for the multiple regression model and  $\varepsilon$ . One of the assumptions is that the mean or expected value of  $\varepsilon$  is zero. A consequence of this assumption is that the mean or expected value of  $Y$ , denoted  $E(Y)$ , is equal to  $\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p$ . The equation that describes how the mean value of  $Y$  is related to  $x_1, x_2, \dots, x_p$  is called the **multiple regression equation**.

#### Multiple regression equation

$$E(Y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p \quad (15.2)$$

### Estimated multiple regression equation

If the values of  $\beta_0, \beta_1, \dots, \beta_p$  were known, equation (15.2) could be used to compute the mean value of  $Y$  at given values of  $x_1, x_2, \dots, x_p$ . Unfortunately, these parameter values will not, in general, be known and must be estimated from sample data. A simple random sample is used to compute sample statistics  $b_0, b_1, \dots, b_p$  that are used as the point estimators of the parameters  $\beta_0, \beta_1, \dots, \beta_p$ . These sample statistics provide the following **estimated multiple regression equation**.

#### Estimated multiple regression equation

$$\hat{y} = b_0 + b_1 x_1 + b_2 x_2 + \dots + b_p x_p \quad (15.3)$$

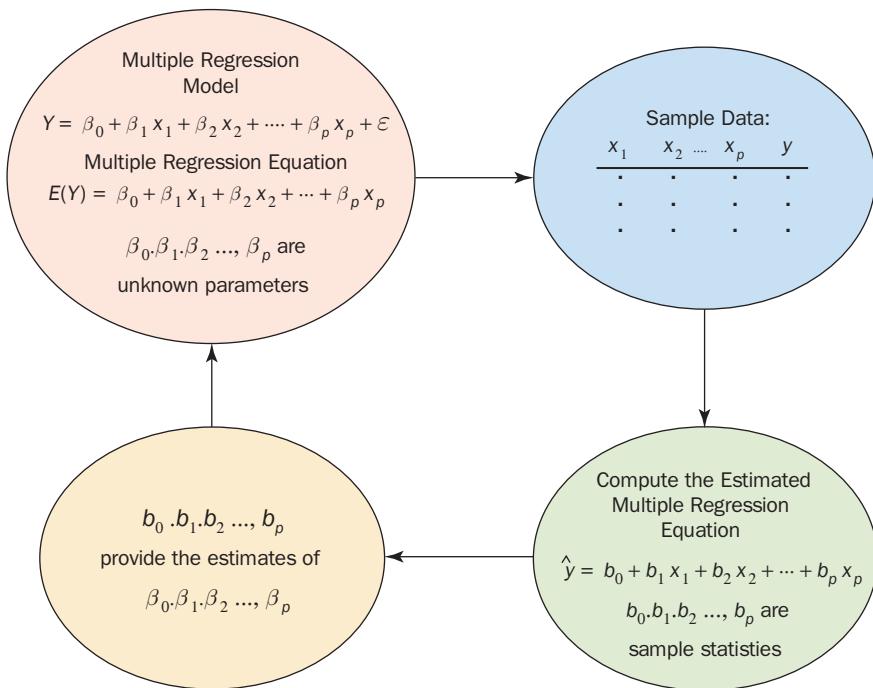
where:

$b_0, b_1, \dots, b_p$  are the estimates of  $\beta_0, \beta_1, \dots, \beta_p$

$\hat{y}$  = estimated value of the dependent variable

**FIGURE 15.1**

The estimation process for multiple regression



The estimation process for multiple regression is shown in Figure 15.1.

## 15.2 LEAST SQUARES METHOD

In Chapter 14 we used the **least squares method** to develop the estimated regression equation that best approximated the straight line relationship between the dependent and independent variables. This same approach is used to develop the estimated multiple regression equation. The least squares criterion is restated as follows.

### Least squares criterion

$$\min \sum (y_i - \hat{y}_i)^2 \quad (15.4)$$

where:

$y_i$  = observed value of the dependent variable for the  $i$ th observation

$\hat{y}_i$  = estimated value of the dependent variable for the  $i$ th observation

The estimated values of the dependent variable are computed by using the estimated multiple regression equation,

$$\hat{y} = b_0 + b_1 x_1 + b_2 x_2 + \dots + b_p x_p$$

As expression (15.4) shows, the least squares method uses sample data to provide the values of  $b_0, b_1, \dots, b_p$  that make the sum of squared residuals {the deviations between the observed values of the dependent variable ( $y_i$ ) and the estimated values of the dependent variable  $\hat{y}_i$ } a minimum.

In Chapter 14 we presented formulae for computing the least squares estimators  $b_0$  and  $b_1$  for the estimated simple linear regression equation  $\hat{y} = b_0 + b_1 x$ . With relatively small data sets, we were able to use those formulae to compute  $b_0$  and  $b_1$  by manual calculations. In multiple regression, however, the

presentation of the formulae for the regression coefficients  $b_0, b_1, \dots, b_p$  involves the use of matrix algebra and is beyond the scope of this text. Therefore, in presenting multiple regression, we focus on how computer software packages can be used to obtain the estimated regression equation and other information. The emphasis will be on how to interpret the computer output rather than on how to make the multiple regression computations.

## An example: Eurodistributor Company

As an illustration of multiple regression analysis, we will consider a problem faced by the Eurodistributor Company, an independent distribution company in the Netherlands. A major portion of Eurodistributor's business involves deliveries throughout its local area. To develop better work schedules, the company's managers want to estimate the total daily travel time for their drivers.

Initially the managers believed that the total daily travel time would be closely related to the distance travelled in making the daily deliveries. A simple random sample of ten driving assignments provided the data shown in Table 15.1 and the scatter diagram shown in Figure 15.2. After reviewing this scatter diagram, the managers hypothesized that the simple linear regression model  $Y = \beta_0 + \beta_1 x_1 + \varepsilon$  could be used to describe the relationship between the total travel time ( $Y$ ) and the distance travelled ( $X_1$ ). To estimate the parameters  $\beta_0$  and  $\beta_1$ , the least squares method was used to develop the estimated regression equation.

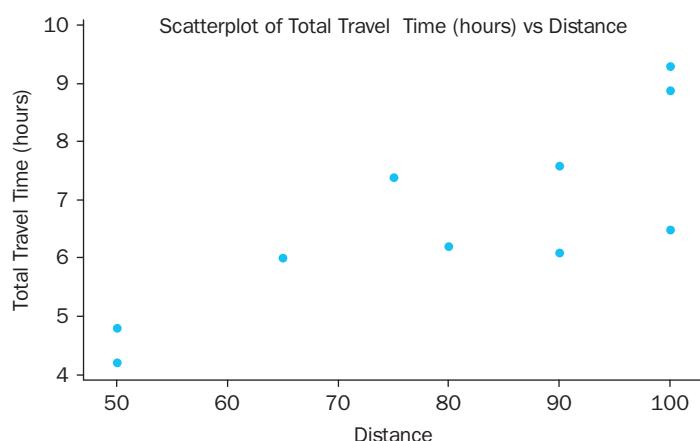
$$\hat{y} = b_0 + b_1 x_1 \quad (15.5)$$

**TABLE 15.1** Preliminary data for Eurodistributor

| Driving assignment | $X_1 = \text{Distance travelled (kilometres)}$ | $Y = \text{Travel time (hours)}$ |
|--------------------|--|----------------------------------|
| 1                  | 100  | 9.3                              |
| 2                  | 50   | 4.8                              |
| 3                  | 100  | 8.9                              |
| 4                  | 100  | 6.5                              |
| 5                  | 50   | 4.2                              |
| 6                  | 80   | 6.2                              |
| 7                  | 75   | 7.4                              |
| 8                  | 65   | 6.0                              |
| 9                  | 90   | 7.6                              |
| 10                 | 90   | 6.1                              |

**FIGURE 15.2**

Scatter diagram of preliminary data for Eurodistributor



**FIGURE 15.3**

MINITAB output for  
Eurodistributor with one  
independent variable

**Regression Analysis: Time versus Distance**

The regression equation is  
 $\text{Time} = 1.27 + 0.0678 \text{ Distance}$

| Predictor | Coeff   | SE Coef | T    | P     |
|-----------|---------|---------|------|-------|
| Constant  | 1.274   | 1.401   | 0.91 | 0.390 |
| Distance  | 0.06783 | 0.01706 | 3.98 | 0.004 |

$$S = 1.00179 \quad R-\text{Sq} = 66.4\% \quad R-\text{Sq}(\text{adj}) = 62.2\%$$

**Analysis of Variance**

| Source         | DF | SS     | MS     | F     | P     |
|----------------|----|--------|--------|-------|-------|
| Regression     | 1  | 15.871 | 15.871 | 15.81 | 0.004 |
| Residual Error | 8  | 8.029  | 1.004  |       |       |
| Total          | 9  | 23.900 |        |       |       |

In Figure 15.3, we show the MINITAB computer output from applying simple linear regression to the data in Table 15.1. The estimated regression equation is:

$$\hat{y} = 1.27 + 0.0678x_1$$

At the 0.05 level of significance, the  $F$  value of 15.81 and its corresponding  $p$ -value of 0.004 indicate that the relationship is significant; that is, we can reject  $H_0: \beta_1 = 0$  because the  $p$ -value is less than  $\alpha = 0.05$ . Thus, we can conclude that the relationship between the total travel time and the distance travelled is significant; longer travel times are associated with more distance. With a coefficient of determination (expressed as a percentage) of  $R-\text{sq} = 66.4$  per cent, we see that 66.4 per cent of the variability in travel time can be explained by the linear effect of the distance travelled. This finding is fairly good, but the managers might want to consider adding a second independent variable to explain some of the remaining variability in the dependent variable.

In attempting to identify another independent variable, the managers felt that the number of deliveries could also contribute to the total travel time. The Eurodistributor data, with the number of deliveries added, are shown in Table 15.2.

**TABLE 15.2** Data for Eurodistributor with distance ( $X_1$ ) and number of deliveries ( $X_2$ ) as the independent variables

| Driving assignment | $X_1 = \text{Distance travelled (kilometres)}$ | $X_2 = \text{Number of deliveries}$ | $Y = \text{Travel time (hours)}$ |
|--------------------|--|-------------------------------------|----------------------------------|
| 1                  | 100  | 4                                   | 9.3                              |
| 2                  | 50   | 3                                   | 4.8                              |
| 3                  | 100  | 4                                   | 8.9                              |
| 4                  | 100  | 2                                   | 6.5                              |
| 5                  | 50   | 2                                   | 4.2                              |
| 6                  | 80   | 2                                   | 6.2                              |
| 7                  | 75   | 3                                   | 7.4                              |
| 8                  | 65   | 4                                   | 6.0                              |
| 9                  | 90   | 3                                   | 7.6                              |
| 10                 | 90   | 2                                   | 6.1                              |



EURO  
DISTRIBUTOR

**FIGURE 15.4**

MINITAB output for  
Eurodistributor with two  
independent variables

**Regression Analysis: Time versus Distance, Deliveries**

The regression equation is  
Time = - 0.869 + 0.0611 Distance + 0.923 Deliveries

| Predictor  | Coef     | SE Coef  | T     | P     |
|------------|----------|----------|-------|-------|
| Constant   | -0.8687  | 0.9515   | -0.91 | 0.392 |
| Distance   | 0.061135 | 0.009888 | 6.18  | 0.000 |
| Deliveries | 0.9234   | 0.2211   | 4.18  | 0.004 |

$$S = 0.573142 \quad R-Sq = 90.4\% \quad R-Sq(\text{adj}) = 87.6\%$$

**Analysis of Variance**

| Source         | DF | SS     | MS     | F     | P     |
|----------------|----|--------|--------|-------|-------|
| Regression     | 2  | 21.601 | 10.800 | 32.88 | 0.000 |
| Residual Error | 7  | 2.299  | 0.328  |       |       |
| Total          | 9  | 23.900 |        |       |       |

| Source     | DF | Seq SS |
|------------|----|--------|
| Distance   | 1  | 15.871 |
| Deliveries | 1  | 5.729  |

The MINITAB computer solution with both distance ( $X_1$ ) and number of deliveries ( $X_2$ ) as independent variables is shown in Figure 15.4. The estimated regression equation is:

$$\hat{y} = -0.869 + 0.0611x_1 + 0.923x_2 \quad (15.6)$$

In the next section we will discuss the use of the coefficient of multiple determination in measuring how good a fit is provided by this estimated regression equation. Before doing so, let us examine more carefully the values of  $b_1 = 0.0611$  and  $b_2 = 0.923$  in equation (15.6).

**Note on interpretation of coefficients**

One observation can be made at this point about the relationship between the estimated regression equation with only the distance as an independent variable and the equation that includes the number of deliveries as a second independent variable. The value of  $b_1$  is not the same in both cases. In simple linear regression, we interpret  $b_1$  as an estimate of the change in  $Y$  for a one-unit change in the independent variable. In multiple regression analysis, this interpretation must be modified somewhat. That is, in multiple regression analysis, we interpret each regression coefficient as follows:  $b_i$  represents an estimate of the change in  $Y$  corresponding to a one-unit change in  $X_i$  when all other independent variables are held constant.

In the Eurodistributor example involving two independent variables,  $b_1 = 0.0611$ . Thus, 0.0611 hours is an estimate of the expected increase in travel time corresponding to an increase of one kilometre in the distance travelled when the number of deliveries is held constant. Similarly, because  $b_2 = 0.923$ , an estimate of the expected increase in travel time corresponding to an increase of one delivery when the distance travelled is held constant is 0.923 hours.

## EXERCISES

Note to student: The exercises involving data in this and subsequent sections were designed to be solved using a computer software package.

### Methods

- 1.** The estimated regression equation for a model involving two independent variables and ten observations follows.

$$\hat{y} = 29.1270 + 0.5906x_1 + 0.4980x_2$$

- a. Interpret  $b_1$  and  $b_2$  in this estimated regression equation.  
 b. Estimate  $Y$  when  $X_1 = 180$  and  $X_2 = 310$ .

- 2.** Consider the following data for a dependent variable  $Y$  and two independent variables,  $X_1$  and  $X_2$ .

| $x_1$ | $x_2$ | $y$ |
|-------|-------|-----|
| 30    | 12    | 94  |
| 47    | 10    | 108 |
| 25    | 17    | 112 |
| 51    | 16    | 178 |
| 40    | 5     | 94  |
| 51    | 19    | 175 |
| 74    | 7     | 170 |
| 36    | 12    | 117 |
| 59    | 13    | 142 |
| 76    | 16    | 211 |

- a. Develop an estimated regression equation relating  $Y$  to  $X_1$ . Estimate  $Y$  if  $X_1 = 45$ .  
 b. Develop an estimated regression equation relating  $Y$  to  $X_2$ . Estimate  $Y$  if  $X_2 = 15$ .  
 c. Develop an estimated regression equation relating  $Y$  to  $X_1$  and  $X_2$ . Estimate  $Y$  if  $X_1 = 45$  and  $X_2 = 15$ .  
**3.** In a regression analysis involving 30 observations, the following estimated regression equation was obtained.

$$\hat{y} = 17.6 + 03.8x_1 - 2.3x_2 + 7.6x_3 + 2.7x_4$$

- a. Interpret  $b_1$ ,  $b_2$ ,  $b_3$  and  $b_4$  in this estimated regression equation.  
 b. Estimate  $Y$  when  $X_1 = 10$ ,  $X_2 = 5$ ,  $X_3 = 1$  and  $X_4 = 2$ .



COMPLETE  
SOLUTIONS

### Applications

- 4.** The stack loss plant data of Brownlee (1965) contains 21 days of measurements from a plant's oxidation of ammonia to nitric acid. The nitric oxide pollutants are captured in an absorption tower. Details of variables are as follows:
- $Y = \text{LOSS}$  = ten times the percentage of ammonia going into the plant that escapes from the absorption column.
  - $X_1 = \text{AIRFLOW}$  = Rate of operation of the plant.
  - $X_2 = \text{TEMP}$  = Cooling water temperature in the absorption tower.
  - $X_3 = \text{ACID}$  = Acid concentration of circulating acid minus 50 times.

The following estimated regression equation relating LOSS to AIRFLOW and TEMP was given.

$$\hat{y} = -50.359 + 0.671x_1 + 1.295x_2$$

- a. Estimate sales resulting from an AIRFLOW of 60 and a TEMP of 20.
  - b. Interpret  $b_1$  and  $b_2$  in this estimated regression equation.
5. The owner of Toulon Theatres would like to estimate weekly gross revenue as a function of advertising expenditures. Historical data for a sample of eight weeks follow.



TOULON

| <i>Weekly gross revenue<br/>(\$'000s)</i> | <i>Television advertising<br/>(\$'000s)</i> | <i>Newspaper advertising<br/>(\$'000s)</i> |
|---|---|--|
| 96  | 5.0   | 1.5  |
| 90  | 2.0   | 2.0  |
| 95  | 4.0   | 1.5  |
| 92  | 2.5   | 2.5  |
| 95  | 3.0   | 3.3  |
| 94  | 3.5   | 2.3  |
| 94  | 2.5   | 4.2  |
| 94  | 3.0   | 2.5  |

- a. Develop an estimated regression equation with the amount of television advertising as the independent variable.
  - b. Develop an estimated regression equation with both television advertising and newspaper advertising as the independent variables.
  - c. Is the estimated regression equation coefficient for television advertising expenditures the same in part (a) and in part (b)? Interpret the coefficient in each case.
  - d. What is the estimate of the weekly gross revenue for a week when €3500 is spent on television advertising and €1800 is spent on newspaper advertising?
6. The following table gives the annual return, the safety rating (0 = riskiest, 10 = safest), and the annual expense ratio for 20 foreign funds.

|                                     | <i>Annual<br/>safety rating</i> | <i>Expense<br/>ratio (%)</i> | <i>Annual<br/>return (%)</i> |
|-------------------------------------|---------------------------------|------------------------------|------------------------------|
| Accessor Int'l Equity 'Adv'         | 7.1                             | 1.59                         | 49                           |
| Aetna 'I' International             | 7.2                             | 1.35                         | 52                           |
| Amer Century Int'l Discovery 'Inv'  | 6.8                             | 1.68                         | 89                           |
| Columbia International Stock        | 7.1                             | 1.56                         | 58                           |
| Concert Inv 'A' Int'l Equity        | 6.2                             | 2.16                         | 131                          |
| Dreyfus Founders Int'l Equity 'F'   | 7.4                             | 1.80                         | 59                           |
| Driehaus International Growth       | 6.5                             | 1.88                         | 99                           |
| Excelsior 'Inst' Int'l Equity       | 7.0                             | 0.90                         | 53                           |
| Julius Baer International Equity    | 6.9                             | 1.79                         | 77                           |
| Marshall International Stock 'Y'    | 7.2                             | 1.49                         | 54                           |
| MassMutual Int'l Equity 'S'         | 7.1                             | 1.05                         | 57                           |
| Morgan Grenfell Int'l Sm Cap 'Inst' | 7.7                             | 1.25                         | 61                           |
| New England 'A' Int'l Equity        | 7.0                             | 1.83                         | 88                           |
| Pilgrim Int'l Small Cap 'A'         | 7.0                             | 1.94                         | 122                          |
| Republic International Equity       | 7.2                             | 1.09                         | 71                           |
| Sit International Growth            | 6.9                             | 1.50                         | 51                           |
| Smith Barney 'A' Int'l Equity       | 7.0                             | 1.28                         | 60                           |



FORFUND

|                                    | <i>Annual<br/>safety rating</i> | <i>Expense<br/>ratio (%)</i> | <i>Annual<br/>return (%)</i> |
|------------------------------------|---------------------------------|------------------------------|------------------------------|
| State St Research 'S' Int'l Equity | 7.1                             | 1.65                         | 50                           |
| Strong International Stock         | 6.5                             | 1.61                         | 93                           |
| Vontobel International Equity      | 7.0                             | 1.50                         | 47                           |

- a. Develop an estimated regression equation relating the annual return to the safety rating and the annual expense ratio.
- b. Estimate the annual return for a firm that has a safety rating of 7.5 and annual expense ratio of 2.

## 15.3 MULTIPLE COEFFICIENT OF DETERMINATION

In simple linear regression we showed that the total sum of squares can be partitioned into two components: the sum of squares due to regression and the sum of squares due to error.

The same procedure applies to the sum of squares in multiple regression.

### Relationship among SST, SSR and SSE

$$\text{SST} = \text{SSR} + \text{SSE} \quad (15.7)$$

where:

$$\text{SST} = \text{total sum of squares} = \sum (y_i - \bar{y})^2$$

$$\text{SSR} = \text{sum of squares due to regression} = \sum (\hat{y}_i - \bar{y})^2$$

$$\text{SSE} = \text{sum of squares due to error} = \sum (y_i - \hat{y}_i)^2$$

Because of the computational difficulty in computing the three sums of squares, we rely on computer packages to determine those values. The analysis of variance part of the MINITAB output in Figure 15.4 shows the three values for the Eurodistributor problem with two independent variables: SST = 23.900, SSR = 21.601 and SSE = 2.299. With only one independent variable (distance travelled), the MINITAB output in Figure 15.3 shows that SST = 23.900, SSR = 15.871 and SSE = 8.029. The value of SST is the same in both cases because it does not depend on  $\hat{y}$  but SSR increases and SSE decreases when a second independent variable (number of deliveries) is added. The implication is that the estimated multiple regression equation provides a better fit for the observed data.

In Chapter 14, we used the coefficient of determination,  $R^2 = \text{SSR/SST}$ , to measure the goodness of fit for the estimated regression equation. The same concept applies to multiple regression. The term **multiple coefficient of determination** indicates that we are measuring the goodness of fit for the estimated multiple regression equation. The multiple coefficient of determination, denoted  $R^2$ , is computed as follows.

### Multiple coefficient of determination

$$R^2 = \frac{\text{SSR}}{\text{SST}} \quad (15.8)$$

The multiple coefficient of determination can be interpreted as the proportion of the variability in the dependent variable that can be explained by the estimated multiple regression equation. Hence, when multiplied by 100, it can be interpreted as the percentage of the variability in  $Y$  that can be explained by the estimated regression equation.

In the two-independent-variable Eurodistributor example, with  $\text{SSR} = 21.601$  and  $\text{SST} = 23.900$ , we have:

$$R^2 = \frac{21.601}{23.900} = 0.904$$

Therefore, 90.4 per cent of the variability in travel time  $Y$  is explained by the estimated multiple regression equation with distance and number of deliveries as the independent variables. In Figure 15.4, we see that the multiple coefficient of determination is also provided by the MINITAB output; it is denoted by  $R\text{-sq} = 90.4$  per cent.

Figure 15.3 shows that the  $R\text{-sq}$  value for the estimated regression equation with only one independent variable, distance travelled ( $X_1$ ), is 66.4 per cent. Thus, the percentage of the variability in travel times that is explained by the estimated regression equation increases from 66.4 per cent to 90.4 per cent when number of deliveries is added as a second independent variable. In general,  $R^2$  increases as independent variables are added to the model.

Many analysts prefer adjusting  $R^2$  for the number of independent variables to avoid overestimating the impact of adding an independent variable on the amount of variability explained by the estimated regression equation. With  $n$  denoting the number of observations and  $p$  denoting the number of independent variables, the **adjusted multiple coefficient of determination** is computed as follows.

### Adjusted multiple coefficient of determination

$$\text{adj } R^2 = 1 - (1 - R^2) \frac{n - 1}{n - p - 1} \quad (15.9)$$

For the Eurodistributor example with  $n = 10$  and  $p = 2$ , we have:

$$\text{adj } R^2 = 1 - (1 - 0.904) \frac{10 - 1}{10 - 2 - 1} = 0.88$$

Therefore, after adjusting for the two independent variables, we have an adjusted multiple coefficient of determination of 0.88. This value, allowing for rounding, corresponds with the value in the MINITAB output in Figure 15.4 of  $R\text{-sq(adj)} = 87.6$  per cent.

## EXERCISES

### Methods

7. In Exercise 1, the following estimated regression equation based on ten observations was presented.

$$\hat{y} = 29.1270 + 0.5906x_1 + 0.4980x_2$$

The values of  $\text{SST}$  and  $\text{SSR}$  are 6724.125 and 6216.375, respectively.

- a. Find SSE.
- b. Compute  $R^2$ .
- c. Compute adj  $R^2$ .
- d. Comment on the goodness of fit.



EXER2

COMPLETE  
SOLUTIONS

TOULON

- 8.** In Exercise 2, ten observations were provided for a dependent variable  $Y$  and two independent variables  $X_1$  and  $X_2$ ; for these data  $SST = 15\,182.9$  and  $SSR = 14\,052.2$ .
- Compute  $R^2$ .
  - Compute adj  $R^2$ .
  - Does the estimated regression equation explain a large amount of the variability in the data? Explain.
- 9.** In Exercise 3, the following estimated regression equation based on 30 observations was presented.

$$\hat{y} = 17.6 + 3.8x_1 - 2.3x_2 + 7.6x_3 + 2.7x_4$$

The values of  $SST$  and  $SSR$  are 1805 and 1760, respectively.

- Compute  $R^2$ .
- Compute adj  $R^2$ .
- Comment on the goodness of fit.

### Applications

- 10.** In Exercise 4, the following estimated regression equation relating LOSS ( $Y$ ) to AIRFLOW ( $X_1$ ) and TEMP ( $X_2$ ) was given.

$$\hat{y} = -50.359 + 0.671x_1 + 1.295x_2$$

For these data  $SST = 2069.238$  and  $SSR = 1880.443$ .

- For the estimated regression equation given, compute  $R^2$ .
  - Compute adj  $R^2$ .
  - Does the model appear to explain a large amount of variability in the data? Explain.
- 11.** In Exercise 5, the owner of Toulon Theatres used multiple regression analysis to predict gross revenue ( $Y$ ) as a function of television advertising ( $X_1$ ) and newspaper advertising ( $X_2$ ). The estimated regression equation was

$$\hat{y} = 83.2 + 2.29x_1 + 1.30x_2$$

The computer solution provided  $SST = 25.5$  and  $SSR = 23.435$ .

- Compute and interpret  $R^2$  and adj  $R^2$ .
- When television advertising was the only independent variable,  $R^2 = 0.653$  and adj  $R^2 = 0.595$ . Do you prefer the multiple regression results? Explain.

## 15.4 MODEL ASSUMPTIONS

In Section 15.1 we introduced the following multiple regression model.

### Multiple regression model

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p + \varepsilon \quad (15.10)$$

The assumptions about the error term  $\varepsilon$  in the multiple regression model parallel those for the simple linear regression model.

### Assumptions about the error term in the multiple regression model

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p + \varepsilon$$

1. The error  $\varepsilon$  is a random variable with mean or expected value of zero; that is,  $E(\varepsilon) = 0$ . *Implication:* For given values of  $X_1, X_2, \dots, X_p$ , the expected, or average, value of  $Y$  is given by:

$$E(Y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p \quad (15.11)$$

Equation (15.11) is the multiple regression equation we introduced in Section 15.1. In this equation,  $E(Y)$  represents the average of all possible values of  $Y$  that might occur for the given values of  $X_1, X_2, \dots, X_p$ .

2. The variance of  $\varepsilon$  is denoted by  $\sigma^2$  and is the same for all values of the independent variables  $X_1, X_2, \dots, X_p$ .

*Implication:* The variance of  $Y$  about the regression line equals  $\sigma^2$  and is the same for all values of  $X_1, X_2, \dots, X_p$ .

3. The values of  $\varepsilon$  are independent.

*Implication:* The size of the error for a particular set of values for the independent variables is not related to the size of the error for any other set of values.

4. The error  $\varepsilon$  is a normally distributed random variable reflecting the deviation between the  $Y$  value and the expected value of  $Y$  given by  $\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p$ .

*Implication:* Because  $\beta_0, \beta_1, \dots, \beta_p$  are constants for the given values of  $x_1, x_2, \dots, x_p$ , the dependent variable  $Y$  is also a normally distributed random variable.

To obtain more insight about the form of the relationship given by equation (15.11), consider the following two-independent-variable multiple regression equation.

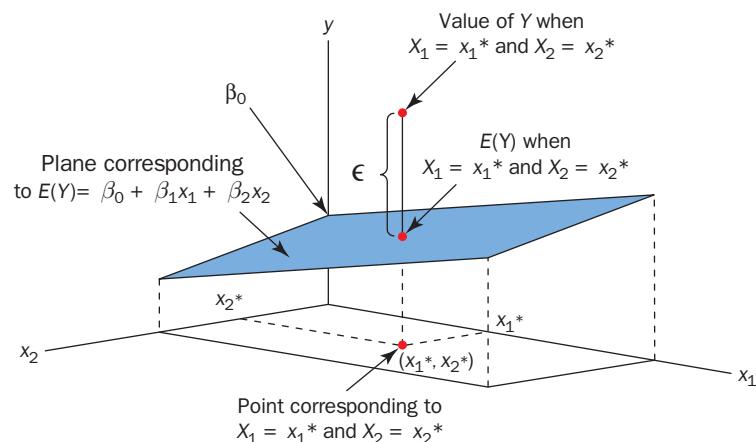
$$E(Y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

The graph of this equation is a plane in three-dimensional space. Figure 15.5 provides an example of such a graph. Note that the value of  $\varepsilon$  shown is the difference between the actual  $Y$  value and the expected value of  $y$ ,  $E(Y)$ , when  $X_1 = x_1^*$  and  $X_2 = x_2^*$ .

In regression analysis, the term *response variable* is often used in place of the term *dependent variable*. Furthermore, since the multiple regression equation generates a plane or surface, its graph is called a *response surface*.

**FIGURE 15.5**

Graph of the regression equation for multiple regression analysis with two independent variables



## 15.5 TESTING FOR SIGNIFICANCE

In this section we show how to conduct significance tests for a multiple regression relationship.

The significance tests we used in simple linear regression were a  $t$  test and an  $F$  test. In simple linear regression, both tests provide the same conclusion: that is, if the null hypothesis is rejected, we conclude that the slope parameter  $\beta_1 \neq 0$ . In multiple regression, the  $t$  test and the  $F$  test have different purposes.

- 1** The  $F$  test is used to determine whether a significant relationship exists between the dependent variable and the set of all the independent variables; we will refer to the  $F$  test as the test for *overall significance*.
- 2** If the  $F$  test shows an overall significance, the  $t$  test is used to determine whether each of the individual independent variables is significant. A separate  $t$  test is conducted for each of the independent variables in the model; we refer to each of these  $t$  tests as a test for *individual significance*.

In the material that follows, we will explain the  $F$  test and the  $t$  test and apply each to the Eurodistributor Company example.

### F test

Given the multiple regression model defined in (15.1)

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p + \varepsilon$$

the hypotheses for the  $F$  test can be written as follows:

$$H_0: \beta_1 = \beta_2 = \dots = \beta_p = 0$$

$H_1$ : One or more of the parameters is not equal to zero

If  $H_0$  is rejected, the test gives us sufficient statistical evidence to conclude that one or more of the parameters is not equal to zero and that the overall relationship between  $Y$  and the set of independent variables  $X_1, X_2, \dots, X_p$  is significant. However, if  $H_0$  cannot be rejected, we deduce there is not sufficient evidence to conclude that a significant relationship is present.

Before confirming the steps involved in performing the  $F$  test, it might be helpful if we first review the concept of *mean square*. A mean square is a sum of squares divided by its corresponding degrees of freedom. In the multiple regression case, the total sum of squares has  $n - 1$  degrees of freedom, the sum of squares due to regression (SSR) has  $p$  degrees of freedom, and the sum of squares due to error has  $n - p - 1$  degrees of freedom. Hence, the mean square due to regression (MSR) is:

#### Mean square regression

$$\text{MSR} = \frac{\text{SSR}}{p} \quad (15.12)$$

and:

#### Mean square error

$$\text{MSE} = s^2 = \frac{\text{SSE}}{n - p - 1} \quad (15.13)$$

As has already been acknowledged in Chapter 14, MSE provides an unbiased estimate of  $\sigma^2$ , the variance of the error term  $\varepsilon$ . If  $H_0: \beta_1 = \beta_2 = \dots = \beta_p = 0$  is true, MSR also provides an unbiased estimate of  $\sigma^2$ , and the value of MSR/MSE should be close to 1. However, if  $H_0$  is false, MSR overestimates  $\sigma^2$  and the value of MSR/MSE becomes larger. To determine how large the value of MSR/MSE must be to reject  $H_0$ , we make use of the fact that if  $H_0$  is true and the assumptions about the multiple regression model are valid, the sampling distribution of MSR/MSE is an  $F$  distribution with  $p$  degrees of freedom in the numerator and  $n - p - 1$  in the denominator. A summary of the  $F$  test for significance in multiple regression follows.

### F test for overall significance

$$H_0: \beta_1 = \beta_2 = \dots = \beta_p = 0$$

$H_1$ : One or more of the parameters is not equal to zero

#### Test statistic

$$F = \frac{\text{MSR}}{\text{MSE}} \quad (15.14)$$

#### Rejection rule

$p$ -value approach: Reject  $H_0$  if  $p$ -value  $\leq \alpha$

Critical value approach: Reject  $H_0$  if  $F \geq F_\alpha$

where  $F_\alpha$  is based on an  $F$  distribution with  $p$  degrees of freedom in the numerator and  $n - p - 1$  degrees of freedom in the denominator.

Applying the  $F$  test to the Eurodistributor Company multiple regression problem with two independent variables, the hypotheses can be written as follows.

$$H_0: \beta_1 = \beta_2 = 0$$

$H_1$ :  $\beta_1$  and/or  $\beta_2$  is not equal to zero

Figure 15.6 shows the MINITAB output for the multiple regression model with distance ( $X_1$ ) and number of deliveries ( $X_2$ ) as the two independent variables. In the analysis of variance part of the output, we see that  $\text{MSR} = 10.8$  and  $\text{MSE} = 0.328$ . Using equation (15.14), we obtain the test statistic.

$$F = \frac{10.8}{0.328} = 32.9$$

Note that the  $F$  value on the MINITAB output is  $F = 32.88$ ; the value we calculated differs because we used rounded values for MSR and MSE in the calculation. Using  $\alpha = 0.01$ , the  $p$ -value = 0.000 in the last column of the analysis of variance table (Figure 15.6) indicates that we can reject  $H_0: \beta_1 = \beta_2 = 0$  because the  $p$ -value is less than  $\alpha = 0.01$ . Alternatively, Table 4 of Appendix B shows that with two degrees of freedom in the numerator and seven degrees of freedom in the denominator,  $F_{0.01} = 9.55$ . With  $32.9 > 9.55$ , we reject  $H_0: \beta_1 = \beta_2 = 0$  and conclude that a significant relationship is present between travel time  $Y$  and the two independent variables, distance and number of deliveries.

As noted previously, the mean square error provides an unbiased estimate of  $\sigma^2$ , the variance of the error term  $\varepsilon$ . Referring to Figure 15.6, we see that the estimate of  $\sigma^2$  is  $\text{MSE} = 0.328$ . The square root of  $\text{MSE}$  is the estimate of the standard deviation of the error term. As defined in Section 14.5, this standard deviation is called the standard error of the estimate and is denoted  $s$ . Hence, we have  $s = \sqrt{\text{MSE}} = \sqrt{0.328} = 0.573$ . Note that the value of the standard error of the estimate appears in the MINITAB output in Figure 15.6.

Table 15.3 is the general analysis of variance (ANOVA) table that provides the  $F$  test results for a multiple regression model.

**FIGURE 15.6**

MINITAB output for Eurodistributor with two independent variables, distance ( $X_1$ ) and number of deliveries ( $X_2$ )

**Regression Analysis: Time versus Distance, Deliveries**

The regression equation is

$$\text{Time} = -0.869 + 0.0611 \text{ Distance} + 0.923 \text{ Deliveries}$$

| Predictor  | Coef     | SE Coef  | T     | P     |
|------------|----------|----------|-------|-------|
| Constant   | -0.8687  | 0.9515   | -0.91 | 0.392 |
| Distance   | 0.061135 | 0.009888 | 6.18  | 0.000 |
| Deliveries | 0.9234   | 0.2211   | 4.18  | 0.004 |

$$S = 0.573142 \quad R-\text{Sq} = 90.4\% \quad R-\text{Sq}(\text{adj}) = 87.6\%$$

**Analysis of Variance**

| Source         | DF | SS     | MS     | F     | P     |
|----------------|----|--------|--------|-------|-------|
| Regression     | 2  | 21.601 | 10.800 | 32.88 | 0.000 |
| Residual Error | 7  | 2.299  | 0.328  |       |       |
| Total          | 9  | 23.900 |        |       |       |

**TABLE 15.3** ANOVA table for a multiple regression model with  $p$  independent variables

| Source     | Degrees of freedom | Sum of squares | Mean square                                 | F                                   |
|------------|--------------------|----------------|---|-------------------------------------|
| Regression | $p$                | SSR            | $\text{MSR} = \frac{\text{SSR}}{p}$         | $F = \frac{\text{MSR}}{\text{MSE}}$ |
| Error      | $n - p - 1$        | SSE            | $\text{MSE} = \frac{\text{SSE}}{n - p - 1}$ |                                     |
| Total      | $n - 1$            | SST            |   |                                     |

The value of the  $F$  test statistic appears in the last column and can be compared to  $F_\alpha$  with  $p$  degrees of freedom in the numerator and  $n - p - 1$  degrees of freedom in the denominator to make the hypothesis test conclusion.

By reviewing the MINITAB output for Eurodistributor Company in Figure 15.6, we see that MINITAB's analysis of variance table contains this information. In addition, MINITAB provides the  $p$ -value corresponding to the  $F$  test statistic.

**t test**

If the  $F$  test shows that the multiple regression relationship is significant, a  $t$  test can be conducted to determine the significance of each of the individual parameters. The  $t$  test for individual significance follows.

**t test for individual significance**

For any parameter  $\beta_i$

$$\begin{aligned} H_0: \beta_i &= 0 \\ H_1: \beta_i &\neq 0 \end{aligned}$$

**Test statistic**

$$t = \frac{b_i}{s_{b_i}} \tag{15.15}$$

**Rejection rule**

*p*-value approach: Reject  $H_0$  if  $p$ -value  $\leq \alpha$

Critical value approach: Reject  $H_0$  if  $t \leq -t_{\alpha/2}$  or if  $t \geq t_{\alpha/2}$

where  $t_{\alpha/2}$  is based on a  $t$  distribution with  $n - p - 1$  degrees of freedom.

In the test statistic,  $s_{b_i}$  is the estimate of the standard deviation of  $b_i$ . The value of  $s_{b_i}$  will be provided by the computer software package.

Let us conduct the  $t$  test for the Eurodistributor regression problem. Refer to the section of Figure 15.6 that shows the MINITAB output for the  $t$ -ratio calculations. Values of  $b_1$ ,  $b_2$ ,  $s_{b_2}$  and  $s_{b_2}$  are as follows.

$$\begin{array}{ll} b_1 = 0.061135 & s_{b_1} = 0.009888 \\ b_2 = 0.9234 & s_{b_2} = 0.2211 \end{array}$$

Using equation (15.15), we obtain the test statistic for the hypotheses involving parameters  $\beta_1$  and  $\beta_2$ .

$$\begin{aligned} t &= 0.061135/0.009888 = 6.18 \\ t &= 0.9234/0.2211 = 4.18 \end{aligned}$$

Note that both of these  $t$ -ratio values and the corresponding  $p$ -values are provided by the MINITAB output in Figure 15.6. Using  $\alpha = 0.01$ , the  $p$ -values of 0.000 and 0.004 from the MINITAB output indicate that we can reject  $H_0: \beta_1 = 0$  and  $H_0: \beta_2 = 0$ . Hence, both parameters are statistically significant. Alternatively, Table 2 of Appendix B shows that with  $n - p - 1 = 10 - 2 - 1 = 7$  degrees of freedom,  $t_{0.005} = 3.499$ . With  $6.18 > 3.499$ , we reject  $H_0: \beta_1 = 0$ . Similarly, with  $4.18 > 3.499$ , we reject  $H_0: \beta_2 = 0$ .

## Multicollinearity

In multiple regression analysis, **multicollinearity** refers to the correlation among the independent variables. We used the term independent variable in regression analysis to refer to any variable being used to predict or explain the value of the dependent variable. The term does not mean, however, that the independent variables themselves are independent in any statistical sense. On the contrary, most independent variables in a multiple regression problem are correlated to some degree with one another. For example, in the Eurodistributor example involving the two independent variables  $X_1$  (distance) and  $X_2$  (number of deliveries), we could treat the distance as the dependent variable and the number of deliveries as the independent variable to determine whether those two variables are themselves related. We could then compute the sample correlation coefficient to determine the extent to which the variables are related. Doing so yields:

Pearson correlation of Distance and Deliveries = 0.162

which suggests only a small degree of linear association exists between the two variables. The implication from this would be that multicollinearity is not a problem for the data. If however the association had been more pronounced the resultant multicollinearity might seriously have jeopardized the estimation of the model.

To provide a better perspective of the potential problems of multicollinearity, let us consider a modification of the Eurodistributor example. Instead of  $X_2$  being the number of deliveries, let  $X_2$  denote the number of litres of petrol consumed. Clearly,  $X_1$  (the distance) and  $X_2$  are related; that is, we know that the number of litres of petrol used depends on the distance travelled. Hence, we would conclude logically that  $X_1$  and  $X_2$  are highly correlated independent variables.

Assume that we obtain the equation  $\hat{y} = b_0 + b_1x_1 + b_2x_2$  and find that the  $F$  test shows the relationship to be significant. Then suppose we conduct a  $t$  test on  $\beta_1$  to determine whether  $\beta_1 = 0$ , and we cannot reject  $H_0: \beta_1 = 0$ . Does this result mean that travel time is not related to distance? Not necessarily. What it probably means is that with  $X_2$  already in the model,  $X_1$  does not make a significant

contribution to determining the value of  $Y$ . This interpretation makes sense in our example; if we know the amount of petrol consumed, we do not gain much additional information useful in predicting  $Y$  by knowing the distance. Similarly, a  $t$  test might lead us to conclude  $\beta_2 = 0$  on the grounds that, with  $X_1$  in the model, knowledge of the amount of petrol consumed does not add much.

One useful way of detecting multicollinearity is to calculate the **variance inflation factor** (VIF) for each independent variable ( $X_j$ ) in the model. The VIF is defined as:

### Variance inflation factor

$$\text{VIF}(X_j) = \frac{1}{1 - R_j^2} \quad (15.16)$$

where  $R_j^2$  is the coefficient of determination obtained when  $X_j$  ( $j = 1, 2, \dots, p$ ) is regressed on all remaining independent variables in the model. If  $X_j$  is not correlated with other predictors  $R_j^2 = 0$  and  $\text{VIF} \approx 1$ . Correspondingly, if  $R_j^2$  is close to 1 the VIF will be very large. Typically VIF values of ten or more are regarded as problematic.

For the Eurodistributor data, the VIF for  $X_1$  (and also  $X_2$  by symmetry) would be:

$$\text{VIF}(X_j) = \frac{1}{1 - 0.162^2} = 1.027$$

signifying, as before, there is no problem with multicollinearity.

To summarize, for  $t$  tests associated with testing for the significance of individual parameters, the difficulty caused by multicollinearity is that it is possible to conclude that none of the individual parameters are significantly different from zero when an  $F$  test on the overall multiple regression equation indicates there is a significant relationship. This problem is avoided, however, when little correlation among the independent variables exists.

If possible, every attempt should be made to avoid including independent variables that are highly correlated. In practice, however, strict adherence to this policy is not always possible. When decision-makers have reason to believe substantial multicollinearity is present, they must realize that separating the effects of the individual independent variables on the dependent variable is difficult.

## EXERCISES

### Methods

- 12.** In Exercise 1, the following estimated regression equation based on ten observations was presented.

$$\hat{y} = 29.1270 + 0.5906x_1 + 0.4980x_2$$

Here  $SST = 6724.125$ ,  $SSR = 6216.375$ ,  $s_{b_1} = 0.0813$  and  $s_{b_2} = 0.0567$

- Compute MSR and MSE.
  - Compute  $F$  and perform the appropriate  $F$  test. Use  $\alpha = 0.05$ .
  - Perform a  $t$  test for the significance of  $\beta_1$ . Use  $\alpha = 0.05$ .
  - Perform a  $t$  test for the significance of  $\beta_2$ . Use  $\alpha = 0.05$ .
- 13.** Refer to the data presented in Exercise 2. The estimated regression equation for these data is

$$\hat{y} = -18.4 + 2.01x_1 + 4.74x_2$$



Here  $SST = 15\ 182.9$ ,  $SSR = 14\ 052.2$ ,  $s_{b_1} = 0.2471$  and  $s_{b_2} = 0.9484$

- Test for a significant relationship among  $X_1$ ,  $X_2$  and  $Y$ . Use  $\alpha = 0.05$ .
  - Is  $\beta_1$  significant? Use  $\alpha = 0.05$ .
  - Is  $\beta_2$  significant? Use  $\alpha = 0.05$ .
- 14.** The following estimated regression equation was developed for a model involving two independent variables.

$$\hat{y} = 40.7 + 8.63x_1 + 2.71x_2$$

After  $X_2$  was dropped from the model, the least squares method was used to obtain an estimated regression equation involving only  $X_1$  as an independent variable.

$$\hat{y} = 42.0 + 9.01x_1$$

- Give an interpretation of the coefficient of  $X_1$  in both models.
- Could multicollinearity explain why the coefficient of  $X_1$  differs in the two models? If so, how?



COMPLETE  
SOLUTIONS

### Applications

- 15.** In Exercise 4, the following estimated regression equation relating LOSS ( $Y$ ) to AIRFLOW ( $X_1$ ) and TEMP ( $X_2$ ) was given.

$$\hat{y} = -50.359 + 0.671x_1 + 1.295x_2$$

For these data  $SST = 2069.238$  and  $SSR = 1880.443$ .

Compute SSE, MSE and MSR.

- Use an  $F$  test and a 0.05 level of significance to determine whether there is a relationship among the variables.
- 16.** Refer to Exercise 5.

- Use  $\alpha = 0.01$  to test the hypotheses

$$\begin{aligned} H_0: \beta_1 &= \beta_2 = 0 \\ H_1: \beta_1 \text{ and/or } \beta_2 &\text{ is not equal to zero} \end{aligned}$$

for the model  $Y = \beta_0 + \beta_1x_1 + \beta_2x_2 + \varepsilon$ , where:

$$\begin{aligned} X_1 &= \text{television advertising (€1000s)} \\ X_2 &= \text{newspaper advertising (€1000s)} \end{aligned}$$

- Use  $\alpha = 0.05$  to test the significance of  $\beta_1$ . Should  $X_1$  be dropped from the model?
- Use  $\alpha = 0.05$  to test the significance of  $\beta_2$ . Should  $X_2$  be dropped from the model?

## 15.6 USING THE ESTIMATED REGRESSION EQUATION FOR ESTIMATION AND PREDICTION

The procedures for estimating the mean value of  $Y$  and predicting an individual value of  $Y$  in multiple regression are similar to those in regression analysis involving one independent variable. First, recall that in Chapter 14 we showed that the point estimate of the expected value of  $Y$  for a given value of  $X$  was the same as the point estimate of an individual value of  $Y$ . In both cases, we used  $\hat{y} = b_0 + b_1x$  as the point estimate.

**TABLE 15.4** The 95 per cent confidence and prediction intervals for Eurodistributor

| Value of $X_1$ | Value of $X_2$ | Confidence interval |             | Prediction interval |             |
|----------------|----------------|---------------------|-------------|---------------------|-------------|
|                |                | Lower limit         | Upper limit | Lower limit         | Upper limit |
| 50             | 2              | 3.146               | 4.924       | 2.414               | 5.656       |
| 50             | 3              | 4.127               | 5.789       | 3.368               | 6.548       |
| 50             | 4              | 4.815               | 6.948       | 4.157               | 7.607       |
| 100            | 2              | 6.258               | 7.926       | 5.500               | 8.683       |
| 100            | 3              | 7.385               | 8.645       | 6.520               | 9.510       |
| 100            | 4              | 8.135               | 9.742       | 7.362               | 10.515      |

In multiple regression we use the same procedure. That is, we substitute the given values of  $X_1, X_2, \dots, X_p$  into the estimated regression equation and use the corresponding value of  $\hat{y}$  as the point estimate. Suppose that for the Eurodistributor example we want to use the estimated regression equation involving  $X_1$  (distance) and  $X_2$  (number of deliveries) to develop two interval estimates:

- 1 A *confidence interval* of the mean travel time for all trucks that travel 100 kilometres and make two deliveries.
- 2 A *prediction interval* of the travel time for *one specific* truck that travels 100 kilometres and makes two deliveries.

Using the estimated regression equation  $\hat{y} = -0.869 + 0.0611x_1 + 0.923x_2$  with  $X_1 = 100$  and  $X_2 = -2$ , we obtain the following value of  $\hat{y}$ .

$$\hat{y} = -0.869 + 0.0611(100) + 0.923(2) = 7.09$$

Hence, the point estimate of travel time in both cases is approximately seven hours.

To develop interval estimates for the mean value of  $Y$  and for an individual value of  $Y$ , we use a procedure similar to that for regression analysis involving one independent variable.

The formulae required are beyond the scope of the text, but computer packages for multiple regression analysis will often provide confidence intervals once the values of  $X_1, X_2, \dots, X_p$  are specified by the user. In Table 15.4 we show the 95 per cent confidence and prediction intervals for the Eurodistributor example for selected values of  $X_1$  and  $X_2$ ; these values were obtained using MINITAB. Note that the interval estimate for an individual value of  $Y$  is wider than the interval estimate for the expected value of  $Y$ . This difference simply reflects the fact that for given values of  $X_1$  and  $X_2$  we can estimate the mean travel time for all trucks with more precision than we can predict the travel time for one specific truck.

## EXERCISES

### Methods

- 17.** In Exercise 1, the following estimated regression equation based on ten observations was presented.

$$\hat{y} = 29.1270 + 0.5906x_1 + 0.4980x_2$$

- a. Develop a point estimate of the mean value of  $Y$  when  $X_1 = 180$  and  $X_2 = 310$ .
- b. Develop a point estimate for an individual value of  $Y$  when  $X_1 = 180$  and  $X_2 = 310$ .

- 18.** Refer to the data in Exercise 2. The estimated regression equation for those data is

$$\hat{y} = -18.4 + 2.01x_1 + 4.74x_2$$

- a. Develop a 95 per cent confidence interval for the mean value of  $Y$  when  $X_1 = 45$  and  $X_2 = 15$ .
- b. Develop a 95 per cent prediction interval for  $Y$  when  $X_1 = 45$  and  $X_2 = 15$ .



EXER2

### Applications

- 19.** In Exercise 5, the owner of Toulon Theatres used multiple regression analysis to predict gross revenue ( $Y$ ) as a function of television advertising ( $X_1$ ) and newspaper advertising ( $X_2$ ). The estimated regression equation was

$$\hat{y} = 83.2 + 2.29x_1 + 1.30x_2$$

- a. What is the gross revenue expected for a week when €3500 is spent on television advertising ( $X_1 = 3.5$ ) and €1800 is spent on newspaper advertising ( $X_2 = 1.8$ )?
- b. Provide a 95 per cent confidence interval for the mean revenue of all weeks with the expenditures listed in part (a).
- c. Provide a 95 per cent prediction interval for next week's revenue, assuming that the advertising expenditures will be allocated as in part (a).



TOULON

## 15.7 QUALITATIVE INDEPENDENT VARIABLES

Thus far, the examples we considered involved quantitative independent variables such as distance travelled and number of deliveries. In many situations, however, we must work with **qualitative independent variables** such as gender (male, female), method of payment (cash, credit card, cheque) and so on. The purpose of this section is to show how qualitative variables are handled in regression analysis. To illustrate the use and interpretation of a qualitative independent variable, we will consider a problem facing the managers of Johansson Filtration.

### An example: Johansson Filtration

Johansson Filtration provides maintenance service for water-filtration systems throughout southern Denmark. Customers contact Johansson with requests for maintenance service on their water-filtration systems. To estimate the service time and the service cost, Johansson's managers wish to predict the repair time necessary for each maintenance request. Hence, repair time in hours is the dependent variable. Repair time is believed to be related to two factors: the number of months since the last maintenance service and the type of repair problem (mechanical or electrical). Data for a sample of ten service calls are reported in Table 15.5.

Let  $Y$  denote the repair time in hours and  $X_1$  denote the number of months since the last maintenance service. The regression model that uses only  $X_1$  to predict  $Y$  is:

$$Y = \beta_0 + \beta_1 x_1 + \varepsilon$$

Using MINITAB to develop the estimated regression equation, we obtained the output shown in Figure 15.7. The estimated regression equation is:

$$\hat{y} = 2.15 + 0.304x_1$$

(15.17)



JOHANSSON

**TABLE 15.5** Data for the Johansson Filtration example

| Service call | Months since last service | Type of repair | Repair time in hours |
|--------------|---------------------------|----------------|----------------------|
| 1            | 2                         | electrical     | 2.9                  |
| 2            | 6                         | mechanical     | 3.0                  |
| 3            | 8                         | electrical     | 4.8                  |
| 4            | 3                         | mechanical     | 1.8                  |
| 5            | 2                         | electrical     | 2.9                  |
| 6            | 7                         | electrical     | 4.9                  |
| 7            | 9                         | mechanical     | 4.2                  |
| 8            | 8                         | mechanical     | 4.8                  |
| 9            | 4                         | electrical     | 4.4                  |
| 10           | 6                         | electrical     | 4.5                  |

**FIGURE 15.7**

MINITAB output for Johansson Filtration with months since last service ( $X_1$ ) as the independent variable

### Regression Analysis: Time versus Months

The regression equation is  
Time = 2.15 + 0.304 Months

| Predictor | Coef   | SE Coef | T    | P     |
|-----------|--------|---------|------|-------|
| Constant  | 2.1473 | 0.6050  | 3.55 | 0.008 |
| Months    | 0.3041 | 0.1004  | 3.03 | 0.016 |

$$S = 0.781022 \quad R-Sq = 53.4\% \quad R-Sq(\text{adj}) = 47.6\%$$

#### Analysis of Variance

| Source         | DF | SS      | MS     | F    | P     |
|----------------|----|---------|--------|------|-------|
| Regression     | 1  | 5.5960  | 5.5960 | 9.17 | 0.016 |
| Residual Error | 8  | 4.8800  | 0.6100 |      |       |
| Total          | 9  | 10.4760 |        |      |       |

At the 0.05 level of significance, the  $p$ -value of 0.016 for the  $t$  (or  $F$ ) test indicates that the number of months since the last service is significantly related to repair time.  $R$ -sq = 53.4 per cent indicates that  $X_1$  alone explains 53.4 per cent of the variability in repair time.

To incorporate the type of failure into the regression model, we define the following variable.

$$\begin{aligned} X_2 &= 0 \text{ if the type of repair is mechanical} \\ X_2 &= 1 \text{ if the type of repair is electrical} \end{aligned}$$

In regression analysis  $X_2$  is called a **dummy variable** or *indicator variable*. Using this dummy variable, we can write the multiple regression model as:

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

Table 15.6 is the revised data set that includes the values of the dummy variable. Using MINITAB and the data in Table 15.6, we can develop estimates of the model parameters. The MINITAB output in Figure 15.8 shows that the estimated multiple regression equation is:

$$\hat{y} = 0.93 + 0.388x_1 + 1.26x_2 \quad (15.18)$$

**TABLE 15.6** Data for the Johansson Filtration example with type of repair indicated by a dummy variable ( $X_2 = 0$  for mechanical;  $X_2 = 1$  for electrical)

| Customer | Months since last service ( $X_1$ ) | Type of repair ( $X_2$ ) | Repair time in hours ( $Y$ ) |
|----------|-------------------------------------|--------------------------|------------------------------|
| 1        | 2                                   | 1                        | 2.9                          |
| 2        | 6                                   | 0                        | 3.0                          |
| 3        | 8                                   | 1                        | 4.8                          |
| 4        | 3                                   | 0                        | 1.8                          |
| 5        | 2                                   | 1                        | 2.9                          |
| 6        | 7                                   | 1                        | 4.9                          |
| 7        | 9                                   | 0                        | 4.2                          |
| 8        | 8                                   | 0                        | 4.8                          |
| 9        | 4                                   | 1                        | 4.4                          |
| 10       | 6                                   | 1                        | 4.5                          |

At the 0.05 level of significance, the  $p$ -value of 0.001 associated with the  $F$  test ( $F = 21.36$ ) indicates that the regression relationship is significant. The  $t$  test part of the printout in Figure 15.8 shows that both months since last service ( $p$ -value = 0.000) and type of repair ( $p$ -value = 0.005) are statistically significant. In addition,  $R$ -sq = 85.9 per cent and  $R$ -sq(adj) = 81.9 per cent indicate that the estimated regression equation does a good job of explaining the variability in repair times. Thus, equation (15.18) should prove helpful in estimating the repair time necessary for the various service calls.

## Interpreting the parameters

The multiple regression equation for the Johansson Filtration example is:

$$E(Y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 \quad (15.19)$$

**FIGURE 15.8**

MINITAB output for Johansson Filtration with months since last service ( $X_1$ ) and type of repair ( $X_2$ ) as the independent variables

### Regression Analysis: Time versus Months, Type

The regression equation is  
Time = 0.930 + 0.388 Months + 1.26 Type

| Predictor | Coef    | SE Coef | T    | P     |
|-----------|---------|---------|------|-------|
| Constant  | 0.9305  | 0.4670  | 1.99 | 0.087 |
| Months    | 0.38762 | 0.06257 | 6.20 | 0.000 |
| Type      | 1.2627  | 0.3141  | 4.02 | 0.005 |

S = 0.459048 R-Sq = 85.9% R-Sq(adj) = 81.9%

### Analysis of Variance

| Source         | DF | SS      | MS     | F     | P     |
|----------------|----|---------|--------|-------|-------|
| Regression     | 2  | 9.0009  | 4.5005 | 21.36 | 0.001 |
| Residual Error | 7  | 1.4751  | 0.2107 |       |       |
| Total          | 9  | 10.4760 |        |       |       |

| Source | DF | Seq SS |
|--------|----|--------|
| Months | 1  | 5.5960 |
| Type   | 1  | 3.4049 |

To understand how to interpret the parameters  $\beta_0$ ,  $\beta_1$  and  $\beta_2$  when a qualitative variable is present, consider the case when  $X_2 = 0$  (mechanical repair). Using  $E(Y | \text{mechanical})$  to denote the mean or expected value of repair time given a mechanical repair, we have:

$$E(Y | \text{mechanical}) = \beta_0 + \beta_1 x_1 + \beta_2(0) = \beta_0 + \beta_1 x_1 \quad (15.20)$$

Similarly, for an electrical repair ( $X_2 = 1$ ), we have:

$$\begin{aligned} E(Y | \text{electrical}) &= \beta_0 + \beta_1 x_1 + \beta_2(1) = \beta_0 + \beta_1 x_1 + \beta_2 \\ &= (\beta_0 + \beta_2) + \beta_1 x_1 \end{aligned} \quad (15.21)$$

Comparing equations (15.20) and (15.21), we see that the mean repair time is a linear function of  $X_1$  for both mechanical and electrical repairs. The slope of both equations is  $\beta_1$ , but the  $y$ -intercept differs. The  $y$ -intercept is  $\beta_0$  in equation (15.20) for mechanical repairs and  $(\beta_0 + \beta_2)$  in equation (15.21) for electrical repairs. The interpretation of  $\beta_2$  is that it indicates the difference between the mean repair time for an electrical repair and the mean repair time for a mechanical repair.

If  $\beta_2$  is positive, the mean repair time for an electrical repair will be greater than that for a mechanical repair; if  $\beta_2$  is negative, the mean repair time for an electrical repair will be less than that for a mechanical repair. Finally, if  $\beta_2 = 0$ , there is no difference in the mean repair time between electrical and mechanical repairs and the type of repair is not related to the repair time.

Using the estimated multiple regression equation  $\hat{y} = 0.93 + 0.388x_1 + 1.26x_2$ , we see that 0.93 is the estimate of  $\beta_0$  and 1.26 is the estimate of  $\beta_2$ . Thus, when  $X_2 = 0$  (mechanical repair):

$$\hat{y} = 0.93 + 0.388x_1 \quad (15.22)$$

and when  $X_2 = 1$  (electrical repair):

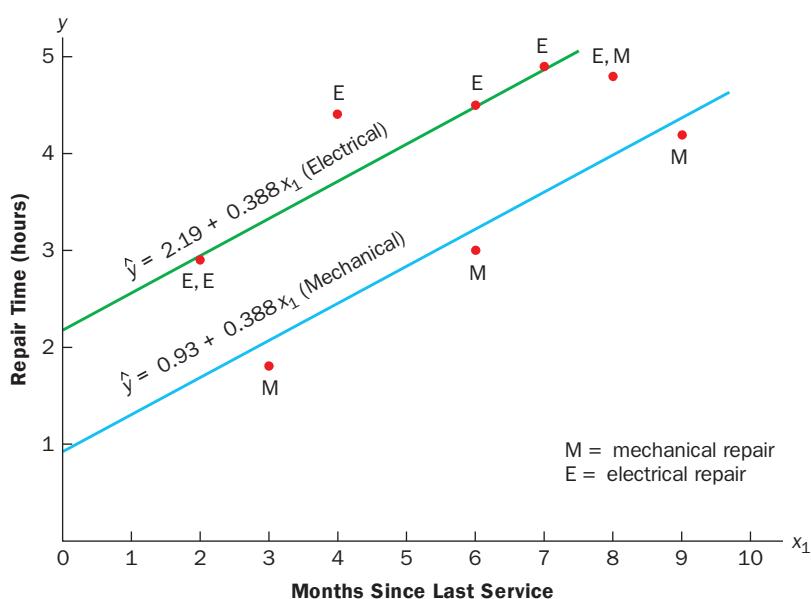
$$\begin{aligned} \hat{y} &= 0.93 + 0.388x_1 + 1.26(1) \\ &= 2.19 + 0.388x_1 \end{aligned} \quad (15.23)$$

In effect, the use of a dummy variable for type of repair provides two equations that can be used to predict the repair time, one corresponding to mechanical repairs and one corresponding to electrical repairs. In addition, with  $b_2 = 1.26$ , we learn that, on average, electrical repairs require 1.26 hours longer than mechanical repairs.

Figure 15.9 is the plot of the Johansson data from Table 15.6. Repair time in hours ( $Y$ ) is represented by the vertical axis and months since last service ( $X_1$ ) is represented by the horizontal axis. A data point for a mechanical repair is indicated by an M and a data point for an electrical repair is indicated by an E.

**FIGURE 15.9**

Scatter diagram for the Johansson Filtration repair data from Table 15.6



Equations (15.22) and (15.23) are plotted on the graph to show graphically the two equations that can be used to predict the repair time, one corresponding to mechanical repairs and one corresponding to electrical repairs.

## More complex qualitative variables

Because the qualitative variable for the Johansson Filtration example had two levels (mechanical and electrical), defining a dummy variable with zero indicating a mechanical repair and one indicating an electrical repair was easy. However, when a qualitative variable has more than two levels, care must be taken in both defining and interpreting the dummy variables. As we will show, if a qualitative variable has  $k$  levels,  $k - 1$  dummy variables are required, with each dummy variable being coded as 0 or 1.

For example, suppose a manufacturer of copy machines organized the sales territories for a particular area into three regions: A, B and C. The managers want to use regression analysis to help predict the number of copiers sold per week. With the number of units sold as the dependent variable, they are considering several independent variables (the number of sales personnel, advertising expenditures and so on). Suppose the managers believe sales region is also an important factor in predicting the number of copiers sold. Because sales region is a qualitative variable with three levels, A, B and C, we will need  $3 - 1 = 2$  dummy variables to represent the sales region. Each variable can be coded 0 or 1 as follows.

$$X_1 = \begin{cases} 1 & \text{if sales region B} \\ 0 & \text{otherwise} \end{cases}$$

$$X_2 = \begin{cases} 1 & \text{if sales region C} \\ 0 & \text{otherwise} \end{cases}$$

With this definition, we have the following values of  $X_1$  and  $X_2$ .

| Region | $X_1$ | $X_2$ |
|--------|-------|-------|
| A      | 0     | 0     |
| B      | 1     | 0     |
| C      | 0     | 1     |

Observations corresponding to region A would be coded  $X_1 = 0, X_2 = 0$ ; observations corresponding to region B would be coded  $X_1 = 1, X_2 = 0$ ; and observations corresponding to region C would be coded  $X_1 = 0, X_2 = 1$ .

The regression equation relating the expected value of the number of units sold,  $E(Y)$ , to the dummy variables would be written as:

$$E(Y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

To help us interpret the parameters  $\beta_0, \beta_1$  and  $\beta_2$ , consider the following three variations of the regression equation.

$$\begin{aligned} E(Y \mid \text{region A}) &= \beta_0 + \beta_1(0) + \beta_2(0) = \beta_0 \\ E(Y \mid \text{region B}) &= \beta_0 + \beta_1(1) + \beta_2(0) = \beta_0 + \beta_1 \\ E(Y \mid \text{region C}) &= \beta_0 + \beta_1(0) + \beta_2(1) = \beta_0 + \beta_2 \end{aligned}$$

Therefore,  $\beta_0$  is the mean or expected value of sales for region A;  $\beta_1$  is the difference between the mean number of units sold in region B and the mean number of units sold in region A; and  $\beta_2$  is the difference between the mean number of units sold in region C and the mean number of units sold in region A.

Two dummy variables were required because sales region is a qualitative variable with three levels. But the assignment of  $X_1 = 0, X_2 = 0$  to indicate region A,  $X_1 = 1, X_2 = 0$  to indicate region B, and  $X_1 = 0, X_2 = 1$  to indicate region C was arbitrary. For example, we could have chosen  $X_1 = 1, X_2 = 0$  to indicate region A,  $X_1 = 0, X_2 = 0$  to indicate region B, and  $X_1 = 0, X_2 = 1$  to indicate region C. In that case,  $\beta_1$  would have been interpreted as the mean difference between regions A and B and  $\beta_2$  as the mean difference between regions C and B.



**COMPLETE  
SOLUTIONS**

## EXERCISES

### Methods

- 20.** Consider a regression study involving a dependent variable  $Y$ , a quantitative independent variable  $X_1$  and a qualitative variable with two levels (level 1 and level 2).
- Write a multiple regression equation relating  $X_1$  and the qualitative variable to  $Y$ .
  - What is the expected value of  $Y$  corresponding to level 1 of the qualitative variable?
  - What is the expected value of  $Y$  corresponding to level 2 of the qualitative variable?
  - Interpret the parameters in your regression equation.
- 21.** Consider a regression study involving a dependent variable  $Y$ , a quantitative independent variable  $X_1$ , and a qualitative independent variable with three possible levels (level 1, level 2 and level 3).
- How many dummy variables are required to represent the qualitative variable?
  - Write a multiple regression equation relating  $X_1$  and the qualitative variable to  $Y$ .
  - Interpret the parameters in your regression equation.

### Applications

- 22.** Management proposed the following regression model to predict the effect of physical exercise on pulse in an experiment involving 92 participants:

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon,$$

where:

$Y$  = Pulse 2 = second pulse reading taken at end of experiment

$x_1$  = Pulse 1 = initial (resting) pulse reading

$x_2$  = Ran = 1 if individual ran on the spot for one minute, 2 if they did not  
(this was decided randomly)

$x_3$  = Sex = 1 if male, 2 if female

The following estimated regression equation was developed using MINITAB:

$$\hat{y} = 42.62 + 0.812x_1 - 20.1x_2 + 7.8x_3$$

- What is the amount of the expected value of Pulse 2 attributable to  $x_3$ ?
  - Predict Pulse 2 for a female who ran on the spot for one minute and had an initial pulse reading of 70 bpm.
  - Predict Pulse 2 for a male who did not run on the spot for one minute and had an initial pulse reading of 60 bpm.
- 23.** Refer to the Johansson Filtration problem introduced in this section. Suppose that in addition to information on the number of months since the machine was serviced and whether a mechanical or an electrical failure had occurred, the managers obtained a list showing which engineer performed the service. The revised data follow.

| Repair time in hours | Months since last service | Type of repair | Engineer      |
|----------------------|---------------------------|----------------|---------------|
| 2.9                  | 2                         | Electrical     | Heinz Kolb    |
| 3.0                  | 6                         | Mechanical     | Heinz Kolb    |
| 4.8                  | 8                         | Electrical     | Wolfgang Linz |
| 1.8                  | 3                         | Mechanical     | Heinz Kolb    |
| 2.9                  | 2                         | Electrical     | Heinz Kolb    |
| 4.9                  | 7                         | Electrical     | Wolfgang Linz |
| 4.2                  | 9                         | Mechanical     | Wolfgang Linz |



REPAIR

| <i>Repair time in hours</i> | <i>Months since last service</i> | <i>Type of repair</i> | <i>Engineer</i> |
|-----------------------------|----------------------------------|-----------------------|-----------------|
| 4.8                         | 8                                | Mechanical            | Wolfgang Linz   |
| 4.4                         | 4                                | Electrical            | Wolfgang Linz   |
| 4.5                         | 6                                | Electrical            | Heinz Kolb      |

- a. Ignore for now the months since the last maintenance service ( $X_1$ ) and the engineer who performed the service. Develop the estimated simple linear regression equation to predict the repair time ( $Y$ ) given the type of repair ( $X_2$ ). Recall that  $X_2 = 0$  if the type of repair is mechanical and 1 if the type of repair is electrical.
- b. Does the equation that you developed in part (a) provide a good fit for the observed data? Explain.
- c. Ignore for now the months since the last maintenance service and the type of repair associated with the machine. Develop the estimated simple linear regression equation to predict the repair time given the engineer who performed the service. Let  $X_3 = 0$  if Heinz Kolb performed the service and  $X_3 = 1$  if Wolfgang Linz performed the service.
- d. Does the equation that you developed in part (c) provide a good fit for the observed data? Explain.
24. In a multiple regression analysis by McIntyre (1994), Tar, Nicotine and Weight are considered as possible predictors of carbon monoxide (CO) content for 25 different brands of cigarette. Details of variables and data follow.

|          |  |
|----------|--|
| Brand    | The cigarette brand                      |
| Tar      | The tar content (in mg)                  |
| Nicotine | The nicotine content (in mg)             |
| Weight   | The weight (in g)                        |
| CO       | The carbon monoxide (CO) content (in mg) |

| <i>Brand</i>       | <i>Tar</i> | <i>Nicotine</i> | <i>Weight</i> | <i>CO</i> |
|--------------------|------------|-----------------|---------------|-----------|
| Alpine             | 14.1       | 0.86            | .9853         | 13.6      |
| Benson & Hedges    | 16.0       | 1.06            | 1.0938        | 16.6      |
| Bull Durham        | 29.8       | 2.03            | 1.1650        | 23.5      |
| Camel Lights       | 8.0        | 0.67            | 0.9280        | 10.2      |
| Carlton            | 4.1        | 0.40            | 0.9462        | 5.4       |
| Chesterfield       | 15.0       | 1.04            | 0.8885        | 15.0      |
| Golden Lights      | 8.8        | 0.76            | 1.0267        | 9.0       |
| Kent               | 12.4       | 0.95            | 0.9225        | 12.3      |
| Kool               | 16.6       | 1.12            | 0.9372        | 16.3      |
| L&M                | 14.9       | 1.02            | 0.8858        | 15.4      |
| Lark Lights        | 13.7       | 1.01            | 0.9643        | 13.0      |
| Marlboro           | 15.1       | 0.90            | 0.9316        | 14.4      |
| Merit              | 7.8        | 0.57            | 0.9705        | 10.0      |
| Multi Filter       | 11.4       | 0.78            | 1.1240        | 10.2      |
| Newport Lights     | 9.0        | 0.74            | 0.8517        | 9.5       |
| Now                | 1.0        | 0.13            | 0.7851        | 1.5       |
| Old Gold           | 17.0       | 1.26            | 0.9186        | 18.5      |
| Pall Mall Light    | 12.8       | 1.08            | 1.0395        | 12.6      |
| Raleigh            | 15.8       | 0.96            | 0.9573        | 17.5      |
| Salem Ultra        | 4.5        | 0.42            | 0.9106        | 4.9       |
| Tareyton           | 14.5       | 1.01            | 1.0070        | 15.9      |
| True               | 7.3        | 0.61            | 0.9806        | 8.5       |
| Viceroy Rich Light | 8.6        | 0.69            | 0.9693        | 10.6      |
| Virginia Slims     | 15.2       | 1.02            | 0.9496        | 13.9      |
| Winston Lights     | 12.0       | 0.82            | 1.1184        | 14.9      |



CIGARETTES



BODYFAT

- a. Examine correlations between variables in the study and hence assess the possibility of problems of multicollinearity affecting any subsequent regression model involving independent variables Tar and Nicotine.
- b. Thus develop an estimated multiple regression equation using an appropriate number of the independent variables featured in the study.
- c. Are your predictors statistically significant? Use  $\alpha = 0.05$ . What explanation can you give for the results observed?

- 25.** The data below (Dunn, 2007) come from a study investigating a new method of measuring body composition. Body fat percentage, age and gender is given for 18 adults aged between 23 and 61.

| Age | Percent.Fat | Gender |
|-----|-------------|--------|
| 23  | 9.5         | M      |
| 23  | 27.9        | F      |
| 27  | 7.8         | M      |
| 27  | 17.8        | M      |
| 39  | 31.4        | F      |
| 41  | 25.9        | F      |
| 45  | 27.4        | M      |
| 49  | 25.2        | F      |
| 50  | 31.1        | F      |
| 53  | 34.7        | F      |
| 53  | 42          | F      |
| 54  | 29.1        | F      |
| 56  | 32.5        | F      |
| 57  | 30.3        | F      |
| 58  | 33          | F      |
| 58  | 33.8        | F      |
| 60  | 41.1        | F      |
| 61  | 34.5        | F      |

- a. Develop an estimated regression equation that relates Age and Gender to Percent.Fat.
- b. Is Age a significant factor in predicting Percent.Fat? Explain. Use  $\alpha = 0.05$ .
- c. What is the estimated body fat percentage for a female aged 45?

## 15.8 RESIDUAL ANALYSIS

In Chapter 14 we pointed out that standardized residuals were frequently used in residuals plots and in the identification of outliers. The general formula for the standardized residual for observation  $i$  follows.

**Standardized residual for observation  $i$**

$$\frac{y_i - \hat{y}_i}{s_{y_i - \hat{y}_i}} \quad (15.24)$$

where:

$s_{y_i - \hat{y}_i}$  = the standard deviation of residual  $i$

The general formula for the standard deviation of residual  $i$  is defined as follows.

### Standard deviation of residual $i$

$$s_{y_i - \hat{y}_i} = s\sqrt{1 - h_i} \quad (15.25)$$

where:

$s$  = standard error of the estimate

$h_i$  = leverage of observation  $i$

The **leverage** of an observation is determined by how far the values of the independent variables are from their means. The computation of  $h_i$ ,  $s_{y_i - \hat{y}_i}$  and hence the standardized residual for observation  $i$  in multiple regression analysis is too complex to be done by hand. However, the standardized residuals can be easily obtained as part of the output from statistical software packages. Table 15.7 lists the predicted values, the residuals and the standardized residuals for the Eurodistributor example presented previously in this chapter; we obtained these values by using the MINITAB statistical software package. The predicted values in the table are based on the estimated regression equation:

$$\hat{y} = -0.869 + 0.0611x_1 + 0.923x_2$$

The standardized residuals and the predicted values of  $Y$  from Table 15.7 are used in the standardized residual plot in Figure 15.10.

This standardized residual plot does not indicate any unusual abnormalities. Also, all of the standardized residuals are between  $-2$  and  $+2$ ; hence, we have no reason to question the assumption that the error term  $\epsilon$  is normally distributed. We conclude that the model assumptions are reasonable.

A normal probability plot also can be used to determine whether the distribution of  $\epsilon$  appears to be normal. The procedure and interpretation for a normal probability plot were discussed in Section 14.8. The same procedure is appropriate for multiple regression. Again, we would use a statistical software package to perform the computations and provide the normal probability plot.

## Detecting outliers

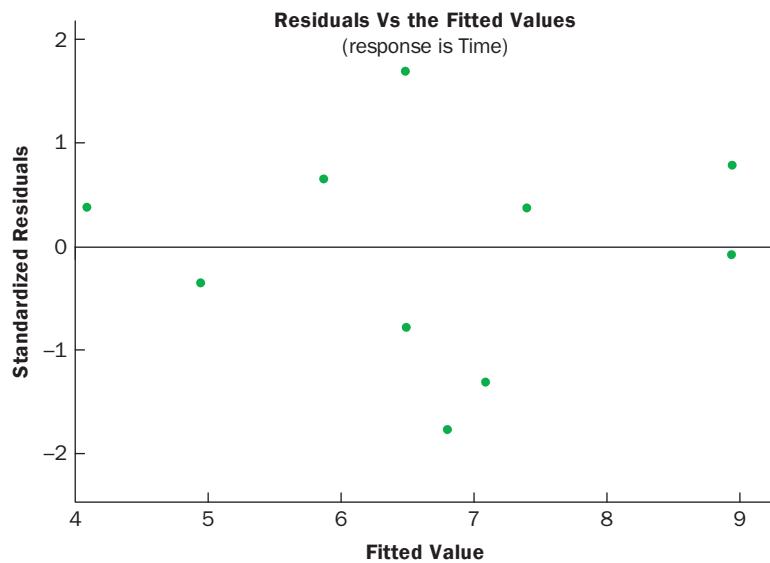
An **outlier** is an observation that is unusual in comparison with the other data; in other words, an outlier does not fit the pattern of the other data. In Chapter 14 we showed an example of an outlier and discussed how standardized residuals can be used to detect outliers.

**TABLE 15.7** Residuals and standardized residuals for the Eurodistributor regression analysis

| Distance travelled ( $X_1$ ) | Deliveries ( $X_2$ ) | Travel time ( $Y$ ) | Predicted value ( $\hat{y}$ ) | Residual ( $y - \hat{y}$ ) | Standardized residual |
|------------------------------|----------------------|---------------------|-------------------------------|----------------------------|-----------------------|
| 100                          | 4                    | 9.3                 | 8.93846                       | 0.361540                   | 0.78344               |
| 50                           | 3                    | 4.8                 | 4.95830                       | -0.158305                  | -0.34962              |
| 100                          | 4                    | 8.9                 | 8.93846                       | -0.038460                  | -0.08334              |
| 100                          | 2                    | 6.5                 | 7.09161                       | -0.591609                  | -1.30929              |
| 50                           | 2                    | 4.2                 | 4.03488                       | 0.165121                   | 0.38167               |
| 80                           | 2                    | 6.2                 | 5.86892                       | 0.331083                   | 0.65431               |
| 75                           | 3                    | 7.4                 | 6.48667                       | 0.913330                   | 1.68917               |
| 65                           | 4                    | 6.0                 | 6.79875                       | -0.798749                  | -1.77372              |
| 90                           | 3                    | 7.6                 | 7.40369                       | 0.196311                   | 0.36703               |
| 90                           | 2                    | 6.1                 | 6.48026                       | -0.380263                  | -0.77639              |

**FIGURE 15.10**

Standardized residual plot for the Eurodistributor multiple regression analysis



MINITAB classifies an observation as an outlier if the value of its standardized residual is less than  $-2$  or greater than  $+2$ . Applying this rule to the standardized residuals for the Eurodistributor example (see Table 15.7), we do not detect any outliers in the data set.

In general, the presence of one or more outliers in a data set tends to increase  $s$ , the standard error of the estimate, and hence increase  $s_{y_i - \hat{y}_i}$ , the standard deviation of residual  $i$ . Because  $s_{y_i - \hat{y}_i}$  appears in the denominator of the formula for the standardized residual (15.24), the size of the standardized residual will decrease as  $s$  increases.

As a result, even though a residual may be unusually large, the large denominator in expression (15.24) may cause the standardized residual rule to fail to identify the observation as being an outlier. We can circumvent this difficulty by using a form of standardized residuals called **studentized deleted residuals**.

### Studentized deleted residuals and outliers

Suppose the  $i$ th observation is deleted from the data set and a new estimated regression equation is developed with the remaining  $n - 1$  observations. Let  $s_{(i)}$  denote the standard error of the estimate based on the data set with the  $i$ th observation deleted. If we compute the standard deviation of residual  $i$  (15.25) using  $s_{(i)}$  instead of  $s$ , and then compute the standardized residual for observation  $i$  (15.24) using the revised value, the resulting standardized residual is called a studentized deleted residual.

If the  $i$ th observation is an outlier,  $s_{(i)}$  will be less than  $s$ . The absolute value of the  $i$ th studentized deleted residual therefore will be larger than the absolute value of the standardized residual. In this sense, studentized deleted residuals may detect outliers that standardized residuals do not detect. Many statistical software packages provide an option for obtaining studentized deleted residuals. Using MINITAB, we obtained the studentized deleted residuals for the Eurodistributor example; the results are reported in Table 15.8. The  $t$  distribution can be used to determine whether the studentized deleted residuals indicate the presence of outliers. Recall that  $p$  denotes the number of independent variables and  $n$  denotes the number of observations. Hence, if we delete the  $i$ th observation, the number of observations in the reduced data set is  $n - 1$ ; in this case the error sum of squares has  $(n - 1) - p - 1$  degrees of freedom. For the Eurodistributor example with  $n = 10$  and  $p = 2$ , the degrees of freedom for the error sum of squares with the  $i$ th observation deleted is  $9 - 2 - 1 = 6$ . At a 0.05 level of significance, the  $t$  distribution (Table 2 of Appendix B) shows that with six degrees of freedom,  $t_{0.025} = 2.447$ . If the value of the  $i$ th studentized deleted residual is less than  $-2.447$  or greater than  $+2.447$ , we can conclude that the  $i$ th observation is an outlier. The studentized deleted residuals in Table 15.8 do not exceed those limits; therefore, we conclude that outliers are not present in the data set.

**TABLE 15.8** Studentized deleted residuals for Eurodistributor

| Distance travelled ( $X_1$ ) | Deliveries ( $X_2$ ) | Travel time ( $Y$ ) | Standardized residual | Studentized deleted residual |
|------------------------------|----------------------|---------------------|-----------------------|------------------------------|
| 100                          | 4                    | 9.3                 | 0.78344               | 0.75938                      |
| 50                           | 3                    | 4.8                 | -0.34962              | -0.32654                     |
| 100                          | 4                    | 8.9                 | -0.08334              | -0.0772                      |
| 100                          | 2                    | 6.5                 | -1.30929              | -1.39494                     |
| 50                           | 2                    | 4.2                 | 0.38167               | 0.35709                      |
| 80                           | 2                    | 6.2                 | 0.65431               | 0.62519                      |
| 75                           | 3                    | 7.4                 | 1.68917               | 2.03187                      |
| 65                           | 4                    | 6.0                 | -1.77372              | -2.21314                     |
| 90                           | 3                    | 7.6                 | 0.36703               | 0.34312                      |
| 90                           | 2                    | 6.1                 | -0.77639              | -0.7519                      |

## Influential observations

In Chapter 14, Section 14.9 we discussed how the leverage of an observation can be used to identify observations for which the value of the independent variable may have a strong influence on the regression results. As we acknowledged, the leverage ( $h_i$ ) of an observation, measures how far the values of the independent variables are from their mean values. The leverage values are easily obtained as part of the output from statistical software packages. MINITAB computes the leverage values and uses the rule of thumb:

$$h_i > 3(p + 1)/n$$

to identify **influential observations**. For the Eurodistributor example with  $p = 2$  independent variables and  $n = 10$  observations, the critical value for leverage is  $3(2 + 1)/10 = 0.9$ . The leverage values for the Eurodistributor example obtained by using MINITAB are reported in Table 15.9. As  $h_i$  does not exceed 0.9, no influential observations in the data set are detected.

## Using Cook's distance measure to identify influential observations

A problem that can arise in using leverage to identify influential observations is that an observation can be identified as having high leverage and not necessarily be influential in terms of the resulting estimated regression equation.

**TABLE 15.9** Leverage and Cook's distance measures for Eurodistributor

| Distance travelled ( $X_1$ ) | Deliveries ( $X_2$ ) | Travel time ( $Y$ ) | Leverage ( $h_i$ ) | Cook's D ( $D_i$ ) |
|------------------------------|----------------------|---------------------|--------------------|--------------------|
| 100                          | 4                    | 9.3                 | 0.351704           | 0.110994           |
| 50                           | 3                    | 4.8                 | 0.375863           | 0.024536           |
| 100                          | 4                    | 8.9                 | 0.351704           | 0.001256           |
| 100                          | 2                    | 6.5                 | 0.378451           | 0.347923           |
| 50                           | 2                    | 4.2                 | 0.430220           | 0.036663           |
| 80                           | 2                    | 6.2                 | 0.220557           | 0.040381           |
| 75                           | 3                    | 7.4                 | 0.110009           | 0.117561           |
| 65                           | 4                    | 6.0                 | 0.382657           | 0.650029           |
| 90                           | 3                    | 7.6                 | 0.129098           | 0.006656           |
| 90                           | 2                    | 6.1                 | 0.269737           | 0.074217           |

**TABLE 15.10** Data set illustrating potential problem using the leverage criterion

| $x_i$ | $y_i$ | Leverage $h_i$ |
|-------|-------|----------------|
| 1     | 18    | 0.204170       |
| 1     | 21    | 0.204170       |
| 2     | 22    | 0.164205       |
| 3     | 21    | 0.138141       |
| 4     | 23    | 0.125977       |
| 4     | 24    | 0.125977       |
| 5     | 26    | 0.127715       |
| 15    | 39    | 0.909644       |

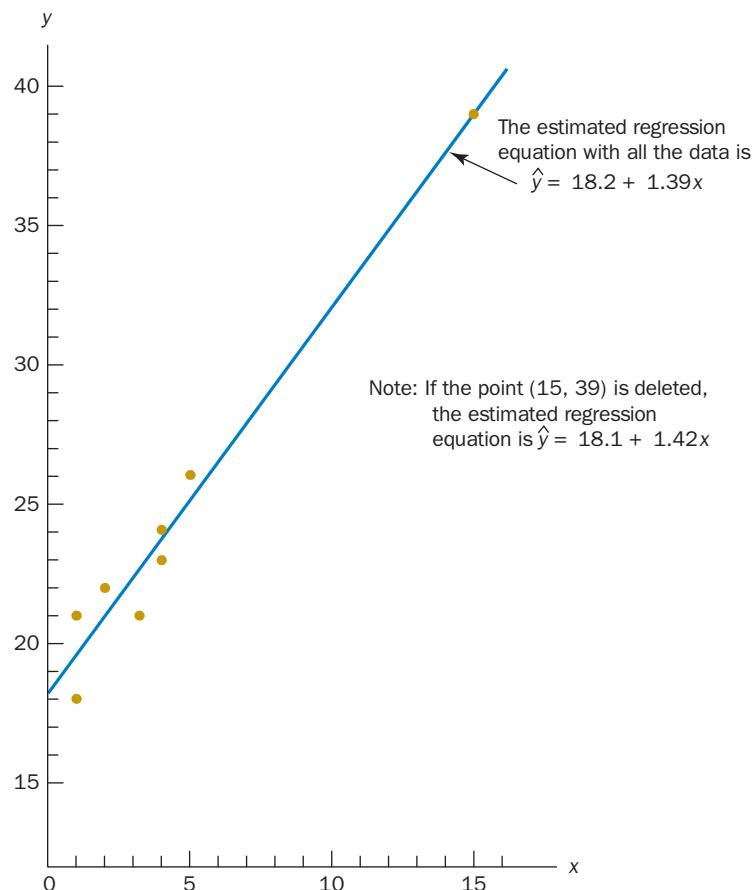
For example, Table 15.10 shows a data set consisting of eight observations and their corresponding leverage values (obtained by using MINITAB). Because the leverage for the eighth observation is  $0.91 > 0.75$  (the critical leverage value), this observation is identified as influential. Before reaching any final conclusions, however, let us consider the situation from a different perspective.

Figure 15.11 shows the scatter diagram and the estimated regression equation corresponding to the data set in Table 15.10. We used MINITAB to develop the following estimated regression equation for these data.

$$\hat{y} = 18.2 + 1.39x$$

**FIGURE 15.11**

Scatter diagram for the data set in Table 15.10



The straight line in Figure 15.11 is the graph of this equation. Now, let us delete the observation  $X = 15$ ,  $Y = 39$  from the data set and fit a new estimated regression equation to the remaining seven observations; the new estimated regression equation is:

$$\hat{y} = 18.1 + 1.42x$$

We note that the  $y$ -intercept and slope of the new estimated regression equation are not fundamentally different from the values obtained by using all the data. Although the leverage criterion identified the eighth observation as influential, this observation clearly had little influence on the results obtained. Thus, in some situations using only leverage to identify influential observations can lead to wrong conclusions.

**Cook's distance measure** uses both the leverage of observation  $i$ ,  $h_i$  and the residual for observation  $i$ ,  $(y_i - \hat{y}_i)$ , to determine whether the observation is influential.

### Cook's distance measure

$$D_i = \frac{(y_i - \hat{y}_i)^2 h_i}{(p-1)s^2(1-h_i)^2} \quad (15.26)$$

where:

- $D_i$  = Cook's distance measure for observation  $i$
- $y_i - \hat{y}_i$  = the residual for observation  $i$
- $h_i$  = the leverage for observation  $i$
- $p$  = the number of independent variables
- $s$  = the standard error of the estimate

The value of Cook's distance measure will be large and indicate an influential observation if the residual or the leverage is large. As a rule of thumb, values of  $D_i > 1$  indicate that the  $i$ th observation is influential and should be studied further. The last column of Table 15.9 provides Cook's distance measure for the Eurodistributor problem as given by MINITAB. Observation 8 with  $D_i = 0.650029$  has the most influence. However, applying the rule  $D_i > 1$ , we should not be concerned about the presence of influential observations in the Eurodistributor data set.

## EXERCISES

### Methods

- 26.** Data for two variables,  $X$  and  $Y$ , follow.

|       |   |   |   |    |    |
|-------|---|---|---|----|----|
| $x_i$ | 1 | 2 | 3 | 4  | 5  |
| $y_i$ | 3 | 7 | 5 | 11 | 14 |

- Develop the estimated regression equation for these data.
- Plot the standardized residuals versus  $\hat{y}$ . Do there appear to be any outliers in these data? Explain.
- Compute the studentized deleted residuals for these data. At the 0.05 level of significance, can any of these observations be classified as an outlier? Explain.



COMPLETE  
SOLUTIONS

- 27.** Data for two variables,  $X$  and  $Y$ , follow.

|       |    |    |    |    |    |
|-------|----|----|----|----|----|
| $x_i$ | 22 | 24 | 26 | 28 | 40 |
| $y_i$ | 12 | 21 | 31 | 35 | 70 |

- a. Develop the estimated regression equation for these data.
- b. Compute the studentized deleted residuals for these data. At the 0.05 level of significance, can any of these observations be classified as an outlier? Explain.
- c. Compute the leverage values for these data. Do there appear to be any influential observations in these data? Explain.
- d. Compute Cook's distance measure for these data. Are any observations influential? Explain.

### Applications

- 28.** Data collected by Montgomery and Peck (see Hawkins, 1991) concern the three variables:  $Y$ , the time taken to service a vending machine,  $X_1$ , the number of items stocked by the machine and  $X_2$ , the distance travelled to reach it.

| $X_1$ | $X_2$ | $Y$   |
|-------|-------|-------|
| 7     | 560   | 16.68 |
| 3     | 220   | 11.5  |
| 3     | 340   | 12.03 |
| 4     | 80    | 14.88 |
| 6     | 150   | 13.75 |
| 7     | 330   | 18.11 |
| 2     | 110   | 8     |
| 7     | 210   | 17.83 |
| 30    | 1460  | 79.24 |
| 5     | 605   | 21.5  |
| 16    | 688   | 40.33 |
| 10    | 215   | 21    |
| 4     | 255   | 13.5  |
| 6     | 462   | 19.75 |
| 9     | 448   | 24    |
| 10    | 776   | 29    |
| 6     | 200   | 15.35 |
| 7     | 132   | 19    |
| 3     | 36    | 9.5   |
| 17    | 770   | 35.1  |
| 10    | 140   | 17.9  |
| 26    | 810   | 52.32 |
| 9     | 450   | 18.75 |
| 8     | 635   | 19.83 |
| 4     | 150   | 10.75 |

- a. Find an estimated regression equation relating the time taken to service a vending machine to the number of items stocked by the machine and the distance travelled to reach it.
- b. Plot the standardized residuals against  $\hat{y}$ . Does the residual plot support the assumptions about  $\varepsilon$ ? Explain.
- c. Check for any outliers in these data. What are your conclusions?
- d. Are there any influential observations? Explain.



CIGARETTES

29. Data (Tufte, 1974) on male deaths per million in 1950 for lung cancer ( $Y$ ) and *per capita* cigarette consumption in 1930 ( $X$ ) are given below:

| Country     | $y$ | $x$  | Country   | $y$ | $x$  |
|-------------|-----|------|-----------|-----|------|
| Ireland     | 58  | 220  | Norway    | 90  | 250  |
| Sweden      | 115 | 310  | Canada    | 150 | 510  |
| Denmark     | 165 | 380  | Australia | 170 | 455  |
| USA         | 190 | 1280 | Holland   | 245 | 460  |
| Switzerland | 250 | 530  | Finland   | 350 | 1115 |
| GB          | 465 | 1145 |           |     |      |

Results from a simple regression analysis of this information are as follows:

### Regression Analysis: $y$ versus $x$

The regression equation is  
 $y = 65.7 + 0.229 x$

| Predictor | Coef    | SE Coef | T    | P     |
|-----------|---------|---------|------|-------|
| Constant  | 65.75   | 48.96   | 1.34 | 0.212 |
| $x$       | 0.22912 | 0.06921 | 3.31 | 0.009 |

$S = 84.1296 \quad R-Sq = 54.9\% \quad R-Sq(adj) = 49.9\%$

#### Analysis of Variance

| Source         | DF | SS     | MS    | F     | P     |
|----------------|----|--------|-------|-------|-------|
| Regression     | 1  | 77554  | 77554 | 10.96 | 0.009 |
| Residual Error | 9  | 63700  | 7078  |       |       |
| Total          | 10 | 141255 |       |       |       |

#### Unusual Observations

| Obs | $x$  | $y$   | Fit   | SE Fit | Residual | St Resid |
|-----|------|-------|-------|--------|----------|----------|
| 4   | 1280 | 190.0 | 359.0 | 53.2   | -169.0   | -2.59R   |

R denotes an observation with a large standardized residual.

Durbin-Watson statistic = 2.07188

Corresponding leverage and cook distance details are as follows.

|          |         |
|----------|---------|
| H1       | COOK1   |
| 0.191237 | 0.06985 |
| 0.149813 | 0.00694 |
| 0.125175 | 0.00172 |
| 0.399306 | 2.23320 |
| 0.094716 | 0.03222 |
| 0.288283 | 0.75365 |
| 0.176211 | 0.02001 |
| 0.097018 | 0.00893 |
| 0.106139 | 0.00000 |
| 0.105140 | 0.05060 |
| 0.266962 | 0.02909 |

Carry out any further statistical tests you deem appropriate, otherwise comment on the effectiveness of the linear modes.

## 15.9 LOGISTIC REGRESSION

In many regression applications the dependent variable may only assume two discrete values. For instance, a bank might like to develop an estimated regression equation for predicting whether a person will be approved for a credit card. The dependent variable can be coded as  $Y = 1$  if the bank approves the request for a credit card and  $Y = 0$  if the bank rejects the request for a credit card. Using logistic regression we can estimate the probability that the bank will approve the request for a credit card given a particular set of values for the chosen independent variables.

Consider an application of logistic regression involving a direct mail promotion being used by Stamm Stores. Stamm owns and operates a national chain of women's fashion stores. Five thousand copies of an expensive four-colour sales catalogue have been printed, and each catalogue includes a coupon that provides a €50 discount on purchases of €200 or more.

The catalogues are expensive and Stamm would like to send them to only those customers who have the highest probability of making a €200 purchase using the discount coupons.

Management thinks that annual spending at Stamm Stores and whether a customer has a Stamm credit card are two variables that might be helpful in predicting whether a customer who receives the catalogue will use the coupon to make a €200 purchase. Stamm conducted a pilot study using a random sample of 50 Stamm credit card customers and 50 other customers who do not have a Stamm credit card. Stamm sent the catalogue to each of the 100 customers selected. At the end of a test period, Stamm noted whether the customer made a purchase (coded 1 if the customer made a purchase and 0 if not). The sample data for the first ten catalogue recipients are shown in Table 15.11. The amount each customer spent last year at Stamm is shown in thousands of euros and the credit card information has been coded as 1 if the customer has a Stamm credit card and 0 if not. In the Purchase column a 1 is recorded if the sampled customer used the €50 discount coupon to make a purchase of €200 or more.

We might think of building a multiple regression model using the data in Table 15.11 to help Stamm predict whether a catalogue recipient will make a purchase. We would use Annual spending and Stamm Card as independent variables and Purchase as the dependent variable.

Because the dependent variable may only assume the values of 0 or 1, however, the ordinary multiple regression model is not applicable. This example shows the type of situation for which logistic regression was developed. Let us see how logistic regression can be used to help Stamm predict which type of customer is most likely to take advantage of their promotion.

### Logistic regression equation

In many ways logistic regression is like ordinary regression. It requires a dependent variable,  $Y$ , and one or more independent variables. In multiple regression analysis, the mean or expected value of  $Y$ , is referred to as the multiple regression equation.

$$E(Y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p \quad (15.27)$$

**TABLE 15.11** Sample data for Stamm Stores

| Customer | Annual spending (€000s) | Stamm card | Purchase |
|----------|-------------------------|------------|----------|
| 1        | 2.291                   | 1          | 0        |
| 2        | 3.215                   | 1          | 0        |
| 3        | 2.135                   | 1          | 0        |
| 4        | 3.924                   | 0          | 0        |
| 5        | 2.528                   | 1          | 0        |
| 6        | 2.473                   | 0          | 1        |
| 7        | 2.384                   | 0          | 0        |
| 8        | 7.076                   | 0          | 0        |
| 9        | 1.182                   | 1          | 1        |
| 10       | 3.345                   | 0          | 0        |

In logistic regression, statistical theory as well as practice has shown that the relationship between  $E(Y)$  and  $X_1, X_2, \dots, X_p$  is better described by the following nonlinear equation.

### Logistic regression equation

$$E(Y) = \frac{e^{\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p}}{1 + e^{\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p}} \quad (15.28)$$

If the two values of the dependent variable  $Y$  are coded as 0 or 1, the value of  $E(Y)$  in equation (15.28) provides the *probability* that  $Y = 1$  given a particular set of values for the independent variables  $X_1, X_2, \dots, X_p$ . Because of the interpretation of  $E(Y)$  as a probability, the **logistic regression equation** is often written as follows.

### Interpretation of $E(Y)$ as a probability in logistic regression

$$E(Y) = P(y = 1 | x_1, x_2, \dots, x_p) \quad (15.29)$$

To provide a better understanding of the characteristics of the logistic regression equation, suppose the model involves only one independent variable  $X$  and the values of the model parameters are  $\beta_0 = -7$  and  $\beta_1 = 3$ . The logistic regression equation corresponding to these parameter values is:

$$E(Y) = P(Y = 1 | x) = \frac{e^{\beta_0 + \beta_1 x}}{1 + e^{\beta_0 + \beta_1 x}} = \frac{e^{-7 + 3x}}{1 + e^{-7 + 3x}} \quad (15.30)$$

Figure 15.12 shows a graph of equation (15.30). Note that the graph is S-shaped. The value of  $E(Y)$  ranges from 0 to 1, with the value of  $E(Y)$  gradually approaching 1 as the value of  $X$  becomes larger and the value of  $E(Y)$  approaching 0 as the value of  $X$  becomes smaller. Note also that the values of  $E(Y)$ , representing probability, increase fairly rapidly as  $X$  increases from 2 to 3. The fact that the values of  $E(Y)$  range from 0 to 1 and that the curve is S-shaped makes equation (15.30) ideally suited to model the probability the dependent variable is equal to 1.

### Estimating the logistic regression equation

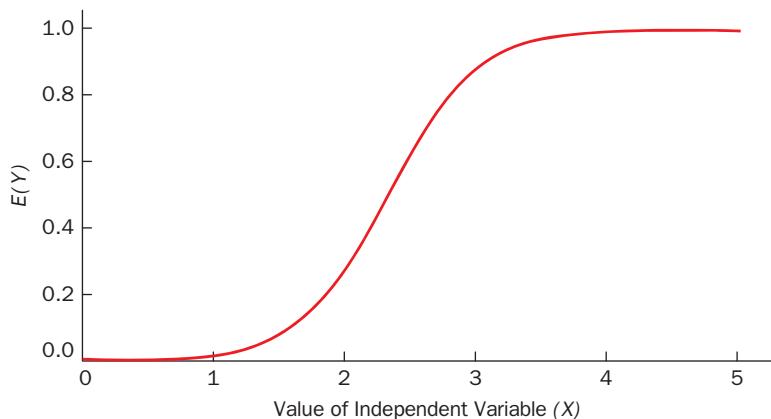
In simple linear and multiple regression the least squares method is used to compute  $b_0, b_1, \dots, b_p$  as estimates of the model parameters ( $\beta_0, \beta_1, \dots, \beta_p$ ). The nonlinear form of the logistic regression equation makes the method of computing estimates more complex and beyond the scope of this text. We will use computer software to provide the estimates. The **estimated logistic regression equation** is:

### Estimated logistic regression equation

$$\hat{y} = \text{estimate of } P(Y = 1 | x_1, x_2, \dots, x_p) = \frac{e^{b_0 + b_1 x_1 + b_2 x_2 + \dots + b_p x_p}}{1 + e^{b_0 + b_1 x_1 + b_2 x_2 + \dots + b_p x_p}} \quad (15.31)$$

**FIGURE 15.12**

Logistic regression equation  
for  $\beta_0 = -7$  and  $\beta_1 = 3$



Here  $\hat{y}$  provides an estimate of the probability that  $Y = 1$ , given a particular set of values for the independent variables.

Let us now return to the Stamm Stores example. The variables in the study are defined as follows:

$$Y = \begin{cases} 0 & \text{if the customer made no purchase during the test period} \\ 1 & \text{if the customer made a purchase during the test period} \end{cases}$$

$X_1$  = annual spending at Stamm Stores (€000s)

$$X_2 = \begin{cases} 0 & \text{if the customer does not have a Stamm credit card} \\ 1 & \text{if the customer has a Stamm credit card} \end{cases}$$

Therefore, we choose a logistic regression equation with two independent variables.

$$E(Y) = \frac{e^{\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p}}{1 + e^{\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p}} \quad (15.32)$$

Using the sample data (see Table 15.11), MINITAB's binary logistic regression procedure was used to compute estimates of the model parameters  $\beta_0$ ,  $\beta_1$  and  $\beta_2$ . A portion of the output obtained is shown in Figure 15.13. We see that  $b_0 = -2.1464$ ,  $b_1 = 0.3416$  and  $b_2 = 1.0987$ . Thus, the estimated logistic regression equation is:

$$\hat{y} = \frac{e^{b_0 + b_1 x_1 + \dots + b_p x_p}}{1 + e^{b_0 + b_1 x_1 + \dots + b_p x_p}} = \frac{e^{-2.1464 + 0.3416x_1 + 1.0987x_2}}{1 + e^{-2.1464 + 0.3416x_1 + 1.0987x_2}} \quad (15.33)$$

We can now use equation (15.33) to estimate the probability of making a purchase for a particular type of customer. For example, to estimate the probability of making a purchase for customers that spend €2000 annually and do not have a Stamm credit card, we substitute  $X_1 = 2$  and  $X_2 = 0$  into equation (15.33).

$$\hat{y} = \frac{e^{-2.1464 + 0.3416(2) + 1.0987(0)}}{1 + e^{-2.1464 + 0.3416(2) + 1.0987(0)}} = \frac{e^{-1.4632}}{1 + e^{-1.4632}} = \frac{0.2315}{1.2315} = 0.1880$$

**FIGURE 15.13**

Partial logistic regression  
output for the Stamm Stores  
example

Logistic Regression Table

| Predictor | Coef     | SE Coef  | Z     | P     | Odds Ratio | 95% CI Lower | 95% CI Upper |
|-----------|----------|----------|-------|-------|------------|--------------|--------------|
| Constant  | -2.14637 | 0.577245 | -3.72 | 0.000 |            |              |              |
| Spending  | 0.341643 | 0.128672 | 2.66  | 0.008 | 1.41       | 1.09         | 1.81         |
| Card      | 1.09873  | 0.444696 | 2.47  | 0.013 | 3.00       | 1.25         | 7.17         |

Log-Likelihood = -60.487

Test that all slopes are zero: G = 13.628, DF = 2, P-Value = 0.001

Thus, an estimate of the probability of making a purchase for this particular group of customers is approximately 0.19. Similarly, to estimate the probability of making a purchase for customers that spent €2000 last year and have a Stamm credit card, we substitute  $X_1 = 2$  and  $X_2 = 1$  into equation (15.33).

$$\hat{y} = \frac{e^{-2.1464 + 0.3416(2) + 1.0987(1)}}{1 + e^{-2.1464 + 0.3416(2) + 1.0987(1)}} = \frac{e^{-0.3645}}{1 + e^{-0.3645}} = \frac{0.6945}{1.6945} = 0.4099$$

Thus, for this group of customers, the probability of making a purchase is approximately 0.41. It appears that the probability of making a purchase is much higher for customers with a Stamm credit card. Before reaching any conclusions, however, we need to assess the statistical significance of our model.

## Testing for significance

Testing for significance in logistic regression is similar to testing for significance in multiple regression. First we conduct a test for overall significance. For the Stamm Stores example, the hypotheses for the test of overall significance follow:

$$\begin{aligned} H_0: \beta_1 &= \beta_2 = 0 \\ H_1: \beta_1 \text{ and/or } \beta_2 &\text{ is not equal to zero} \end{aligned}$$

The test for overall significance is based upon the value of a  $G$  test statistic. This is commonly referred to as the 'Deviance Statistic'. If the null hypothesis is true, the sampling distribution of  $G$  follows a chi-square distribution with degrees of freedom equal to the number of independent variables in the model. Although the computation of  $G$  is beyond the scope of this book, the value of  $G$  and its corresponding  $p$ -value are provided as part of MINITAB's binary logistic regression output. Referring to the last line in Figure 15.13, we see that the value of  $G$  is 13.628, its degrees of freedom are 2, and its  $p$ -value is 0.001. Thus, at any level of significance  $\alpha \geq 0.001$ , we would reject the null hypothesis and conclude that the overall model is significant.

If the  $G$  test shows an overall significance, a  $z$  test can be used to determine whether each of the individual independent variables is making a significant contribution to the overall model. For the independent variables  $X_i$ , the hypotheses are:

$$\begin{aligned} H_0: \beta_i &= 0 \\ H_1: \beta_i &\neq 0 \end{aligned}$$

If the null hypothesis is true, the value of the estimated coefficient divided by its standard error follows a standard normal probability distribution. The column labelled  $Z$  in the MINITAB output contains the values of  $z_i = b_i / s_{b_i}$  for each of the estimated coefficients and the column labelled  $p$  contains the corresponding  $p$ -values. The  $z_i$  ratio is also known as a 'Wald Statistic'. Suppose we use  $\alpha = 0.05$  to test for the significance of the independent variables in the Stamm model. For the independent variable  $X_1$  the  $z$  value is 2.66 and the corresponding  $p$ -value is 0.008. Thus, at the 0.05 level of significance we can reject  $H_0: \beta_1 = 0$ . In a similar fashion we can also reject  $H_0: \beta_2 = 0$  because the  $p$ -value corresponding to  $z = 2.47$  is 0.013. Hence, at the 0.05 level of significance, both independent variables are statistically significant.

## Managerial use

We now use the estimated logistic regression equation to make a decision recommendation concerning the Stamm Stores catalogue promotion. For Stamm Stores, we already computed:

$$P(Y = 1 | X_1 = 2, X_2 = 1) = 0.4099 \text{ and } P(Y = 1 | X_1 = 2, X_2 = 0) = 0.1880$$

These probabilities indicate that for customers with annual spending of €2000 the presence of a Stamm credit card increases the probability of making a purchase using the discount coupon. In Table 15.12 we show estimated probabilities for values of annual spending ranging from €1000 to €7000 for both customers who have a Stamm credit card and customers who do not have a Stamm credit card. How can Stamm use this information to better target customers for the new promotion?

**TABLE 15.12** Estimated probabilities for Stamm Stores

|             |     | Annual spending |        |        |        |        |        |        |
|-------------|-----|-----------------|--------|--------|--------|--------|--------|--------|
|             |     | €1000           | €2000  | €3000  | €4000  | €5000  | €6000  | €7000  |
| Credit card | Yes | 0.3305          | 0.4099 | 0.4943 | 0.5790 | 0.6593 | 0.7314 | 0.7931 |
|             | No  | 0.1413          | 0.1880 | 0.2457 | 0.3143 | 0.3921 | 0.4758 | 0.5609 |

Suppose Stamm wants to send the promotional catalogue only to customers who have a 0.40 or higher probability of making a purchase. Using the estimated probabilities in Table 15.12, Stamm promotion strategy would be:

**Customers who have a Stamm credit card:** Send the catalogue to every customer that spent €2000 or more last year.

**Customers who do not have a Stamm credit card:** Send the catalogue to every customer that spent €6000 or more last year.

Looking at the estimated probabilities further, we see that the probability of making a purchase for customers who do not have a Stamm credit card, but spend €5000 annually is 0.3921. Thus, Stamm may want to consider revising this strategy by including those customers who do not have a credit card as long as they spent €5000 or more last year.

## Interpreting the logistic regression equation

Interpreting a regression equation involves relating the independent variables to the business question that the equation was developed to answer. With logistic regression, it is difficult to interpret the relation between the independent variables and the probability that  $Y = 1$  directly because the logistic regression equation is nonlinear. However, statisticians have shown that the relationship can be interpreted indirectly using a concept called the odds ratio.

The **odds in favour of an event occurring** is defined as the probability the event will occur divided by the probability the event will not occur. In logistic regression the event of interest is always  $Y = 1$ . Given a particular set of values for the independent variables, the odds in favour of  $Y = 1$  can be calculated as follows:

$$\text{Odds} = \frac{P(Y = 1 | X_1, X_2, \dots, X_y)}{P(Y = 0 | X_1, X_2, \dots, X_y)} = \frac{P(Y = 1 | X_1, X_2, \dots, X_y)}{1 - P(Y = 1 | X_1, X_2, \dots, X_y)} \quad (15.34)$$

...

The **odds ratio** measures the impact on the odds of a one-unit increase in only one of the independent variables. The odds ratio is the odds that  $Y = 1$  given that one of the independent variables has been increased by one unit ( $\text{odds}_1$ ) divided by the odds that  $Y = 1$  given no change in the values for the independent variables ( $\text{odds}_0$ ).

### Odds ratio

$$\text{Odds ratio} = \frac{\text{Odds}_1}{\text{Odds}_0} \quad (15.35)$$

For example, suppose we want to compare the odds of making a purchase for customers who spend €2000 annually and have a Stamm credit card ( $X_1 = 2$  and  $X_2 = 1$ ) to the odds of making a purchase for

customers who spend €2000 annually and do not have a Stamm credit card ( $X_1 = 2$  and  $X_2 = 0$ ). We are interested in interpreting the effect of a one-unit increase in the independent variable  $X_2$ . In this case:

$$\text{Odds}_1 = \frac{P(Y = 1 | X_1 = 2, X_2 = 1)}{1 - P(Y = 1 | X_1 = 2, X_2 = 1)}$$

and:

$$\text{Odds}_0 = \frac{P(Y = 1 | X_1 = 2, X_2 = 0)}{1 - P(Y = 1 | X_1 = 2, X_2 = 0)}$$

Previously we showed that an estimate of the probability that  $Y = 1$  given  $X_1 = 2$  and  $X_2 = 1$  is 0.4099, and an estimate of the probability that  $Y = 1$  given  $X_1 = 2$  and  $X_2 = 0$  is 0.1880. Thus,

$$\text{Estimate of odds}_1 = \frac{0.4099}{1 - 0.4099} = 0.6946$$

and:

$$\text{Estimate of odds}_0 = \frac{0.1880}{1 - 0.1880} = 0.2315$$

The estimated odds ratio is:

$$\text{Estimate odds ratio} = \frac{0.6946}{0.2315} = 3.00$$

Thus, we can conclude that the estimated odds in favour of making a purchase for customers who spent €2000 last year and have a Stamm credit card are three times greater than the estimated odds in favour of making a purchase for customers who spent €2000 last year and do not have a Stamm credit card.

The odds ratio for each independent variable is computed while holding all the other independent variables constant. But it does not matter what constant values are used for the other independent variables. For instance, if we computed the odds ratio for the Stamm credit card variable ( $X_2$ ) using €3000, instead of €2000, as the value for the annual spending variable ( $X_1$ ), we would still obtain the same value for the estimated odds ratio (3.00). Thus, we can conclude that the estimated odds of making a purchase for customers who have a Stamm credit card are three times greater than the estimated odds of making a purchase for customers who do not have a Stamm credit card.

The odds ratio is standard output for logistic regression software packages. Refer to the MINITAB output in Figure 15.13. The column with the heading Odds Ratio contains the estimated odds ratios for each of the independent variables. The estimated odds ratio for  $X_1$  is 1.41 and the estimated odds ratio for  $X_2$  is 3.00. We already showed how to interpret the estimated odds ratio for the binary independent variable  $X_2$ . Let us now consider the interpretation of the estimated odds ratio for the continuous independent variable  $X_1$ .

The value of 1.41 in the Odds Ratio column of the MINITAB output tells us that the estimated odds in favour of making a purchase for customers who spent €3000 last year is 1.41 times greater than the estimated odds in favour of making a purchase for customers who spent €2000 last year. Moreover, this interpretation is true for any one-unit change in  $X_1$ .

For instance, the estimated odds in favour of making a purchase for someone who spent €5000 last year is 1.41 times greater than the odds in favour of making a purchase for a customer who spent €4000 last year. But suppose we are interested in the change in the odds for an increase of more than one unit for an independent variable. Note that  $X_1$  can range from 1 to 7. The odds ratio as printed by the MINITAB output does not answer this question.

To answer this question we must explore the relationship between the odds ratio and the regression coefficients.

A unique relationship exists between the odds ratio for a variable and its corresponding regression coefficient. For each independent variable in a logistic regression equation it can be shown that:

$$\text{Odds ratio} = e^{b_i}$$

To illustrate this relationship, consider the independent variable  $X_1$  in the Stamm example. The estimated odds ratio for  $X_1$  is:

$$\text{Estimated odds ratio} = e^{b_1} = e^{0.3416} = 1.41$$

Similarly, the estimated odds ratio for  $X_2$  is:

$$\text{Estimated odds ratio} = e^{b_2} = e^{1.0987} = 3.00$$

This relationship between the odds ratio and the coefficients of the independent variables makes it easy to compute estimates of the odds ratios once we develop estimates of the model parameters. Moreover, it also provides us with the ability to investigate changes in the odds ratio of more than or less than one unit for a continuous independent variable.

The odds ratio for an independent variable represents the change in the odds for a one unit change in the independent variable holding all the other independent variables constant. Suppose that we want to consider the effect of a change of more than one unit, say  $c$  units. For instance, suppose in the Stamm example that we want to compare the odds of making a purchase for customers who spend €5000 annually ( $X_1 = 5$ ) to the odds of making a purchase for customers who spend €2000 annually ( $X_1 = 2$ ). In this case  $c = 5 - 2 = 3$  and the corresponding estimated odds ratio is:

$$e^{cb} = e^{3(0.3416)} = e^{1.0248} = 2.79$$

This result indicates that the estimated odds of making a purchase for customers who spend €5000 annually is 2.79 times greater than the estimated odds of making a purchase for customers who spend €2000 annually. In other words, the estimated odds ratio for an increase of €3000 in annual spending is 2.79.

In general, the odds ratio enables us to compare the odds for two different events. If the value of the odds ratio is 1, the odds for both events are the same. Thus, if the independent variable we are considering (such as Stamm credit card status) has a positive impact on the probability of the event occurring, the corresponding odds ratio will be greater than 1. Most logistic regression software packages provide a confidence interval for the odds ratio. The MINITAB output in Figure 15.13 provides a 95 per cent confidence interval for each of the odds ratios. For example, the point estimate of the odds ratio for  $X_1$  is 1.41 and the 95 per cent confidence interval is 1.09 to 1.81. Because the confidence interval does not contain the value of 1, we can conclude that  $X_1$  has a significant effect on the odds ratio. Similarly, the 95 per cent confidence interval for the odds ratio for  $X_2$  is 1.25 to 7.17. Because this interval does not contain the value of 1, we can also conclude that  $X_2$  has a significant effect on the odds ratio.

## Logit transformation

An interesting relationship can be observed between the odds in favour of  $Y = 1$  and the exponent for  $e$  in the logistic regression equation. It can be shown that:

$$\ln(\text{odds}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p$$

This equation shows that the natural logarithm of the odds in favour of  $Y = 1$  is a linear function of the independent variables. This linear function is called the **logit**. We will use the notation  $g(x_1, x_2, \dots, x_p)$  to denote the logit.

### Logit

$$g(x_1, x_2, \dots, x_p) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p$$

(15.36)

Substituting  $g(x_1, x_2, \dots, x_p)$  for  $\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p$  in equation (15.28), we can write the logistic regression equation as:

$$E(Y) = \frac{e^{g(x_1, x_2, \dots, x_p)}}{1 + e^{g(x_1, x_2, \dots, x_p)}} \quad (15.37)$$

Once we estimate the parameters in the logistic regression equation, we can compute an estimate of the logit. Using  $\hat{g}(x_1, x_2, \dots, x_p)$  to denote the **estimated logit**, we obtain:

### Estimated logit

$$\hat{g}(x_1, x_2, \dots, x_p) = b_0 + b_1 x_1 + b_2 x_2 + \dots + b_p x_p \quad (15.38)$$

Therefore, in terms of the estimated logit, the estimated regression equation is:

$$\hat{y} = \frac{e^{b_0 + b_1 x_1 + b_2 x_2 + \dots + b_p x_p}}{1 + e^{b_0 + b_1 x_1 + b_2 x_2 + \dots + b_p x_p}} = \frac{e^{\hat{g}(x_1, x_2, \dots, x_p)}}{1 + e^{\hat{g}(x_1, x_2, \dots, x_p)}}$$

For the Stamm Stores example, the estimated logit is:

$$\hat{g}(x_1, x_2) = -2.1464 + 0.3416x_1 + 1.0987x_2$$

and the estimated regression equation is:

$$\hat{y} = \frac{e^{\hat{g}(x_1, x_2)}}{1 + e^{\hat{g}(x_1, x_2)}} = \frac{e^{-2.1464 + 0.3416x_1 + 1.0987x_2}}{1 + e^{-2.1464 + 0.3416x_1 + 1.0987x_2}}$$

Therefore, because of the unique relationship between the estimated logit and the estimated logistic regression equation, we can compute the estimated probabilities for Stamm Stores by dividing  $e^{\hat{g}(x_1, x_2)}$  by  $1 + e^{\hat{g}(x_1, x_2)}$ .

## EXERCISES

### Applications

- 30.** Refer to the Stamm Stores example introduced in this section. The dependent variable is coded as  $Y = 1$  if the customer makes a purchase and 0 if not.

Suppose that the only information available to help predict whether the customer will make a purchase is the customer's credit card status, coded as  $X = 1$  if the customer has a Stamm credit card and  $X = 0$  if not.

- Write the logistic regression equation relating  $X$  to  $Y$ .
- What is the interpretation of  $E(Y)$  when  $X = 0$ ?
- For the Stamm data in Table 15.11, use MINITAB to compute the estimated logit.
- Use the estimated logit computed in part (c) to compute an estimate of the probability of making a purchase for customers who do not have a Stamm credit card and an estimate of the probability of making a purchase for customers who have a Stamm credit card.
- What is the estimate of the odds ratio? What is its interpretation?



COMPLETE  
SOLUTIONS

- 31.** In Table 15.12 we provided estimates of the probability of a purchase in the Stamm Stores catalogue promotion. A different value is obtained for each combination of values for the independent variables.
- Compute the odds in favour of a purchase for a customer with annual spending of €4000 who does not have a Stamm credit card ( $X_1 = 4$ ,  $X_2 = 0$ ).
  - Use the information in Table 15.12 and part (a) to compute the odds ratio for the Stamm credit card variable  $X_2$  holding annual spending constant at  $X_1 = 4$ .
  - In the text, the odds ratio for the credit card variable was computed using the information in the €2000 column of Table 15.12. Did you get the same value for the odds ratio in part (b)?
- 32.** Community Bank would like to increase the number of customers who use payroll direct deposit. Management is considering a new sales campaign that will require each branch manager to call each customer who does not currently use payroll direct deposit. As an incentive to sign up for payroll direct deposit, each customer contacted will be offered free banking for two years. Because of the time and cost associated with the new campaign, management would like to focus their efforts on customers who have the highest probability of signing up for payroll direct deposit. Management believes that the average monthly balance in a customer's current account may be a useful predictor of whether the customer will sign up for direct payroll deposit. To investigate the relationship between these two variables, Community Bank tried the new campaign using a sample of 50 current account customers that do not currently use payroll direct deposit. The sample data show the average monthly current account balance (in hundreds of euros) and whether the customer contacted signed up for payroll direct deposit (coded 1 if the customer signed up for payroll direct deposit and 0 if not). The data are contained in the data set named 'Bank' on the companion online platform; a portion of the data follows.

| Customer | X Monthly balance | Y Direct deposit |
|----------|-------------------|------------------|
| 1        | 1.22              | 0                |
| 2        | 1.56              | 0                |
| 3        | 2.10              | 0                |
| 4        | 2.25              | 0                |
| 5        | 2.89              | 0                |
| 6        | 3.55              | 0                |
| 7        | 3.56              | 0                |
| 8        | 3.65              | 1                |
| .        | .                 | .                |
| .        | .                 | .                |
| 48       | 18.45             | 1                |
| 49       | 24.98             | 0                |
| 50       | 26.05             | 1                |

- 
- BANK
- Write the logistic regression equation relating  $X$  to  $Y$ .
  - For the Community Bank data, use MINITAB to compute the estimated logistic regression equation.
  - Conduct a test of significance using the  $G$  test statistic. Use  $\alpha = 0.05$ .
  - Estimate the probability that customers with an average monthly balance of €1000 will sign up for direct payroll deposit.
  - Suppose Community Bank only wants to contact customers who have a 0.50 or higher probability of signing up for direct payroll deposit. What is the average monthly balance required to achieve this level of probability?
  - What is the estimate of the odds ratio? What is its interpretation?

- 33.** Prior to the *Challenger* tragedy on 28 January 1986, after each launch of the space shuttle the solid rocket boosters were recovered from the ocean and inspected. Of the previous 24 shuttle launches, seven had incidents of damage to the joints, 16 had no incidents of damage and one was unknown because the boosters were not recovered after launch.

In trying to explain the damage to joints it was thought that temperature at the time of launch could be a contributing factor.

For the data that follow, a 1 represents damage to field joints, and a 0 represents no damage.

| Temp | Damage | Temp | Damage | Temp | Damage |
|------|--------|------|--------|------|--------|
| 66   | 0      | 57   | 1      | 70   | 0      |
| 70   | 1      | 63   | 1      | 81   | 0      |
| 69   | 0      | 70   | 1      | 76   | 0      |
| 68   | 0      | 78   | 0      | 79   | 0      |
| 67   | 0      | 67   | 0      | 75   | 1      |
| 72   | 0      | 53   | 1      | 76   | 0      |
| 73   | 0      | 67   | 0      | 58   | 1      |
| 70   | 0      | 75   | 0      |      |        |

- Fit a logistic regression model to these data and obtain a plot of the data and fitted curve.
- Conduct a test of significance using the  $G$  test statistic. Use  $\alpha = 0.05$ .
- Estimate the probability of damage for a temperature of 50.
- What is the estimate of the odds ratio? How would you interpret it?



SHUTTLE

## ONLINE RESOURCES

For data files, additional online summary, questions, answers and software section visit the online platform.



## SUMMARY

In this chapter, we introduced multiple regression analysis as an extension of simple linear regression analysis presented in Chapter 14. Multiple regression analysis enables us to understand how a dependent variable is related to two or more independent variables. The regression equation  $E(Y) = \beta_0 + \beta_1x_1 + \beta_2x_2 + \dots + \beta_px_p$  shows that the expected value or mean value of the dependent variable  $Y$  is related to the values of the independent variables  $X_1, X_2, \dots, X_p$ . Sample data and the least squares method are used to develop the estimated regression equation  $\hat{y} = b_0 + b_1x_1 + b_2x_2 + \dots + b_px_p$ . In effect  $b_0, b_1, b_2, \dots, b_p$  are sample statistics used to estimate the unknown model parameters  $\beta_0, \beta_1, \beta_2, \dots, \beta_p$ . Computer printouts were used throughout the chapter to emphasize the fact that statistical software packages are the only realistic means of performing the numerous computations required in multiple regression analysis.

The multiple coefficient of determination was presented as a measure of the goodness of fit of the estimated regression equation. It determines the proportion of the variation of  $Y$  that can be explained by the estimated regression equation. The adjusted multiple coefficient of determination is a similar measure of goodness of fit that adjusts for the number of independent variables and thus avoids overestimating the impact of adding more independent variables. Model assumptions for multiple regression are shown to parallel those for simple regression analysis.

An *F* test and a *t* test were presented as ways of determining statistically whether the relationship among the variables is significant. The *F* test is used to determine whether there is a significant overall relationship between the dependent variable and the set of all independent variables. The *t* test is used to determine whether there is a significant relationship between the dependent variable and an individual independent variable given the other independent variables in the regression model. Correlation among the independent variables, known as multicollinearity, was discussed.

The section on qualitative independent variables showed how dummy variables can be used to incorporate qualitative data into multiple regression analysis. The section on residual analysis showed how residual analysis can be used to validate the model assumptions, detect outliers and identify influential observations. Standardized residuals, leverage, studentized deleted residuals and Cook's distance measure were discussed. The chapter concluded with a section on how logistic regression can be used to model situations in which the dependent variable may only assume two values.

## KEY TERMS

**Adjusted multiple coefficient of determination**  
**Cook's distance measure**  
**Dummy variable**  
**Estimated logistic regression equation**  
**Estimated logit**  
**Estimated multiple regression equation**  
**Influential observation**  
**Least squares method**  
**Leverage**  
**Logistic regression equation**  
**Logit**

**Multicollinearity**  
**Multiple coefficient of determination**  
**Multiple regression analysis**  
**Multiple regression equation**  
**Multiple regression model**  
**Odds in favour of an event occurring**  
**Odds ratio**  
**Outlier**  
**Qualitative independent variable**  
**Studentized deleted residuals**  
**Variance inflation factor**

## KEY FORMULAE

### Multiple regression model

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p + \varepsilon \quad (15.1)$$

### Multiple regression equation

$$E(Y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p \quad (15.2)$$

### Estimated multiple regression equation

$$\hat{y} = b_0 + b_1 x_1 + b_2 x_2 + \dots + b_p x_p \quad (15.3)$$

### Least squares criterion

$$\min \Sigma(y_i - \hat{y}_i)^2 \quad (15.4)$$

### Relationship among SST, SSR and SSE

$$\text{SST} = \text{SSR} + \text{SSE} \quad (15.7)$$

### Multiple coefficient of determination

$$R^2 = \frac{\text{SSR}}{\text{SST}} \quad (15.8)$$

### Adjusted multiple coefficient of determination

$$\text{adj } R^2 = 1 - (1 - R^2) \frac{n - 1}{n - p - 1} \quad (15.9)$$

### Mean square regression

$$\text{MSR} = \frac{\text{SSR}}{P} \quad (15.12)$$

### Mean square error

$$\text{MSE} = s^2 = \frac{\text{SSE}}{n - p - 1} \quad (15.13)$$

### F test statistic

$$F = \frac{\text{MSR}}{\text{MSE}} \quad (15.14)$$

### t test statistic

$$t = \frac{b_i}{s_{b_i}} \quad (15.15)$$

### Variance Inflation Factor

$$\text{VIF}(X_j) = \frac{1}{1 - R_j^2} \quad (15.16)$$

### Standardized residual for observation *i*

$$\frac{y_i - \hat{y}_i}{s_{y_i - \hat{y}_i}} \quad (15.24)$$

### Standard deviation of residual *i*

$$s_{y_i - \hat{y}_i} = s \sqrt{l - h_j} \quad (15.25)$$

**Cook's distance measure**

$$D_i = \frac{(y_i - \hat{y}_i)^2 h_i}{(p-1)s^2(l-h_i)^2} \quad (15.26)$$

**Logistic regression equation**

$$E(Y) = \frac{e^{\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p}}{1 + e^{\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p}} \quad (15.28)$$

**Interpretation of  $E(Y)$  as a probability in logistic regression**

$$E(Y) = P(Y = 1 | x_1, x_2, \dots, x_p) \quad (15.29)$$

**Estimated logistic regression equation**

$$\hat{y} = \text{estimate of } P(Y=1 | x_1, x_2, \dots, x_p) = \frac{e^{b_0 + b_1 x_1 + b_2 x_2 + \dots + b_p x_p}}{1 + e^{b_0 + b_1 x_1 + b_2 x_2 + \dots + b_p x_p}} \quad (15.31)$$

**Odds ratio**

$$\text{Odds ratio} = \frac{\text{odds}_1}{\text{odds}_0} \quad (15.35)$$

**Logit**

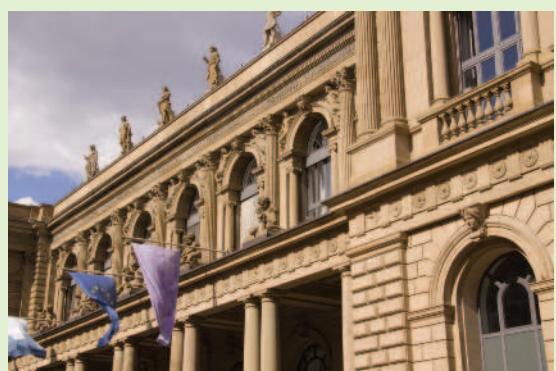
$$g(x_1, x_2, \dots, x_p) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p \quad (15.36)$$

**Estimated logits**

$$\hat{g}(x_1, x_2, \dots, x_p) = b_0 + b_1 x_1 + b_2 x_2 + \dots + b_p x_p \quad (15.38)$$

**CASE PROBLEM****P/E ratios**

Valuation is one of the most important aspects of business. Frequently, although an absolute valuation (e.g. \$100 million) would be desirable, relative valuation (e.g. company A is better than Company B) is enough for investment decision-making. When deciding to perform a relative valuation, it is necessary to decide on what attributes to compare; of the many



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possibilities, Price-to-Earnings ratios (PEs) are perhaps the most frequently used. This ratio typically is calculated using data on a per share basis:

$$\text{PE ratio} = \frac{\text{Market price per share}}{\text{Earnings per share}}$$

Other things being equal, the higher the price-to-earnings ratio, the higher the expected future income relative to the reported income.

### **Managerial report**

A portfolio manager in a leading brokerage firm has asked you to develop a model that can help them to allocate funds between the various international markets. Theoretically, the job is easy – invest in undervalued markets and sell any assets

in overvalued markets. PE ratios can be used to identify over-/undervalued markets.

Three variables thought to influence the PE ratio are:

1. Price-to-Book Value (PBV)<sup>1</sup>
2. Return on Equity (ROE)<sup>2</sup>
3. The Effective Tax Rate (Tax)<sup>3</sup>

Formulate and estimate a multiple regression model using the data provided. In your report, you should help the manager understand each of the estimated regression coefficients, the standard error of estimate, and the co-efficient of determination.

Data are available in a file called 'Funds' on the online platform. Below is a part of the table.

*Criteria for inclusion: Publicly traded firms with \$ market cap > \$ 50 million*

| Country   | Number of firms | PE    | PBV  | Return on Equity | Effective Tax Rate |
|-----------|-----------------|-------|------|------------------|--------------------|
| Argentina | 43              | 14.10 | 1.67 | -11.48%          | 10.30%             |
| Australia | 419             | 28.93 | 4.78 | 11.32%           | 22.37%             |
| Austria   | 68              | 41.81 | 2.00 | 7.54%            | 22.41%             |

<sup>1</sup> A ratio used to compare a stock's market value to its book value. It is calculated by dividing the current closing price of the stock by the latest quarter's book value (book value is simply total assets minus intangible assets and liabilities). A lower PBV ratio could mean that the stock is undervalued. However, it could also mean that something is fundamentally wrong with the company.

<sup>2</sup> Essentially, ROE reveals how much profit a company generates with the money shareholders have invested in it. The ROE is useful for comparing the profitability of a company to that of other firms in the same industry. Investors usually look for companies with ROEs that are high and growing.

<sup>3</sup> Actual income tax paid divided by net taxable income before taxes.



FUNDS



# 16

# Regression

# Analysis: Model

# Building

## CHAPTER CONTENTS

Statistics in Practice Selecting a university

- 16.1 General linear model
- 16.2 Determining when to add or delete variables
- 16.3 Analysis of a larger problem
- 16.4 Variable selection procedures

**LEARNING OBJECTIVES** After reading this chapter and doing the exercises, you should be able to:

- 1 Appreciate how the general linear model can be used to model problems involving curvilinear relationships.
- 2 Understand the concept of interaction and how it can be accounted for in the general linear model.
- 3 Understand how an  $F$  test can be used to determine when to add or delete one or more variables.
- 4 Appreciate the complexities involved in solving larger regression analysis problems.
- 5 Understand how variable selection procedures can be used to choose a set of independent variables for an estimated regression equation.

**M**odel building in regression analysis is the process of developing an estimated regression equation that describes the relationship between a dependent variable and one or more independent variables. The major issues in model building are finding an effective functional form of the relationship and selecting the independent variables to be included in the model. In Section 16.1 we establish the framework for model building by introducing the concept of a general linear model. Section 16.2, which provides the foundation for the more sophisticated computer-based procedures, introduces a general approach for determining when to add or delete independent variables. In Section 16.3 we consider a larger regression problem involving eight independent variables and 25 observations; this problem is used to illustrate the variable selection procedures presented in Section 16.4, including stepwise regression, the forward selection procedure, the backward elimination procedure and best-subsets regression.



## STATISTICS IN PRACTICE

Selecting a university

To demonstrate an application of their new decision analysis methodology, Sutton and Green (2002) consider a school-leaver, Jenny, who is hoping to go to university. Before applying, however, she wishes to prioritize alternatives from the 97 choices available. She undertakes an analysis based on the nine classifications used in *The Times Good University Guide* (2000) to construct the 2000 League Table, a section of which is summarized below.

Jenny's priorities vary by each of these variables but on degree quality she is not so impressed by how students perform at university overall so much as the amount of 'gain' that takes place for a given intake standard. Having noted that degree quality (Deg) is related to entry standards, she therefore estimated the effect of this using a polynomial regression analysis and then removed it to create a new variable 'degree quality gain' which was used instead of Deg. The relevant equation was as follows:

| University     | T  | R  | As | St | L  | Fac | Deg | Des | Com |
|----------------|----|----|----|----|----|-----|-----|-----|-----|
| Aberdeen       | 86 | 66 | 72 | 53 | 70 | 78  | 76  | 95  | 86  |
| Abertay Dundee | 76 | 23 | 30 | 50 | 72 | 62  | 59  | 86  | 90  |
| Aberystwyth    | 81 | 61 | 60 | 38 | 68 | 78  | 66  | 88  | 93  |
| Anglia         | 76 | 20 | 42 | 47 | 61 | 67  | 66  | 86  | 80  |
| Aston          | 87 | 58 | 70 | 42 | 69 | 79  | 74  | 96  | 93  |
| Bangor         | 81 | 53 | 55 | 44 | 72 | 71  | 59  | 90  | 93  |
| Bath           | 82 | 81 | 82 | 50 | 84 | 94  | 80  | 93  | 96  |
| Birmingham     | 89 | 73 | 82 | 57 | 68 | 79  | 78  | 94  | 94  |
| Bournemouth    | 63 | 19 | 43 | 50 | 62 | 62  | 52  | 89  | 87  |
| Bradford       | 68 | 67 | 58 | 40 | 67 | 84  | 55  | 90  | 89  |

where T denotes teaching quality, R, research assessment, As, 'A' level examination points (required for entry), St, student-staff ratio, L, library and computing facilities spending and Fac, student facilities spending.

## 16.1 GENERAL LINEAR MODEL

Suppose we collected data for one dependent variable  $Y$  and  $k$  independent variables  $X_1, X_2, \dots, X_k$ . Our objective is to use these data to develop an estimated regression equation that provides the best relationship between the dependent and independent variables. As a general framework for developing more complex relationships among the independent variables we introduce the concept of the **general linear model** involving  $p$  independent variables.

### General linear model

$$Y = \beta_0 + \beta_1 z_1 + \beta_2 z_2 + \dots + \beta_p z_p + \varepsilon \quad (16.1)$$

In equation (16.1), each of the independent variables  $Z_j$  (where  $j = 1, 2, \dots, p$ ) is a function of  $X_1, X_2, \dots, X_k$  (the variables for which data are collected). In some cases, each  $Z_j$  may be a function of only one  $X$  variable. The simplest case is when we collect data for just one variable  $X_1$  and want to estimate  $Y$  by using a straight-line relationship. In this case  $Z_1 = X_1$  and equation (16.1) becomes:

$$Y = \beta_0 + \beta_1 x_1 + \varepsilon \quad (16.2)$$



King's College, The University of Aberdeen

Degree quality gain,

$$DQG = \text{Deg} - 0.0047\text{As}^2 - 0.0088\text{As} - 43.8$$

Similarly she developed new variables in place of Des (subsequent employment), and Com (course completion), 'removing' the effects of entry standards from Com, and degree quality from Des, as follows:

$$\begin{aligned} \text{Destination gain, DG} &= \text{Des} - 0.246\text{Deg} - 74.1 \\ \text{Completion gain, CG} &= \text{Com} - 0.292\text{As} - 69.3 \end{aligned}$$

These decision variables were then later combined with the other original variables, to create the function:

$$\text{Relative Value} = 0.00218T + 0.00182/(St + DQG + L + Fac) + 0.0011(R + As + CG + DG)$$

which could then be used to represent Jenny's distinctive outlook.

Sources: Sutton, P.P. and Green, R.H. (2002) 'A data envelope approach to decision analysis'. *J. Opl. Res. Soc.* 53: 1215–1224. *The Times, the Good University Guide 14 April 2000*

Equation (16.2) is the simple linear regression model introduced in Chapter 14 with the exception that the independent variable is labelled  $X_1$  instead of  $X$ . In the statistical modelling literature, this model is called a *simple first-order model with one predictor variable*.

## Modelling curvilinear relationships

More complex types of relationships can be modelled with equation (16.1). To illustrate, let us consider the problem facing Reynard Ltd, a manufacturer of industrial scales and laboratory equipment. Managers at Reynard want to investigate the relationship between length of employment of their salespeople and the number of electronic laboratory scales sold. Table 16.1 gives the number of scales sold by 15 randomly selected salespeople for the most recent sales period and the number of months each salesperson has been employed by the firm. Figure 16.1 is the scatter diagram for these data. The scatter diagram indicates a possible curvilinear relationship between the length of time employed and the number of units sold. Before considering how to develop a curvilinear relationship for Reynard, let us consider the MINITAB output in Figure 16.2 corresponding to a simple first-order model; the estimated regression is:

$$\text{Sales} = 111 + 2.38 \text{ Months}$$

where:

Sales = number of electronic laboratory scales sold

Months = the number of months the salesperson has been employed

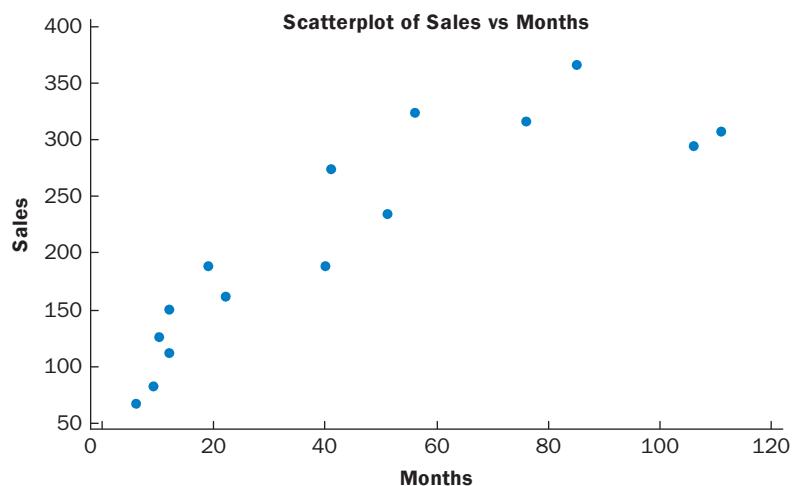
Figure 16.3 is the corresponding standardized residual plot. Although the computer output shows that the relationship is significant ( $p$ -value = 0.000) and that a linear relationship explains a high percentage

**TABLE 16.1** Data for the Reynard example

| Months employed | Scales sold |
|-----------------|-------------|
| 41              | 275         |
| 106             | 296         |
| 76              | 317         |
| 104             | 376         |
| 22              | 162         |
| 12              | 150         |
| 85              | 367         |
| 111             | 308         |
| 40              | 189         |
| 51              | 235         |
| 9               | 83          |
| 12              | 112         |
| 6               | 67          |
| 56              | 325         |
| 19              | 189         |

**FIGURE 16.1**

Scatter diagram for the Reynard example



of the variability in sales ( $R^2 = 78.1$  per cent), the standardized residual plot suggests that a curvilinear relationship is needed.

To account for the curvilinear relationship, we set  $Z_1 = X_1$  and  $Z_2 = X_1^2$  in equation (16.1) to obtain the model:

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_1^2 + \epsilon \quad (16.3)$$

This model is called a *second-order model with one predictor variable*. To develop an estimated regression equation corresponding to this second-order model, the statistical software package we are using needs the original data in Table 16.1, as well as the data corresponding to adding a second independent variable that is the square of the number of months the employee has been with the firm. In Figure 16.4 we show the MINITAB output corresponding to the second-order model; the estimated regression equation is:

$$\text{Sales} = 45.3 + 6.34 \text{ Months} - 0.0345 \text{ MonthsSq}$$

where:

MonthsSq = the square of the number of months the salesperson has been employed

**FIGURE 16.2**

MINITAB output for the Reynard example: first-order model

### Regression Analysis: Sales versus Months

The regression equation is  
Sales = 111 + 2.38 Months

| Predictor | Coef   | SE Coef | T    | P     |
|-----------|--------|---------|------|-------|
| Constant  | 111.23 | 21.63   | 5.14 | 0.000 |
| Months    | 2.3768 | 0.3489  | 6.81 | 0.000 |

S = 49.5158 R-Sq = 78.1% R-Sq(adj) = 76.4%

#### Analysis of Variance

| Source         | DF | SS     | MS     | F     | P     |
|----------------|----|--------|--------|-------|-------|
| Regression     | 1  | 113783 | 113783 | 46.41 | 0.000 |
| Residual Error | 13 | 31874  | 2452   |       |       |
| Total          | 14 | 145657 |        |       |       |

**FIGURE 16.3**

Standardized residual plot for the Reynard example: first-order model

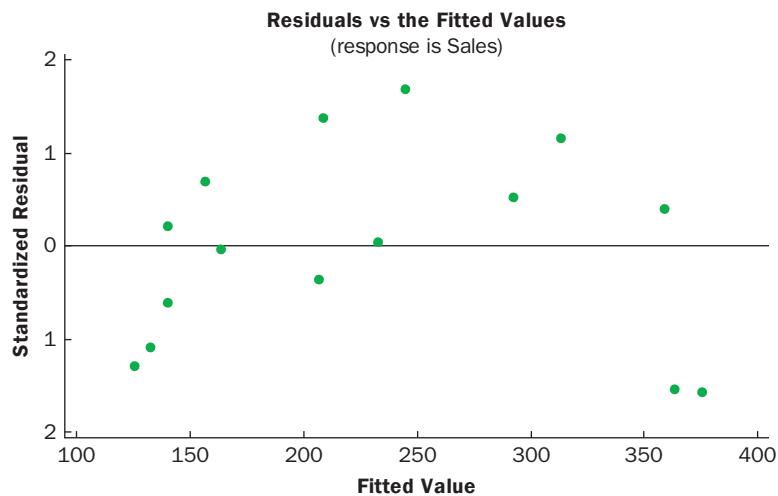


Figure 16.5 is the corresponding standardized residual plot. It shows that the previous curvilinear pattern has been removed. At the 0.05 level of significance, the computer output shows that the overall model is significant (*p*-value for the *F* test is 0.000); note also that the *p*-value corresponding to the *t*-ratio for MonthsSq (*p*-value = 0.002) is less than 0.05, and hence we can conclude that adding MonthsSq to the model involving Months is significant. With an *R*-sq(adj) value of 88.6 per cent, we should be pleased with the fit provided by this estimated regression equation. More important, however, is seeing how easy it is to handle curvilinear relationships in regression analysis.

Clearly, many types of relationships can be modelled by using equation (16.1). The regression techniques with which we have been working are definitely not limited to linear, or straight-line, relationships. In multiple regression analysis the word *linear* in the term ‘general linear model’ refers only to the fact that  $\beta_0, \beta_1, \dots, \beta_p$  all have exponents of 1; it does not imply that the relationship between *Y* and the *X*s is linear. Indeed, in this section we have seen one example of how equation (16.1) can be used to model a curvilinear relationship.

**FIGURE 16.4**

MINITAB output for the Reynard example: second-order model

### Regression Analysis: Sales versus Months, MonthsSq

The regression equation is

$$\text{Sales} = 45.3 + 6.34 \text{ Months} - 0.0345 \text{ MonthsSq}$$

| Predictor | Coef      | SE Coef  | T     | P     |
|-----------|-----------|----------|-------|-------|
| Constant  | 45.35     | 22.77    | 1.99  | 0.070 |
| Months    | 6.345     | 1.058    | 6.00  | 0.000 |
| MonthsSq  | -0.034486 | 0.008948 | -3.85 | 0.002 |

$$S = 34.4528 \quad R\text{-Sq} = 90.2\% \quad R\text{-Sq(adj)} = 88.6\%$$

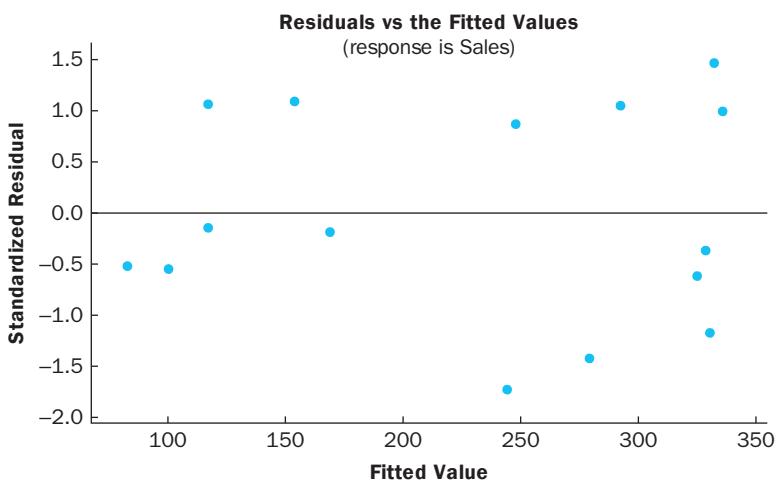
#### Analysis of Variance

| Source         | DF | SS     | MS    | F     | P     |
|----------------|----|--------|-------|-------|-------|
| Regression     | 2  | 131413 | 65707 | 55.36 | 0.000 |
| Residual Error | 12 | 14244  | 1187  |       |       |
| Total          | 14 | 145657 |       |       |       |

| Source   | DF | Seq SS |
|----------|----|--------|
| Months   | 1  | 113783 |
| MonthsSq | 1  | 17630  |

**FIGURE 16.5**

Standardized residual plot for the Reynard example: second-order model



## Interaction

To provide an illustration of **interaction** and what it means, let us review the regression study conducted by Veneto Care for one of its new shampoo products. Two factors believed to have the most influence on sales are unit selling price and advertising expenditure.

To investigate the effects of these two variables on sales, prices of €2.00, €2.50 and €3.00 were paired with advertising expenditures of €50 000 and €100 000 in 24 test markets. See Figure 16.6.

The unit sales (in thousands) that were observed are reported in Table 16.2. See Figure 16.6.

Table 16.3 is a summary of these data. Note that the mean sales corresponding to a price of €2.00 and an advertising expenditure of €50 000 is 461 000, and the mean sales corresponding to a price of €2.00 and an advertising expenditure of €100 000 is 808 000. Hence, with price held constant at €2.00, the difference in mean sales between advertising expenditures of €50 000 and €100 000 is 808 000 – 461 000 = 347 000 units.

**TABLE 16.2** Data for the Veneto Care example

| Price | Advertising expenditure<br>(€000s) | Sales<br>(000s) | Price | Advertising expenditure<br>(€000s) | Sales<br>(000s) |
|-------|------------------------------------|-----------------|-------|------------------------------------|-----------------|
| €2.00 | 50                                 | 478             | €2.00 | 100                                | 810             |
| €2.50 | 50                                 | 373             | €2.50 | 100                                | 653             |
| €3.00 | 50                                 | 335             | €3.00 | 100                                | 345             |
| €2.00 | 50                                 | 473             | €2.00 | 100                                | 832             |
| €2.50 | 50                                 | 358             | €2.50 | 100                                | 641             |
| €3.00 | 50                                 | 329             | €3.00 | 100                                | 372             |
| €2.00 | 50                                 | 456             | €2.00 | 100                                | 800             |
| €2.50 | 50                                 | 360             | €2.50 | 100                                | 620             |
| €3.00 | 50                                 | 322             | €3.00 | 100                                | 390             |
| €2.00 | 50                                 | 437             | €2.00 | 100                                | 790             |
| €2.50 | 50                                 | 365             | €2.50 | 100                                | 670             |
| €3.00 | 50                                 | 342             | €3.00 | 100                                | 393             |

**TABLE 16.3** Mean unit sales (1000s) for the Veneto Care example

|   | Price    |       |       |
|---|----------|-------|-------|
|   | €2.00    | €2.50 | €3.00 |
| Advertising Expenditure   | €50 000  | 461   | 364   |
|   | €100 000 | 808   | 646   |
| Mean sales of 808 000 units when price = €2.00 and advertising expenditure = €100 000 |          |       |       |

When the price of the product is €2.50, the difference in mean sales is  $646\,000 - 364\,000 = 282\,000$  units. Finally, when the price is €3.00, the difference in mean sales is  $375\,000 - 332\,000 = 43\,000$  units. Clearly, the difference in mean sales between advertising expenditures of €50 000 and €100 000 depends on the price of the product. In other words, at higher selling prices, the effect of increased advertising expenditure diminishes. These observations provide evidence of interaction between the price and advertising expenditure variables (Figure 16.6).

When interaction between two variables is present, we cannot study the effect of one variable on the response  $Y$  independently of the other variable. In other words, meaningful conclusions can be developed only if we consider the joint effect that both variables have on the response.

To account for the effect of interaction, we will use the following regression model.

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2 + \varepsilon \quad (16.4)$$

where:

$$\begin{aligned} Y &= \text{unit sales (000s)} \\ X_1 &= \text{price (\euro)} \\ X_2 &= \text{advertising expenditure (\euro000s)} \end{aligned}$$

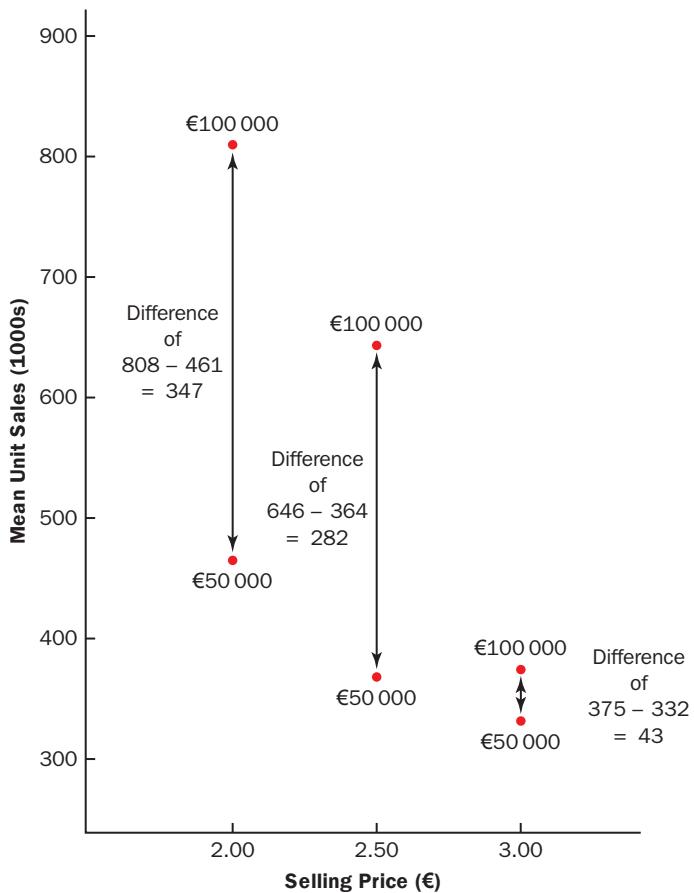
Note that equation (16.4) reflects Veneto's belief that the number of units sold depends linearly on selling price and advertising expenditure (accounted for by the  $\beta_1 x_1$  and  $\beta_2 x_2$  terms), and that there is interaction between the two variables (accounted for by the  $\beta_3 x_1 x_2$  term).

To develop an estimated regression equation, a general linear model involving three independent variables ( $Z_1$ ,  $Z_2$  and  $Z_3$ ) was used.

$$Y = \beta_0 + \beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3 + \varepsilon \quad (16.5)$$

**FIGURE 16.6**

Mean unit sales (1000s) as a function of selling price and advertising expenditure



where:

$$z_1 = x_1$$

$$z_2 = x_2$$

$$z_3 = x_1 x_2$$

Figure 16.7 is the MINITAB output corresponding to the interaction model for the Veneto Care example. The resulting estimated regression equation is:

$$\text{Sales} = -276 + 175 \text{ Price} + 19.7 \text{ AdvExp} - 6.08 \text{ PriceAdv}$$

where:

Sales = unit sales (000s)

Price = price of the product (€)

AdvExp = advertising expenditure (€000s)

PriceAdv = interaction term (Price times AdvExp)

Because the model is significant ( $p$ -value for the  $F$  test is 0.000) and the  $p$ -value corresponding to the  $t$  test for PriceAdv is 0.000, we conclude that interaction is significant given the linear effect of the price of the product and the advertising expenditure. Thus, the regression results show that the effect of advertising expenditure on sales depends on the price.

### Transformations involving the dependent variable

In showing how the general linear model can be used to model a variety of possible relationships between the independent variables and the dependent variable, we have focused attention on transformations involving one or more of the independent variables. Often it is worthwhile to consider transformations involving the dependent variable  $Y$ . As an illustration of when we might want to transform the dependent variable, consider the data in Table 16.4, which shows the kilometres-per-litre ratings and weights (kg) for 12 cars.

**FIGURE 16.7**

MINITAB output for the Veneto Care example

### Regression Analysis: Sales versus Price, AdvExpen, PriceAdv

The regression equation is

$$\text{Sales} = -276 + 175 \text{ Price} + 19.7 \text{ AdvExpen} - 6.08 \text{ PriceAdv}$$

| Predictor | Coef    | SE Coef | T      | P     |
|-----------|---------|---------|--------|-------|
| Constant  | -275.8  | 112.8   | -2.44  | 0.024 |
| Price     | 175.00  | 44.55   | 3.93   | 0.001 |
| AdvExpen  | 19.680  | 1.427   | 13.79  | 0.000 |
| PriceAdv  | -6.0800 | 0.5635  | -10.79 | 0.000 |

$$S = 28.1739 \quad R-\text{Sq} = 97.8\% \quad R-\text{Sq}(\text{adj}) = 97.5\%$$

#### Analysis of Variance

| Source         | DF | SS     | MS     | F      | P     |
|----------------|----|--------|--------|--------|-------|
| Regression     | 3  | 709316 | 236439 | 297.87 | 0.000 |
| Residual Error | 20 | 15875  | 794    |        |       |
| Total          | 23 | 725191 |        |        |       |

| Source   | DF | Seq SS |
|----------|----|--------|
| Price    | 1  | 315844 |
| AdvExpen | 1  | 301056 |
| PriceAdv | 1  | 92416  |

#### Unusual Observations

| Obs | Price | Sales  | Fit    | SE Fit | Residual | St Resid |
|-----|-------|--------|--------|--------|----------|----------|
| 23  | 2.50  | 670.00 | 609.67 | 8.13   | 60.33    | 2.24R    |

R denotes an observation with a large standardized residual.

**TABLE 16.4** Kilometres-per-litre ratings and weights for 12 cars

| Weight | Kilometres per litre |
|--------|----------------------|
| 1038   | 10.2                 |
| 958    | 10.3                 |
| 989    | 12.1                 |
| 1110   | 9.9                  |
| 919    | 11.8                 |
| 1226   | 9.3                  |
| 1205   | 8.5                  |
| 955    | 10.8                 |
| 1463   | 6.4                  |
| 1457   | 6.9                  |
| 1636   | 5.1                  |
| 1310   | 7.4                  |

The scatter diagram in Figure 16.8 indicates a negative linear relationship between these two variables. Therefore, we use a simple first-order model to relate the two variables. The MINITAB output is shown in Figure 16.9; the resulting estimated regression equation is:

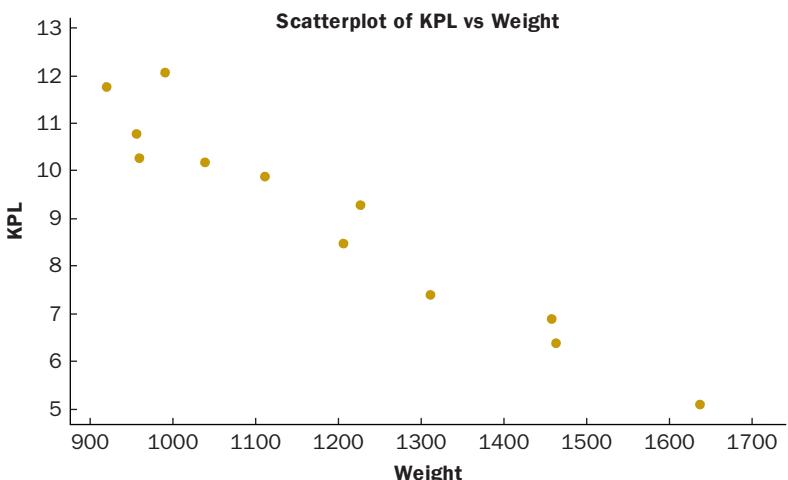
$$\text{KPL} = 19.8 - 0.00907 \text{ Weight}$$

where:

KPL = kilometres-per-litre rating  
Weight = weight of the car in kilograms

**FIGURE 16.8**

Scatter diagram for the  
kilometres-per-litre problem

**FIGURE 16.9**

MINITAB output for the  
kilometres-per-litre problem

### Regression Analysis: KPL versus Weight

The regression equation is  
 $KPL = 19.8 - 0.00907 \text{ Weight}$

| Predictor | Coef       | SE Coef   | T      | P     |
|-----------|------------|-----------|--------|-------|
| Constant  | 19.8381    | 0.9099    | 21.80  | 0.000 |
| Weight    | -0.0090675 | 0.0007519 | -12.06 | 0.000 |

S = 0.588710 R-Sq = 93.6% R-Sq(adj) = 92.9%

### Analysis of Variance

| Source         | DF | SS     | MS     | F      | P     |
|----------------|----|--------|--------|--------|-------|
| Regression     | 1  | 50.403 | 50.403 | 145.43 | 0.000 |
| Residual Error | 10 | 3.466  | 0.347  |        |       |
| Total          | 11 | 53.869 |        |        |       |

### Unusual Observations

| Obs | Weight | KPL    | Fit    | SE Fit | Residual | St Resid |
|-----|--------|--------|--------|--------|----------|----------|
| 3   | 989    | 12.100 | 10.870 | 0.227  | 1.230    | 2.26R    |

R denotes an observation with a large standardized residual.

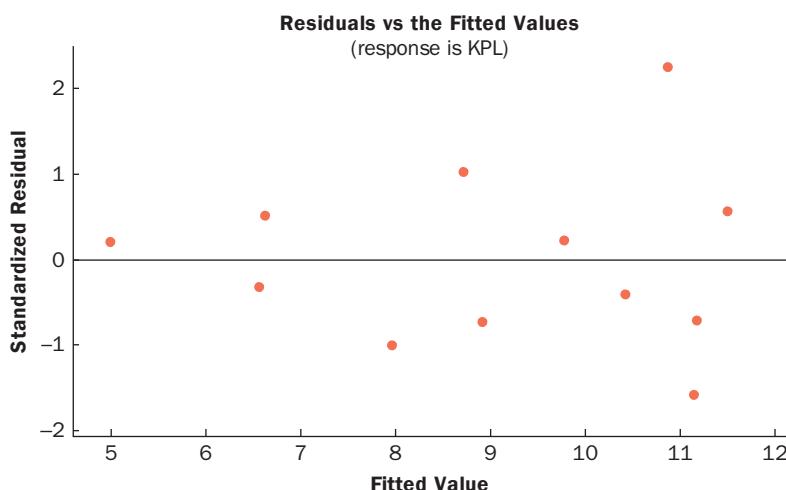
The model is significant ( $p$ -value for the  $F$  test is 0.000) and the fit is very good ( $R$ -sq = 93.6 per cent). However, we note in Figure 16.9 that observation 3 is identified as having a large standardized residual.

Figure 16.10 is the standardized residual plot corresponding to the first-order model. The pattern we observe does not look like the horizontal band we should expect to find if the assumptions about the error term are valid. Instead, the variability in the residuals appears to increase as the value of increases. In other words, we see the wedge-shaped pattern referred to in Chapters 14 and 15 as being indicative of a nonconstant variance. We are not justified in reaching any conclusions about the statistical significance of the resulting estimated regression equation when the underlying assumptions for the tests of significance do not appear to be satisfied.

Often the problem of non-constant variance can be corrected by transforming the dependent variable to a different scale.

**FIGURE 16.10**

Standardized residual plot for the kilometres-per-litre problem



For instance, if we work with the logarithm of the dependent variable instead of the original dependent variable, the effect will be to compress the values of the dependent variable and thus diminish the effects of non-constant variance.

Most statistical packages provide the ability to apply logarithmic transformations using either the base 10 (common logarithm) or the base  $e = 2.71828 \dots$  (natural logarithm). We applied a natural logarithmic transformation to the kilometres-per-litre data and developed the estimated regression equation relating weight to the natural logarithm of kilometres-per-litre. The regression results obtained by using the natural logarithm of kilometres-per-litre as the dependent variable, labelled LogeKPL in the output, are shown in Figure 16.11; Figure 16.12 is the corresponding standardized residual plot.

Looking at the residual plot in Figure 16.12, we see that the wedge-shaped pattern has now disappeared. Moreover, none of the observations are identified as having a large standardized residual. The model with the logarithm of kilometres per litre as the dependent variable is statistically significant and provides an excellent fit to the observed data. Hence, we would recommend using the estimated regression equation:

$$\text{Log}_e \text{ KPL} = 3.49 - 0.00110 \text{ Weight}$$

To estimate the kilometres-per-litre rating for an car that weighs 1500 kilograms, we first develop an estimate of the logarithm of the kilometres-per-litre rating.

$$\text{Log}_e \text{ KPL} = 3.49 - 0.00110 (1500) = 1.84$$

**FIGURE 16.11**

MINITAB output for the kilometres-per-litre problem: logarithmic transformation

### Regression Analysis: LogeKPL versus Weight

The regression equation is  
 $\text{LogeKPL} = 3.48 - 0.00110 \text{ Weight}$

| Predictor | Coef        | SE Coef    | T      | P     |
|-----------|-------------|------------|--------|-------|
| Constant  | 3.48115     | 0.09780    | 35.60  | 0.000 |
| Weight    | -0.00110033 | 0.00008082 | -13.62 | 0.000 |

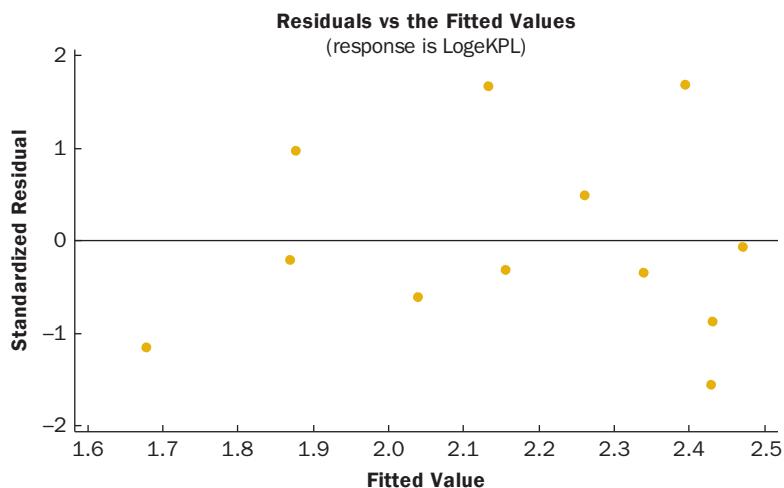
$$S = 0.0632759 \quad R-\text{Sq} = 94.9\% \quad R-\text{Sq}(\text{adj}) = 94.4\%$$

### Analysis of Variance

| Source         | DF | SS      | MS      | F      | P     |
|----------------|----|---------|---------|--------|-------|
| Regression     | 1  | 0.74221 | 0.74221 | 185.37 | 0.000 |
| Residual Error | 10 | 0.04004 | 0.00400 |        |       |
| Total          | 11 | 0.78225 |         |        |       |

**FIGURE 16.12**

Standardized residual plot for the kilometres-per-litre problem: logarithmic transformation



The kilometres-per-litre estimate is obtained by finding the number whose natural logarithm is 1.84. Using a calculator with an exponential function, or raising  $e$  to the power 1.84, we obtain 6.2 kilometres per litre.

Another approach to problems of non-constant variance is to use  $1/Y$  as the dependent variable instead of  $Y$ . This type of transformation is called a *reciprocal transformation*. For instance, if the dependent variable is measured in kilometres per litre, the reciprocal transformation would result in a new dependent variable whose units would be  $1/(\text{kilometres per litre})$  or litres per kilometre. In general, there is no way to determine whether a logarithmic transformation or a reciprocal transformation will perform better without actually trying each of them.

## Nonlinear models that are intrinsically linear

Models in which the parameters  $(\beta_0, \beta_1, \dots, \beta_p)$  have exponents other than 1 are called nonlinear models. The exponential model involves the following regression equation.

$$E(Y) = \beta_0 \beta_1^x \quad (16.6)$$

This model is appropriate when the dependent variable  $Y$  increases or decreases by a constant percentage, instead of by a fixed amount, as  $X$  increases.

As an example, suppose sales for a product  $Y$  are related to advertising expenditure  $X$  (in thousands of euros) according to the following exponential model.

$$E(Y) = 500(1.2)^x$$

Thus: for  $X = 1$ ,  $E(Y) = 500(1.2)^1 = 600$ ; for  $X = 2$ ,  $E(Y) = 500(1.2)^2 = 720$ ; and for  $X = 3$ ,  $E(Y) = 500(1.2)^3 = 864$ . Note that  $E(Y)$  is not increasing by a constant amount in this case, but by a constant percentage; the percentage increase is 20 per cent.

We can transform this nonlinear model to a linear model by taking the logarithm of both sides of equation (16.6).

$$\log E(Y) = \log \beta_0 + x \log \beta_1 \quad (16.7)$$

Now if we let  $y' = \log E(Y)$ ,  $\beta_0' = \log \beta_0$ , and  $\beta_1' = \log \beta_1$ , we can rewrite equation (16.7) as:

$$y' = \beta_0' + \beta_1' x \quad (16.8)$$

It is clear that the formulae for simple linear regression can now be used to develop estimates of  $\beta_0'$  and  $\beta_1'$ . Denoting the estimates as  $b_0'$  and  $b_1'$  leads to the following estimated regression equation.

$$y' = b_0' + b_1' x \quad (16.9)$$

To obtain predictions of the original dependent variable  $Y$  given a value of  $X$ , we would first substitute the value of  $X$  into equation (16.8) and compute  $\hat{y}$ . The antilog of  $\hat{y}$  would be the prediction of  $Y$ , or the expected value of  $Y$ .

Many nonlinear models cannot be transformed into an equivalent linear model. However, such models have had limited use in business and economic applications. Furthermore, the mathematical background needed for study of such models is beyond the scope of this text.

## EXERCISES

### Methods

- 1.** Consider the following data for two variables,  $X$  and  $Y$ .

|     |    |    |    |    |    |    |
|-----|----|----|----|----|----|----|
| $x$ | 22 | 24 | 26 | 30 | 35 | 40 |
| $y$ | 12 | 21 | 33 | 35 | 40 | 36 |

- a. Develop an estimated regression equation for the data of the form  $\hat{y} = b_0 + b_1x$ .
- b. Use the results from part (a) to test for a significant relationship between  $X$  and  $Y$ . Use  $\alpha = 0.05$ .
- c. Develop a scatter diagram for the data. Does the scatter diagram suggest an estimated regression equation of the form  $\hat{y} = b_0 + b_1x + b_2x^2$ ? Explain.
- d. Develop an estimated regression equation for the data of the form  $\hat{y} = b_0 + b_1x + b_2x^2$ .
- e. Refer to part (d). Is the relationship between  $X$ ,  $X^2$  and  $Y$  significant? Use  $\alpha = 0.05$ .
- f. Predict the value of  $Y$  when  $X = 25$ .

- 2.** Consider the following data for two variables,  $X$  and  $Y$ .

|     |    |    |    |    |    |
|-----|----|----|----|----|----|
| $x$ | 9  | 32 | 18 | 15 | 26 |
| $y$ | 10 | 20 | 21 | 16 | 22 |

- a. Develop an estimated regression equation for the data of the form  $\hat{y} = b_0 + b_1x$ . Comment on the adequacy of this equation for predicting  $Y$ .
- b. Develop an estimated regression equation for the data of the form  $\hat{y} = b_0 + b_1x + b_2x^2$ . Comment on the adequacy of this equation for predicting  $Y$ .
- c. Predict the value of  $Y$  when  $X = 20$ .

- 3.** Consider the following data for two variables,  $X$  and  $Y$ .

|     |   |   |   |   |   |   |   |   |    |
|-----|---|---|---|---|---|---|---|---|----|
| $x$ | 2 | 3 | 4 | 5 | 7 | 7 | 7 | 8 | 9  |
| $y$ | 4 | 5 | 4 | 6 | 4 | 6 | 9 | 5 | 11 |

- a. Does there appear to be a linear relationship between  $X$  and  $Y$ ? Explain.
- b. Develop the estimated regression equation relating  $X$  and  $Y$ .
- c. Plot the standardized residuals versus for the estimated regression equation developed in part (b). Do the model assumptions appear to be satisfied? Explain.
- d. Perform a logarithmic transformation on the dependent variable  $Y$ . Develop an estimated regression equation using the transformed dependent variable. Do the model assumptions appear to be satisfied by using the transformed dependent variable? Does a reciprocal transformation work better in this case? Explain.

### Applications

- 4.** The table below lists the total estimated numbers of AIDS cases, by year of diagnosis from 1999 to 2003 in the United States. (Source: US Dept of Health and Human Services, Centers for Disease Control and Prevention, HIV/AIDS Surveillance, 2003.)



COMPLETE  
SOLUTIONS



COMPLETE  
SOLUTIONS

| Year | AIDS cases |
|------|------------|
| 1999 | 41 356     |
| 2000 | 41 267     |
| 2001 | 40 833     |
| 2002 | 41 289     |
| 2003 | 43 171     |

- a. Plot the data, letting  $x = 0$  correspond to the year 1998,  
     Find a linear  $\hat{y} = b_0 + b_1x$  that models the data,  
 b. Plot the function on the graph with the data and determine how well the graph fits the data.
5. In working further with the problem of Exercise 4, statisticians suggested the use of the following curvilinear estimated regression equation.

$$\hat{y} = b_0 + b_1x + b_2x^2$$

- a. Use the data of Exercise 4 to determine estimated regression equation.  
 b. Use  $\alpha = 0.01$  to test for a significant relationship.
6. An international study of life expectancy by Ross (1994) covers variables:

|                |                                 |
|----------------|---------------------------------|
| LifeExp        | Life expectancy in years        |
| People.per.TV  | Average number of people per TV |
| LifeExp.Male   | Male life expectancy in years   |
| LifeExp.Female | Female life expectancy in years |

With data details as follows:

|             | LifeExp | People.per.TV | People.per.Dr | LifeExp.Male | LifeExp.Female |
|-------------|---------|---------------|---------------|--------------|----------------|
| Argentina   | 70.5    | 4             | 370           | 74           | 67             |
| Bangladesh  | 53.5    | 315           | 6 166         | 53           | 54             |
| Brazil      | 65      | 4             | 684           | 68           | 62             |
| Canada      | 76.5    | 1.7           | 449           | 80           | 73             |
| China       | 70      | 8             | 643           | 72           | 68             |
| Colombia    | 71      | 5.6           | 1 551         | 74           | 68             |
| Egypt       | 60.5    | 15            | 616           | 61           | 60             |
| Ethiopia    | 51.5    | 503           | 36 660        | 53           | 50             |
| France      | 78      | 2.6           | 403           | 82           | 74             |
| Germany     | 76      | 2.6           | 346           | 79           | 73             |
| India       | 57.5    | 44            | 2 471         | 58           | 57             |
| Indonesia   | 61      | 24            | 7 427         | 63           | 59             |
| Iran        | 64.5    | 23            | 2 992         | 65           | 64             |
| Italy       | 78.5    | 3.8           | 233           | 82           | 75             |
| Japan       | 79      | 1.8           | 609           | 82           | 76             |
| Kenya       | 61      | 96            | 7 615         | 63           | 59             |
| Korea.North | 70      | 90            | 370           | 73           | 67             |
| Korea.South | 70      | 4.9           | 1 066         | 73           | 67             |
| Mexico      | 72      | 6.6           | 600           | 76           | 68             |
| Morocco     | 64.5    | 21            | 4 873         | 66           | 63             |
| Burma       | 54.5    | 592           | 3 485         | 56           | 53             |
| Pakistan    | 56.5    | 73            | 2 364         | 57           | 56             |

|              | <i>LifeExp</i> | <i>People.per.TV</i> | <i>People.per.Dr</i> | <i>LifeExp.Male</i> | <i>LifeExp.Female</i> |
|--------------|----------------|----------------------|----------------------|---------------------|-----------------------|
| Peru         | 64.5           | 14                   | 1 016                | 67                  | 62                    |
| Philippines  | 64.5           | 8.8                  | 1 062                | 67                  | 62                    |
| Poland       | 73             | 3.9                  | 480                  | 77                  | 69                    |
| Romania      | 72             | 6                    | 559                  | 75                  | 69                    |
| Russia       | 69             | 3.2                  | 259                  | 74                  | 64                    |
| South.Africa | 64             | 11                   | 1 340                | 67                  | 61                    |
| Spain        | 78.5           | 2.6                  | 275                  | 82                  | 75                    |
| Sudan        | 53             | 23                   | 12 550               | 54                  | 52                    |
| Taiwan       | 75             | 3.2                  | 965                  | 78                  | 72                    |
| Tanzania     | 52.5           | NA                   | 25 229               | 55                  | 50                    |
| Thailand     | 68.5           | 11                   | 4 883                | 71                  | 66                    |
| Turkey       | 70             | 5                    | 1 189                | 72                  | 68                    |
| Ukraine      | 70.5           | 3                    | 226                  | 75                  | 66                    |
| UK           | 76             | 3                    | 611                  | 79                  | 73                    |
| USA          | 75.5           | 1.3                  | 404                  | 79                  | 72                    |
| Venezuela    | 74.5           | 5.6                  | 576                  | 78                  | 71                    |
| Vietnam      | 65             | 29                   | 3 096                | 67                  | 63                    |
| Zaire        | 54             | NA                   | 23 193               | 56                  | 52                    |

(Note that the average number of people per TV is not given for Tanzania and Zaire.)

- Develop scatter diagrams for these data, treating *LifeExp* as the dependent variable.
  - Does a simple linear model appear to be appropriate? Explain.
  - Estimate simple regression equations for the data accordingly. Which do you prefer and why?
7. To assess the reliability of computer media, *Choice* magazine ([www.choice.com.au](http://www.choice.com.au)) has obtained data by:

|              |   |
|--------------|---|
| price (AU\$) | Paid in April 2005                              |
| pack         | the number of disks in the pack                 |
| media        | one of CD (CD), DVD (DVD-R) or DVDRW (DVD+/-RW) |

with details as follows:

| <i>Price</i> | <i>Pack</i> | <i>Media</i> | <i>Price</i> | <i>Pack</i> | <i>Media</i> |
|--------------|-------------|--------------|--------------|-------------|--------------|
| 0.48         | 50          | CD           | 1.85         | 10          | DVD          |
| 0.60         | 25          | CD           | 0.72         | 25          | DVD          |
| 0.64         | 25          | CD           | 2.28         | 10          | DVD          |
| 0.50         | 50          | CD           | 2.34         | 5           | DVD          |
| 0.89         | 10          | CD           | 2.40         | 10          | DVD          |
| 0.89         | 10          | CD           | 1.49         | 5           | DVD          |
| 1.20         | 10          | CD           | 3.60         | 5           | DVDRW        |
| 1.30         | 10          | CD           | 5.00         | 10          | DVDRW        |
| 1.29         | 10          | CD           | 2.79         | 5           | DVDRW        |
| 0.50         | 10          | CD           | 2.79         | 10          | DVDRW        |
| 0.57         | 50          | DVD          | 4.37         | 5           | DVDRW        |
| 2.60         | 10          | DVD          | 1.50         | 10          | DVDRW        |
| 1.59         | 10          | DVD          | 2.50         | 5           | DVDRW        |
| 1.85         | 10          | DVD          | 3.90         | 10          | DVDRW        |

- Develop scatter diagrams for these data with pack and media as potential independent variables.
- Does a simple or multiple linear regression model appear to be appropriate?
- Develop an estimated regression equation for the data you believe will best explain the relationship between these variables.



LIFE  
EXPECTANCY



MEDIA

8. In Europe the number of Internet users varies widely from country to country. In 1999, 44.3 per cent of all Swedes used the Internet, while in France the audience was less than 10 per cent. The disparities are expected to persist even though Internet usage is expected to grow dramatically over the next several years. The following table shows the number of Internet users in 1999 and in 2011 for selected European countries. ([www.internetworldstats.com/top25.htm](http://www.internetworldstats.com/top25.htm))

|             | % Internet users |      |
|-------------|------------------|------|
|             | 1999             | 2011 |
| Austria     | 12.6             | 74.8 |
| Belgium     | 24.2             | 81.4 |
| Denmark     | 40.4             | 89.0 |
| Finland     | 40.9             | 88.6 |
| France      | 9.7              | 77.2 |
| Germany     | 15.0             | 82.7 |
| Ireland     | 12.1             | 66.8 |
| Netherlands | 18.6             | 89.5 |
| Norway      | 38.0             | 97.2 |
| Spain       | 7.4              | 65.6 |
| Sweden      | 44.3             | 92.9 |
| Switzerland | 28.1             | 84.2 |
| UK          | 23.6             | 84.5 |



INTERNET  
2011

- a. Develop a scatter diagram of the data using the 1999 Internet user percentage as the independent variable. Does a simple linear regression model appear to be appropriate? Discuss.
- b. Develop an estimated multiple regression equation with  $X =$  the number of 1999 Internet users and  $X^2$  as the two independent variables.
- c. Consider the nonlinear relationship shown by equation (16.6). Use logarithms to develop an estimated regression equation for this model.
- d. Do you prefer the estimated regression equation developed in part (b) or part (c)? Explain.

## 16.2 DETERMINING WHEN TO ADD OR DELETE VARIABLES

To illustrate the use of this  $F$  statistic, let us return to the Eurodistributor data introduced in Chapter 15. Recall that the managers were trying to develop a regression model to predict total daily travel time for trucks using two independent variables: distance travelled ( $X_1$ ) and number of deliveries ( $X_2$ ). With one model using only  $X_1$  as an independent variable the error sum of squares was found to be 8.029. For the second, however, using both  $X_1$  and  $X_2$ , the error sum of squares was 2.299. The question is, did the addition of the second independent variable  $X_2$  result in a significant reduction in the error sum of squares?

Using formula 16.14 with  $n = 10$ ,  $q = 1$  and  $p = 2$  it is easily shown the test statistic is:

$$F = \frac{\frac{8.029 - 2.299}{1}}{\frac{2.299}{7}} = 17.47$$

which is statistically significant since  $17.47 > F_{0.05}(1, 7) = 5.59$ .

## Use of *p*-values

Note also that the *p*-value associated with  $F(1, 7) = 17.47$  is 0.004. As this is less than  $\alpha = 0.05$  we can conclude once again that the addition of the second independent variable is statistically significant. In general *p*-values cannot be looked up directly from tables of the *F* distribution, but can be straightforwardly obtained using computer software packages, such as MINITAB, SPSS or EXCEL.

## General case

Consider the following multiple regression model involving  $q$  independent variables, where  $q < p$ .

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_q x_q + \varepsilon \quad (16.10)$$

If we add variables  $X_{q+1}, X_q, \dots, X_p$  to this model, we obtain a model involving  $p$  independent variables.

$$\begin{aligned} Y = & \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_q x_q \\ & + \beta_{q+1} x_{q+1} + \beta_{q+2} x_{q+2} + \cdots + \beta_p x_p + \varepsilon \end{aligned} \quad (16.11)$$

To test whether the addition of  $X_{q+1}, X_q, \dots, X_p$  is statistically significant, the null and alternative hypotheses can be stated as follows.

$$H_0: \beta_{q+1} = \beta_{q+2} = \cdots = \beta_p = 0$$

$H_1$ : One or more of the parameters is not equal to zero

The following *F* statistic provides the basis for testing whether the additional independent variables are statistically significant.

### **F** test statistic for adding or deleting $p-q$ variables

$$F = \frac{\frac{\text{SSE}(x_1, x_2, \dots, x_q) - \text{SSE}(x_1, x_2, \dots, x_q, x_{q+1}, \dots, x_p)}{p - q}}{\frac{\text{SSE}(x_1, x_2, \dots, x_q, x_{q+1}, \dots, x_p)}{n - p - 1}} \quad (16.12)$$

This computed *F* value is then compared with  $F_{\alpha}$ , the table value with  $p - q$  numerator degrees of freedom and  $n - p - 1$  denominator degrees of freedom. If  $F > F$  we reject  $H_0$  and conclude that the set of additional independent variables is statistically significant.

Many students find equation (16.12) somewhat complex. To provide a simpler description of this *F* ratio, we can refer to the model with the smaller number of independent variables as the reduced model and the model with the larger number of independent variables as the full model. If we let SSE (reduced) denote the error sum of squares for the reduced model and SSE (full) denote the error sum of squares for the full model, we can write the numerator of (16.12) as:

$$\frac{\text{SSE}(\text{reduced}) - \text{SSE}(\text{full})}{\text{number of extra terms}} \quad (16.13)$$

Note that ‘number of extra terms’ denotes the difference between the number of independent variables in the full model and the number of independent variables in the reduced model. The denominator of equation (16.12) is the error sum of squares for the full model divided by the corresponding degrees of freedom; in other words, the denominator is the mean square error for the full model. Denoting the mean square error for the full model as MSE(full) enables us to write it as:

$$F = \frac{\frac{\text{SSE}(\text{reduced}) - \text{SSE}(\text{full})}{\text{number of extra terms}}}{\text{MSE}(\text{full})} \quad (16.14)$$

## EXERCISES

### Methods

- 9.** In a regression analysis involving 27 observations, the following estimated regression equation was developed.

$$\hat{y} = 25.2 + 5.5x_1$$

For this estimated regression equation  $SST = 1550$  and  $SSE = 520$ .

- a. At  $\alpha = 0.05$ , test whether  $X_1$  is significant.

Suppose that variables  $X_2$  and  $X_3$  are added to the model and the following regression equation is obtained.

$$\hat{y} = 16.3 + 2.3x_1 + 12.1x_2 - 5.8x_3$$

For this estimated regression equation  $SST = 1550$  and  $SSE = 100$ .

- b. Use an  $F$  test and a 0.05 level of significance to determine whether  $X_2$  and  $X_3$  contribute significantly to the model.

- 10.** In a regression analysis involving 30 observations, the following estimated regression equation was obtained.

$$\hat{y} = 17.6 + 3.8x_1 - 2.3x_2 + 7.6x_3 + 2.7x_4$$

For this estimated regression equation  $SST = 1805$  and  $SSR = 1760$ .

- a. At  $\alpha = 0.05$ , test the significance of the relationship among the variables.

Suppose variables  $X_1$  and  $X_4$  are dropped from the model and the following estimated regression equation is obtained.

$$\hat{y} = 11.1 - 3.6x_2 + 8.1x_3$$

For this model  $SST = 1805$  and  $SSR = 1705$ .

- a. Compute  $SSE(x_1, x_2, x_3, x_4)$ .

- b. Compute  $SSE(x_2, x_3)$ .

- c. Use an  $F$  test and a 0.05 level of significance to determine whether  $X_1$  and  $X_4$  contribute significantly to the model.

### Applications

- 11.** In an experiment involving measurements of heat production (calories) at various body masses (kgs) and work levels (calories/hour) on a stationary bike, the following results were obtained:

| Body mass ( $M$ ) | Work level ( $W$ ) | Heat production ( $H$ ) |
|-------------------|--------------------|-------------------------|
| 43.7              | 19                 | 177                     |
| 43.7              | 43                 | 279                     |
| 43.7              | 56                 | 346                     |
| 54.6              | 13                 | 160                     |
| 54.6              | 19                 | 193                     |
| 54.6              | 56                 | 335                     |
| 55.7              | 13                 | 169                     |
| 55.7              | 26                 | 212                     |
| 55.7              | 34.5               | 244                     |
| 55.7              | 43                 | 285                     |



COMPUTER  
SOLUTIONS



COMPUTER  
SOLUTIONS

| <i>Body mass (M)</i> | <i>Work level (W)</i> | <i>Heat production (H)</i> |
|----------------------|-----------------------|----------------------------|
| 58.8                 | 13                    | 181                        |
| 58.8                 | 43                    | 298                        |
| 60.5                 | 19                    | 212                        |
| 60.5                 | 43                    | 317                        |
| 60.5                 | 56                    | 347                        |
| 61.9                 | 13                    | 186                        |
| 61.9                 | 19                    | 216                        |
| 61.9                 | 34.5                  | 265                        |
| 61.9                 | 43                    | 306                        |
| 61.9                 | 56                    | 348                        |
| 66.7                 | 13                    | 209                        |
| 66.7                 | 43                    | 324                        |
| 66.7                 | 56                    | 352                        |



MUSCLE

- a. Develop an estimated regression equation that can be used to predict heat production for a given body mass and work level.
- b. Consider adding an independent variable to the model developed in part (a) for the interaction between body mass and work level. Develop an estimated regression equation using these three independent variables.
- c. At a 0.05 level of significance, test to see whether the addition of the interaction term contributes significantly to the estimated regression equation developed in part (a).
12. Failure data obtained in the course of the development of a silver-zinc battery for a NASA programme were analyzed by Sidik, Leibecki and Bozek in 1980. Relevant variables were as follows:

|                      |  |
|----------------------|--|
| <i>x<sub>1</sub></i> | charge rate (amps):                            |
| <i>x<sub>2</sub></i> | discharge rate (amps)                          |
| <i>x<sub>3</sub></i> | depth of discharge (% of rated ampere – hours) |
| <i>x<sub>4</sub></i> | temperature (°C)                               |
| <i>x<sub>5</sub></i> | end of charge voltage (volts)                  |
| <i>y</i>             | cycles to failure                              |

Adopting  $\ln(Y)$  as the response variable, a number of regression models were estimated for the data using MINITAB:

#### Regression Analysis: $\ln(y)$ versus $x_1, x_2, x_3, x_4, x_5$

The regression equation is:

$$\ln(y) = 63.7 - 0.459 x_1 - 0.327 x_2 - 0.0111 x_3 + 0.116 x_4 + 33.8 x_5$$

| Predictor            | Coef     | SE Coef | T     | P     | VIF |
|----------------------|----------|---------|-------|-------|-----|
| Constant             | -63.68   | 51.18   | -1.24 | 0.234 |     |
| <i>x<sub>1</sub></i> | -0.4593  | 0.5493  | -0.84 | 0.417 | 1.1 |
| <i>x<sub>2</sub></i> | -0.3267  | 0.1761  | -1.85 | 0.085 | 1.0 |
| <i>x<sub>3</sub></i> | -0.01113 | 0.01699 | -0.66 | 0.523 | 1.1 |
| <i>x<sub>4</sub></i> | 0.11577  | 0.02499 | 4.63  | 0.000 | 1.0 |
| <i>x<sub>5</sub></i> | 33.81    | 25.59   | 1.32  | 0.208 | 1.0 |

$$S = 1.070 \quad R-Sq = 66.3\% \quad R-Sq(adj) = 54.3\%$$

**Analysis of Variance**

| Source         | DF | SS     | MS    | F    | P     |
|----------------|----|--------|-------|------|-------|
| Regression     | 5  | 31.578 | 6.316 | 5.52 | 0.005 |
| Residual Error | 14 | 16.032 | 1.145 |      |       |
| Total          | 19 | 47.610 |       |      |       |
| Source         | DF | Seq SS |       |      |       |
| x1             | 1  | 1.464  |       |      |       |
| x2             | 1  | 4.512  |       |      |       |
| x3             | 1  | 0.291  |       |      |       |
| x4             | 1  | 23.311 |       |      |       |
| x5             | 1  | 1.999  |       |      |       |

**Unusual Observations**

| Obs | x1   | Iny   | Fit   | StDev Fit | Residual | St Resid |
|-----|------|-------|-------|-----------|----------|----------|
| 1   | 0.38 | 4.615 | 6.708 | 0.651     | -2.093   | -2.46R   |

R denotes an observation with a large standardized residual Durbin–Watson statistic = 1.72

**Regression Analysis: Iny versus x4**

The regression equation is:

$$\text{Iny} = 1.78 + 0.114 \text{ x4}$$

| Predictor | Coef    | SE Coef      | T    | P                 |
|-----------|---------|--------------|------|-------------------|
| Constant  | 1.7777  | 0.5660       | 3.14 | 0.006             |
| x4        | 0.11395 | 0.02597      | 4.39 | 0.000             |
| S = 1.130 |         | R-Sq = 51.7% |      | R-Sq(adj) = 49.0% |

**Analysis of Variance**

| Source         | DF | SS     | MS     | F     | P     |
|----------------|----|--------|--------|-------|-------|
| Regression     | 1  | 24.607 | 24.607 | 19.26 | 0.000 |
| Residual Error | 18 | 23.002 | 1.278  |       |       |
| Total          | 19 | 47.610 |        |       |       |

**Unusual Observations**

| Ob | x4   | Iny   | Fit   | StDev Fit | Residual | St Resid |
|----|------|-------|-------|-----------|----------|----------|
| 12 | 10.0 | 0.693 | 2.917 | 0.353     | -2.224   | 97       |

R denotes an observation with a large standardized residual.

- Explain this computer output, carrying out any additional tests you think necessary or appropriate.
- Is the first model significantly better than the second?
- Which model do you prefer and why?

13. A section of MINITAB output from an analysis of data relating to truck exhaust emissions under different atmospheric conditions (Hare and Bradow, 1977) is as follows:

### Regression Analysis: nox versus humi, temp, HT

The regression equation is  
 $\text{nox} = 1.61 - 0.014572 \text{ humi} - 0.006806 \text{ temp} + 0.00014985 \text{ HT}$

| Predictor | Coef       | SE Coef    | T     | P     |
|-----------|------------|------------|-------|-------|
| Constant  | 1.6104     | 0.2287     | 7.04  | 0.000 |
| humi      | -0.014572  | 0.003091   | -4.71 | 0.000 |
| temp      | -0.006806  | 0.002889   | -2.36 | 0.023 |
| HT        | 0.00014985 | 0.00003733 | 4.01  | 0.000 |

S = 0.0595096 R-Sq = 71.5% R-Sq(adj) = 69.4%

### Analysis of Variance

| Source         | DF | SS      | MS      | F     | P     |
|----------------|----|---------|---------|-------|-------|
| Regression     | 3  | 0.35544 | 0.11848 | 33.46 | 0.000 |
| Residual Error | 40 | 0.14166 | 0.00354 |       |       |
| Total          | 43 | 0.49710 |         |       |       |

| Source | DF | Seq SS  |
|--------|----|---------|
| humi   | 1  | 0.28446 |
| temp   | 1  | 0.01392 |
| HT     | 1  | 0.05706 |

### Unusual Observations

| Obs | humi | nox     | Fit     | SE Fit  | Residual | St Resid |
|-----|------|---------|---------|---------|----------|----------|
| 6   | 13   | 1.11000 | 1.09407 | 0.03316 | 0.01593  | 0.32 X   |
| 14  | 11   | 1.10000 | 0.99555 | 0.03105 | 0.10445  | 2.06R    |

R denotes an observation with a large standardized residual.  
X denotes an observation whose X value gives it large leverage.

Durbin-Watson statistic = 1.63335

Variables used in this analysis are defined as follows:

|      |   |
|------|---|
| nox  | Nitrous oxides, NO and NO <sub>2</sub> , (grams/km) |
| humi | Humidity (grains H <sub>2</sub> O/lbm dry air)      |
| temp | Temperature (°F)                                    |
| HT   | humi × temp   |

- Provide a descriptive summary of this information, carrying out any further calculations or statistical tests you think relevant or necessary.
- It has been argued that the inclusion of quadratic terms

$$\text{HH} = \text{humi} \times \text{humi}$$

$$\text{TT} = \text{temp} \times \text{temp}$$

on the right-hand side of the model will lead to a significantly improved R-square outcome. Details of the revised analysis are shown below. Is the claim justified?



EMISSIONS

### Regression Analysis: nox versus humi, temp, HT, HH, TT

The regression equation is  
 $\text{nox} = 2.69 - 0.0102 \text{ humi} - 0.0371 \text{ temp} + 0.000057 \text{ HT} + 0.000022 \text{ HH}$   
 $+ 0.000222 \text{ TT}$

| Predictor | Coef       | SE Coef    | T     | P     |
|-----------|------------|------------|-------|-------|
| Constant  | 2.685      | 1.306      | 2.06  | 0.047 |
| humi      | -0.010167  | 0.003015   | -3.37 | 0.002 |
| temp      | -0.03714   | 0.03414    | -1.09 | 0.284 |
| HT        | 0.00005662 | 0.00004073 | 1.39  | 0.173 |
| HH        | 0.00002209 | 0.00000592 | 3.73  | 0.001 |
| TT        | 0.0002221  | 0.0002224  | 1.00  | 0.324 |

S = 0.0515260 R-Sq = 79.7% R-Sq(adj) = 77.0%

#### Analysis of Variance

| Source         | DF | SS       | MS       | F     | P     |
|----------------|----|----------|----------|-------|-------|
| Regression     | 5  | 0.396213 | 0.079243 | 29.85 | 0.000 |
| Residual Error | 38 | 0.100887 | 0.002655 |       |       |
| Total          | 43 | 0.497100 |          |       |       |

| Source | DF | Seq SS   |
|--------|----|----------|
| humi   | 1  | 0.284462 |
| temp   | 1  | 0.013924 |
| HT     | 1  | 0.057058 |
| HH     | 1  | 0.038121 |
| TT     | 1  | 0.002648 |

#### Unusual Observations

| Obs | humi | nox     | Fit     | SE Fit  | Residual | St Resid |
|-----|------|---------|---------|---------|----------|----------|
| 5   | 10   | 0.99000 | 1.08738 | 0.02324 | -0.09738 | -2.12R   |
| 14  | 11   | 1.10000 | 1.08224 | 0.03654 | 0.01776  | 0.49 X   |
| 40  | 139  | 0.70000 | 0.82314 | 0.01759 | -0.12314 | -2.54R   |

R denotes an observation with a large standardized residual.

X denotes an observation whose X value gives it large leverage.

Durbin-Watson statistic = 1.77873

## 16.3 ANALYSIS OF A LARGER PROBLEM

In introducing multiple regression analysis, we used the Eurodistributor example extensively. The small size of this problem was an advantage in exploring introductory concepts, but would make it difficult to illustrate some of the variable selection issues involved in model building. To provide an illustration of the variable selection procedures discussed in the next section, we introduce a data set consisting of 25 observations on eight independent variables. Permission to use these data was provided by Dr David W. Cravens of the Department of Marketing at Texas Christian University. Consequently, we refer to the data set as the Cravens data.\*

\*For details see David W. Cravens, Robert B. Woodruff and Joe C. Stamper, 'Analytical Approach for Evaluating Sales Territory Performance', *Journal of Marketing*, 36 (January 1972): 31-37. Copyright © 1972 American Marketing Association.

**TABLE 16.5** Cravens data

| Sales   | Time   | Poten   | AdvExp  | Share | Change | Accounts | Work  | Rating |
|---------|--------|---------|---------|-------|--------|----------|-------|--------|
| 3669.88 | 43.10  | 74065.1 | 4582.9  | 2.51  | 0.34   | 74.86    | 15.05 | 4.9    |
| 3473.95 | 108.13 | 58117.3 | 5539.8  | 5.51  | 0.15   | 107.32   | 19.97 | 5.1    |
| 2295.10 | 13.82  | 21118.5 | 2950.4  | 10.91 | -0.72  | 96.75    | 17.34 | 2.9    |
| 4675.56 | 186.18 | 68521.3 | 2243.1  | 8.27  | 0.17   | 195.12   | 13.40 | 3.4    |
| 6125.96 | 161.79 | 57805.1 | 7747.1  | 9.15  | 0.50   | 180.44   | 17.64 | 4.6    |
| 2134.94 | 8.94   | 37806.9 | 402.4   | 5.51  | 0.15   | 104.88   | 16.22 | 4.5    |
| 5031.66 | 365.04 | 50935.3 | 3140.6  | 8.54  | 0.55   | 256.10   | 18.80 | 4.6    |
| 3367.45 | 220.32 | 35602.1 | 2086.2  | 7.07  | -0.49  | 126.83   | 19.86 | 2.3    |
| 6519.45 | 127.64 | 46176.8 | 8846.2  | 12.54 | 1.24   | 203.25   | 17.42 | 4.9    |
| 4876.37 | 105.69 | 42053.2 | 5673.1  | 8.85  | 0.31   | 119.51   | 21.41 | 2.8    |
| 2468.27 | 57.72  | 36829.7 | 2761.8  | 5.38  | 0.37   | 116.26   | 16.32 | 3.1    |
| 2533.31 | 23.58  | 33612.7 | 1991.8  | 5.43  | -0.65  | 142.28   | 14.51 | 4.2    |
| 2408.11 | 13.82  | 21412.8 | 1971.5  | 8.48  | 0.64   | 89.43    | 19.35 | 4.3    |
| 2337.38 | 13.82  | 20416.9 | 1737.4  | 7.80  | 1.01   | 84.55    | 20.02 | 4.2    |
| 4586.95 | 86.99  | 36272.0 | 10694.2 | 10.34 | 0.11   | 119.51   | 15.26 | 5.5    |
| 2729.24 | 165.85 | 23093.3 | 8618.6  | 5.15  | 0.04   | 80.49    | 15.87 | 3.6    |
| 3289.40 | 116.26 | 26878.6 | 7747.9  | 6.64  | 0.68   | 136.58   | 7.81  | 3.4    |
| 2800.78 | 42.28  | 39572.0 | 4565.8  | 5.45  | 0.66   | 78.86    | 16.00 | 4.2    |
| 3264.20 | 52.84  | 51866.1 | 6022.7  | 6.31  | -0.10  | 136.58   | 17.44 | 3.6    |
| 3453.62 | 165.04 | 58749.8 | 3721.1  | 6.35  | -0.03  | 138.21   | 17.98 | 3.1    |
| 1741.45 | 10.57  | 23990.8 | 861.0   | 7.37  | -1.63  | 75.61    | 20.99 | 1.6    |
| 2035.75 | 13.82  | 25694.9 | 3571.5  | 8.39  | -0.43  | 102.44   | 21.66 | 3.4    |
| 1578.00 | 8.13   | 23736.3 | 2845.5  | 5.15  | 0.04   | 76.42    | 21.46 | 2.7    |
| 4167.44 | 58.44  | 34314.3 | 5060.1  | 12.88 | 0.22   | 136.58   | 24.78 | 2.8    |
| 2799.97 | 21.14  | 22809.5 | 3552.0  | 9.14  | -0.74  | 88.62    | 24.96 | 3.9    |



CRAVENS

The Cravens data are for a company that sells products in several sales territories, each of which is assigned to a single sales representative. A regression analysis was conducted to determine whether a variety of predictor (independent) variables could explain sales in each territory. A random sample of 25 sales territories resulted in the data in Table 16.5; the variable definitions are given in Table 16.6.

As a preliminary step, let us consider the sample correlation coefficients between each pair of variables. Figure 16.13 is the correlation matrix obtained using MINITAB. Note that the sample correlation coefficient between Sales and Time is 0.623, between Sales and Poten is 0.598 and so on.

Looking at the sample correlation coefficients between the independent variables, we see that the correlation between Time and Accounts is 0.758 and significant; hence, if Accounts were used as an independent variable, Time would not add much more explanatory power to the model. Recall that inclusion of highly correlated independent variables, as discussed in the Section 15.4 on multicollinearity, can cause problems for the model. If possible, then, we should avoid including both Time and Accounts in the same regression model. The sample correlation coefficient of 0.549 between Change and Rating is also significant ( $p$ -value  $< 0.05$ ) and this may also prove problematic.

Looking at the sample correlation coefficients between Sales and each of the independent variables can give us a quick indication of which independent variables are, by themselves, good predictors. We see that the single best predictor of Sales is Accounts, because it has the highest sample correlation coefficient (0.754). Recall that for the case of one independent variable, the square of the sample correlation coefficient is the coefficient of determination.

Thus, Accounts can explain  $(0.754)^2(100)$ , or 56.85 per cent, of the variability in Sales. The next most important independent variables are Time, Poten and AdvExp, each with a sample correlation coefficient of approximately 0.6.

**TABLE 16.6** Variable definitions for the Cravens data

| Variable | Definition  |
|----------|---|
| Sales    | Total sales credited to the sales representative  |
| Time     | Length of time employed in months   |
| Poten    | Market potential; total industry sales in units for the sales territory*                                |
| AdvExp   | Advertising expenditure in the sales territory  |
| Share    | Market share; weighted average for the past four years  |
| Change   | Change in the market share over the previous four years   |
| Accounts | Number of accounts assigned to the sales representative*  |
| Work     | Workload; a weighted index based on annual purchases and concentrations of accounts                     |
| Rating   | Sales representative overall rating on eight performance dimensions; an aggregate rating on a 1–7 scale |

\* These data were coded to preserve confidentiality.

**FIGURE 16.13****Correlations: Sales, Time, Poten, AdvExp, Share, Change, Accounts, Work, Rating**

| Sample correlation coefficients for the Cravens data |          | Sales          | Time           | Poten          | AdvExp         | Share          | Change         | Accounts        |
|--|----------|----------------|----------------|----------------|----------------|----------------|----------------|-----------------|
|  | Time     | 0.623<br>0.001 |                |                |                |                |                |                 |
|  | Poten    |                | 0.454<br>0.002 | 0.023          |                |                |                |                 |
|  | AdvExp   |                |                | 0.174<br>0.002 | 0.405          |                |                |                 |
|  | Share    |                |                |                | 0.264<br>0.201 |                |                |                 |
|  | Change   |                |                |                |                | 0.085<br>0.685 |                |                 |
|  | Accounts |                |                |                |                |                | 0.327<br>0.110 |                 |
|  | Work     |                |                |                |                |                |                | -0.199<br>0.341 |
|  | Rating   |                |                |                |                |                |                | 0.229<br>0.272  |

Cell Contents: Pearson correlation  
P-Value

Although there are potential multicollinearity problems, let us consider developing an estimated regression equation using all eight independent variables. The MINITAB computer package provided the results in Figure 16.14. The eight-variable multiple regression model has an adjusted coefficient of determination of 88.3 per cent. Note, however, that the *p*-values for the *t* tests of individual parameters show that only Poten, AdvExp and Share are significant at the  $\alpha = 0.05$  level, given the effect of all the other variables. Hence, we might be inclined to investigate the results that would be obtained if we used just those three variables. Figure 16.15 shows the MINITAB results obtained for the estimated regression equation with those three variables. We see that the estimated regression equation has an adjusted coefficient of determination of 82.7 per cent, which, although not quite as good as that for the eight-independent-variable estimated regression equation, is high.

How can we find an estimated regression equation that will do the best job given the data available? One approach is to compute all possible regressions. That is, we could develop eight one-variable estimated regression equations (each of which corresponds to one of the independent variables), 28 two-variable estimated regression equations (the number of combinations of eight variables taken two at a time), and so on.

### Regression Analysis: Sales versus Time, Poten, ...

The regression equation is

$$\text{Sales} = -1508 + 2.01 \text{ Time} + 0.0372 \text{ Poten} + 0.151 \text{ AdvExp} + 199 \text{ Share} \\ + 291 \text{ Change} + 5.55 \text{ Accounts} + 19.8 \text{ Work} + 8 \text{ Rating}$$

| Predictor | Coef     | SE Coef  | T     | P     |
|-----------|----------|----------|-------|-------|
| Constant  | -1507.8  | 778.6    | -1.94 | 0.071 |
| Time      | 2.010    | 1.931    | 1.04  | 0.313 |
| Poten     | 0.037206 | 0.008202 | 4.54  | 0.000 |
| AdvExp    | 0.15098  | 0.04711  | 3.21  | 0.006 |
| Share     | 199.04   | 67.03    | 2.97  | 0.009 |
| Change    | 290.9    | 186.8    | 1.56  | 0.139 |
| Accounts  | 5.550    | 4.775    | 1.16  | 0.262 |
| Work      | 19.79    | 33.68    | 0.59  | 0.565 |
| Rating    | 8.2      | 128.5    | 0.06  | 0.950 |

$$S = 449.015 \quad R-\text{Sq} = 92.2\% \quad R-\text{Sq}(\text{adj}) = 88.3\%$$

#### Analysis of Variance

| Source         | DF | SS       | MS      | F     | P     |
|----------------|----|----------|---------|-------|-------|
| Regression     | 8  | 38153712 | 4769214 | 23.66 | 0.000 |
| Residual Error | 16 | 3225837  | 201615  |       |       |
| Total          | 24 | 41379549 |         |       |       |

| Source   | DF | Seq SS   |
|----------|----|----------|
| Time     | 1  | 16054463 |
| Poten    | 1  | 5173018  |
| AdvExp   | 1  | 7701943  |
| Share    | 1  | 8145442  |
| Change   | 1  | 788073   |
| Accounts | 1  | 219388   |
| Work     | 1  | 70567    |
| Rating   | 1  | 819      |

**FIGURE 16.14**

MINITAB output for the model involving all eight independent variables

In all, for the Cravens data, 255 different estimated regression equations involving one or more independent variables would have to be fitted to the data.

With the excellent computer packages available today, it is possible to compute all possible regressions. But doing so involves a great amount of computation and requires the model builder to review a large volume of computer output, much of which is associated with obviously poor models. Statisticians prefer a more systematic approach to selecting the subset of independent variables that provide the best estimated regression equation. In the next section, we introduce some of the more popular approaches.

## 16.4 VARIABLE SELECTION PROCEDURES

In this section we discuss four **variable selection procedures**: stepwise regression, forward selection, backward elimination and best-subsets regression. Given a data set with several possible independent variables, we can use these procedures to identify which independent variables provide the best model. The first three procedures are iterative; at each step of the procedure a single independent variable is added or deleted and the new model is evaluated.

**FIGURE 16.15**

MINITAB output for the model involving Poten, AdvExp and Share

**Regression Analysis: Sales versus Poten, AdvExp, Share**

The regression equation is

$$\text{Sales} = -1604 + 0.0543 \text{ Poten} + 0.167 \text{ AdvExp} + 283 \text{ Share}$$

| Predictor | Coef     | SE Coef  | T     | P     |
|-----------|----------|----------|-------|-------|
| Constant  | -1603.6  | 505.6    | -3.17 | 0.005 |
| Poten     | 0.054286 | 0.007474 | 7.26  | 0.000 |
| AdvExp    | 0.16748  | 0.04427  | 3.78  | 0.001 |
| Share     | 282.75   | 48.76    | 5.80  | 0.000 |

$$S = 545.515 \quad R-\text{Sq} = 84.9\% \quad R-\text{Sq}(\text{adj}) = 82.7\%$$

**Analysis of Variance**

| Source         | DF | SS       | MS       | F     | P     |
|----------------|----|----------|----------|-------|-------|
| Regression     | 3  | 35130228 | 11710076 | 39.35 | 0.000 |
| Residual Error | 21 | 6249321  | 297587   |       |       |
| Total          | 24 | 41379549 |          |       |       |

| Source | DF | Seq SS   |
|--------|----|----------|
| Poten  | 1  | 14788203 |
| AdvExp | 1  | 10333728 |
| Share  | 1  | 10008297 |

The process continues until a stopping criterion indicates that the procedure cannot find a better model. The last procedure (best subsets) is not a one-variable-at-a-time procedure; it evaluates regression models involving different subsets of the independent variables.

In the stepwise regression, forward selection and backward elimination procedures, the criterion for selecting an independent variable to add or delete from the model at each step is based on the *F* statistic introduced in Section 16.2.

**Stepwise regression**

Based on this statistic, the stepwise regression procedure begins each step by determining whether any of the variables *already in the model* should be removed. If none of the independent variables can be removed from the model, the procedure checks to see whether any of the independent variables that are not currently in the model can be entered.

Because of the nature of the stepwise regression procedure, an independent variable can enter the model at one step, be removed at a subsequent step, and then enter the model at a later step. The procedure stops when no independent variables can be removed from or entered into the model.

Figure 16.16 shows the results obtained by using the MINITAB stepwise regression procedure for the Cravens data using values of 0.05 for *Alpha to remove* and 0.05 for *Alpha to enter*. (These are the technical settings used by the software for deciding whether an independent variable should be removed or entered into the model.) The stepwise procedure terminated after four steps. The estimated regression equation identified by the MINITAB stepwise regression procedure is:

$$\hat{y} = -1441.93 + 9.2 \text{ Accounts} + 0.175 \text{ AdvExp} + 0.0382 \text{ Poten} + 190 \text{ Share}$$

Note also in Figure 16.16 that  $s = \sqrt{\text{MSE}}$  MSE has been reduced from 881 with the best one variable model (using Accounts) to 454 after four steps. The value of *R-sq* has been increased from 56.85 per cent to 90.04 per cent, and the recommended estimated regression equation has an *R-sq(adj)* value of 88.05 per cent.

**FIGURE 16.16**

MINITAB stepwise regression output for the Cravens data

### Stepwise Regression: Sales versus Time, Poten, ...

Alpha-to-Enter: 0.05 Alpha-to-Remove: 0.05

Response is Sales on 8 predictors, with N = 25

| Step       | 1      | 2     | 3       | 4        |
|------------|--------|-------|---------|----------|
| Constant   | 709.32 | 50.29 | -327.24 | -1441.93 |
| Accounts   | 21.7   | 19.0  | 15.6    | 9.2      |
| T-Value    | 5.50   | 6.41  | 5.19    | 3.22     |
| P-Value    | 0.000  | 0.000 | 0.000   | 0.004    |
| AdvExp     |        | 0.227 | 0.216   | 0.175    |
| T-Value    |        | 4.50  | 4.77    | 4.74     |
| P-Value    |        | 0.000 | 0.000   | 0.000    |
| Poten      |        |       | 0.0219  | 0.0382   |
| T-Value    |        |       | 2.53    | 4.79     |
| P-Value    |        |       | 0.019   | 0.000    |
| Share      |        |       |         | 190      |
| T-Value    |        |       |         | 3.82     |
| P-Value    |        |       |         | 0.001    |
| S          | 881    | 650   | 583     | 454      |
| R-Sq       | 56.85  | 77.51 | 82.77   | 90.04    |
| R-Sq(adj)  | 54.97  | 75.47 | 80.31   | 88.05    |
| Mallows Cp | 67.6   | 27.2  | 18.4    | 5.4      |

### Forward selection

The forward selection procedure starts with no independent variables. It adds variables one at a time using the same procedure as stepwise regression for determining whether an independent variable should be entered into the model. However, the forward selection procedure does not permit a variable to be removed from the model once it has been entered.

The estimated regression equation obtained using MINITAB's forward selection procedure is:

$$\hat{y} = -1441.93 + 9.2 \text{ Accounts} + 0.175 \text{ AdvExp} + 0.0382 \text{ Poten} + 190 \text{ Share}$$

Thus, for the Cravens data, the forward selection procedure leads to the same estimated regression equation as the stepwise procedure.

### Backward elimination

The backward elimination procedure begins with a model that includes all the independent variables. It then deletes one independent variable at a time using the same procedure as stepwise regression. However, the backward elimination procedure does not permit an independent variable to be re-entered once it has been removed.

The estimated regression equation obtained using MINITAB's backward elimination procedure for the Cravens data is:

$$\hat{y} = -1312 + 3.8 \text{ Time} + 0.0444 \text{ Poten} + 0.152 \text{ AdvExp} + 259 \text{ share}$$

Comparing the estimated regression equation identified using the backward elimination procedure to the estimated regression equation identified using the forward selection procedure, we see that three independent variables – AdvExp, Poten and Share – are common to both. However, the backward elimination procedure has included Time instead of Accounts.

Forward selection and backward elimination are the two extremes of model building; the forward selection procedure starts with no independent variables in the model and adds independent variables one at a time, whereas the backward elimination procedure starts with all independent variables in the model and deletes variables one at a time. The two procedures may lead to the same estimated regression equation. It is possible, however, for them to lead to two different estimated regression equations, as we saw with the Cravens data. Deciding which estimated regression equation to use remains a topic for discussion. Ultimately, the analyst's judgement must be applied. The best-subsets model-building procedure we discuss next provides additional model-building information to be considered before a final decision is made.

## Best-subsets regression

Stepwise regression, forward selection and backward elimination are approaches to choosing the regression model by adding or deleting independent variables one at a time. None of them guarantees that the best model for a given number of variables will be found. Hence, these one-variable-at-a-time methods are properly viewed as heuristics for selecting a good regression model.

Some software packages use a procedure called best-subsets regression that enables the user to find, given a specified number of independent variables, the best regression model. MINITAB has such a procedure. Figure 16.17 is a portion of the computer output obtained by using the best-subsets procedure for the Cravens data set.

This output identifies the two best one-variable estimated regression equations, the two best two-variable equations, the two best three-variable equations and so on. The criterion used in determining which estimated regression equations are best for any number of predictors is the value of the coefficient of determination ( $R^2$ ). For instance, Accounts, with an  $R^2 = 56.8$  per cent, provides the best estimated regression equation using only one independent variable; AdvExp and Accounts, with an  $R^2 = 77.5$  per cent, provides the best estimated regression equation using two independent variables; and Poten, AdvExp and Share, with an  $R^2 = 84.9$  per cent, provides the best estimated regression equation with three independent variables.

**FIGURE 16.17**

Portion of MINITAB best-subsets regression output

**Best Subsets Regression: Sales versus Time, Poten, ...**

| Vars | R-Sq | R-Sq(adj) | Cp    | S      | Mallows |   |   |   |   |   |   |
|------|------|-----------|-------|--------|---------|---|---|---|---|---|---|
|      |      |           |       |        | A       | C | c | R | P | d | S |
| 1    | 56.8 | 55.0      | 67.6  | 881.09 |         |   |   |   |   |   | X |
| 1    | 38.8 | 36.1      | 104.6 | 1049.3 | X       |   |   |   |   |   |   |
| 2    | 77.5 | 75.5      | 27.2  | 650.39 |         | X |   | X |   |   |   |
| 2    | 74.6 | 72.3      | 33.1  | 691.11 |         | X | X |   |   |   |   |
| 3    | 84.9 | 82.7      | 14.0  | 545.52 |         | X | X | X |   |   |   |
| 3    | 82.8 | 80.3      | 18.4  | 582.64 |         | X | X |   |   |   | X |
| 4    | 90.0 | 88.1      | 5.4   | 453.84 |         | X | X | X |   |   | X |
| 4    | 89.6 | 87.5      | 6.4   | 463.93 | X       | X | X | X |   |   |   |
| 5    | 91.5 | 89.3      | 4.4   | 430.21 | X       | X | X | X |   |   |   |
| 5    | 91.2 | 88.9      | 5.0   | 436.75 |         | X | X | X | X |   |   |
| 6    | 92.0 | 89.4      | 5.4   | 427.99 | X       | X | X | X |   |   |   |
| 6    | 91.6 | 88.9      | 6.1   | 438.20 |         | X | X | X | X |   |   |
| 7    | 92.2 | 89.0      | 7.0   | 435.66 | X       | X | X | X | X |   |   |
| 7    | 92.0 | 88.8      | 7.3   | 440.29 | X       | X | X | X | X |   | X |
| 8    | 92.2 | 88.3      | 9.0   | 449.02 | X       | X | X | X | X | X | X |

For the Cravens data, the adjusted coefficient of determination ( $R\text{-sq(adj)}$ ) = 89.4 per cent is largest for the model with six independent variables: Time, Poten, AdvExp, Share, Change and Accounts. However, the best model with four independent variables (Poten, AdvExp, Share, Accounts) has an adjusted coefficient of determination almost as high (88.1 per cent). All other things being equal, a simpler model with fewer variables is usually preferred.

## Making the final choice

The analysis performed on the Cravens data to this point is good preparation for choosing a final model, but more analysis should be conducted before the final choice is made. As we noted in Chapters 14 and 15, a careful analysis of the residuals should be undertaken. We want the residual plot for the chosen model to resemble approximately a horizontal band. Let us assume the residuals are not a problem and that we want to use the results of the best-subsets procedure to help choose the model.

The best-subsets procedure shows us that the best four-variable model contains the independent variables Poten, AdvExp, Share and Accounts. This result also happens to be the four-variable model identified with the stepwise regression procedure. Note also that the  $S$  and  $R\text{-sq(adj)}$  results are virtually identical between the two models. Also there is very little difference between the corresponding  $R\text{-sq}$  values.

## EXERCISES

### Applications



COMPLETE  
SOLUTIONS

- 14.** Brownlee (1965)<sup>1</sup> presents stack loss data for a chemical plant involving 21 observations on four variables, namely:

Airflow: Flow of cooling air

Temp: Cooling Water Inlet Temperature

Acid: Concentration of acid [per 1000, minus 500]

Loss: Stack loss (the dependent variable) is 10 times the percentage of the ingoing ammonia to the plant that escapes from the absorption column unabsorbed; that is, an (inverse) measure of the over-all efficiency of the plant

| Loss | Airflow | Temp | Acid |
|------|---------|------|------|
| 42   | 80      | 27   | 89   |
| 37   | 80      | 27   | 88   |
| 37   | 75      | 25   | 90   |
| 28   | 62      | 24   | 87   |
| 18   | 62      | 22   | 87   |
| 18   | 62      | 23   | 87   |
| 19   | 62      | 24   | 93   |
| 20   | 62      | 24   | 93   |
| 15   | 58      | 23   | 87   |
| 14   | 58      | 18   | 80   |
| 14   | 58      | 18   | 89   |
| 13   | 58      | 17   | 88   |
| 11   | 58      | 18   | 82   |

<sup>1</sup> Brownlee, K.A. (1960, 2nd ed. 1965) *Statistical Theory and Methodology in Science and Engineering*. New York: Wiley. pp. 491–500

| Loss | Airflow | Temp | Acid |
|------|---------|------|------|
| 12   | 58      | 19   | 93   |
| 8    | 50      | 18   | 89   |
| 7    | 50      | 18   | 86   |
| 8    | 50      | 19   | 72   |
| 8    | 50      | 19   | 79   |
| 9    | 50      | 20   | 80   |
| 15   | 56      | 20   | 82   |
| 15   | 70      | 20   | 91   |

Develop an estimated regression equation that can be used to predict loss. Briefly discuss the process you used to develop a recommended estimated regression equation for these data.

15. A sales executive is interested in predicting sales of a newly released record (Field, 2005). Details are available for 200 individual past recordings as follows:

*airplay* = number of times a record is played on Radio 1

*sales* = record sales (thousands)

*advert* = advertising budget (£000s)

*attract* = attractiveness rating (1–10) of recording act

Selective modelling details using MINITAB are given below:

**Correlations: adverts, sales, airplay, attract**

|         | adverts | sales | airplay |
|---------|---------|-------|---------|
| sales   | 0.578   |       |         |
|         | 0.000   |       |         |
| airplay | 0.102   | 0.599 |         |
|         | 0.151   | 0.000 |         |
| attract | 0.081   | 0.326 | 0.182   |
|         | 0.256   | 0.000 | 0.010   |

Cell Contents: Pearson correlation

P-Value

**Stepwise Regression: sales versus adverts, airplay, attract**

Alpha to Enter: 0.15    Alpha to Remove: 0.15

Response is sales on three predictors, with  $N = 200$

| Step     | 1      | 2      | 3      |
|----------|--------|--------|--------|
| Constant | 84.87  | 41.12  | -26.61 |
| airplay  | 3.94   | 3.59   | 3.37   |
| T-value  | 10.52  | 12.51  | 12.12  |
| P-value  | 0.000  | 0.000  | 0.000  |
| adverts  | 0.0869 | 0.0849 |        |
| T-value  | 11.99  | 12.26  |        |
| P-value  | 0.000  | 0.000  |        |

|            |       |       |       |
|------------|-------|-------|-------|
| attract    |       |       | 11.1  |
| T-value    |       |       | 4.55  |
| P-value    |       |       | 0.000 |
| S          | 64.8  | 49.4  | 47.1  |
| R-sq       | 35.87 | 62.93 | 66.47 |
| R-sq(adj)  | 35.55 | 62.55 | 65.95 |
| Mallows Cp | 178.8 | 22.7  | 4.0   |

**Regression Analysis: sales versus adverts, airplay, attract**

The regression equation is:

$$\text{sales} = -26.6 + 0.0849 \text{ adverts} + 3.37 \text{ airplay} + 11.1 \text{ attract}$$

| Predictor | Coef     | SE Coef  | T     | P     | VIF   |
|-----------|----------|----------|-------|-------|-------|
| Constant  | -26.61   | 17.35    | -1.53 | 0.127 |       |
| adverts   | 0.084885 | 0.006923 | 12.26 | 0.000 | 1.015 |
| airplay   | 3.3674   | 0.2778   | 12.12 | 0.000 | 1.043 |
| attract   | 11.086   | 2.438    | 4.55  | 0.000 | 1.038 |

$$S = 47.0873 \quad R\text{-sq} = 66.5\% \quad R\text{-sq(adj)} = 66.0\%$$

**Analysis of Variance**

| Source         | DF  | SS      | MS     | F      | P     |
|----------------|-----|---------|--------|--------|-------|
| Regression     | 3   | 861377  | 287126 | 129.50 | 0.000 |
| Residual Error | 196 | 434575  | 2217   |        |       |
| Total          | 199 | 1295952 |        |        |       |

| Source  | DF | Seq SS |
|---------|----|--------|
| adverts | 1  | 433688 |
| airplay | 1  | 381836 |
| attract | 1  | 45853  |

**Unusual Observations**

| Obs | adverts | sales  | Fit    | SE Fit | Residual | St Resid |
|-----|---------|--------|--------|--------|----------|----------|
| 1   | 10      | 330.00 | 229.92 | 10.23  | 100.08   | 2.18R    |
| 2   | 986     | 120.00 | 228.95 | 4.21   | -108.95  | -2.32R   |
| 7   | 472     | 70.00  | 91.87  | 14.21  | -21.87   | -0.49 X  |
| 10  | 174     | 300.00 | 200.47 | 5.85   | 99.53    | 2.13R    |
| 12  | 611     | 70.00  | 114.81 | 11.92  | -44.81   | -0.98 X  |
| 47  | 103     | 40.00  | 154.97 | 5.90   | -114.97  | -2.46R   |
| 52  | 406     | 190.00 | 92.60  | 8.05   | 97.40    | 2.10R    |
| 55  | 1542    | 190.00 | 304.12 | 7.61   | -114.12  | -2.46R   |
| 61  | 579     | 300.00 | 201.19 | 3.44   | 98.81    | 2.10R    |
| 68  | 57      | 70.00  | 180.42 | 5.90   | -110.42  | -2.36R   |
| 100 | 1000    | 250.00 | 152.71 | 7.85   | 97.29    | 2.10R    |
| 138 | 30      | 60.00  | 81.34  | 14.79  | -21.34   | -0.48 X  |
| 164 | 9       | 120.00 | 241.32 | 9.34   | -121.32  | -2.63R   |
| 169 | 146     | 360.00 | 215.87 | 6.79   | 144.13   | 3.09R    |

| Obs | adverts | sales  | Fit    | SE Fit | Residual | St Resid |
|-----|---------|--------|--------|--------|----------|----------|
| 181 | 179     | 70.00  | 63.65  | 14.33  | 6.35     | 0.14 X   |
| 184 | 2272    | 320.00 | 326.06 | 12.97  | -6.06    | -0.13 X  |
| 200 | 786     | 110.00 | 207.21 | 7.07   | -97.21   | -2.09R   |

R denotes an observation with a large standardized residual.

X denotes an observation whose  $X$  value gives it large leverage.

Durbin-Watson statistic = 1.94982

### Best Subsets Regression: sales versus adverts, airplay, attract

Response is sales

| Vars | R-sq | R-sq(adj) | Mallows |        |   | s | y | t |
|------|------|-----------|---------|--------|---|---|---|---|
|      |      |           | Cp      | S      | t |   |   |   |
| 1    | 35.9 | 35.5      | 178.8   | 64.787 |   |   |   | X |
| 1    | 33.5 | 33.1      | 192.9   | 65.991 | X |   |   |   |
| 2    | 62.9 | 62.6      | 22.7    | 49.383 | X | X |   |   |
| 2    | 41.3 | 40.7      | 149.0   | 62.129 | X |   |   | X |
| 3    | 66.5 | 66.0      | 4.0     | 47.087 | X | X | X |   |

- a. How would you interpret this information?
- b. Which of the various models shown here do you favour and why?

16. In a study of car ownership in 24 countries, data (OECD, 1982) have been collected on the following variables:

|     |  |
|-----|--|
| ao  | cars per person                              |
| pop | population (millions)                        |
| den | population density                           |
| gdp | per capita income (\$)                       |
| pr  | petrol price (cents per litre)               |
| con | petrol consumption (tonnes per car per year) |
| tr  | bus and rail use (passenger km per person)   |

Selective results from a linear modelling analysis (ao is the dependent variable) are as follows:

### Best Subsets Regression: ao versus pop, den, gdp, pr, con, tr

Response is ao

| Vars | R-Sq | R-Sq(adj) | Mallows |          |             | p         | d   | g | c |
|------|------|-----------|---------|----------|-------------|-----------|-----|---|---|
|      |      |           | Cp      | S        | o e d p o t |           |     |   |   |
| 1    | 53.0 | 50.9      | 41.2    | 0.085534 |             | X         |     |   |   |
| 1    | 10.7 | 6.7       | 96.4    | 0.11791  |             | X         |     |   |   |
| 2    | 67.8 | 64.7      | 24.0    | 0.072526 |             | X X       |     |   |   |
| 2    | 67.3 | 64.2      | 24.6    | 0.073035 |             | X         | X   |   |   |
| 3    | 72.5 | 68.4      | 19.8    | 0.068579 |             | X X       | X   |   |   |
| 3    | 72.1 | 68.0      | 20.3    | 0.069090 | X           | X         | X   |   |   |
| 4    | 83.0 | 79.5      | 8.1     | 0.055298 |             | X X X X   |     |   |   |
| 4    | 77.1 | 72.3      | 15.8    | 0.064197 | X           | X         | X X |   |   |
| 5    | 86.2 | 82.4      | 6.0     | 0.051208 | X           | X X X X   |     |   |   |
| 5    | 83.2 | 78.5      | 9.9     | 0.056611 |             | X X X X X |     |   |   |
| 6    | 87.0 | 82.4      | 7.0     | 0.051270 | X X X X X X |           |     |   |   |



OECD CARS

**Correlations: ao, pop, den, gdp, pr, con, tr**

|     | ao     | pop    | den    | gdp    | pr     | con    |
|-----|--------|--------|--------|--------|--------|--------|
| pop | 0.278  |        |        |        |        |        |
|     | 0.188  |        |        |        |        |        |
| den | -0.042 | 0.109  |        |        |        |        |
|     | 0.846  | 0.612  |        |        |        |        |
| gdp | 0.728  | 0.057  | 0.193  |        |        |        |
|     | 0.000  | 0.791  | 0.365  |        |        |        |
| pr  | -0.327 | -0.437 | 0.338  | 0.076  |        |        |
|     | 0.118  | 0.033  | 0.106  | 0.724  |        |        |
| con | 0.076  | 0.342  | -0.357 | -0.085 | -0.723 |        |
|     | 0.723  | 0.101  | 0.087  | 0.694  | 0.000  |        |
| tr  | -0.119 | -0.025 | 0.397  | 0.328  | 0.483  | -0.602 |
|     | 0.581  | 0.906  | 0.055  | 0.118  | 0.017  | 0.002  |

Cell Contents: Pearson correlation  
P-Value

**Regression Analysis: ao versus pop, gdp, pr, con, tr**

The regression equation is

$$ao = 0.472 + 0.000521 \text{ pop} + 0.0319 \text{ gdp} - 0.00429 \text{ pr} - 0.104 \text{ con} - 0.0735 \text{ tr}$$

| Predictor | Coef      | SE Coef   | T     | P     |
|-----------|-----------|-----------|-------|-------|
| Constant  | 0.47190   | 0.09081   | 5.20  | 0.000 |
| pop       | 0.0005211 | 0.0002556 | 2.04  | 0.056 |
| gdp       | 0.031889  | 0.003423  | 9.32  | 0.000 |
| pr        | -0.004289 | 0.001245  | -3.44 | 0.003 |
| con       | -0.10449  | 0.02626   | -3.98 | 0.001 |
| tr        | -0.07354  | 0.01733   | -4.24 | 0.000 |

$$S = 0.0512085 \quad R-Sq = 86.2\% \quad R-Sq(adj) = 82.4\%$$

- a. Which of the various model options considered here do you prefer and why?
  - b. Corresponding stepwise output from MINITAB terminates after two stages, gdp being the first independent variable selected and pr the second. How does this latest information reconcile with that summarized earlier?
  - c. Does it alter in any way, your inferences for (a)? If so, why, and if not, why not?
17. In an analysis of the effects of rainfall, temperature and time of exposure on the ret loss of flax, the following MINITAB output has been obtained:

(Note:  $X_1$  = Mean daily rainfall (0.01 inches per day))

$X_2$  = Retting period (days)

$X_3$  = Mean maximum daily temperature ( $^{\circ}$ F)

$Y$  = per cent ret loss of flax

**Regression Analysis: y versus x1, x2, x3**

The regression equation is

$$y = 10.8 + 1.81 x_1 + 0.109 x_2 + 0.0926 x_3$$

| Predictor | Coef    | SE Coef | T    | P     | VIF |
|-----------|---------|---------|------|-------|-----|
| Constant  | 10.819  | 7.258   | 1.49 | 0.150 |     |
| x1        | 1.8101  | 0.5451  | 3.32 | 0.003 | 1.2 |
| x2        | 0.10887 | 0.05858 | 1.86 | 0.076 | 1.5 |
| x3        | 0.09263 | 0.09296 | 1.00 | 0.329 | 1.7 |

$$S = 2.197 \quad R-sq = 42.3\% \quad R-sq(adj) = 34.7\%$$

**Analysis of Variance**

| SOURCE     | DF | SS      | MS     | F    | P     |
|------------|----|---------|--------|------|-------|
| Regression | 3  | 81.285  | 27.095 | 5.61 | 0.005 |
| Error      | 23 | 111.045 | 4.828  |      |       |
| Total      | 26 | 192.330 |        |      |       |
| SOURCE     | DF | SEQ SS  |        |      |       |
| x1         | 1  | 37.060  |        |      |       |
| x2         | 1  | 39.430  |        |      |       |
| x3         | 1  | 4.795   |        |      |       |

**Unusual Observations**

| Obs. | x1   | y      | Fit    | SE Fit | Residual | St. Resid |
|------|------|--------|--------|--------|----------|-----------|
| 21   | 4.80 | 29.500 | 34.004 | 1.013  | -4.504   | -2.31R    |
| 24   | 5.40 | 38.900 | 34.050 | 0.890  | 4.850    | 2.41R     |

R denotes an obs. with a large st. resid.

Durbin-Watson statistic = 1.64

**Stepwise Regression:  $y$  versus  $x_1, x_2, x_3$** 

Stepwise regression of  $y$  on three predictors, with  $N = 27$

| STEP     | 1     | 2     |
|----------|-------|-------|
| CONSTANT | 27.39 | 16.42 |
| x1       | 1.36  | 1.59  |
| T-RATIO  | 2.44  | 3.20  |
| x2       |       | 0.141 |
| T-RATIO  |       | 2.86  |
| S        | 2.49  | 2.20  |
| R-SQ     | 19.27 | 39.77 |

**Best Subsets Regression:  $y$  versus  $x_1, x_2, x_3$** 

Best Subsets

Regression of  $y$

| Vars | R-sq | Adj.<br>R-sq |      |        | s | x<br>1    x    x<br>2    x    x<br>3    x    x |   |   |
|------|------|--------------|------|--------|---|--|---|---|
|      |      | C-p          |      |        |   | 1  | 2 | 3 |
| 1    | 19.3 | 16.0         | 9.2  | 2.4921 | X |  |   |   |
| 1    | 14.1 | 10.7         | 11.2 | 2.5700 |   | X  |   |   |
| 2    | 39.8 | 34.8         | 3.0  | 2.1970 | X | X  |   |   |
| 2    | 33.6 | 28.1         | 5.5  | 2.3069 | X |  | X |   |
| 3    | 42.3 | 34.7         | 4.0  | 2.1973 | X | X  | X |   |

- a. How would you interpret this information?
- b. Confirm details of any tests you carry out to support your inferences.
- c. Which is your preferred model of those covered here?

18. A senior police manager is reviewing manpower allocation of police officers to a number of geographical districts which fall under their responsibility (Wisniewski, 2002). Detailed regression analysis results have been obtained involving the following variables:



POLICE

|              |  |
|--------------|--|
| Crimes       | number of reported crimes                          |
| Officers     | number of full-time equivalent police officers     |
| Support      | number of civilian support staff                   |
| Unemployment | unemployment rate (%) for the area                 |
| Retired      | percentage of the local population who are retired |

Selected MINITAB output is given below:

#### Correlations: Crimes, Officers, Support, Unemployment, Retired

|              | Crimes          | Officers        | Support         | Unemployment    |
|--------------|-----------------|-----------------|-----------------|-----------------|
| Officers     | -0.735<br>0.000 |                 |                 |                 |
| Support      | 0.259<br>0.202  | -0.345<br>0.085 |                 |                 |
| Unemployment | 0.760<br>0.000  | -0.434<br>0.027 | 0.128<br>0.535  |                 |
| Retired      | -0.867<br>0.000 | 0.655<br>0.000  | -0.138<br>0.501 | -0.661<br>0.000 |

Cell Contents: Pearson correlation  
P-Value

#### Best Subsets Regression: Crimes versus Officers, Support, ...

Response is Crimes

| Vars | R-Sq | R-Sq(adj) | Cp   | S      | Mallows   | U       |
|------|------|-----------|------|--------|-----------|---------|
| 1    | 75.1 | 74.0      | 16.8 | 100.55 | X         | n       |
| 1    | 57.7 | 55.9      | 43.9 | 131.05 | X         | e       |
| 2    | 81.3 | 79.7      | 9.2  | 89.041 | XX        | m       |
| 2    | 80.0 | 78.3      | 11.1 | 91.970 | X X       | o       |
| 3    | 86.2 | 84.3      | 3.5  | 78.170 | X X X     | p       |
| 3    | 82.9 | 80.6      | 8.6  | 86.946 | X X X X   | f S l R |
| 4    | 86.5 | 83.9      | 5.0  | 79.085 | X X X X X | f u o e |

#### Stepwise Regression: Crimes versus Officers, Support, ...

Backward elimination. Alpha-to-Remove: 0.1

Response is Crimes on 4 predictors, with N = 26

| Step     | 1     | 2     |
|----------|-------|-------|
| Constant | 1344  | 1411  |
| Officers | -14.1 | -15.5 |
| T-Value  | -2.36 | -2.80 |
| P-Value  | 0.028 | 0.010 |
| Support  | 10    |       |
| T-Value  | 0.70  |       |
| P-Value  | 0.490 |       |

If a new variable  $\text{total staff} = \text{Officers} + \text{Support}$  is created and a further analysis undertaken, the following results are obtained.

### Regression Analysis: Crimes versus Unemployment, Retired, Total staff

The regression equation is

$\text{Crimes} = 1433 + 17.4 \text{ Unemployment} - 21.5 \text{ Retired} - 13.4 \text{ Total staff}$

| Predictor    | Coef    | SE Coef | T     | P     |
|--------------|---------|---------|-------|-------|
| Constant     | 1433.0  | 164.6   | 8.71  | 0.000 |
| Unemployment | 17.412  | 5.786   | 3.01  | 0.006 |
| Retired      | -21.511 | 5.883   | -3.66 | 0.001 |
| Total staff  | -13.398 | 6.205   | -2.16 | 0.042 |

$S = 82.7009$     $R-\text{Sq} = 84.6\%$     $R-\text{Sq}(\text{adj}) = 82.4\%$

#### Analysis of Variance

| Source         | DF | SS     | MS     | F     | P     |
|----------------|----|--------|--------|-------|-------|
| Regression     | 3  | 823591 | 274530 | 40.14 | 0.000 |
| Residual Error | 22 | 150468 | 6839   |       |       |
| Total          | 25 | 974059 |        |       |       |

| Source       | DF | Seq SS |
|--------------|----|--------|
| Unemployment | 1  | 561901 |
| Retired      | 1  | 229809 |
| Total staff  | 1  | 31881  |

Durbin-Watson statistic = 2.22341

- Explain this computer output, carrying out any additional tests you think necessary or appropriate.
- Is the last model a significant improvement on the corresponding two predictor model (best subsets option with  $R^2 = 81.3$  per cent) for which details were summarized earlier?
- Which of the various models shown do you prefer and why?

### ONLINE RESOURCES

For the associated data files, additional online summary, questions and answers, visit the online platform.



### SUMMARY

In this chapter we discussed several concepts used by model builders in identifying the best estimated regression equation. First, we introduced the concept of a general linear model to show how the methods discussed in Chapters 14 and 15 could be extended to handle curvilinear

relationships and interaction effects. Then we discussed how transformations involving the dependent variable could be used to account for problems such as non-constant variance in the error term.

In many applications of regression analysis, a large number of independent variables are considered. We presented a general approach based on an  $F$  statistic for adding or deleting variables from a regression model. We then introduced a larger problem involving 25 observations and eight independent variables. We saw that one issue encountered in solving larger problems is finding the best subset of the independent variables. To help in that task, we discussed several variable selection procedures: stepwise regression, forward selection, backward elimination and best-subsets regression.

## KEY TERMS

**General linear model**  
**Interaction**

**Variable selection procedures**

## KEY FORMULAE

### General linear model

$$Y = \beta_0 + \beta_1 Z_1 + \beta_2 Z_2 + \dots + \beta_p Z_p + \varepsilon \quad (16.1)$$

### $F$ test statistic for adding or deleting $p-q$ variables

$$F = \frac{\frac{\text{SSE}(x_1, x_2, \dots, x_q) - \text{SSE}(x_1, x_2, \dots, x_q, x_{q+1}, \dots, x_p)}{p-q}}{\frac{\text{SSE}(x_1, x_2, \dots, x_q, x_{q+1}, \dots, x_p)}{n-p-1}} \quad (16.12)$$

## CASE PROBLEM 1



### House prices

The data relate to bungalow and two-story homes located in ten selected neighbourhoods of Canada. Each home was listed and sold individually through the Multiple Listing System.

Apart from the dependent variable list price, basic house descriptive variables were categorized into two groups as shown in Table 1, which cover house attributes and lot attributes.

### Managerial report

Use the methods presented in this and previous chapters to analyze this data set. Present a summary



of your analysis, including key statistical results, conclusions and recommendations, in a managerial report. Include any appropriate technical material (computer output, residual plots, etc.) in an appendix



HOMESALES

**TABLE 1** Definition of variables

| Variable                | Definition   |
|-------------------------|--|
| <i>House attributes</i> |  |
| STYLE                   | 1 if bungalow, 2 if two storey                           |
| R                       | Number of rooms  |
| B                       | Number of bathrooms                                      |
| BR                      | Number of bedrooms                                       |
| S                       | Living area (square meters)                              |
| A                       | Age (years)  |
| BAS                     | Basement (from 1 (open) to 3 (finished))                 |
| G                       | Number of garage space                                   |
| ATT                     | Dummy variable, 1 if attached, 0 detached                |
| F                       | Number of fireplaces (woodburning)                       |
| C                       | Number of chattels (appliances e.g. stove, fridge, etc.) |
| <i>Lot attributes</i>   |  |
| LOTS                    | Lot size (square metres)                                 |
| CO                      | 1 if corner lot, 0 otherwise                             |
| CUL                     | 1 if cul-de-sac, 0 otherwise                             |
| LA                      | 1 if lane behind, 0 otherwise                            |
| E                       | Exposure of yard (N, NE, E = 1, otherwise 0)             |
| Z0                      | Dummy variable represents zone 0                         |
| Z1                      | Dummy variable represents zone 1                         |
| Z2                      | Dummy variable represents zone 2                         |
| Z3                      | Dummy variable represents zone 3                         |
| Z4                      | Dummy variable represents zone 4                         |
| Z5                      | Dummy variable represents zone 5                         |
| Z6                      | Dummy variable represents zone 6                         |
| Z7                      | Dummy variable represents zone 7                         |
| Z8                      | Dummy variable represents zone 8                         |
| Z9                      | Dummy variable represents zone 9                         |
| TIME                    | Month of sale  |

## CASE PROBLEM 2



### Treating obesity\*

Obesity is a major health risk throughout Europe and the USA, leading to a number of possibly life-threatening diseases. Developing a successful treatment for obesity is therefore important, as a reduc-

tion in weight can greatly reduce the risk of illness. A sustained weight loss of 5–10 per cent of initial body weight reduces the health risks associated with obesity. Diet and exercise are useful in weight control but may not always be successful in the long term. An integrated programme of diet, exercise and drug treatment may be beneficial for obese patients.

### The study

In 1998 Knoll Pharmaceuticals received authorization to market sibutramine for the treatment of obesity in the USA. One of their suite of studies

\*Source: STARS ([www.stars.ac.uk](http://www.stars.ac.uk))



involved 37 obese patients who followed a treatment regime comprising a combination of diet, exercise and drug treatment. Patients taking part in this study were healthy adults (aged 18 to 65 years) and were between 30 per cent and 80 per cent above their ideal body weight. Rigorous criteria were defined to ensure that only otherwise-healthy individuals took part.

Patients received either the new drug or placebo for an eight-week period and body weight was recorded at the start (week 0, also known as baseline) and at week eight. The information recorded for each patient was:

- Age (years)
- Gender (F: female, M: male)

- Height (cm)
- Family history of obesity? (N: no, Y: yes)  
Missing for patient number 134
- Motivation rating (1: some, 2: moderate, 3: great)
- Number of previous weight loss attempts
- Age of onset of obesity (1: 11 years, 2: 12–17 years, 3: 18–65 years)
- Weight at week 0 (kg)
- Weight at week 8 (kg)
- Treatment group (1 = placebo, 2 = new drug)  
Results are shown below for a selection of ten of the 37 patients that took part in the study:

| Age | Gender | Height | Family history? | Motivation rating | Previous weight loss attempts |   | Age of onset | Weight at week 0 | Weight at week 8 | Treatment group |
|-----|--------|--------|-----------------|-------------------|-------------------------------|---|--------------|------------------|------------------|-----------------|
|     |        |        |                 |                   |                               |   |              |                  |                  |                 |
| 40  | F      | 170    | N               | 2                 | 1                             | 3 | 83.4         | 75.0             | 2                |                 |
| 50  | F      | 164    | Y               | 2                 | 5                             | 2 | 102.2        | 96.3             | 1                |                 |
| 39  | F      | 154    | Y               | 2                 | 1                             | 3 | 84.0         | 82.6             | 1                |                 |
| 40  | F      | 169    | Y               | 1                 | 7                             | 3 | 103.7        | 95.7             | 2                |                 |
| 44  | F      | 169    | N               | 2                 | 1                             | 1 | 99.2         | 99.2             | 2                |                 |
| 44  | M      | 177    | Y               | 2                 | 2                             | 2 | 126.0        | 123.2            | 2                |                 |
| 38  | M      | 171    | Y               | 1                 | 1                             | 1 | 103.7        | 95.5             | 2                |                 |
| 42  | M      | 175    | N               | 2                 | 4                             | 3 | 117.9        | 117.0            | 1                |                 |
| 53  | M      | 177    | Y               | 2                 | 3                             | 3 | 112.4        | 111.8            | 1                |                 |
| 52  | F      | 166    | Y               | 1                 | 3                             | 3 | 85.0         | 80.0             | 2                |                 |

### Clinical trials

The study is an example of a clinical trial commonly used to assess the effectiveness of a new treatment. Clinical trials are subject to rigorous controls to ensure that individuals are not unnecessarily put at risk and that they are fully informed and give their consent to take part in the study. As giving any patient a treatment may have a psychological effect, many studies compare a new drug with a dummy treatment (placebo) where, to avoid bias, neither the patient nor the doctor recording information knows whether the patient is on the new treatment or placebo as the tablets/

capsules look identical; this approach is known as double-blinding. Bias could also occur if the treatment given to a patient was based on their characteristics; for example, if the more-overweight patients were given the new treatment rather than the placebo they would have a greater chance of weight loss. To avoid such bias the decision as to which individuals will receive the new treatment or placebo must be made using a process known as randomization. Using this approach each individual has the same chance of being given either the new treatment or the placebo.



### *Managerial report*

- 1.** Use the methods presented in this and previous chapters to analyze this data set. The priority is to use regression modelling to help determine which variables most influence weight loss. The treatment group variable is a particular concern in this respect.
- 2.** Present a summary of your analysis, including key statistical results, conclusions and recommendations, in a managerial report. Include any appropriate technical material (computer output, residual plots, etc.) in an appendix.



# 17

# Time Series Analysis and Forecasting

## CHAPTER CONTENTS

Statistics in Practice Asylum applications

- 17.1 Time series patterns
- 17.2 Forecast accuracy
- 17.3 Moving averages and exponential smoothing
- 17.4 Trend projection
- 17.5 Seasonality and trend
- 17.6 Time series decomposition

**LEARNING OBJECTIVES** After reading this chapter and doing the exercises you should be able to:

- 1 Understand that the long-run success of an organization is often closely related to how well management is able to predict future aspects of the operation.
- 2 Know the various components of a time series.
- 3 Use smoothing techniques such as moving averages and exponential smoothing.
- 4 Use either least squares or the Holt's smoothing method to identify the trend component of a time series.
- 5 Understand how the classical time series model can be used to explain the pattern or behaviour of the data in a time series and to develop a forecast for the time series.
- 6 Be able to determine and use seasonal indices for a time series.
- 7 Know how regression models can be used in forecasting.
- 8 Know the definition of the following terms: time series; forecast; trend component; cyclical component; seasonal component; irregular component; mean squared error; moving averages; weighted moving averages; smoothing constants; seasonal constant.



## STATISTICS IN PRACTICE

### Asylum applications

**A**sylum applications to the UK have been a major concern for the authorities for a number of years (Langham, 2005). In the autumn of 2002 the monthly rate of applicants seeking political asylum in the UK exceeded 7500 for the first time in history. Respond-



ing to charges that immigration was running out of control, the Labour government of the time introduced a series of initiatives with the aim of drastically reducing the numbers of asylum seekers coming into the country. The effect of these was dramatic, the number of asylum applications halving between October 2002 and September 2003. In a report\* commissioned by the Home Office subsequently, relevant datasets were checked and analyzed using regression (trend) and correlation analysis to see if the reduction in the number of asylum applications had had a significant impact on other forms of migration. Although no clear connection was found it was accepted that reasons for migration were extremely complex. The report also recognized that government measures to manage down the intake of asylum seekers had played a part in reducing the number of asylum applications.

Source: Langham, Alison (2005) Asylum and migration: A review of Home Office statistics. *Significance*, Vol 2 Issue 2 pp 78–80.

\*Can be obtained from: [www.nao.org.uk](http://www.nao.org.uk)

**T**he purpose of this chapter is to provide an introduction to time series analysis and forecasting. Suppose we are asked to provide quarterly **forecasts** of sales for one of our company's products over the coming one-year period. Production schedules, raw material purchasing, inventory policies and sales quotas will all be affected by the quarterly forecasts we provide. Consequently, poor forecasts may result in poor planning and increased costs for the company. How should we go about providing the quarterly sales forecasts? Good judgement, intuition and an awareness of the state of the economy may give us a rough idea or 'feeling' of what is likely to happen in the future, but converting that feeling into a number that can be used as next year's sales forecast is difficult.

Forecasting methods can be classified as qualitative or quantitative. Qualitative methods generally involve the use of expert judgement to develop forecasts. Such methods are appropriate when historical data on the variable being forecast are either not applicable or unavailable. Quantitative forecasting methods can be used when (1) past information about the variable being forecast is available, (2) the information can be quantified and (3) it is reasonable to assume that the pattern of the past will continue into the future. In such cases, a forecast can be developed using a time series method or a causal method. We will focus exclusively on quantitative forecasting methods in this chapter.

If the historical data are restricted to past values of the variable to be forecast, the forecasting procedure is called a *time series method* and the historical data are referred to as a time series. The objective of time series analysis is to discover a pattern in the historical data or time series and then extrapolate the pattern into the future; the forecast is based solely on past values of the variable and/or on past forecast errors.

**Causal forecasting methods** are based on the assumption that the variable we are forecasting has a cause–effect relationship with one or more other variables. In the discussion of regression analysis in Chapters 14, 15 and 16, we showed how one or more independent variables could be used to predict the value of a single dependent variable. Looking at regression analysis as a forecasting tool, we can view the time series value that we want to forecast as the dependent variable. Hence, if we can identify a good set of related independent, or explanatory variables, we may be able to develop an estimated regression equation for predicting or forecasting the time series. For instance, the sales for many products are influenced by advertising expenditures, so regression analysis may be used to develop an equation showing how sales and advertising are related. Once the advertising budget for the next period is determined, we could

substitute this value into the equation to develop a prediction or forecast of the sales volume for that period. Note that if a time series method were used to develop the forecast, advertising expenditures would not be considered; that is, a time series method would base the forecast solely on past sales.

By treating time as the independent variable and the time series as a dependent variable, regression analysis can also be used as a time series method. To help differentiate the application of regression analysis in these two cases, we use the terms *cross-sectional regression* and *time series regression*. Thus, time series regression refers to the use of regression analysis when the independent variable is time. Because our focus in this chapter is on time series methods, we leave the discussion of the application of regression analysis as a causal forecasting method to more advanced texts on forecasting.



PETROL

## 17.1 TIME SERIES PATTERNS

A **time series** is a sequence of observations on a variable measured at successive points in time or over successive periods of time. The measurements may be taken every hour, day, week, month or year, or at any other regular interval.\* The pattern of the data is an important factor in understanding how the time series has behaved in the past. If such behaviour can be expected to continue in the future, we can use the past pattern to guide us in selecting an appropriate forecasting method.

To identify the underlying pattern in the data, a useful first step is to construct a **time series plot**. A time series plot is a graphical presentation of the relationship between time and the time series variable; time is on the horizontal axis and the time series values are shown on the vertical axis. Let us review some of the common types of data patterns that can be identified when examining a time series plot.

### Horizontal pattern

A **horizontal pattern** exists when the data fluctuate around a constant mean. To illustrate a time series with a horizontal pattern, consider the 12 weeks of data in Table 17.1. These data show the number of litres of petrol sold by a petrol distributor in Sitges, Spain over the past 12 weeks. The average value or mean for this time series is 19.25 or 19 250 litres per week. Figure 17.1 shows a time series plot for these data. Note how the data fluctuate around the sample mean of 19 250 litres. Although random variability is present, we would say that these data follow a horizontal pattern.

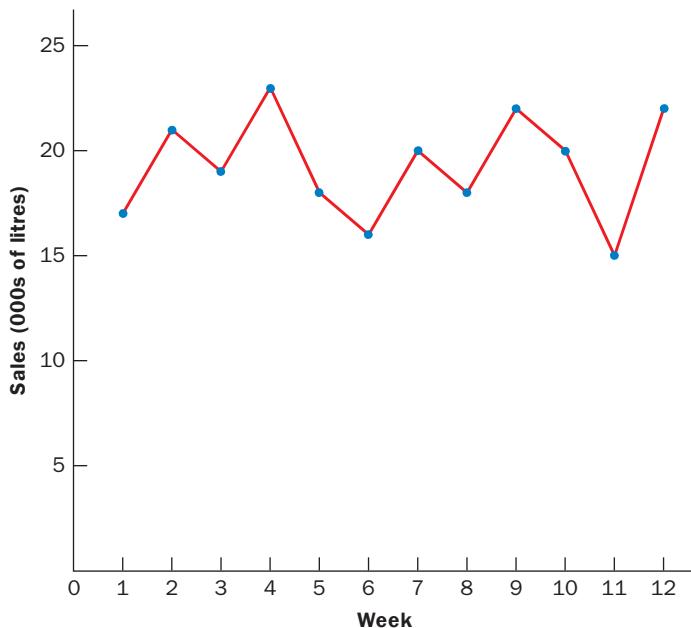
**TABLE 17.1** Petrol sales time series

| Week | Sales (000s of litres) |
|------|------------------------|
| 1    | 17                     |
| 2    | 21                     |
| 3    | 19                     |
| 4    | 23                     |
| 5    | 18                     |
| 6    | 16                     |
| 7    | 20                     |
| 8    | 18                     |
| 9    | 22                     |
| 10   | 20                     |
| 11   | 15                     |
| 12   | 22                     |

\*We limit our discussion to time series in which the values of the series are recorded at equal intervals. Cases in which the observations are made at unequal intervals are beyond the scope of this text.

**FIGURE 17.1**

Petrol sales time series plot



The term **stationary time series**\* is used to denote a time series whose statistical properties are independent of time. In particular this means that:

- 1 The process generating the data has a constant mean.
- 2 The variability of the time series is constant over time.

A time series plot for a stationary time series will always exhibit a horizontal pattern. But simply observing a horizontal pattern is not sufficient evidence to conclude that the time series is stationary. More advanced texts on forecasting discuss procedures for determining if a time series is stationary and provide methods for transforming a time series that is not stationary into a stationary series.

Changes in business conditions can often result in a time series that has a horizontal pattern shifting to a new level. For instance, suppose the petrol distributor signs a contract with the Guardia Civil to provide petrol for police cars located in northern Spain. With this new contract, the distributor expects to see a major increase in weekly sales starting in week 13. Table 17.2 shows the number of litres of petrol sold for the original time series and for the ten weeks after signing the new contract. Figure 17.2 shows the corresponding time series plot. Note the increased level of the time series beginning in week 13. This change in the level of the time series makes it more difficult to choose an appropriate forecasting method. Selecting a forecasting method that adapts well to changes in the level of a time series is an important consideration in many practical applications.

## Trend pattern

Although time series data generally exhibit random fluctuations, a time series may also show gradual shifts or movements to relatively higher or lower values over a longer period of time. If a time series plot exhibits this type of behaviour, we say that a **trend pattern** exists. A trend is usually the result of long-term factors such as population increases or decreases, changing demographic characteristics of the population, technology, and/or consumer preferences.

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\*For a formal definition of stationarity see G.E.P. Box, G.M. Jenkins and G.C. Reinsel (1994), *Time Series Analysis: Forecasting and Control*, 3rd ed. Englewood Cliffs, NJ: Prentice Hall, p. 23.



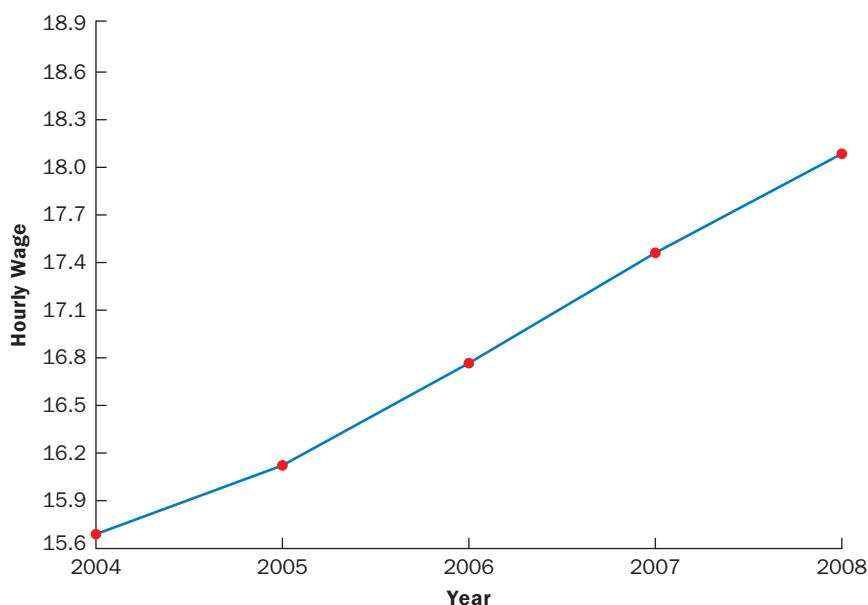
PETROL  
REVISED

**TABLE 17.2** Petrol sales time series after obtaining the contract with the Guardia Civil

| Week | Sales (000s of litres) |
|------|------------------------|
| 1    | 17                     |
| 2    | 21                     |
| 3    | 19                     |
| 4    | 23                     |
| 5    | 18                     |
| 6    | 16                     |
| 7    | 20                     |
| 8    | 18                     |
| 9    | 22                     |
| 10   | 20                     |
| 11   | 15                     |
| 12   | 22                     |
| 13   | 31                     |
| 14   | 34                     |
| 15   | 31                     |
| 16   | 33                     |
| 17   | 28                     |
| 18   | 32                     |
| 19   | 30                     |
| 20   | 29                     |
| 21   | 34                     |
| 22   | 33                     |

**FIGURE 17.2**

Petrol sales time series plot after obtaining the contract with the Guardia Civil



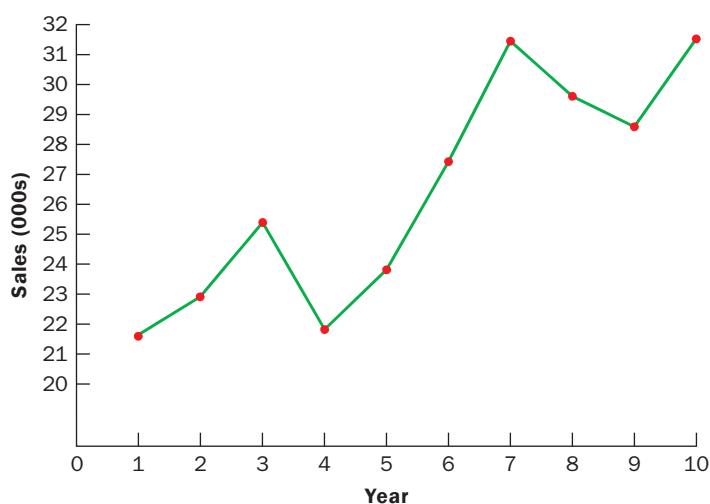
To illustrate a time series with a trend pattern, consider the time series of bicycle sales for a particular manufacturer over the past ten years, as shown in Table 17.3 and Figure 17.3. Note that 21 600 bicycles were sold in year one, 22 900 were sold in year two and so on. In year 10, the most recent year, 31 400 bicycles were sold. Visual inspection of the time series plot shows some up and down movement over the past ten years, but the time series also seems to have a systematically increasing or upward trend.

**TABLE 17.3** Bicycle sales time series

| Year | Sales ('000s) |
|------|---------------|
| 1    | 21.6          |
| 2    | 22.9          |
| 3    | 25.5          |
| 4    | 21.9          |
| 5    | 23.9          |
| 6    | 27.5          |
| 7    | 31.5          |
| 8    | 29.7          |
| 9    | 28.6          |
| 10   | 31.4          |

**FIGURE 17.3**

Bicycle sales time series plot

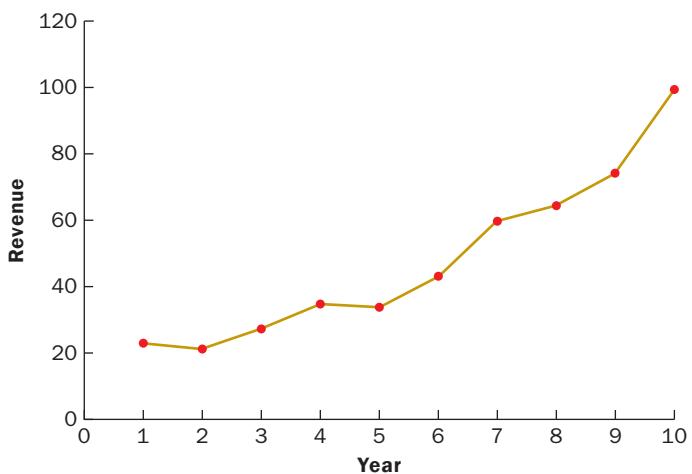
**TABLE 17.4** Cholesterol revenue time series (€ millions)

| Year | Revenue |
|------|---------|
| 1    | 23.1    |
| 2    | 21.3    |
| 3    | 27.4    |
| 4    | 34.6    |
| 5    | 33.8    |
| 6    | 43.2    |
| 7    | 59.5    |
| 8    | 64.4    |
| 9    | 74.2    |
| 10   | 99.3    |

The trend for the bicycle sales time series appears to be linear and increasing over time, but sometimes a trend can be described better by other types of patterns. For instance, the data in Table 17.4 and the corresponding time series plot in Figure 17.4 show the sales for a cholesterol drug since the company won government approval for it ten years ago. The time series increases in a nonlinear fashion; that is, the rate of change of revenue does not increase by a constant amount from one year to the next.

**FIGURE 17.4**

Cholesterol revenue time series plot  
(\$ millions)



In fact, the revenue appears to be growing in an exponential fashion. Exponential relationships such as this are appropriate when the percentage change from one period to the next is relatively constant.

### Seasonal pattern

The trend of a time series can be identified by analyzing multiyear movements in historical data. Seasonal patterns are recognized by seeing the same repeating patterns over successive periods of time. For example, a manufacturer of swimming pools expects low sales activity in the fall and winter months, with peak sales in the spring and summer months. Manufacturers of snow removal equipment and heavy clothing, however, expect just the opposite yearly pattern. Not surprisingly, the pattern for a time series plot that exhibits a repeating pattern over a one-year period due to seasonal influences is called a **seasonal pattern**. While we generally think of seasonal movement in a time series as occurring within one year, time series data can also exhibit seasonal patterns of less than one year in duration. For example, daily traffic volume shows within-the-day ‘seasonal’ behaviour, with peak levels occurring during rush hours, moderate flow during the rest of the day and early evening, and light flow from midnight to early morning.

As an example of a seasonal pattern, consider the number of umbrellas sold at a clothing store over the past five years. Table 17.5 shows the time series and Figure 17.5 shows the corresponding time series plot. The time series plot does not indicate any long-term trend in sales. In fact, unless you look carefully at the data, you might conclude that the data follow a horizontal pattern. But closer inspection of the time series plot reveals a regular pattern in the data. That is, the first and third quarters have moderate sales, the second quarter has the highest sales, and the fourth quarter tends to have the lowest sales volume. Thus, we would conclude that a quarterly seasonal pattern is present.

### Trend and seasonal pattern

Some time series include a combination of a trend and seasonal pattern. For instance, the data in Table 17.6 and the corresponding time series plot in Figure 17.6 show television set sales for a particular manufacturer over the past four years. Clearly, an increasing trend is present. But, Figure 17.6 also indicates that sales are lowest in the second quarter of each year and increase in quarters 3 and 4. Thus, we conclude that a seasonal pattern also exists for television set sales. In such cases we need to use a forecasting method that has the capability to deal with both trend and seasonality.

### Cyclical pattern

A **cyclical pattern** exists if the time series plot shows an alternating sequence of points below and above the trend line lasting more than one year. Many economic time series exhibit cyclical behaviour with regular runs of observations below and above the trend line. Often, the cyclical component of a time series is due to multiyear business cycles.

**TABLE 17.5** Umbrella sales time series

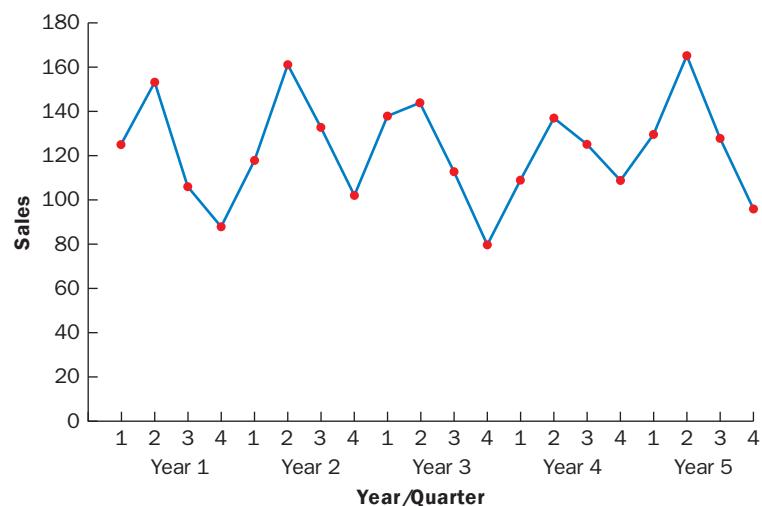
| Year | Quarter | Sales |
|------|---------|-------|
| 1    | 1       | 125   |
|      | 2       | 153   |
|      | 3       | 106   |
|      | 4       | 88    |
| 2    | 1       | 118   |
|      | 2       | 161   |
|      | 3       | 133   |
|      | 4       | 102   |
| 3    | 1       | 138   |
|      | 2       | 144   |
|      | 3       | 113   |
|      | 4       | 80    |
| 4    | 1       | 109   |
|      | 2       | 137   |
|      | 3       | 125   |
|      | 4       | 109   |
| 5    | 1       | 130   |
|      | 2       | 165   |
|      | 3       | 128   |
|      | 4       | 96    |



UMBRELLA

**FIGURE 17.5**

Umbrella sales time series plot



For example, periods of moderate inflation followed by periods of rapid inflation can lead to time series that alternate below and above a generally increasing trend line (e.g. a time series for housing costs). Business cycles are extremely difficult, if not impossible, to forecast. As a result, cyclical effects are often combined with long-term trend effects and referred to as trend-cycle effects. In this chapter we do not deal with cyclical effects that may be present in the time series.

## Selecting a forecasting method

The underlying pattern in the time series is an important factor in selecting a forecasting method. Thus, a time series plot should be one of the first things developed when trying to determine what forecasting method to use. If we see a horizontal pattern, then we need to select a method appropriate for this type of pattern.

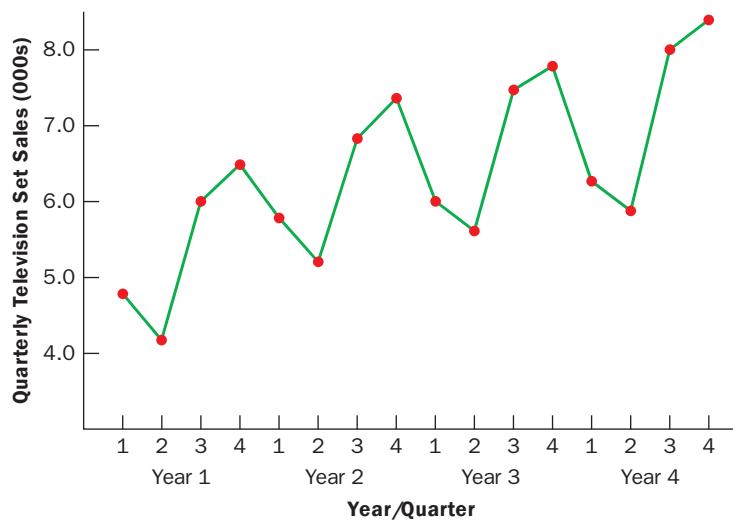
**TABLE 17.6** Quarterly television set sales time series

TVSALES

| Year | Quarter | Sales (000s) |
|------|---------|--------------|
| 1    | 1       | 4.8          |
|      | 2       | 4.1          |
|      | 3       | 6.0          |
|      | 4       | 6.5          |
| 2    | 1       | 5.8          |
|      | 2       | 5.2          |
|      | 3       | 6.8          |
|      | 4       | 7.4          |
| 3    | 1       | 6.0          |
|      | 2       | 5.6          |
|      | 3       | 7.5          |
|      | 4       | 7.8          |
| 4    | 1       | 6.3          |
|      | 2       | 5.9          |
|      | 3       | 8.0          |
|      | 4       | 8.4          |

**FIGURE 17.6**

Quarterly television set sales time series plot



Similarly, if we observe a trend in the data, then we need to use a forecasting method that has the capability to handle trend effectively. The next two sections illustrate methods that can be used in situations where the underlying pattern is horizontal; in other words, no trend or seasonal effects are present. We then consider methods appropriate when trend and/or seasonality are present in the data.

## 17.2 FORECAST ACCURACY

In this section we begin by developing forecasts for the petrol time series shown in Table 17.1 using the simplest of all the forecasting methods: an approach that uses the most recent week's sales volume as the forecast for the next week. For instance, the distributor sold 17 thousand litres of petrol in week 1; this value is used as the forecast for week 2. Next, we use 21, the actual value of sales in week 2, as the forecast for week 3 and so on.

**TABLE 17.7** Computing forecasts and measures of forecast accuracy using the most recent value as the forecast for the next period

| Week   | Time series value | Forecast | Forecast error | Absolute value of forecast error | Squared forecast error | Percentage error | Absolute value of percentage error |
|--------|-------------------|----------|----------------|----------------------------------|------------------------|------------------|------------------------------------|
| 1      | 17                |          |                |                                  |                        |                  |                                    |
| 2      | 21                | 17       | 4              | 4                                | 16                     | 19.05            | 19.05                              |
| 3      | 19                | 21       | -2             | 2                                | 4                      | -10.53           | 10.53                              |
| 4      | 23                | 19       | 4              | 4                                | 16                     | 17.39            | 17.39                              |
| 5      | 18                | 23       | -5             | 5                                | 25                     | -27.78           | 27.78                              |
| 6      | 16                | 18       | -2             | 2                                | 4                      | -12.50           | 12.50                              |
| 7      | 20                | 16       | 4              | 4                                | 16                     | 20.00            | 20.00                              |
| 8      | 18                | 20       | -2             | 2                                | 4                      | -11.11           | 11.11                              |
| 9      | 22                | 18       | 4              | 4                                | 16                     | 18.18            | 18.18                              |
| 10     | 20                | 22       | -2             | 2                                | 4                      | -10.00           | 10.00                              |
| 11     | 15                | 20       | -5             | 5                                | 25                     | -33.33           | 33.33                              |
| 12     | 22                | 15       | 7              | 7                                | 49                     | 31.82            | 31.82                              |
| Totals |                   |          | 5              | 41                               | 179                    | 1.19             | 211.69                             |

The forecasts obtained for the historical data using this method are shown in Table 17.7 in the column labelled Forecast. Because of its simplicity, this method is often referred to as a *naive* forecasting method.

How accurate are the forecasts obtained using this *naive* forecasting method? To answer this question we will introduce several measures of forecast accuracy. These measures are used to determine how well a particular forecasting method is able to reproduce the time series data that are already available. By selecting the method that has the best accuracy for the data already known, we hope to increase the likelihood that we will obtain better forecasts for future time periods.

The key concept associated with measuring forecast accuracy is **forecast error**, defined as:

$$\text{Forecast Error} = \text{Actual Value} - \text{Forecast}$$

For instance, because the distributor actually sold 21 thousand litres of petrol in week 2 and the forecast, using the sales volume in week 1, was 17 thousand litres, the forecast error in week 2 is:

$$\text{Forecast Error in week 2} = 21 - 17 = 4$$

The fact that the forecast error is positive indicates that in week 2 the forecasting method underestimated the actual value of sales. Next, we use 21, the actual value of sales in week 2, as the forecast for week 3. Since the actual value of sales in week 3 is 19, the forecast error for week 3 is  $19 - 21 = -2$ . In this case, the negative forecast error indicates that in week 3 the forecast overestimated the actual value. Thus, the forecast error may be positive or negative, depending on whether the forecast is too low or too high. A complete summary of the forecast errors for this naive forecasting method is shown in Table 17.7 in the column labelled forecast error.

A simple measure of forecast accuracy is the mean or average of the forecast errors. Table 17.7 shows that the sum of the forecast errors for the petrol sales time series is 5; thus, the mean or average forecast error is  $5/11 = 0.45$ . Note that although the petrol time series consists of 12 values, to compute the mean error we divided the sum of the forecast errors by 11 because there are only 11 forecast errors. Because the mean forecast error is positive, the method is under-forecasting; in other words, the observed values tend to be greater than the forecasted values. Because positive and negative forecast errors tend to offset one another, the mean error is likely to be small; thus, the mean error is not a very useful measure of forecast accuracy.

The **mean absolute error**, denoted MAE, is a measure of forecast accuracy that avoids the problem of positive and negative forecast errors offsetting one another. As you might expect given its name, MAE is the average of the absolute values of the forecast errors. Table 17.7 shows that the sum of the absolute values of the forecast errors is 41; thus,

$$\text{MAE} = \text{average of the absolute value of forecast errors} = \frac{41}{11} = 3.73$$

Another measure that avoids the problem of positive and negative forecast errors offsetting each other is obtained by computing the average of the squared forecast errors. This measure of forecast accuracy, referred to as the **mean squared error**, is denoted MSE. From Table 17.7, the sum of the squared errors is 179; hence,

$$\text{MSE} = \text{average of the sum of squared forecast errors} = \frac{179}{11} = 16.27$$

The size of MAE and MSE depends upon the scale of the data. As a result, it is difficult to make comparisons for different time intervals, such as comparing a method of forecasting monthly petrol sales to a method of forecasting weekly sales, or to make comparisons across different time series. To make comparisons like these we need to work with relative or percentage error measures. The **mean absolute percentage error**, denoted MAPE, is such a measure. To compute MAPE we must first compute the percentage error for each forecast. For example, the percentage error corresponding to the forecast of 17 in week 2 is computed by dividing the forecast error in week 2 by the actual value in week 2 and multiplying the result by 100. For week 2 the percentage error is computed as follows:

$$\text{percentage error for week 2} = \frac{4}{21} (100) = 19.05\%$$

Thus, the forecast error for week 2 is 19.05 per cent of the observed value in week 2. A complete summary of the percentage errors is shown in Table 17.7 in the column labelled percentage error. In the next column, we show the absolute value of the percentage error.

Table 17.7 shows that the sum of the absolute values of the percentage errors is 211.69; thus,

$$\text{MAPE} = \text{average of the absolute value of percentage forecast errors} = \frac{211.69}{11} = 19.24\%$$

Summarizing, using the naive (most recent observation) forecasting method, we obtained the following measures of forecast accuracy:

$$\begin{aligned}\text{MAE} &= 3.73 \\ \text{MSE} &= 16.27 \\ \text{MAPE} &= 19.24\%\end{aligned}$$

These measures of forecast accuracy simply measure how well the forecasting method is able to forecast historical values of the time series. Now, suppose we want to forecast sales for a future time period, such as week 13. In this case the forecast for week 13 is 22, the actual value of the time series in week 12. Is this an accurate estimate of sales for week 13? Unfortunately, there is no way to address the issue of accuracy associated with forecasts for future time periods. But, if we select a forecasting method that works well for the historical data, and we think that the historical pattern will continue into the future, we should obtain results that will ultimately be shown to be good.

Before closing this section, let's consider another method for forecasting the petrol sales time series in Table 17.1. Suppose we use the average of all the historical data available as the forecast for the next period. We begin by developing a forecast for week 2. Since there is only one historical value available prior to week 2, the forecast for week 2 is just the time series value in week 1; thus, the forecast for week 2 is 17 thousand litres of petrol. To compute the forecast for week 3, we take the average of the sales values in weeks 1 and 2. Thus,

$$\text{Forecast for week 3} = \frac{17 + 21}{2} = 19$$

Similarly, the forecast for week 4 is:

$$\text{Forecast for week 4} = \frac{17 + 21 + 19}{3} = 19$$

The forecasts obtained using this method for the petrol time series are shown in Table 17.8 in the column labelled forecast. Using the results shown in Table 17.8, we obtained the following values of MAE, MSE and MAPE:

$$\begin{aligned}\text{MAE} &= \frac{26.81}{11} = 2.44 \\ \text{MSE} &= \frac{89.07}{11} = 8.10 \\ \text{MAPE} &= \frac{141.34}{11} = 12.85\%\end{aligned}$$

We can now compare the accuracy of the two forecasting methods we have considered in this section by comparing the values of MAE, MSE and MAPE for each method.

|      | Naive method | Average of past values |
|------|--------------|------------------------|
| MAE  | 3.73         | 2.44                   |
| MSE  | 16.27        | 8.10                   |
| MAPE | 19.24%       | 12.85%                 |

For every measure, the average of past values provides more accurate forecasts than using the most recent observation as the forecast for the next period. In general, if the underlying time series is stationary, the average of all the historical data will always provide the best results.

**TABLE 17.8** Computing forecasts and measures of forecast accuracy using the average of all the historical data as the forecast for the next period

| Week | Time series value | Forecast | Forecast error | Absolute value of forecast error | Squared forecast error | Percentage error | Absolute value of percentage error |
|------|-------------------|----------|----------------|----------------------------------|------------------------|------------------|------------------------------------|
| 1    | 17                |          |                |                                  |                        |                  |                                    |
| 2    | 21                | 17.00    | 4.00           | 4.00                             | 16.00                  | 19.05            | 19.05                              |
| 3    | 19                | 19.00    | 0.00           | 0.00                             | 0.00                   | 0.00             | 0.00                               |
| 4    | 23                | 19.00    | 4.00           | 4.00                             | 16.00                  | 17.39            | 17.39                              |
| 5    | 18                | 20.00    | -2.00          | 2.00                             | 4.00                   | -11.11           | 11.11                              |
| 6    | 16                | 19.60    | -3.60          | 3.60                             | 12.96                  | -22.50           | 22.50                              |
| 7    | 20                | 19.00    | 1.00           | 1.00                             | 1.00                   | 5.00             | 5.00                               |
| 8    | 18                | 19.14    | -1.14          | 1.14                             | 1.31                   | -6.35            | 6.35                               |
| 9    | 22                | 19.00    | 3.00           | 3.00                             | 9.00                   | 13.64            | 13.64                              |
| 10   | 20                | 19.33    | 0.67           | 0.67                             | 0.44                   | 3.33             | 3.33                               |
| 11   | 15                | 19.40    | -4.40          | 4.40                             | 19.36                  | -29.33           | 29.33                              |
| 12   | 22                | 19.00    | 3.00           | 3.00                             | 9.00                   | 13.64            | 13.64                              |
|      | Totals            | 4.53     |                | 26.81                            | 89.07                  | 2.76             | 141.34                             |

But suppose that the underlying time series is not stationary. In Section 17.1 we mentioned that changes in business conditions can often result in a time series that has a horizontal pattern shifting to a new level. We discussed a situation in which the petrol distributor signed a contract with the Guardia Civil to provide petrol for police cars located in northern Spain. Table 17.2 shows the number of litres of petrol sold for the original time series and the ten weeks after signing the new contract, and Figure 17.2 shows the corresponding time series plot. Note the change in level in week 13 for the resulting time series. When a shift to a new level like this occurs, it takes a long time for the forecasting method that uses the average of all the historical data to adjust to the new level of the time series. But, in this case, the simple naive method adjusts very rapidly to the change in level because it uses the most recent observation available as the forecast.

Measures of forecast accuracy are important factors in comparing different forecasting methods, but we have to be careful not to rely upon them too heavily. Good judgement and knowledge about business conditions that might affect the forecast also have to be carefully considered when selecting a method. And historical forecast accuracy is not the only consideration, especially if the time series is likely to change in the future.

In the next section we will introduce more sophisticated methods for developing forecasts for a time series that exhibits a horizontal pattern. Using the measures of forecast accuracy developed here, we will be able to determine if such methods provide more accurate forecasts than we obtained using the simple approaches illustrated in this section. The methods that we will introduce also have the advantage of adapting well in situations where the time series changes to a new level. The ability of a forecasting method to adapt quickly to changes in level is an important consideration, especially in short-term forecasting situations.

## EXERCISES

### Methods

- Consider the following time series data.

| Week  | 1  | 2  | 3  | 4  | 5  | 6  |
|-------|----|----|----|----|----|----|
| Value | 18 | 13 | 16 | 11 | 17 | 14 |

Using the naive method (most recent value) as the forecast for the next week, compute the following measures of forecast accuracy.

- a. Mean absolute error.
  - b. Mean squared error.
  - c. Mean absolute percentage error.
  - d. What is the forecast for week 7?
- Refer to the time series data in Exercise 1. Using the average of all the historical data as a forecast for the next period, compute the following measures of forecast accuracy.
    - a. Mean absolute error.
    - b. Mean squared error.
    - c. Mean absolute percentage error.
    - d. What is the forecast for week 7?
  - Exercises 1 and 2 used different forecasting methods. Which method appears to provide the more accurate forecasts for the historical data? Explain.



COMPLETE  
SOLUTIONS

4. Consider the following time series data.

| Month | 1  | 2  | 3  | 4  | 5  | 6  | 7  |
|-------|----|----|----|----|----|----|----|
| Value | 24 | 13 | 20 | 12 | 19 | 23 | 15 |

- a. Compute MSE using the most recent value as the forecast for the next period. What is the forecast for month 8?
- b. Compute MSE using the average of all the data available as the forecast for the next period. What is the forecast for month 8?
- c. Which method appears to provide the better forecast?

## 17.3 MOVING AVERAGES AND EXPONENTIAL SMOOTHING

In this section we discuss three forecasting methods that are appropriate for a time series with a horizontal pattern: moving averages, weighted moving averages and exponential smoothing. These methods also adapt well to changes in the level of a horizontal pattern such as we saw with the extended petrol sales time series (Table 17.2 and Figure 17.2). However, without modification they are not appropriate when significant trend, cyclical or seasonal effects are present. Because the objective of each of these methods is to ‘smooth out’ the random fluctuations in the time series, they are referred to as smoothing methods. These methods are easy to use and generally provide a high level of accuracy for short-range forecasts, such as a forecast for the next time period.

### Moving averages

The **moving averages** method uses the average of the most recent  $k$  data values in the time series as the forecast for the next period. Mathematically, a moving average forecast of order  $k$  is as follows:

#### Moving average forecast of order $k$

$$F_{t+1} = \frac{\sum(\text{most recent } k \text{ data values})}{k} = \frac{Y_t + Y_{t-1} + \dots + Y_{t-k+1}}{k} \quad (17.1)$$

where  
 $F_{t+1}$  = forecast of the times series for period  $t+1$   
 $Y_t$  = actual value of the time series in period  $t$

The term *moving* is used because every time a new observation becomes available for the time series, it replaces the oldest observation in the equation and a new average is computed. As a result, the average will change, or move, as new observations become available.

To illustrate the moving averages method, let us return to the petrol sales data in Table 17.1 and Figure 17.1. The time series plot in Figure 17.1 indicates that the petrol sales time series has a horizontal pattern. Thus, the smoothing methods of this section are applicable.

To use moving averages to forecast a time series, we must first select the order, or number of time series values, to be included in the moving average. If only the most recent values of the time series are considered relevant, a small value of  $k$  is preferred. If more past values are considered relevant, then a larger value of  $k$  is better. As mentioned earlier, a time series with a horizontal pattern can shift to a new level over time. A moving average will adapt to the new level of the series and resume providing good forecasts in  $k$  periods. Thus, a smaller value of  $k$  will track shifts in a time series more quickly. But larger values of  $k$  will be more effective in smoothing out the random fluctuations over time. So managerial judgement based on an understanding of the behaviour of a time series is helpful in choosing a good value for  $k$ .

To illustrate how moving averages can be used to forecast petrol sales, we will use a three-week moving average ( $k = 3$ ). We begin by computing the forecast of sales in week 4 using the average of the time series values in weeks 1–3.

$$F_4 = \text{average of weeks } 1 - 3 = \frac{17 + 21 + 19}{3} = 19$$

Thus, the moving average forecast of sales in week 4 is 19 or 19 000 litres of petrol. Because the actual value observed in week 4 is 23, the forecast error in week 4 is  $23 - 19 = 4$ .

Next, we compute the forecast of sales in week 5 by averaging the time series values in weeks 2–4.

$$F_5 = \text{average of weeks } 2 - 4 = \frac{21 + 19 + 23}{3} = 21$$

Hence, the forecast of sales in week 5 is 21 and the error associated with this forecast is  $18 - 21 = -3$ . A complete summary of the three-week moving average forecasts for the petrol sales time series is provided in Table 17.9. Figure 17.7 shows the original time series plot and the three-week moving average forecasts. Note how the graph of the moving average forecasts has tended to smooth out the random fluctuations in the time series.

To forecast sales in week 13, the next time period in the future, we simply compute the average of the time series values in weeks 10, 11, and 12.

$$F_{13} = \text{average of weeks } 10 - 12 = \frac{20 + 15 + 22}{3} = 19$$

Thus, the forecast for week 13 is 19 or 19 000 litres of petrol.

### Forecast accuracy

In Section 17.2 we discussed three measures of forecast accuracy: MAE, MSE and MAPE. Using the three-week moving average calculations in Table 17.9, the values for these three measures of forecast accuracy are:

$$\text{MAE} = \frac{24}{9} = 2.67$$

$$\text{MSE} = \frac{92}{9} = 10.22$$

$$\text{MAPE} = \frac{129.21}{9} = 14.36\%$$

In Section 17.2 we also showed that using the most recent observation as the forecast for the next week (a moving average of order  $k = 1$ ) resulted in values of  $\text{MAE} = 3.73$ ,  $\text{MSE} = 16.27$  and  $\text{MAPE} = 19.24\%$ . Thus, in each case the three-week moving average approach provided more accurate forecasts than simply using the most recent observation as the forecast.

To determine if a moving average with a different order  $k$  can provide more accurate forecasts, we recommend using trial and error to determine the value of  $k$  that minimizes MSE. For the petrol sales time series, it can be shown that the minimum value of MSE corresponds to a moving average of order  $k = 6$  with  $\text{MSE} = 6.79$ . If we are willing to assume that the order of the moving average that is best for the historical data will also be best for future values of the time series, the most accurate moving average forecasts of petrol sales can be obtained using a moving average of order  $k = 6$ .

### Weighted moving averages

With the moving averages method, each observation in the moving average calculation receives the same weight. One variation, known as **weighted moving averages**, involves selecting a different weight for each data value and then computing a weighted average of the most recent  $k$  values as the forecast. In most cases, the most recent observation receives the most weight, and the weight decreases for older data values. Let us use the petrol sales time series to illustrate the computation of a weighted three-week moving average. We assign a weight of  $3/6$  to the most recent observation, a weight of  $2/6$  to the second most recent observation and a weight of  $1/6$  to the third most recent observation. Using this weighted average, our forecast for week 4 is computed as follows:

$$\text{Forecast for week 4} = \frac{1}{6} (17) + \frac{2}{6} (21) + \frac{3}{6} (19) = 19.33$$

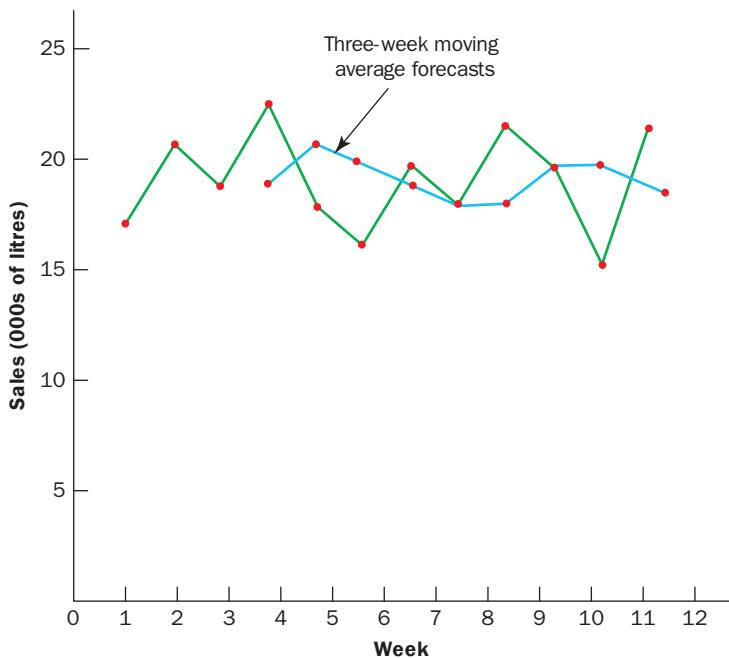
Note that for the weighted moving average method the sum of the weights is equal to 1.

**TABLE 17.9** Summary of three-week moving average calculations

| Week | Time series value | Forecast | Forecast error | Absolute value of forecast error | Squared forecast error | Percentage error | Absolute value of percentage error |
|------|-------------------|----------|----------------|----------------------------------|------------------------|------------------|------------------------------------|
| 1    | 17                |          |                |                                  |                        |                  |                                    |
| 2    | 21                |          |                |                                  |                        |                  |                                    |
| 3    | 19                |          |                |                                  |                        |                  |                                    |
| 4    | 23                | 19       | 4              | 4                                | 16                     | 17.39            | 17.39                              |
| 5    | 18                | 21       | -3             | 3                                | 9                      | -16.67           | 16.67                              |
| 6    | 16                | 20       | -4             | 4                                | 16                     | -25.00           | 25.00                              |
| 7    | 20                | 19       | 1              | 1                                | 1                      | 5.00             | 5.00                               |
| 8    | 18                | 18       | 0              | 0                                | 0                      | 0.00             | 0.00                               |
| 9    | 22                | 18       | 4              | 4                                | 16                     | 18.18            | 18.18                              |
| 10   | 20                | 20       | 0              | 0                                | 0                      | 0.00             | 0.00                               |
| 11   | 15                | 20       | -5             | 5                                | 25                     | -33.33           | 33.33                              |
| 12   | 22                | 19       | 3              | 3                                | 9                      | 13.64            | 13.64                              |
|      | Totals            |          | 0              | 24                               | 92                     | -20.79           | 129.21                             |

**FIGURE 17.7**

Petrol sales time series plot and three-week moving average forecasts



### Forecast accuracy

To use the weighted moving averages method, we must first select the number of data values to be included in the weighted moving average and then choose weights for each of the data values. In general, if we believe that the recent past is a better predictor of the future than the distant past, larger weights should be given to the more recent observations. However, when the time series is highly variable, selecting approximately equal weights for the data values may be best. The only requirement in selecting the weights is that their sum must equal 1. To determine whether one particular combination of number of data values and weights provides a more accurate forecast than another combination, we recommend using MSE as the measure of forecast accuracy. That is, if we assume that the combination that is best for the past will also be best for the future, we would use the combination of number of data values and weights that minimizes MSE for the historical time series to forecast the next value in the time series.

### Exponential smoothing

**Exponential smoothing** also uses a weighted average of past time series values as a forecast; it is a special case of the weighted moving averages method in which we select only one weight – the weight for the most recent observation. The weights for the other data values are computed automatically and become smaller as the observations move farther into the past. The exponential smoothing equation follows.

#### Exponential smoothing forecast

$$F_{t+1} = \alpha Y_t + (1 - \alpha)F_t \quad (17.2)$$

where:

- $F_{t+1}$  = forecast of the time series for period  $t+1$
- $Y_t$  = actual value of the time series in period  $t$
- $F_t$  = forecast of the time series for period  $t$
- $\alpha$  = smoothing constant ( $0 \leq \alpha \leq 1$ )

Equation (17.2) shows that the forecast for period  $t + 1$  is a weighted average of the actual value in period  $t$  and the forecast for period  $t$ . The weight given to the actual value in period  $t$  is the **smoothing constant**  $\alpha$  and the weight given to the forecast in period  $t$  is  $1 - \alpha$ . It turns out that the exponential smoothing forecast for any period is actually a weighted average of *all the previous actual values* of the time series. Let us illustrate by working with a time series involving only three periods of data:  $Y_1$ ,  $Y_2$  and  $Y_3$ .

To initiate the calculations, we let  $F_1$  equal the actual value of the time series in period 1; that is,  $F_1 = Y_1$ . Hence, the forecast for period 2 is:

$$\begin{aligned} F_2 &= \alpha Y_1 + (1 - \alpha)F_1 \\ &= \alpha Y_1 + (1 - \alpha)Y_1 \\ &= Y_1 \end{aligned}$$

We see that the exponential smoothing forecast for period 2 is equal to the actual value of the time series in period 1.

The forecast for period 3 is:

$$F_3 = \alpha Y_2 + (1 - \alpha)F_2 = \alpha Y_2 + (1 - \alpha)Y_1$$

Finally, substituting this expression for  $F_3$  in the expression for  $F_4$ , we obtain:

$$\begin{aligned} F_4 &= \alpha Y_3 + (1 - \alpha)F_3 \\ &= \alpha Y_3 + (1 - \alpha)[\alpha Y_2 + (1 - \alpha)Y_1] \\ &= \alpha Y_3 + (1 - \alpha)\alpha Y_2 + (1 - \alpha)^2 Y_1 \end{aligned}$$

We now see that  $F_4$  is a weighted average of the first three time series values. The sum of the coefficients, or weights, for  $Y_1$ ,  $Y_2$  and  $Y_3$  equals 1. A similar argument can be made to show that, in general, any forecast  $F_{t+1}$  is a weighted average of all the previous time series values.

Despite the fact that exponential smoothing provides a forecast that is a weighted average of all previous observations, past data do not all need to be saved to compute the forecast for the next period. In fact, equation (17.2) shows that once the value for the smoothing constant  $\alpha$  is selected, only two pieces of information are needed to compute the forecast  $F_{t+1}$ :  $Y_t$ , the actual value of the time series in period  $t$ , and  $F_t$ , the forecast for period  $t$ .

To illustrate the exponential smoothing approach, let us again consider the petrol sales time series in Table 17.1 and Figure 17.1. As indicated previously, to start the calculations we set the exponential smoothing forecast for period 2 equal to the actual value of the time series in period 1. Thus, with  $Y_1 = 17$ , we set  $F_2 = 17$  to initiate the computations. Referring to the time series data in Table 17.1, we find an actual time series value in period 2 of  $Y_2 = 21$ . Thus, period 2 has a forecast error of  $21 - 17 = 4$ .

Continuing with the exponential smoothing computations using a smoothing constant of  $\alpha = 0.2$ , we obtain the following forecast for period 3:

$$F_3 = 0.2Y_2 + 0.8F_2 = 0.2(21) + 0.8(17) = 17.8$$

Once the actual time series value in period 3,  $Y_3 = 19$ , is known, we can generate a forecast for period 4 as follows:

$$F_4 = 0.2Y_3 + 0.8F_3 = 0.2(19) + 0.8(17.8) = 18.04$$

Continuing the exponential smoothing calculations, we obtain the weekly forecast values shown in Table 17.10. Note that we have not shown an exponential smoothing forecast or a forecast error for week 1 because no forecast was made. For week 12, we have  $Y_{12} = 22$  and  $F_{12} = 18.48$ . We can we use this information to generate a forecast for week 13.

$$F_{13} = 0.2Y_{12} + 0.8F_{12} = 0.2(22) + 0.8(18.48) = 19.18$$

Thus, the exponential smoothing forecast of the amount sold in week 13 is 19.18, or 19 180 litres of petrol. With this forecast, the firm can make plans and decisions accordingly.

**TABLE 17.10** Summary of the exponential smoothing forecasts and forecast errors for the petrol sales time series with smoothing constant  $\alpha = 0.2$

| Week | Time series value | Forecast | Forecast error | Squared forecast error |
|------|-------------------|----------|----------------|------------------------|
| 1    | 17                |          |                |                        |
| 2    | 21                | 17.00    | 4.00           | 16.00                  |
| 3    | 19                | 17.80    | 1.20           | 1.44                   |
| 4    | 23                | 18.04    | 4.96           | 24.60                  |
| 5    | 18                | 19.03    | -1.03          | 1.06                   |
| 6    | 16                | 18.83    | -2.83          | 8.01                   |
| 7    | 20                | 18.26    | 1.74           | 3.03                   |
| 8    | 18                | 18.61    | -0.61          | 0.37                   |
| 9    | 22                | 18.49    | 3.51           | 12.32                  |
| 10   | 20                | 19.19    | 0.81           | 0.66                   |
| 11   | 15                | 19.35    | -4.35          | 18.92                  |
| 12   | 22                | 18.48    | 3.52           | 12.39                  |
|      | Totals            |          | 10.92          | 98.80                  |

**FIGURE 17.8**

Actual and forecast petrol sales time series with smoothing constant  $\alpha = 0.2$

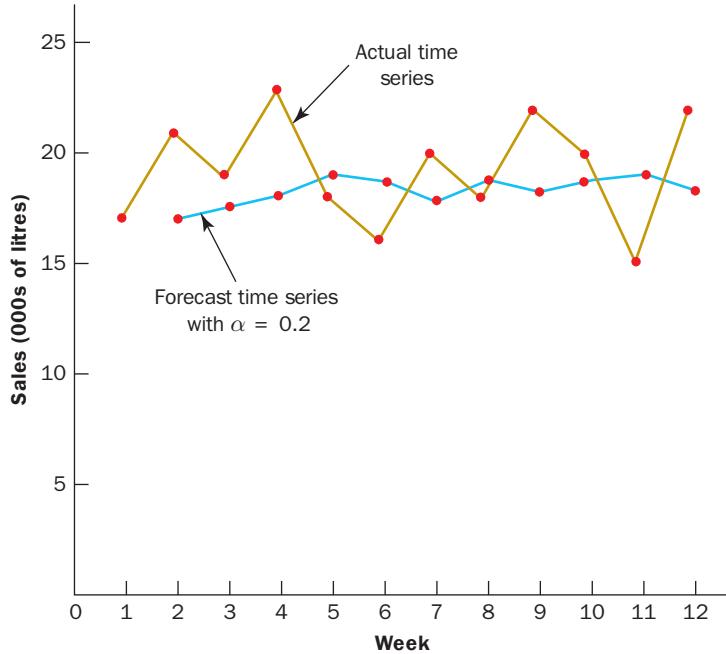


Figure 17.8 shows the time series plot of the actual and forecast time series values. Note in particular how the forecasts ‘smooth out’ the irregular or random fluctuations in the time series.

### Forecast accuracy

In the preceding exponential smoothing calculations, we used a smoothing constant of  $\alpha = 0.2$ . Although any value of  $\alpha$  between 0 and 1 is acceptable, some values will yield better forecasts than others. Insight into choosing a good value for  $\alpha$  can be obtained by rewriting the basic exponential smoothing model as follows:

$$F_{t+1} = \alpha Y_t + (1 - \alpha) F_t \quad (17.3)$$

$$F_{t+1} = \alpha Y_t + F_t - \alpha F_t$$

$$F_{t+1} = F_t + \alpha(Y_t - F_t)$$

**TABLE 17.11** Summary of the exponential smoothing forecasts and forecast errors for the petrol sales time series with smoothing constant  $\alpha = 0.3$

| Week   | Time series value | Forecast | Forecast error | Squared forecast error |
|--------|-------------------|----------|----------------|------------------------|
| 1      | 17                |          |                |                        |
| 2      | 21                | 17.00    | 4.00           | 16.00                  |
| 3      | 19                | 18.20    | 0.80           | 0.64                   |
| 4      | 23                | 18.44    | 4.56           | 20.79                  |
| 5      | 18                | 19.81    | -1.81          | 3.28                   |
| 6      | 16                | 19.27    | -3.27          | 10.69                  |
| 7      | 20                | 18.29    | 1.71           | 2.92                   |
| 8      | 18                | 18.80    | -0.80          | 0.64                   |
| 9      | 22                | 18.56    | 3.44           | 11.83                  |
| 10     | 20                | 19.59    | 0.41           | 0.17                   |
| 11     | 15                | 19.71    | -4.71          | 22.18                  |
| 12     | 22                | 18.30    | 3.70           | 13.69                  |
| Totals |                   |          | 8.03           | 102.83                 |

Thus, the new forecast  $F_{t+1}$  is equal to the previous forecast  $F_t$  plus an adjustment, which is the smoothing constant  $\alpha$  times the most recent forecast error,  $Y_t - F_t$ . That is, the forecast in period  $t + 1$  is obtained by adjusting the forecast in period  $t$  by a fraction of the forecast error. If the time series contains substantial random variability, a small value of the smoothing constant is preferred. The reason for this choice is that if much of the forecast error is due to random variability, we do not want to overreact and adjust the forecasts too quickly. For a time series with relatively little random variability, forecast errors are more likely to represent a change in the level of the series. Thus, larger values of the smoothing constant provide the advantage of quickly adjusting the forecasts; this allows the forecasts to react more quickly to changing conditions.

The criterion we will use to determine a desirable value for the smoothing constant  $\alpha$  is the same as the criterion we proposed for determining the order or number of periods of data to include in the moving averages calculation. That is, we choose the value of  $\alpha$  that minimizes the MSE. A summary of the MSE calculations for the exponential smoothing forecast of petrol sales with  $\alpha = 0.2$  is shown in Table 17.10. Note that there is one less squared error term than the number of time periods because we had no past values with which to make a forecast for period 1. The value of the sum of squared forecast errors is 98.80; hence  $MSE = 98.80/11 = 8.98$ . Would a different value of  $\alpha$  provide better results in terms of a lower MSE value? Perhaps the most straightforward way to answer this question is simply to try another value for  $\alpha$ . We will then compare its mean squared error with the MSE value of 8.98 obtained by using a smoothing constant of  $\alpha = 0.2$ .

The exponential smoothing results with  $\alpha = 0.3$  are shown in Table 17.11. The value of the sum of squared forecast errors is 102.83; hence  $MSE = 102.83/11 = 9.35$ . With  $MSE = 9.35$ , we see that, for the current data set, a smoothing constant of  $\alpha = 0.3$  results in less forecast accuracy than a smoothing constant of  $\alpha = 0.2$ . Thus, we would be inclined to prefer the original smoothing constant of  $\alpha = 0.2$ . Using a trial-and-error calculation with other values of  $\alpha$ , we can find a ‘good’ value for the smoothing constant. This value can be used in the exponential smoothing model to provide forecasts for the future. At a later date, after new time series observations are obtained, we analyze the newly collected time series data to determine whether the smoothing constant should be revised to provide better forecasting results.

## EXERCISES

### Methods

5. Consider the following time series data.

| Week  | 1  | 2  | 3  | 4  | 5  | 6  |
|-------|----|----|----|----|----|----|
| Value | 18 | 13 | 16 | 11 | 17 | 14 |

- a. Construct a time series plot. What type of pattern exists in the data?
- b. Develop the three-week moving average forecasts for this time series. Compute MSE and a forecast for week 7.
- c. Use  $\alpha = 0.2$  to compute the exponential smoothing forecasts for the time series. Compute MSE and a forecast for week 7.
- d. Compare the three-week moving average approach with the exponential smoothing approach using  $\alpha = 0.2$ . Which appears to provide more accurate forecasts based on MSE? Explain.
- e. Use a smoothing constant of  $\alpha = 0.4$  to compute the exponential smoothing forecasts. Does a smoothing constant of 0.2 or 0.4 appear to provide more accurate forecasts based on MSE? Explain.

6. Consider the following time series data.

| Month | 1  | 2  | 3  | 4  | 5  | 6  | 7  |
|-------|----|----|----|----|----|----|----|
| Value | 24 | 13 | 20 | 12 | 19 | 23 | 15 |

Construct a time series plot. What type of pattern exists in the data?

- a. Develop the three-week moving average forecasts for this time series. Compute MSE and a forecast for week 8.
  - b. Use  $\alpha = 0.2$  to compute the exponential smoothing forecasts for the time series. Compute MSE and a forecast for week 8.
  - c. Compare the three-week moving average approach with the exponential smoothing approach using  $\alpha = 0.2$ . Which appears to provide more accurate forecasts based on MSE?
  - d. Use a smoothing constant of  $\alpha = 0.4$  to compute the exponential smoothing forecasts. Does a smoothing constant of 0.2 or 0.4 appear to provide more accurate forecasts based on MSE? Explain.
7. Refer to the petrol sales time series data in Table 17.1.
- a. Compute four-week and five-week moving averages for the time series.
  - b. Compute the MSE for the four-week and five-week moving average forecasts.
  - c. What appears to be the best number of weeks of past data (three, four or five) to use in the moving average computation? Recall that MSE for the three-week moving average is 10.22.
8. Refer again to the petrol sales time series data in Table 17.1.
- a. Using a weight of  $1/2$  for the most recent observation,  $1/3$  for the second most recent observation, and  $1/6$  for third most recent observation, compute a three-week weighted moving average for the time series.
  - b. Compute the MSE for the weighted moving average in part (a). Do you prefer this weighted moving average to the unweighted moving average? Remember that the MSE for the unweighted moving average is 10.22.
  - c. Suppose you are allowed to choose any weights as long as they sum to 1. Could you always find a set of weights that would make the MSE at least as small for a weighted moving average than for an unweighted moving average? Why or why not?



PETROL

- 9.** With the petrol time series data from Table 17.1, show the exponential smoothing forecasts using  $\alpha = 0.1$ .
- Applying the MSE measure of forecast accuracy, would you prefer a smoothing constant of  $\alpha = 0.1$  or  $\alpha = 0.2$  for the petrol sales time series?
  - Are the results the same if you apply MAE as the measure of accuracy?
  - What are the results if MAPE is used?
- 10.** With a smoothing constant of  $\alpha = 0.2$ , equation (17.2) shows that the forecast for week 13 of the petrol sales data from Table 17.1 is given by  $F_{13} = 0.2Y_{12} + 0.8F_{12}$ . However, the forecast for week 12 is given by  $F_{12} = 0.2Y_{11} + 0.8F_{11}$ . Thus, we could combine these two results to show that the forecast for week 13 can be written:

$$F_{13} = 0.2Y_{12} + 0.8(0.2Y_{11} + 0.8F_{11}) = 0.2Y_{12} + 0.16Y_{11} + 0.64F_{11}$$

- Making use of the fact that  $F_{11} = 0.2Y_{10} + 0.8F_{10}$  (and similarly for  $F_{10}$  and  $F_9$ ), continue to expand the expression for  $F_{13}$  until it is written in terms of the past data values  $Y_{12}$ ,  $Y_{11}$ ,  $Y_{10}$ ,  $Y_9$ ,  $Y_8$ , and the forecast for period 8.
- Refer to the coefficients or weights for the past values  $Y_{12}$ ,  $Y_{11}$ ,  $Y_{10}$ ,  $Y_9$ ,  $Y_8$ . What observation can you make about how exponential smoothing weights past data values in arriving at new forecasts? Compare this weighting pattern with the weighting pattern of the moving averages method.

### Applications

- 11.** For SIS Cargo Services in Dubai, the monthly percentages of all shipments received on time over the past 12 months are 80, 82, 84, 83, 83, 84, 85, 84, 82, 83, 84 and 83.
- Construct a time series plot. What type of pattern exists in the data?
  - Compare the three-month moving average approach with the exponential smoothing approach for  $\alpha = 0.2$ . Which provides more accurate forecasts using MSE as the measure of forecast accuracy?
  - What is the forecast for next month?
- 12.** The values of Austrian building contracts (in millions of euros) for a 12-month period follow.
- |     |     |     |     |     |     |     |     |     |     |     |     |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 240 | 350 | 230 | 260 | 280 | 320 | 220 | 310 | 240 | 310 | 240 | 230 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
- Construct a time series plot. What type of pattern exists in the data?
  - Compare the three-month moving average approach with the exponential smoothing forecast using  $\alpha = 0.2$ . Which provides more accurate forecasts based on MSE?
  - What is the forecast for the next month?
- 13.** The following data represent indices for the seasonally adjusted merchandise trade volumes for New Zealand from 2005–2008.

|      | Year | Quarter | Index | Year | Quarter | Index |
|------|------|---------|-------|------|---------|-------|
| 2005 |      | Mar     | 999   | 2007 | Mar     | 1046  |
|      |      | Jun     | 998   |      | Jun     | 1057  |
|      |      | Sep     | 981   |      | Sep     | 1052  |
|      |      | Dec     | 1007  |      | Dec     | 1157  |
| 2006 |      | Mar     | 993   | 2008 | Mar     | 1111  |
|      |      | Jun     | 1004  |      | Jun     | 1068  |
|      |      | Sep     | 1062  |      | Sep     | 1043  |
|      |      | Dec     | 1005  |      |         |       |



COMPLETE  
SOLUTIONS

- a. Compute three- and four-quarter moving averages for this time series. Which moving average provides the better forecast for the fourth quarter of 2008?
- b. Plot the data. Do you think the exponential smoothing model would be appropriate for forecasting in this case?

## 17.4 TREND PROJECTION

We present three forecasting methods in this section that are appropriate for time series exhibiting a trend pattern. First, we show how simple linear regression can be used to forecast a time series with a linear trend. We then illustrate how to develop forecasts using Holt's **linear exponential smoothing**, an extension of single exponential smoothing that uses two smoothing constants: one to account for the level of the time series and a second to account for the linear trend in the data. Finally, we show how the curve-fitting capability of regression analysis can also be used to forecast time series with a curvilinear or nonlinear trend.

### Linear trend regression

In Section 17.1 we used the bicycle sales time series in Table 17.3 and Figure 17.3 to illustrate a time series with a trend pattern. Let us now use this time series to illustrate how simple linear regression can be used to forecast a time series with a linear trend. The data for the bicycle time series are repeated in Table 17.12 and Figure 17.9.

Although the time series plot in Figure 17.9 shows some up and down movement over the past ten years, we might agree that the linear trend line shown in Figure 17.10 provides a reasonable approximation of the long-run movement in the series. We can use the methods of simple linear regression (see Chapter 14) to develop such a linear trend line for the bicycle sales time series.

In Chapter 14, the estimated regression equation describing a straight-line relationship between an independent variable  $x$  and a dependent variable  $y$  is written as:

$$\hat{y} = b_0 + b_1x$$

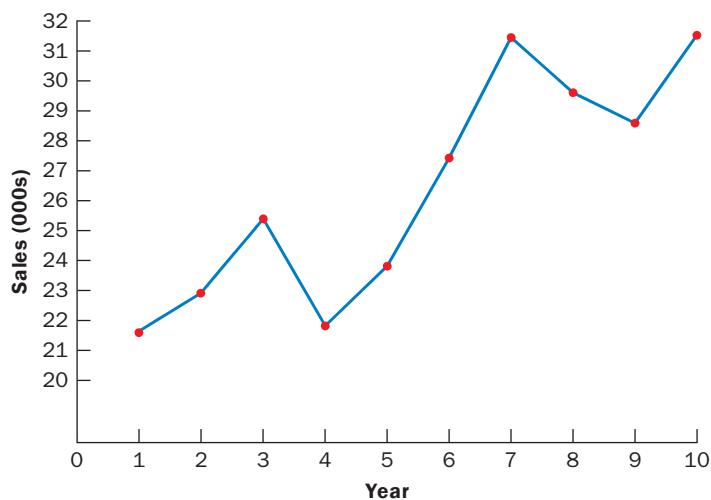
where  $\hat{y}$  is the estimated or predicted value of  $y$ . To emphasize the fact that in forecasting the independent variable is time, we will replace  $x$  with  $t$  and  $\hat{y}$  with  $T_t$  to emphasize that we are estimating the trend for a time series. Thus, for estimating the linear trend in a time series we will use the following estimated regression equation.

**TABLE 17.12** Bicycle sales time series

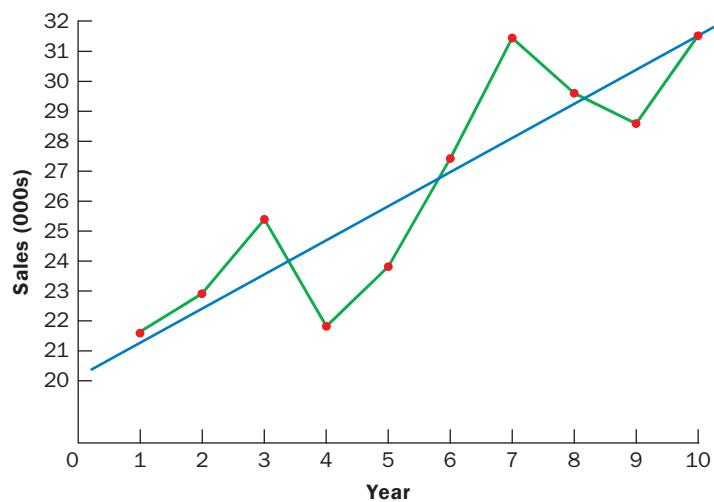
| Year | Sales (000s) |
|------|--------------|
| 1    | 21.6         |
| 2    | 22.9         |
| 3    | 25.5         |
| 4    | 21.9         |
| 5    | 23.9         |
| 6    | 27.5         |
| 7    | 31.5         |
| 8    | 29.7         |
| 9    | 28.6         |
| 10   | 31.4         |

**FIGURE 17.9**

Bicycle sales time series plot

**FIGURE 17.10**

Trend represented by a linear function for bicycle sales



### Linear trend equation

where:

$$T_t = b_0 + b_1 t \quad (17.4)$$

 $T_t$  = linear trend forecast in period  $t$  $b_0$  = intercept of the linear trend line $b_1$  = slope of the linear trend line $t$  = time period

In equation (17.4), the time variable begins at  $t = 1$  corresponding to the first time series observation (year 1 for the bicycle sales time series) and continues until  $t = n$  corresponding to the most recent time series observation (year 10 for the bicycle sales time series). Thus the bicycle sales time series  $t = 1$  corresponds to the oldest time series value and  $t = 10$  corresponds to the most recent year.

Formulae for computing the estimated regression coefficients ( $b_1$  and  $b_0$ ) in equation (17.4) follow.

### Computing the slope and intercept for a linear trend\*

$$b_1 = \frac{\sum_{t=1}^n (t - \bar{t})(Y_t - \bar{Y})}{\sum_{t=1}^n (t - \bar{t})^2} \quad (17.5)$$

$$b_0 = \bar{Y} - b_1 \bar{t} \quad (17.6)$$

where:

$Y_t$  = value of the time series in period  $t$

$n$  = number of time periods (number of observations)

$\bar{Y}$  = average value of the time series

$\bar{t}$  = average value of  $t$

This form of equation (17.5) is often recommended when using a calculator to compute  $b_1$ .

To compute the linear trend equation for the bicycle sales time series, we begin the calculations by computing  $\bar{t}$  and  $\bar{Y}$  using the information in Table 17.12.

$$\bar{t} = \frac{\sum_{t=1}^n t}{n} = \frac{55}{10} = 5.5$$

$$\bar{Y} = \frac{\sum_{t=1}^n Y_t}{n} = \frac{264.5}{10} = 26.45$$

Using these values, and the information in Table 17.13, we can compute the slope and intercept of the trend line for the bicycle sales time series.

$$b_1 = \frac{\sum_{t=1}^n (t - \bar{t})(Y_t - \bar{Y})}{\sum_{t=1}^n (t - \bar{t})^2} = \frac{90.75}{82.5} = 1.1$$

$$b_0 = \bar{Y} - b_1 \bar{t} = 26.45 - 1.1(5.5) = 20.4$$

Therefore, the linear trend equation is:

$$T_t = 20.4 + 1.1t$$

The slope of 1.1 indicates that over the past ten years the firm experienced an average growth in sales of about 1100 units per year. If we assume that the past ten-year trend in sales is a good indicator of the future, this trend equation can be used to develop forecasts for future time periods. For example, substituting  $t = 11$  into the equation yields next year's trend projection or forecast,  $T_{11}$ .

$$T_{11} = 20.4 + 1.1(11) = 32.5$$

\*An alternate formula for  $b_1$  is:

$$b_1 = \frac{\sum_{t=1}^n t Y_t - \left( \sum_{t=1}^n t \right) \left( \sum_{t=1}^n Y_t \right) / n}{\sum_{t=1}^n t^2 - \left( \sum_{t=1}^n t \right)^2 / n}$$

**TABLE 17.13** Summary of linear trend calculations for the bicycle sales time series

| <b>t</b> | <b>Y<sub>t</sub></b> | <b>t – <math>\bar{t}</math></b> | <b>Y<sub>t</sub> – <math>\bar{Y}</math></b> | <b>(t – <math>\bar{t}</math>)(Y<sub>t</sub> – <math>\bar{Y}</math>)</b> | <b>(t – <math>\bar{t}</math>)<sup>2</sup></b> |
|----------|----------------------|---------------------------------|---|---|---|
| 1        | 21.6                 | -4.5                            | -4.85                                       | 21.825  | 20.25   |
| 2        | 22.9                 | -3.5                            | -3.55                                       | 12.425  | 12.25   |
| 3        | 25.5                 | -2.5                            | -0.95                                       | 2.375   | 6.25  |
| 4        | 21.9                 | -1.5                            | -4.55                                       | 6.825   | 2.25  |
| 5        | 23.9                 | -0.5                            | -2.55                                       | 1.275   | 0.25  |
| 6        | 27.5                 | 0.5                             | 1.05  | 0.525   | 0.25  |
| 7        | 31.5                 | 1.5                             | 5.05  | 7.575   | 2.25  |
| 8        | 29.7                 | 2.5                             | 3.25  | 8.125   | 6.25  |
| 9        | 28.6                 | 3.5                             | 2.15  | 7.525   | 12.25   |
| 10       | 31.4                 | 4.5                             | 4.95  | 22.275  | 20.25   |
| Totals   | 55                   | 264.5                           |   | 90.750  | 82.50   |

Thus, using trend projection, we would forecast sales of 32 500 bicycles next year.

To compute the accuracy associated with the trend projection forecasting method, we will use the MSE. Table 17.14 shows the computation of the sum of squared errors for the bicycle sales time series. Thus, for the bicycle sales time series,

$$\text{MSE} = \frac{\sum_{t=1}^n (Y_t - F_t)^2}{n} = \frac{30.7}{10} = 3.07$$

Because linear trend regression in forecasting uses the same regression analysis procedure introduced in Chapter 14, we can use the standard regression analysis procedures in MINITAB or EXCEL to perform the calculations. Figure 17.11 shows the computer output for the bicycle sales time series obtained using MINITAB's regression analysis module.

In Figure 17.11 the value of MSE in the ANOVA table is:

$$\text{MSE} = \frac{\text{Sum of Squares Due to Error}}{\text{Degrees of Freedom}} = \frac{30.7}{8} = 3.837$$

This value of MSE differs from the value of MSE that we computed previously because the sum of squared errors is divided by 8 instead of 10; thus, MSE in the regression output is not the average of the squared forecast errors.

**TABLE 17.14** Summary of the linear trend forecasts and forecast errors for the bicycle sales time series

| <b>Year</b> | <b>Sales (000s) Y<sub>t</sub></b> | <b>Forecast T<sub>t</sub></b> | <b>Forecast error</b> | <b>Squared forecast error</b> |
|-------------|-----------------------------------|-------------------------------|-----------------------|-------------------------------|
| 1           | 21.6                              | 21.5                          | 0.1                   | 0.01                          |
| 2           | 22.9                              | 22.6                          | 0.3                   | 0.09                          |
| 3           | 25.5                              | 23.7                          | 1.8                   | 3.24                          |
| 4           | 21.9                              | 24.8                          | -2.9                  | 8.41                          |
| 5           | 23.9                              | 25.9                          | -2.0                  | 4.00                          |
| 6           | 27.5                              | 27.0                          | 0.5                   | 0.25                          |
| 7           | 31.5                              | 28.1                          | 3.4                   | 11.56                         |
| 8           | 29.7                              | 29.2                          | 0.5                   | 0.25                          |
| 9           | 28.6                              | 30.3                          | -1.7                  | 2.89                          |
| 10          | 31.4                              | 31.4                          | 0.0                   | 0.00                          |
|             |                                   | Total                         |                       | 30.70                         |

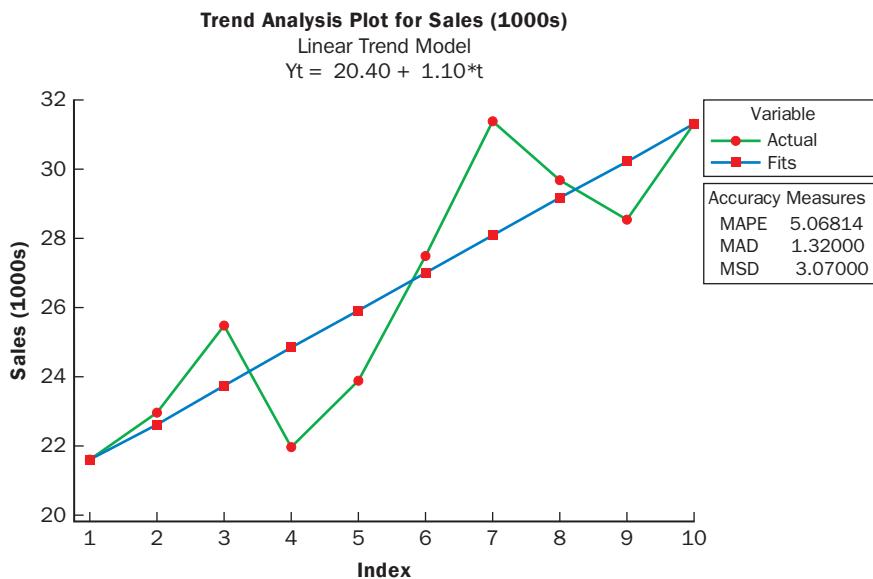
|                                   |              |                |           |                   |          |
|-----------------------------------|--------------|----------------|-----------|-------------------|----------|
| <b>The regression equation is</b> |              |                |           |                   |          |
| <b>Y = 20.4 + 1.10 t</b>          |              |                |           |                   |          |
| <b>Predictor</b>                  | <b>Coeff</b> | <b>SE Coef</b> | <b>T</b>  | <b>p</b>          |          |
| Constant                          | 20.400       | 1.338          | 15.24     | 0.000             |          |
| t                                 | 1.1000       | 0.2157         | 5.10      | 0.001             |          |
| S = 1.95895                       | R-sq = 76.5% |                |           | R-sq(adj) = 73.5% |          |
| <b>Analysis of Variance</b>       |              |                |           |                   |          |
| <b>SOURCE</b>                     | <b>DF</b>    | <b>SS</b>      | <b>MS</b> | <b>F</b>          | <b>p</b> |
| Regression                        | 1            | 99.825         | 99.825    | 26.01             | 0.001    |
| Residual Error                    | 8            | 30.700         | 3.837     |                   |          |
| Total                             | 9            | 130.525        |           |                   |          |

**FIGURE 17.11**

MINITAB regression output for the bicycle sales time series

**FIGURE 17.12**

MINITAB time series linear trend analysis output for the bicycle sales time series



Most forecasting packages, however, compute MSE by taking the average of the squared errors. Thus, when using time series packages to develop a trend equation, the value of MSE that is reported may differ slightly from the value you would obtain using a general regression approach. For instance, in Figure 17.12, we show the graphical portion of the computer output obtained using MINITAB's Trend Analysis time series procedure. Note that  $MSD = 3.07$  is the average of the squared forecast errors. ( $MSD$  in MINITAB's Trend Analysis output is the mean squared deviation and equates to  $MSE$  defined earlier.).

## Holt's linear exponential smoothing

Charles Holt developed a version of exponential smoothing that can be used to forecast a time series with a linear trend. Recall that the exponential smoothing procedure discussed in Section 17.3 uses the smoothing constant  $\alpha$  to 'smooth out' the randomness or irregular fluctuations in a time series; and, forecasts for time period  $t + 1$  are obtained using the equation:

$$F_{t+1} = \alpha Y_t + (1 - \alpha)F_t$$

Forecasts for Holt's linear exponential smoothing method are obtained using two smoothing constants,  $\alpha$  and  $\beta$ , and three equations.

### Equations for Holt's linear exponential smoothing

$$L_t = \alpha Y_t + (1 - \alpha)(L_{t-1} + b_{t-1}) \quad (17.7)$$

$$b_t = \beta(L_t - L_{t-1}) + (1 - \beta)b_{t-1} \quad (17.8)$$

$$F_{t+k} = L_t + b_t k \quad (17.9)$$

where:

$L_t$  = estimate of the level of the time series in period  $t$

$b_t$  = estimate of the slope of the time series in period  $t$

$\alpha$  = smoothing constant for the level of the time series

$\beta$  = smoothing constant for the slope of the time series

$F_{t+k}$  = forecast for  $k$  periods ahead

$k$  = the number of periods ahead to be forecast

Let us apply Holt's method to the bicycle sales time series in Table 17.12 using  $\alpha = 0.1$  and  $\beta = 0.2$ . To get the method started, we need values for  $L_1$ , the estimate of the level of the time series in year 1, and  $b_1$ , the estimate of the slope of the time series in year 1. A commonly used approach is to set  $L_1 = Y_1$  and  $b_1 = Y_2 - Y_1$ . Using this start up procedure, we obtain:

$$\begin{aligned} L_1 &= Y_1 = 21.6 \\ b_1 &= Y_2 - Y_1 = 22.9 - 21.6 = 1.3 \end{aligned}$$

Using equation (17.9) with  $k = 1$ , the forecast of sales in year 2 is  $F_2 = L_1 + b_1 = 21.6 + 1.3(1) = 22.9$ . Then we move on using equations (17.7) to (17.9) to compute estimates of the level and trend for year 2 as well as a forecast for year 3.

First we use equation (17.7) and the smoothing constant  $\alpha = 0.1$  to compute an estimate of the level of the time series in year 2.

$$L_2 = 0.1(22.9) + 0.9(21.6 + 1.3) = 22.9$$

Note that  $21.6 + 1.3$  is the forecast of sales for year 2. Thus, the estimate of the level of the time series in year 2 obtained using equation (17.7) is simply a weighted average of the observed value in year 2 (using a weight of  $\alpha = 0.1$ ) and the forecast for year 2 (using a weight of  $1 - \alpha = 1 - 0.1 = 0.9$ ). In general, large values of  $\alpha$  place more weight on the observed value ( $Y_t$ ), whereas smaller values place more weight on the forecasted value ( $L_{t-1} + b_{t-1}$ ).

Next we use equation (17.8) and the smoothing constant  $\beta = 0.2$  to compute an estimate of the slope of the time series in year 2.

$$b_2 = 0.2(22.9 - 21.6) + (1 - 0.2)(1.3) = 1.3$$

The estimate of the slope of the time series in year 2 is a weighted average of the difference in the estimated level of the time series between year 2 and year 1 (using a weight of  $\beta = 0.2$ ) and the estimate of the slope in year 1 (using a weight of  $1 - \beta = 1 - 0.2 = 0.8$ ). In general, higher values of  $\beta$  place more weight on the difference between the estimated levels, whereas smaller values place more weight on the estimate of the slope from the last period.

Using the estimates of  $L_2$  and  $b_2$  just obtained, the forecast of sales for year 3 is computed using equation (17.9):

$$F_3 = L_2 + b_2 = 22.9 + 1.3(1) = 24.2$$

The other calculations are made in a similar manner and are shown in Table 17.15. The sum of the squared forecast errors is 39.678; hence  $MSE = 39.678/9 = 4.41$ .

**TABLE 17.15** Summary calculations for Holt's linear exponential smoothing for the bicycle sales time series using  $\alpha = 0.1$  and  $\beta = 0.2$

| Year | Sales<br>(000s)<br>$y_t$ | Estimated<br>level $L_t$ | Estimated<br>trend $b_t$ | Forecast<br>$F_t$ | Forecast<br>error | Squared<br>forecast<br>error |
|------|--------------------------|--------------------------|--------------------------|-------------------|-------------------|------------------------------|
| 1    | 21.6                     | 21.600                   | 1.300                    |                   |                   |                              |
| 2    | 22.9                     | 22.900                   | 1.300                    | 22.900            | 0.000             | 0.000                        |
| 3    | 25.5                     | 24.330                   | 1.326                    | 24.200            | 1.300             | 1.690                        |
| 4    | 21.9                     | 25.280                   | 1.251                    | 25.656            | -3.756            | 14.108                       |
| 5    | 23.9                     | 26.268                   | 1.198                    | 26.531            | -2.631            | 6.924                        |
| 6    | 27.5                     | 27.470                   | 1.199                    | 27.466            | 0.034             | 0.001                        |
| 7    | 31.5                     | 28.952                   | 1.256                    | 28.669            | 2.831             | 8.016                        |
| 8    | 29.7                     | 30.157                   | 1.245                    | 30.207            | -0.507            | 0.257                        |
| 9    | 28.6                     | 31.122                   | 1.189                    | 31.402            | -2.802            | 7.851                        |
| 10   | 31.4                     | 32.220                   | 1.171                    | 32.311            | -0.911            | 0.830                        |
|      |                          |                          |                          | Total             | 36.678            |                              |

Will different values for the smoothing constants  $\alpha$  and  $\beta$  provide more accurate forecasts? To answer this question we would have to try different combinations of  $\alpha$  and  $\beta$ , to determine if a combination can be found that will provide a value of MSE lower than 4.41, the value we obtained using smoothing constants  $\alpha = 0.1$  and  $\beta = 0.2$ . Searching for good values of  $\alpha$  and  $\beta$  can be done by trial and error or using more advanced statistical software packages that have an option for selecting the optimal set of smoothing constants.

Note that the estimate of the level of the time series in year 10 is  $L_{10} = 32.220$  and the estimate of the slope in year 10 is  $b_{10} = 1.171$ . If we assume that the past ten-year trend in sales is a good indicator of the future, equation (17.9) can be used to develop forecasts for future time periods. For example, substituting  $t = 11$  into equation (17.9) yields next year's trend projection or forecast,  $F_{11}$ .

$$F_{11} = L_{10} + b_{10}(1) = 32.220 + 1.171 = 33.391$$

Thus, using Holt's linear exponential smoothing we would forecast sales of 33.391 bicycles next year.

## Nonlinear trend regression

The use of a linear function to model trend is common. However, as we discussed previously, sometimes time series have a curvilinear or nonlinear trend. As an example, consider the annual revenue in millions of dollars for a cholesterol drug for the first ten years of sales. Table 17.16 shows the time series and Figure 17.13 shows the corresponding time series plot. For instance, revenue in year 1 was \$23.1 million; revenue in year 2 was \$21.3 million; and so on. The time series plot indicates an overall increasing or upward trend. But, unlike the bicycle sales time series, a linear trend does not appear to be appropriate. Instead, a curvilinear function appears to be needed to model the long-term trend.



CHOLESTEROL

### Quadratic trend equation

A variety of nonlinear functions can be used to develop an estimate of the trend for the cholesterol time series. For instance, consider the following quadratic trend equation:

$$T_t = b_0 + b_1 t + b_2 t^2 \quad (17.10)$$

For the cholesterol time series,  $t = 1$  corresponds to year 1,  $t = 2$  corresponds to year 2 and so on.

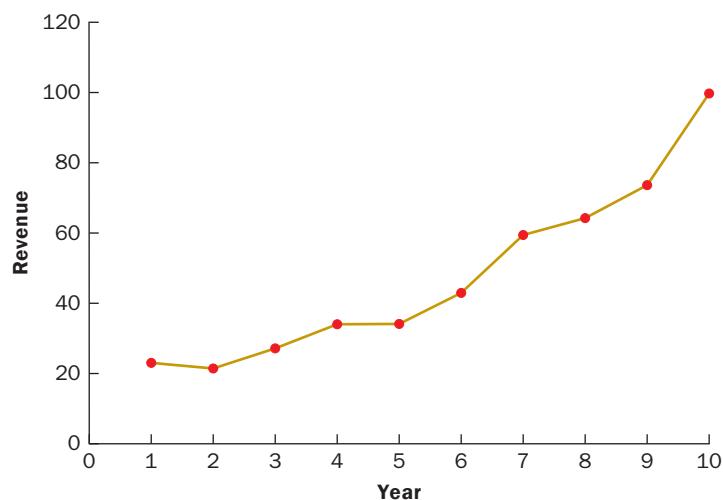
The general linear model discussed in Section 16.1 can be used to compute the values of  $b_0$ ,  $b_1$  and  $b_2$ . There are two independent variables, year and year squared, and the dependent variable is the sales revenue in millions of dollars. Thus, the first observation is 1, 1, 23.1; the second observation is 2, 4, 21.3; the third observation is 3, 9, 27.4; and so on. Figure 17.14 shows the MINITAB multiple regression output for the quadratic trend model.

**TABLE 17.16** Cholesterol revenue time series (\$ millions)

| Year ( <i>t</i> ) | Revenue (\$ millions) |
|-------------------|-----------------------|
| 1                 | 23.1                  |
| 2                 | 21.3                  |
| 3                 | 27.4                  |
| 4                 | 34.6                  |
| 5                 | 33.8                  |
| 6                 | 43.2                  |
| 7                 | 59.5                  |
| 8                 | 64.4                  |
| 9                 | 74.2                  |
| 10                | 99.3                  |

**FIGURE 17.13**

Cholesterol revenue times series plot (\$ millions)



The regression equation is

$$\text{Revenue} = 24.2 - 2.11 \text{ Year} + 0.922 \text{ YearSq}$$

| Predictor   | Coef         | SE Coef | T                 | p     |
|-------------|--------------|---------|-------------------|-------|
| Constant    | 24.182       | 4.676   | 5.17              | 0.001 |
| Year        | -2.106       | 1.953   | -1.08             | 0.317 |
| YearSq      | 0.9216       | 0.1730  | 5.33              | 0.001 |
| S = 3.97578 | R-Sq = 98.1% |         | R-Sq(adj) = 97.6% |       |

Analysis of Variance

| SOURCE         | DF | SS     | MS     | F      | p     |
|----------------|----|--------|--------|--------|-------|
| Regression     | 2  | 5770.1 | 2885.1 | 182.52 | 0.000 |
| Residual Error | 7  | 110.6  | 15.8   |        |       |
| Total          | 9  | 5880.8 |        |        |       |

**FIGURE 17.14**

MINITAB quadratic trend regression output for the bicycle sales time series

The estimated regression equation is:

$$\text{Revenue} (\$ \text{ millions}) = 24.2 - 2.11 \text{ Year} + 0.922 \text{ YearSq}$$

where:

$$\begin{aligned}\text{Year} &= 1, 2, 3, \dots, 10 \\ \text{YearSq} &= 1, 4, 9, \dots, 100\end{aligned}$$

Using the standard multiple regression procedure requires us to compute the values for year squared as a second independent variable. Alternatively, we can use MINITAB's Time Series – Trend Analysis procedure to provide the same results. It does not require developing values for year squared and is easier to use. We recommend using this approach when solving exercises that involve using quadratic trends.

### Exponential trend equation

Another alternative that can be used to model the nonlinear pattern exhibited by the cholesterol time series is to fit an exponential model to the data. For instance, consider the following exponential trend equation:

$$T_t = b_0(b_1)^t \quad (17.11)$$

To better understand this exponential trend equation, suppose  $b_0 = 20$  and  $b_1 = 1.2$ . Then, for  $t = 1$ ,  $T_1 = 20(1.2)^1 = 24$ ; for  $t = 2$ ,  $T_2 = 20(1.2)^2 = 28.8$ ; and for  $t = 3$ ,  $T_3 = 20(1.2)^3 = 34.56$ . Note that  $T_t$  is not increasing by a constant amount as in the case of the linear trend model, but by a constant percentage; the percentage increase is 20 per cent.

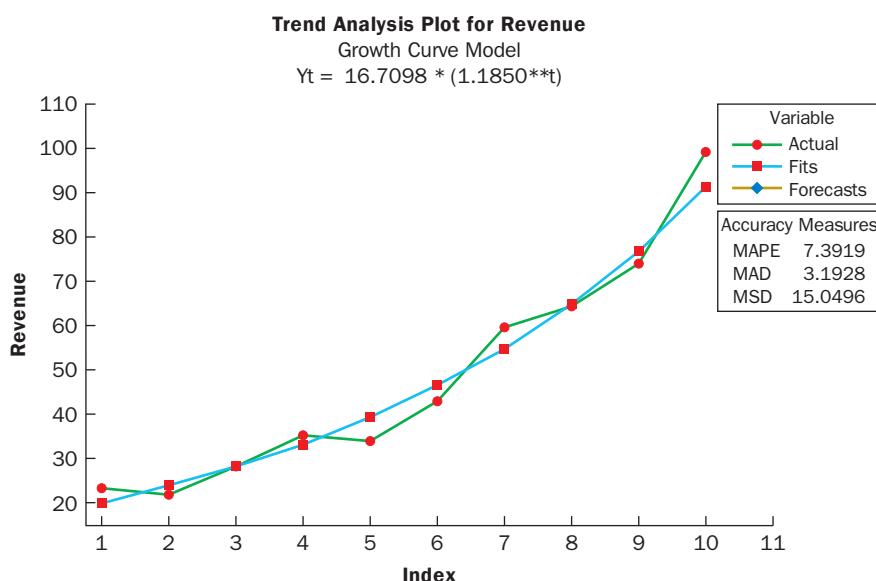
MINITAB has the capability in its time series module to compute an exponential trend equation and it can then be used for forecasting. Unfortunately, EXCEL does not have this capability. But, in Chapter 16, Section 16.1, we do describe how, by taking logarithms of the terms in equation (17.11), the general linear model methodology can be used to compute an exponential trend equation.

MINITAB's time series module is quite easy to use to develop an exponential trend equation. There is no need to deal with logarithms and use regression analysis to compute the exponential trend equation. In Figure 17.15, we show the graphical portion of the computer output obtained using MINITAB's Trend Analysis time series procedure to fit an exponential trend equation.

Linear trend regression is based upon finding the estimated regression equation that minimizes the sum of squared forecast errors and therefore MSE. So, we would expect linear trend regression to outperform Holt's linear exponential smoothing in terms of MSE. For example, for the bicycle sales time series, the value of MSE using linear trend regression is 3.07 as compared to a value of 3.97 using Holt's linear exponential smoothing. Linear trend regression also provides a more accurate forecast using the MAE measure of forecast accuracy; for the bicycle sales time series, linear trend regression results in a value of MAE of 1.32 versus a value of 1.67 using Holt's linear method.

**FIGURE  
17.15**

MINITAB time series exponential growth trend analysis output for the cholesterol sales time series



However, based on MAPE, Holt's linear exponential smoothing (MAPE = 5.07%) outperforms linear trend regression (6.42%). Hence, for the bicycle sales time series, deciding which method provides the more accurate forecasts depends upon which measure of forecast accuracy is used.

## EXERCISES

### Methods

- 14.** Consider the following time series data.



COMPLETE  
SOLUTIONS

|       |   |    |   |    |    |
|-------|---|----|---|----|----|
| $t$   | 1 | 2  | 3 | 4  | 5  |
| $Y_t$ | 6 | 11 | 9 | 14 | 15 |

- a. Construct a time series plot. What type of pattern exists in the data?
  - b. Develop the linear trend equation for this time series.
  - c. What is the forecast for  $t = 6$ ?
- 15.** Refer to the time series in Exercise 14. Use Holt's linear exponential smoothing method with  $\alpha = 0.3$  and  $\beta = 0.5$  to develop a forecast for  $t = 6$ .
- 16.** Consider the following time series.

|       |     |     |     |    |    |    |    |
|-------|-----|-----|-----|----|----|----|----|
| $t$   | 1   | 2   | 3   | 4  | 5  | 6  | 7  |
| $Y_t$ | 120 | 110 | 100 | 96 | 94 | 92 | 88 |

- a. Construct a time series plot. What type of pattern exists in the data?
  - b. Develop the linear trend equation for this time series.
  - c. What is the forecast for  $t = 8$ ?
- 17.** Consider the following time series.

|       |    |    |    |    |    |    |    |
|-------|----|----|----|----|----|----|----|
| $t$   | 1  | 2  | 3  | 4  | 5  | 6  | 7  |
| $Y_t$ | 82 | 60 | 44 | 35 | 30 | 29 | 35 |

- a. Construct a time series plot. What type of pattern exists in the data?
- b. Using MINITAB or EXCEL, develop the quadratic trend equation for the time series.
- c. What is the forecast for  $t = 8$ ?

### Applications

- 18.** Car sales at Perez Motors provided the following ten-year time series.



COMPLETE  
SOLUTIONS

| Year | Sales | Year | Sales |
|------|-------|------|-------|
| 1    | 400   | 6    | 260   |
| 2    | 390   | 7    | 300   |
| 3    | 320   | 8    | 320   |
| 4    | 340   | 9    | 340   |
| 5    | 270   | 10   | 370   |

Plot the time series and comment on the appropriateness of a linear trend. What type of functional form do you believe would be most appropriate for the trend pattern of this time series?

- 19.** Numbers of overseas visitors to Ireland (000s) estimated by the Central Statistics Office for the years 2001–2007 are as follows:

| 2001 | 2002 | 2003 | 2004 | 2005 | 2006 | 2007 |
|------|------|------|------|------|------|------|
| 5990 | 6065 | 6369 | 6574 | 6977 | 7709 | 8012 |

- a. Graph the data and assess its suitability for linear trend projection.
- b. Use a linear trend projection to forecast this time series for 2008–2009.

### GDP

- 20.** GDP (Singapore \$) for 1990–2007 are tabulated below (*Statistics Singapore*, 2009).

| Year | S\$     | Year | S\$     |
|------|---------|------|---------|
| 1990 | 66 778  | 1999 | 140 022 |
| 1991 | 74 570  | 2000 | 159 840 |
| 1992 | 80 984  | 2001 | 153 398 |
| 1993 | 93 971  | 2002 | 158 047 |
| 1994 | 107 957 | 2003 | 162 288 |
| 1995 | 119 470 | 2004 | 184 508 |
| 1996 | 130 502 | 2005 | 199 375 |
| 1997 | 142 341 | 2006 | 216 995 |
| 1998 | 137 902 | 2007 | 243 169 |

- a. Graph this time series. Does a linear trend appear to be present?
  - b. Develop a linear trend equation for this time series.
  - c. Use the trend equation to estimate the GDP for the years 2008–2010.
- 21.** Gross revenue data (in millions of euros) for Hispanic Airlines for a ten-year period follow.

| Year | Revenue | Year | Revenue |
|------|---------|------|---------|
| 1    | 2428    | 6    | 4264    |
| 2    | 2951    | 7    | 4738    |
| 3    | 3533    | 8    | 4460    |
| 4    | 3618    | 9    | 5318    |
| 5    | 3616    | 10   | 6915    |

- a. Develop a linear trend equation for this time series. Comment on what the equation tells about the gross revenue for Hispanic Airlines for the ten-year period.
- b. Provide the forecasts for gross revenue for years 11 and 12.



## 17.5 SEASONALITY AND TREND

In this section we show how to develop forecasts for a time series that has a seasonal pattern. To the extent that seasonality exists, we need to incorporate it into our forecasting models to ensure accurate forecasts. We begin by considering a seasonal time series with no trend and then discuss how to model seasonality with trend.



UMBRELLA

## Seasonality without trend

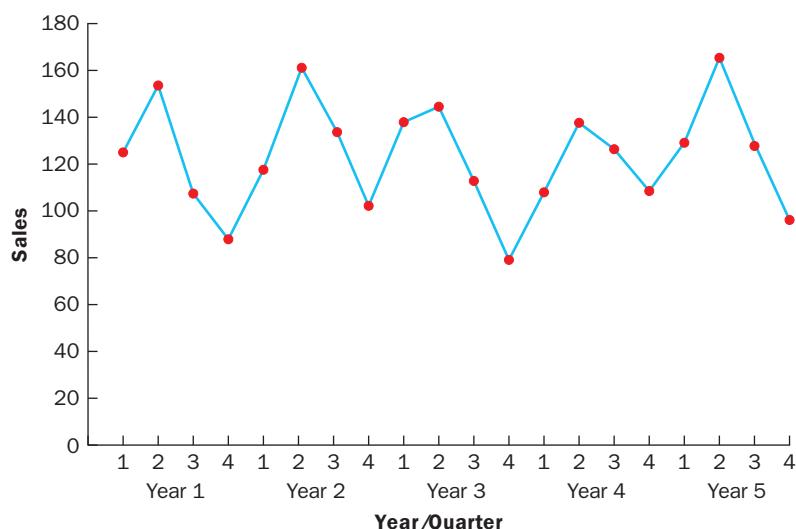
As an example, consider the number of umbrellas sold at a clothing store over the past five years. Table 17.17 shows the time series and Figure 17.16 shows the corresponding time series plot. The time series plot does not indicate any long-term trend in sales. In fact, unless you look carefully at the data, you might conclude that the data follow a horizontal pattern and that single exponential smoothing could be used to forecast sales. But closer inspection of the time series plot reveals a pattern in the data. That is, the first and third quarters have moderate sales, the second quarter has the highest sales and the fourth quarter tends to be the lowest quarter in terms of sales volume. Thus, we would conclude that a quarterly seasonal pattern is present.

**TABLE 17.17** Umbrella sales time series

| Year | Quarter | Sales |
|------|---------|-------|
| 1    | 1       | 125   |
|      | 2       | 153   |
|      | 3       | 106   |
|      | 4       | 88    |
| 2    | 1       | 118   |
|      | 2       | 161   |
|      | 3       | 133   |
|      | 4       | 102   |
| 3    | 1       | 138   |
|      | 2       | 144   |
|      | 3       | 113   |
|      | 4       | 80    |
| 4    | 1       | 109   |
|      | 2       | 137   |
|      | 3       | 125   |
|      | 4       | 109   |
| 5    | 1       | 130   |
|      | 2       | 165   |
|      | 3       | 128   |
|      | 4       | 96    |

**FIGURE 17.16**

Umbrella sales time series plot



In Chapter 15 we showed how dummy variables can be used to deal with categorical independent variables in a multiple regression model. We can use the same approach to model a time series with a seasonal pattern by treating the season as a categorical variable. Recall that when a categorical variable has  $k$  levels,  $k - 1$  dummy variables are required. So, if there are four seasons, we need three dummy variables. For instance, in the umbrella sales time series season is a categorical variable with four levels: quarter 1, quarter 2, quarter 3 and quarter 4. Thus, to model the seasonal effects in the umbrella time series we need  $4 - 1 = 3$  dummy variables. The three dummy variables can be coded as follows:

$$\text{Qtr1} = \begin{cases} 1 & \text{if Quarter 1} \\ 0 & \text{otherwise} \end{cases} \quad \text{Qtr2} = \begin{cases} 1 & \text{if Quarter 2} \\ 0 & \text{otherwise} \end{cases} \quad \text{Qtr3} = \begin{cases} 1 & \text{if Quarter 3} \\ 0 & \text{otherwise} \end{cases}$$

Using  $\hat{Y}$  to denote the estimated or forecasted value of sales, the general form of the estimated regression equation relating the number of umbrellas sold to the quarter the sales take place follows:

$$\hat{y} = b_0 + b_1 \text{ Qtr1} + b_2 \text{ Qtr2} + b_3 \text{ Qtr3}$$

Table 17.18 is the umbrella sales time series with the coded values of the dummy variables shown. Using the data in Table 17.18 and MINITAB's regression procedure, we obtained the computer output shown in Figure 17.17. The estimated multiple regression equation obtained is:

$$\text{Sales} = 95.0 + 29.0 \text{ Qtr1} + 57.0 \text{ Qtr2} + 26.0 \text{ Qtr3}$$

We can use this equation to forecast quarterly sales for next year.

$$\begin{aligned} \text{Quarter 1 : Sales} &= 95.0 + 29.0(1) + 57.0(0) + 26.0(0) = 124 \\ \text{Quarter 2 : Sales} &= 95.0 + 29.0(0) + 57.0(1) + 26.0(0) = 152 \\ \text{Quarter 3 : Sales} &= 95.0 + 29.0(0) + 57.0(0) + 26.0(1) = 121 \\ \text{Quarter 4 : Sales} &= 95.0 + 29.0(0) + 57.0(1) + 26.0(0) = 95 \end{aligned}$$

It is interesting to note that we could have obtained the quarterly forecasts for next year simply by computing the average number of umbrellas sold in each quarter, as shown in the following table.

**TABLE 17.18** Umbrella sales time series with dummy variables

| Year | Quarter | Qtr1 | Qtr2 | Qtr3 | Sales |
|------|---------|------|------|------|-------|
| 1    | 1       | 1    | 0    | 0    | 125   |
|      | 2       | 0    | 1    | 0    | 153   |
|      | 3       | 0    | 0    | 1    | 106   |
|      | 4       | 0    | 0    | 0    | 88    |
| 2    | 1       | 1    | 0    | 0    | 118   |
|      | 2       | 0    | 1    | 0    | 161   |
|      | 3       | 0    | 0    | 1    | 133   |
|      | 4       | 0    | 0    | 0    | 102   |
| 3    | 1       | 1    | 0    | 0    | 138   |
|      | 2       | 0    | 1    | 0    | 144   |
|      | 3       | 0    | 0    | 1    | 113   |
|      | 4       | 0    | 0    | 0    | 80    |
| 4    | 1       | 1    | 0    | 0    | 109   |
|      | 2       | 0    | 1    | 0    | 137   |
|      | 3       | 0    | 0    | 1    | 125   |
|      | 4       | 0    | 0    | 0    | 109   |
| 5    | 1       | 1    | 0    | 0    | 130   |
|      | 2       | 0    | 1    | 0    | 165   |
|      | 3       | 0    | 0    | 1    | 128   |
|      | 4       | 0    | 0    | 0    | 96    |

| The regression equation is<br><b>Sales = 95.0 + 29.0 Qtr1 + 57.0 Qtr2 + 26.0 Qtr3</b> |        |         |       |       |
|---|--------|---------|-------|-------|
| Predictor   | Coef   | SE Coef | T     | P     |
| Constant  | 95.000 | 5.065   | 18.76 | 0.000 |
| Qtr1  | 29.000 | 7.162   | 4.05  | 0.001 |
| Qtr2  | 57.000 | 7.162   | 7.96  | 0.000 |
| Qtr3  | 26.000 | 7.162   | 3.63  | 0.002 |

**FIGURE 17.17**

MINITAB regression output for the umbrella sales time series

| Year    | Quarter 1 | Quarter 2 | Quarter 3 | Quarter 4 |
|---------|-----------|-----------|-----------|-----------|
| 1       | 125       | 153       | 106       | 88        |
| 2       | 118       | 161       | 133       | 102       |
| 3       | 138       | 144       | 113       | 80        |
| 4       | 109       | 137       | 125       | 109       |
| 5       | 130       | 165       | 128       | 96        |
| Average | 124       | 152       | 121       | 95        |

Nonetheless, the regression output shown in Figure 17.17 provides additional information that can be used to assess the accuracy of the forecast and determine the significance of the results. And, for more complex types of problem situations, such as dealing with a time series that has both trend and seasonal effects, this simple averaging approach will not work.

## Seasonality and trend

Let us now extend the regression approach to include situations where the time series contains both a seasonal effect and a linear trend by showing how to forecast the quarterly television set sales time series introduced in Section 17.1. The data for the television set time series are shown in Table 17.19.



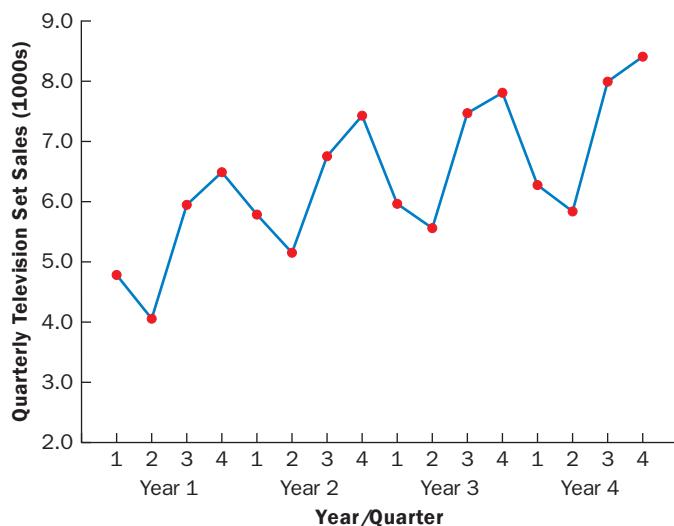
TVSALES

**TABLE 17.19** Television set sales time series

| Year | Quarter | Sales ('000s) |
|------|---------|---------------|
| 1    | 1       | 4.8           |
|      | 2       | 4.1           |
|      | 3       | 6.0           |
|      | 4       | 6.5           |
| 2    | 1       | 5.8           |
|      | 2       | 5.2           |
|      | 3       | 6.8           |
|      | 4       | 7.4           |
| 3    | 1       | 6.0           |
|      | 2       | 5.6           |
|      | 3       | 7.5           |
|      | 4       | 7.8           |
| 4    | 1       | 6.3           |
|      | 2       | 5.9           |
|      | 3       | 8.0           |
|      | 4       | 8.4           |

**FIGURE 17.18**

Television set sales time series plot



The time series plot in Figure 17.18 indicates that sales are lowest in the second quarter of each year and increase in quarters 3 and 4. Thus, we conclude that a seasonal pattern exists for television set sales. But the time series also has an upward linear trend that will need to be accounted for in order to develop accurate forecasts of quarterly sales. This is easily handled by combining the dummy variable approach for seasonality with the time series regression approach we discussed in Section 17.3 for handling linear trend.

The general form of the estimated multiple regression equation for modelling both the quarterly seasonal effects and the linear trend in the television set time series is as follows:

$$\hat{y}_t = b_0 + b_1 \text{Qtr1} + b_2 \text{Qtr2} + b_3 \text{Qtr3} + b_4 t$$

where:

$\hat{y}_t$  = estimate or forecast of sales in period  $t$

Qtr1 = 1 if time period  $t$  corresponds to the first quarter of the year; 0 otherwise

Qtr2 = 1 if time period  $t$  corresponds to the second quarter of the year; 0 otherwise

Qtr3 = 1 if time period  $t$  corresponds to the third quarter of the year; 0 otherwise

$t$  = time period

Table 17.20 is the revised television set sales time series that includes the coded values of the dummy variables and the time period  $t$ . Using the data in Table 17.20, and MINITAB's regression procedure, we obtained the computer output shown in Figure 17.19. The estimated multiple regression equation is:

$$\text{Sales} = 6.07 - 1.36 \text{Qtr1} - 2.03 \text{Qtr2} - 0.304 \text{Qtr3} - 0.146t \quad (17.12)$$

We can now use equation (17.12) to forecast quarterly sales for next year. Next year is year 5 for the television set sales time series; that is, time periods 17, 18, 19 and 20.

Forecast for Time Period 17 (Quarter 1 in Year 5):

$$\text{Sales} = 6.07 - 1.36(1) - 2.03(0) - 0.304(0) + 0.146(17) = 7.19$$

Forecast for Time Period 18 (Quarter 2 in Year 5):

$$\text{Sales} = 6.07 - 1.36(0) - 2.03(1) - 0.304(0) + 0.146(18) = 6.67$$

Forecast for Time Period 19 (Quarter 3 in Year 5):

$$\text{Sales} = 6.07 - 1.36(0) - 2.03(0) - 0.304(1) + 0.146(19) = 8.54$$

**TABLE 17.20** Television set sales time series with dummy variables and time period

| Year | Quarter | Qtr1 | Qtr2 | Qtr3 | Period | Sales (000s) |
|------|---------|------|------|------|--------|--------------|
| 1    | 1       | 1    | 0    | 0    | 1      | 4.8          |
|      | 2       | 0    | 1    | 0    | 2      | 4.1          |
|      | 3       | 0    | 0    | 1    | 3      | 6.0          |
|      | 4       | 0    | 0    | 0    | 4      | 6.5          |
| 2    | 1       | 1    | 0    | 0    | 5      | 5.8          |
|      | 2       | 0    | 1    | 0    | 6      | 5.2          |
|      | 3       | 0    | 0    | 1    | 7      | 6.8          |
|      | 4       | 0    | 0    | 0    | 8      | 7.4          |
| 3    | 1       | 1    | 0    | 0    | 9      | 6.0          |
|      | 2       | 0    | 1    | 0    | 10     | 5.6          |
|      | 3       | 0    | 0    | 1    | 11     | 7.5          |
|      | 4       | 0    | 0    | 0    | 12     | 7.8          |
| 4    | 1       | 1    | 0    | 0    | 13     | 6.3          |
|      | 2       | 0    | 1    | 0    | 14     | 5.9          |
|      | 3       | 0    | 0    | 1    | 15     | 8.0          |
|      | 4       | 0    | 0    | 0    | 16     | 8.4          |

The regression equation is

$$\text{Sales (1000s)} = 6.07 - 1.36 \text{ Qtr1} - 2.03 \text{ Qtr2} - 0.304$$

Qtr3 + 0.146 Period

| Predictor | Coef    | SE Coef | T      | P     |
|-----------|---------|---------|--------|-------|
| Constant  | 6.0688  | 0.1625  | 37.35  | 0.000 |
| Qtr1      | -1.3631 | 0.1575  | -8.66  | 0.000 |
| Qtr2      | -2.0337 | 0.1551  | -13.11 | 0.000 |
| Qtr3      | -0.3044 | 0.1537  | -1.98  | 0.073 |
| Period    | 0.14562 | 0.01211 | 12.02  | 0.000 |

**FIGURE 17.9**

MINITAB regression output for the umbrella sales time series

Forecast for Time Period 20 (Quarter 4 in Year 5):

$$\text{Sales} = 6.07 - 1.36(0) - 2.03(0) - 0.304(0) + 0.146(20) = 8.99$$

Thus, accounting for the seasonal effects and the linear trend in television set sales, the estimates of quarterly sales in year 5 are 7190, 6670, 8540 and 8990.

The dummy variables in the estimated multiple regression equation actually provide four estimated multiple regression equations, one for each quarter. For instance, if time period  $t$  corresponds to quarter 1, the estimate of quarterly sales is:

$$\text{Quarter 1 : Sales} = 6.07 - 1.36(1) - 2.03(0) - 0.304(0) + 0.146t = 4.71 + 0.146t$$

Similarly, if time period  $t$  corresponds to quarters 2, 3 and 4, the estimates of quarterly sales are:

$$\text{Quarter 2 : Sales} = 6.07 + 1.36(0) - 2.03(1) - 0.304(0) + 0.146t = 4.04 + 0.146t$$

$$\text{Quarter 3 : Sales} = 6.07 + 1.36(0) - 2.03(0) - 0.304(1) + 0.146t = 5.77 + 0.146t$$

$$\text{Quarter 4 : Sales} = 6.07 + 1.36(0) - 2.03(0) - 0.304(0) + 0.146t = 6.07 + 0.146t$$

The slope of the trend line for each quarterly forecast equation is 0.146, indicating a growth in sales of about 146 sets per quarter. The only difference in the four equations is that they have different intercepts. For

instance, the intercept for the quarter 1 equation is 4.71 and the intercept for the quarter 4 equation is 6.07. Thus, sales in quarter 1 are  $4.71 - 6.07 = -1.36$  or 1360 sets less than in quarter 4. In other words, the estimated regression coefficient for Qtr1 in equation (17.12) provides an estimate of the difference in sales between quarter 1 and quarter 4. Similar interpretations can be provided for -2.03, the estimated regression coefficient for dummy variable Qtr2, and -0.304, the estimated regression coefficient for dummy variable Qtr3.

## Models based on monthly data

In the preceding television set sales example, we showed how dummy variables can be used to account for the quarterly seasonal effects in the time series. Because there were four levels for the categorical variable season, three dummy variables were required. However, many businesses use monthly rather than quarterly forecasts. For monthly data, season is a categorical variable with 12 levels and thus  $12 - 1 = 11$  dummy variables are required. For example, the 11 dummy variables could be coded as follows:

$$\text{Month1} = \begin{cases} 1 & \text{if January} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Month2} = \begin{cases} 1 & \text{if February} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Month11} = \begin{cases} 1 & \text{if November} \\ 0 & \text{otherwise} \end{cases}$$

Other than this change, the multiple regression approach for handling seasonality remains the same.

## EXERCISES

### Methods

- 22.** Consider the following time series.

| Quarter | Year 1 | Year 2 | Year 3 |
|---------|--------|--------|--------|
| 1       | 71     | 68     | 62     |
| 2       | 49     | 41     | 51     |
| 3       | 58     | 60     | 53     |
| 4       | 78     | 81     | 72     |

- a. Construct a time series plot. What type of pattern exists in the data?
- b. Use the following dummy variables to develop an estimated regression equation to account for seasonal effects in the data: Qtr1 = 1 if Quarter 1, 0 otherwise; Qtr2 = 1 if Quarter 2, 0 otherwise; Qtr3 = 1 if Quarter 3, 0 otherwise.
- c. Compute the quarterly forecasts for next year.

- 23.** Consider the following time series data.

| Quarter | Year 1 | Year 2 | Year 3 |
|---------|--------|--------|--------|
| 1       | 4      | 6      | 7      |
| 2       | 2      | 3      | 6      |
| 3       | 3      | 5      | 6      |
| 4       | 5      | 7      | 8      |



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SOLUTIONS

- Construct a time series plot. What type of pattern exists in the data?
- Use the following dummy variables to develop an estimated regression equation to account for any seasonal and linear trend effects in the data:  $Qtr1 = 1$  if Quarter 1, 0 otherwise;  $Qtr2 = 1$  if Quarter 2, 0 otherwise;  $Qtr3 = 1$  if Quarter 3, 0 otherwise.
- Compute the quarterly forecasts for next year

### Applications

- 24.** The quarterly sales data (number of copies sold) for a college textbook over the past three years follow.

| Quarter | Year 1 | Year 2 | Year 3 |
|---------|--------|--------|--------|
| 1       | 1690   | 1800   | 1850   |
| 2       | 940    | 900    | 1100   |
| 3       | 2625   | 2900   | 2930   |
| 4       | 2500   | 2360   | 2615   |

- Construct a time series plot. What type of pattern exists in the data?
  - Use the following dummy variables to develop an estimated regression equation to account for any seasonal effects in the data:  $Qtr1 = 1$  if Quarter 1, 0 otherwise;  $Qtr2 = 1$  if Quarter 2, 0 otherwise;  $Qtr3 = 1$  if Quarter 3, 0 otherwise.
  - Compute the quarterly forecasts for next year.
  - Let  $t = 1$  to refer to the observation in quarter 1 of year 1;  $t = 2$  to refer to the observation in quarter 2 of year 1; ... and  $t = 12$  to refer to the observation in quarter 4 of year 3. Using the dummy variables defined in part (b) and  $t$ , develop an estimated regression equation to account for seasonal effects and any linear trend in the time series. Based upon the seasonal effects in the data and linear trend, compute the quarterly forecasts for next year.
- 25.** Air pollution control specialists in northern Poland monitor the amount of ozone, carbon dioxide and nitrogen dioxide in the air on an hourly basis. The hourly time series data exhibit seasonality, with the levels of pollutants showing patterns that vary over the hours in the day. On July 15, 16 and 17, the following levels of nitrogen dioxide were observed for the 12 hours from 6:00 a.m. to 6:00 p.m.

|          |    |    |    |    |    |    |    |    |    |    |    |    |
|----------|----|----|----|----|----|----|----|----|----|----|----|----|
| July 15: | 25 | 28 | 35 | 50 | 60 | 60 | 40 | 35 | 30 | 25 | 25 | 20 |
| July 16: | 28 | 30 | 35 | 48 | 60 | 65 | 50 | 40 | 35 | 25 | 20 | 20 |
| July 17: | 35 | 42 | 45 | 70 | 72 | 75 | 60 | 45 | 40 | 25 | 25 | 25 |

- 
- POLLUTION
- Construct a time series plot. What type of pattern exists in the data?
  - Use the following dummy variables to develop an estimated regression equation to account for the seasonal effects in the data.
    - $Hour1 = 1$  if the reading was made between 6:00 a.m. and 7:00 a.m.; 0 otherwise.
    - $Hour2 = 1$  if the reading was made between 7:00 a.m. and 8:00 a.m.; 0 otherwise.
    - $Hour11 = 1$  if the reading was made between 4:00 p.m. and 5:00 p.m.; 0 otherwise.

Note that when the values of the 11 dummy variables are equal to 0, the observation corresponds to the 5:00 p.m. to 6:00 p.m. hour.

- Using the estimated regression equation developed in part (a), compute estimates of the levels of nitrogen dioxide for July 18.
- Let  $t = 1$  to refer to the observation in hour 1 on July 15;  $t = 2$  to refer to the observation in hour 2 of July 15; 0... and  $t = 36$  to refer to the observation in hour 12 of July 17. Using the dummy variables defined in part (b) and  $t$ , develop an estimated regression equation to account for seasonal effects and any linear trend in the time series. Based upon the seasonal effects in the data and linear trend, compute estimates of the levels of nitrogen dioxide for July 18.

## 17.6 TIME SERIES DECOMPOSITION

In this section we turn our attention to what is called **time series decomposition**. Time series decomposition can be used to separate or decompose a time series into seasonal, trend and **irregular components**. While this method can be used for forecasting, its primary applicability is to obtain a better understanding of the time series. Many business and economic time series are maintained and published by agencies such as Eurostat and the OECD. These agencies use time series decomposition to create deseasonalized time series.

Understanding what is really going on with a time series often depends upon the use of deseasonalized data. For instance, we might be interested in learning whether electrical power consumption is increasing in our area. Suppose we learn that electric power consumption in September is down 3 per cent from the previous month. Care must be exercised in using such information, because whenever a seasonal influence is present, such comparisons may be misleading if the data have not been deseasonalized. The fact that electric power consumption is down 3 per cent from August to September might be only the seasonal effect associated with a decrease in the use of air conditioning and not because of a long-term decline in the use of electric power. Indeed, after adjusting for the seasonal effect, we might even find that the use of electric power increased. Many other time series, such as unemployment statistics, home sales and retail sales, are subject to strong seasonal influences. It is important to deseasonalize such data before making a judgement about any long-term trend.

Time series decomposition methods assume that  $Y_t$ , the actual time series value at period  $t$ , is a function of three components: a trend component; a seasonal component; and an irregular or error component. How these three components are combined to generate the observed values of the time series depends upon whether we assume the relationship is best described by an additive or a multiplicative model.

An **additive decomposition model** takes the following form:

$$Y_t = \text{Trend}_t + \text{Seasonal}_t + \text{Irregular}_t \quad (17.13)$$

where:

$\text{Trend}_t$  = trend value at time period  $t$

$\text{Seasonal}_t$  = seasonal value at time period  $t$

$\text{Irregular}_t$  = irregular value at time period  $t$

In an additive model the values for the three components are simply added together to obtain the actual time series value  $Y_t$ . The irregular or error component accounts for the variability in the time series that cannot be explained by the trend and seasonal components.

An additive model is appropriate in situations where the seasonal fluctuations do not depend upon the level of the time series. The regression model for incorporating seasonal and trend effects in Section 17.5 is an additive model. If the sizes of the seasonal fluctuations in earlier time periods are about the same as the sizes of the seasonal fluctuations in later time periods, an additive model is appropriate. However, if the seasonal fluctuations change over time, growing larger as the sales volume increases because of a long-term linear trend, then a multiplicative model should be used. Many business and economic time series follow this pattern.

A **multiplicative decomposition model** takes the following form:

$$Y_t = \text{Trend}_t \times \text{Seasonal}_t \times \text{Irregular}_t \quad (17.14)$$

where:

$\text{Trend}_t$  = trend value at time period  $t$

$\text{Seasonal}_t$  = seasonal index at time period  $t$

$\text{Irregular}_t$  = irregular index at time period  $t$

In this model, the trend and seasonal and irregular components are multiplied to give the value of the time series. Trend is measured in units of the item being forecast. However, the seasonal and irregular components are measured in relative terms, with values above 1.00 indicating effects above the trend and values below 1.00 indicating effects below the trend.

Because this is the method most often used in practice, we will restrict our discussion of time series decomposition to showing how to develop estimates of the trend and seasonal components for a multiplicative model. As an illustration we will work with the quarterly television set sales time series introduced in Section 17.5; the quarterly sales data are shown in Table 17.19 and the corresponding time series plot is presented in Figure 17.18. After demonstrating how to decompose a time series using the multiplicative model, we will show how the seasonal indices and trend component can be recombined to develop a forecast.

## Calculating the seasonal indices

Figure 17.18 indicates that sales are lowest in the second quarter of each year and increase in quarters 3 and 4. Thus, we conclude that a seasonal pattern exists for the television set sales time series. The computational procedure used to identify each quarter's seasonal influence begins by computing a moving average to remove the combined seasonal and irregular effects from the data, leaving us with a time series that contains only trend and any remaining random variation not removed by the moving average calculations.

Because we are working with a quarterly series, we will use four data values in each moving average. The moving average calculation for the first four quarters of the television set sales data is:

$$\text{First moving average} = \frac{4.8 + 4.1 + 6.0 + 6.5}{4} = \frac{21.4}{4} = 5.35$$

Note that the moving average calculation for the first four quarters yields the average quarterly sales over year 1 of the time series. Continuing the moving average calculations, we next add the 5.8 value for the first quarter of year 2 and drop the 4.8 for the first quarter of year 1. Thus, the second moving average is:

$$\text{Second moving average} = \frac{4.1 + 6.0 + 6.5 + 5.8}{4} = \frac{22.4}{4} = 5.60$$

Similarly, the third moving average calculation is  $(6.0 + 6.5 + 5.8 + 5.2)/4 = 5.875$ .

Before we proceed with the moving average calculations for the entire time series, let us return to the first moving average calculation, which resulted in a value of 5.35. The 5.35 value is the average quarterly sales volume for year 1. As we look back at the calculation of the 5.35 value, associating 5.35 with the 'middle' of the moving average group makes sense. Note, however, that with four quarters in the moving average, there is no middle period. The 5.35 value really corresponds to period 2.5, the last half of quarter 2 and the first half of quarter 3. Similarly, if we go to the next moving average value of 5.60, the middle period corresponds to period 3.5, the last half of quarter 3 and the first half of quarter 4.

The two moving average values we computed do not correspond directly to the original quarters of the time series. We can resolve this difficulty by computing the average of the two moving averages. Since the centre of the first moving average is period 2.5 (half a period or quarter early) and the centre of the second moving average is period 3.5 (half a period or quarter late), the average of the two moving averages is centred at quarter 3, exactly where it should be. This moving average is referred to as a *centred moving average*. Thus, the centred moving average for period 3 is  $(5.35 + 5.60)/2 = 5.475$ . Similarly, the centred moving average value for period 4 is  $(5.60 + 5.875)/2 = 5.738$ . Table 17.21 shows a complete summary of the moving average and centred moving average calculations for the television set sales data.

What do the centred moving averages in Table 17.21 tell us about this time series? Figure 17.20 shows a time series plot of the actual time series values and the centred moving average values. Note particularly how the centred moving average values tend to 'smooth out' both the seasonal and irregular fluctuations in the time series. The centred moving averages represent the trend in the data and any random variation that was not removed by using moving averages to smooth the data.

By dividing each side of equation (17.14) by the trend component  $T_t$ , we can identify the combined seasonal-irregular effect in the time series.

$$\frac{Y_t}{\text{Trend}_t} = \frac{\text{Trend}_t \times \text{Seasonal}_t \times \text{Irregular}_t}{\text{Trend}_t} = \text{Seasonal}_t \times \text{Irregular}_t$$

For example, the third quarter of year 1 shows a trend value of 5.475 (the centred moving average). So  $6.0/5.475 = 1.096$  is the combined seasonal-irregular value. Table 17.22 summarizes the seasonal-irregular ('detrended') values for the entire time series.

Consider the seasonal-irregular values for the third quarter: 1.096, 1.075 and 1.109. Seasonal-irregular values greater than 1.00 indicate effects above the trend estimate and values below 1.00 indicate effects below the trend estimate. Thus, the three seasonal-irregular values for quarter 3 show an above-average effect in the third quarter. Since the year-to-year fluctuations in the seasonal-irregular values are primarily due to random error, we can average the computed values to eliminate the irregular influence and obtain an estimate of the third-quarter seasonal influence.

**TABLE 17.21** Centred moving average calculations for the television set sales time series

| Year | Quarter | Sales (000s) | Four-quarter moving average | Centred moving average |
|------|---------|--------------|-----------------------------|------------------------|
| 1    | 1       | 4.8          |                             |                        |
| 1    | 2       | 4.1          | 5.350                       |                        |
| 1    | 3       | 6.0          | 5.600                       | 5.475                  |
| 1    | 4       | 6.5          | 5.875                       | 5.738                  |
| 2    | 1       | 5.8          | 6.075                       | 5.975                  |
| 2    | 2       | 5.2          | 6.300                       | 6.188                  |
| 2    | 3       | 6.8          | 6.350                       | 6.325                  |
| 2    | 4       | 7.4          | 6.450                       | 6.400                  |
| 3    | 1       | 6.0          | 6.625                       | 6.538                  |
| 3    | 2       | 5.6          | 6.725                       | 6.675                  |
| 3    | 3       | 7.5          | 6.800                       | 6.763                  |
| 3    | 4       | 7.8          | 6.875                       | 6.838                  |
| 4    | 1       | 6.3          | 7.000                       | 6.938                  |
| 4    | 2       | 5.9          | 7.150                       | 7.075                  |
| 4    | 3       | 8.0          |                             |                        |
| 4    | 4       | 8.4          |                             |                        |

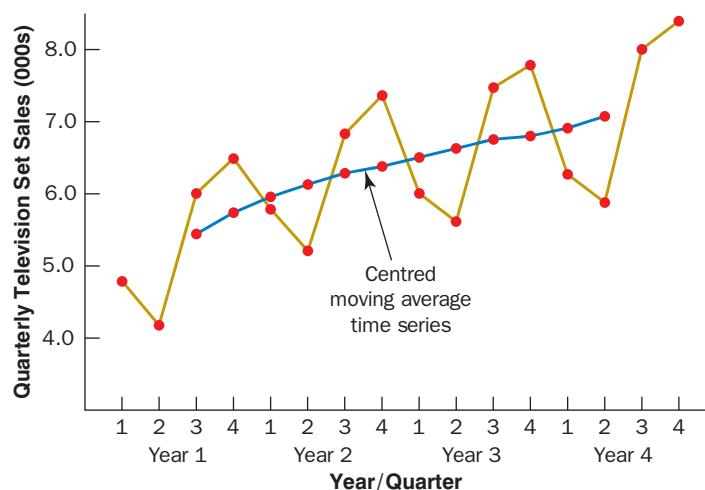
$$\text{Seasonal effect of quarter 3} = \frac{1.096 + 1.075 + 1.109}{3} = 1.09$$

We refer to 1.09 as the *seasonal index* for the third quarter. Table 17.23 summarizes the calculations involved in computing the seasonal indices for the television set sales time series. The seasonal indices for the four quarters are 0.93, 0.84, 1.09 and 1.14.

Interpretation of the seasonal indices in Table 17.23 provides some insight about the seasonal component in television set sales. The best sales quarter is the fourth quarter, with sales averaging 14 per cent above the trend estimate. The worst, or slowest, sales quarter is the second quarter; its seasonal index of 0.84 shows that the sales average is 16 per cent below the trend estimate. The seasonal component corresponds clearly to the intuitive expectation that television viewing interest and thus television purchase patterns tend to peak in the fourth quarter because of the coming winter season and reduction in outdoor activities. The low second-quarter sales reflect the reduced interest in television viewing due to the spring and pre-summer activities of potential customers.

**FIGURE 17.20**

Quarterly television set sales time series and centred moving average



**TABLE 17.22** Seasonal irregular values for the television set sales time series

| Year | Quarter | Sales (000s) | Centred moving average | Seasonal-irregular value |
|------|---------|--------------|------------------------|--------------------------|
| 1    | 1       | 4.8          |                        |                          |
| 1    | 2       | 4.1          |                        |                          |
| 1    | 3       | 6.0          | 5.475                  | 1.096                    |
| 1    | 4       | 6.5          | 5.738                  | 1.133                    |
| 2    | 1       | 5.8          | 5.975                  | 0.971                    |
| 2    | 2       | 5.2          | 6.188                  | 0.840                    |
| 2    | 3       | 6.8          | 6.325                  | 1.075                    |
| 2    | 4       | 7.4          | 6.400                  | 1.156                    |
| 3    | 1       | 6.0          | 6.538                  | 0.918                    |
| 3    | 2       | 5.6          | 6.675                  | 0.839                    |
| 3    | 3       | 7.5          | 6.763                  | 1.109                    |
| 3    | 4       | 7.8          | 6.838                  | 1.141                    |
| 4    | 1       | 6.3          | 6.938                  | 0.908                    |
| 4    | 2       | 5.9          | 7.075                  | 0.834                    |
| 4    | 3       | 8.0          |                        |                          |
| 4    | 4       | 8.4          |                        |                          |

**TABLE 17.23** Seasonal index calculations for the television set sales time series

| Quarter | Seasonal-irregular values |       | Seasonal index |
|---------|---------------------------|-------|----------------|
| 1       | 0.971                     | 0.918 | 0.908          |
| 2       | 0.840                     | 0.839 | 0.834          |
| 3       | 1.096                     | 1.075 | 1.109          |
| 4       | 1.133                     | 1.156 | 1.141          |

One final adjustment is sometimes necessary in obtaining the seasonal indices. Because the multiplicative model requires that the average seasonal index equal 1.00, the sum of the four seasonal indices in Table 17.23 must equal 4.00. In other words, the seasonal effects must even out over the year. The average of the seasonal indices in our example is equal to 1.00, and hence this type of adjustment is not necessary. In other cases, a slight adjustment may be necessary. To make the adjustment, multiply each seasonal index by the number of seasons divided by the sum of the unadjusted seasonal indices. For instance, for quarterly data, multiply each seasonal index by 4/(sum of the unadjusted seasonal indices). Some of the exercises will require this adjustment to obtain the appropriate seasonal indices.

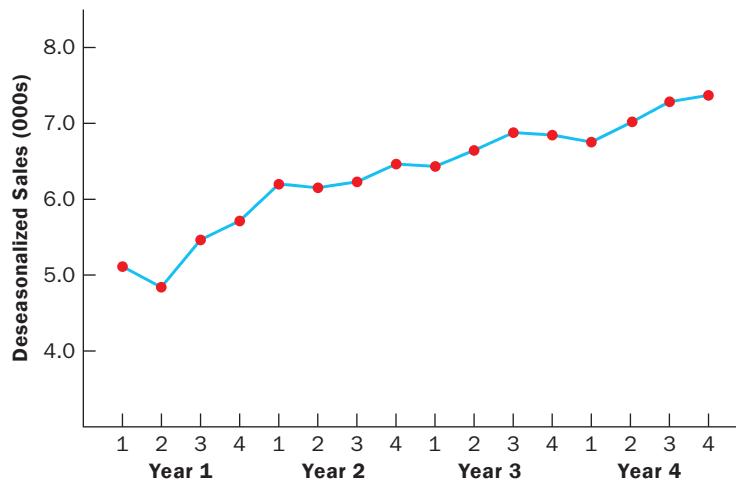
### Deseasonalizing the time series

A time series that has had the seasonal effects removed is referred to as a **deseasonalized time series**, and the process of using the seasonal indices to remove the seasonal effects from a time series is referred to as deseasonalizing the time series. Using a multiplicative decomposition model, we deseasonalize a time series by dividing each observation by its corresponding seasonal index.

By dividing each time series observation ( $Y_t$ ) in equation (17.14) by its corresponding seasonal index, the resulting data show only trend and random variability (the irregular component). The deseasonalized time series for television set sales is summarized in Table 17.24. A graph of the deseasonalized time series is shown in Figure 17.21.

**FIGURE 17.21**

Deseasonalized television set sales time series



### Using the deseasonalized time series to identify trend

The graph of the deseasonalized television set sales time series shown in Figure 17.21 appears to have an upward linear trend. To identify this trend, we will fit a linear trend equation to the deseasonalized time series using the same method shown in Section 17.4. The only difference is that we will be fitting a trend line to the deseasonalized data instead of the original data.

Recall that for a linear trend the estimated regression equation can be written as:

$$T_t = b_0 + b_1 t$$

where:

$T_t$  = linear trend forecast in period  $t$

$b_0$  = intercept of the linear trend line

$b_1$  = slope of the trend line

$t$  = time period

**TABLE 17.24** Deseasonalized values for the television set sales time series

| Year | Quarter | Time period | Sales (000s) | Seasonal index | Deseasonalized sales |
|------|---------|-------------|--------------|----------------|----------------------|
| 1    | 1       | 1           | 4.8          | 0.93           | 5.16                 |
|      | 2       | 2           | 4.1          | 0.84           | 4.88                 |
|      | 3       | 3           | 6.0          | 1.09           | 5.50                 |
|      | 4       | 4           | 6.5          | 1.14           | 5.70                 |
| 2    | 1       | 5           | 5.8          | 0.93           | 6.24                 |
|      | 2       | 6           | 5.2          | 0.84           | 6.19                 |
|      | 3       | 7           | 6.8          | 1.09           | 6.24                 |
|      | 4       | 8           | 7.4          | 1.14           | 6.49                 |
| 3    | 1       | 9           | 6.0          | 0.93           | 6.45                 |
|      | 2       | 10          | 5.6          | 0.84           | 6.67                 |
|      | 3       | 11          | 7.5          | 1.09           | 6.88                 |
|      | 4       | 12          | 7.8          | 1.14           | 6.84                 |
| 4    | 1       | 13          | 6.3          | 0.93           | 6.77                 |
|      | 2       | 14          | 5.9          | 0.84           | 7.02                 |
|      | 3       | 15          | 8.0          | 1.09           | 7.34                 |
|      | 4       | 16          | 8.4          | 1.14           | 7.37                 |

| <b>The regression equation is</b>                 |              |                |                   |          |
|---|--------------|----------------|-------------------|----------|
| <b>Deseasonalized Sales = 5.10 + 0.148 Period</b> |              |                |                   |          |
| <b>Predictor</b>                                  | <b>Coeff</b> | <b>SE Coef</b> | <b>T</b>          | <b>P</b> |
| Constant  | 5.1050       | 0.1133         | 45.07             | 0.000    |
| Period  | 0.14760      | 0.01171        | 12.60             | 0.000    |
| S = 0.215985                                      | R-Sq = 91.9% |                | R-Sq(adj) = 91.3% |          |
| <b>Analysis of Variance</b>                       |              |                |                   |          |
| <b>Source</b>                                     | <b>DF</b>    | <b>SS</b>      | <b>MS</b>         | <b>F</b> |
| Regression  | 1            | 7.4068         | 7.4068            | 158.78   |
| Residual Error                                    | 14           | 0.6531         | 0.0466            |          |
| Total   | 15           | 8.0599         |                   |          |

**FIGURE 17.22**

MINITAB regression output for the deseasonalized television set sales time series

In Section 17.4 we provided formulae for computing the values of  $b_0$  and  $b_1$ . To fit a linear trend line to the deseasonalized data in Table 17.24, the only change is that the deseasonalized time series values are used instead of the observed values  $Y_t$  in computing  $b_0$  and  $b_1$ .

Figure 17.22 shows the computer output obtained using MINITAB's regression analysis procedure to estimate the trend line for the deseasonalized television set time series. The estimated linear trend equation is:

$$\text{Deseasonalized sales} = 5.10 + 0.148 t$$

The slope of 0.148 indicates that over the past 16 quarters, the firm averaged a deseasonalized growth in sales of about 148 sets per quarter. If we assume that the past 16-quarter trend in sales data is a reasonably good indicator of the future, this equation can be used to develop a trend projection for future quarters. For example, substituting  $t = 17$  into the equation yields next quarter's deseasonalized trend projection,  $T_{17}$ .

$$T_{17} = 5.10 + 0.148(17) = 7.616$$

Thus, using the deseasonalized data, the linear trend forecast for next quarter (period 17) is 7616 television sets. Similarly, the deseasonalized trend forecasts for the next three quarters (periods 18, 19 and 20) are 7764, 7912 and 8060 television sets, respectively.

## Seasonal adjustments

The final step in developing the forecast when both trend and seasonal components are present is to use the seasonal indices to adjust the deseasonalized trend projections. Returning to the television set sales example, we have a deseasonalized trend projection for the next four quarters. Now we must adjust the forecast for the seasonal effect. The seasonal index for the first quarter of year 5 ( $t = 17$ ) is 0.93, so we obtain the quarterly forecast by multiplying the deseasonalized forecast based on trend ( $T_{17} = 7616$ ) by the seasonal index (0.93). Thus, the forecast for the next quarter is  $7616(0.93) = 7083$ .

**TABLE 17.25** Quarterly forecasts for the television set sales time series

| <b>Year</b> | <b>Quarter</b> | <b>Deseasonalized trend forecast</b> | <b>Seasonal index</b> | <b>Quarterly forecast</b> |
|-------------|----------------|--------------------------------------|-----------------------|---------------------------|
| 5           | 1              | 7616                                 | 0.93                  | (7616)(0.93) = 7083       |
|             | 2              | 7764                                 | 0.84                  | (7764)(0.84) = 6522       |
|             | 3              | 7912                                 | 1.09                  | (7912)(1.09) = 8624       |
|             | 4              | 8060                                 | 1.14                  | (8060)(1.14) = 9188       |

Table 17.25 shows the quarterly forecast for quarters 17 through 20. The high-volume fourth quarter has a 9188-unit forecast, and the low-volume second quarter has a 6522-unit forecast.

## Models based on monthly data

In the preceding television set sales example, we used quarterly data to illustrate the computation of seasonal indices. However, many businesses use monthly rather than quarterly forecasts. In such cases, the procedures introduced in this section can be applied with minor modifications. First, a 12-month moving average replaces the four-quarter moving average; second, 12 monthly seasonal indices, rather than four quarterly seasonal indices, must be computed. Other than these changes, the computational and forecasting procedures are identical.

## Cyclical component

Mathematically, the multiplicative model of equation (17.14) can be expanded to include a cyclical component.

$$Y_t = \text{Trend}_t \times \text{Cyclical}_t \times \text{Seasonal}_t \times \text{Irregular}_t \quad (17.15)$$

The cyclical component, like the seasonal component, is expressed as a percentage of trend. As mentioned in Section 17.1, this component is attributable to multiyear cycles in the time series. It is analogous to the seasonal component, but over a longer period of time. However, because of the length of time involved, obtaining enough relevant data to estimate the cyclical component is often difficult. Another difficulty is that cycles usually vary in length. Because it is so difficult to identify and/or separate cyclical effects from long-term trend effects, in practice these effects are often combined and referred to as a combined trend-cycle component. We leave further discussion of the cyclical component to specialized texts on forecasting methods.

## EXERCISES

### Methods

**26.** Consider the following time series data.

| Quarter | Year 1 | Year 2 | Year 3 |
|---------|--------|--------|--------|
| 1       | 4      | 6      | 7      |
| 2       | 2      | 3      | 6      |
| 3       | 3      | 5      | 6      |
| 4       | 5      | 7      | 8      |

- a. Construct a time series plot. What type of pattern exists in the data?
  - b. Show the four-quarter and centred moving average values for this time series.
  - c. Compute seasonal indices and adjusted seasonal indices for the four quarters.
- 27.** Refer to Exercise 26.
- a. Deseasonalize the time series using the adjusted seasonal indices computed in (c) of Exercise 26.
  - b. Using MINITAB or EXCEL, compute the linear trend regression equation for the deseasonalized data.
  - c. Compute the deseasonalized quarterly trend forecast for Year 4.
  - d. Use the seasonal indices to adjust the deseasonalized trend forecasts computed in (c).



COMPLETE  
SOLUTIONS



**COMPLETE  
SOLUTIONS**

## Applications

- 28.** The quarterly sales data (number of copies sold) for a college textbook over the past three years follow.

### WEB file text sales

| Quarter | Year 1 | Year 2 | Year 3 |
|---------|--------|--------|--------|
| 1       | 1690   | 1800   | 1850   |
| 2       | 940    | 900    | 1100   |
| 3       | 2625   | 2900   | 2930   |
| 4       | 2500   | 2360   | 2615   |

- a. Construct a time series plot. What type of pattern exists in the data?
  - b. Show the four-quarter and centred moving average values for this time series.
  - c. Compute the seasonal and adjusted seasonal indices for the four quarters.
  - d. When does the publisher have the largest seasonal index? Does this result appear reasonable? Explain.
  - e. Deseasonalize the time series.
  - f. Compute the linear trend equation for the deseasonalized data and forecast sales using the linear trend equation.
  - g. Adjust the linear trend forecasts using the adjusted seasonal indices computed in (c).
- 29.** Quarterly sales data for the number of houses sold over the past four years or so by a national chain are as follows:

| Year | Q1  | Q2  | Q3  | Q4  |
|------|-----|-----|-----|-----|
| 1    | 200 | 212 | 229 | 207 |
| 2    | 195 | 204 | 216 | 202 |
| 3    | 201 | 209 | 221 | 205 |
| 4    | 208 | 217 | 231 | 213 |
| 5    | 218 |     |     |     |

- a. Decompose the series into trend, seasonal and random components using a multiplicative model.
  - b. Hence derive forecasts of the number of houses that will be sold in the next four quarters.
  - c. Comment on the quality of your modelling results.
- 30.** The following table shows the number of passengers per quarter (in thousands) who flew with MBI Junior for the first quarter of this year and the three years preceding:

| Year | Q1 | Q2  | Q3  | Q4  |
|------|----|-----|-----|-----|
| 1    | 44 | 92  | 156 | 68  |
| 2    | 60 | 112 | 180 | 80  |
| 3    | 64 | 124 | 200 | 104 |
| 4    | 76 |     |     |     |

- a. Decompose the series into trend, seasonal and random components using an additive model.
- b. Hence derive forecasts of the passenger numbers in the next four quarters.
- c. Comment on the quality of your modelling.

- 31.** The data below relates to the UK and show the number of marriages (000s) over a recent four-year period.

| Year | Quarter | Marriages | Year | Quarter | Marriages |
|------|---------|-----------|------|---------|-----------|
| 1    | 1       | 52.9      | 3    | 1       | 41.7      |
|      | 2       | 114.3     |      | 2       | 100.5     |
|      | 3       | 138.7     |      | 3       | 138.5     |
|      | 4       | 62.7      |      | 4       | 60.9      |
| 2    | 1       | 45.6      | 4    | 1       | 41.7      |
|      | 2       | 101.9     |      | 2       | 100.5     |
|      | 3       | 146.2     |      | 3       | 138.5     |
|      | 4       | 62.3      |      | 4       | 60.9      |

- a. Using the decomposition method, forecast marriages for the next four quarters in the series.

## ONLINE RESOURCES

For the associated data files, additional online summary, questions and answers and software section for Chapter 17, visit the online platform.



## SUMMARY

This chapter provided an introduction to the basic methods of time series analysis and forecasting. First, we showed that the underlying pattern in the time series can often be identified by constructing a time series plot. Several types of data patterns can be distinguished, including a horizontal pattern, a trend pattern and a seasonal pattern. The forecasting methods we have discussed are based on which of these patterns are present in the time series.

For a time series with a horizontal pattern, we showed how moving averages and exponential smoothing can be used to develop a forecast. The moving averages method consists of computing an average of past data values and then using that average as the forecast for the next period. In the exponential smoothing method, a weighted average of past time series values is used to compute a forecast. These methods also adapt well when a horizontal pattern shifts to a different level and resumes a horizontal pattern.

An important factor in determining what forecasting method to use involves the accuracy of the method. We discussed three measures of forecast accuracy: mean absolute error (MAE), mean squared error (MSE) and mean absolute percentage error (MAPE). Each of these measures is designed to determine how well a particular forecasting method is able to reproduce the time series data that are already available. By selecting a method that has the best accuracy for the data already known, we hope to increase the likelihood that we will obtain better forecasts for future time periods.

For time series that have only a long-term linear trend, we showed how simple time series regression can be used to make trend projections. We also discussed how an extension of single exponential smoothing, referred to as Holt's linear exponential smoothing, can be used to forecast a time series with a linear trend. For a time series with a curvilinear or nonlinear trend, we showed how multiple regression can be used to fit a quadratic trend equation or an exponential trend equation to the data.

For a time series with a seasonal pattern, we showed how the use of dummy variables in a multiple regression model can be used to develop an estimated regression equation with seasonal

effects. We then extended the regression approach to include situations where the time series contains both a seasonal and a linear trend effect by showing how to combine the dummy variable approach for handling seasonality with the time series regression approach for handling linear trend.

In the last section of the chapter we showed how time series decomposition can be used to separate or decompose a time series into seasonal and trend components and then to deseasonalize the time series. We showed how to compute seasonal indices for a multiplicative model, how to use the seasonal indices to deseasonalize the time series and how to use regression analysis on the deseasonalized data to estimate the trend component. The final step in developing a forecast when both trend and seasonal components are present is to use the seasonal indices to adjust the trend projections.

## KEY TERMS

**Additive decomposition model**

**Cyclical pattern**

**Causal forecasting methods**

**Deseasonalized time series**

**Exponential smoothing**

**Forecast**

**Forecast error**

**Horizontal pattern**

**Irregular component**

**Linear exponential smoothing**

**Mean absolute error (MAE)**

**Mean absolute percentage error (MAPE)**

**Mean squared error (MSE)**

**Moving averages**

**Multiplicative decomposition model**

**Seasonal pattern**

**Smoothing constant**

**Stationary time series**

**Time series**

**Time series decomposition**

**Time series plot**

**Trend pattern**

**Weighted moving averages**

## KEY FORMULAE

### Moving average forecast of order $k$

$$F_{t+1} = \frac{\sum(\text{most recent } k \text{ data values})}{k} = \frac{Y_t + Y_{t-1} + \dots + Y_{t-k+1}}{k} \quad (17.1)$$

### Exponential smoothing forecast

$$F_{t+1} = \alpha Y_t + (1 - \alpha)F_t \quad (17.2)$$

### Linear trend equation

$$T_t = b_0 + b_1 t \quad (17.4)$$

where

$$b_1 = \frac{\sum_{t=1}^n (t - \bar{t})(Y_t - \bar{Y})}{\sum_{t=1}^n (t - \bar{t})^2} \quad (17.5)$$

$$b_0 = \bar{Y} - b_1 \bar{t} \quad (17.6)$$

### Holt's linear exponential smoothing

$$L_t = \alpha Y_t + (1 - \alpha)(L_{t-1} + b_{t-1}) \quad (17.7)$$

$$b_t = \beta(L_t - L_{t-1}) + (1 - \beta)b_{t-1} \quad (17.8)$$

$$F_{t+k} = L_t + b_t k \quad (17.9)$$

### Quadratic trend equation

$$T_t = b_0 + b_1 t + b_2 t^2 \quad (17.10)$$

### Exponential trend equation

$$T_t = b_0(b_1)^t \quad (17.11)$$

### Additive decomposition model

$$Y_t = \text{Trend}_t + \text{Seasonal}_t + \text{Irregular}_t \quad (17.13)$$

### Multiplicative decomposition model

$$Y_t = \text{Trend}_t \times \text{Seasonal}_t \times \text{Irregular}_t \quad (17.27)$$

## CASE PROBLEM 1



### Forecasting food and beverage sales

The Vesuvius Restaurant near Naples, Italy, is owned and operated by Luigi Marconi. The restaurant has just completed its third year of operation. During that time, Luigi sought to establish a reputation for the restaurant as a high-quality dining establishment that specializes in fresh seafood. Through the efforts of Luigi and his staff, his restaurant has become one of the best and fastest growing restaurants in the area.

Luigi believes that, to plan for the growth of the restaurant in the future, he needs to develop a system that will enable him to forecast food and beverage sales by month for up to one year in advance.

Luigi compiled the following data (in thousands of euros) on total food and beverage sales for the three years of operation.

### Managerial report

Perform an analysis of the sales data for the Vesuvius Restaurant. Prepare a report for Luigi that summarizes your findings, forecasts and recommendations. Include the following:

1. A graph of the time series.
2. An analysis of the seasonality of the data. Indicate the seasonal indices for each month, and comment on the high and low seasonal sales months. Do the seasonal indices make intuitive sense? Discuss.
3. A forecast of sales for January through December of the fourth year.



| Month     | First year | Second year | Third year |
|-----------|------------|-------------|------------|
| January   | 242        | 263         | 282        |
| February  | 235        | 238         | 255        |
| March     | 232        | 247         | 265        |
| April     | 178        | 193         | 205        |
| May       | 184        | 193         | 210        |
| June      | 140        | 149         | 160        |
| July      | 145        | 157         | 166        |
| August    | 152        | 161         | 174        |
| September | 110        | 122         | 126        |
| October   | 130        | 130         | 148        |
| November  | 152        | 167         | 173        |
| December  | 206        | 230         | 235        |

4. Recommendations as to when the system that you develop should be updated to account for new sales data.

5. Any detailed calculations of your analysis in the appendix of your report.

Assume that January sales for the fourth year turn out to be €295 000. What was your forecast error? If this error is large, Luigi may be puzzled about the difference between your forecast and the actual sales value. What can you do to resolve his uncertainty in the forecasting procedure?



## CASE PROBLEM 2



### Allocating patrols to meet future demand for vehicle rescue

The data below summarize actual monthly demands for RAC rescue services over a five-year time period.

(The Royal Automobile Club is one of the major motoring organizations that offer emergency breakdown cover in the UK.)

To meet the national demand for its services in the coming year, the RAC's human resources planning department forecasts the number of members expected, using historical data and market forecasts. It then predicts the average number of breakdowns

and number of rescue calls expected, by referring to the probability of a member's vehicle breaking down each year. In 2003, an establishment of approximately 1400 patrols was available to deal with the expected workload. Note that this figure had to be

reviewed monthly since it was an average for the year and did not take into account, fluctuations in demand 'in different seasons'.

Monthly demand for RAC rescue services 1999–2003

| Month     | Year    |         |         |         |         |
|-----------|---------|---------|---------|---------|---------|
|           | 2003    | 2002    | 2001    | 2000    | 1999    |
| January   | 270 093 | 248 658 | 253 702 | 220 332 | 241 489 |
| February  | 216 050 | 210 591 | 216 575 | 189 223 | 193 794 |
| March     | 211 154 | 208 969 | 220 903 | 188 950 | 206 068 |
| April     | 194 909 | 191 840 | 191 415 | 196 343 | 191 359 |
| May       | 200 148 | 194 654 | 190 436 | 189 627 | 179 592 |
| June      | 195 608 | 189 892 | 175 512 | 177 653 | 183 712 |
| July      | 208 493 | 203 275 | 193 900 | 182 219 | 193 306 |
| August    | 215 145 | 213 357 | 197 628 | 190 538 | 199 947 |
| September | 200 477 | 196 811 | 183 912 | 183 481 | 191 231 |
| October   | 216 821 | 225 182 | 213 909 | 214 009 | 198 514 |
| November  | 222 128 | 244 498 | 219 336 | 239 104 | 202 219 |
| December  | 250 866 | 257 704 | 246 780 | 254 041 | 254 217 |



### Managerial report

1. By undertaking an appropriate statistical analysis of the information provided, describe how you would advise the RAC on its patrol allocation in 2004.
2. State your assumptions.
3. Comment on the validity of your results or otherwise.



RAC



# 18

## Non-Parametric Methods

### CHAPTER CONTENTS

Statistics in Practice Coffee lovers' preferences: Costa, Starbucks and Caffè Nero

- 18.1 Sign test
- 18.2 Wilcoxon signed-rank test
- 18.3 Mann–Whitney–Wilcoxon test
- 18.4 Kruskal–Wallis test
- 18.5 Rank correlation

**LEARNING OBJECTIVES** After studying this chapter and doing the exercises, you should be able to:

- 1 Explain the essential differences between parametric and non-parametric methods of inference.
- 2 Recognize the circumstances when it is appropriate to apply the following non-parametric statistical procedures; calculate the appropriate sample statistics; use these statistics to carry out a hypothesis test; interpret the results.
- 2.1 Sign test.
- 2.2 Wilcoxon signed-rank test.
- 2.3 Mann–Whitney–Wilcoxon test.
- 2.4 Kruskal–Wallis test.
- 2.5 Spearman rank correlation.

The inferential methods presented previously in the text are generally known as **parametric methods**. These methods begin with an assumption about the distribution of the population, which is often that the population has a normal distribution. Based on this assumption, statisticians are able to derive the sampling distribution that can be used to make inferences about one or more parameters of the population, such as the population mean  $\mu$  or the population standard deviation  $\sigma$ . For example, in Chapter 9 we presented a method for making an inference about a population mean based on an assumption that the population had a normal distribution with unknown parameters  $\mu$  and  $\sigma$ . Using the sample standard deviation  $s$  to estimate the population standard deviation  $\sigma$ , the test statistic for making an inference about the population mean was shown to have a  $t$  distribution. As a result, the  $t$  distribution was used to compute confidence intervals and do hypothesis tests about the mean of a normally distributed population.



## STATISTICS IN PRACTICE

Coffee lovers' preferences: Costa, Starbucks and Caffè Nero

**A** few years ago, Costa Coffee ran a vigorous promotional campaign in the UK under the headline **SORRY STARBUCKS THE PEOPLE HAVE VOTED**. The by-line was 'In head-to-head tests, seven out of ten coffee lovers preferred Costa cappuccino to Starbucks'. At the bottom of the advertisements (some were nearly full-page in broadsheet newspapers), the small print noted that 70 per cent of respondents who identified themselves as coffee lovers preferred Costa cappuccino, and that the total sample size of coffee lovers was 174.

The market research behind the claim was carried out by an independent market research organization, Tangible Branding Limited, in three UK towns (High Wycombe, Glasgow and Sheffield). Each participant was asked to undertake a two-way blind tasting test: either Costa versus Starbucks or Costa versus Caffè Nero. 'Runners' transported the coffees to the tasting venue from nearby coffee houses. The order of

tasting was rotated. Over the three tasting venues, the total Costa versus Starbucks sample was of size 166, and the Costa versus Caffè Nero sample was 168.

In the Costa versus Caffè Nero comparisons, 64 per cent of tasters preferred Costa. In the Costa versus Starbucks tests, 66 per cent preferred Costa. Among self-identified 'coffee lovers', 69 per cent preferred the Costa coffee to Caffè Nero coffee, and 72 per cent preferred Costa to Starbucks. Among Caffè Nero regulars, 72 per cent expressed a preference for Costa's cappuccino, while 67 per cent of Starbucks regulars preferred Costa's cappuccino. The Costa website noted that 'All results are significant at the 95 per cent confidence level'.

The data on which the results are based are qualitative data: a simple expression of preference between two options. The kind of statistical test needed for data such as these is known as a non-parametric test. Non-parametric tests are the subject of the present chapter. The chapter begins with a discussion of the sign test, a test particularly appropriate for the research situation described by Costa in its advertising.



In this chapter we present **non-parametric methods** that can be used to make inferences about a population without requiring an assumption about the specific form of the population distribution. For this reason, these non-parametric methods are also called **distribution-free methods**.

Most of the statistical methods referred to as parametric methods require quantitative data, whereas non-parametric methods allow inferences based sometimes on categorical data, and sometimes on either ranked or quantitative data. In the first section of the chapter, we show how the binomial distribution can be used (as a sampling distribution) to make an inference about a population median. In the following three sections, we show how rank-ordered data are used in non-parametric tests about two or more populations. In the final section, we use rank-ordered data to compute the rank correlation for two variables.

## 18.1 SIGN TEST

The **sign test** is a versatile non-parametric method for hypothesis testing that uses the binomial distribution with  $\pi = 0.50$  as the sampling distribution. It does not require an assumption about the distribution of the population. In this section we present two applications of the sign test: one involving a hypothesis test about a population median and one involving a matched-sample test about the difference between two populations.

### Hypothesis test about a population median

In Chapter 9 we described hypothesis tests about a population mean. In this section we show how the sign test can be used to do a hypothesis test about a population median. If we consider a population where no data value is exactly equal to the median, the median divides the population such that 50 per cent of the values are greater than the median and 50 per cent of the values are less than the median. When a population distribution is skewed, the median is often preferred over the mean as the best measure of central location for the population. The sign test provides a non-parametric procedure for testing a hypothesis about the value of a population median.

To demonstrate the sign test, we consider the weekly sales of MotherEarth Potato Snacks by the Lineker chain of convenience stores. Lineker's management decided to stock the new product on the basis of the manufacturer's estimate that median sales would be €450 per week per store. After stocking the product for three months, Lineker's management requested the following hypothesis test regarding the population median weekly sales.

$$\begin{aligned} H_0: \text{Median} &= 450 \\ H_1: \text{Median} &\neq 450 \end{aligned}$$

Data showing one-week sales at 12 randomly selected Lineker's stores are in Table 18.1.

In the sign test, we compare each sample observation to the hypothesized value of the population median. If the observation is greater than the hypothesized value, we record a plus sign '+'.' If the observation is less than the hypothesized value, we record a minus sign '-'.

**TABLE 18.1** Lineker sample data for the sign test about the population median weekly sales

| Store ID | One-week sales (€) | Sign | Store ID | One-week sales (€) | Sign |
|----------|--------------------|------|----------|--------------------|------|
| 56       | 485                | +    | 63       | 474                | +    |
| 19       | 562                | +    | 39       | 662                | +    |
| 93       | 499                | +    | 21       | 492                | +    |
| 36       | 415                | -    | 84       | 380                | -    |
| 128      | 860                | +    | 102      | 515                | +    |
| 12       | 426                | -    | 44       | 721                | +    |

If an observation is exactly equal to the hypothesized value, the observation is eliminated from the sample and the analysis proceeds with the smaller sample size, using only the observations where a plus sign or a minus sign has been recorded. The conversion of the sample data to either a plus sign or a minus sign gives the non-parametric method its name: the sign test.

Consider the sample data in Table 18.1. The first observation, 485, is greater than the hypothesized median 450; a plus sign is recorded. The second observation, 562, is greater than the hypothesized median 450; a plus sign is recorded. Continuing with the 12 observations in the sample provides the plus and minus signs as shown in Table 18.1. Note that there are nine plus signs and three minus signs.

Assigning the plus signs and minus signs has made the situation a binomial distribution application. The sample size  $n = 12$  is the number of trials. There are two possible outcomes per trial, a plus sign or a minus sign, and the trials are independent. Let  $\pi$  denote the probability of a plus sign. If the population median is 450,  $\pi$  would equal 0.50 as there should be 50 per cent plus signs and 50 per cent minus signs in the population. So, in terms of the binomial probability  $\pi$ , the sign test hypotheses regarding the population median:

$$\begin{aligned} H_0: \text{Median} &= 450 \\ H_1: \text{Median} &\neq 450 \end{aligned}$$

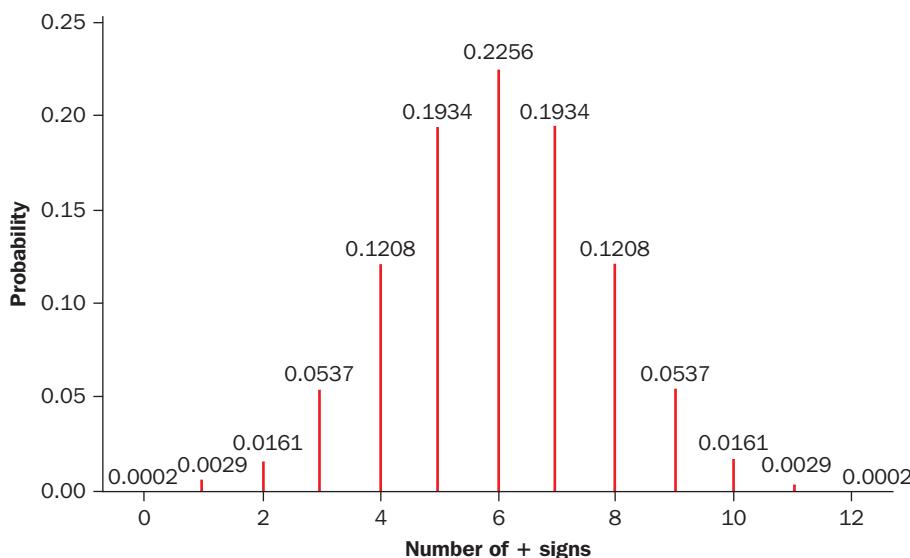
are converted to the following hypotheses about the binomial probability  $\pi$ .

$$\begin{aligned} H_0: \pi &= 0.50 \\ H_1: \pi &\neq 0.50 \end{aligned}$$

If  $H_0$  is not rejected, we cannot conclude that  $\pi$  is different from 0.50 and so we cannot conclude that the population median is different from 450. However, if  $H_0$  is rejected, we can conclude that  $\pi$  is not equal to 0.50 and that the population median is not equal to 450.

With  $n = 12$  stores or trials and  $\pi = 0.50$ , we use Table 5 in Appendix B to obtain the binomial probabilities for the number of plus signs under the assumption  $H_0$  is true. Figure 18.1 shows a graphical representation of this binomial distribution.

We now use the binomial distribution to test the hypothesis about the population median. We shall use a 0.10 level of significance for the test. Since the observed number of plus signs for the sample data (nine) is in the upper tail of the binomial distribution, we begin by computing the probability of obtaining nine or more plus signs, i.e. the probability of nine, ten, 11 or 12 plus signs. Adding these probabilities, we have  $0.0537 + 0.0161 + 0.0029 + 0.0002 = 0.0729$ . Since we are using a two-tailed hypothesis test, this upper-tail probability is doubled to obtain the  $p$ -value  $2(0.0729) = 0.1548$ . With  $p$ -value  $> \alpha$ , we cannot reject  $H_0$ . In terms of the binomial probability  $\pi$ , we cannot reject  $H_0: \pi = 0.50$ , and so we cannot reject the hypothesis that the population median is €450.



**FIGURE 18.1**

Binomial sampling distribution for the number of plus signs when  $n = 12$  and  $\pi = 0.50$

In this example, the hypothesis test was formulated as a two-tailed test. One-tailed sign tests about a population median are also possible. For example, suppose the test had been formulated as an upper-tail test with the following null and alternative hypotheses:

$$\begin{aligned} H_0: \text{Median} &\leq 450 \\ H_1: \text{Median} &> 450 \end{aligned}$$

The appropriate  $p$ -value is the binomial probability that the number of plus signs is greater than or equal to nine found in the sample. This one-tailed  $p$ -value would have been  $0.0537 + 0.0161 + 0.0029 + 0.0002 = 0.0729$ .

The application we have just described makes use of the binomial distribution with  $\pi = 0.50$ . The binomial probabilities provided in Table 5 of Appendix B can be used to compute the  $p$ -value when the sample size is 20 or less. With larger sample sizes, we can use a computer program such as EXCEL, MINITAB or SPSS to calculate the binomial probabilities. Alternatively, we can rely on the normal distribution approximation of the binomial distribution to compute the  $p$ -value. A large-sample application of the sign test is illustrated next.

Suppose that one year ago the median price of a new home was €236 000. However, a current downturn in the economy prompts an estate agent to use sample data on recent home sales to determine if the population median price of a new home is less today than it was a year ago. The hypothesis test about the population median price of a new home is as follows:

$$\begin{aligned} H_0: \text{Median} &\geq 236\,000 \\ H_1: \text{Median} &< 236\,000 \end{aligned}$$

We will use a 0.05 level of significance to do this test.

A random sample of 61 recent new home sales found 22 homes sold for more than €236 000, 38 homes sold for less than €236 000 and one home sold for €236 000. After deleting the home that sold for the hypothesized median price of €236 000, the sign test continues with 22 plus signs, 38 minus signs and a sample of 60 homes.

The null hypothesis that the population median is greater than or equal to €236 000 is expressed by the binomial distribution hypothesis  $H_0: \pi \geq 0.50$ . If  $H_0$  were true as an equality, we would expect  $0.50(60) = 30$  homes to have a plus sign. The sample result showing 22 plus signs is in the lower tail of the binomial distribution. So the  $p$ -value is the probability of 22 or fewer plus signs when  $\pi = 0.50$ . Although it is possible to compute the exact binomial probabilities for 0, 1, 2 ... to 22 and sum these probabilities, we will use the normal distribution approximation of the binomial distribution to make this computation easier. For this approximation, the mean and standard deviation of the normal distribution are as follows.

#### Normal approximation of the sampling distribution of the number of plus signs for $H_0: \pi = 0.50$ .

$$\text{Mean: } \mu = 0.50n \quad (18.1)$$

$$\text{Standard deviation: } \sigma = \sqrt{0.25n} \quad (18.2)$$

Distribution form: approximately normal for  $n > 20$ .

Using equations (18.1) and (18.2) with  $n = 60$  homes and  $\pi = 0.50$ , the sampling distribution of the number of plus signs can be approximated by a normal distribution with:

$$\begin{aligned} \mu &= 0.50n = 0.50(60) = 30 \\ \sigma &= \sqrt{0.25n} = \sqrt{0.25(60)} = 3.783 \end{aligned}$$

We now use this distribution to approximate the binomial probability of 22 or fewer plus signs. Remember that the binomial probability distribution is discrete and the normal probability distribution is continuous. To take account of this, the binomial probability of 22 is computed by the normal probability interval 21.5 to 22.5. The 0.5 added to and subtracted from 22 is called the continuity correction factor.



HOMESALES

To compute the  $p$ -value for 22 or fewer plus signs we use the normal distribution with  $\mu = 30$  and  $\sigma = 3.873$  to compute the probability that the normal random variable,  $X$ , has a value less than or equal to 22.5:

$$p\text{-value} = P(X \leq 22.5) = P\left(Z \leq \frac{22.5 - 30}{3.873}\right) = P(Z \leq -1.94)$$

Using the normal probability distribution table, we see that the cumulative probability for  $z = -1.94$  provides the  $p$ -value = 0.0262. With  $0.0262 < 0.05$  we reject the null hypothesis and conclude that the median price of a new home is less than the €236 000 median price a year ago.

## Hypothesis test with matched samples

In Chapter 10 we introduced a matched-sample experimental design where each of  $n$  experimental units provided a pair of observations, one from population 1 and one from population 2. Using quantitative data and assuming that the differences between the pairs of matched observations were normally distributed, the  $t$  distribution was used to make an inference about the difference between the means of the two populations.

In the following example we use the non-parametric sign test to analyze matched-sample data. Unlike the  $t$  distribution procedure, which required quantitative data and the assumption that the differences were normally distributed, the sign test enables us to analyze categorical as well as quantitative data and requires no assumption about the distribution of the differences. This type of matched-sample design occurs in market research when a sample of  $n$  potential customers is asked to compare two brands of a product such as coffee, soft drinks or detergents (see Statistics in Practice at the beginning of the chapter). Without obtaining a quantitative measure of each individual's preference for the brands, each individual is asked to state a brand preference. Consider the following example.

Sunny Vale Farms produces an orange juice product marketed under the name Citrus Delight. A competitor produces an orange juice product known as Tropical Orange. In a study of consumer preferences for the two brands, 14 individuals were given unmarked samples of each product. The brand each individual tasted first was selected randomly. After tasting the two products, the individuals were asked to state a preference for one of the two brands. The purpose of the study is to determine whether consumers in general prefer one product over the other.

If the individual selected Citrus Delight as the more preferred, a plus sign was recorded. If the individual selected Tropical Orange as the more preferred, a minus sign was recorded. If the individual was unable to express a difference in preference for the two products, no sign was recorded. The data for the 14 individuals in the study are shown in Table 18.2.

Deleting the two individuals who could not express a preference for either brand, the data have been converted to a sign test with two plus signs and ten minus signs for the  $n = 12$  individuals who could express a preference for one of the two brands. Letting  $\pi$  indicate the proportion of the population of customers who prefer Citrus Delight orange juice, we want to test the hypotheses that there is no difference between the preferences for the two brands as follows:

$$\begin{aligned} H_0: \pi &= 0.50 \\ H_1: \pi &\neq 0.50 \end{aligned}$$

**TABLE 18.2** Preference data for the Sunny Vale Farms taste test

| Individual | Preference      | Sign | Individual | Preference      | Sign |
|------------|-----------------|------|------------|-----------------|------|
| 1          | Tropical Orange | -    | 8          | Tropical Orange | -    |
| 2          | Tropical Orange | -    | 9          | Tropical Orange | -    |
| 3          | Citrus Valley   | +    | 10         | No Preference   |      |
| 4          | Tropical Orange | -    | 11         | Tropical Orange | -    |
| 5          | Tropical Orange | -    | 12         | Citrus Valley   | +    |
| 6          | No Preference   |      | 13         | Tropical Orange | -    |
| 7          | Tropical Orange | -    | 14         | Tropical Orange | -    |

If  $H_0$  cannot be rejected, we shall have no evidence indicating a difference in preference for the two brands of orange juice. However, if  $H_0$  can be rejected, we can conclude that the consumer preferences are different for the two brands. In that case, the brand selected by the greater number of consumers can be considered the preferred brand. We shall use a 0.05 level of significance.

We conduct the sign test exactly as we did earlier in this section. The sampling distribution for the number of plus signs is a binomial distribution with  $\pi = 0.50$  and  $n = 12$ . Using Table 5 in Appendix B we obtain the binomial probabilities for the number of plus signs (the same ones shown in Figure 18.1). Under the assumption  $H_0$  is true, we would expect  $0.50n = 0.50(12) = 6$  plus signs. With only two plus signs in the sample, the results are in the lower tail of the binomial distribution. To compute the  $p$ -value for this two-tailed test, we first compute the probability of two or fewer plus signs and then double this value. Using the binomial probabilities of 0, 1 and 2 shown in Figure 18.1, the  $p$ -value is  $2(0.0002 + 0.0029 + 0.0161) = 0.0384$ . With  $0.0384 < 0.05$ , we reject  $H_0$ . The taste test provides evidence that consumer preference differs significantly for the two brands of orange juice. We would advise Sunny Vale Farms of this result and conclude that the competitor's Tropical Orange product is the more preferred. Sunny Vale Farms can then pursue a strategy to address this issue.

As with other uses of the sign test, one-tailed tests may be used depending upon the application. Also, as the sample size becomes large, the normal distribution approximation of the binomial distribution will ease the computations as shown earlier in this section. While the Sunny Vale Farms sign test for matched samples used categorical preference data, the sign test for matched samples can be used with quantitative data as well. This would be particularly helpful if the paired differences are not normally distributed and are skewed. In this case a positive difference is assigned a plus sign, a negative difference is assigned a negative sign, and a zero difference is removed from the sample. The sign test computations proceed as before.

## EXERCISES

### Methods

- The following table lists the preferences indicated by ten individuals in taste tests involving two brands of a product. A plus indicates a preference for Brand A over Brand B.

| Individual | <i>Brand A versus Brand B</i> | Individual | <i>Brand A versus Brand B</i> |
|------------|-------------------------------|------------|-------------------------------|
| 1          | +                             | 6          | +                             |
| 2          | +                             | 7          | –                             |
| 3          | +                             | 8          | +                             |
| 4          | –                             | 9          | –                             |
| 5          | +                             | 10         | +                             |

With  $\alpha = 0.05$ , test for a significant difference in the preferences for the two brands.

- The following hypothesis test is to be conducted.

$$\begin{aligned} H_0: \text{Median} &\leq 150 \\ H_1: \text{Median} &> 150 \end{aligned}$$

A sample of size 30 yields 22 cases in which a value greater than 150 is obtained, three cases in which a value of exactly 150 is obtained, and five cases in which a value less than 150 is obtained. Conduct the hypothesis test using  $\alpha = 0.01$ .



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### Applications

- A poll asked 1253 adults a series of questions about the state of the economy and their children's future. One question was, 'Do you expect your children to have a better life than you have had, a worse life or a life about as good as yours?' The responses were 34 per cent better, 29 per cent

worse, 33 per cent about the same and 4 per cent not sure. Use the sign test and a 0.05 level of significance to determine whether more adults feel their children will have a better future than feel their children will have a worse future. What is your conclusion?

4. Previous research by SNL Securities suggested that stock splits in the banking industry tended to increase the value of an individual's stock holding. Assume that of a sample of 20 recent stock splits, 14 led to an increase in value, four led to a decrease in value and two resulted in no change. Suppose a sign test is to be used to determine whether stock splits continue to be beneficial for holders of bank stocks.
  - a. What are the null and alternative hypotheses?
  - b. With  $\alpha = 0.05$ , what is your conclusion?
5. An opinion survey asked the following question regarding a proposed educational policy. 'Do you favour or oppose providing tax-funded vouchers or tax deductions to parents who send their children to private fee-paying schools?' Of the 2010 individuals surveyed, 905 favoured the support, 1045 opposed the support and 60 offered no opinion. Do the data indicate a significant tendency towards favouring or opposing the proposed policy? Use a 0.05 level of significance.
6. Suppose a national survey in France has shown that the median annual income adults say would make their dreams come true is €152 000. Suppose further that, of a sample of 225 individuals in Calais, 122 individuals report that the amount of income needed to make their dreams come true is less than €152 000 and 103 report that the amount needed is more than €152 000. Test the null hypothesis that the median amount of annual income needed to make dreams come true in Calais is €152 000. Use  $\alpha = 0.05$ . What is your conclusion?
7. The median number of part-time employees at fast-food restaurants in a particular city was known to be 15 last year. The city council thinks the use of part-time employees may have increased this year. A sample of nine fast-food restaurants showed that more than 15 part-time employees worked at seven of the restaurants, one restaurant had exactly 15 part-time employees and one had fewer than 15 part-time employees. Test at  $\alpha = 0.05$  to see whether the median number of part-time employees has increased.
8. Land Registry figures for late 2011 show the median selling price of houses in England as £185 000. Assume that the following data were obtained for sales of houses in Greater Manchester and in Oxfordshire.

|                    | <i>Greater than<br/>£185 000</i> | <i>Equal to<br/>£185 000</i> | <i>Less than<br/>£185 000</i> |
|--------------------|----------------------------------|------------------------------|-------------------------------|
| Greater Manchester | 11                               | 2                            | 32                            |
| Oxfordshire        | 27                               | 1                            | 13                            |

- a. Is the median selling price in Greater Manchester lower than the national median of £185 000? Use a statistical test with  $\alpha = 0.05$  to support your conclusion.
- b. Is the median selling price in Oxfordshire higher than the national median of £185 000? Use a statistical test with  $\alpha = 0.05$  to support your conclusion.



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## 18.2 WILCOXON SIGNED-RANK TEST

The **Wilcoxon signed-rank test** is a non-parametric procedure for analyzing data from a matched-sample experiment. The test uses quantitative data but does not require the assumption that the differences between the paired observations are normally distributed. It requires only the assumption that the differences between the paired observations have a symmetrical distribution, and examines whether the population differences are centred on the value zero (i.e. have a mean or median equal to zero). We demonstrate the Wilcoxon signed-rank test with the following example.



Let  $T$  denote the sum of the signed-rank values. The procedure assumes that the distribution of differences between matched pairs is symmetrical, but not necessarily normal in shape. It can be shown that if the two populations are identical and the number of matched pairs of data is ten or more, the sampling distribution of  $T$  can be approximated by a normal distribution as follows.

### Sampling distribution of $T$ for identical populations

$$\text{Mean: } \mu_T = 0 \quad (18.3)$$

$$\text{Standard deviation: } \sigma_T = \sqrt{\frac{n(n+1)(2n+1)}{6}} \quad (18.4)$$

Distribution form: approximately normal provided  $n \geq 10$ .

For the example, we have  $n = 10$  after discarding the observation with the difference of zero (worker 8). Using equation (18.4), we have:

$$\text{Standard deviation: } \sigma_T = \sqrt{\frac{(10)(11)(21)}{6}} = 19.62$$

We shall use a 0.05 level of significance to draw a conclusion. With the sum of the signed-rank values  $T = 44$ , we calculate the following value for the test statistic.

$$z = \frac{T - \mu_T}{\sigma_T} = \frac{44 - 0}{19.62} = 2.24$$

Using the standard normal distribution table and  $z = 2.24$ , we find the two-tailed  $p$ -value  $= 2(1 - 0.9875) = 0.025$ . With  $p$ -value  $< \alpha = 0.05$ , we reject  $H_0$  and conclude that the two populations are not identical and that the methods differ in task completion time. Method 2's shorter completion times for eight of the workers lead us to conclude that method 2 is the preferred production method.

## EXERCISES

### Applications

9. Two fuel additives are tested on family cars to determine their effect on litres of fuel consumed per 100 kilometres travelled. Test results for 12 cars follow. Each car was tested with both fuel additives. Use  $\alpha = 0.05$  and the Wilcoxon signed-rank test to see whether there is a significant difference in the additives.

| Car | Additive |      | Car | Additive |      |
|-----|----------|------|-----|----------|------|
|     | 1        | 2    |     | 1        | 2    |
| 1   | 7.02     | 7.82 | 7   | 8.74     | 8.21 |
| 2   | 6.00     | 6.49 | 8   | 7.62     | 9.43 |
| 3   | 6.41     | 6.26 | 9   | 6.46     | 7.05 |
| 4   | 7.37     | 8.28 | 10  | 5.83     | 6.68 |
| 5   | 6.65     | 6.65 | 11  | 6.09     | 6.20 |
| 6   | 5.70     | 5.93 | 12  | 5.65     | 5.96 |


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- 10.** A sample of ten men was used in a study to test the effects of a relaxant on the time required to fall asleep for male adults. Data for ten participants showing the number of minutes required to fall asleep with and without the relaxant follow. Use a 0.05 level of significance to determine whether the relaxant reduces the time required to fall asleep. What is your conclusion?

| Participant | Without relaxant | With relaxant | Participant | Without relaxant | With relaxant |
|-------------|------------------|---------------|-------------|------------------|---------------|
| 1           | 15               | 10            | 6           | 7                | 5             |
| 2           | 12               | 10            | 7           | 8                | 10            |
| 3           | 22               | 12            | 8           | 10               | 7             |
| 4           | 8                | 11            | 9           | 14               | 11            |
| 5           | 10               | 9             | 10          | 9                | 6             |

- 11.** A test was conducted of two overnight mail delivery services. Two samples of identical deliveries were set up so that both delivery services were notified of the need for a delivery at the same time. The hours required to make each delivery follow. Do the data shown suggest a difference in the delivery times for the two services? Use a 0.05 level of significance for the test.

| Delivery | Service |      |
|----------|---------|------|
|          | 1       | 2    |
| 1        | 24.5    | 18.0 |
| 2        | 26.0    | 25.5 |
| 3        | 28.0    | 32.0 |
| 4        | 21.0    | 20.0 |
| 5        | 18.0    | 19.5 |
| 6        | 36.0    | 28.0 |
| 7        | 25.0    | 29.0 |
| 8        | 21.0    | 22.0 |
| 9        | 24.0    | 23.5 |
| 10       | 26.0    | 29.5 |
| 11       | 31.0    | 30.0 |

- 12.** Ten test-market cities in France were selected as part of a market research study designed to evaluate the effectiveness of a particular advertising campaign. The sales in euros for each city were recorded for the week prior to the promotional programme. Then the campaign was conducted for two weeks and new sales data were collected for the week immediately after the campaign. The two sets of sales data (in thousands of euros) follow.

| City       | Pre-campaign sales | Post-campaign sales |
|------------|--------------------|---------------------|
| Bordeaux   | 130                | 160                 |
| Strasbourg | 100                | 105                 |
| Nantes     | 120                | 140                 |
| St Etienne | 95                 | 90                  |
| Lyon       | 140                | 130                 |
| Rennes     | 80                 | 82                  |
| Le Havre   | 65                 | 55                  |
| Amiens     | 90                 | 105                 |
| Toulouse   | 140                | 152                 |
| Marseilles | 125                | 140                 |

Use  $\alpha = 0.05$ . What conclusion would you draw about the value of the advertising programme?

## 18.3 MANN-WHITNEY-WILCOXON TEST

In Chapter 10 we introduced a procedure for doing a hypothesis test about the difference between the means of two populations using two independent samples: one from population 1 and one from population 2. This parametric test required quantitative data and the assumption that both populations had a normal distribution. In the case where the population standard deviations  $\sigma_1$  and  $\sigma_2$  were unknown, the sample standard deviations  $s_1$  and  $s_2$  provided estimates of  $\sigma_1$  and  $\sigma_2$  and the  $t$  distribution was used to make an inference about the difference between the means of the two populations.

In this section we present another non-parametric method that can be used to determine whether a difference exists between two populations. This test, unlike the signed-rank test, is not based on matched samples. Two independent samples are used, one from each population. The test was developed jointly by Mann and Whitney and by Wilcoxon. It is sometimes called the *Mann–Whitney test* and sometimes the *Wilcoxon rank-sum test*. The Mann–Whitney and Wilcoxon versions of this test are equivalent. We refer to it as the **Mann–Whitney–Wilcoxon (MWW) test**.

The MWW test does not require interval data nor the assumption that the populations are normally distributed. The only requirement of the MWW test is that the measurement scale for the data is at least ordinal. The MWW test examines whether the two populations are identical:

$$\begin{aligned} H_0: & \text{The two populations are identical} \\ H_1: & \text{The two populations are not identical} \end{aligned}$$

If  $H_0$  is rejected, we are using the test to conclude that the populations are not identical and that population 1 tends to provide either smaller or larger values than population 2.

We shall first illustrate the MWW test using small samples with rank-ordered data. This will give you an understanding of how the rank-sum statistic is computed and how it is used to determine whether to reject the null hypothesis that the two populations are identical. Later in the section, we will introduce a large-sample approximation based on the normal distribution that will simplify the calculations required by the MWW test.

Consider on-the-job performance ratings for employees at a CineMax 20-screen multiplex. During an employee performance review, the multiplex manager rated all 35 employees from best (highest rating of 35) to worst (lowest rating of 1). Knowing that the part-time employees were primarily university and senior school students, the multiplex manager asked if there was evidence of a difference in performance for university students compared to senior school students. In terms of the population of university students and the population of senior school students who could be considered for employment at the multiplex, the hypotheses were stated as follows:

$$\begin{aligned} H_0: & \text{University and senior school student populations are identical in terms of performance} \\ H_0: & \text{University and senior school student populations are not identical in terms of performance} \end{aligned}$$

We will use a 0.05 level of significance for this test.

We begin by selecting a random sample of four university students and a random sample of five senior school students working at the CineMax multiplex (these sample numbers are chosen arbitrarily for illustrative purposes). The multiplex manager's performance rating based on all 35 employees was recorded for each of these employees, as shown in Table 18.4. The first university student selected was given a rating of 21, the university student selected was given a rating of 33 and so on.

The next step in the MWW procedure is to rank the *combined* samples from low to high. Since there is a total of nine students, we rank the performance rating data in Table 18.4 from 1 to 9. The lowest value of 4 for senior school student 3 receives a rank of 1 and the second lowest value of 11 for senior school student 5 receives a rank of 2. The highest value of 33 for university student 2 receives a rank of 9. The combined-sample ranks for all nine students are shown in Table 18.4.

Next we sum the ranks for each sample as shown in Table 18.4. The MWW procedure may use the sum of the ranks for either sample. In our application of the MWW test we will follow the common practice of using the first sample, which is the sample of four university students. The sum of ranks for the first sample will be the test statistic  $W$  for the MWW test. This sum, as shown in Table 18.4, is  $W = 6 + 9 + 3 + 8 = 26$ .

**TABLE 18.4** Ranks for the nine students in the CineMax combined samples

| University student | Manager's performance rating | Rank      | Senior school student | Manager's performance rating | Rank      |
|--------------------|------------------------------|-----------|-----------------------|------------------------------|-----------|
| 1                  | 21                           | 6         | 1                     | 18                           | 5         |
| 2                  | 33                           | 9         | 2                     | 16                           | 4         |
| 3                  | 13                           | 3         | 3                     | 4                            | 1         |
| 4                  | 28                           | 8         | 4                     | 27                           | 7         |
|                    | <b>Sum of ranks</b>          | <b>26</b> | 5                     | 11                           | 2         |
|                    |                              |           |                       | <b>Sum of ranks</b>          | <b>19</b> |

Let us consider why the sum of the ranks will help us select between the two hypotheses:  $H_0$ , the two populations are identical and  $H_1$ , the two populations are not identical. Letting U denote a university student and S denote a senior school student, suppose the ranks of the nine students had the following order with the four university students having the four lowest ranks.

| Rank    | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---------|---|---|---|---|---|---|---|---|---|
| Student | U | U | U | U | S | S | S | S | S |

This permutation or ordering separates the two samples, with the university students all having a lower rank than the senior school students. This is a strong indication that the two populations are not identical. The sum of ranks for the college students in this case is  $W = 1 + 2 + 3 + 4 = 10$ .

Now consider a ranking where the four university students have the four highest ranks.

| Rank    | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---------|---|---|---|---|---|---|---|---|---|
| Student | S | S | S | S | S | U | U | U | U |

This permutation or ordering separates the two samples again, but this time the university students all have a higher rank than the senior school students. This is another strong indication that the two populations are not identical. The sum of ranks for the university students in this case is  $W = 6 + 7 + 8 + 9 = 30$ . So we see that the sum of the ranks for the university students must be between 10 and 30. Values of  $W$  near 10 imply that university students have lower ranks than the senior school students, whereas values of  $W$  near 30 imply that university students have higher ranks than the senior school students. Either of these extremes would signal the two populations are not identical. However, if the two populations are identical, we would expect a mix in the ordering of the U's and S's so that the sum of ranks  $W$  is closer to the average of the two extremes, or nearer to  $(10 + 30)/2 = 20$ .

Evaluation of the exact sampling distribution of the  $W$  statistic is not straightforward, and needs a computer program. However, there are published tables of critical values, such as those in Table 8 of Appendix B for cases in which both sample sizes are less than or equal to ten. In that table,  $n_1$  refers to the sample size corresponding to the sample whose rank sum is being used in the test. The value of  $W_L$  is read directly from the table and the value of  $W_U$  is computed from equation (18.5).

$$W_U = n_1(n_1 + n_2 + 1) - W_L \quad (18.5)$$

The null hypothesis of identical populations should be rejected only if  $W$  is strictly less than  $W_L$  or strictly greater than  $W_U$ .

Using Table 8 of Appendix B with a 0.05 level of significance, we see that the lower-tail critical value for the MWW statistic with  $n_1 = 4$  (university students) and  $n_2 = 5$  (senior school students) is  $W_L = 12$ . The upper-tail critical value for the MWW statistic computed by using equation (18.5) is:

$$W_U = 4(4 + 5 + 1) - 12 = 28$$

The MW<sub>W</sub> decision rule indicates that the null hypothesis of identical populations can be rejected if the sum of the ranks for the first sample (university students) is less than 12 or greater than 28. The rejection rule can be written as:

$$\text{Reject } H_0 \text{ if } W < 12 \text{ or if } W > 28$$

Referring to Table 18.4, we see that  $W = 26$ . The MW<sub>W</sub> test conclusion is that we cannot reject the null hypothesis that the populations of university and senior school students are identical. The sample of four university students and the sample of five senior school students did not provide statistical evidence to conclude there is a difference between the two populations. Further study with larger samples should be considered before drawing a final conclusion.

As noted above, the exact sampling distribution of the  $W$  statistic is not straightforward to evaluate. Some statistical programs are able to do this and give an exact  $p$ -value. For example, IBM SPSS includes exact versions of several non-parametric methods. For the CineMax employee ratings illustration, SPSS gives a two-tailed  $p$ -value of 0.190 (confirming our conclusion that we do not have sufficient evidence to reject  $H_0$ ).

Most applications of the MW<sub>W</sub> test involve larger sample sizes than shown in this first example. For such applications, a large-sample approximation of the sampling distribution of  $W$  based on the normal distribution can be used. We will use the same combined-sample ranking procedure that we used in the previous example but will use the normal distribution approximation to compute the  $p$ -value and draw the conclusion rather than using the tables of critical values for  $W$ . We illustrate the large sample case by considering a situation at People's Bank.

People's Bank has two branch offices. Data collected from two independent simple random samples, one from each branch, are given in Table 18.5 (the rankings in this table are explained below). What do the data indicate regarding the hypothesis that the populations of current account balances at the two branch banks are identical?

The first step in the MW<sub>W</sub> test is to rank the *combined* data from the lowest to the highest values. Using the combined set of 22 observations in Table 18.5, we find the lowest data value of €750 (sixth item of sample 2) and assign to it a rank of 1. Continuing the ranking gives us the following list.

| Balance (€) | Item            | Assigned rank |
|-------------|-----------------|---------------|
| 750         | 6th of sample 2 | 1             |
| 800         | 5th of sample 2 | 2             |
| 805         | 7th of sample 1 | 3             |
| 850         | 2nd of sample 2 | 4             |
| :           |                 |               |
| 1195        | 4th of sample 1 | 21            |
| 1200        | 3rd of sample 1 | 22            |

In ranking the combined data, we may find that two or more data values are the same. In that case, the tied values are given the *average* ranking of their positions in the combined data set. For example, the balance of €945 (eighth item of sample 1) will be assigned the rank of 11. However, the next two values in the data set are tied with values of €950 (see the sixth item of sample 1 and the fourth item of sample 2). These two values would be assigned ranks of 12 and 13 if they were distinct, so they are both assigned the rank of 12.5. The next data value of €955 is then assigned the rank of 14. Table 18.5 shows the assigned rank of each observation.

The next step in the MW<sub>W</sub> test is to sum the ranks for each sample. The sums are given in Table 18.5. The test procedure can be based on the sum of the ranks for either sample. We use the sum of the ranks for the sample from branch 1. So, for this example,  $W = 169.5$ .

Given that the sample sizes are  $n_1 = 12$  and  $n_2 = 10$ , we can use the normal approximation to the sampling distribution of the rank sum  $T$ . The appropriate sampling distribution is given by the following expressions.

**TABLE 18.5** Cheque account balances for two branches of People's Bank, and combined ranking of the data

| Branch 1            |             |              | Branch 2            |             |             |
|---------------------|-------------|--------------|---------------------|-------------|-------------|
| Account             | Balance (€) | Rank         | Account             | Balance (€) | Rank        |
| 1                   | 1095        | 20           | 1                   | 885         | 7           |
| 2                   | 955         | 14           | 2                   | 850         | 4           |
| 3                   | 1200        | 22           | 3                   | 915         | 8           |
| 4                   | 1195        | 21           | 4                   | 950         | 12.5        |
| 5                   | 925         | 9            | 5                   | 800         | 2           |
| 6                   | 950         | 12.5         | 6                   | 750         | 1           |
| 7                   | 805         | 3            | 7                   | 865         | 5           |
| 8                   | 945         | 11           | 8                   | 1000        | 16          |
| 9                   | 875         | 6            | 9                   | 1050        | 18          |
| 10                  | 1055        | 19           | 10                  | 935         | 10          |
| 11                  | 1025        | 17           | <b>Sum of ranks</b> |             | <b>83.5</b> |
| 12                  | 975         | 15           |                     |             |             |
| <b>Sum of ranks</b> |             | <b>169.5</b> |                     |             |             |

### Sampling distribution of W for identical populations

$$\text{Mean: } \mu_W = 0.5n_1(n_1 + n_2 + 1) \quad (18.6)$$

$$\text{Standard deviation: } \sigma_W = \sqrt{\frac{n_1 n_2 (n_1 + n_2 + 1)}{12}} \quad (18.7)$$

Distribution form: approximately normal provided  $n_1 \geq 10$  and  $n_2 \geq 10$ .

For branch 1, we have:

$$\mu_W = 0.5(12)(12 + 10 + 1) = 138$$

$$\sigma_W = \sqrt{\frac{(12)(10)(12 + 10 + 1)}{12}} = 15.17$$

We shall use a 0.05 level of significance to draw a conclusion. With the sum of the ranks for branch 1,  $W = 169.5$ , we calculate the following value for the test statistic.

$$z = \frac{W - \mu_W}{\sigma_W} = \frac{169.5 - 138}{15.17} = 2.08$$

Using the standard normal distribution table and  $z = 2.08$ , we find the two-tailed  $p$ -value =  $2(1 - 0.9812) = 0.0376$ . With  $p$ -value <  $\alpha = 0.05$ , we reject  $H_0$  and conclude that the two populations are not identical; that is, the populations of current account balances at the branch banks are not the same. The evidence suggests that the balances at branch 1 tend to be higher (and therefore be assigned higher ranks) than the balances at branch 2.

In summary, the Mann–Whitney–Wilcoxon rank-sum test consists of the following steps to determine whether two independent random samples are selected from identical populations:

- 1 Rank the combined sample observations from lowest to highest, with tied values being assigned the average of the tied rankings.
- 2 Compute  $W$ , the sum of the ranks for the first sample.

- 3** In the large-sample case, make the test for significant differences between the two populations by using the observed value of  $W$  and comparing it with the sampling distribution of  $W$  for identical populations using equations (18.6) and (18.7). The value of the standardized test statistic  $z$  and the  $p$ -value provide the basis for deciding whether to reject  $H_0$ . In the small-sample case, use Table 9 in Appendix B to find the critical values for the test.

The parametric statistical tests described in Chapter 10 test the equality of two population means. When we reject the hypothesis that the means are equal, we conclude that the populations differ in their means. When we reject the hypothesis that the populations are identical by using the MWW test, we cannot state how they differ. The populations could have different means, different medians, different variances or different forms. Nonetheless, if we believe that the populations are the same in every aspect but the means, a rejection of  $H_0$  by the non-parametric method implies that the means differ.

## EXERCISES

### Applications

- 13.** Two fuel additives are being tested to determine their effect on petrol consumption. Seven cars were tested with additive 1 and nine cars were tested with additive 2. The following data show the litres of fuel used per 100 kilometres with the two additives. Use  $\alpha = 0.05$  and the MWW test to see whether there is a significant difference in petrol consumption for the two additives.

| Additive 1 | Additive 2 |
|------------|------------|
| 8.20       | 7.52       |
| 7.69       | 7.94       |
| 7.41       | 6.62       |
| 8.47       | 6.71       |
| 7.75       | 6.41       |
| 7.58       | 7.52       |
| 8.06       | 7.14       |
|            | 6.80       |
|            | 6.99       |

- 14.** A company's price/earnings (P/E) ratio is the company's current stock price divided by the latest 12 months' earnings per share. Listed below are the P/E ratios for a sample of ten Japanese and twelve US companies. Is the difference in P/E ratios between the two countries significant? Use the MWW test and  $\alpha = 0.01$  to support your conclusion.

| Japan             |           | US                      |           |
|-------------------|-----------|-------------------------|-----------|
| Company           | P/E ratio | Company                 | P/E ratio |
| Sumitomo Corp.    | 153       | Gannet                  | 19        |
| Kinden            | 21        | Motorola                | 24        |
| Heiwa             | 18        | Schlumberger            | 24        |
| NCR Japan         | 125       | Oracle Systems          | 43        |
| Suzuki Motor      | 31        | Gap                     | 22        |
| Fuji Bank         | 213       | Winn-Dixie              | 14        |
| Sumitomo Chemical | 64        | Ingersoll-Rand          | 21        |
| Seibu Railway     | 666       | American Electric Power | 14        |
| Shiseido          | 33        | Hercules                | 21        |
| Toho Gas          | 68        | Times Mirror            | 38        |
|                   |           | WellPoint Health        | 15        |
|                   |           | Northern States Power   | 14        |



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- 15.** Samples of annual starting salaries for individuals entering the public accounting and financial planning professions follow. Annual salaries are shown in thousands of euros.

| Public accountant | Public accountant | Financial planner | Financial planner |
|-------------------|-------------------|-------------------|-------------------|
| 45.2              | 50.0              | 44.0              | 48.6              |
| 53.8              | 45.9              | 44.2              | 44.7              |
| 51.3              | 54.5              | 48.1              | 48.9              |
| 53.2              | 52.0              | 50.9              | 46.8              |
| 49.2              | 46.9              | 46.9              | 43.9              |

- a. Using  $\alpha = 0.05$ , test the hypothesis that there is no difference between the starting annual salaries of public accountants and financial planners. What is your conclusion?
- b. What are the sample mean annual salaries for the two professions?
- 16.** A confederation of house builders provided data on the cost (in £) of the most popular home re-modelling projects. Use the Mann–Whitney–Wilcoxon test to see whether it can be concluded that the cost of kitchen re-modelling differs from the cost of master bedroom re-modelling. Use a 0.05 level of significance.

| Kitchen | Master bedroom |
|---------|----------------|
| 13 200  | 6 000          |
| 5 400   | 10 900         |
| 10 800  | 14 400         |
| 9 900   | 12 800         |
| 7 700   | 14 900         |
| 11 000  | 5 800          |
| 7 700   | 12 600         |
| 4 900   | 9 000          |
| 9 800   |                |
| 11 600  |                |

- 17.** The gap between the earnings of men and women with equal education is narrowing in many countries but has not closed. Sample data from the United Arab Emirates for seven men and seven women with Bachelor's degrees are as follows. Data of monthly earnings are shown in thousands of Dirham.

|       |      |      |      |      |      |      |      |
|-------|------|------|------|------|------|------|------|
| Men   | 12.2 | 30.2 | 18.1 | 24.9 | 15.3 | 20.0 | 22.1 |
| Women | 17.8 | 14.2 | 11.2 | 16.2 | 10.3 | 19.0 | 9.9  |

- a. What is the median salary for men? For women?
- b. Use  $\alpha = 0.05$  and conduct the hypothesis test for identical populations. What is your conclusion?

## 18.4 KRUSKAL–WALLIS TEST

The MWW test in Section 18.3 can be used to test whether two populations are identical. Kruskal and Wallis extended the test to the case of three or more populations. The hypotheses for the **Kruskal–Wallis test** with  $k \geq 3$  populations can be written as follows.

$$\begin{aligned} H_0: & \text{ All } k \text{ populations are identical} \\ H_1: & \text{ Not all } k \text{ populations are identical} \end{aligned}$$

The Kruskal–Wallis test is based on the analysis of independent random samples from each of the  $k$  populations.

In Chapter 13 we showed that analysis of variance (ANOVA) can be used to test for the equality of means among three or more populations. The ANOVA procedure requires interval- or ratio-level data

and the assumption that the  $k$  populations are normally distributed. The non-parametric Kruskal–Wallis test can be used with ordinal data as well as with interval or ratio data. In addition, the Kruskal–Wallis test does not require the assumption of normally distributed populations. We demonstrate the Kruskal–Wallis test by using it in an employee selection application.

Williams Manufacturing hires employees for its management staff from three local colleges. Recently the company's personnel department began collecting and reviewing annual performance ratings in an attempt to determine whether there are differences in performance among the managers hired from these colleges. Performance rating data are available from independent samples of seven employees from college A, six employees from college B and seven employees from college C. These data are summarized in Table 18.6; the overall performance rating of each manager is given on a 0–100 scale, with 100 being the highest possible performance rating (the rankings are explained below).

Suppose we want to test whether the three populations are identical in terms of performance evaluations. We shall use a 0.05 level of significance. The Kruskal–Wallis test statistic, which is based on the sum of ranks for each of the samples, can be computed as follows.

### Kruskal–Wallis test statistic

where:

$$W = \left[ \frac{12}{n_T(n_T + 1)} \sum_{i=1}^k \frac{R_i^2}{n_i} \right] - 3(n_T + 1) \quad (18.8)$$

$k$  = the number of populations

$n_i$  = the number of items in sample  $i$

$n_T = \sum n_i$  = total number of items in all samples

$R_i$  = sum of the ranks for sample  $i$

Kruskal and Wallis were able to show that, under the null hypothesis that the populations are identical, the sampling distribution of  $W$  can be approximated by a chi-squared distribution with  $k - 1$  degrees of freedom. This approximation is generally acceptable if each of the sample sizes is greater than or equal to five. The null hypothesis of identical populations will be rejected if the test statistic is large. As a result, the procedure uses an upper-tail test.

To compute the  $W$  statistic for our example, we first rank all 20 data items. The lowest data value of 15 from college B sample receives a rank of 1, whereas the highest data value of 95 from college A sample receives a rank of 20. The ranks and the sums of the ranks for the three samples are given in Table 18.6. Note that we assign the average rank to tied items;\* for example, the data values of 60, 70, 80 and 90 had ties.

The sample sizes are:

$$n_1 = 7 \quad n_2 = 6 \quad n_3 = 7$$

and:

$$n_T = \sum n_i = 7 + 6 + 7 = 20$$

We compute the  $W$  statistic by using equation (18.8).

$$W = \left[ \frac{12}{(20)(21)} \right] \left[ \frac{(95)^2}{7} + \frac{(27)^2}{6} + \frac{(88)^2}{7} \right] - 3(20 + 1) = 8.92$$

We can now use the chi-squared distribution table (Table 3 of Appendix B) to determine the  $p$ -value for the test. Using  $k - 1 = 3 - 1 = 2$  degrees of freedom, we find  $\chi^2 = 7.378$  has an area of 0.025 in the upper tail of the distribution and  $\chi^2 = 9.21$  has an area of 0.01 in the upper tail. For  $W = 8.92$ , between 7.378 and 9.21, the area in the upper tail is between 0.025 and 0.01. Because it is an upper tail test, we can conclude that the  $p$ -value is between 0.025 and 0.01. (A calculation in MINITAB, IBM SPSS or EXCEL shows  $p$ -value = 0.0116.)

\*If numerous tied ranks are observed, equation (18.8) must be modified. The modified formula is given in W. J. Conover (1999) *Practical Non parametric Statistics*, 3rd ed. Wiley.

**TABLE 18.6** Performance evaluation ratings for 20 Williams employees

| College A           | Rank      | College B | Rank      | College C | Rank      |
|---------------------|-----------|-----------|-----------|-----------|-----------|
| 25                  | 3         | 60        | 9         | 50        | 7         |
| 70                  | 12        | 20        | 2         | 70        | 12        |
| 60                  | 9         | 30        | 4         | 60        | 9         |
| 85                  | 17        | 15        | 1         | 80        | 15.5      |
| 95                  | 20        | 40        | 6         | 90        | 18.5      |
| 90                  | 18.5      | 35        | 5         | 70        | 12        |
| 80                  | 15.5      |           |           | 75        | 14        |
| <b>Sum of ranks</b> | <b>95</b> |           | <b>27</b> |           | <b>88</b> |

Because  $p\text{-value} < \alpha = 0.05$ , we reject  $H_0$  and conclude that the three populations are not identical. Manager performance differs significantly depending on the college attended. Furthermore, because the performance ratings are lowest for college B, it would be reasonable for the company to either cut back recruiting from college B or at least evaluate its graduates more thoroughly.

## EXERCISES



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### Applications

- 18.** Three college admission test preparation programmes are being evaluated. The scores obtained by a sample of 20 people who used the test preparation programmes provided the following data. Use the Kruskal–Wallis test to determine whether there is a significant difference among the three test preparation programmes. Use  $\alpha = 0.01$ .

| Programme |     |     |
|-----------|-----|-----|
| A         | B   | C   |
| 540       | 450 | 600 |
| 400       | 540 | 630 |
| 490       | 400 | 580 |
| 530       | 410 | 490 |
| 490       | 480 | 590 |
| 610       | 370 | 620 |
|           | 550 | 570 |

- 19.** Forty-minute workouts of one of the following activities three days a week may lead to a loss of weight. The following sample data show the number of calories burned during 40-minute workouts for three different activities. Do these data indicate differences in the amount of calories burned for the three activities? Use a 0.05 level of significance. What is your conclusion?

| Swimming | Tennis | Cycling |
|----------|--------|---------|
| 408      | 415    | 385     |
| 380      | 485    | 250     |
| 425      | 450    | 295     |
| 400      | 420    | 402     |
| 427      | 530    | 268     |

- 20.** Condé Nast Traveler magazine conducts an annual survey of its readers in order to rate the top 80 cruise ships in the world. With 100 the highest possible rating, the overall ratings for a sample of ships from the Holland America, Princess and Royal Caribbean cruise lines are shown here. Use the Kruskal–Wallis test with  $\alpha = 0.05$  to determine whether the overall ratings among the three cruise lines differ significantly.

| <i>Holland America</i> |               | <i>Princess</i> |               | <i>Royal Caribbean</i> |               |
|------------------------|---------------|-----------------|---------------|------------------------|---------------|
| <i>Ship</i>            | <i>Rating</i> | <i>Ship</i>     | <i>Rating</i> | <i>Ship</i>            | <i>Rating</i> |
| Amsterdam              | 84.5          | Coral           | 85.1          | Adventure              | 84.8          |
| Maasdam                | 81.4          | Dawn            | 79.0          | Jewel                  | 81.8          |
| Oosterdam              | 84.0          | Island          | 83.9          | Mariner                | 84.0          |
| Volendam               | 78.5          | Princess        | 81.1          | Navigator              | 85.9          |
| Westerdam              | 80.9          | Star            | 83.7          | Serenade               | 87.4          |

- 21.** Course-evaluation ratings for four instructors follow. Use the Kruskal–Wallis procedure with  $\alpha = 0.05$  to test for a significant difference in teaching abilities.

| <i>Instructor</i> | <i>Course-evaluation rating</i> |    |    |    |    |    |    |    |    |
|-------------------|---------------------------------|----|----|----|----|----|----|----|----|
| Black             | 88                              | 80 | 79 | 68 | 96 | 69 |    |    |    |
| Jennings          | 87                              | 78 | 82 | 85 | 99 | 99 | 85 | 94 | 81 |
| Swanson           | 88                              | 76 | 68 | 82 | 85 | 82 | 84 | 83 |    |
| Wilson            | 80                              | 85 | 56 | 71 | 89 | 87 |    |    |    |

## 18.5 RANK CORRELATION

The Pearson product-moment correlation coefficient (see Chapter 3, Section 3.5 and Chapter 14, Section 14.3) is a measure of the linear association between two variables for which interval or ratio data are available. In this section, we consider the **Spearman rank-correlation coefficient**  $r_s$ , which is a measure of association between two variables applicable when only ordinal data are available.

### Spearman rank-correlation coefficient

$$r_s = 1 - \frac{6\sum d_i^2}{n(n^2 - 1)} \quad (18.9)$$

where:

$n$  = the number of items or individuals being ranked

$x_i$  = the rank of item  $i$  with respect to one variable

$y_i$  = the rank of item  $i$  with respect to the second variable

$d_i = x_i - y_i$

Suppose a company wants to determine whether individuals who were expected at the time of employment to be better salespersons actually turn out to have better sales records. To investigate this question, the personnel manager carefully reviewed the original job interview summaries, academic records and letters of recommendation for ten current members of the firm's sales force. After the review, the personnel manager ranked the ten individuals in terms of their potential for success, basing the assessment solely on the information available at the time of employment. Then a list was obtained of the number of units sold by each salesperson over the first two years.

**TABLE 18.7** Sales potential and actual two-year sales data for ten salespeople, and computation of the Spearman rank-correlation coefficient

| Salesperson   | $x_i = \text{Ranking of potential}$ | Two-year sales (units) | $y_i = \text{Ranking of sales performance}$ | $d_i = x_i - y_i$ | $d_i^2$ |
|---|-------------------------------------|------------------------|---|-------------------|---------|
| A   | 9                                   | 400                    | 10  | -1                | 1       |
| B   | 7                                   | 360                    | 8   | -1                | 1       |
| C   | 4                                   | 300                    | 6   | -2                | 4       |
| D   | 10                                  | 295                    | 5   | 5                 | 25      |
| E   | 5                                   | 280                    | 4   | 1                 | 1       |
| F   | 8                                   | 350                    | 7   | 1                 | 1       |
| G   | 1                                   | 200                    | 1   | 0                 | 0       |
| H   | 2                                   | 260                    | 3   | -1                | 1       |
| I   | 3                                   | 220                    | 2   | 1                 | 1       |
| J   | 6                                   | 385                    | 9   | -3                | 9       |
| $\sum d_i^2 = 44$   |                                     |                        |   |                   |         |
| $r_s = 1 - \frac{6\sum d_i^2}{n(n^2 - 1)} = 1 - \frac{(6)(44)}{(10)(100 - 1)} = 0.73$ |                                     |                        |   |                   |         |

On the basis of actual sales performance, a second ranking of the ten salespersons was carried out. Table 18.7 gives the relevant data and the two rankings. In the ranking of potential, rank 1 means lowest potential, rank 2 next lowest and so on. The statistical question is whether there is agreement between the ranking of potential at the time of employment and the ranking based on the actual sales performance over the first two years.

The computations for the Spearman rank-correlation coefficient are summarized in Table 18.7. We see that the rank-correlation coefficient is a positive 0.73. The Spearman rank-correlation coefficient ranges from  $-1.0$  to  $+1.0$  and its interpretation is similar to that of the Pearson correlation coefficient, in that positive values near  $1.0$  indicate a strong association between the rankings; as one rank increases, the other rank increases. Rank correlations near  $-1.0$  indicate a strong negative association between the rankings; as one rank increases, the other rank decreases. The value  $r_s = 0.73$  indicates a positive correlation between potential and actual performance. Individuals ranked high on potential tend to rank high on performance.

At this point, we may want to use the sample results to make an inference about the population rank correlation  $\rho_s$ . To do this, we test the following hypotheses.

$$\begin{aligned} H_0: \rho_s &= 0 \\ H_1: \rho_s &\neq 0 \end{aligned}$$

Under the null hypothesis of no rank correlation ( $\rho_s = 0$ ), the rankings are independent, and the sampling distribution of  $r_s$  is as follows.

#### Sampling distribution of $r_s$

$$\text{Mean: } \mu_{r_s} = 0 \quad (18.10)$$

$$\text{Standard deviation: } \sigma_{r_s} = \sqrt{\frac{1}{n-1}} \quad (18.11)$$

Distribution form: approximately normal provided  $n \geq 10$ .

The sample rank-correlation coefficient for sales potential and sales performance is  $r_S = 0.73$ . From equation (18.10) we have  $\mu_{r_S} = 0$  and from (18.11) we have:

$$\sigma_{r_S} = \sqrt{1/(10-1)} = 0.33$$

Using the test statistic, we have:

$$z = \frac{r_S - \mu_{r_S}}{\sigma_{r_S}} = \frac{0.73 - 0}{0.33} = 2.20$$

Using the standard normal distribution table and  $z = 2.20$ , we find the  $p$ -value =  $2(1 - 0.9861) = 0.0278$ . With a 0.05 level of significance,  $p$ -value <  $\alpha = 0.05$  leads to the rejection of the hypothesis that the rank correlation is zero. We can conclude that there is a positive rank correlation between sales potential and sales performance.

## EXERCISES

### Methods

- 22.** Consider the following set of rankings for a sample of ten elements.

| Element | $x_i$ | $y_i$ | Element | $x_i$ | $y_i$ |
|---------|-------|-------|---------|-------|-------|
| 1       | 10    | 8     | 6       | 2     | 7     |
| 2       | 6     | 4     | 7       | 8     | 6     |
| 3       | 7     | 10    | 8       | 5     | 3     |
| 4       | 3     | 2     | 9       | 1     | 1     |
| 5       | 4     | 5     | 10      | 9     | 9     |

- a. Compute the Spearman rank-correlation coefficient for the data.
  - b. Use  $\alpha = 0.05$  and test for significant rank correlation. What is your conclusion?
- 23.** Consider the following two sets of rankings for six items.

| Case One |               |                | Case Two |               |                |
|----------|---------------|----------------|----------|---------------|----------------|
| Item     | First ranking | Second ranking | Item     | First ranking | Second ranking |
| A        | 1             | 1              | A        | 1             | 6              |
| B        | 2             | 2              | B        | 2             | 5              |
| C        | 3             | 3              | C        | 3             | 4              |
| D        | 4             | 4              | D        | 4             | 3              |
| E        | 5             | 5              | E        | 5             | 2              |
| F        | 6             | 6              | F        | 6             | 1              |

Note that in the first case the rankings are identical, whereas in the second case the rankings are exactly opposite. What value should you expect for the Spearman rank-correlation coefficient for each of these cases? Explain. Calculate the rank-correlation coefficient for each case.

### Applications

- 24.** The following two lists show how ten IT companies ranked in a national survey, in terms of reputation and percentage of respondents who said they would purchase the company's shares. A positive rank correlation is anticipated because it seems reasonable to expect that a company with a higher reputation would be a more desirable purchase.



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| Company           | Reputation | Probable purchase |
|-------------------|------------|-------------------|
| Microsoft         | 1          | 3                 |
| Intel             | 2          | 4                 |
| Dell              | 3          | 1                 |
| Lucent            | 4          | 2                 |
| Texas Instruments | 5          | 9                 |
| Cisco Systems     | 6          | 5                 |
| Hewlett-Packard   | 7          | 10                |
| IBM               | 8          | 6                 |
| Motorola          | 9          | 7                 |
| Yahoo             | 10         | 8                 |

- a. Compute the rank correlation between reputation and probable purchase.
- b. Test for a significant positive rank correlation. What is the  $p$ -value?
- c. At  $\alpha = 0.05$ , what is your conclusion?
25. A student organization surveyed both recent graduates and current students to obtain information on the quality of teaching at a particular university. An analysis of the responses provided the following teaching-ability rankings. Do the rankings given by the current students agree with the rankings given by the recent graduates? Use  $\alpha = 0.10$  and test for a significant rank correlation.

| Professor | <i>Ranking by</i>       |                         |
|-----------|-------------------------|-------------------------|
|           | <i>Current students</i> | <i>Recent graduates</i> |
| A         | 4                       | 6                       |
| B         | 6                       | 8                       |
| C         | 8                       | 5                       |
| D         | 3                       | 1                       |
| E         | 1                       | 2                       |
| F         | 2                       | 3                       |
| G         | 5                       | 7                       |
| H         | 10                      | 9                       |
| J         | 7                       | 4                       |
| K         | 9                       | 10                      |

26. A sample of 15 students received the following rankings on mid-term and final examinations in a statistics course.

| <i>Mid-term</i> | <i>Rank</i>  |                 | <i>Rank</i>  |                 | <i>Rank</i>  |  |
|-----------------|--------------|-----------------|--------------|-----------------|--------------|--|
|                 | <i>Final</i> | <i>Mid-term</i> | <i>Final</i> | <i>Mid-term</i> | <i>Final</i> |  |
| 1               | 4            | 6               | 2            | 11              | 14           |  |
| 2               | 7            | 7               | 5            | 12              | 15           |  |
| 3               | 1            | 8               | 12           | 13              | 11           |  |
| 4               | 3            | 9               | 6            | 14              | 10           |  |
| 5               | 8            | 10              | 9            | 15              | 13           |  |

Compute the Spearman rank-correlation coefficient for the data and test for a significant correlation, with  $\alpha = 0.10$ .

## ONLINE RESOURCES

For the data sets, online summary, additional questions and answers, and the software section for Chapter 18, visit the online platform.



## SUMMARY

In this chapter we presented several statistical procedures that are classified as non-parametric methods. Because non-parametric methods can be applied to ordinal and in some cases nominal data, as well as interval and ratio data, and because they require less restrictive population distribution assumptions, they expand the class of problems that can be subjected to statistical analysis.

The sign test is a non-parametric procedure for identifying differences between two populations when the data available are nominal data. In the small-sample case, the binomial probability distribution can be used to determine the critical values for the sign test; in the large-sample case, a normal approximation can be used. The Wilcoxon signed-rank test is a procedure for analyzing matched-sample data whenever interval- or ratio-scaled data are available for each matched pair. The procedure tests the hypothesis that the two populations being considered are identical. The procedure assumes that the distribution of differences between matched pairs is symmetrical, but not necessarily normal in shape.

The Mann–Whitney–Wilcoxon test is a non-parametric method for testing for a difference between two populations based on two independent random samples. Tables were presented for the small-sample case, and a normal approximation was provided for the large-sample case. The Kruskal–Wallis test extends the Mann–Whitney–Wilcoxon test to the case of three or more populations. The Kruskal–Wallis test is the non-parametric analogue of the parametric ANOVA for differences among population means.

We introduced the Spearman rank-correlation coefficient as a measure of association for two ordinal or rank-ordered sets of items.

## KEY TERMS

**Distribution-free methods**

**Kruskal–Wallis test**

**Mann–Whitney–Wilcoxon (MWW) test**

**Non-parametric methods**

**Parametric methods**

**Sign test**

**Spearman rank-correlation coefficient**

**Wilcoxon signed-rank test**

## KEY FORMULAE

### Sign test (large-sample case)

$$\text{Mean: } \mu = 0.50n \quad (18.1)$$

$$\text{Standard deviation: } \sigma = \sqrt{0.25n} \quad (18.2)$$

**Wilcoxon signed-rank test**

$$\text{Mean: } \mu_T = 0 \quad (18.3)$$

$$\text{Standard deviation: } \sigma_T = \sqrt{\frac{n(n+1)(2n+1)}{6}} \quad (18.4)$$

**Mann–Whitney–Wilcoxon test (large-sample)**

$$\text{Mean: } \mu_T = 0.5n_1(n_1 + n_2 + 1) \quad (18.6)$$

$$\text{Standard deviation: } \sigma_T = \sqrt{\frac{n_1 n_2 (n_1 + n_2 + 1)}{12}} \quad (18.7)$$

**Kruskal–Wallis test statistic**

$$W = \left[ \frac{12}{n_T(n_T + 1)} \sum_{i=1}^k \frac{R_i^2}{n_i} \right] - 3(n_T + 1) \quad (18.8)$$

**Spearman rank-correlation coefficient**

$$r_S = 1 - \frac{6\sum d_i^2}{n(n^2 - 1)} \quad (18.9)$$

**Sampling distribution of  $r_S$  in test for significant rank correlation**

$$\text{Mean: } \mu_{r_S} = 0 \quad (18.10)$$

$$\text{Standard deviation: } \sigma_{r_S} = \sqrt{\frac{1}{n-1}} \quad (18.11)$$

**CASE PROBLEM****Company profiles II**

The file ‘Companies 2012’ on the online platform contains a data set compiled mid-year 2012. It comprises figures relating to samples of companies whose shares are traded on the stock exchanges in Germany, France, South Africa and Israel. The data contained in the file are:

Name of company

Country of stock exchange where the shares are traded

Return on shareholders’ funds in 2011 (%)

Profit margin in 2011 (%)



Return on total assets in 2011 (%)

Current ratio, 2011

Solvency ratio, 2011

Price/earnings ratio, 2011

The first few rows of data are shown below.

| <i>Company name</i> | <i>Country</i> | <i>Return on<br/>share holders'<br/>funds, 2011<br/>(%)</i> | <i>Profit<br/>margin,<br/>2011 (%)</i> | <i>Return on<br/>total assets,<br/>2011 (%)</i> | <i>Current<br/>ratio,<br/>2011</i> | <i>Solvency<br/>ratio,<br/>2011</i> | <i>Price/<br/>earnings<br/>ratio, 2011</i> |
|---------------------|----------------|---|--|---|------------------------------------|-------------------------------------|--|
| Adidas AG           | Germany        | 17.40   | 6.85                                   | 8.15  | 1.50                               | 46.81                               | 15.72                                      |
| Allianz SE          | Germany        | 10.79   | 6.99                                   | 0.77  |                                    | 7.15                                | 11.92                                      |
| Altana AG           | Germany        | 3.32  | 3.28                                   | 2.28  | 2.40                               | 68.77                               | 200.13                                     |
| BASF SE             | Germany        | 37.16   | 11.90                                  | 14.66   | 1.64                               | 39.46                               | 7.96                                       |
| Bayer AG            | Germany        | 17.50   | 9.04                                   | 6.37  | 1.50                               | 39.41                               | 16.47                                      |
| BMW AG              | Germany        | 27.31   | 10.69                                  | 5.98  | 1.04                               | 21.91                               | 6.52                                       |
| Commerzbank         | Germany        | 2.04  | 4.09                                   | 0.08  | 0.41                               | 3.75                                | 8.92                                       |
| Continental AG      | Germany        | 26.05   | 6.06                                   | 7.15  | 1.06                               | 27.44                               | 7.71                                       |
| Daimler AG          | Germany        | 21.32   | 7.84                                   | 5.70  | 1.11                               | 26.75                               | 6.35                                       |
| Deutsche Bank AG    | Germany        | 9.86  | 16.16                                  | 0.25  | 0.82                               | 2.53                                | 6.23                                       |

### **Analyst's report**

Using non-parametric methods of testing, investigate the following:

1. Is there any evidence of differences between the companies traded on the five different stock exchanges in respect of the return on shareholders' funds, in respect of the profit margins and in respect of the price/earnings ratios?
2. Is there any evidence of differences between the companies traded on the French and German stock markets in respect of the distribution of current ratios and solvency ratios?
3. Is there any evidence of a relationship between return on shareholders' funds and profit margin?
4. Is there any evidence of a relationship between return on total assets and solvency ratio?
5. Is there any evidence of a relationship between return on total assets and price/earnings ratio?



# APPENDIX A

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### EXPERIMENTAL DESIGN

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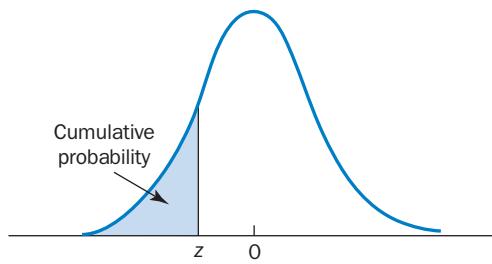
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# APPENDIX B

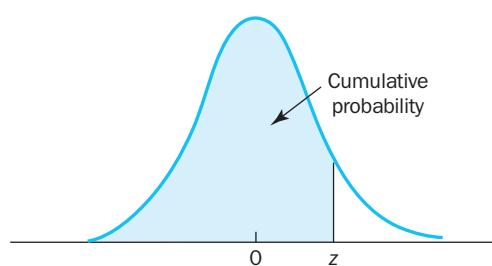
## Tables

**TABLE 1** Cumulative Probabilities for the Standard Normal Distribution



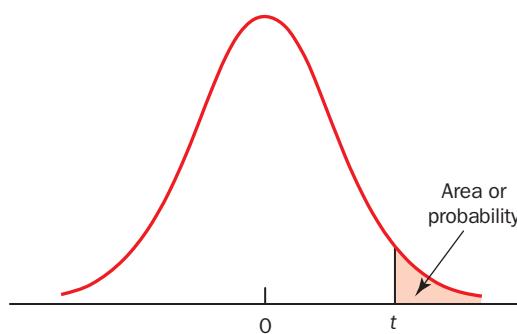
Entries in the table give the area under the curve to the left of the  $z$  value. For example, for  $z = -.85$ , the cumulative probability is .1977.

| $z$  | .00   | .01   | .02   | .03   | .04   | .05   | .06   | .07   | .08   | .09   |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| -3.0 | .0013 | .0013 | .0013 | .0012 | .0012 | .0011 | .0011 | .0011 | .0010 | .0010 |
| -2.9 | .0019 | .0018 | .0018 | .0017 | .0016 | .0016 | .0015 | .0015 | .0014 | .0014 |
| -2.8 | .0026 | .0025 | .0024 | .0023 | .0023 | .0022 | .0021 | .0021 | .0020 | .0019 |
| -2.7 | .0035 | .0034 | .0033 | .0032 | .0031 | .0030 | .0029 | .0028 | .0027 | .0026 |
| -2.6 | .0047 | .0045 | .0044 | .0043 | .0041 | .0040 | .0039 | .0038 | .0037 | .0036 |
| -2.5 | .0062 | .0060 | .0059 | .0057 | .0055 | .0054 | .0052 | .0051 | .0049 | .0048 |
| -2.4 | .0082 | .0080 | .0078 | .0075 | .0073 | .0071 | .0069 | .0068 | .0066 | .0064 |
| -2.3 | .0107 | .0104 | .0102 | .0099 | .0096 | .0094 | .0091 | .0089 | .0087 | .0084 |
| -2.2 | .0139 | .0136 | .0132 | .0129 | .0125 | .0122 | .0119 | .0116 | .0113 | .0110 |
| -2.1 | .0179 | .0174 | .0170 | .0166 | .0162 | .0158 | .0154 | .0150 | .0146 | .0143 |
| -2.0 | .0228 | .0222 | .0217 | .0212 | .0207 | .0202 | .0197 | .0192 | .0188 | .0183 |
| -1.9 | .0287 | .0281 | .0274 | .0268 | .0262 | .0256 | .0250 | .0244 | .0239 | .0233 |
| -1.8 | .0359 | .0351 | .0344 | .0336 | .0329 | .0322 | .0314 | .0307 | .0301 | .0294 |
| -1.7 | .0446 | .0436 | .0427 | .0418 | .0409 | .0401 | .0392 | .0384 | .0375 | .0367 |
| -1.6 | .0548 | .0537 | .0526 | .0516 | .0505 | .0495 | .0485 | .0475 | .0465 | .0455 |
| -1.5 | .0668 | .0655 | .0643 | .0630 | .0618 | .0606 | .0594 | .0582 | .0571 | .0559 |
| -1.4 | .0808 | .0793 | .0778 | .0764 | .0749 | .0735 | .0721 | .0708 | .0694 | .0681 |
| -1.3 | .0968 | .0951 | .0934 | .0918 | .0901 | .0885 | .0869 | .0853 | .0838 | .0823 |
| -1.2 | .1151 | .1131 | .1112 | .1093 | .1075 | .1056 | .1038 | .1020 | .1003 | .0985 |
| -1.1 | .1357 | .1335 | .1314 | .1292 | .1271 | .1251 | .1230 | .1210 | .1190 | .1170 |
| -1.0 | .1587 | .1562 | .1539 | .1515 | .1492 | .1469 | .1446 | .1423 | .1401 | .1379 |
| -.9  | .1841 | .1814 | .1788 | .1762 | .1736 | .1711 | .1685 | .1660 | .1635 | .1611 |
| -.8  | .2119 | .2090 | .2061 | .2033 | .2005 | .1977 | .1949 | .1922 | .1894 | .1867 |
| -.7  | .2420 | .2389 | .2358 | .2327 | .2296 | .2266 | .2236 | .2206 | .2177 | .2148 |
| -.6  | .2743 | .2709 | .2676 | .2643 | .2611 | .2578 | .2546 | .2514 | .2483 | .2451 |
| -.5  | .3085 | .3050 | .3015 | .2981 | .2946 | .2912 | .2877 | .2843 | .2810 | .2776 |
| -.4  | .3446 | .3409 | .3372 | .3336 | .3300 | .3264 | .3228 | .3192 | .3156 | .3121 |
| -.3  | .3821 | .3783 | .3745 | .3707 | .3669 | .3632 | .3594 | .3557 | .3520 | .3483 |
| -.2  | .4207 | .4168 | .4129 | .4090 | .4052 | .4013 | .3974 | .3936 | .3897 | .3859 |
| -.1  | .4602 | .4562 | .4522 | .4483 | .4443 | .4404 | .4364 | .4325 | .4286 | .4247 |
| -.0  | .5000 | .4960 | .4920 | .4880 | .4840 | .4801 | .4761 | .4721 | .4681 | .4641 |

**TABLE 1** (Continued)

Entries in the table give the area under the curve to the left of the  $z$  value. For example, for  $z = 1.25$ , the cumulative probability is .8944.

| $z$ | .00   | .01   | .02   | .03   | .04   | .05   | .06   | .07   | .08   | .09   |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.0 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| 0.1 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| 0.2 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| 0.3 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| 0.4 | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| 0.5 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| 0.6 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| 0.7 | .7580 | .7611 | .7642 | .7673 | .7704 | .7734 | .7764 | .7794 | .7823 | .7852 |
| 0.8 | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 |
| 0.9 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 |
| 1.1 | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 |
| 1.2 | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |
| 1.3 | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 |
| 1.4 | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9279 | .9292 | .9306 | .9319 |
| 1.5 | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| 1.6 | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| 1.7 | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 |
| 1.8 | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |
| 2.0 | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 |
| 2.1 | .9821 | .9826 | .9830 | .9834 | .9838 | .9842 | .9846 | .9850 | .9854 | .9857 |
| 2.2 | .9861 | .9864 | .9868 | .9871 | .9875 | .9878 | .9881 | .9884 | .9887 | .9890 |
| 2.3 | .9893 | .9896 | .9898 | .9901 | .9904 | .9906 | .9909 | .9911 | .9913 | .9913 |
| 2.4 | .9918 | .9920 | .9922 | .9925 | .9927 | .9929 | .9931 | .9932 | .9934 | .9936 |
| 2.5 | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |
| 2.6 | .9953 | .9955 | .9956 | .9957 | .9959 | .9960 | .9961 | .9962 | .9963 | .9964 |
| 2.7 | .9965 | .9966 | .9967 | .9968 | .9969 | .9970 | .9971 | .9972 | .9973 | .9974 |
| 2.8 | .9974 | .9975 | .9976 | .9977 | .9977 | .9978 | .9979 | .9979 | .9980 | .9981 |
| 2.9 | .9981 | .9982 | .9982 | .9983 | .9984 | .9984 | .9985 | .9985 | .9986 | .9986 |
| 3.0 | .9986 | .9987 | .9987 | .9988 | .9988 | .9989 | .9989 | .9989 | .9990 | .9990 |

**TABLE 2** *t* distribution

Entries in the table give *t* values for an area or probability in the upper tail of the *t* distribution. For example, with ten degrees of freedom and 0.05 area in the upper tail,  $t_{.05} = 1.812$ .

| Degrees of freedom | Area in upper tail |       |       |        |        |        |
|--------------------|--------------------|-------|-------|--------|--------|--------|
|                    | .20                | .10   | .05   | .025   | .01    | .005   |
| 1                  | 1.376              | 3.078 | 6.314 | 12.706 | 31.821 | 63.656 |
| 2                  | 1.061              | 1.886 | 2.920 | 4.303  | 6.965  | 9.925  |
| 3                  | .978               | 1.638 | 2.353 | 3.182  | 4.541  | 5.841  |
| 4                  | .941               | 1.533 | 2.132 | 2.776  | 3.747  | 4.604  |
| 5                  | .920               | 1.476 | 2.015 | 2.571  | 3.365  | 4.032  |
| 6                  | .906               | 1.440 | 1.943 | 2.447  | 3.143  | 3.707  |
| 7                  | .896               | 1.415 | 1.895 | 2.365  | 2.998  | 3.499  |
| 8                  | .889               | 1.397 | 1.860 | 2.306  | 2.896  | 3.355  |
| 9                  | .883               | 1.383 | 1.833 | 2.262  | 2.821  | 3.250  |
| 10                 | .879               | 1.372 | 1.812 | 2.228  | 2.764  | 3.169  |
| 11                 | .876               | 1.363 | 1.796 | 2.201  | 2.718  | 3.106  |
| 12                 | .873               | 1.356 | 1.782 | 2.179  | 2.681  | 3.055  |
| 13                 | .870               | 1.350 | 1.771 | 2.160  | 2.650  | 3.012  |
| 14                 | .868               | 1.345 | 1.761 | 2.145  | 2.624  | 2.977  |
| 15                 | .866               | 1.341 | 1.753 | 2.131  | 2.602  | 2.947  |
| 16                 | .865               | 1.337 | 1.746 | 2.120  | 2.583  | 2.921  |
| 17                 | .863               | 1.333 | 1.740 | 2.110  | 2.567  | 2.898  |
| 18                 | .862               | 1.330 | 1.734 | 2.101  | 2.552  | 2.878  |
| 19                 | .861               | 1.328 | 1.729 | 2.093  | 2.539  | 2.861  |
| 20                 | .860               | 1.325 | 1.725 | 2.086  | 2.528  | 2.845  |
| 21                 | .859               | 1.323 | 1.721 | 2.080  | 2.518  | 2.831  |
| 22                 | .858               | 1.321 | 1.717 | 2.074  | 2.508  | 2.819  |
| 23                 | .858               | 1.319 | 1.714 | 2.069  | 2.500  | 2.807  |
| 24                 | .857               | 1.318 | 1.711 | 2.064  | 2.492  | 2.797  |
| 25                 | .856               | 1.316 | 1.708 | 2.060  | 2.485  | 2.787  |
| 26                 | .856               | 1.315 | 1.706 | 2.056  | 2.479  | 2.779  |
| 27                 | .855               | 1.314 | 1.703 | 2.052  | 2.473  | 2.771  |
| 28                 | .855               | 1.313 | 1.701 | 2.048  | 2.467  | 2.763  |
| 29                 | .854               | 1.311 | 1.699 | 2.045  | 2.462  | 2.756  |

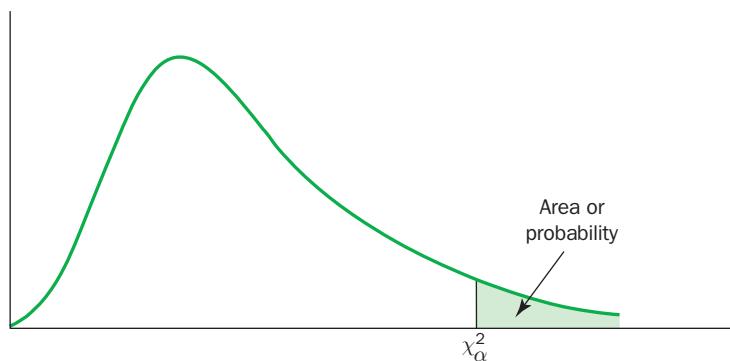
**TABLE 2** (Continued)

| Degrees of freedom | Area in upper tail |       |       |       |       |       |
|--------------------|--------------------|-------|-------|-------|-------|-------|
|                    | .20                | .10   | .05   | .025  | .01   | .005  |
| 30                 | .854               | 1.310 | 1.697 | 2.042 | 2.457 | 2.750 |
| 31                 | .853               | 1.309 | 1.696 | 2.040 | 2.453 | 2.744 |
| 32                 | .853               | 1.309 | 1.694 | 2.037 | 2.449 | 2.738 |
| 33                 | .853               | 1.308 | 1.692 | 2.035 | 2.445 | 2.733 |
| 34                 | .852               | 1.307 | 1.691 | 2.032 | 2.441 | 2.728 |
| 35                 | .852               | 1.306 | 1.690 | 2.030 | 2.438 | 2.724 |
| 36                 | .852               | 1.306 | 1.688 | 2.028 | 2.434 | 2.719 |
| 37                 | .851               | 1.305 | 1.687 | 2.026 | 2.431 | 2.715 |
| 38                 | .851               | 1.304 | 1.686 | 2.024 | 2.429 | 2.712 |
| 39                 | .851               | 1.304 | 1.685 | 2.023 | 2.426 | 2.708 |
| 40                 | .851               | 1.303 | 1.684 | 2.021 | 2.423 | 2.704 |
| 41                 | .850               | 1.303 | 1.683 | 2.020 | 2.421 | 2.701 |
| 42                 | .850               | 1.302 | 1.682 | 2.018 | 2.418 | 2.698 |
| 43                 | .850               | 1.302 | 1.681 | 2.017 | 2.416 | 2.695 |
| 44                 | .850               | 1.301 | 1.680 | 2.015 | 2.414 | 2.692 |
| 45                 | .850               | 1.301 | 1.679 | 2.014 | 2.412 | 2.690 |
| 46                 | .850               | 1.300 | 1.679 | 2.013 | 2.410 | 2.687 |
| 47                 | .849               | 1.300 | 1.678 | 2.012 | 2.408 | 2.685 |
| 48                 | .849               | 1.299 | 1.677 | 2.011 | 2.407 | 2.682 |
| 49                 | .849               | 1.299 | 1.677 | 2.010 | 2.405 | 2.680 |
| 50                 | .849               | 1.299 | 1.676 | 2.009 | 2.403 | 2.678 |
| 51                 | .849               | 1.298 | 1.675 | 2.008 | 2.402 | 2.676 |
| 52                 | .849               | 1.298 | 1.675 | 2.007 | 2.400 | 2.674 |
| 53                 | .848               | 1.298 | 1.674 | 2.006 | 2.399 | 2.672 |
| 54                 | .848               | 1.297 | 1.674 | 2.005 | 2.397 | 2.670 |
| 55                 | .848               | 1.297 | 1.673 | 2.004 | 2.396 | 2.668 |
| 56                 | .848               | 1.297 | 1.673 | 2.003 | 2.395 | 2.667 |
| 57                 | .848               | 1.297 | 1.672 | 2.002 | 2.394 | 2.665 |
| 58                 | .848               | 1.296 | 1.672 | 2.002 | 2.392 | 2.663 |
| 59                 | .848               | 1.296 | 1.671 | 2.001 | 2.391 | 2.662 |
| 60                 | .848               | 1.296 | 1.671 | 2.000 | 2.390 | 2.660 |
| 61                 | .848               | 1.296 | 1.670 | 2.000 | 2.389 | 2.659 |
| 62                 | .847               | 1.295 | 1.670 | 1.999 | 2.388 | 2.657 |
| 63                 | .847               | 1.295 | 1.669 | 1.998 | 2.387 | 2.656 |
| 64                 | .847               | 1.295 | 1.669 | 1.998 | 2.386 | 2.655 |
| 65                 | .847               | 1.295 | 1.669 | 1.997 | 2.385 | 2.654 |
| 66                 | .847               | 1.295 | 1.668 | 1.997 | 2.384 | 2.652 |
| 67                 | .847               | 1.294 | 1.668 | 1.996 | 2.383 | 2.651 |
| 68                 | .847               | 1.294 | 1.668 | 1.995 | 2.382 | 2.650 |
| 69                 | .847               | 1.294 | 1.667 | 1.995 | 2.382 | 2.649 |

(continued)

**TABLE 2** (*Continued*)

| Degrees of freedom | Area in upper tail |       |       |       |       |       |
|--------------------|--------------------|-------|-------|-------|-------|-------|
|                    | .20                | .10   | .05   | .025  | .01   | .005  |
| 70                 | .847               | 1.294 | 1.667 | 1.994 | 2.381 | 2.648 |
| 71                 | .847               | 1.294 | 1.667 | 1.994 | 2.380 | 2.647 |
| 72                 | .847               | 1.293 | 1.666 | 1.993 | 2.379 | 2.646 |
| 73                 | .847               | 1.293 | 1.666 | 1.993 | 2.379 | 2.645 |
| 74                 | .847               | 1.293 | 1.666 | 1.993 | 2.378 | 2.644 |
| 75                 | .846               | 1.293 | 1.665 | 1.992 | 2.377 | 2.643 |
| 76                 | .846               | 1.293 | 1.665 | 1.992 | 2.376 | 2.642 |
| 77                 | .846               | 1.293 | 1.665 | 1.991 | 2.376 | 2.641 |
| 78                 | .846               | 1.292 | 1.665 | 1.991 | 2.375 | 2.640 |
| 79                 | .846               | 1.292 | 1.664 | 1.990 | 2.374 | 2.639 |
| 80                 | .846               | 1.292 | 1.664 | 1.990 | 2.374 | 2.639 |
| 81                 | .846               | 1.292 | 1.664 | 1.990 | 2.373 | 2.638 |
| 82                 | .846               | 1.292 | 1.664 | 1.989 | 2.373 | 2.637 |
| 83                 | .846               | 1.292 | 1.663 | 1.989 | 2.372 | 2.636 |
| 84                 | .846               | 1.292 | 1.663 | 1.989 | 2.372 | 2.636 |
| 85                 | .846               | 1.292 | 1.663 | 1.988 | 2.371 | 2.635 |
| 86                 | .846               | 1.291 | 1.663 | 1.988 | 2.370 | 2.634 |
| 87                 | .846               | 1.291 | 1.663 | 1.988 | 2.370 | 2.634 |
| 88                 | .846               | 1.291 | 1.662 | 1.987 | 2.369 | 2.633 |
| 89                 | .846               | 1.291 | 1.662 | 1.987 | 2.369 | 2.632 |
| 90                 | .846               | 1.291 | 1.662 | 1.987 | 2.368 | 2.632 |
| 91                 | .846               | 1.291 | 1.662 | 1.986 | 2.368 | 2.631 |
| 92                 | .846               | 1.291 | 1.662 | 1.986 | 2.368 | 2.630 |
| 93                 | .846               | 1.291 | 1.661 | 1.986 | 2.367 | 2.630 |
| 94                 | .845               | 1.291 | 1.661 | 1.986 | 2.367 | 2.629 |
| 95                 | .845               | 1.291 | 1.661 | 1.985 | 2.366 | 2.629 |
| 96                 | .845               | 1.290 | 1.661 | 1.985 | 2.366 | 2.628 |
| 97                 | .845               | 1.290 | 1.661 | 1.985 | 2.365 | 2.627 |
| 98                 | .845               | 1.290 | 1.661 | 1.984 | 2.365 | 2.627 |
| 99                 | .845               | 1.290 | 1.660 | 1.984 | 2.364 | 2.626 |
| 100                | .845               | 1.290 | 1.660 | 1.984 | 2.364 | 2.626 |
| $\infty$           | .842               | 1.282 | 1.645 | 1.960 | 2.326 | 2.576 |

**TABLE 3** Chi-squared distribution

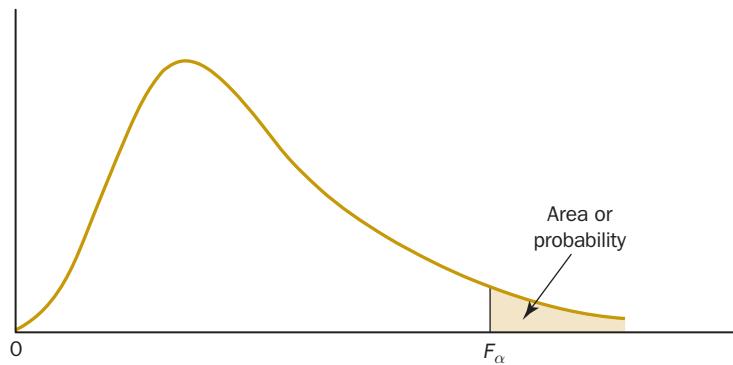
Entries in the table give  $\chi^2_\alpha$  values, where  $\alpha$  is the area or probability in the upper tail of the chi-squared distribution. For example, with ten degrees of freedom and 0.01 area in the upper tail,  $\chi^2_{0.01}=23.209$

| Degrees of freedom | Area in upper tail |        |        |        |        |        |        |        |        |        |
|--------------------|--------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
|                    | .995               | .99    | .975   | .95    | .90    | .10    | .05    | .025   | .01    | .005   |
| 1                  | .000               | .000   | .001   | .004   | .016   | 2.706  | 3.841  | 5.024  | 6.635  | 7.879  |
| 2                  | .010               | .020   | .051   | .103   | .211   | 4.605  | 5.991  | 7.378  | 9.210  | 10.597 |
| 3                  | .072               | .115   | .216   | .352   | .584   | 6.251  | 7.815  | 9.348  | 11.345 | 12.838 |
| 4                  | .207               | .297   | .484   | .711   | 1.064  | 7.779  | 9.488  | 11.143 | 13.277 | 14.860 |
| 5                  | .412               | .554   | .831   | 1.145  | 1.610  | 9.236  | 11.070 | 12.832 | 15.086 | 16.750 |
| 6                  | .676               | .872   | 1.237  | 1.635  | 2.204  | 10.645 | 12.592 | 14.449 | 16.812 | 18.548 |
| 7                  | .989               | 1.239  | 1.690  | 2.167  | 2.833  | 12.017 | 14.067 | 16.013 | 18.475 | 20.278 |
| 8                  | 1.344              | 1.647  | 2.180  | 2.733  | 3.490  | 13.362 | 15.507 | 17.535 | 20.090 | 21.955 |
| 9                  | 1.735              | 2.088  | 2.700  | 3.325  | 4.168  | 14.684 | 16.919 | 19.023 | 21.666 | 23.589 |
| 10                 | 2.156              | 2.558  | 3.247  | 3.940  | 4.865  | 15.987 | 18.307 | 20.483 | 23.209 | 25.188 |
| 11                 | 2.603              | 3.053  | 3.816  | 4.575  | 5.578  | 17.275 | 19.675 | 21.920 | 24.725 | 26.757 |
| 12                 | 3.074              | 3.571  | 4.404  | 5.226  | 6.304  | 18.549 | 21.026 | 23.337 | 26.217 | 28.300 |
| 13                 | 3.565              | 4.107  | 5.009  | 5.892  | 7.041  | 19.812 | 22.362 | 24.736 | 27.688 | 29.819 |
| 14                 | 4.075              | 4.660  | 5.629  | 6.571  | 7.790  | 21.064 | 23.685 | 26.119 | 29.141 | 31.319 |
| 15                 | 4.601              | 5.229  | 6.262  | 7.261  | 8.547  | 22.307 | 24.996 | 27.488 | 30.578 | 32.801 |
| 16                 | 5.142              | 5.812  | 6.908  | 7.962  | 9.312  | 23.542 | 26.296 | 28.845 | 32.000 | 34.267 |
| 17                 | 5.697              | 6.408  | 7.564  | 8.672  | 10.085 | 24.769 | 27.587 | 30.191 | 33.409 | 35.718 |
| 18                 | 6.265              | 7.015  | 8.231  | 9.390  | 10.865 | 25.989 | 28.869 | 31.526 | 34.805 | 37.156 |
| 19                 | 6.844              | 7.633  | 8.907  | 10.117 | 11.651 | 27.204 | 30.144 | 32.852 | 36.191 | 38.582 |
| 20                 | 7.434              | 8.260  | 9.591  | 10.851 | 12.443 | 28.412 | 31.410 | 34.170 | 37.566 | 39.997 |
| 21                 | 8.034              | 8.897  | 10.283 | 11.591 | 13.240 | 29.615 | 32.671 | 35.479 | 38.932 | 41.401 |
| 22                 | 8.643              | 9.542  | 10.982 | 12.338 | 14.041 | 30.813 | 33.924 | 36.781 | 40.289 | 42.796 |
| 23                 | 9.260              | 10.196 | 11.689 | 13.091 | 14.848 | 32.007 | 35.172 | 38.076 | 41.638 | 44.181 |
| 24                 | 9.886              | 10.856 | 12.401 | 13.848 | 15.659 | 33.196 | 36.415 | 39.364 | 42.980 | 45.558 |
| 25                 | 10.520             | 11.524 | 13.120 | 14.611 | 16.473 | 34.382 | 37.652 | 40.646 | 44.314 | 46.928 |
| 26                 | 11.160             | 12.198 | 13.844 | 15.379 | 17.292 | 35.563 | 38.885 | 41.923 | 45.642 | 48.290 |
| 27                 | 11.808             | 12.878 | 14.573 | 16.151 | 18.114 | 36.741 | 40.113 | 43.195 | 46.963 | 49.645 |

(continued)

**TABLE 3** (*Continued*)

| Degrees of freedom | Area in upper tail |        |        |        |        |         |         |         |         |         |
|--------------------|--------------------|--------|--------|--------|--------|---------|---------|---------|---------|---------|
|                    | .995               | .99    | .975   | .95    | .90    | .10     | .05     | .025    | .01     | .005    |
| 28                 | 12.461             | 13.565 | 15.308 | 16.928 | 18.939 | 37.916  | 41.337  | 44.461  | 48.278  | 50.994  |
| 29                 | 13.121             | 14.256 | 16.047 | 17.708 | 19.768 | 39.087  | 42.557  | 45.722  | 49.588  | 52.335  |
| 30                 | 13.787             | 14.953 | 16.791 | 18.493 | 20.599 | 40.256  | 43.773  | 46.979  | 50.892  | 53.672  |
| 35                 | 17.192             | 18.509 | 20.569 | 22.465 | 24.797 | 46.059  | 49.802  | 53.203  | 57.342  | 60.275  |
| 40                 | 20.707             | 22.164 | 24.433 | 26.509 | 29.051 | 51.805  | 55.758  | 59.342  | 63.691  | 66.766  |
| 45                 | 24.311             | 25.901 | 28.366 | 30.612 | 33.350 | 57.505  | 61.656  | 65.410  | 69.957  | 73.166  |
| 50                 | 27.991             | 29.707 | 32.357 | 34.764 | 37.689 | 63.167  | 67.505  | 71.420  | 76.154  | 79.490  |
| 55                 | 31.735             | 33.571 | 36.398 | 38.958 | 42.060 | 68.796  | 73.311  | 77.380  | 82.292  | 85.749  |
| 60                 | 35.534             | 37.485 | 40.482 | 43.188 | 46.459 | 74.397  | 79.082  | 83.298  | 88.379  | 91.952  |
| 65                 | 39.383             | 41.444 | 44.603 | 47.450 | 50.883 | 79.973  | 84.821  | 89.177  | 94.422  | 98.105  |
| 70                 | 43.275             | 45.442 | 48.758 | 51.739 | 55.329 | 85.527  | 90.531  | 95.023  | 100.425 | 104.215 |
| 75                 | 47.206             | 49.475 | 52.942 | 56.054 | 59.795 | 91.061  | 96.217  | 100.839 | 106.393 | 110.285 |
| 80                 | 51.172             | 53.540 | 57.153 | 60.391 | 64.278 | 96.578  | 101.879 | 106.629 | 112.329 | 116.321 |
| 85                 | 55.170             | 57.634 | 61.389 | 64.749 | 68.777 | 102.079 | 107.522 | 112.393 | 118.236 | 122.324 |
| 90                 | 59.196             | 61.754 | 65.647 | 69.126 | 73.291 | 107.565 | 113.145 | 118.136 | 124.116 | 128.299 |
| 95                 | 63.250             | 65.898 | 69.925 | 73.520 | 77.818 | 113.038 | 118.752 | 123.858 | 129.973 | 134.247 |
| 100                | 67.328             | 70.065 | 74.222 | 77.929 | 82.358 | 118.498 | 124.342 | 129.561 | 135.807 | 140.170 |

**TABLE 4** *F* distribution

Entries in the table give  $F_\alpha$  values, where  $\alpha$  is the area or probability in the upper tail of the *F* distribution. For example, with four numerator degrees of freedom, eight denominator degrees of freedom, and 0.05 area in the upper tail,  $F_{.05} = 3.84$ .

| Denominator degrees of freedom | Area in upper tail | Numerator degrees of freedom |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |         |
|--------------------------------|--------------------|------------------------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
|                                |                    | 1                            | 2       | 3       | 4       | 5       | 6       | 7       | 8       | 9       | 10      | 15      | 20      | 25      | 30      | 40      | 60      | 100     | 1000    |
| 1                              | .10                | 39.86                        | 49.50   | 53.59   | 55.83   | 57.24   | 58.20   | 58.91   | 59.44   | 59.86   | 60.19   | 61.22   | 61.74   | 62.05   | 62.26   | 62.53   | 62.79   | 63.01   | 63.30   |
|                                | .05                | 161.45                       | 199.50  | 215.71  | 224.58  | 230.16  | 233.99  | 236.77  | 238.88  | 240.54  | 241.88  | 245.95  | 248.02  | 249.26  | 250.10  | 251.14  | 252.20  | 253.04  | 254.19  |
|                                | .025               | 647.79                       | 799.48  | 864.15  | 899.60  | 921.83  | 937.11  | 948.20  | 956.64  | 963.28  | 968.63  | 984.87  | 993.08  | 998.09  | 1001.40 | 1005.60 | 1009.79 | 1013.16 | 1017.76 |
|                                | .01                | 4052.18                      | 4999.34 | 5403.53 | 5624.26 | 5763.96 | 5858.95 | 5928.33 | 5980.95 | 6022.40 | 6055.93 | 6156.97 | 6208.66 | 6239.86 | 6260.35 | 6286.43 | 6312.97 | 6333.92 | 6362.80 |
| 2                              | .10                | 8.53                         | 9.00    | 9.16    | 9.24    | 9.29    | 9.33    | 9.35    | 9.37    | 9.38    | 9.39    | 9.42    | 9.44    | 9.45    | 9.46    | 9.47    | 9.47    | 9.48    | 9.49    |
|                                | .05                | 18.51                        | 19.00   | 19.16   | 19.25   | 19.30   | 19.33   | 19.35   | 19.37   | 19.38   | 19.40   | 19.43   | 19.45   | 19.46   | 19.46   | 19.47   | 19.48   | 19.49   | 19.49   |
|                                | .025               | 38.51                        | 39.00   | 39.17   | 39.25   | 39.30   | 39.33   | 39.36   | 39.37   | 39.39   | 39.40   | 39.43   | 39.45   | 39.46   | 39.46   | 39.47   | 39.48   | 39.49   | 39.50   |
|                                | .01                | 98.50                        | 99.00   | 99.16   | 99.25   | 99.30   | 99.33   | 99.36   | 99.38   | 99.39   | 99.40   | 99.43   | 99.45   | 99.46   | 99.47   | 99.48   | 99.48   | 99.49   | 99.50   |
| 3                              | .10                | 5.54                         | 5.46    | 5.39    | 5.34    | 5.31    | 5.28    | 5.27    | 5.25    | 5.24    | 5.23    | 5.20    | 5.18    | 5.17    | 5.17    | 5.16    | 5.15    | 5.14    | 5.13    |
|                                | .05                | 10.13                        | 9.55    | 9.28    | 9.12    | 9.01    | 8.94    | 8.89    | 8.85    | 8.81    | 8.79    | 8.70    | 8.66    | 8.63    | 8.62    | 8.59    | 8.57    | 8.55    | 8.53    |
|                                | .025               | 17.44                        | 16.04   | 15.44   | 15.10   | 14.88   | 14.73   | 14.62   | 14.54   | 14.47   | 14.42   | 14.25   | 14.17   | 14.12   | 14.08   | 14.04   | 13.99   | 13.96   | 13.91   |
|                                | .01                | 34.12                        | 30.82   | 29.46   | 28.71   | 28.24   | 27.91   | 27.67   | 27.49   | 27.34   | 27.23   | 26.87   | 26.69   | 26.58   | 26.50   | 26.41   | 26.32   | 26.24   | 26.14   |
| 4                              | .10                | 4.54                         | 4.32    | 4.19    | 4.11    | 4.05    | 4.01    | 3.98    | 3.95    | 3.94    | 3.92    | 3.87    | 3.84    | 3.83    | 3.82    | 3.80    | 3.79    | 3.78    | 3.76    |
|                                | .05                | 7.71                         | 6.94    | 6.59    | 6.39    | 6.26    | 6.16    | 6.09    | 6.04    | 6.00    | 5.96    | 5.86    | 5.80    | 5.77    | 5.75    | 5.72    | 5.69    | 5.66    | 5.63    |
|                                | .025               | 12.22                        | 10.65   | 9.98    | 9.60    | 9.36    | 9.20    | 9.07    | 8.98    | 8.90    | 8.84    | 8.66    | 8.56    | 8.50    | 8.46    | 8.41    | 8.36    | 8.32    | 8.26    |
|                                | .01                | 21.20                        | 18.00   | 16.69   | 15.98   | 15.52   | 15.21   | 14.98   | 14.80   | 14.66   | 14.55   | 14.20   | 14.02   | 13.91   | 13.84   | 13.75   | 13.65   | 13.58   | 13.47   |

(continued)

**TABLE 4** (Continued)

| Denominator degrees of freedom | Area in upper tail | Numerator degrees of freedom |       |       |       |       |       |       |       |       |       |       |      |      |      |      |      |      |      |
|--------------------------------|--------------------|------------------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|------|------|------|------|------|------|------|
|                                |                    | 1                            | 2     | 3     | 4     | 5     | 6     | 7     | 8     | 9     | 10    | 15    | 20   | 25   | 30   | 40   | 60   | 100  | 1000 |
| 5                              | .10                | 4.06                         | 3.78  | 3.62  | 3.52  | 3.45  | 3.40  | 3.37  | 3.34  | 3.32  | 3.30  | 3.324 | 3.21 | 3.19 | 3.17 | 3.16 | 3.14 | 3.13 | 3.11 |
|                                | .05                | 6.61                         | 5.79  | 5.41  | 5.19  | 5.05  | 4.95  | 4.88  | 4.82  | 4.77  | 4.74  | 4.62  | 4.56 | 4.52 | 4.50 | 4.46 | 4.43 | 4.41 | 4.37 |
|                                | .025               | 10.01                        | 8.43  | 7.76  | 7.39  | 7.15  | 6.98  | 6.85  | 6.76  | 6.68  | 6.62  | 6.43  | 6.33 | 6.27 | 6.23 | 6.18 | 6.12 | 6.08 | 6.02 |
|                                | .01                | 16.26                        | 13.27 | 12.06 | 11.39 | 10.97 | 10.67 | 10.46 | 10.29 | 10.16 | 10.05 | 9.72  | 9.55 | 9.45 | 9.38 | 9.29 | 9.20 | 9.13 | 9.03 |
| 6                              | .10                | 3.78                         | 3.46  | 3.29  | 3.18  | 3.11  | 3.05  | 3.01  | 2.98  | 2.96  | 2.94  | 2.87  | 2.84 | 2.81 | 2.80 | 2.78 | 2.76 | 2.75 | 2.72 |
|                                | .05                | 5.99                         | 5.14  | 4.76  | 4.53  | 4.39  | 4.28  | 4.21  | 4.15  | 4.10  | 4.06  | 3.94  | 3.87 | 3.83 | 3.81 | 3.77 | 3.74 | 3.71 | 3.67 |
|                                | .025               | 8.81                         | 7.26  | 6.60  | 6.23  | 5.99  | 5.82  | 5.70  | 5.60  | 5.52  | 5.46  | 5.27  | 5.17 | 5.11 | 5.07 | 5.01 | 4.96 | 4.92 | 4.86 |
|                                | .01                | 13.75                        | 10.92 | 9.78  | 9.15  | 8.75  | 8.47  | 8.26  | 8.10  | 7.98  | 7.87  | 7.56  | 7.40 | 7.30 | 7.23 | 7.14 | 7.06 | 6.99 | 6.89 |
| 7                              | .10                | 3.59                         | 3.26  | 3.07  | 2.96  | 2.88  | 2.83  | 2.78  | 2.75  | 2.72  | 2.70  | 2.63  | 2.59 | 2.57 | 2.56 | 2.54 | 2.51 | 2.50 | 2.47 |
|                                | .05                | 5.59                         | 4.74  | 4.35  | 4.12  | 3.97  | 3.87  | 3.79  | 3.73  | 3.68  | 3.64  | 3.51  | 3.44 | 3.40 | 3.38 | 3.34 | 3.30 | 3.27 | 3.23 |
|                                | .025               | 8.07                         | 6.54  | 5.89  | 5.52  | 5.29  | 5.12  | 4.99  | 4.90  | 4.82  | 4.76  | 4.57  | 4.47 | 4.40 | 4.36 | 4.31 | 4.25 | 4.21 | 4.15 |
|                                | .01                | 12.25                        | 9.55  | 8.45  | 7.85  | 7.46  | 7.19  | 6.99  | 6.84  | 6.72  | 6.62  | 6.31  | 6.16 | 6.06 | 5.99 | 5.91 | 5.82 | 5.75 | 5.66 |
| 8                              | .10                | 3.46                         | 3.11  | 2.92  | 2.81  | 2.73  | 2.67  | 2.62  | 2.59  | 2.56  | 2.54  | 2.46  | 2.42 | 2.40 | 2.38 | 2.36 | 2.34 | 2.32 | 2.30 |
|                                | .05                | 5.32                         | 4.46  | 4.07  | 3.84  | 3.69  | 3.58  | 3.50  | 3.44  | 3.39  | 3.35  | 3.22  | 3.15 | 3.11 | 3.08 | 3.04 | 3.01 | 2.97 | 2.93 |
|                                | .025               | 7.57                         | 6.06  | 5.42  | 5.05  | 4.82  | 4.65  | 4.53  | 4.43  | 4.36  | 4.30  | 4.10  | 4.00 | 3.94 | 3.89 | 3.84 | 3.78 | 3.74 | 3.68 |
|                                | .01                | 11.26                        | 8.65  | 7.59  | 7.01  | 6.63  | 6.37  | 6.18  | 6.03  | 5.91  | 5.81  | 5.52  | 5.36 | 5.26 | 5.20 | 5.12 | 5.03 | 4.96 | 4.87 |
| 9                              | .10                | 3.36                         | 3.01  | 2.81  | 2.69  | 2.61  | 2.55  | 2.51  | 2.47  | 2.44  | 2.42  | 2.34  | 2.30 | 2.27 | 2.25 | 2.23 | 2.21 | 2.19 | 2.16 |
|                                | .05                | 5.12                         | 4.26  | 3.86  | 3.63  | 3.48  | 3.37  | 3.29  | 3.23  | 3.18  | 3.14  | 3.01  | 2.94 | 2.89 | 2.86 | 2.83 | 2.79 | 2.76 | 2.71 |
|                                | .025               | 7.21                         | 5.71  | 5.08  | 4.72  | 4.48  | 4.32  | 4.20  | 4.10  | 4.03  | 3.96  | 3.77  | 3.67 | 3.60 | 3.56 | 3.51 | 3.45 | 3.40 | 3.34 |
|                                | .01                | 10.56                        | 8.02  | 6.99  | 6.42  | 6.06  | 5.80  | 5.61  | 5.47  | 5.35  | 5.26  | 4.96  | 4.81 | 4.71 | 4.65 | 4.57 | 4.48 | 4.41 | 4.32 |
| 10                             | .10                | 3.29                         | 2.92  | 2.73  | 2.61  | 2.52  | 2.46  | 2.41  | 2.38  | 2.35  | 2.32  | 2.24  | 2.20 | 2.17 | 2.16 | 2.13 | 2.11 | 2.09 | 2.06 |
|                                | .05                | 4.96                         | 4.10  | 3.71  | 3.48  | 3.33  | 3.22  | 3.14  | 3.07  | 3.02  | 2.98  | 2.85  | 2.77 | 2.73 | 2.70 | 2.66 | 2.62 | 2.59 | 2.54 |
|                                | .025               | 6.94                         | 5.46  | 4.83  | 4.47  | 4.24  | 4.07  | 3.95  | 3.85  | 3.78  | 3.72  | 3.52  | 3.42 | 3.35 | 3.31 | 3.26 | 3.20 | 3.15 | 3.09 |
|                                | .01                | 10.04                        | 7.56  | 6.55  | 5.99  | 5.64  | 5.39  | 5.20  | 5.06  | 4.94  | 4.85  | 4.56  | 4.41 | 4.31 | 4.25 | 4.17 | 4.08 | 4.01 | 3.92 |
| 11                             | .10                | 3.23                         | 2.86  | 2.66  | 2.54  | 2.45  | 2.39  | 2.34  | 2.30  | 2.27  | 2.25  | 2.17  | 2.12 | 2.10 | 2.08 | 2.05 | 2.03 | 2.01 | 1.98 |
|                                | .05                | 4.84                         | 3.98  | 3.59  | 3.36  | 3.20  | 3.09  | 3.01  | 2.95  | 2.90  | 2.85  | 2.72  | 2.65 | 2.60 | 2.57 | 2.53 | 2.49 | 2.46 | 2.41 |
|                                | .025               | 6.72                         | 5.26  | 4.63  | 4.28  | 4.04  | 3.88  | 3.76  | 3.66  | 3.59  | 3.53  | 3.33  | 3.23 | 3.16 | 3.12 | 3.06 | 3.00 | 2.96 | 2.89 |
|                                | .01                | 9.65                         | 7.21  | 6.22  | 5.67  | 5.32  | 5.07  | 4.89  | 4.74  | 4.63  | 4.54  | 4.25  | 4.10 | 4.01 | 3.94 | 3.86 | 3.78 | 3.71 | 3.61 |

**TABLE 4** (Continued)

| Denominator degrees of freedom | Area in upper tail | Numerator degrees of freedom |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
|--------------------------------|--------------------|------------------------------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
|                                |                    | 1                            | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   | 15   | 20   | 25   | 30   | 40   | 60   | 100  | 1000 |
| 12                             | .10                | 3.18                         | 2.81 | 2.61 | 2.48 | 2.39 | 2.33 | 2.28 | 2.24 | 2.21 | 2.19 | 2.10 | 2.06 | 2.03 | 2.01 | 1.99 | 1.96 | 1.94 | 1.91 |
|                                | .05                | 4.75                         | 3.89 | 3.49 | 3.26 | 3.11 | 3.00 | 2.91 | 2.85 | 2.80 | 2.75 | 2.62 | 2.54 | 2.50 | 2.47 | 2.43 | 2.38 | 2.35 | 2.30 |
|                                | .025               | 6.55                         | 5.10 | 4.47 | 4.12 | 3.89 | 3.73 | 3.61 | 3.51 | 3.44 | 3.37 | 3.18 | 3.07 | 3.01 | 2.96 | 2.91 | 2.85 | 2.80 | 2.73 |
|                                | .01                | 9.33                         | 6.93 | 5.95 | 5.41 | 5.06 | 4.82 | 4.64 | 4.50 | 4.39 | 4.30 | 4.01 | 3.86 | 3.76 | 3.70 | 3.62 | 3.54 | 3.47 | 3.37 |
| 13                             | .10                | 3.14                         | 2.76 | 2.56 | 2.43 | 2.35 | 2.28 | 2.23 | 2.20 | 2.16 | 2.14 | 2.05 | 2.01 | 1.98 | 1.96 | 1.93 | 1.90 | 1.88 | 1.85 |
|                                | .05                | 4.67                         | 3.81 | 3.41 | 3.18 | 3.03 | 2.92 | 2.83 | 2.77 | 2.71 | 2.67 | 2.53 | 2.46 | 2.41 | 2.38 | 2.34 | 2.30 | 2.26 | 2.21 |
|                                | .025               | 6.41                         | 4.97 | 4.35 | 4.00 | 3.77 | 3.60 | 3.48 | 3.39 | 3.31 | 3.25 | 3.05 | 2.95 | 2.88 | 2.84 | 2.78 | 2.72 | 2.67 | 2.60 |
|                                | .01                | 9.07                         | 6.70 | 5.74 | 5.21 | 4.86 | 4.62 | 4.44 | 4.30 | 4.19 | 4.10 | 3.82 | 3.66 | 3.57 | 3.51 | 3.43 | 3.34 | 3.27 | 3.18 |
| 14                             | .10                | 3.10                         | 2.73 | 2.52 | 2.39 | 2.31 | 2.24 | 2.19 | 2.15 | 2.12 | 2.10 | 2.01 | 1.96 | 1.93 | 1.99 | 1.89 | 1.86 | 1.83 | 1.80 |
|                                | .05                | 4.60                         | 3.74 | 3.34 | 3.11 | 2.96 | 2.85 | 2.76 | 2.70 | 2.65 | 2.60 | 2.46 | 2.39 | 2.34 | 2.31 | 2.27 | 2.22 | 2.19 | 2.14 |
|                                | .025               | 6.30                         | 4.86 | 4.24 | 3.89 | 3.66 | 3.50 | 3.38 | 3.29 | 3.21 | 3.15 | 2.95 | 2.84 | 2.78 | 2.73 | 2.67 | 2.61 | 2.56 | 2.50 |
|                                | .01                | 8.86                         | 6.51 | 5.56 | 5.04 | 4.69 | 4.46 | 4.28 | 4.14 | 4.03 | 3.94 | 3.66 | 3.51 | 3.41 | 3.35 | 3.27 | 3.18 | 3.11 | 3.02 |
| 15                             | .10                | 3.07                         | 2.70 | 2.49 | 2.36 | 2.27 | 2.21 | 2.16 | 2.12 | 2.09 | 2.06 | 1.97 | 1.92 | 1.89 | 1.87 | 1.85 | 1.82 | 1.79 | 1.76 |
|                                | .05                | 4.54                         | 3.68 | 3.29 | 3.06 | 2.90 | 2.79 | 2.71 | 2.64 | 2.59 | 2.54 | 2.40 | 2.33 | 2.28 | 2.25 | 2.20 | 2.16 | 2.12 | 2.07 |
|                                | .025               | 6.20                         | 4.77 | 4.15 | 3.80 | 3.58 | 3.41 | 3.29 | 3.20 | 3.12 | 3.06 | 2.86 | 2.76 | 2.69 | 2.64 | 2.59 | 2.52 | 2.47 | 2.40 |
|                                | .01                | 8.68                         | 6.36 | 5.42 | 4.89 | 4.56 | 4.32 | 4.14 | 4.00 | 3.89 | 3.80 | 3.52 | 3.37 | 3.28 | 3.21 | 3.13 | 3.05 | 2.98 | 2.88 |
| 16                             | .10                | 3.05                         | 2.67 | 2.46 | 2.33 | 2.24 | 2.18 | 2.13 | 2.09 | 2.06 | 2.03 | 1.94 | 1.89 | 1.86 | 1.84 | 1.81 | 1.78 | 1.76 | 1.72 |
|                                | .05                | 4.49                         | 3.63 | 3.24 | 3.01 | 2.85 | 2.74 | 2.66 | 2.59 | 2.54 | 2.49 | 2.35 | 2.28 | 2.23 | 2.19 | 2.15 | 2.11 | 2.07 | 2.02 |
|                                | .025               | 6.12                         | 4.69 | 4.08 | 3.73 | 3.50 | 3.34 | 3.22 | 3.12 | 3.05 | 2.99 | 2.79 | 2.68 | 2.61 | 2.57 | 2.51 | 2.45 | 2.40 | 2.32 |
|                                | .01                | 8.53                         | 6.23 | 5.29 | 4.77 | 4.44 | 4.20 | 4.03 | 3.89 | 3.78 | 3.69 | 3.41 | 3.26 | 3.16 | 3.10 | 3.02 | 2.93 | 2.86 | 2.76 |
| 17                             | .10                | 3.03                         | 2.64 | 2.44 | 2.31 | 2.22 | 2.15 | 2.10 | 2.06 | 2.03 | 2.00 | 1.91 | 1.86 | 1.83 | 1.81 | 1.78 | 1.75 | 1.73 | 1.69 |
|                                | .05                | 4.45                         | 3.59 | 3.20 | 2.96 | 2.81 | 2.70 | 2.61 | 2.55 | 2.49 | 2.45 | 2.31 | 2.23 | 2.18 | 2.15 | 2.10 | 2.06 | 2.02 | 1.97 |
|                                | .025               | 6.04                         | 4.62 | 4.01 | 3.66 | 3.44 | 3.28 | 3.16 | 3.06 | 2.98 | 2.92 | 2.72 | 2.62 | 2.55 | 2.50 | 2.44 | 2.38 | 2.33 | 2.26 |
|                                | .01                | 8.40                         | 6.11 | 5.19 | 4.67 | 4.34 | 4.10 | 3.93 | 3.79 | 3.68 | 3.59 | 3.31 | 3.16 | 3.07 | 3.00 | 2.92 | 2.83 | 2.76 | 2.66 |
| 18                             | .10                | 3.01                         | 2.62 | 2.42 | 2.29 | 2.20 | 2.13 | 2.08 | 2.04 | 2.00 | 1.98 | 1.89 | 1.84 | 1.80 | 1.78 | 1.75 | 1.72 | 1.70 | 1.66 |
|                                | .05                | 4.41                         | 3.55 | 3.16 | 2.93 | 2.77 | 2.66 | 2.58 | 2.51 | 2.46 | 2.41 | 2.27 | 2.19 | 2.14 | 2.11 | 2.06 | 2.02 | 1.98 | 1.92 |
|                                | .025               | 5.98                         | 4.56 | 3.95 | 3.61 | 3.38 | 3.22 | 3.10 | 3.01 | 2.93 | 2.87 | 2.67 | 2.56 | 2.49 | 2.44 | 2.38 | 2.32 | 2.27 | 2.20 |
|                                | .01                | 8.29                         | 6.01 | 5.09 | 4.58 | 4.25 | 4.01 | 3.84 | 3.71 | 3.60 | 3.51 | 3.23 | 3.08 | 2.98 | 2.92 | 2.84 | 2.75 | 2.68 | 2.58 |
| 19                             | .10                | 2.99                         | 2.61 | 2.40 | 2.27 | 2.18 | 2.11 | 2.06 | 2.02 | 1.98 | 1.96 | 1.86 | 1.81 | 1.78 | 1.76 | 1.73 | 1.70 | 1.67 | 1.64 |
|                                | .05                | 4.38                         | 3.52 | 3.13 | 2.90 | 2.74 | 2.63 | 2.54 | 2.48 | 2.42 | 2.38 | 2.23 | 2.16 | 2.11 | 2.07 | 2.03 | 1.98 | 1.94 | 1.88 |
|                                | .025               | 5.92                         | 4.51 | 3.90 | 3.56 | 3.33 | 3.17 | 3.05 | 2.96 | 2.88 | 2.82 | 2.62 | 2.51 | 2.44 | 2.39 | 2.33 | 2.27 | 2.22 | 2.14 |
|                                | .01                | 8.18                         | 5.93 | 5.01 | 4.50 | 4.17 | 3.94 | 3.77 | 3.63 | 3.52 | 3.43 | 3.15 | 3.00 | 2.91 | 2.84 | 2.76 | 2.67 | 2.60 | 2.50 |

(continued)

**TABLE 4** (Continued)

| Denominator degrees of freedom | Area in upper tail | Numerator degrees of freedom |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
|--------------------------------|--------------------|------------------------------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
|                                |                    | 1                            | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   | 15   | 20   | 25   | 30   | 40   | 60   | 100  | 1000 |
| 20                             | .10                | 2.97                         | 2.59 | 2.38 | 2.25 | 2.16 | 2.09 | 2.04 | 2.00 | 1.96 | 1.94 | 1.84 | 1.79 | 1.76 | 1.74 | 1.71 | 1.68 | 1.65 | 1.61 |
|                                | .05                | 4.35                         | 3.49 | 3.10 | 2.87 | 2.71 | 2.60 | 2.51 | 2.45 | 2.39 | 2.35 | 2.20 | 2.12 | 2.07 | 2.04 | 1.99 | 1.95 | 1.91 | 1.85 |
|                                | .025               | 5.87                         | 4.46 | 3.86 | 3.51 | 3.29 | 3.13 | 3.01 | 2.91 | 2.84 | 2.77 | 2.57 | 2.46 | 2.40 | 2.35 | 2.29 | 2.22 | 2.17 | 2.09 |
|                                | .01                | 8.10                         | 5.85 | 4.94 | 4.43 | 4.10 | 3.87 | 3.70 | 3.56 | 3.46 | 3.37 | 3.09 | 2.94 | 2.84 | 2.78 | 2.69 | 2.61 | 2.54 | 2.43 |
| 21                             | .10                | 2.96                         | 2.57 | 2.36 | 2.23 | 2.14 | 2.08 | 2.02 | 1.98 | 1.95 | 1.92 | 1.83 | 1.78 | 1.74 | 1.72 | 1.69 | 1.66 | 1.63 | 1.59 |
|                                | .05                | 4.32                         | 3.47 | 3.07 | 2.84 | 2.68 | 2.57 | 2.49 | 2.42 | 2.37 | 2.32 | 2.18 | 2.10 | 2.05 | 2.01 | 1.96 | 1.92 | 1.88 | 1.82 |
|                                | .025               | 5.83                         | 4.42 | 3.82 | 3.48 | 3.25 | 3.09 | 2.97 | 2.87 | 2.80 | 2.73 | 2.53 | 2.42 | 2.36 | 2.31 | 2.25 | 2.18 | 2.13 | 2.05 |
|                                | .01                | 8.02                         | 5.78 | 4.87 | 4.37 | 4.04 | 3.81 | 3.64 | 3.51 | 3.40 | 3.31 | 3.03 | 2.88 | 2.79 | 2.72 | 2.64 | 2.55 | 2.48 | 2.37 |
| 22                             | .10                | 2.95                         | 2.56 | 2.35 | 2.22 | 2.13 | 2.06 | 2.01 | 1.97 | 1.93 | 1.90 | 1.81 | 1.76 | 1.73 | 1.70 | 1.67 | 1.64 | 1.61 | 1.57 |
|                                | .05                | 4.30                         | 3.44 | 3.05 | 2.82 | 2.66 | 2.55 | 2.46 | 2.40 | 2.34 | 2.30 | 2.15 | 2.07 | 2.02 | 1.98 | 1.94 | 1.89 | 1.85 | 1.79 |
|                                | .025               | 5.79                         | 4.38 | 3.78 | 3.44 | 3.22 | 3.05 | 2.93 | 2.84 | 2.76 | 2.70 | 2.50 | 2.39 | 2.32 | 2.27 | 2.21 | 2.14 | 2.09 | 2.01 |
|                                | .01                | 7.95                         | 5.72 | 4.82 | 4.31 | 3.99 | 3.76 | 3.59 | 3.45 | 3.35 | 3.26 | 2.98 | 2.83 | 2.73 | 2.67 | 2.58 | 2.50 | 2.42 | 2.32 |
| 23                             | .10                | 2.94                         | 2.55 | 2.34 | 2.21 | 2.11 | 2.05 | 1.99 | 1.95 | 1.92 | 1.89 | 1.80 | 1.74 | 1.71 | 1.69 | 1.66 | 1.62 | 1.59 | 1.55 |
|                                | .05                | 4.28                         | 3.42 | 3.03 | 2.80 | 2.64 | 2.53 | 2.44 | 2.37 | 2.32 | 2.27 | 2.13 | 2.05 | 2.00 | 1.96 | 1.91 | 1.86 | 1.82 | 1.76 |
|                                | .025               | 5.75                         | 4.35 | 3.75 | 3.41 | 3.18 | 3.02 | 2.90 | 2.81 | 2.73 | 2.67 | 2.47 | 2.36 | 2.29 | 2.24 | 2.18 | 2.11 | 2.06 | 1.98 |
|                                | .01                | 7.88                         | 5.66 | 4.76 | 4.26 | 3.94 | 3.71 | 3.54 | 3.41 | 3.30 | 3.21 | 2.93 | 2.78 | 2.69 | 2.62 | 2.54 | 2.45 | 2.37 | 2.27 |
| 24                             | .10                | 2.93                         | 2.54 | 2.33 | 2.19 | 2.10 | 2.04 | 1.98 | 1.94 | 1.91 | 1.88 | 1.78 | 1.73 | 1.70 | 1.67 | 1.64 | 1.61 | 1.58 | 1.54 |
|                                | .05                | 4.26                         | 3.40 | 3.01 | 2.78 | 2.62 | 2.51 | 2.42 | 2.36 | 2.30 | 2.25 | 2.11 | 2.03 | 1.97 | 1.94 | 1.89 | 1.84 | 1.80 | 1.74 |
|                                | .025               | 5.72                         | 4.32 | 3.72 | 3.38 | 3.15 | 2.99 | 2.87 | 2.78 | 2.70 | 2.64 | 2.44 | 2.33 | 2.26 | 2.21 | 2.15 | 2.08 | 2.02 | 1.94 |
|                                | .01                | 7.82                         | 5.61 | 4.72 | 4.22 | 3.90 | 3.67 | 3.50 | 3.36 | 3.26 | 3.17 | 2.89 | 2.74 | 2.64 | 2.58 | 2.49 | 2.40 | 2.33 | 2.22 |
| 25                             | .10                | 2.92                         | 2.53 | 2.32 | 2.18 | 2.09 | 2.02 | 1.97 | 1.93 | 1.89 | 1.87 | 1.77 | 1.72 | 1.68 | 1.66 | 1.63 | 1.59 | 1.56 | 1.52 |
|                                | .05                | 4.24                         | 3.39 | 2.99 | 2.76 | 2.60 | 2.49 | 2.40 | 2.34 | 2.28 | 2.24 | 2.09 | 2.01 | 1.96 | 1.92 | 1.87 | 1.82 | 1.78 | 1.72 |
|                                | .025               | 5.69                         | 4.29 | 3.69 | 3.35 | 3.13 | 2.97 | 2.85 | 2.75 | 2.68 | 2.61 | 2.41 | 2.30 | 2.23 | 2.18 | 2.12 | 2.05 | 2.00 | 1.91 |
|                                | .01                | 7.77                         | 5.57 | 4.68 | 4.18 | 3.85 | 3.63 | 3.46 | 3.32 | 3.22 | 3.13 | 2.85 | 2.70 | 2.60 | 2.54 | 2.45 | 2.36 | 2.29 | 2.18 |
| 26                             | .10                | 2.91                         | 2.52 | 2.31 | 2.17 | 2.08 | 2.01 | 1.96 | 1.92 | 1.88 | 1.86 | 1.76 | 1.71 | 1.67 | 1.65 | 1.61 | 1.58 | 1.55 | 1.51 |
|                                | .05                | 4.23                         | 3.37 | 2.98 | 2.74 | 2.59 | 2.47 | 2.39 | 2.32 | 2.27 | 2.22 | 2.07 | 1.99 | 1.94 | 1.90 | 1.85 | 1.80 | 1.76 | 1.70 |
|                                | .025               | 5.66                         | 4.27 | 3.67 | 3.33 | 3.10 | 2.94 | 2.82 | 2.73 | 2.65 | 2.59 | 2.39 | 2.28 | 2.21 | 2.16 | 2.09 | 2.03 | 1.97 | 1.89 |
|                                | .01                | 7.72                         | 5.53 | 4.64 | 4.14 | 3.82 | 3.59 | 3.42 | 3.29 | 3.18 | 3.09 | 2.81 | 2.66 | 2.57 | 2.50 | 2.42 | 2.33 | 2.25 | 2.14 |
| 27                             | .10                | 2.90                         | 2.51 | 2.30 | 2.17 | 2.07 | 2.00 | 1.95 | 1.91 | 1.87 | 1.85 | 1.75 | 1.70 | 1.66 | 1.64 | 1.60 | 1.57 | 1.54 | 1.50 |
|                                | .05                | 4.21                         | 3.35 | 2.96 | 2.73 | 2.57 | 2.46 | 2.37 | 2.31 | 2.25 | 2.20 | 2.06 | 1.97 | 1.92 | 1.88 | 1.84 | 1.79 | 1.74 | 1.68 |
|                                | .025               | 5.63                         | 4.24 | 3.65 | 3.31 | 3.08 | 2.92 | 2.80 | 2.71 | 2.63 | 2.57 | 2.36 | 2.25 | 2.18 | 2.13 | 2.07 | 2.00 | 1.94 | 1.86 |
|                                | .01                | 7.68                         | 5.49 | 4.60 | 4.11 | 3.78 | 3.56 | 3.39 | 3.26 | 3.15 | 3.06 | 2.78 | 2.63 | 2.54 | 2.47 | 2.38 | 2.29 | 2.22 | 2.11 |

**TABLE 4** (Continued)

| Denominator degrees of freedom | Area in upper tail | Numerator degrees of freedom |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
|--------------------------------|--------------------|------------------------------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
|                                |                    | 1                            | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   | 15   | 20   | 25   | 30   | 40   | 60   | 100  | 1000 |
| 28                             | .10                | 2.89                         | 2.50 | 2.29 | 2.16 | 2.06 | 2.00 | 1.94 | 1.90 | 1.87 | 1.84 | 1.74 | 1.69 | 1.65 | 1.63 | 1.59 | 1.56 | 1.53 | 1.48 |
|                                | .05                | 4.20                         | 3.34 | 2.95 | 2.71 | 2.56 | 2.45 | 2.36 | 2.29 | 2.24 | 2.19 | 2.04 | 1.96 | 1.91 | 1.87 | 1.82 | 1.77 | 1.73 | 1.66 |
|                                | .025               | 5.61                         | 4.22 | 3.63 | 3.29 | 3.06 | 2.90 | 2.78 | 2.69 | 2.61 | 2.55 | 2.34 | 2.23 | 2.16 | 2.11 | 2.05 | 1.98 | 1.92 | 1.84 |
|                                | .01                | 7.64                         | 5.45 | 4.57 | 4.07 | 3.75 | 3.53 | 3.36 | 3.23 | 3.12 | 3.03 | 2.75 | 2.60 | 2.51 | 2.44 | 2.35 | 2.26 | 2.19 | 2.08 |
| 29                             | .10                | 2.89                         | 2.50 | 2.28 | 2.15 | 2.06 | 1.99 | 1.93 | 1.89 | 1.86 | 1.83 | 1.73 | 1.68 | 1.64 | 1.62 | 1.58 | 1.55 | 1.52 | 1.47 |
|                                | .05                | 4.18                         | 3.33 | 2.93 | 2.70 | 2.55 | 2.43 | 2.35 | 2.28 | 2.22 | 2.18 | 2.03 | 1.94 | 1.89 | 1.85 | 1.81 | 1.75 | 1.71 | 1.65 |
|                                | .025               | 5.59                         | 4.20 | 3.61 | 3.27 | 3.04 | 2.88 | 2.76 | 2.67 | 2.59 | 2.53 | 2.32 | 2.21 | 2.14 | 2.09 | 2.03 | 1.96 | 1.90 | 1.82 |
|                                | .01                | 7.60                         | 5.42 | 4.54 | 4.04 | 3.73 | 3.50 | 3.33 | 3.20 | 3.09 | 3.00 | 2.73 | 2.57 | 2.48 | 2.41 | 2.33 | 2.23 | 2.16 | 2.05 |
| 30                             | .10                | 2.88                         | 2.49 | 2.28 | 2.14 | 2.05 | 1.98 | 1.93 | 1.88 | 1.85 | 1.82 | 1.72 | 1.67 | 1.63 | 1.61 | 1.57 | 1.54 | 1.51 | 1.46 |
|                                | .05                | 4.17                         | 3.32 | 2.92 | 2.69 | 2.53 | 2.42 | 2.33 | 2.27 | 2.21 | 2.16 | 2.01 | 1.93 | 1.88 | 1.84 | 1.79 | 1.74 | 1.70 | 1.63 |
|                                | .025               | 5.57                         | 4.18 | 3.59 | 3.25 | 3.03 | 2.87 | 2.75 | 2.65 | 2.57 | 2.51 | 2.31 | 2.20 | 2.12 | 2.07 | 2.01 | 1.94 | 1.88 | 1.80 |
|                                | .01                | 7.56                         | 5.39 | 4.51 | 4.02 | 3.70 | 3.47 | 3.30 | 3.17 | 3.07 | 2.98 | 2.70 | 2.55 | 2.45 | 2.39 | 2.30 | 2.21 | 2.13 | 2.02 |
| 40                             | .10                | 2.84                         | 2.44 | 2.23 | 2.09 | 2.00 | 1.93 | 1.87 | 1.83 | 1.79 | 1.76 | 1.66 | 1.61 | 1.57 | 1.54 | 1.51 | 1.47 | 1.43 | 1.38 |
|                                | .05                | 4.08                         | 3.23 | 2.84 | 2.61 | 2.45 | 2.34 | 2.25 | 2.18 | 2.12 | 2.08 | 1.92 | 1.84 | 1.78 | 1.74 | 1.69 | 1.64 | 1.59 | 1.52 |
|                                | .025               | 5.42                         | 4.05 | 3.46 | 3.13 | 2.90 | 2.74 | 2.62 | 2.53 | 2.45 | 2.39 | 2.18 | 2.07 | 1.99 | 1.94 | 1.88 | 1.80 | 1.74 | 1.65 |
|                                | .01                | 7.31                         | 5.18 | 4.31 | 3.83 | 3.51 | 3.29 | 3.12 | 2.99 | 2.89 | 2.80 | 2.52 | 2.37 | 2.27 | 2.20 | 2.11 | 2.02 | 1.94 | 1.82 |
| 60                             | .10                | 2.79                         | 2.39 | 2.18 | 2.04 | 1.95 | 1.87 | 1.82 | 1.77 | 1.74 | 1.71 | 1.60 | 1.54 | 1.50 | 1.48 | 1.44 | 1.40 | 1.36 | 1.30 |
|                                | .05                | 4.00                         | 3.15 | 2.76 | 2.53 | 2.37 | 2.25 | 2.17 | 2.10 | 2.04 | 1.99 | 1.84 | 1.75 | 1.69 | 1.65 | 1.59 | 1.53 | 1.48 | 1.40 |
|                                | .025               | 5.29                         | 3.93 | 3.34 | 3.01 | 2.79 | 2.63 | 2.51 | 2.41 | 2.33 | 2.27 | 2.06 | 1.94 | 1.87 | 1.82 | 1.74 | 1.67 | 1.60 | 1.49 |
|                                | .01                | 7.08                         | 4.98 | 4.13 | 3.65 | 3.34 | 3.12 | 2.95 | 2.82 | 2.72 | 2.63 | 2.35 | 2.20 | 2.10 | 2.03 | 1.94 | 1.84 | 1.75 | 1.62 |
| 100                            | .10                | 2.76                         | 2.36 | 2.14 | 2.00 | 1.91 | 1.83 | 1.78 | 1.73 | 1.69 | 1.66 | 1.56 | 1.49 | 1.45 | 1.42 | 1.38 | 1.34 | 1.29 | 1.22 |
|                                | .05                | 3.94                         | 3.09 | 2.70 | 2.46 | 2.31 | 2.19 | 2.10 | 2.03 | 1.97 | 1.93 | 1.77 | 1.68 | 1.62 | 1.57 | 1.52 | 1.45 | 1.39 | 1.30 |
|                                | .025               | 5.18                         | 3.83 | 3.25 | 2.92 | 2.70 | 2.54 | 2.42 | 2.32 | 2.24 | 2.18 | 1.97 | 1.85 | 1.77 | 1.71 | 1.64 | 1.56 | 1.48 | 1.36 |
|                                | .01                | 6.90                         | 4.82 | 3.98 | 3.51 | 3.21 | 2.99 | 2.82 | 2.69 | 2.59 | 2.50 | 2.22 | 2.07 | 1.97 | 1.89 | 1.80 | 1.69 | 1.60 | 1.45 |
| 1000                           | .10                | 2.71                         | 2.31 | 2.09 | 1.95 | 1.85 | 1.78 | 1.72 | 1.68 | 1.64 | 1.61 | 1.49 | 1.43 | 1.38 | 1.35 | 1.30 | 1.25 | 1.20 | 1.08 |
|                                | .05                | 3.85                         | 3.00 | 2.61 | 2.38 | 2.22 | 2.11 | 2.02 | 1.95 | 1.89 | 1.84 | 1.68 | 1.58 | 1.52 | 1.47 | 1.41 | 1.33 | 1.26 | 1.11 |
|                                | .025               | 5.04                         | 3.70 | 3.13 | 2.80 | 2.58 | 2.42 | 2.30 | 2.20 | 2.13 | 2.06 | 1.85 | 1.72 | 1.64 | 1.58 | 1.50 | 1.41 | 1.32 | 1.13 |
|                                | .01                | 6.66                         | 4.63 | 3.80 | 3.34 | 3.04 | 2.82 | 2.66 | 2.53 | 2.43 | 2.34 | 2.06 | 1.90 | 1.79 | 1.72 | 1.61 | 1.50 | 1.38 | 1.16 |

**TABLE 5** Binomial probabilities

Entries in the table give the probability of  $x$  successes in  $n$  trials of a binomial experiment, where  $\pi$  is the probability of a success on one trial. For example, with six trials and  $\pi = 0.05$ , the probability of two successes is 0.0305.

**TABLE 5** (Continued)

| n  | x  | $\pi$ |       |       |       |       |       |       |       |       |
|----|----|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|    |    | .01   | .02   | .03   | .04   | .05   | .06   | .07   | .08   | .09   |
|    | 7  | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 |
|    | 8  | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 |
| 9  | 0  | .9135 | .8337 | .7602 | .6925 | .6302 | .5730 | .5204 | .4722 | .4279 |
|    | 1  | .0830 | .1531 | .2116 | .2597 | .2985 | .3292 | .3525 | .3695 | .3809 |
|    | 2  | .0034 | .0125 | .0262 | .0433 | .0629 | .0840 | .1061 | .1285 | .1507 |
|    | 3  | .0001 | .0006 | .0019 | .0042 | .0077 | .0125 | .0186 | .0261 | .0348 |
|    | 4  | .0000 | .0000 | .0001 | .0003 | .0006 | .0012 | .0021 | .0034 | .0052 |
|    | 5  | .0000 | .0000 | .0000 | .0000 | .0000 | .0001 | .0002 | .0003 | .0005 |
|    | 6  | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 |
|    | 7  | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 |
|    | 8  | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 |
|    | 9  | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 |
| 10 | 0  | .9044 | .8171 | .7374 | .6648 | .5987 | .5386 | .4840 | .4344 | .3894 |
|    | 1  | .0914 | .1667 | .2281 | .2770 | .3151 | .3438 | .3643 | .3777 | .3851 |
|    | 2  | .0042 | .0153 | .0317 | .0519 | .0746 | .0988 | .1234 | .1478 | .1714 |
|    | 3  | .0001 | .0008 | .0026 | .0058 | .0105 | .0168 | .0248 | .0343 | .0452 |
|    | 4  | .0000 | .0000 | .0001 | .0004 | .0010 | .0019 | .0033 | .0052 | .0078 |
|    | 5  | .0000 | .0000 | .0000 | .0000 | .0001 | .0001 | .0003 | .0005 | .0009 |
|    | 6  | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0001 |
|    | 7  | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 |
|    | 8  | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 |
|    | 9  | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 |
|    | 10 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 |
| 12 | 0  | .8864 | .7847 | .6938 | .6127 | .5404 | .4759 | .4186 | .3677 | .3225 |
|    | 1  | .1074 | .1922 | .2575 | .3064 | .3413 | .3645 | .3781 | .3837 | .3827 |
|    | 2  | .0060 | .0216 | .0438 | .0702 | .0988 | .1280 | .1565 | .1835 | .2082 |
|    | 3  | .0002 | .0015 | .0045 | .0098 | .0173 | .0272 | .0393 | .0532 | .0686 |
|    | 4  | .0000 | .0001 | .0003 | .0009 | .0021 | .0039 | .0067 | .0104 | .0153 |
|    | 5  | .0000 | .0000 | .0000 | .0001 | .0002 | .0004 | .0008 | .0014 | .0024 |
|    | 6  | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0001 | .0001 | .0003 |
|    | 7  | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 |
|    | 8  | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 |
|    | 9  | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 |
|    | 10 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 |
|    | 11 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 |
|    | 12 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 |
| 15 | 0  | .8601 | .7386 | .6333 | .5421 | .4633 | .3953 | .3367 | .2863 | .2430 |
|    | 1  | .1303 | .2261 | .2938 | .3388 | .3658 | .3785 | .3801 | .3734 | .3605 |
|    | 2  | .0092 | .0323 | .0636 | .0988 | .1348 | .1691 | .2003 | .2273 | .2496 |
|    | 3  | .0004 | .0029 | .0085 | .0178 | .0307 | .0468 | .0653 | .0857 | .1070 |
|    | 4  | .0000 | .0002 | .0008 | .0022 | .0049 | .0090 | .0148 | .0223 | .0317 |
|    | 5  | .0000 | .0000 | .0001 | .0002 | .0006 | .0013 | .0024 | .0043 | .0069 |
|    | 6  | .0000 | .0000 | .0000 | .0000 | .0000 | .0001 | .0003 | .0006 | .0011 |
|    | 7  | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0001 | .0001 |
|    | 8  | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 |
|    | 9  | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 |
|    | 10 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 |

(continued)

**TABLE 5** (*Continued*)

**TABLE 5** (Continued)

| n | x | $\pi$ |       |       |       |       |       |       |       |       |  |
|---|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|--|
|   |   | .10   | .15   | .20   | .25   | .30   | .35   | .40   | .45   | .50   |  |
| 2 | 0 | .8100 | .7225 | .6400 | .5625 | .4900 | .4225 | .3600 | .3025 | .2500 |  |
|   | 1 | .1800 | .2550 | .3200 | .3750 | .4200 | .4550 | .4800 | .4950 | .5000 |  |
|   | 2 | .0100 | .0225 | .0400 | .0625 | .0900 | .1225 | .1600 | .2025 | .2500 |  |
| 3 | 0 | .7290 | .6141 | .5120 | .4219 | .3430 | .2746 | .2160 | .1664 | .1250 |  |
|   | 1 | .2430 | .3251 | .3840 | .4219 | .4410 | .4436 | .4320 | .4084 | .3750 |  |
|   | 2 | .0270 | .0574 | .0960 | .1406 | .1890 | .2389 | .2880 | .3341 | .3750 |  |
|   | 3 | .0010 | .0034 | .0080 | .0156 | .0270 | .0429 | .0640 | .0911 | .1250 |  |
| 4 | 0 | .6561 | .5220 | .4096 | .3164 | .2401 | .1785 | .1296 | .0915 | .0625 |  |
|   | 1 | .2916 | .3685 | .4096 | .4219 | .4116 | .3845 | .3456 | .2995 | .2500 |  |
|   | 2 | .0486 | .0975 | .1536 | .2109 | .2646 | .3105 | .3456 | .3675 | .3750 |  |
|   | 3 | .0036 | .0115 | .0256 | .0469 | .0756 | .1115 | .1536 | .2005 | .2500 |  |
|   | 4 | .0001 | .0005 | .0016 | .0039 | .0081 | .0150 | .0256 | .0410 | .0625 |  |
| 5 | 0 | .5905 | .4437 | .3277 | .2373 | .1681 | .1160 | .0778 | .0503 | .0312 |  |
|   | 1 | .3280 | .3915 | .4096 | .3955 | .3602 | .3124 | .2592 | .2059 | .1562 |  |
|   | 2 | .0729 | .1382 | .2048 | .2637 | .3087 | .3364 | .3456 | .3369 | .3125 |  |
|   | 3 | .0081 | .0244 | .0512 | .0879 | .1323 | .1811 | .2304 | .2757 | .3125 |  |
|   | 4 | .0004 | .0022 | .0064 | .0146 | .0284 | .0488 | .0768 | .1128 | .1562 |  |
|   | 5 | .0000 | .0001 | .0003 | .0010 | .0024 | .0053 | .0102 | .0185 | .0312 |  |
| 6 | 0 | .5314 | .3771 | .2621 | .1780 | .1176 | .0754 | .0467 | .0277 | .0156 |  |
|   | 1 | .3543 | .3993 | .3932 | .3560 | .3025 | .2437 | .1866 | .1359 | .0938 |  |
|   | 2 | .0984 | .1762 | .2458 | .2966 | .3241 | .3280 | .3110 | .2780 | .2344 |  |
|   | 3 | .0146 | .0415 | .0819 | .1318 | .1852 | .2355 | .2765 | .3032 | .3125 |  |
|   | 4 | .0012 | .0055 | .0154 | .0330 | .0595 | .0951 | .1382 | .1861 | .2344 |  |
|   | 5 | .0001 | .0004 | .0015 | .0044 | .0102 | .0205 | .0369 | .0609 | .0938 |  |
|   | 6 | .0000 | .0000 | .0001 | .0002 | .0007 | .0018 | .0041 | .0083 | .0156 |  |
| 7 | 0 | .4783 | .3206 | .2097 | .1335 | .0824 | .0490 | .0280 | .0152 | .0078 |  |
|   | 1 | .3720 | .3960 | .3670 | .3115 | .2471 | .1848 | .1306 | .0872 | .0547 |  |
|   | 2 | .1240 | .2097 | .2753 | .3115 | .3177 | .2985 | .2613 | .2140 | .1641 |  |
|   | 3 | .0230 | .0617 | .1147 | .1730 | .2269 | .2679 | .2903 | .2918 | .2734 |  |
|   | 4 | .0026 | .0109 | .0287 | .0577 | .0972 | .1442 | .1935 | .2388 | .2734 |  |
|   | 5 | .0002 | .0012 | .0043 | .0115 | .0250 | .0466 | .0774 | .1172 | .1641 |  |
|   | 6 | .0000 | .0001 | .0004 | .0013 | .0036 | .0084 | .0172 | .0320 | .0547 |  |
|   | 7 | .0000 | .0000 | .0000 | .0001 | .0002 | .0006 | .0016 | .0037 | .0078 |  |
| 8 | 0 | .4305 | .2725 | .1678 | .1001 | .0576 | .0319 | .0168 | .0084 | .0039 |  |
|   | 1 | .3826 | .3847 | .3355 | .2670 | .1977 | .1373 | .0896 | .0548 | .0312 |  |
|   | 2 | .1488 | .2376 | .2936 | .3115 | .2965 | .2587 | .2090 | .1569 | .1094 |  |
|   | 3 | .0331 | .0839 | .1468 | .2076 | .2541 | .2786 | .2787 | .2568 | .2188 |  |
|   | 4 | .0046 | .0185 | .0459 | .0865 | .1361 | .1875 | .2322 | .2627 | .2734 |  |
|   | 5 | .0004 | .0026 | .0092 | .0231 | .0467 | .0808 | .1239 | .1719 | .2188 |  |
|   | 6 | .0000 | .0002 | .0011 | .0038 | .0100 | .0217 | .0413 | .0703 | .1094 |  |
|   | 7 | .0000 | .0000 | .0001 | .0004 | .0012 | .0033 | .0079 | .0164 | .0313 |  |
|   | 8 | .0000 | .0000 | .0000 | .0000 | .0001 | .0002 | .0007 | .0017 | .0039 |  |

(continued)

**TABLE 5** (*Continued*)

| n  | x  | $\pi$ |       |       |       |       |       |       |       |       |  |
|----|----|-------|-------|-------|-------|-------|-------|-------|-------|-------|--|
|    |    | .10   | .15   | .20   | .25   | .30   | .35   | .40   | .45   | .50   |  |
| 9  | 0  | .3874 | .2316 | .1342 | .0751 | .0404 | .0207 | .0101 | .0046 | .0020 |  |
|    | 1  | .3874 | .3679 | .3020 | .2253 | .1556 | .1004 | .0605 | .0339 | .0176 |  |
|    | 2  | .1722 | .2597 | .3020 | .3003 | .2668 | .2162 | .1612 | .1110 | .0703 |  |
|    | 3  | .0446 | .1069 | .1762 | .2336 | .2668 | .2716 | .2508 | .2119 | .1641 |  |
|    | 4  | .0074 | .0283 | .0661 | .1168 | .1715 | .2194 | .2508 | .2600 | .2461 |  |
|    | 5  | .0008 | .0050 | .0165 | .0389 | .0735 | .1181 | .1672 | .2128 | .2461 |  |
|    | 6  | .0001 | .0006 | .0028 | .0087 | .0210 | .0424 | .0743 | .1160 | .1641 |  |
|    | 7  | .0000 | .0000 | .0003 | .0012 | .0039 | .0098 | .0212 | .0407 | .0703 |  |
|    | 8  | .0000 | .0000 | .0000 | .0001 | .0004 | .0013 | .0035 | .0083 | .0176 |  |
|    | 9  | .0000 | .0000 | .0000 | .0000 | .0000 | .0001 | .0003 | .0008 | .0020 |  |
| 10 | 0  | .3487 | .1969 | .1074 | .0563 | .0282 | .0135 | .0060 | .0025 | .0010 |  |
|    | 1  | .3874 | .3474 | .2684 | .1877 | .1211 | .0725 | .0403 | .0207 | .0098 |  |
|    | 2  | .1937 | .2759 | .3020 | .2816 | .2335 | .1757 | .1209 | .0763 | .0439 |  |
|    | 3  | .0574 | .1298 | .2013 | .2503 | .2668 | .2522 | .2150 | .1665 | .1172 |  |
|    | 4  | .0112 | .0401 | .0881 | .1460 | .2001 | .2377 | .2508 | .2384 | .2051 |  |
|    | 5  | .0015 | .0085 | .0264 | .0584 | .1029 | .1536 | .2007 | .2340 | .2461 |  |
|    | 6  | .0001 | .0012 | .0055 | .0162 | .0368 | .0689 | .1115 | .1596 | .2051 |  |
|    | 7  | .0000 | .0001 | .0008 | .0031 | .0090 | .0212 | .0425 | .0746 | .1172 |  |
|    | 8  | .0000 | .0000 | .0001 | .0004 | .0014 | .0043 | .0106 | .0229 | .0439 |  |
|    | 9  | .0000 | .0000 | .0000 | .0000 | .0001 | .0005 | .0016 | .0042 | .0098 |  |
|    | 10 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0001 | .0003 | .0010 |  |
| 12 | 0  | .2824 | .1422 | .0687 | .0317 | .0138 | .0057 | .0022 | .0008 | .0002 |  |
|    | 1  | .3766 | .3012 | .2062 | .1267 | .0712 | .0368 | .0174 | .0075 | .0029 |  |
|    | 2  | .2301 | .2924 | .2835 | .2323 | .1678 | .1088 | .0639 | .0339 | .0161 |  |
|    | 3  | .0853 | .1720 | .2362 | .2581 | .2397 | .1954 | .1419 | .0923 | .0537 |  |
|    | 4  | .0213 | .0683 | .1329 | .1936 | .2311 | .2367 | .2128 | .1700 | .1208 |  |
|    | 5  | .0038 | .0193 | .0532 | .1032 | .1585 | .2039 | .2270 | .2225 | .1934 |  |
|    | 6  | .0005 | .0040 | .0155 | .0401 | .0792 | .1281 | .1766 | .2124 | .2256 |  |
|    | 7  | .0000 | .0006 | .0033 | .0115 | .0291 | .0591 | .1009 | .1489 | .1934 |  |
|    | 8  | .0000 | .0001 | .0005 | .0024 | .0078 | .0199 | .0420 | .0762 | .1208 |  |
|    | 9  | .0000 | .0000 | .0001 | .0004 | .0015 | .0048 | .0125 | .0277 | .0537 |  |
|    | 10 | .0000 | .0000 | .0000 | .0000 | .0002 | .0008 | .0025 | .0068 | .0161 |  |
|    | 11 | .0000 | .0000 | .0000 | .0000 | .0000 | .0001 | .0003 | .0010 | .0029 |  |
|    | 12 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0001 | .0002 |  |
| 15 | 0  | .2059 | .0874 | .0352 | .0134 | .0047 | .0016 | .0005 | .0001 | .0000 |  |
|    | 1  | .3432 | .2312 | .1319 | .0668 | .0305 | .0126 | .0047 | .0016 | .0005 |  |
|    | 2  | .2669 | .2856 | .2309 | .1559 | .0916 | .0476 | .0219 | .0090 | .0032 |  |
|    | 3  | .1285 | .2184 | .2501 | .2252 | .1700 | .1110 | .0634 | .0318 | .0139 |  |
|    | 4  | .0428 | .1156 | .1876 | .2252 | .2186 | .1792 | .1268 | .0780 | .0417 |  |
|    | 5  | .0105 | .0449 | .1032 | .1651 | .2061 | .2123 | .1859 | .1404 | .0916 |  |
|    | 6  | .0019 | .0132 | .0430 | .0917 | .1472 | .1906 | .2066 | .1914 | .1527 |  |
|    | 7  | .0003 | .0030 | .0138 | .0393 | .0811 | .1319 | .1771 | .2013 | .1964 |  |
|    | 8  | .0000 | .0005 | .0035 | .0131 | .0348 | .0710 | .1181 | .1647 | .1964 |  |
|    | 10 | .0000 | .0000 | .0001 | .0007 | .0030 | .0096 | .0245 | .0515 | .0916 |  |

**TABLE 5** (*Continued*)

**TABLE 6** Poisson probabilities

Entries in the table give the probability of  $x$  occurrences for a Poisson process with a mean  $\mu$ . For example, when  $\mu = 2.5$ , the probability of four occurrences is .1336.

**TABLE 6** (Continued)

| x  | $\mu$ |       |       |       |       |       |       |       |       |       |
|----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|    | 3.1   | 3.2   | 3.3   | 3.4   | 3.5   | 3.6   | 3.7   | 3.8   | 3.9   | 4.0   |
| 0  | .0450 | .0408 | .0369 | .0344 | .0302 | .0273 | .0247 | .0224 | .0202 | .0183 |
| 1  | .1397 | .1304 | .1217 | .1135 | .1057 | .0984 | .0915 | .0850 | .0789 | .0733 |
| 2  | .2165 | .2087 | .2008 | .1929 | .1850 | .1771 | .1692 | .1615 | .1539 | .1465 |
| 3  | .2237 | .2226 | .2209 | .2186 | .2158 | .2125 | .2087 | .2046 | .2001 | .1954 |
| 4  | .1734 | .1781 | .1823 | .1858 | .1888 | .1912 | .1931 | .1944 | .1951 | .1954 |
| 5  | .1075 | .1140 | .1203 | .1264 | .1322 | .1377 | .1429 | .1477 | .1522 | .1563 |
| 6  | .0555 | .0608 | .0662 | .0716 | .0771 | .0826 | .0881 | .0936 | .0989 | .1042 |
| 7  | .0246 | .0278 | .0312 | .0348 | .0385 | .0425 | .0466 | .0508 | .0551 | .0595 |
| 8  | .0095 | .0111 | .0129 | .0148 | .0169 | .0191 | .0215 | .0241 | .0269 | .0298 |
| 9  | .0033 | .0040 | .0047 | .0056 | .0066 | .0076 | .0089 | .0102 | .0116 | .0132 |
| 10 | .0010 | .0013 | .0016 | .0019 | .0023 | .0028 | .0033 | .0039 | .0045 | .0053 |
| 11 | .0003 | .0004 | .0005 | .0006 | .0007 | .0009 | .0011 | .0013 | .0016 | .0019 |
| 12 | .0001 | .0001 | .0001 | .0002 | .0002 | .0003 | .0003 | .0004 | .0005 | .0006 |
| 13 | .0000 | .0000 | .0000 | .0000 | .0001 | .0001 | .0001 | .0001 | .0002 | .0002 |
| 14 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0001 |

| x  | $\mu$ |       |       |       |       |       |       |       |       |       |
|----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|    | 4.1   | 4.2   | 4.3   | 4.4   | 4.5   | 4.6   | 4.7   | 4.8   | 4.9   | 5.0   |
| 0  | .0166 | .0150 | .0136 | .0123 | .0111 | .0101 | .091  | .0082 | .0074 | .0067 |
| 1  | .0679 | .0630 | .0583 | .0540 | .0500 | .0462 | .0427 | .0395 | .0365 | .0337 |
| 2  | .1393 | .1323 | .1254 | .1188 | .1125 | .1063 | .1005 | .0948 | .0894 | .0842 |
| 3  | .1904 | .1852 | .1798 | .1743 | .1687 | .1631 | .1574 | .1517 | .1460 | .1404 |
| 4  | .1951 | .1944 | .1933 | .1917 | .1898 | .1875 | .1849 | .1820 | .1789 | .1755 |
| 5  | .1600 | .1633 | .1662 | .1687 | .1708 | .1725 | .1738 | .1747 | .1753 | .1755 |
| 6  | .1093 | .1143 | .1191 | .1237 | .1281 | .1323 | .1362 | .1398 | .1432 | .1462 |
| 7  | .0640 | .0686 | .0732 | .0778 | .0824 | .0869 | .0914 | .0959 | .1002 | .1044 |
| 8  | .0328 | .0360 | .0393 | .0428 | .0463 | .0500 | .0537 | .0575 | .0614 | .0653 |
| 9  | .0150 | .0168 | .0188 | .0209 | .0232 | .0255 | .0280 | .0307 | .0334 | .0363 |
| 10 | .0061 | .0071 | .0081 | .0092 | .0104 | .0118 | .0132 | .0147 | .0164 | .0181 |
| 11 | .0023 | .0027 | .0032 | .0037 | .0043 | .0049 | .0056 | .0064 | .0073 | .0082 |
| 12 | .0008 | .0009 | .0011 | .0014 | .0016 | .0019 | .0022 | .0026 | .0030 | .0034 |
| 13 | .0002 | .0003 | .0004 | .0005 | .0006 | .0007 | .0008 | .0009 | .0011 | .0013 |
| 14 | .0001 | .0001 | .0001 | .0001 | .0002 | .0002 | .0003 | .0003 | .0004 | .0005 |
| 15 | .0000 | .0000 | .0000 | .0000 | .0001 | .0001 | .0001 | .0001 | .0001 | .0002 |

| x | $\mu$ |       |       |       |       |       |       |       |       |       |
|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|   | 5.1   | 5.2   | 5.3   | 5.4   | 5.5   | 5.6   | 5.7   | 5.8   | 5.9   | 6.0   |
| 0 | .0061 | .0055 | .0050 | .0045 | .0041 | .0037 | .0033 | .0030 | .0027 | .0025 |
| 1 | .0311 | .0287 | .0265 | .0244 | .0225 | .0207 | .0191 | .0176 | .0162 | .0149 |
| 2 | .0793 | .0746 | .0701 | .0659 | .0618 | .0580 | .0544 | .0509 | .0477 | .0446 |
| 3 | .1348 | .1293 | .1239 | .1185 | .1133 | .1082 | .1033 | .0985 | .0938 | .0892 |
| 4 | .1719 | .1681 | .1641 | .1600 | .1558 | .1515 | .1472 | .1428 | .1383 | .1339 |

(continued)

**TABLE 6** (*Continued*)

| x  | $\mu$ |       |       |       |       |       |       |       |       |       |
|----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|    | 5.1   | 5.2   | 5.3   | 5.4   | 5.5   | 5.6   | 5.7   | 5.8   | 5.9   | 6.0   |
| 5  | .1753 | .1748 | .1740 | .1728 | .1714 | .1697 | .1678 | .1656 | .1632 | .1606 |
| 6  | .1490 | .1515 | .1537 | .1555 | .1571 | .1587 | .1594 | .1601 | .1605 | .1606 |
| 7  | .1086 | .1125 | .1163 | .1200 | .1234 | .1267 | .1298 | .1326 | .1353 | .1377 |
| 8  | .0692 | .0731 | .0771 | .0810 | .0849 | .0887 | .0925 | .0962 | .0998 | .1033 |
| 9  | .0392 | .0423 | .0454 | .0486 | .0519 | .0552 | .0586 | .0620 | .0654 | .0688 |
| 10 | .0200 | .0220 | .0241 | .0262 | .0285 | .0309 | .0334 | .0359 | .0386 | .0413 |
| 11 | .0093 | .0104 | .0116 | .0129 | .0143 | .0157 | .0173 | .0190 | .0207 | .0225 |
| 12 | .0039 | .0045 | .0051 | .0058 | .0065 | .0073 | .0082 | .0092 | .0102 | .0113 |
| 13 | .0015 | .0018 | .0021 | .0024 | .0028 | .0032 | .0036 | .0041 | .0046 | .0052 |
| 14 | .0006 | .0007 | .0008 | .0009 | .0011 | .0013 | .0015 | .0017 | .0019 | .0022 |
| 15 | .0002 | .0002 | .0003 | .0003 | .0004 | .0005 | .0006 | .0007 | .0008 | .0009 |
| 16 | .0001 | .0001 | .0001 | .0001 | .0001 | .0002 | .0002 | .0002 | .0003 | .0003 |
| 17 | .0000 | .0000 | .0000 | .0000 | .0000 | .0001 | .0001 | .0001 | .0001 | .0001 |

| x  | $\mu$ |       |       |       |       |       |       |       |       |       |
|----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|    | 6.1   | 6.2   | 6.3   | 6.4   | 6.5   | 6.6   | 6.7   | 6.8   | 6.9   | 7.0   |
| 0  | .0022 | .0020 | .0018 | .0017 | .0015 | .0014 | .0012 | .0011 | .0010 | .0009 |
| 1  | .0137 | .0126 | .0116 | .0106 | .0098 | .0090 | .0082 | .0076 | .0070 | .0064 |
| 2  | .0417 | .0390 | .0364 | .0340 | .0318 | .0296 | .0276 | .0258 | .0240 | .0223 |
| 3  | .0848 | .0806 | .0765 | .0726 | .0688 | .0652 | .0617 | .0584 | .0552 | .0521 |
| 4  | .1294 | .1249 | .1205 | .1162 | .1118 | .1076 | .1034 | .0992 | .0952 | .0912 |
| 5  | .1579 | .1549 | .1519 | .1487 | .1454 | .1420 | .1385 | .1349 | .1314 | .1277 |
| 6  | .1605 | .1601 | .1595 | .1586 | .1575 | .1562 | .1546 | .1529 | .1511 | .1490 |
| 7  | .1399 | .1418 | .1435 | .1450 | .1462 | .1472 | .1480 | .1486 | .1489 | .1490 |
| 8  | .1066 | .1099 | .1130 | .1160 | .1188 | .1215 | .1240 | .1263 | .1284 | .1304 |
| 9  | .0723 | .0757 | .0791 | .0825 | .0858 | .0891 | .0923 | .0954 | .0985 | .1014 |
| 10 | .0441 | .0469 | .0498 | .0528 | .0558 | .0588 | .0618 | .0649 | .0679 | .0710 |
| 11 | .0245 | .0265 | .0285 | .0307 | .0330 | .0353 | .0377 | .0401 | .0426 | .0452 |
| 12 | .0124 | .0137 | .0150 | .0164 | .0179 | .0194 | .0210 | .0227 | .0245 | .0264 |
| 13 | .0058 | .0065 | .0073 | .0081 | .0089 | .0098 | .0108 | .0119 | .0130 | .0142 |
| 14 | .0025 | .0029 | .0033 | .0037 | .0041 | .0046 | .0052 | .0058 | .0064 | .0071 |
| 15 | .0010 | .0012 | .0014 | .0016 | .0018 | .0020 | .0023 | .0026 | .0029 | .0033 |
| 16 | .0004 | .0005 | .0005 | .0006 | .0007 | .0008 | .0010 | .0011 | .0013 | .0014 |
| 17 | .0001 | .0002 | .0002 | .0002 | .0003 | .0003 | .0004 | .0004 | .0005 | .0006 |
| 18 | .0000 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0002 | .0002 | .0002 |
| 19 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0001 | .0001 | .0001 |

**TABLE 6** (Continued)

| x  | $\mu$ |       |       |       |       |       |       |       |       |       |
|----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|    | 7.1   | 7.2   | 7.3   | 7.4   | 7.5   | 7.6   | 7.7   | 7.8   | 7.9   | 8.0   |
| 0  | .0008 | .0007 | .0007 | .0006 | .0006 | .0005 | .0005 | .0004 | .0004 | .0003 |
| 1  | .0059 | .0054 | .0049 | .0045 | .0041 | .0038 | .0035 | .0032 | .0029 | .0027 |
| 2  | .0208 | .0194 | .0180 | .0167 | .0156 | .0145 | .0134 | .0125 | .0116 | .0107 |
| 3  | .0492 | .0464 | .0438 | .0413 | .0389 | .0366 | .0345 | .0324 | .0305 | .0286 |
| 4  | .0874 | .0836 | .0799 | .0764 | .0729 | .0696 | .0663 | .0632 | .0602 | .0573 |
| 5  | .1241 | .1204 | .1167 | .1130 | .1094 | .1057 | .1021 | .0986 | .0951 | .0916 |
| 6  | .1468 | .1445 | .1420 | .1394 | .1367 | .1339 | .1311 | .1282 | .1252 | .1221 |
| 7  | .1489 | .1486 | .1481 | .1474 | .1465 | .1454 | .1442 | .1428 | .1413 | .1396 |
| 8  | .1321 | .1337 | .1351 | .1363 | .1373 | .1382 | .1388 | .1392 | .1395 | .1396 |
| 9  | .1042 | .1070 | .1096 | .1121 | .1144 | .1167 | .1187 | .1207 | .1224 | .1241 |
| 10 | .0740 | .0770 | .0800 | .0829 | .0858 | .0887 | .0914 | .0941 | .0967 | .0993 |
| 11 | .0478 | .0504 | .0531 | .0558 | .0585 | .0613 | .0640 | .0667 | .0695 | .0722 |
| 12 | .0283 | .0303 | .0323 | .0344 | .0366 | .0388 | .0411 | .0434 | .0457 | .0481 |
| 13 | .0154 | .0168 | .0181 | .0196 | .0211 | .0227 | .0243 | .0260 | .0278 | .0296 |
| 14 | .0078 | .0086 | .0095 | .0104 | .0113 | .0123 | .0134 | .0145 | .0157 | .0169 |
| 15 | .0037 | .0041 | .0046 | .0051 | .0057 | .0062 | .0069 | .0075 | .0083 | .0090 |
| 16 | .0016 | .0019 | .0021 | .0024 | .0026 | .0030 | .0033 | .0037 | .0041 | .0045 |
| 17 | .0007 | .0008 | .0009 | .0010 | .0012 | .0013 | .0015 | .0017 | .0019 | .0021 |
| 18 | .0003 | .0003 | .0004 | .0004 | .0005 | .0006 | .0006 | .0007 | .0008 | .0009 |
| 19 | .0001 | .0001 | .0001 | .0002 | .0002 | .0002 | .0003 | .0003 | .0003 | .0004 |
| 20 | .0000 | .0000 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0002 |
| 21 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0001 | .0001 |

| x  | $\mu$ |       |       |       |       |       |       |       |       |       |
|----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|    | 8.1   | 8.2   | 8.3   | 8.4   | 8.5   | 8.6   | 8.7   | 8.8   | 8.9   | 9.0   |
| 0  | .0003 | .0003 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 | .0001 | .0001 |
| 1  | .0025 | .0023 | .0021 | .0019 | .0017 | .0016 | .0014 | .0013 | .0012 | .0011 |
| 2  | .0100 | .0092 | .0086 | .0079 | .0074 | .0068 | .0063 | .0058 | .0054 | .0050 |
| 3  | .0269 | .0252 | .0237 | .0222 | .0208 | .0195 | .0183 | .0171 | .0160 | .0150 |
| 4  | .0544 | .0517 | .0491 | .0466 | .0443 | .0420 | .0398 | .0377 | .0357 | .0337 |
| 5  | .0882 | .0849 | .0816 | .0784 | .0752 | .0722 | .0692 | .0663 | .0635 | .0607 |
| 6  | .1191 | .1160 | .1128 | .1097 | .1066 | .1034 | .1003 | .0972 | .0941 | .0911 |
| 7  | .1378 | .1358 | .1338 | .1317 | .1294 | .1271 | .1247 | .1222 | .1197 | .1171 |
| 8  | .1395 | .1392 | .1388 | .1382 | .1375 | .1366 | .1356 | .1344 | .1332 | .1318 |
| 9  | .1256 | .1269 | .1280 | .1290 | .1299 | .1306 | .1311 | .1315 | .1317 | .1318 |
| 10 | .1017 | .1040 | .1063 | .1084 | .1104 | .1123 | .1140 | .1157 | .1172 | .1186 |
| 11 | .0749 | .0776 | .0802 | .0828 | .0853 | .0878 | .0902 | .0925 | .0948 | .0970 |
| 12 | .0505 | .0530 | .0555 | .0579 | .0604 | .0629 | .0654 | .0679 | .0703 | .0728 |

(continued)

**TABLE 6** (*Continued*)

| x  | $\mu$ |       |       |       |       |       |       |       |       |       |
|----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|    | 8.1   | 8.2   | 8.3   | 8.4   | 8.5   | 8.6   | 8.7   | 8.8   | 8.9   | 9.0   |
| 13 | .0315 | .0334 | .0354 | .0374 | .0395 | .0416 | .0438 | .0459 | .0481 | .0504 |
| 14 | .0182 | .0196 | .0210 | .0225 | .0240 | .0256 | .0272 | .0289 | .0306 | .0324 |
| 15 | .0098 | .0107 | .0116 | .0126 | .0136 | .0147 | .0158 | .0169 | .0182 | .1094 |
| 16 | .0050 | .0055 | .0060 | .0066 | .0072 | .0079 | .0086 | .0093 | .0101 | .0109 |
| 17 | .0024 | .0026 | .0029 | .0033 | .0036 | .0040 | .0044 | .0048 | .0053 | .0058 |
| 18 | .0011 | .0012 | .0014 | .0015 | .0017 | .0019 | .0021 | .0024 | .0026 | .0029 |
| 19 | .0005 | .0005 | .0006 | .0007 | .0008 | .0009 | .0010 | .0011 | .0012 | .0014 |
| 20 | .0002 | .0002 | .0002 | .0003 | .0003 | .0004 | .0004 | .0005 | .0005 | .0006 |
| 21 | .0001 | .0001 | .0001 | .0001 | .0001 | .0002 | .0002 | .0002 | .0002 | .0003 |
| 22 | .0000 | .0000 | .0000 | .0000 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 |

| x  | $\mu$ |       |       |       |       |       |       |       |       |       |
|----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|    | 9.1   | 9.2   | 9.3   | 9.4   | 9.5   | 9.6   | 9.7   | 9.8   | 9.9   | 10    |
| 0  | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0000 |
| 1  | .0010 | .0009 | .0009 | .0008 | .0007 | .0007 | .0006 | .0005 | .0005 | .0005 |
| 2  | .0046 | .0043 | .0040 | .0037 | .0034 | .0031 | .0029 | .0027 | .0025 | .0023 |
| 3  | .0140 | .0131 | .0123 | .0115 | .0107 | .0100 | .0093 | .0087 | .0081 | .0076 |
| 4  | .0319 | .0302 | .0285 | .0269 | .0254 | .0240 | .0226 | .0213 | .0201 | .0189 |
| 5  | .0581 | .0555 | .0530 | .0506 | .0483 | .0460 | .0439 | .0418 | .0398 | .0378 |
| 6  | .0881 | .0851 | .0822 | .0793 | .0764 | .0736 | .0709 | .0682 | .0656 | .0631 |
| 7  | .1145 | .1118 | .1091 | .1064 | .1037 | .1010 | .0982 | .0955 | .0928 | .0901 |
| 8  | .1302 | .1286 | .1269 | .1251 | .1232 | .1212 | .1191 | .1170 | .1148 | .1126 |
| 9  | .1317 | .1315 | .1311 | .1306 | .1300 | .1293 | .1284 | .1274 | .1263 | .1251 |
| 10 | .1198 | .1210 | .1219 | .1228 | .1235 | .1241 | .1245 | .1249 | .1250 | .1251 |
| 11 | .0991 | .1012 | .1031 | .1049 | .1067 | .1083 | .1098 | .1112 | .1125 | .1137 |
| 12 | .0752 | .0776 | .0799 | .0822 | .0844 | .0866 | .0888 | .0908 | .0928 | .0948 |
| 13 | .0526 | .0549 | .0572 | .0594 | .0617 | .0640 | .0662 | .0685 | .0707 | .0729 |
| 14 | .0342 | .0361 | .0380 | .0399 | .0419 | .0439 | .0459 | .0479 | .0500 | .0521 |
| 15 | .0208 | .0221 | .0235 | .0250 | .0265 | .0281 | .0297 | .0313 | .0330 | .0347 |
| 16 | .0118 | .0127 | .0137 | .0147 | .0157 | .0168 | .0180 | .0192 | .0204 | .0217 |
| 17 | .0063 | .0069 | .0075 | .0081 | .0088 | .0095 | .0103 | .0111 | .0119 | .0128 |
| 18 | .0032 | .0035 | .0039 | .0042 | .0046 | .0051 | .0055 | .0060 | .0065 | .0071 |
| 19 | .0015 | .0017 | .0019 | .0021 | .0023 | .0026 | .0028 | .0031 | .0034 | .0037 |
| 20 | .0007 | .0008 | .0009 | .0010 | .0011 | .0012 | .0014 | .0015 | .0017 | .0019 |
| 21 | .0003 | .0003 | .0004 | .0004 | .0005 | .0006 | .0006 | .0007 | .0008 | .0009 |
| 22 | .0001 | .0001 | .0002 | .0002 | .0002 | .0002 | .0003 | .0003 | .0004 | .0004 |
| 23 | .0000 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0002 | .0002 |
| 24 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0001 | .0001 | .0001 |

**TABLE 6** (*Continued*)

**TABLE 7** Critical values for the Durbin–Watson test for autocorrelation

Entries in the table give the critical values for a one-tailed Durbin–Watson test for autocorrelation. For a two-tailed test, the level of significance is doubled.

| Significance points of $d_L$ and $d_U$ : $\alpha = .05$ |      |       |       |       |       |       |       |       |       |       |       |
|---|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| Number of independent variables                         |      |       |       |       |       |       |       |       |       |       |       |
| n   | k    | 1     |       | 2     |       | 3     |       | 4     |       | 5     |       |
|   |      | $d_L$ | $d_U$ |
| 15  | 1.08 | 1.36  | 0.95  | 1.54  | 0.82  | 1.75  | 0.69  | 1.97  | 0.56  | 2.21  |       |
| 16  | 1.10 | 1.37  | 0.98  | 1.54  | 0.86  | 1.73  | 0.74  | 1.93  | 0.62  | 2.15  |       |
| 17  | 1.13 | 1.38  | 1.02  | 1.54  | 0.90  | 1.71  | 0.78  | 1.90  | 0.67  | 2.10  |       |
| 18  | 1.16 | 1.39  | 1.05  | 1.53  | 0.93  | 1.69  | 0.82  | 1.87  | 0.71  | 2.06  |       |
| 19  | 1.18 | 1.40  | 1.08  | 1.53  | 0.97  | 1.68  | 0.86  | 1.85  | 0.75  | 2.02  |       |
| 20  | 1.20 | 1.41  | 1.10  | 1.54  | 1.00  | 1.68  | 0.90  | 1.83  | 0.79  | 1.99  |       |
| 21  | 1.22 | 1.42  | 1.13  | 1.54  | 1.03  | 1.67  | 0.93  | 1.81  | 0.83  | 1.96  |       |
| 22  | 1.24 | 1.43  | 1.15  | 1.54  | 1.05  | 1.66  | 1.96  | 1.80  | 0.86  | 1.94  |       |
| 23  | 1.26 | 1.44  | 1.17  | 1.54  | 1.08  | 1.66  | 0.99  | 1.79  | 0.90  | 1.92  |       |
| 24  | 1.27 | 1.45  | 1.19  | 1.55  | 1.10  | 1.66  | 1.01  | 1.78  | 0.93  | 1.90  |       |
| 25  | 1.29 | 1.45  | 1.21  | 1.55  | 1.12  | 1.66  | 1.04  | 1.77  | 0.95  | 1.89  |       |
| 26  | 1.30 | 1.46  | 1.22  | 1.55  | 1.14  | 1.65  | 1.06  | 1.76  | 0.98  | 1.88  |       |
| 27  | 1.32 | 1.47  | 1.24  | 1.56  | 1.16  | 1.65  | 1.08  | 1.76  | 1.01  | 1.86  |       |
| 28  | 1.33 | 1.48  | 1.26  | 1.56  | 1.18  | 1.65  | 1.10  | 1.75  | 1.03  | 1.85  |       |
| 29  | 1.34 | 1.48  | 1.27  | 1.56  | 1.20  | 1.65  | 1.12  | 1.74  | 1.05  | 1.84  |       |
| 30  | 1.35 | 1.49  | 1.28  | 1.57  | 1.21  | 1.65  | 1.14  | 1.74  | 1.07  | 1.83  |       |
| 31  | 1.36 | 1.50  | 1.30  | 1.57  | 1.23  | 1.65  | 1.16  | 1.74  | 1.09  | 1.83  |       |
| 32  | 1.37 | 1.50  | 1.31  | 1.57  | 1.24  | 1.65  | 1.18  | 1.73  | 1.11  | 1.82  |       |
| 33  | 1.38 | 1.51  | 1.32  | 1.58  | 1.26  | 1.65  | 1.19  | 1.73  | 1.13  | 1.81  |       |
| 34  | 1.39 | 1.51  | 1.33  | 1.58  | 1.27  | 1.65  | 1.21  | 1.73  | 1.15  | 1.81  |       |
| 35  | 1.40 | 1.52  | 1.34  | 1.58  | 1.28  | 1.65  | 1.22  | 1.73  | 1.16  | 1.80  |       |
| 36  | 1.41 | 1.52  | 1.35  | 1.59  | 1.29  | 1.65  | 1.24  | 1.73  | 1.18  | 1.80  |       |
| 37  | 1.42 | 1.53  | 1.36  | 1.59  | 1.31  | 1.66  | 1.25  | 1.72  | 1.19  | 1.80  |       |
| 38  | 1.43 | 1.54  | 1.37  | 1.59  | 1.32  | 1.66  | 1.26  | 1.72  | 1.21  | 1.79  |       |
| 39  | 1.43 | 1.54  | 1.38  | 1.60  | 1.33  | 1.66  | 1.27  | 1.72  | 1.22  | 1.79  |       |
| 40  | 1.44 | 1.54  | 1.39  | 1.60  | 1.34  | 1.66  | 1.29  | 1.72  | 1.23  | 1.79  |       |
| 45  | 1.48 | 1.57  | 1.43  | 1.62  | 1.38  | 1.67  | 1.34  | 1.72  | 1.29  | 1.78  |       |
| 50  | 1.50 | 1.59  | 1.46  | 1.63  | 1.42  | 1.67  | 1.38  | 1.72  | 1.34  | 1.77  |       |
| 55  | 1.53 | 1.60  | 1.49  | 1.64  | 1.45  | 1.68  | 1.41  | 1.72  | 1.38  | 1.77  |       |
| 60  | 1.55 | 1.62  | 1.51  | 1.65  | 1.48  | 1.69  | 1.44  | 1.73  | 1.41  | 1.77  |       |
| 65  | 1.57 | 1.63  | 1.54  | 1.66  | 1.50  | 1.70  | 1.47  | 1.73  | 1.44  | 1.77  |       |
| 70  | 1.58 | 1.64  | 1.55  | 1.67  | 1.52  | 1.70  | 1.49  | 1.74  | 1.46  | 1.77  |       |
| 75  | 1.60 | 1.65  | 1.57  | 1.68  | 1.54  | 1.71  | 1.51  | 1.74  | 1.49  | 1.77  |       |
| 80  | 1.61 | 1.66  | 1.59  | 1.69  | 1.56  | 1.72  | 1.53  | 1.74  | 1.51  | 1.77  |       |
| 85  | 1.62 | 1.67  | 1.60  | 1.70  | 1.57  | 1.72  | 1.55  | 1.75  | 1.52  | 1.77  |       |
| 90  | 1.63 | 1.68  | 1.61  | 1.70  | 1.59  | 1.73  | 1.57  | 1.75  | 1.54  | 1.78  |       |
| 95  | 1.64 | 1.69  | 1.62  | 1.71  | 1.60  | 1.73  | 1.58  | 1.75  | 1.56  | 1.78  |       |
| 100   | 1.65 | 1.69  | 1.63  | 1.72  | 1.61  | 1.74  | 1.59  | 1.76  | 1.57  | 1.78  |       |

**TABLE 7** (Continued)

| n   | k    | Significance points of $d_L$ and $d_U$ : $\alpha = .05$ |       |       |       |       |       |       |       |       |       |
|-----|------|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|     |      | Number of independent variables                         |       |       |       |       |       |       |       |       |       |
|     |      | 1   |       | 2     |       | 3     |       | 4     |       | 5     |       |
| n   | k    | $d_L$   | $d_U$ | $d_L$ | $d_U$ | $d_L$ | $d_U$ | $d_L$ | $d_U$ | $d_L$ | $d_U$ |
| 15  | 0.95 | 1.23  | 0.83  | 1.40  | 0.71  | 1.61  | 0.59  | 1.84  | 0.48  | 2.09  |       |
| 16  | 0.98 | 1.24  | 0.86  | 1.40  | 0.75  | 1.59  | 0.64  | 1.80  | 0.53  | 2.03  |       |
| 17  | 1.01 | 1.25  | 0.90  | 1.40  | 0.79  | 1.58  | 0.68  | 1.77  | 0.57  | 1.98  |       |
| 18  | 1.03 | 1.26  | 0.93  | 1.40  | 0.82  | 1.56  | 0.72  | 1.74  | 0.62  | 1.93  |       |
| 19  | 1.06 | 1.28  | 0.96  | 1.41  | 0.86  | 1.55  | 0.76  | 1.72  | 0.66  | 1.90  |       |
| 20  | 1.08 | 1.28  | 0.99  | 1.41  | 0.89  | 1.55  | 0.79  | 1.70  | 0.70  | 1.87  |       |
| 21  | 1.10 | 1.30  | 1.01  | 1.41  | 0.92  | 1.54  | 0.83  | 1.69  | 0.73  | 1.84  |       |
| 22  | 1.12 | 1.31  | 1.04  | 1.42  | 0.95  | 1.54  | 0.86  | 1.68  | 0.77  | 1.82  |       |
| 23  | 1.14 | 1.32  | 1.06  | 1.42  | 0.97  | 1.54  | 0.89  | 1.67  | 0.80  | 1.80  |       |
| 24  | 1.16 | 1.33  | 1.08  | 1.43  | 1.00  | 1.54  | 0.91  | 1.66  | 0.83  | 1.79  |       |
| 25  | 1.18 | 1.34  | 1.10  | 1.43  | 1.02  | 1.54  | 0.94  | 1.65  | 0.86  | 1.77  |       |
| 26  | 1.19 | 1.35  | 1.12  | 1.44  | 1.04  | 1.54  | 0.96  | 1.65  | 0.88  | 1.76  |       |
| 27  | 1.21 | 1.36  | 1.13  | 1.44  | 1.06  | 1.54  | 0.99  | 1.64  | 0.91  | 1.75  |       |
| 28  | 1.22 | 1.37  | 1.15  | 1.45  | 1.08  | 1.54  | 1.01  | 1.64  | 0.93  | 1.74  |       |
| 29  | 1.24 | 1.38  | 1.17  | 1.45  | 1.10  | 1.54  | 1.03  | 1.63  | 0.96  | 1.73  |       |
| 30  | 1.25 | 1.38  | 1.18  | 1.46  | 1.12  | 1.54  | 1.05  | 1.63  | 0.98  | 1.73  |       |
| 31  | 1.26 | 1.39  | 1.20  | 1.47  | 1.13  | 1.55  | 1.07  | 1.63  | 1.00  | 1.72  |       |
| 32  | 1.27 | 1.40  | 1.21  | 1.47  | 1.15  | 1.55  | 1.08  | 1.63  | 1.02  | 1.71  |       |
| 33  | 1.28 | 1.41  | 1.22  | 1.48  | 1.16  | 1.55  | 1.10  | 1.63  | 1.04  | 1.71  |       |
| 34  | 1.29 | 1.41  | 1.24  | 1.48  | 1.17  | 1.55  | 1.12  | 1.63  | 1.06  | 1.70  |       |
| 35  | 1.30 | 1.42  | 1.25  | 1.48  | 1.19  | 1.55  | 1.13  | 1.63  | 1.07  | 1.70  |       |
| 36  | 1.31 | 1.43  | 1.26  | 1.49  | 1.20  | 1.56  | 1.15  | 1.63  | 1.09  | 1.70  |       |
| 37  | 1.32 | 1.43  | 1.27  | 1.49  | 1.21  | 1.56  | 1.16  | 1.62  | 1.10  | 1.70  |       |
| 38  | 1.33 | 1.44  | 1.28  | 1.50  | 1.23  | 1.56  | 1.17  | 1.62  | 1.12  | 1.70  |       |
| 39  | 1.34 | 1.44  | 1.29  | 1.50  | 1.24  | 1.56  | 1.19  | 1.63  | 1.13  | 1.69  |       |
| 40  | 1.35 | 1.45  | 1.30  | 1.51  | 1.25  | 1.57  | 1.20  | 1.63  | 1.15  | 1.69  |       |
| 45  | 1.39 | 1.48  | 1.34  | 1.53  | 1.30  | 1.58  | 1.25  | 1.63  | 1.21  | 1.69  |       |
| 50  | 1.42 | 1.50  | 1.38  | 1.54  | 1.34  | 1.59  | 1.30  | 1.64  | 1.26  | 1.69  |       |
| 55  | 1.45 | 1.52  | 1.41  | 1.56  | 1.37  | 1.60  | 1.33  | 1.64  | 1.30  | 1.69  |       |
| 60  | 1.47 | 1.54  | 1.44  | 1.57  | 1.40  | 1.61  | 1.37  | 1.65  | 1.33  | 1.69  |       |
| 65  | 1.49 | 1.55  | 1.46  | 1.59  | 1.43  | 1.62  | 1.40  | 1.66  | 1.36  | 1.69  |       |
| 70  | 1.51 | 1.57  | 1.48  | 1.60  | 1.45  | 1.63  | 1.42  | 1.66  | 1.39  | 1.70  |       |
| 75  | 1.53 | 1.58  | 1.50  | 1.61  | 1.47  | 1.64  | 1.45  | 1.67  | 1.42  | 1.70  |       |
| 80  | 1.54 | 1.59  | 1.52  | 1.62  | 1.49  | 1.65  | 1.47  | 1.67  | 1.44  | 1.70  |       |
| 85  | 1.56 | 1.60  | 1.53  | 1.63  | 1.51  | 1.65  | 1.49  | 1.68  | 1.46  | 1.71  |       |
| 90  | 1.57 | 1.61  | 1.55  | 1.64  | 1.53  | 1.66  | 1.50  | 1.69  | 1.48  | 1.71  |       |
| 95  | 1.58 | 1.62  | 1.56  | 1.65  | 1.54  | 1.67  | 1.52  | 1.69  | 1.50  | 1.71  |       |
| 100 | 1.59 | 1.63  | 1.57  | 1.65  | 1.55  | 1.67  | 1.53  | 1.70  | 1.51  | 1.72  |       |

(continued)

**TABLE 7** (Continued)

| n   | k    | Significance points of $d_L$ and $d_U$ : $\alpha = .01$ |       |       |       |       |       |       |       |       |       |
|-----|------|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|     |      | Number of independent variables                         |       |       |       |       |       |       |       |       |       |
|     |      | 1   | 2     |       | 3     |       | 4     |       | 5     |       |       |
| n   | k    | $d_L$   | $d_U$ | $d_L$ | $d_U$ | $d_L$ | $d_U$ | $d_L$ | $d_U$ | $d_L$ | $d_U$ |
| 15  | 0.81 | 1.07  | 0.70  | 1.25  | 0.59  | 1.46  | 0.49  | 1.70  | 0.39  | 1.96  |       |
| 16  | 0.84 | 1.09  | 0.74  | 1.25  | 0.63  | 1.44  | 0.53  | 1.66  | 0.44  | 1.90  |       |
| 17  | 0.87 | 1.10  | 0.77  | 1.25  | 0.67  | 1.43  | 0.57  | 1.63  | 0.48  | 1.85  |       |
| 18  | 0.90 | 1.12  | 0.80  | 1.26  | 0.71  | 1.42  | 0.61  | 1.60  | 0.52  | 1.80  |       |
| 19  | 0.93 | 1.13  | 0.83  | 1.26  | 0.74  | 1.41  | 0.65  | 1.58  | 0.56  | 1.77  |       |
| 20  | 0.95 | 1.15  | 0.86  | 1.27  | 0.77  | 1.41  | 0.68  | 1.57  | 0.60  | 1.74  |       |
| 21  | 0.97 | 1.16  | 0.89  | 1.27  | 0.80  | 1.41  | 0.72  | 1.55  | 0.63  | 1.71  |       |
| 22  | 1.00 | 1.17  | 0.91  | 1.28  | 0.83  | 1.40  | 0.75  | 1.54  | 0.66  | 1.69  |       |
| 23  | 1.02 | 1.19  | 0.94  | 1.29  | 0.86  | 1.40  | 0.77  | 1.53  | 0.70  | 1.67  |       |
| 24  | 1.04 | 1.20  | 0.96  | 1.30  | 0.88  | 1.41  | 0.80  | 1.53  | 0.72  | 1.66  |       |
| 25  | 1.05 | 1.21  | 0.98  | 1.30  | 0.90  | 1.41  | 0.83  | 1.52  | 0.75  | 1.65  |       |
| 26  | 1.07 | 1.22  | 1.00  | 1.31  | 0.93  | 1.41  | 0.85  | 1.52  | 0.78  | 1.64  |       |
| 27  | 1.09 | 1.23  | 1.02  | 1.32  | 0.95  | 1.41  | 0.88  | 1.51  | 0.81  | 1.63  |       |
| 28  | 1.10 | 1.24  | 1.04  | 1.32  | 0.97  | 1.41  | 0.90  | 1.51  | 0.83  | 1.62  |       |
| 29  | 1.12 | 1.25  | 1.05  | 1.33  | 0.99  | 1.42  | 0.92  | 1.51  | 0.85  | 1.61  |       |
| 30  | 1.13 | 1.26  | 1.07  | 1.34  | 1.01  | 1.42  | 0.94  | 1.51  | 0.88  | 1.61  |       |
| 31  | 1.15 | 1.27  | 1.08  | 1.34  | 1.02  | 1.42  | 0.96  | 1.51  | 0.90  | 1.60  |       |
| 32  | 1.16 | 1.28  | 1.10  | 1.35  | 1.04  | 1.43  | 0.98  | 1.51  | 0.92  | 1.60  |       |
| 33  | 1.17 | 1.29  | 1.11  | 1.36  | 1.05  | 1.43  | 1.00  | 1.51  | 0.94  | 1.59  |       |
| 34  | 1.18 | 1.30  | 1.13  | 1.36  | 1.07  | 1.43  | 1.01  | 1.51  | 0.95  | 1.59  |       |
| 35  | 1.19 | 1.31  | 1.14  | 1.37  | 1.08  | 1.44  | 1.03  | 1.51  | 0.97  | 1.59  |       |
| 36  | 1.21 | 1.32  | 1.15  | 1.38  | 1.10  | 1.44  | 1.04  | 1.51  | 0.99  | 1.59  |       |
| 37  | 1.22 | 1.32  | 1.16  | 1.38  | 1.11  | 1.45  | 1.06  | 1.51  | 1.00  | 1.59  |       |
| 38  | 1.23 | 1.33  | 1.18  | 1.39  | 1.12  | 1.45  | 1.07  | 1.52  | 1.02  | 1.58  |       |
| 39  | 1.24 | 1.34  | 1.19  | 1.39  | 1.14  | 1.45  | 1.09  | 1.52  | 1.03  | 1.58  |       |
| 40  | 1.25 | 1.34  | 1.20  | 1.40  | 1.15  | 1.46  | 1.10  | 1.52  | 1.05  | 1.58  |       |
| 45  | 1.29 | 1.38  | 1.24  | 1.42  | 1.20  | 1.48  | 1.16  | 1.53  | 1.11  | 1.58  |       |
| 50  | 1.32 | 1.40  | 1.28  | 1.45  | 1.24  | 1.49  | 1.20  | 1.54  | 1.16  | 1.59  |       |
| 55  | 1.36 | 1.43  | 1.32  | 1.47  | 1.28  | 1.51  | 1.25  | 1.55  | 1.21  | 1.59  |       |
| 60  | 1.38 | 1.45  | 1.35  | 1.48  | 1.32  | 1.52  | 1.28  | 1.56  | 1.25  | 1.60  |       |
| 65  | 1.41 | 1.47  | 1.38  | 1.50  | 1.35  | 1.53  | 1.31  | 1.57  | 1.28  | 1.61  |       |
| 70  | 1.43 | 1.49  | 1.40  | 1.52  | 1.37  | 1.55  | 1.34  | 1.58  | 1.31  | 1.61  |       |
| 75  | 1.45 | 1.50  | 1.42  | 1.53  | 1.39  | 1.56  | 1.37  | 1.59  | 1.34  | 1.62  |       |
| 80  | 1.47 | 1.52  | 1.44  | 1.54  | 1.42  | 1.57  | 1.39  | 1.60  | 1.36  | 1.62  |       |
| 85  | 1.48 | 1.53  | 1.46  | 1.55  | 1.43  | 1.58  | 1.41  | 1.60  | 1.39  | 1.63  |       |
| 90  | 1.50 | 1.54  | 1.47  | 1.56  | 1.45  | 1.59  | 1.43  | 1.61  | 1.41  | 1.64  |       |
| 95  | 1.51 | 1.55  | 1.49  | 1.57  | 1.47  | 1.60  | 1.45  | 1.62  | 1.42  | 1.64  |       |
| 100 | 1.52 | 1.56  | 1.50  | 1.58  | 1.48  | 1.60  | 1.46  | 1.63  | 1.44  | 1.65  |       |

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**TABLE 8**  $T_L$  Values for the Mann–Whitney–Wilcoxon test

Reject the hypothesis of identical populations if the sum of the ranks for the  $n_1$  items is *less* than the value  $T_L$  shown in the following table or if the sum of the ranks for the  $n_1$  items is *greater* than the value  $T_U$  where:

$$T_U = n_1(n_1 + n_2 + 1) - T_L$$

| $\alpha = 0.10$ |  | $n_2$ |    |    |    |    |    |    |    |    |
|-----------------|--|-------|----|----|----|----|----|----|----|----|
|                 |  | 2     | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
| $n_1$           |  | 2     | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
| 2               |  | 3     | 3  | 3  | 4  | 4  | 4  | 5  | 5  | 5  |
| 3               |  | 6     | 7  | 7  | 8  | 9  | 9  | 10 | 11 | 11 |
| 4               |  | 10    | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| 5               |  | 16    | 17 | 18 | 20 | 21 | 22 | 24 | 25 | 27 |
| 6               |  | 22    | 24 | 25 | 27 | 29 | 30 | 32 | 34 | 36 |
| 7               |  | 29    | 31 | 33 | 35 | 37 | 40 | 42 | 44 | 46 |
| 8               |  | 38    | 40 | 42 | 45 | 47 | 50 | 52 | 55 | 57 |
| 9               |  | 47    | 50 | 52 | 55 | 58 | 61 | 64 | 67 | 70 |
| 10              |  | 57    | 60 | 63 | 67 | 70 | 73 | 76 | 80 | 83 |

| $\alpha = 0.05$ |  | $n_2$ |    |    |    |    |    |    |    |    |
|-----------------|--|-------|----|----|----|----|----|----|----|----|
|                 |  | 2     | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
| $n_1$           |  | 2     | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
| 2               |  | 3     | 3  | 3  | 3  | 3  | 3  | 4  | 4  | 4  |
| 3               |  | 6     | 6  | 6  | 7  | 8  | 8  | 9  | 9  | 10 |
| 4               |  | 10    | 10 | 11 | 12 | 13 | 14 | 15 | 15 | 16 |
| 5               |  | 15    | 16 | 17 | 18 | 19 | 21 | 22 | 23 | 24 |
| 6               |  | 21    | 23 | 24 | 25 | 27 | 28 | 30 | 32 | 33 |
| 7               |  | 28    | 30 | 32 | 34 | 35 | 37 | 39 | 41 | 43 |
| 8               |  | 37    | 39 | 41 | 43 | 45 | 47 | 50 | 52 | 54 |
| 9               |  | 46    | 48 | 50 | 53 | 56 | 58 | 61 | 63 | 66 |
| 10              |  | 56    | 59 | 61 | 64 | 67 | 70 | 73 | 76 | 79 |

**TABLE 9** Critical values for the studentized range

| Degrees<br>of<br>freedom | $\alpha = .05$        |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |  |
|--------------------------|-----------------------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|--|
|                          | Number of populations |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |  |
|                          | 2                     | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   | 11   | 12   | 13   | 14   | 15   | 16   | 17   | 18   | 19   | 20   |  |
| 1                        | 18.0                  | 27.0 | 32.8 | 37.1 | 40.4 | 43.1 | 45.4 | 47.4 | 49.1 | 50.6 | 52.0 | 53.2 | 54.3 | 55.4 | 56.3 | 57.2 | 58.0 | 58.8 | 59.6 |  |
| 2                        | 6.08                  | 8.33 | 9.80 | 10.9 | 11.7 | 12.4 | 13.0 | 13.5 | 14.0 | 14.4 | 14.7 | 15.1 | 15.4 | 15.7 | 15.9 | 16.1 | 16.4 | 16.6 | 16.8 |  |
| 3                        | 4.50                  | 5.91 | 6.82 | 7.50 | 8.04 | 8.48 | 8.85 | 9.18 | 9.46 | 9.72 | 9.95 | 10.2 | 10.3 | 10.5 | 10.7 | 10.8 | 11.0 | 11.1 | 11.2 |  |
| 4                        | 3.93                  | 5.04 | 5.76 | 6.29 | 6.71 | 7.05 | 7.35 | 7.60 | 7.83 | 8.03 | 8.21 | 8.37 | 8.52 | 8.66 | 8.79 | 8.91 | 9.03 | 9.13 | 9.23 |  |
| 5                        | 3.64                  | 4.60 | 5.22 | 5.67 | 6.03 | 6.33 | 6.58 | 6.80 | 6.99 | 7.17 | 7.32 | 7.47 | 7.60 | 7.72 | 7.83 | 7.93 | 8.03 | 8.12 | 8.21 |  |
| 6                        | 3.46                  | 4.34 | 4.90 | 5.30 | 5.63 | 5.90 | 6.12 | 6.32 | 6.49 | 6.65 | 6.79 | 6.92 | 7.03 | 7.14 | 7.24 | 7.34 | 7.43 | 7.51 | 7.59 |  |
| 7                        | 3.34                  | 4.16 | 4.68 | 5.06 | 5.36 | 5.61 | 5.82 | 6.00 | 6.16 | 6.30 | 6.43 | 6.55 | 6.66 | 6.76 | 6.85 | 6.94 | 7.02 | 7.10 | 7.17 |  |
| 8                        | 3.26                  | 4.04 | 4.53 | 4.89 | 5.17 | 5.40 | 5.60 | 5.77 | 5.92 | 6.05 | 6.18 | 6.29 | 6.39 | 6.48 | 6.57 | 6.65 | 6.73 | 6.80 | 6.87 |  |
| 9                        | 3.20                  | 3.95 | 4.41 | 4.76 | 5.02 | 5.24 | 5.43 | 5.59 | 5.74 | 5.87 | 5.98 | 6.09 | 6.19 | 6.28 | 6.36 | 6.44 | 6.51 | 6.58 | 6.64 |  |
| 10                       | 3.15                  | 3.88 | 4.33 | 4.65 | 4.91 | 5.12 | 5.30 | 5.46 | 5.60 | 5.72 | 5.83 | 5.93 | 6.03 | 6.11 | 6.19 | 6.27 | 6.34 | 6.40 | 6.47 |  |
| 11                       | 3.11                  | 3.82 | 4.26 | 4.57 | 4.82 | 5.03 | 5.20 | 5.35 | 5.49 | 5.61 | 5.71 | 5.81 | 5.90 | 5.98 | 6.06 | 6.13 | 6.20 | 6.27 | 6.33 |  |
| 12                       | 3.08                  | 3.77 | 4.20 | 4.51 | 4.75 | 4.95 | 5.12 | 5.27 | 5.39 | 5.51 | 5.61 | 5.71 | 5.80 | 5.88 | 5.95 | 6.02 | 6.09 | 6.15 | 6.21 |  |
| 13                       | 3.06                  | 3.73 | 4.15 | 4.45 | 4.69 | 4.88 | 5.05 | 5.19 | 5.32 | 5.43 | 5.53 | 5.63 | 5.71 | 5.79 | 5.86 | 5.93 | 5.99 | 6.05 | 6.11 |  |
| 14                       | 3.03                  | 3.70 | 4.11 | 4.41 | 4.64 | 4.83 | 4.99 | 5.13 | 5.25 | 5.36 | 5.46 | 5.55 | 5.64 | 5.71 | 5.79 | 5.85 | 5.91 | 5.97 | 6.03 |  |
| 15                       | 3.01                  | 3.67 | 4.08 | 4.37 | 4.59 | 4.78 | 4.94 | 5.08 | 5.20 | 5.31 | 5.40 | 5.49 | 5.57 | 5.65 | 5.72 | 5.78 | 5.85 | 5.90 | 5.96 |  |
| 16                       | 3.00                  | 3.65 | 4.05 | 4.33 | 4.56 | 4.74 | 4.90 | 5.03 | 5.15 | 5.26 | 5.35 | 5.44 | 5.52 | 5.59 | 5.66 | 5.73 | 5.79 | 5.84 | 5.90 |  |
| 17                       | 2.98                  | 3.63 | 4.02 | 4.30 | 4.52 | 4.70 | 4.86 | 4.99 | 5.11 | 5.21 | 5.31 | 5.39 | 5.47 | 5.54 | 5.61 | 5.67 | 5.73 | 5.79 | 5.84 |  |
| 18                       | 2.97                  | 3.61 | 4.00 | 4.28 | 4.49 | 4.67 | 4.82 | 4.96 | 5.07 | 5.17 | 5.27 | 5.35 | 5.43 | 5.50 | 5.57 | 5.63 | 5.69 | 5.74 | 5.79 |  |
| 19                       | 2.96                  | 3.59 | 3.98 | 4.25 | 4.47 | 4.65 | 4.79 | 4.92 | 5.04 | 5.14 | 5.23 | 5.31 | 5.39 | 5.46 | 5.53 | 5.59 | 5.65 | 5.70 | 5.75 |  |
| 20                       | 2.95                  | 3.58 | 3.96 | 4.23 | 4.45 | 4.62 | 4.77 | 4.90 | 5.01 | 5.11 | 5.20 | 5.28 | 5.36 | 5.43 | 5.49 | 5.55 | 5.61 | 5.66 | 5.71 |  |
| 24                       | 2.92                  | 3.53 | 3.90 | 4.17 | 4.37 | 4.54 | 4.68 | 4.81 | 4.92 | 5.01 | 5.10 | 5.18 | 5.25 | 5.32 | 5.38 | 5.44 | 5.49 | 5.55 | 5.59 |  |
| 30                       | 2.89                  | 3.49 | 3.85 | 4.10 | 4.30 | 4.46 | 4.60 | 4.72 | 4.82 | 4.92 | 5.00 | 5.08 | 5.15 | 5.21 | 5.27 | 5.33 | 5.38 | 5.43 | 5.47 |  |
| 40                       | 2.86                  | 3.44 | 3.79 | 4.04 | 4.23 | 4.39 | 4.52 | 4.63 | 4.73 | 4.82 | 4.90 | 4.98 | 5.04 | 5.11 | 5.16 | 5.22 | 5.27 | 5.31 | 5.36 |  |
| 60                       | 2.83                  | 3.40 | 3.74 | 3.98 | 4.16 | 4.31 | 4.44 | 4.55 | 4.65 | 4.73 | 4.81 | 4.88 | 4.94 | 5.00 | 5.06 | 5.11 | 5.15 | 5.20 | 5.24 |  |
| 120                      | 2.80                  | 3.36 | 3.68 | 3.92 | 4.10 | 4.24 | 4.36 | 4.47 | 4.56 | 4.64 | 4.71 | 4.78 | 4.84 | 4.90 | 4.95 | 5.00 | 5.04 | 5.09 | 5.13 |  |
| $\infty$                 | 2.77                  | 3.31 | 3.63 | 3.86 | 4.03 | 4.17 | 4.29 | 4.39 | 4.47 | 4.55 | 4.62 | 4.68 | 4.74 | 4.80 | 4.85 | 4.89 | 4.93 | 4.97 | 5.01 |  |

**TABLE 9** (Continued)

|                          |      | $\alpha = .05$        |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |    |
|--------------------------|------|-----------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----|
| Degrees<br>of<br>freedom |      | Number of populations |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |    |
|                          |      | 2                     | 3     | 4     | 5     | 6     | 7     | 8     | 9     | 10    | 11    | 12    | 13    | 14    | 15    | 16    | 17    | 18    | 19    | 20 |
| 1                        | 90.0 | 135.0                 | 164.0 | 186.0 | 202.0 | 216.0 | 227.0 | 237.0 | 246.0 | 253.0 | 260.0 | 266.0 | 272.0 | 277.0 | 282.0 | 286.0 | 290.0 | 294.0 | 298.0 |    |
| 2                        | 14.0 | 19.0                  | 22.3  | 24.7  | 26.6  | 28.2  | 29.5  | 30.7  | 31.7  | 32.6  | 33.4  | 34.1  | 34.8  | 35.4  | 36.0  | 36.5  | 37.0  | 37.5  | 37.9  |    |
| 3                        | 8.26 | 10.6                  | 12.2  | 13.3  | 14.2  | 15.0  | 15.6  | 16.2  | 16.7  | 17.1  | 17.5  | 17.9  | 18.2  | 18.5  | 18.8  | 19.1  | 19.3  | 19.5  | 19.8  |    |
| 4                        | 6.51 | 8.12                  | 9.17  | 9.96  | 10.6  | 11.1  | 11.5  | 11.9  | 12.3  | 12.6  | 12.8  | 13.1  | 13.3  | 13.5  | 13.7  | 13.9  | 14.1  | 14.2  | 14.4  |    |
| 5                        | 5.70 | 6.97                  | 7.80  | 8.42  | 8.91  | 9.32  | 9.67  | 9.97  | 10.2  | 10.5  | 10.7  | 10.9  | 11.1  | 11.2  | 11.4  | 11.6  | 11.7  | 11.8  | 11.9  |    |
| 6                        | 5.24 | 6.33                  | 7.03  | 7.56  | 7.97  | 8.32  | 8.61  | 8.87  | 9.10  | 9.30  | 9.49  | 9.65  | 9.81  | 9.95  | 10.1  | 10.2  | 10.3  | 10.4  | 10.5  |    |
| 7                        | 4.95 | 5.92                  | 6.54  | 7.01  | 7.37  | 7.68  | 7.94  | 8.17  | 8.37  | 8.55  | 8.71  | 8.86  | 9.00  | 9.12  | 9.24  | 9.35  | 9.46  | 9.55  | 9.65  |    |
| 8                        | 4.74 | 5.63                  | 6.20  | 6.63  | 6.96  | 7.24  | 7.47  | 7.68  | 7.87  | 8.03  | 8.18  | 8.31  | 8.44  | 8.55  | 8.66  | 8.76  | 8.85  | 8.94  | 9.03  |    |
| 9                        | 4.60 | 5.43                  | 5.96  | 6.35  | 6.66  | 6.91  | 7.13  | 7.32  | 7.49  | 7.65  | 7.78  | 7.91  | 8.03  | 8.13  | 8.23  | 8.32  | 8.41  | 8.49  | 8.57  |    |
| 10                       | 4.48 | 5.27                  | 5.77  | 6.14  | 6.43  | 6.67  | 6.87  | 7.05  | 7.21  | 7.36  | 7.48  | 7.60  | 7.71  | 7.81  | 7.91  | 7.99  | 8.07  | 8.15  | 8.22  |    |
| 11                       | 4.39 | 5.14                  | 5.62  | 5.97  | 6.25  | 6.48  | 6.67  | 6.84  | 6.99  | 7.13  | 7.25  | 7.36  | 7.46  | 7.56  | 7.65  | 7.73  | 7.81  | 7.88  | 7.95  |    |
| 12                       | 4.32 | 5.04                  | 5.50  | 5.84  | 6.10  | 6.32  | 6.51  | 6.67  | 6.81  | 6.94  | 7.06  | 7.17  | 7.26  | 7.36  | 7.44  | 7.52  | 7.59  | 7.66  | 7.73  |    |
| 13                       | 4.26 | 4.96                  | 5.40  | 5.73  | 5.98  | 6.19  | 6.37  | 6.53  | 6.67  | 6.79  | 6.90  | 7.01  | 7.10  | 7.19  | 7.27  | 7.34  | 7.42  | 7.48  | 7.55  |    |
| 14                       | 4.21 | 4.89                  | 5.32  | 5.63  | 5.88  | 6.08  | 6.26  | 6.41  | 6.54  | 6.66  | 6.77  | 6.87  | 6.96  | 7.05  | 7.12  | 7.20  | 7.27  | 7.33  | 7.39  |    |
| 15                       | 4.17 | 4.83                  | 5.25  | 5.56  | 5.80  | 5.99  | 6.16  | 6.31  | 6.44  | 6.55  | 6.66  | 6.76  | 6.84  | 6.93  | 7.00  | 7.07  | 7.14  | 7.20  | 7.26  |    |
| 16                       | 4.13 | 4.78                  | 5.19  | 5.49  | 5.72  | 5.92  | 6.08  | 6.22  | 6.35  | 6.46  | 6.56  | 6.66  | 6.74  | 6.82  | 6.90  | 6.97  | 7.03  | 7.09  | 7.15  |    |
| 17                       | 4.10 | 4.74                  | 5.14  | 5.43  | 5.66  | 5.85  | 6.01  | 6.15  | 6.27  | 6.38  | 6.48  | 6.57  | 6.66  | 6.73  | 6.80  | 6.87  | 6.94  | 7.00  | 7.05  |    |
| 18                       | 4.07 | 4.70                  | 5.09  | 5.38  | 5.60  | 5.79  | 5.94  | 6.08  | 6.20  | 6.31  | 6.41  | 6.50  | 6.58  | 6.65  | 6.72  | 6.79  | 6.85  | 6.91  | 6.96  |    |
| 19                       | 4.05 | 4.67                  | 5.05  | 5.33  | 5.55  | 5.73  | 5.89  | 6.02  | 6.14  | 6.25  | 6.34  | 6.43  | 6.51  | 6.58  | 6.65  | 6.72  | 6.78  | 6.84  | 6.89  |    |
| 20                       | 4.02 | 4.64                  | 5.02  | 5.29  | 5.51  | 5.69  | 5.84  | 5.97  | 6.09  | 6.19  | 6.29  | 6.37  | 6.45  | 6.52  | 6.59  | 6.65  | 6.71  | 6.76  | 6.82  |    |
| 24                       | 3.96 | 4.54                  | 4.91  | 5.17  | 5.37  | 5.54  | 5.69  | 5.81  | 5.92  | 6.02  | 6.11  | 6.19  | 6.26  | 6.33  | 6.39  | 6.45  | 6.51  | 6.56  | 6.61  |    |
| 30                       | 3.89 | 4.45                  | 4.80  | 5.05  | 5.24  | 5.40  | 5.54  | 5.65  | 5.76  | 5.85  | 5.93  | 6.01  | 6.08  | 6.14  | 6.20  | 6.26  | 6.31  | 6.36  | 6.41  |    |
| 40                       | 3.82 | 4.37                  | 4.70  | 4.93  | 5.11  | 5.27  | 5.39  | 5.50  | 5.60  | 5.69  | 5.77  | 5.84  | 5.90  | 5.96  | 6.02  | 6.07  | 6.12  | 6.17  | 6.21  |    |
| 60                       | 3.76 | 4.28                  | 4.60  | 4.82  | 4.99  | 5.13  | 5.25  | 5.36  | 5.45  | 5.53  | 5.60  | 5.67  | 5.73  | 5.79  | 5.84  | 5.89  | 5.93  | 5.98  | 6.02  |    |
| 120                      | 3.70 | 4.20                  | 4.50  | 4.71  | 4.87  | 5.01  | 5.12  | 5.21  | 5.30  | 5.38  | 5.44  | 5.51  | 5.56  | 5.61  | 5.66  | 5.71  | 5.75  | 5.79  | 5.83  |    |
| $\infty$                 | 3.64 | 4.12                  | 4.40  | 4.60  | 4.76  | 4.88  | 4.99  | 5.08  | 5.16  | 5.23  | 5.29  | 5.35  | 5.40  | 5.45  | 5.49  | 5.54  | 5.57  | 5.61  | 5.65  |    |

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# GLOSSARY



**Acceptance criterion** The maximum number of defective items that can be found in the sample and still indicate an acceptable lot.

**Acceptance sampling** A statistical method in which the number of defective items found in a sample is used to determine whether a lot should be accepted or rejected.

**Addition law** A probability law used to compute the probability of the union of two events. It is  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ . For mutually exclusive events,  $P(A \cap B) = 0$ ; in this case the addition law reduces to  $P(A \cup B) = P(A) + P(B)$ .

**Additive decomposition model** In an additive model the values for the Trend, Seasonal and Irregular components are simply added together to obtain the actual time series value,  $Y_t$ .

**Adjusted multiple coefficient of determination** A measure of the goodness of fit of the estimated multiple regression equation that adjusts for the number of independent variables in the model and thus avoids overestimating the impact of adding more independent variables.

**Aggregate price index** A composite price index based on the prices of a group of items.

**Alternative hypothesis** The hypothesis concluded to be true if the null hypothesis is rejected.

**ANOVA table** A table used to summarize the analysis of variance computations and results. It contains columns showing the source of variation, the sum of squares, the degrees of freedom, the mean square and the  $F$  value(s).

**Assignable causes** Variations in process outputs that are due to factors such as machine tools wearing out, incorrect machine settings, poor-quality raw materials, operator error and so on. Corrective action should be taken when assignable causes of output variation are detected.

**Autocorrelation** Correlation in the errors that arises when the error terms at successive points in time are related.

**Bar graph, Bar chart** A graphical device for depicting qualitative data that have been summarized in a frequency, relative frequency or percentage frequency distribution.

**Basic requirements for assigning probabilities** Two requirements that restrict the manner in which probability assignments can be made: (1) for each experimental outcome  $E_i$  we must have  $0 \leq P(E_i) \leq 1$ ; (2) considering all experimental outcomes, we must have  $P(E_1) + P(E_2) + \dots + P(E_n) = 1.0$ .

**Bayes' theorem** A theorem that enables the use of sample information to revise prior probabilities.

**Binomial experiment** An experiment having the four properties stated at the beginning of Section 5.4.

**Binomial probability distribution** A probability distribution showing the probability of  $x$  successes in  $n$  trials of binomial experiments.

**Binomial probability function** The function used to compute binomial probabilities.

**Blocking** The process of using the same or similar experimental units for all treatments. The purpose of blocking is to remove a source of variation from the error term and hence provide a more powerful test for a difference in population or treatment means.

**Bound on the sampling error** A number added to and subtracted from a point estimate to create an approximate 95 per cent confidence interval. It is given by two times the standard error of the point estimator.

**Box plot** A graphical summary of data based on a five-number summary.

**Branch** Lines showing the alternatives from decision nodes and the outcomes from chance nodes.

**Categorical data** Non-numeric data which include labels or names used to identify an attribute of each element of a data set.

**Categorical variable** A variable with categorical data.

**Causal forecasting methods** Forecasting methods that relate a time series to other variables that are believed to explain or cause its behaviour.

**Census** A survey to collect data on the entire population.

**Central limit theorem** A theorem that enables one to use the normal probability distribution to approximate the sampling distribution of  $\bar{X}$  when the sample size is large.

**Chance event** An uncertain future event affecting the consequence, or payoff, associated with a decision.

**Chance nodes** Nodes indicating points where an uncertain event will occur.

**Chebyshev's theorem** A theorem that can be used to make statements about the proportion of data values that must be within a specified number of standard deviations of the mean.

**Class midpoint** The value halfway between the lower and upper class limits in a frequency distribution.

**Classical method** A method of assigning probabilities that is appropriate when all the experimental outcomes are equally likely.

**Cluster sampling** A probabilistic method of sampling in which the population is first divided into clusters and then one or more clusters are selected for sampling. In single-stage cluster sampling, every element in each selected cluster is sampled; in two-stage cluster sampling, a sample of the elements in each selected cluster is collected.

**Coefficient of determination** A measure of the goodness of fit of the estimated regression equation. It can be interpreted as the proportion of the variability in the dependent variable  $y$  that is explained by the estimated regression equation.

**Coefficient of variation** A measure of relative variability computed by dividing the standard deviation by the mean and multiplying by 100.

**Common causes** Normal or natural variations in process outputs that are due purely to chance. No corrective action is necessary when output variations are due to common causes.

**Comparisonwise Type I error rate** The probability of a Type I error associated with a single pairwise comparison.

**Complement of A** The event consisting of all sample points that are not in  $A$ .

**Completely randomized design** An experimental design in which the treatments are randomly assigned to the experimental units.

**Conditional probability** The probability of an event given that another event already occurred. The conditional probability of  $A$  given  $B$  is  $P(A | B) = P(A \cap B)/P(B)$ .

**Confidence coefficient** The confidence level expressed as a decimal value. For example, 0.95 is the confidence coefficient for a 95 per cent confidence level.

**Confidence interval** The interval estimate of the mean value of  $Y$  for a given value of  $X$ .

**Confidence level** The confidence associated with an interval estimate. For example, if an interval estimation procedure provides intervals such that 95 per cent of the intervals formed using the procedure will include the population parameter, the interval estimate is said to be constructed at the 95 per cent confidence level.

**Consequence** The result obtained when a decision alternative is chosen and a chance event occurs. A measure of the consequence is often called a payoff.

**Consumer Price Index** A price index that uses the price changes in a market basket of consumer goods and services to measure the changes in consumer prices over time.

**Consumer's risk** The risk of accepting a poor-quality lot; a Type II error.

**Contingency table** A frequency table resulting from the cross-classification of two or more categorical variables.

**Continuity correction factor** A value of 0.5 that is added to or subtracted from a value of  $X$  when the continuous normal distribution is used to approximate the discrete binomial distribution.

**Continuous random variable** A random variable that may assume any numerical value in an interval or collection of intervals.

**Control chart** A graphical tool used to help determine whether a process is in control or out of control.

**Convenience sampling** A non-probabilistic method of sampling whereby elements are selected on the basis of convenience.

**Cook's distance measure** A measure of the influence of an observation based on both the leverage of observation  $i$  and the residual for observation  $i$ .

**Correlation coefficient** A measure of association between two variables that takes on values between  $-1$  and  $+1$ . Values near  $+1$  indicate a strong positive relationship, values near  $-1$  indicate a strong negative relationship. Values near zero indicate the lack of a relationship. Pearson's product-moment correlation coefficient measures linear association between two variables.

**Covariance** A measure of linear association between two variables. Positive values indicate a positive relationship; negative values indicate a negative relationship.

**Critical value** A value that is compared with the test statistic to determine whether  $H_0$  should be rejected.

**Cross-sectional data** Data collected at the same or approximately the same point in time.

**Cross-tabulation** A tabular summary of data for two variables. The classes for one variable are represented by the rows; the classes for the other variable are represented by the columns.

**Cumulative frequency distribution** A tabular summary of quantitative data showing the number of items with values less than or equal to the upper class limit of each class.

**Cumulative percentage frequency distribution** A tabular summary of quantitative data showing the percentage of items with values less than or equal to the upper class limit of each class.

**Cumulative relative frequency distribution** A tabular summary of quantitative data showing the fraction or proportion of items with values less than or equal to the upper class limit of each class.

**Cyclical component** The component of the time series that results in periodic above-trend and below-trend behaviour of the time series lasting more than one year.

**Cyclical pattern** One that shows an alternating sequence of points below and above a trend line lasting more than one year.

**Data** The facts and figures collected, analyzed and summarized for presentation and interpretation.

**Data mining** The process of converting data in a warehouse into useful information using a combination of procedures from statistics, mathematics and computer science.

**Data set** All the data collected in a particular study.

**Decision nodes** Nodes indicating points where a decision is made.

**Decision strategy** A strategy involving a sequence of decisions and chance outcomes to provide the optimal solution to a decision problem.

**Decision tree** A graphical representation of the decision problem that shows the sequential nature of the decision-making process.

**Degrees of freedom** A parameter of the  $t$  distribution. When the  $t$  distribution is used in the computation of an interval estimate of a population mean, the appropriate  $t$  distribution has  $n - 1$  degrees of freedom, where  $n$  is the size of the simple random sample. (Also a parameter of the  $\chi^2$  distribution.)

**Dependent variable** The variable that is being predicted or explained. It is denoted by  $Y$ .

**Descriptive statistics** Tabular, graphical and numerical summaries of data.

**Deseasonalized time series** A time series from which the effect of season has been removed by dividing each original time series observation by the corresponding seasonal index.

**Discrete random variable** A random variable that may assume either a finite number of values or an infinite sequence of values.

**Discrete uniform probability distribution** A probability distribution for which each possible value of the random variable has the same probability.

**Distribution-free methods** Statistical methods that make no assumption about the distributional form of the population.

**Dot plot** A graphical device that summarizes data by the number of dots above each data value on the horizontal axis.

**Dummy variable** A variable used to model the effect of qualitative independent variables. A dummy variable may take only the value zero or one.

**Durbin–Watson test** A test to determine whether first-order correlation is present.

**Element** The entity on which data are collected.

**Empirical rule** A rule that can be used to approximate the percentage of data values that are within one, two and three standard deviations of the mean for data that exhibit a bell-shaped distribution.

**Estimated logistic regression equation** The estimate of the logistic regression equation based on sample data: that is  $\hat{y} = \text{estimate}$  of

$$P(Y = 1 | x_1, x_2, \dots, x_p) = \frac{e^{\beta_0 + \beta_1 x_1 + \dots + \beta_p x_p}}{1 + e^{\beta_0 + \beta_1 x_1 + \dots + \beta_p x_p}}$$

**Estimated logit** An estimate of the logit based on sample data: that is,

$$\hat{g}(x_1, x_2, \dots, x_p) = b_0 + b_1 x_1 + b_2 x_2 + \dots + b_p x_p$$

**Estimated multiple regression equation** The estimate of the multiple regression equation based on sample data and the least squares method: it is

$$\hat{y} = b_0 + b_1 x_1 + b_2 x_2 + \dots + b_p x_p$$

**Estimated regression equation** The estimate of the regression equation developed from sample data by using the least squares method. For simple linear regression, the estimated regression equation is

$$\hat{y} = b_0 + b_1 x$$

**Event** A collection of sample points.

**Expected value** A measure of the central location of a random variable. For a chance node, it is the weighted average of the payoffs. The weights are the state-of-nature probabilities.

**Expected value approach** An approach to choosing a decision alternative that is based on the expected value of each decision alternative. The recommended decision alternative is the one that provides the best expected value.

**Expected value of perfect information (EVPI)** The expected value of information that would tell the decision-maker exactly which state of nature is going to occur (i.e. perfect information).

**Expected value of sample information (EVSI)** The difference between the expected value of an optimal strategy based on sample information and the ‘best’ expected value without any sample information.

**Experiment** A process that generates well-defined outcomes.

**Experimental units** The objects of interest in the experiment.

**Experimentwise Type I error rate** The probability of making a Type I error on at least one of several pairwise comparisons.

**Exploratory data analysis** Methods that use simple arithmetic and easy-to-draw graphs to summarize data quickly.

**Exponential probability distribution** A continuous probability distribution that is useful in computing probabilities for the time it takes to complete a task.

**Exponential smoothing** A forecasting technique that uses a weighted average of past time series values as the forecast.

**Factor** Another word for the independent variable of interest.

**Factorial experiment** An experimental design that allows statistical conclusions about two or more factors.

**Finite population correction factor** The term  $\sqrt{(N-n)/(N-1)}$  that is used in the formulae for  $\sigma_x$  and  $\sigma_p$  when a finite population, rather than an infinite population, is being sampled. The generally accepted rule of thumb is to ignore the finite population correction factor whenever  $n/N \leq 0.05$ .

**Five-number summary** An exploratory data analysis technique that uses five numbers to summarize the data: smallest value, first quartile, median, third quartile and largest value.

**Forecast** A prediction of future values of a time series.

**Forecast error** The forecast error is the difference between the actual value of a time series and its forecast.

**Frame** A list of the sampling units for a study. The sample is drawn by selecting units from the frame.

**Frequency distribution** A tabular summary of data showing the number (frequency) of items in each of several non-overlapping classes.

**General linear model** A model of the form  $Y = \beta_0 + \beta_1 z_1 + \beta_2 z_2 + \dots + \beta_p z_p + \varepsilon$ , where each of the independent variables  $z_j$  ( $j = 1, 2, \dots, p$ ) is a function of  $x_1, x_2, \dots, x_k$ , the variables for which data have been collected.

**Goodness of fit test** A statistical test conducted to determine whether to reject a hypothesized probability distribution for a population.

**Grouped data** Data available in class intervals as summarized by a frequency distribution. Individual values of the original data are not available.

**High leverage points** Observations with extreme values for the independent variables.

**Histogram** A graphical presentation of a frequency distribution, relative frequency distribution or percentage frequency distribution of quantitative data, constructed by placing the class intervals on the horizontal axis and the frequencies, relative frequencies or percentage frequencies on the vertical axis.

**Horizontal pattern** A horizontal pattern exists when the data fluctuate around a constant mean.

**Hypergeometric probability distribution** A probability distribution showing the probability of  $x$  successes in  $n$  trials from a population with  $r$  successes and  $N - r$  failures.

**Hypergeometric probability function** The function used to compute hypergeometric probabilities.

**Independent events** Two events  $A$  and  $B$  where  $P(A \mid B) = P(A)$  or  $P(B \mid A) = P(B)$ ; that is, the events have no influence on each other.

**Independent samples** Where, e.g., two groups of workers are selected and each group uses a different method to collect production time data.

**Independent variable** The variable that is doing the predicting or explaining. It is denoted by  $X$ .

**Influential observation** An observation that has a strong influence or effect on the regression results.

**Interaction** The effect of two independent variables acting together.

**Interquartile range (IQR)** A measure of variability, defined to be the difference between the third and first quartiles.

**Intersection of A and B** The event containing the sample points belonging to both  $A$  and  $B$ . The intersection is denoted  $A \cap B$ .

**Interval estimate** An estimate of a population parameter that provides an interval believed to contain the value of the parameter.

**Interval scale** The scale of measurement for a variable if the data demonstrate the properties of ordinal data and the interval between values is expressed in terms of a fixed unit of measure. Interval data are always numeric.

**Irregular component** The component of the time series that reflects the random variation of the time series values beyond what can be explained by the trend, cyclical and seasonal components.

**$i$ th residual** The difference between the observed value of the dependent variable and the value predicted using the

estimated regression equation; for the  $i$ th observation the  $i$ th residual is  $y_i - \hat{y}_i$ .

**Joint probability** The probability of two events both occurring; that is, the probability of the intersection of two events.

**Judgement sampling** A non-probabilistic method of sampling whereby element selection is based on the judgement of the person doing the study.

**Kruskal-Wallis test** A non-parametric test for identifying differences among three or more populations on the basis of independent samples.

**Laspeyres' price index** A weighted aggregate price index in which the weight for each item is its base-period quantity.

**Least squares method** The method used to develop the estimated regression equation. It minimizes the sum of squared residuals (the deviations between the observed values of the dependent variable,  $y$ , and the estimated values of the dependent variable,  $\hat{y}_i$ ).

**Level of significance** The probability of making a Type I error when the null hypothesis is true as an equality.

**Leverage** A measure of how far the values of the independent variables are from their mean values.

**Linear exponential smoothing** This is a version of exponential smoothing that can be used to forecast a time series with a linear trend.

**Logistic regression equation** The mathematical equation relating  $E(Y)$ , the probability that  $Y = 1$ , to the values of the independent variables: that is,

$$E(Y) = P(Y = 1|x_1, x_2, \dots, x_p) = \frac{e^{\beta_0 + \beta_1 x_1 + \dots + \beta_p x_p}}{1 + e^{\beta_0 + \beta_1 x_1 + \dots + \beta_p x_p}}$$

**Logit** The natural logarithm of the odds in favour of  $Y = 1$ : that is,  $g(x_1, x_2, \dots, x_p) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p$

**Lot** A group of items such as incoming shipments of raw materials or purchased parts as well as finished goods from final assembly.

**Mann-Whitney-Wilcoxon (MWW) test** A non-parametric statistical test for identifying differences between two populations based on the analysis of two independent samples.

**Margin of error** The value added to and subtracted from a point estimate in order to construct an interval estimate of a population parameter.

**Marginal probability** The values in the margins of a joint probability table that provide the probabilities of each event separately.

**Matched samples** Where, e.g., only a sample of workers is selected and each worker uses first one and then the other method, with each worker providing a pair of data values.

**Mean** A measure of central location computed by summing the data values and dividing by the number of observations.

**Mean absolute error (MAE)** The average of the absolute forecast errors.

**Mean absolute percentage error (MAPE)** The average of the ratios of absolute forecast errors to actual values expressed as a percentage.

**Mean squared error (MSE)** A measure of the accuracy of a forecasting method. This measure is the average of the sum of the squared differences between the forecast values and the actual time series values.

**Median** A measure of central location provided by the value in the middle when the data are arranged in ascending order.

**Mode** A measure of location, defined as the value that occurs with greatest frequency.

**Moving averages** A method of forecasting or smoothing a time series that uses the average of the most recent  $n$  data values in the time series as the forecast for the next period.

**Multicollinearity** The term used to describe the correlation among the independent variables.

**Multinomial population** A population in which each element is assigned to one and only one of several categories. The multinomial distribution extends the binomial distribution from two to three or more outcomes.

**Multiple coefficient of determination** A measure of the goodness of fit of the estimated multiple regression equation. It can be interpreted as the proportion of the variability in the dependent variable that is explained by the estimated regression equation.

**Multiple comparison procedures** Statistical procedures that can be used to conduct statistical comparisons between pairs of population means.

**Multiple regression analysis** Regression analysis involving two or more independent variables.

**Multiple regression equation** The mathematical equation relating the expected value or mean value of the dependent variable to the values of the independent variables; that is

$$E(Y) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p$$

**Multiple regression model** The mathematical equation that describes how the dependent variable  $Y$  is related to the independent variables  $x_1, x_2, \dots, x_p$  and an error term  $\varepsilon$ .

**Multiple sampling plan** A form of acceptance sampling in which more than one sample or stage is used. On the basis of the number of defective items found in a sample, a decision will be made to accept the lot, reject the lot or continue sampling.

**Multiplication law** A probability law used to compute the probability of the intersection of two events. It is  $P(A \cap B) = P(B|A)P(A)$  or  $P(A \cap B) = P(A)P(B|A)$ . For independent events it reduces to  $P(A \cap B) = P(A)P(B)$ .

**Multiplicative decomposition model** In an multiplicative model the values for the trend, seasonal and irregular components are simply multiplied together to obtain the actual time series value.

**Mutually exclusive events** Events that have no sample points in common: that is,  $A \cap B$  is empty and  $P(A \cap B) = 0$ .

**Node** An intersection or junction point of an influence diagram or a decision tree.

**Nominal scale** The scale of measurement for a variable when the data use labels or names to identify an attribute of an element. Nominal data may be non-numeric or numeric.

**Non-parametric methods** Statistical methods that require relatively few assumptions about the population probability distributions and about the level of measurement. These methods can be applied when nominal or ordinal data are available.

**Non-probabilistic sampling** Any method of sampling for which the probability of selecting a sample of any given configuration cannot be computed.

**Non-sampling error** All types of errors other than sampling error, such as measurement error, interviewer error and processing error.

**Normal probability distribution** A continuous probability distribution. Its probability density function is bell shaped and determined by its mean  $\mu$  and standard deviation  $\sigma$ .

**Normal probability plot** A graph of the standardized residuals plotted against values of the normal scores. This plot helps

- determine whether the assumption that the error term has a normal probability distribution appears to be valid.
- np chart** A control chart used to monitor the quality of the output of a process in terms of the number of defective items.
- Null hypothesis** The hypothesis tentatively assumed true in the hypothesis testing procedure.
- Observation** The set of measurements obtained for a particular element.
- Odds in favour of an event occurring** The probability the event will occur divided by the probability the event will not occur.
- Odds ratio** The odds that  $Y=1$  given that one of the independent variables increased by one unit ( $\text{odds}_1$ ) divided by the odds that  $Y=1$  given no change in the values for the independent variables ( $\text{odds}_0$ ): that is, Odds ratio =  $\text{odds}_1/\text{odds}_0$ .
- Ogive** A graph of a cumulative distribution.
- One-tailed test** A hypothesis test in which rejection of the null hypothesis occurs for values of the test statistic in one tail of its sampling distribution.
- Operating characteristic curve** A graph showing the probability of accepting the lot as a function of the percentage defective in the lot. This curve can be used to help determine whether a particular acceptance sampling plan meets both the producer's and the consumer's risk requirements.
- Ordinal scale** The scale of measurement for a variable if the data exhibit the properties of nominal data and the order or rank of the data is meaningful. Ordinal data may be non-numeric or numeric.
- Outlier** A data point or observation that does not fit the pattern shown by the remaining data, often unusually small or unusually large.
- p chart** A control chart used when the quality of the output of a process is measured in terms of the proportion defective.
- p-value** A probability, computed using the test statistic, that measures the support (or lack of support) provided by the sample for the null hypothesis. For a lower tail test, the p-value is the probability of obtaining a value for the test statistic at least as small as that provided by the sample. For an upper tail test, the p-value is the probability of obtaining a value for the test statistic at least as large as that provided by the sample. For a two-tailed test, the p-value is the probability of obtaining a value for the test statistic at least as unlikely as that provided by the sample.
- Paasche price index** A weighted aggregate price index in which the weight for each item is its current-period quantity.
- Parameter** A numerical characteristic of a population, such as a population mean  $\mu$ , a population standard deviation  $\sigma$ , a population proportion  $\pi$  and so on.
- Parametric methods** Statistical methods that begin with an assumption about the distributional shape of the population. This is often that the population follows a normal distribution.
- Partitioning** The process of allocating the total sum of squares and degrees of freedom to the various components.
- Payoff** A measure of the consequence of a decision such as profit, cost or time. Each combination of a decision alternative and a state of nature has an associated payoff (consequence).
- Payoff table** A tabular representation of the payoffs for a decision problem.
- Percentage frequency distribution** A tabular summary of data showing the percentage of items in each of several non-overlapping classes.
- Percentile** A value such that at least  $p$  per cent of the observations are less than or equal to this value and at least  $(100 - p)$  per cent of the observations are greater than or equal to this value. The 50th percentile is the median.
- Pie chart** A graphical device for presenting data summaries based on subdivision of a circle into sectors that correspond to the relative frequency for each class.
- Point estimate** The value of a point estimator used in a particular instance as an estimate of a population parameter.
- Point estimator** The sample statistic, such as  $\bar{X}$ ,  $S$  or  $P$ , that provides the point estimate of the population parameter.
- Poisson probability distribution** A probability distribution showing the probability of  $x$  occurrences of an event over a specified interval of time or space.
- Poisson probability function** The function used to compute Poisson probabilities.
- Pooled estimator of  $\pi$**  A weighted average of  $P_1$  and  $P_2$ .
- Population** The set of all elements of interest in a particular study.
- Population parameter** A numerical value used as a summary measure for a population (e.g. the population mean  $\mu$ , the population variance  $\sigma^2$  and the population standard deviation  $\sigma$ ).
- Posterior probabilities** Revised probabilities of events based on additional information.
- Posterior (revised) probabilities** The probabilities of the states of nature after revising the prior probabilities based on sample information.
- Power** The probability of correctly rejecting  $H_0$  when it is false.
- Power curve** A graph of the probability of rejecting  $H_0$  for all possible values of the population parameter not satisfying the null hypothesis. The power curve provides the probability of correctly rejecting the null hypothesis.
- Prediction interval** The interval estimate of an individual value of  $Y$  for a given value of  $X$ .
- Price relative** A price index for a given item that is computed by dividing a current unit price by a base-period unit price and multiplying the result by 100.
- Prior probabilities** The probabilities of the states of nature prior to obtaining sample information.
- Probabilistic sampling** Any method of sampling for which the probability of each possible sample can be computed.
- Probability** A numerical measure of the likelihood that an event will occur.
- Probability density function** A function used to compute probabilities for a continuous random variable. The area under the graph of a probability density function over an interval represents probability.
- Probability distribution** A description of how the probabilities are distributed over the values of the random variable.
- Probability function** A function, denoted by  $p(x)$ , that provides the probability that  $X$  assumes a particular value for a discrete random variable.
- Producer Price Index** A price index designed to measure changes in prices of goods sold in primary markets (i.e. first purchase of a commodity in non-retail markets).
- Producer's risk** The risk of rejecting a good-quality lot; a Type I error.
- Qualitative data** Labels or names used to identify an attribute of each element. Qualitative data use either the nominal or ordinal scale of measurement and may be non-numeric or numeric.
- Qualitative independent variable** An independent variable with qualitative data.
- Qualitative variable** A variable with qualitative data.
- Quality control** A series of inspections and measurements that determine whether quality standards are being met.

- Quantitative data** Numerical values that indicate how much or how many of something.
- Quantitative variable** A variable with quantitative data.
- Quantity index** An index designed to measure changes in quantities over time.
- Quartiles** The 25th, 50th and 75th percentiles, referred to as the first quartile, the second quartile (median) and third quartile, respectively. The quartiles can be used to divide a data set into four parts, with each part containing approximately 25 per cent of the data.
- R chart** A control chart used when the quality of the output of a process is measured in terms of the range of a variable.
- Random variable** A numerical description of the outcome of an experiment.
- Randomized block design** An experimental design employing blocking.
- Range** A measure of variability, defined to be the largest value minus the smallest value.
- Ratio scale** The scale of measurement for a variable if the data demonstrate all the properties of interval data and the ratio of two values is meaningful. Ratio data are always numeric.
- Regression equation** The equation that describes how the mean or expected value of the dependent variable is related to the independent variable; in simple linear regression,  $E(Y) = \beta_0 + \beta_1 X$
- Regression model** The equation describing how  $Y$  is related to  $X$  and an error term; in simple linear regression, the regression model is  $y = \beta_0 + \beta_1 x + \varepsilon$
- Relative frequency distribution** A tabular summary of data showing the fraction or proportion of data items in each of several non-overlapping classes.
- Relative frequency method** A method of assigning probabilities that is appropriate when data are available to estimate the proportion of the time the experimental outcome will occur if the experiment is repeated a large number of times.
- Replications** The number of times each experimental condition is repeated in an experiment.
- Residual analysis** The analysis of the residuals used to determine whether the assumptions made about the regression model appear to be valid. Residual analysis is also used to identify outliers and influential observations.
- Residual plot** Graphical representation of the residuals that can be used to determine whether the assumptions made about the regression model appear to be valid.
- Response variable** Another term for dependent variable.
- Sample** A subset of the population.
- Sample information** New information obtained through research or experimentation that enables an updating or revision of the state-of-nature probabilities.
- Sample point** An element of the sample space. A sample point represents an experimental outcome.
- Sample space** The set of all experimental outcomes.
- Sample statistic** A numerical value used as a summary measure for a sample (e.g. the sample mean  $\bar{X}$ , the sample variance  $S^2$  and the sample standard deviation  $S$ ).
- Sample survey** A survey to collect data on a sample.
- Sampled population** The population from which the sample is taken.
- Sampling distribution** A probability distribution consisting of all possible values of a sample statistic.
- Sampling error** The error that occurs because a sample, and not the entire population, is used to estimate a population parameter.
- Sampling frame** A list of the sampling units for a study. The sample is drawn by selecting units from the sampling frame.
- Sampling unit** The units selected for sampling. A sampling unit may include several elements.
- Sampling with replacement** Once an element has been included in the sample, it is returned to the population. A previously selected element can be selected again and therefore may appear in the sample more than once.
- Sampling without replacement** Once an element has been included in the sample, it is removed from the population and cannot be selected a second time.
- Scatter diagram** A graphical presentation of the relationship between two quantitative variables. One variable is shown on the horizontal axis and the other variable is shown on the vertical axis.
- Seasonal component** The component of the time series that shows a periodic pattern over one year or less.
- Seasonal pattern** The same repeating pattern in observations over successive periods of time.
- Serial correlation** Same as autocorrelation.
- $\sigma$  (sigma) known** The condition existing when historical data or other information provide a good estimate or value for the population standard deviation prior to taking a sample. The interval estimation procedure uses this known value of  $\sigma$  in computing the margin of error.
- $\sigma$  (sigma) unknown** The condition existing when no good basis exists for estimating the population standard deviation prior to taking the sample. The interval estimation procedure uses the sample standard deviation  $S$  in computing the margin of error.
- Sign test** A non-parametric statistical test for identifying differences between two populations based on the analysis of nominal data.
- Simple linear regression** Regression analysis involving one independent variable and one dependent variable in which the relationship between the variables is approximated by a straight line.
- Simple random sampling** Finite population: a sample selected such that each possible sample of size  $n$  has the same probability of being selected. Infinite population: a sample selected such that each element comes from the same population and the elements are selected independently.
- Simpson's paradox** Conclusions drawn from two or more separate cross-tabulations that can be reversed when the data are aggregated into a single cross-tabulation.
- Single-factor experiment** An experiment involving only one factor with  $k$  populations or treatments.
- Skewness** A measure of the shape of a data distribution. Data skewed to the left result in negative skewness; a symmetrical data distribution results in zero skewness; and data skewed to the right result in positive skewness.
- Smoothing constant** A parameter of the exponential smoothing model that provides the weight given to the most recent time series value in the calculation of the forecast value.
- Spearman rank-correlation coefficient** A correlation measure based on rank-ordered data for two variables.
- Standard deviation** A measure of variability computed by taking the positive square root of the variance.
- Standard error** The standard deviation of a point estimator.
- Standard error of the estimate** The square root of the mean square error, denoted by  $s$ . It is the estimate of  $\sigma$ , the standard deviation of the error term  $\varepsilon$ .
- Standard normal probability distribution** A normal distribution with a mean of zero and a standard deviation of one.

- Standardized residual** The value obtained by dividing a residual by its standard deviation.
- States of nature** The possible outcomes for chance events that affect the payoff associated with a decision alternative.
- Stationary time series** One whose statistical properties are independent of time.
- Statistical inference** The process of using data obtained from a sample to make estimates or test hypotheses about the characteristics of a population.
- Statistics** The art and science of collecting, analyzing, presenting and interpreting data.
- Stem-and-leaf display** An exploratory data analysis technique that simultaneously rank orders quantitative data and provides insight about the shape of the distribution.
- Stratified random sampling** A probabilistic method of selecting a sample in which the population is first divided into strata and a simple random sample is then taken from each stratum.
- Studentized deleted residuals** Standardized residuals that are based on a revised standard error of the estimate obtained by deleting observation  $i$  from the data set and then performing the regression analysis and computations.
- Subjective method** A method of assigning probabilities on the basis of judgement.
- Systematic sampling** A method of choosing a sample by randomly selecting the first element and then selecting every  $k$ th element thereafter.
- t distribution** A family of probability distributions that can be used to develop an interval estimate of a population mean whenever the population standard deviation  $\sigma$  is unknown and is estimated by the sample standard deviation  $s$ .
- Target population** The population about which inferences are made.
- Test statistic** A statistic whose value helps determine whether a null hypothesis can be rejected.
- Time series** A set of observations on a variable measured at successive points in time or over successive periods of time.
- Time series data** Data collected over several time periods.
- Time series decomposition** This technique can be used to separate or decompose a time series into seasonal, trend and irregular components.
- Time series plot** A graphical presentation of the relationship between time and the time series variable; time is on the horizontal axis and the time series values are shown on the vertical axis.
- Treatments** Different levels of a factor.
- Tree diagram** A graphical representation that helps in visualizing a multiple-step experiment.
- Trend** The long-run shift or movement in the time series observable over several periods of time.
- Trend line** A line that provides an approximation of the relationship between two variables.
- Trend pattern** Gradual shifts or movements to relatively higher or lower values over a longer period of time.
- Two-tailed test** A hypothesis test in which rejection of the null hypothesis occurs for values of the test statistic in either tail of its sampling distribution.
- Type I error** The error of rejecting  $H_0$  when it is true.
- Type II error** The error of accepting  $H_0$  when it is false.
- Unbiasedness** A property of a point estimator that is present when the expected value of the point estimator is equal to the population parameter it estimates.
- Uniform probability distribution** A continuous probability distribution for which the probability that the random variable will assume a value in any interval is the same for each interval of equal length.
- Union of A and B** The event containing all sample points belonging to A or B or both. The union is denoted  $A \cup B$ .
- Variable** A characteristic of interest for the elements.
- Variable selection procedures** Methods for selecting a subset of the independent variables for a regression model.
- Variance** A measure of variability based on the squared deviations of the data values about the mean.
- Variance inflation factor** A measure of how correlated an independent variable is with all other independent predictors in a multiple regression model.
- Venn diagram** A graphical representation for showing symbolically the sample space and operations involving events in which the sample space is represented by a rectangle and events are represented as circles within the sample space.
- Weighted aggregate price index** A composite price index in which the prices of the items in the composite are weighted by their relative importance.
- Weighted mean** The mean obtained by assigning each observation a weight that reflects its importance.
- Weighted moving averages** A method of forecasting or smoothing a time series by computing a weighted average of past data values. The sum of the weights must equal one.
- Wilcoxon signed-rank test** A non-parametric statistical test for identifying differences between two populations based on the analysis of two matched or paired samples.
- $\bar{x}$  chart** A control chart used when the quality of the output of a process is measured in terms of the mean value of a variable such as a length, weight, temperature and so on.
- z-score** A value computed by dividing the deviation about the mean ( $x_i - \bar{x}$ ) by the standard deviation  $s$ . A z-score is referred to as a standardized value and denotes the number of standard deviations  $x_i$  is from the mean.

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