

# Digital Signal Processing

Module 3: from Euclid to Hilbert

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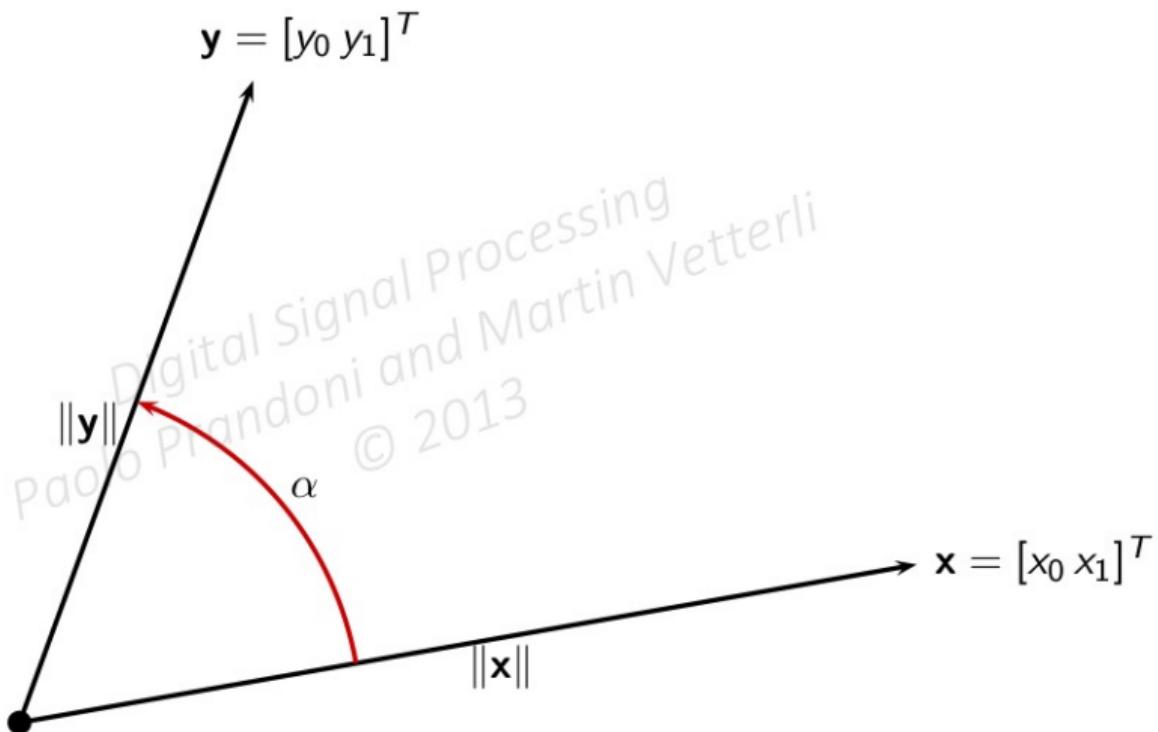
- ▶ **Module 3.1:** Signal processing as geometry or from Euclid to Hilbert spaces
- ▶ **Module 3.2:** Vectors, vector spaces, inner products, and Hilbert spaces
- ▶ **Module 3.3:** Bases for Hilbert spaces

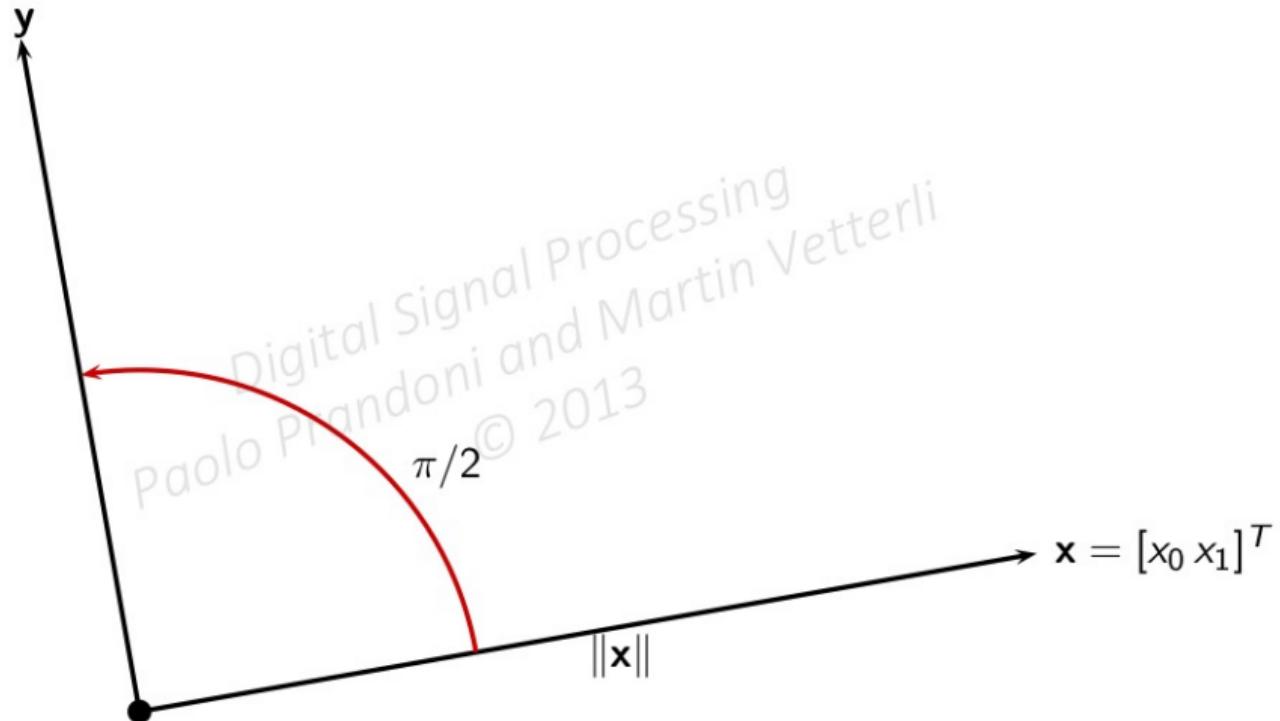
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# Digital Signal Processing

Module 3.1: a tale of two (and more) vectors

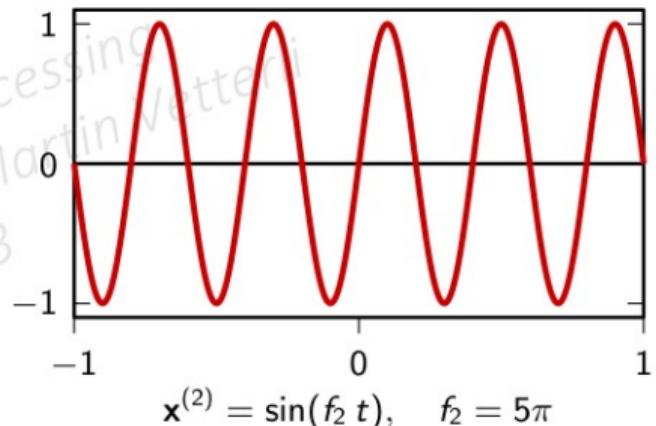
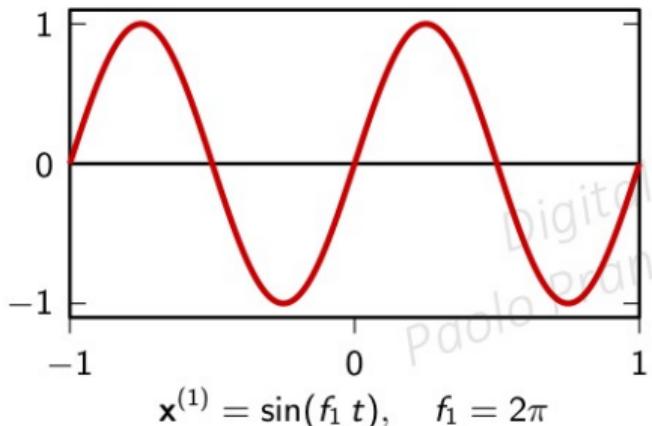
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# Vectors can be very general objects!

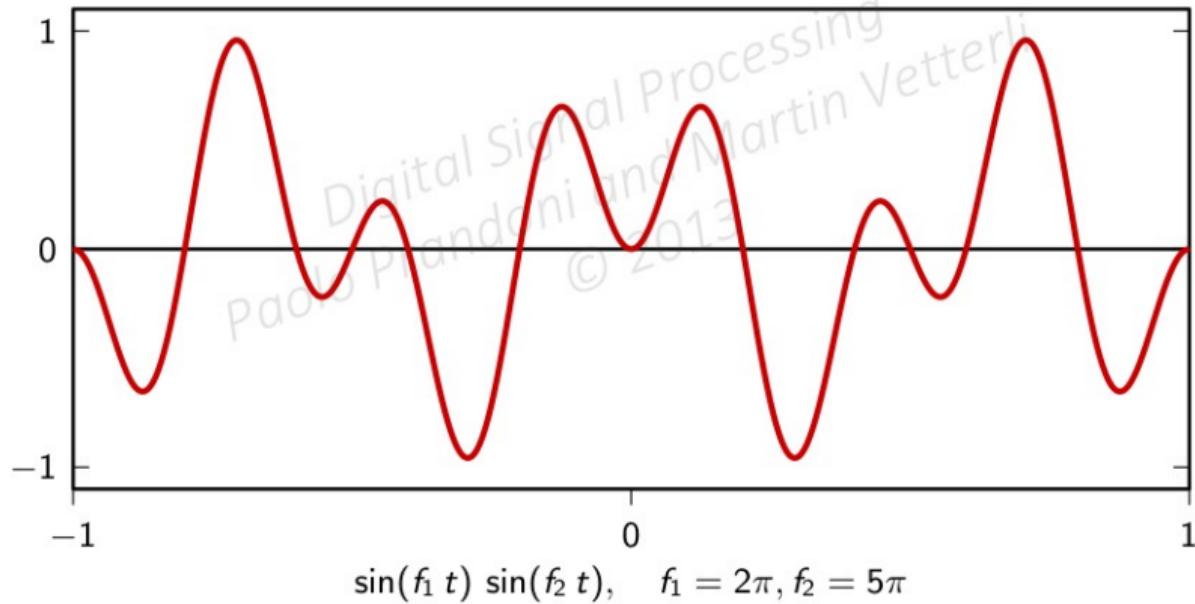
Example: space of square-integrable functions over  $[-1, 1]$ :  $L_2([-1, 1])$



$$\langle \mathbf{x}^{(1)}, \mathbf{x}^{(2)} \rangle = \int_{-1}^1 \sin(f_1 t) \sin(f_2 t) dt$$

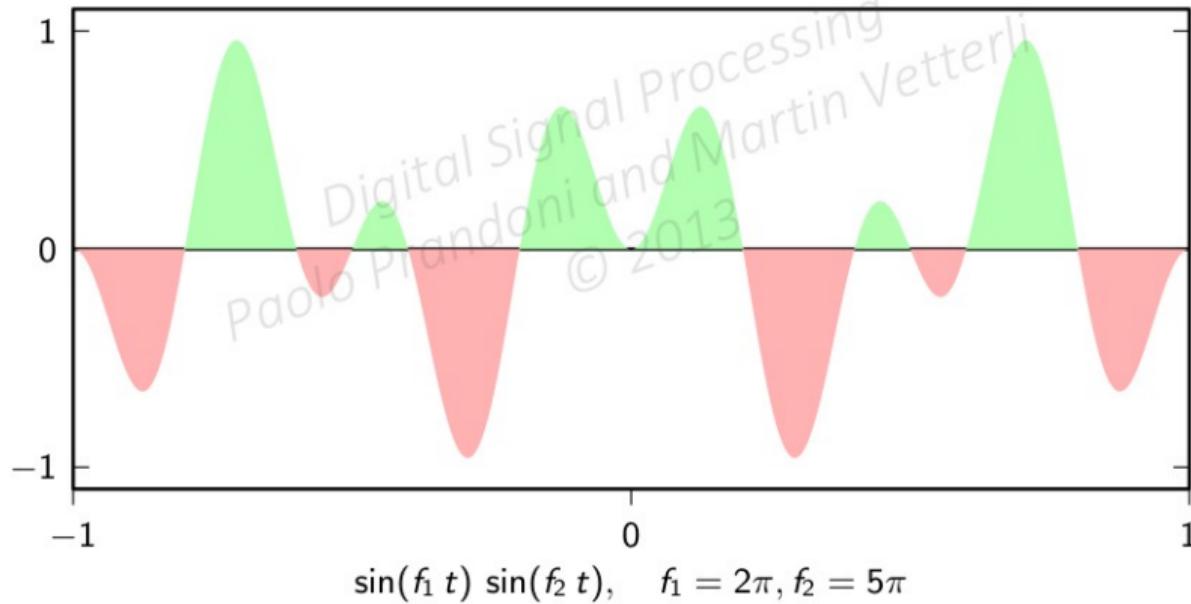
# Orthogonality in a functional vector space.

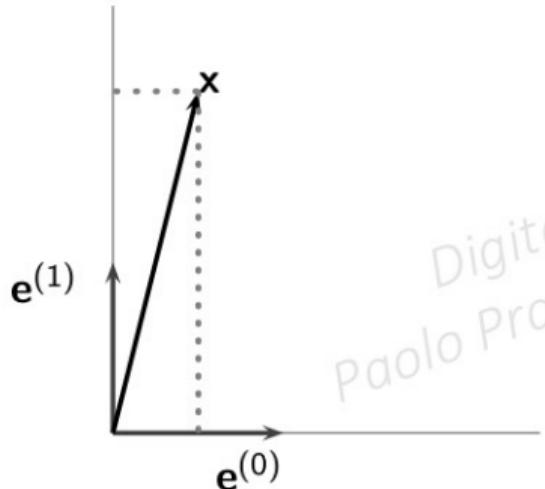
$\mathbf{x}^{(1)} \perp \mathbf{x}^{(2)}$  if  $f_1 \neq f_2$  and  $f_1, f_2$  integer multiples of a fundamental (harmonically related)



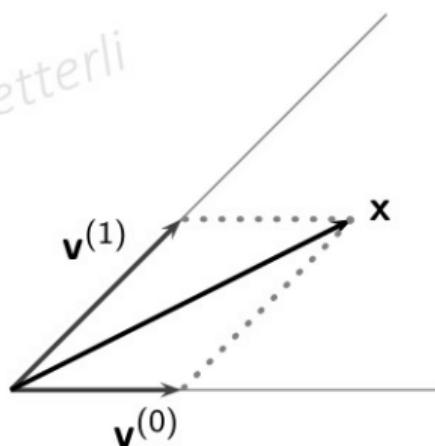
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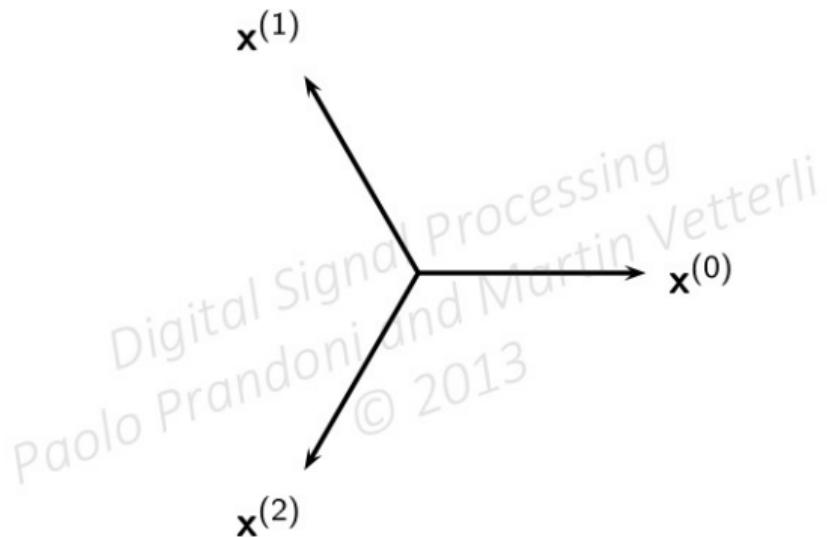




orthogonal basis

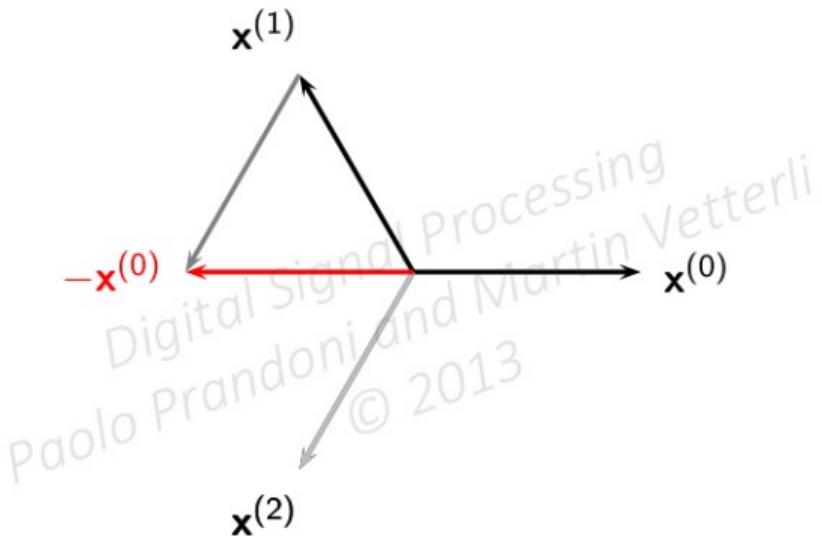


biorthogonal basis



Linear dependence:

$$\exists \{a_0, a_1, a_2\} \text{ s.t. } a_0\mathbf{x}^{(0)} + a_1\mathbf{x}^{(1)} + a_2\mathbf{x}^{(2)} = 0$$

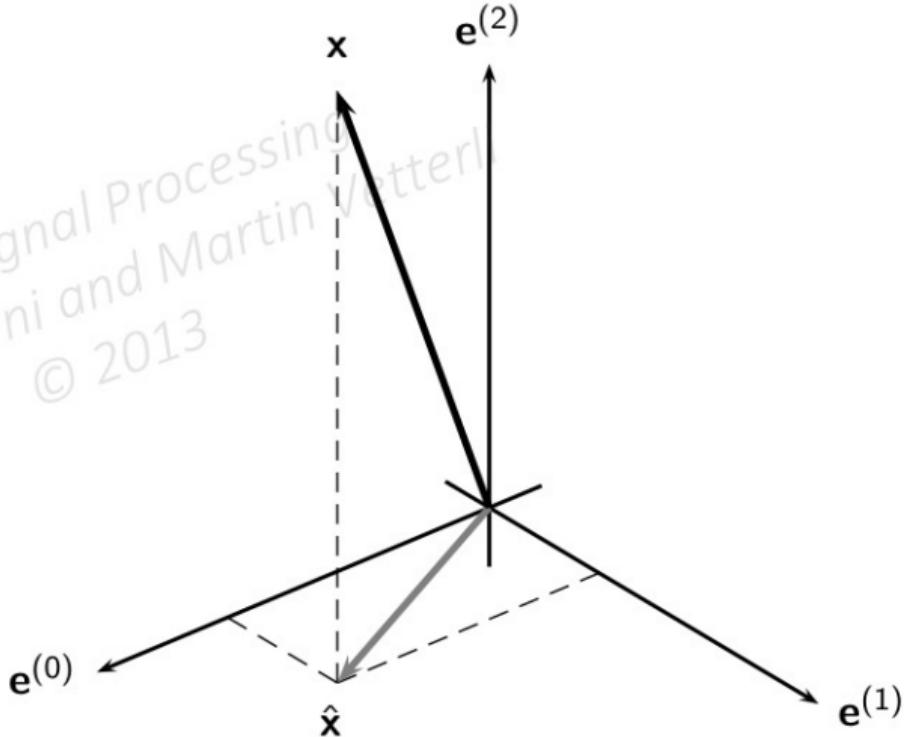


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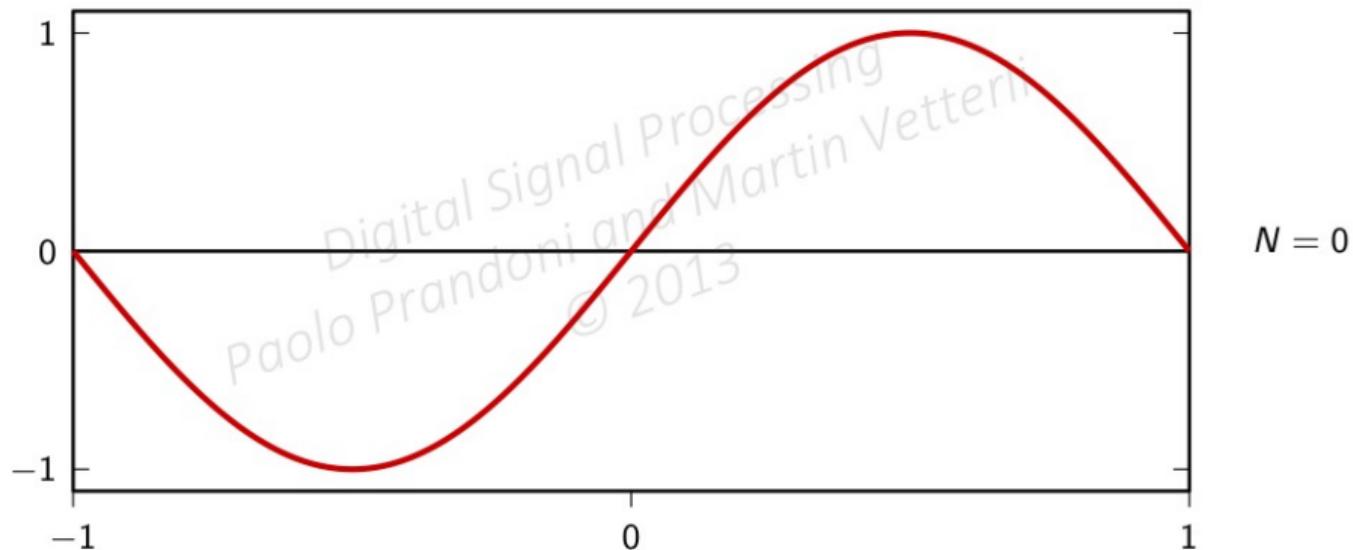
# Not enough vectors for the space

subspace projection:  $\hat{x}$  is the closest approximation to  $x$  in the space spanned by  $\{e^{(0)}, e^{(1)}\}$



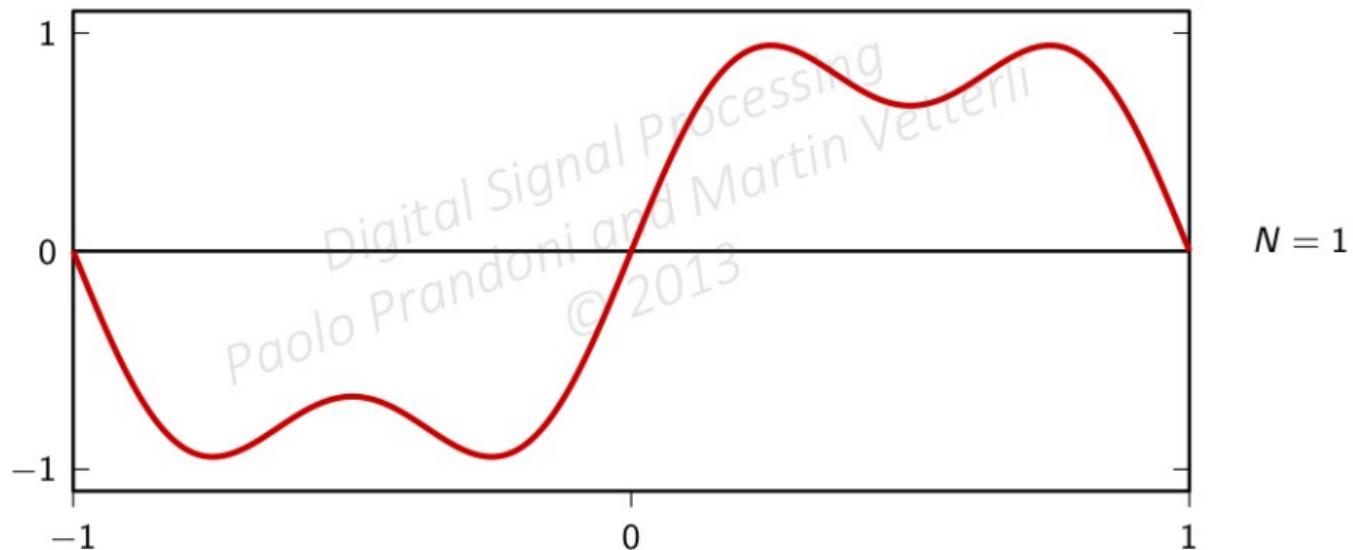
# Interesting questions when the space has $\infty$ dimensions

$$\sum_{k=0}^N \mathbf{x}^{(2k+1)}, \quad \mathbf{x}^{(n)} = \sin(\pi n t)/n, \quad t \in [-1, 1]$$



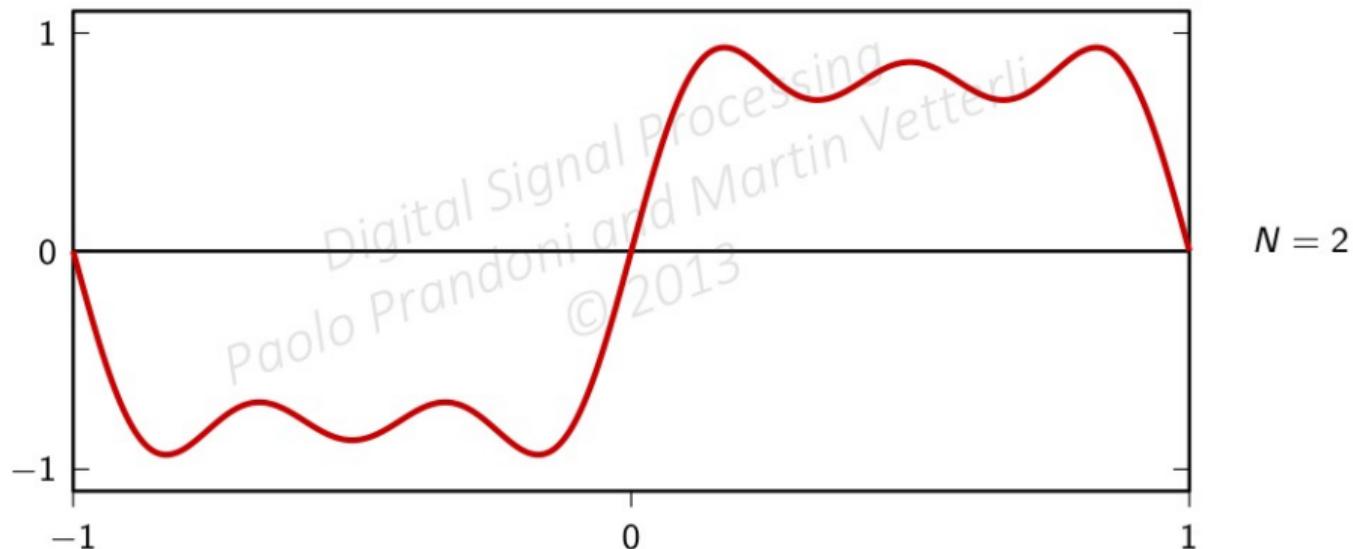
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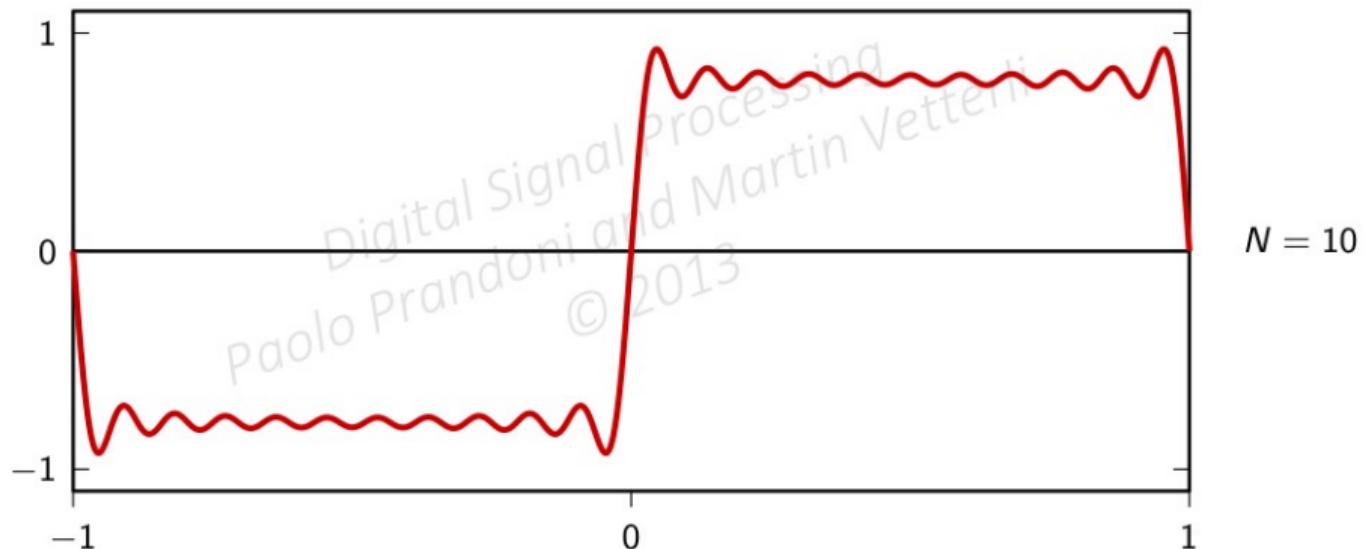
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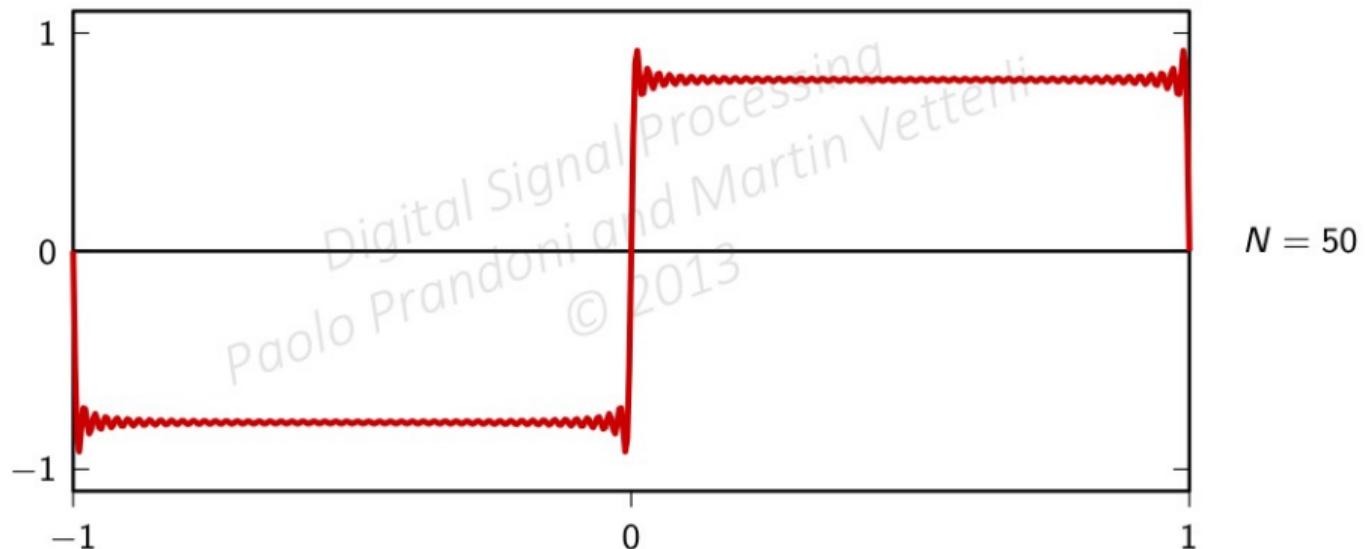
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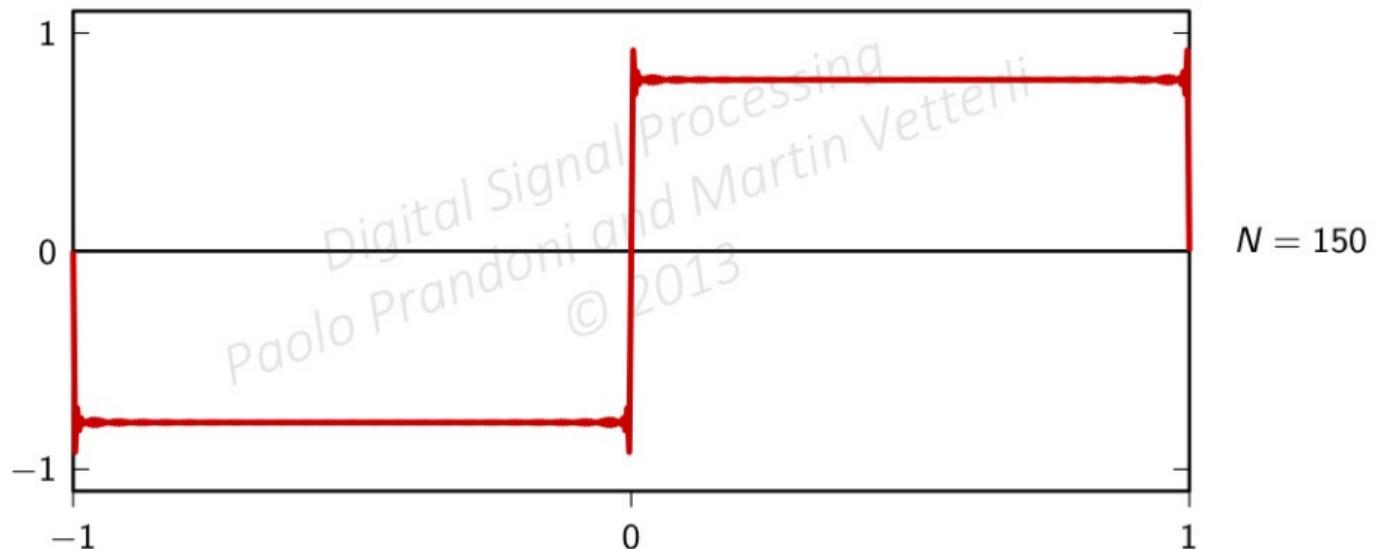
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# Digital Signal Processing

Module 3.2: Hilbert Space, properties and bases

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- ▶ Definition of Hilbert space
- ▶ Examples

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1. a vector space:  $H(V, \mathbb{C})$
2. an inner product:  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$
3. completeness

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# 1) Vector space

We need *at least* to:

- ▶ resize vectors: scalar multiplication
- ▶ combine vectors together: addition

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# Scalar multiplication in $\mathbb{R}^2$

$$\mathbf{x} = [x_0 \quad x_1]^T$$

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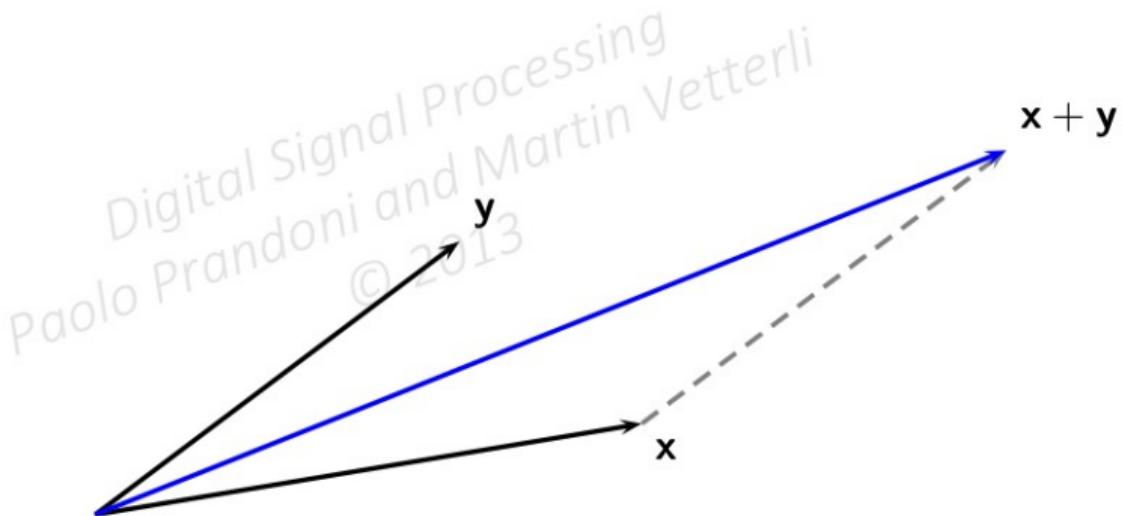


$$a\mathbf{x} = [ax_0 \quad ax_1]^T$$

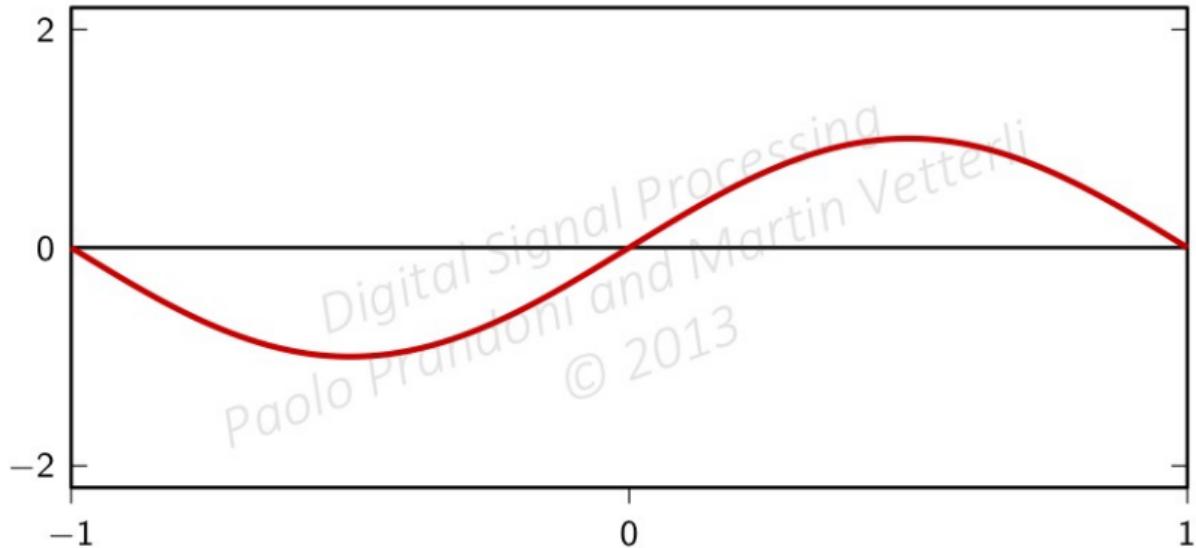
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$$\mathbf{x} + \mathbf{y} = [x_0 + y_0 \quad x_1 + y_1]^T$$

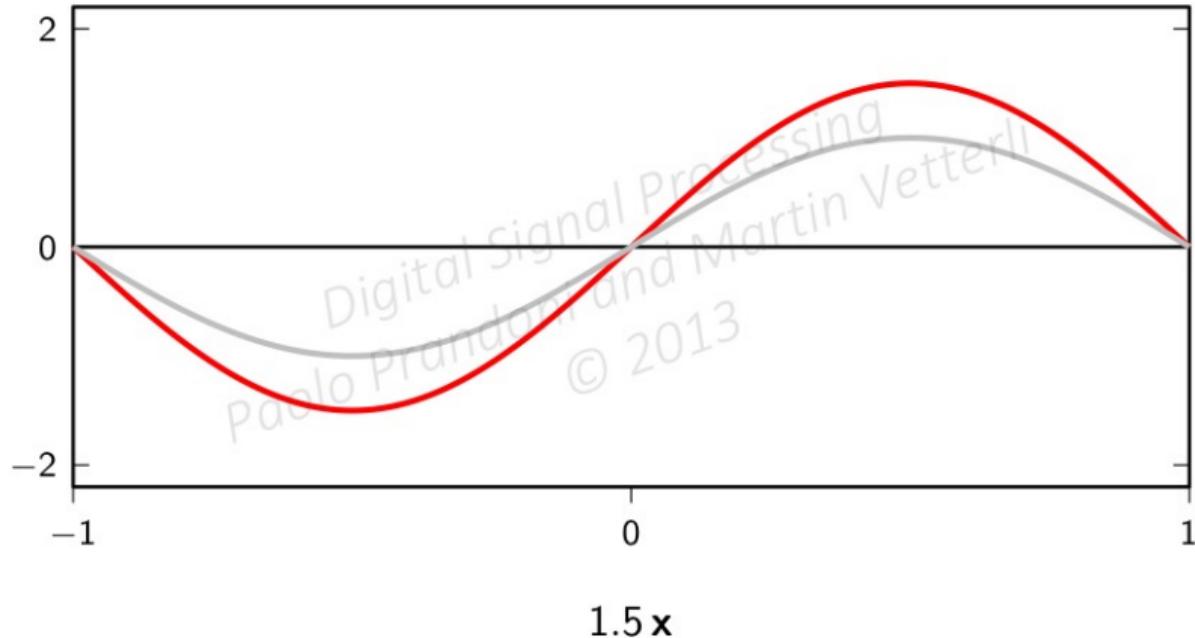


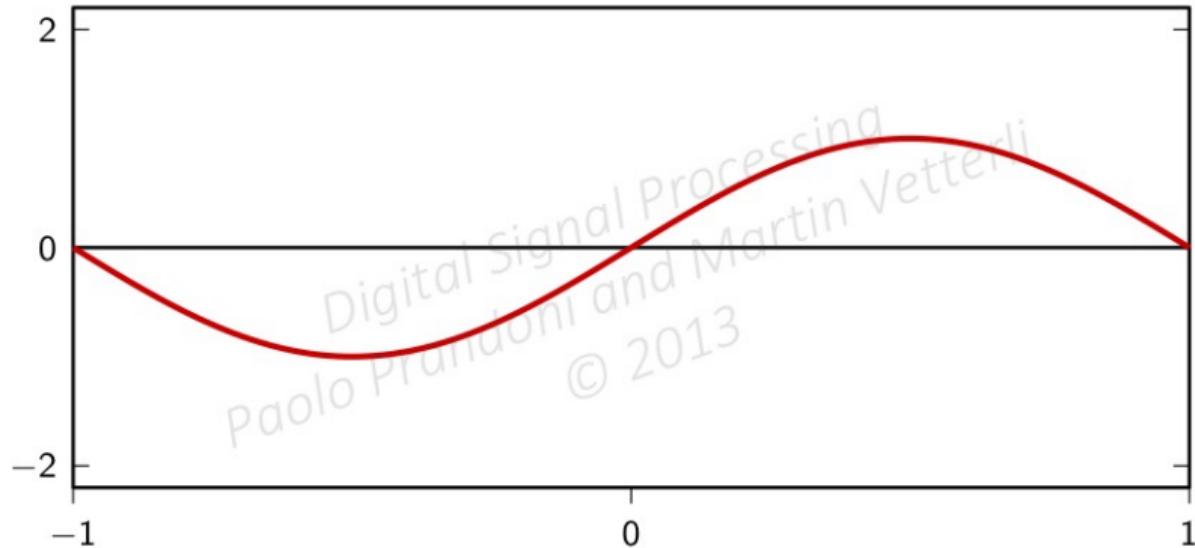
## Scalar multiplication in $L_2[-1, 1]$



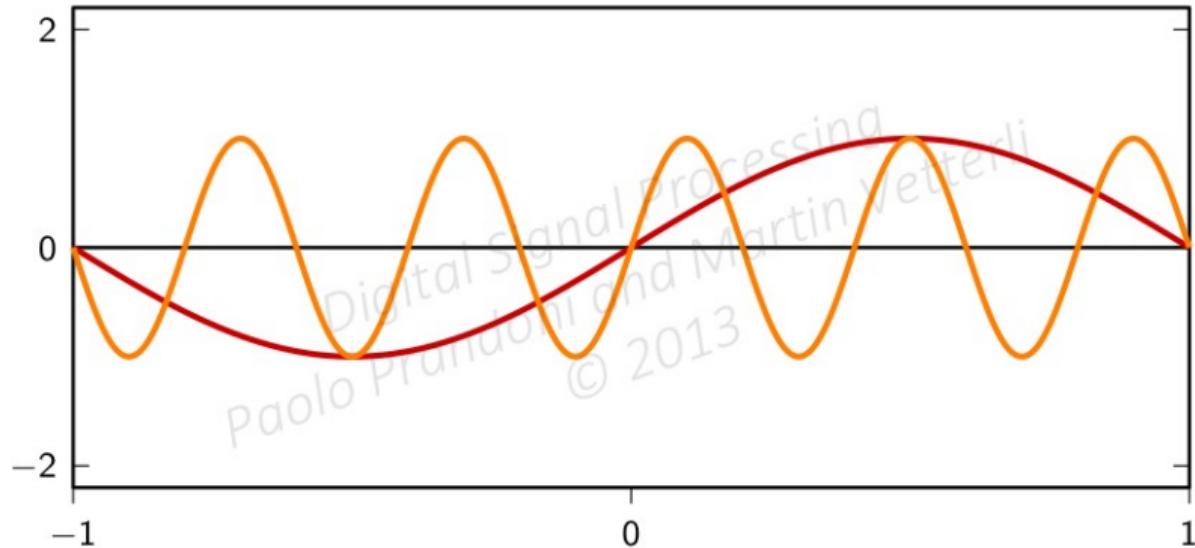
$$x = \sin(\pi t)$$

# Scalar multiplication in $L_2[-1, 1]$



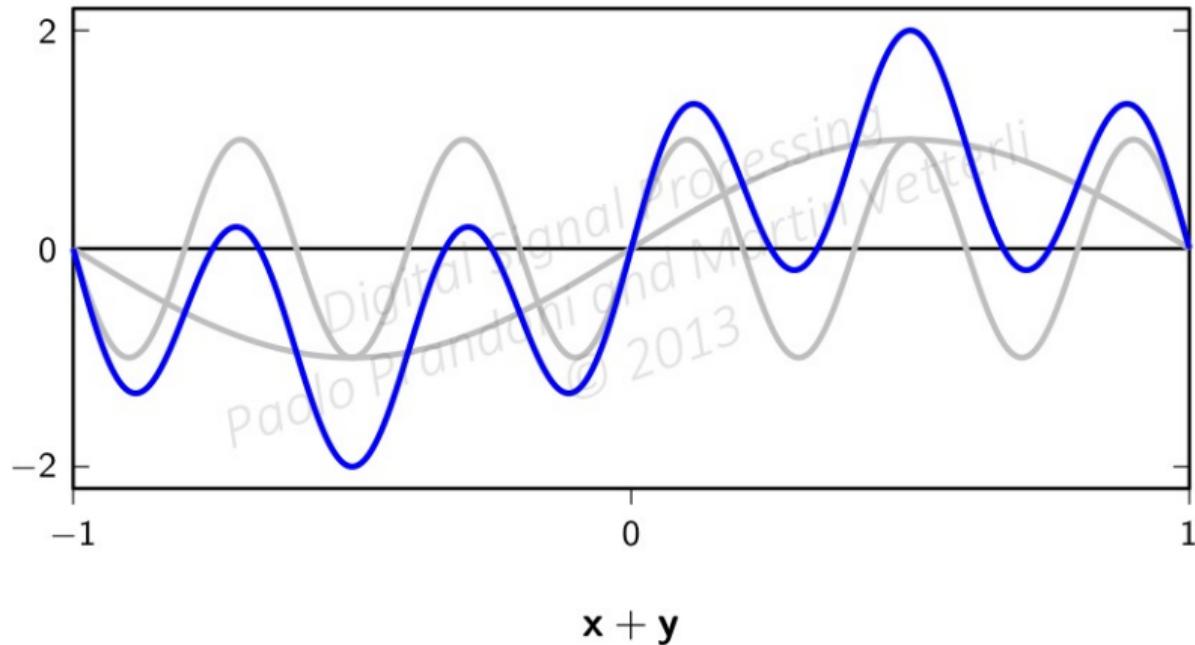


$$x = \sin(\pi t)$$



$$\mathbf{y} = \sin(5\pi t)$$

# Addition in $L_2[-1, 1]$



## Formal properties of a vector space:

For  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$  and  $\alpha, \beta \in \mathbb{C}$ :

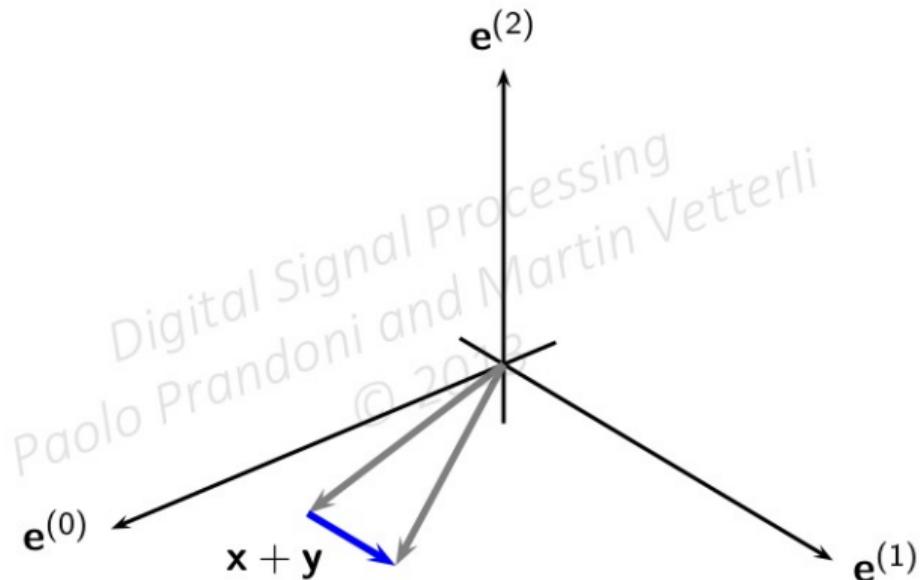
- ▶  $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
- ▶  $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
- ▶  $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{y} + \alpha\mathbf{x}$
- ▶  $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$
- ▶  $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$
- ▶  $\exists \mathbf{0} \in V \quad | \quad \mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$
- ▶  $\forall \mathbf{x} \in V \exists (-\mathbf{x}) \quad | \quad \mathbf{x} + (-\mathbf{x}) = \mathbf{0}$

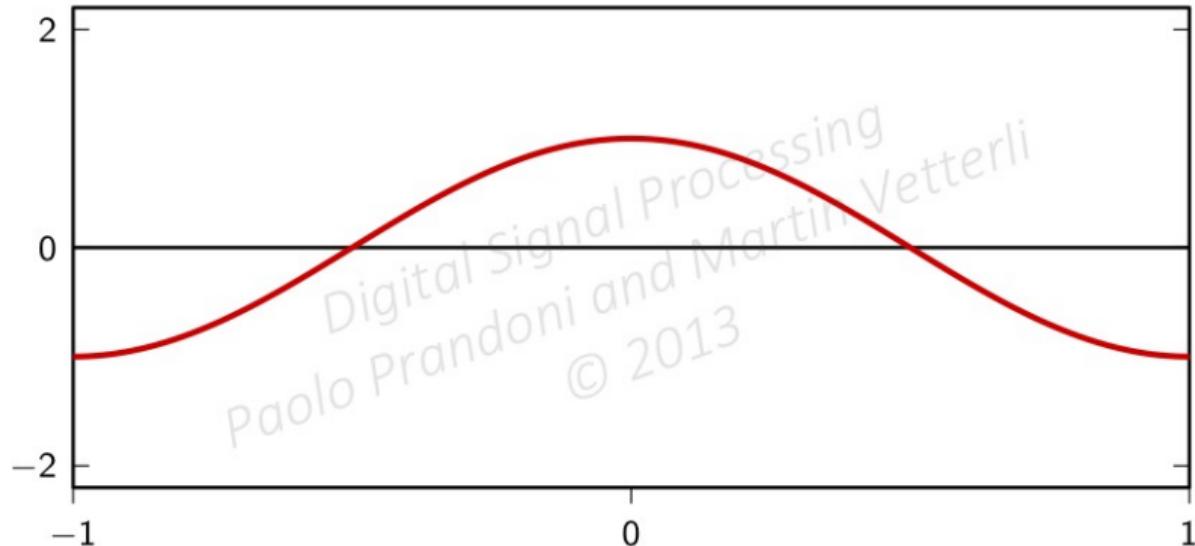
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- ▶ intuition:  $\mathbb{R}^2 \subset \mathbb{R}^3$
- ▶ addition and scaling in subspace remain in subspace

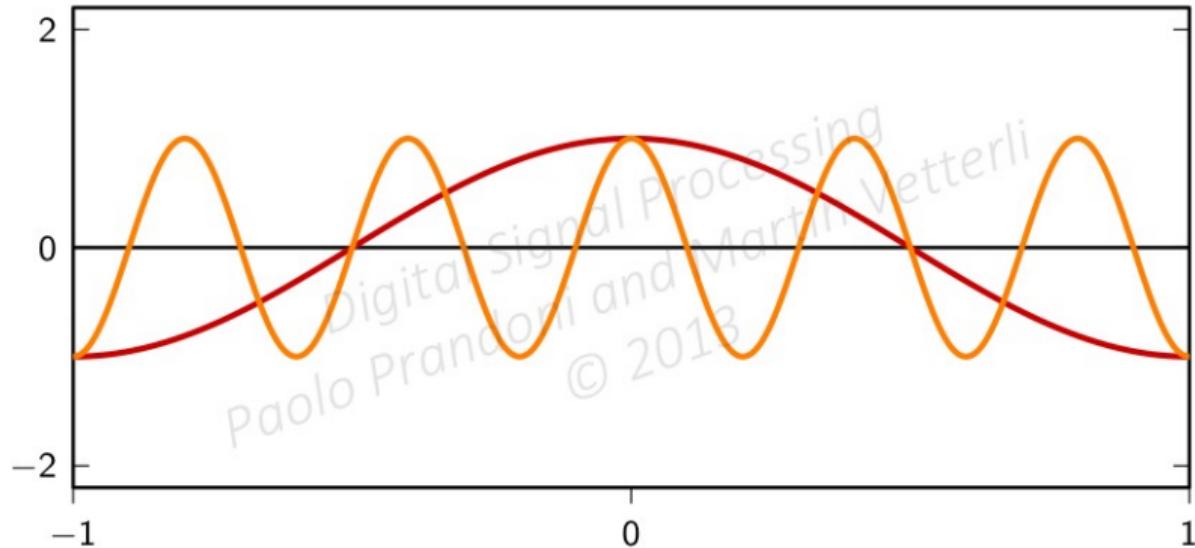
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## Addition in subspace:



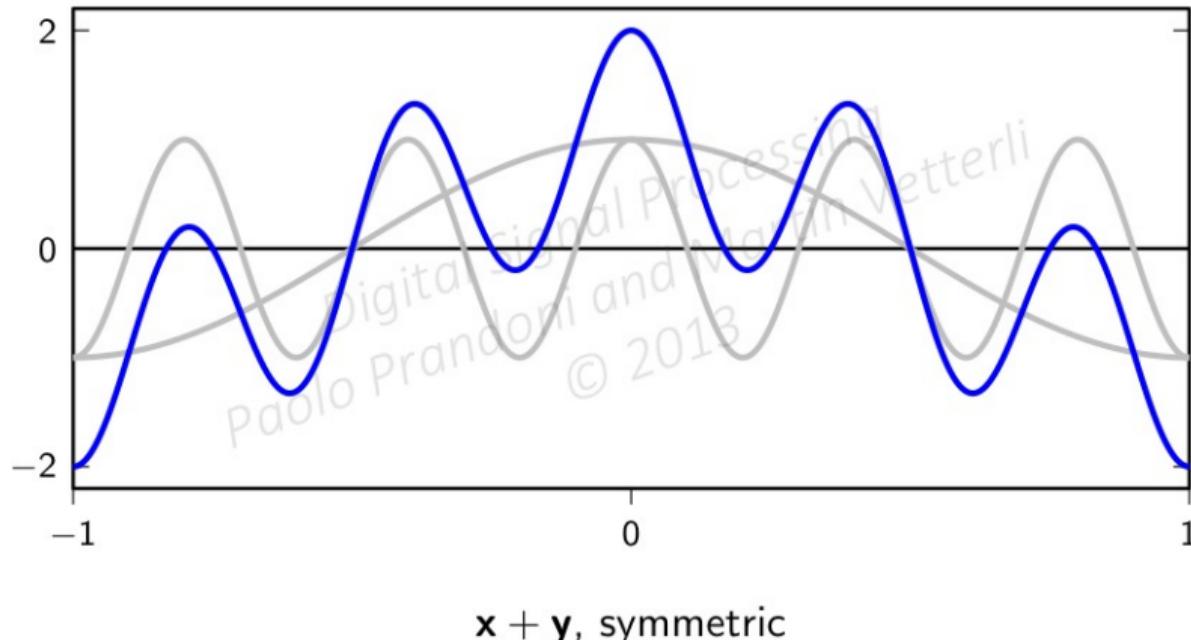


$$x = \cos(\pi t)$$



$$y = \cos(5\pi t)$$

# Subspace of symmetric functions over $L_2[-1, 1]$

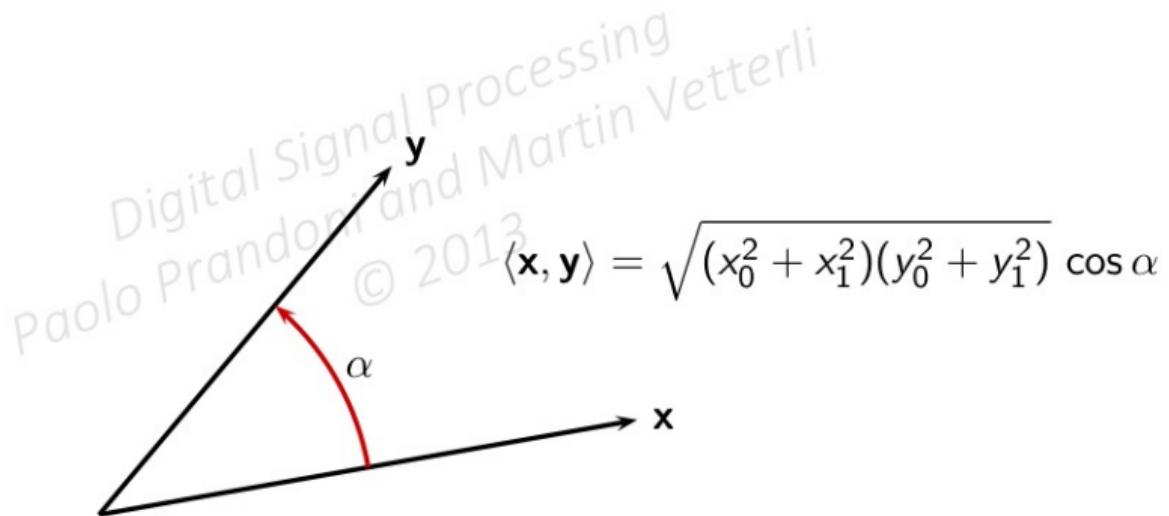


## 2) Inner product

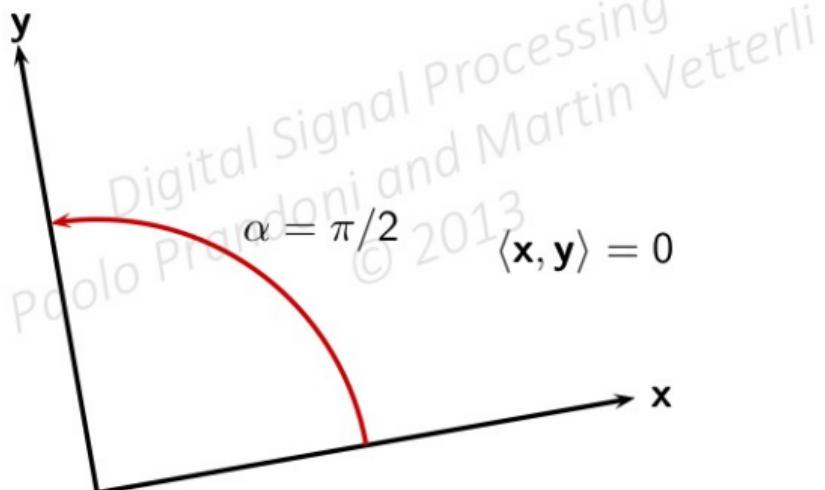
- ▶ measure of similarity between vectors
- ▶ when inner product is zero vectors are most different: orthogonal vectors

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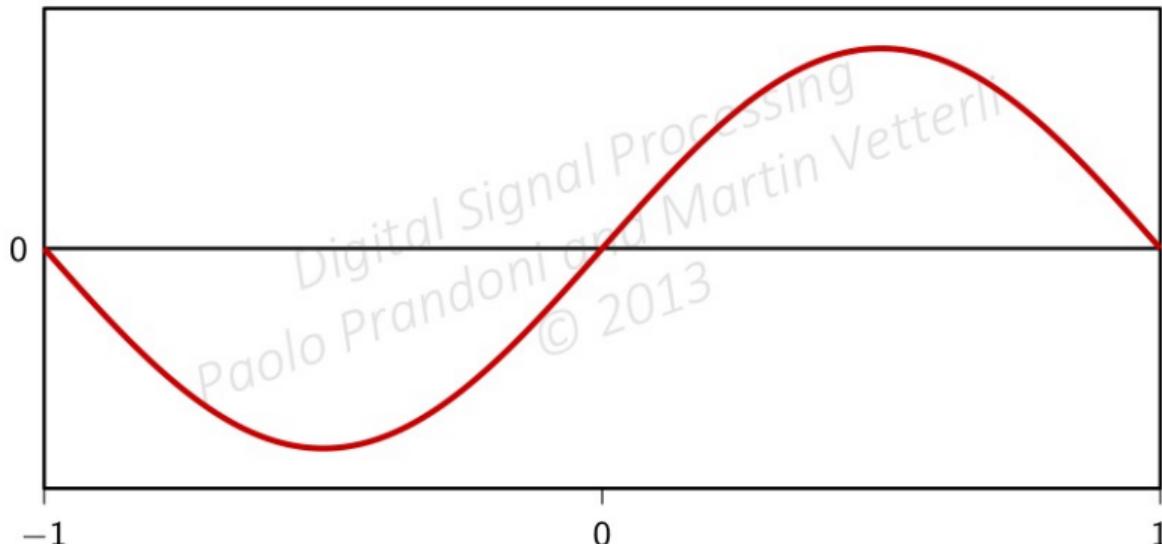
$$\langle \mathbf{x}, \mathbf{y} \rangle = x_0 y_0 + x_1 y_1$$



$$\langle \mathbf{x}, \mathbf{y} \rangle = x_0 y_0 + x_1 y_1$$

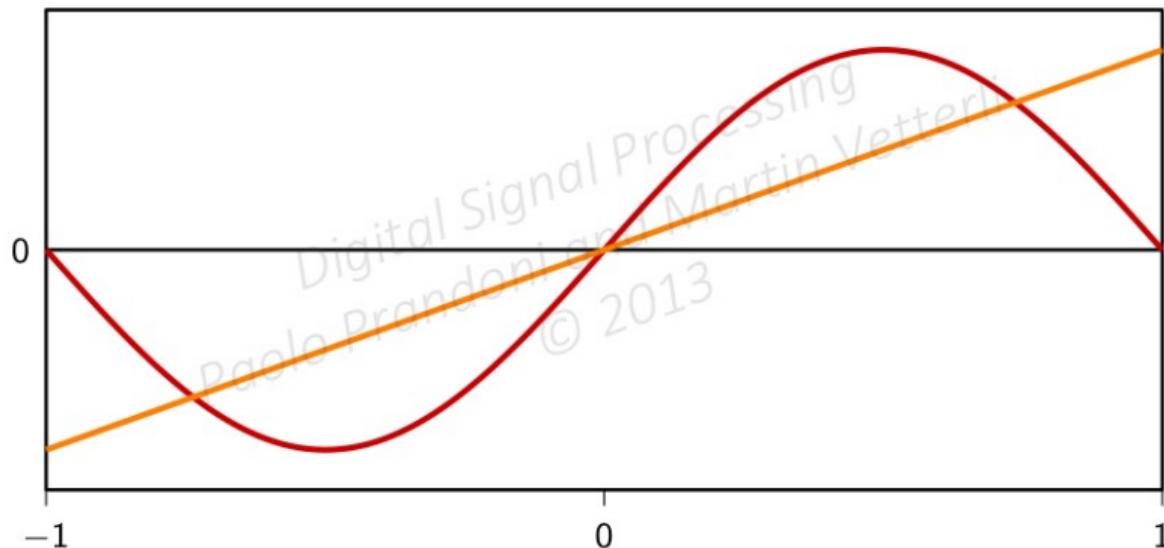


$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{-1}^1 x(t)y(t)dt$$



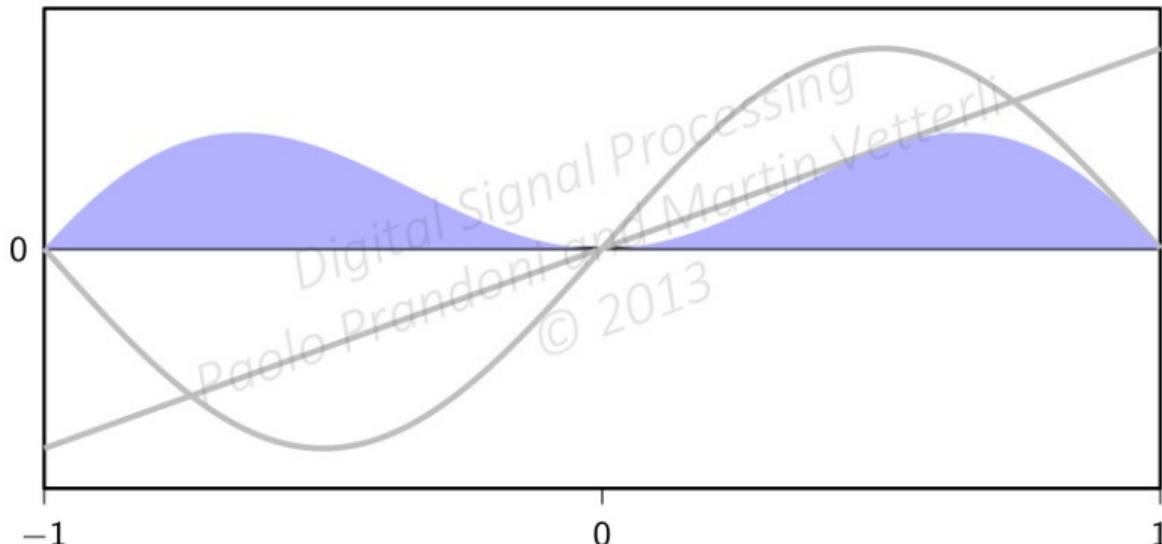
$$\mathbf{x} = \sin(\pi t)$$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{-1}^1 x(t)y(t)dt$$



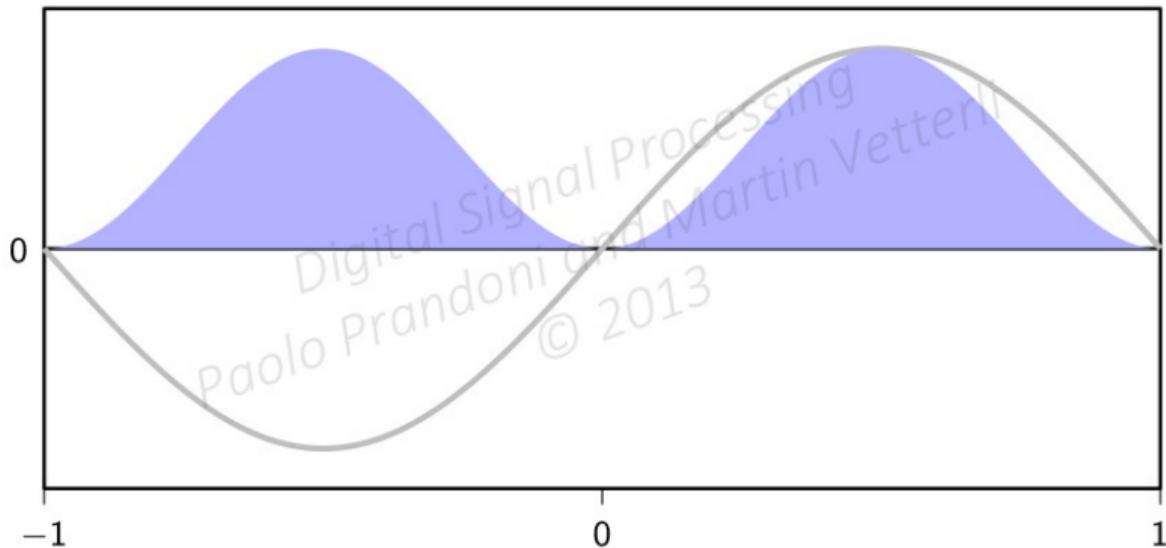
$$\mathbf{y} = t$$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{-1}^1 t \sin(\pi t) dt$$



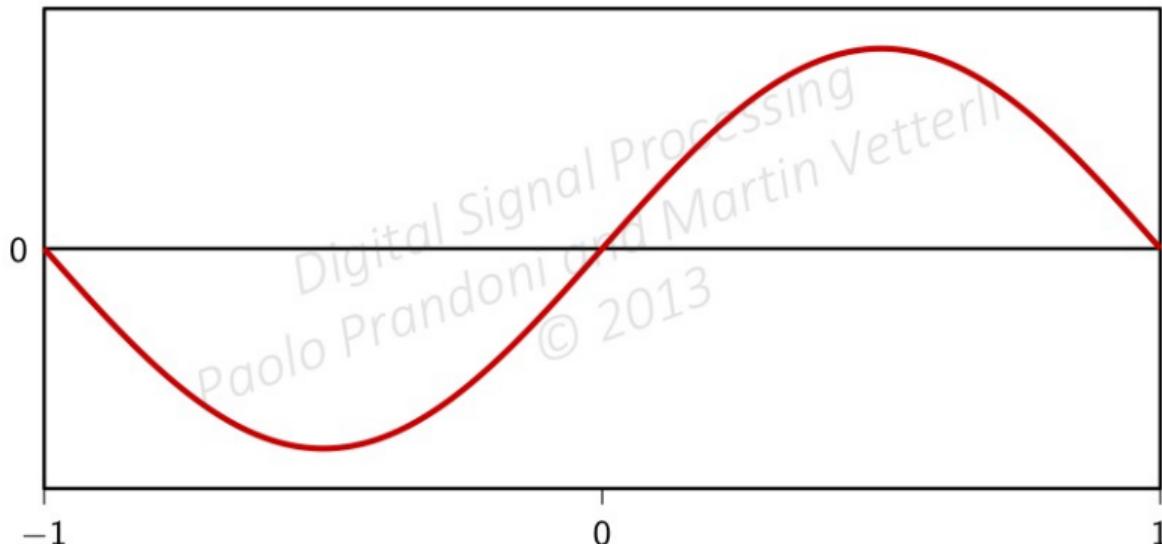
$$\langle \mathbf{x}, \mathbf{y} \rangle = 2/\pi \approx 0.6367$$

$$\langle \mathbf{x}, \mathbf{x} \rangle$$



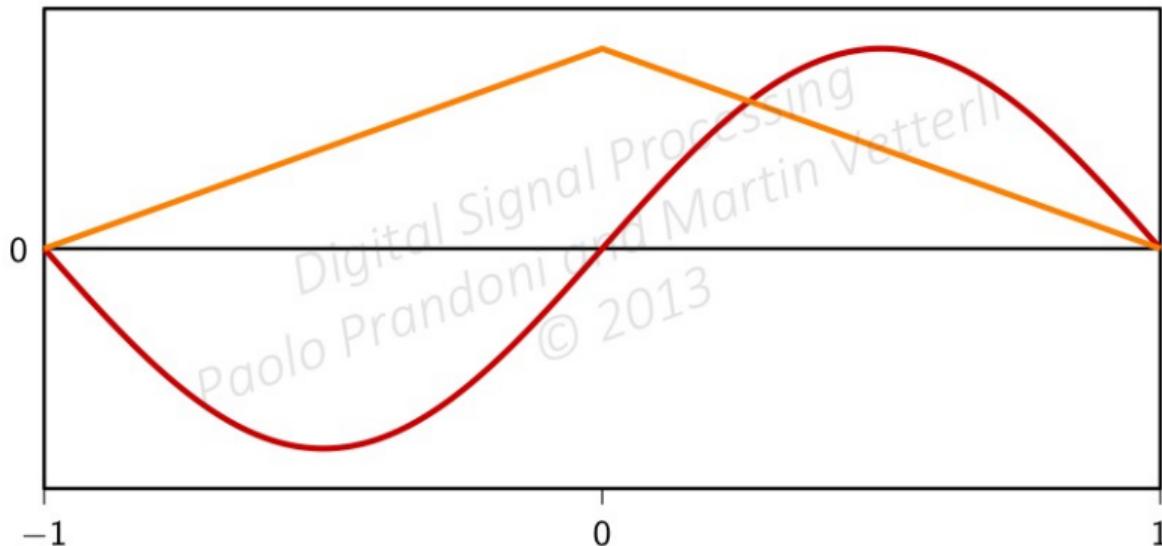
$$\mathbf{x} = \sin(\pi t), \langle \mathbf{x}, \mathbf{x} \rangle = 1$$

$\mathbf{x}, \mathbf{y}$  from orthogonal subspaces



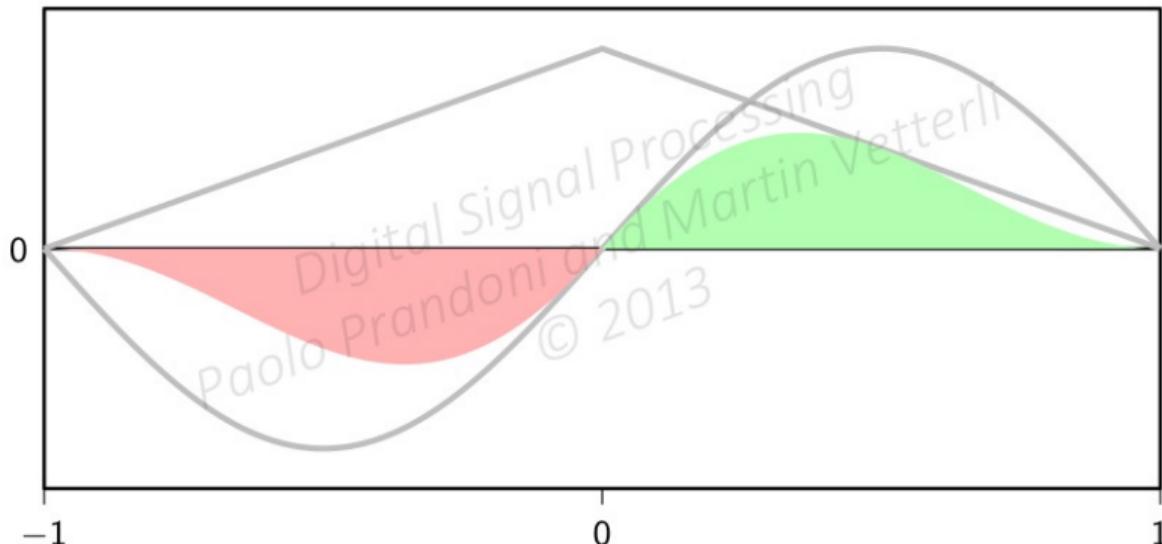
$\mathbf{x} = \sin(\pi t)$ , antisymmetric

$\mathbf{x}, \mathbf{y}$  from orthogonal subspaces



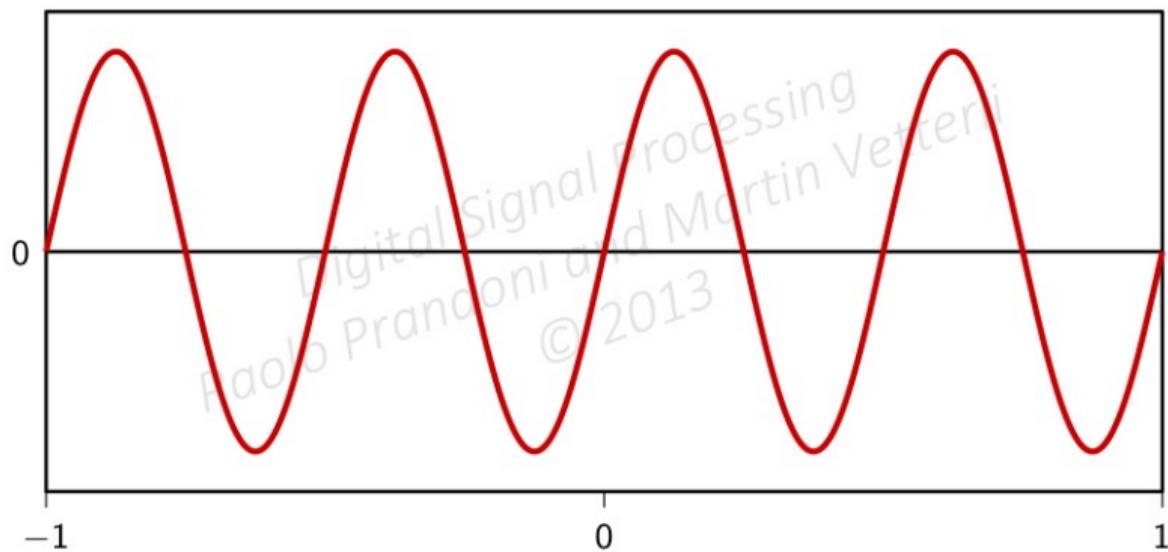
$$\mathbf{y} = 1 - |t|, \text{ symmetric}$$

$\mathbf{x}, \mathbf{y}$  from orthogonal subspaces



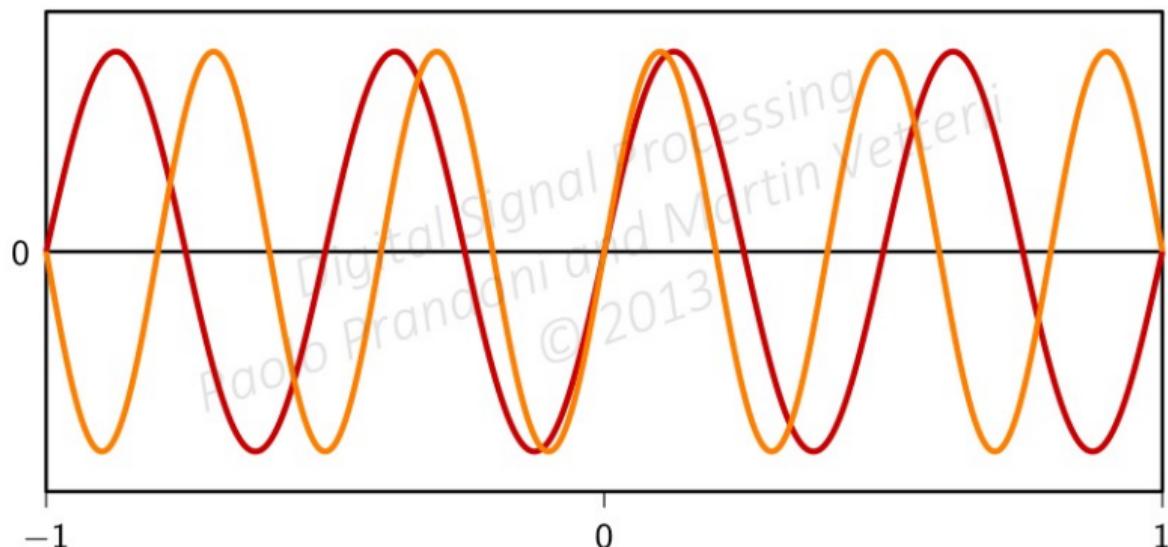
$$\langle \mathbf{x}, \mathbf{y} \rangle = 0$$

sinusoids with frequencies integer multiples of a fundamental



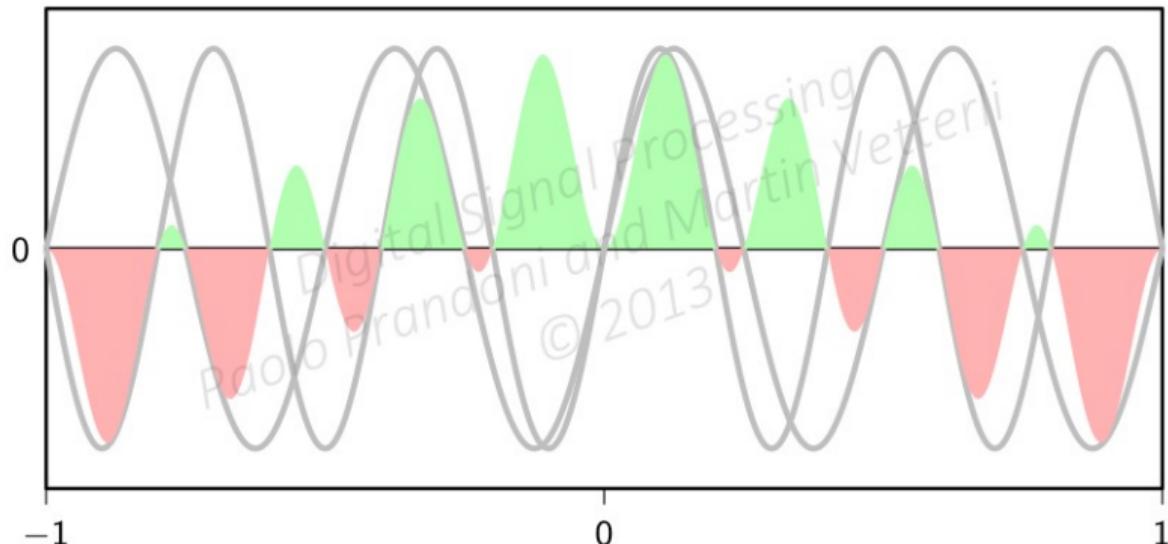
$$x = \sin(4\pi t)$$

sinusoids with frequencies integer multiples of a fundamental



$$x = \sin(4\pi t), \quad y = \sin(5\pi t)$$

sinusoids with frequencies integer multiples of a fundamental



$$\mathbf{x} = \sin(4\pi t), \quad \mathbf{y} = \sin(5\pi t), \quad \langle \mathbf{x}, \mathbf{y} \rangle = 0$$

For  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$  and  $\alpha \in \mathbb{C}$ :

- ▶  $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
- ▶  $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle^*$
- ▶  $\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha^* \langle \mathbf{x}, \mathbf{y} \rangle$   
 $\langle \mathbf{x}, \alpha \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$
- ▶  $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$
- ▶  $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$
- ▶ if  $\langle \mathbf{x}, \mathbf{y} \rangle = 0$  and  $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$  then  $\mathbf{x}$  and  $\mathbf{y}$  are called orthogonal

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$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=0}^{N-1} x^*[n]y[n]$$

well defined for all finite-length vectors (i.e. vectors in  $\mathbb{C}^N$ )

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=-\infty}^{\infty} x^*[n]y[n]$$

careful: sum may explode!

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=-\infty}^{\infty} x^*[n]y[n]$$

We require sequences to be *square-summable*:  $\sum |x[n]|^2 < \infty$

Space of square-summable sequences:  $\ell_2(\mathbb{Z})$

- ▶ inner product defines a norm:  $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$
- ▶ norm defines a distance:  $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$

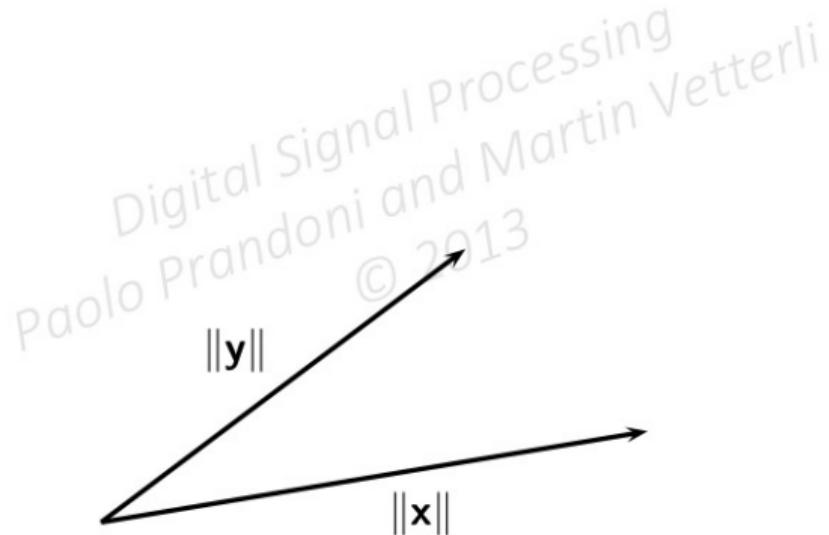
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$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{x_0^2 + x_1^2}$$

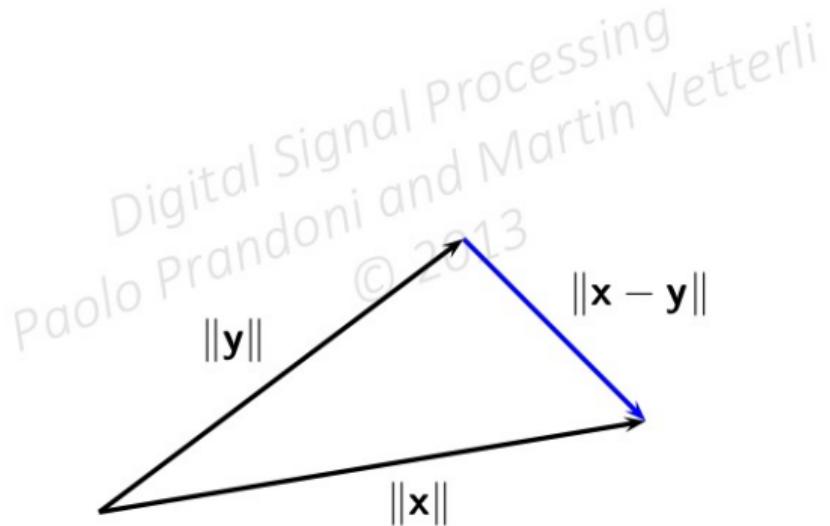
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$$\|\mathbf{y}\| = \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle} = \sqrt{y_0^2 + y_1^2}$$



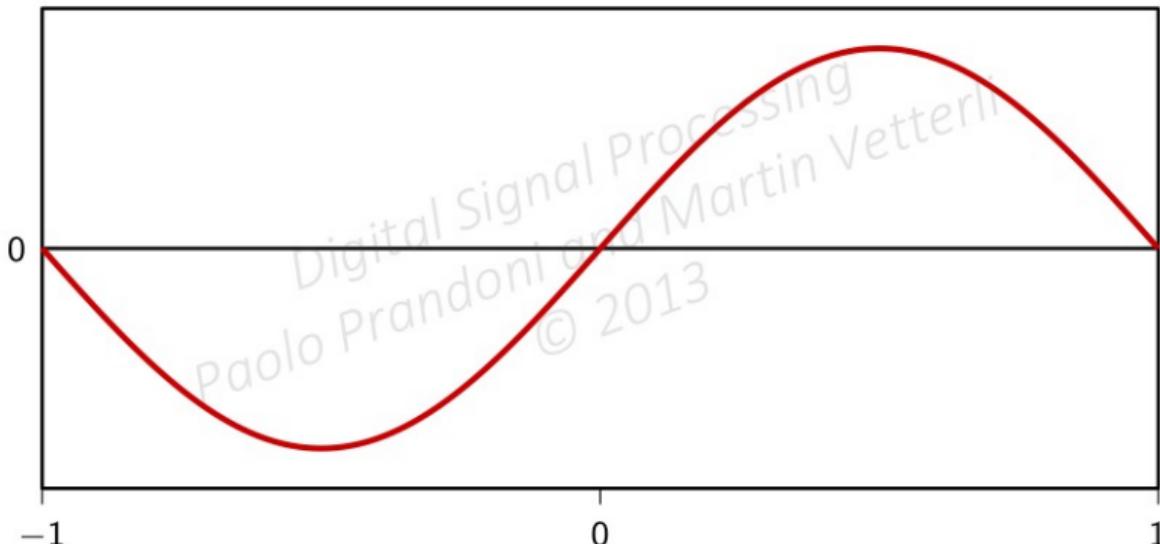
$$\|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_0 - y_0)^2 + (x_1 - y_1)^2}$$



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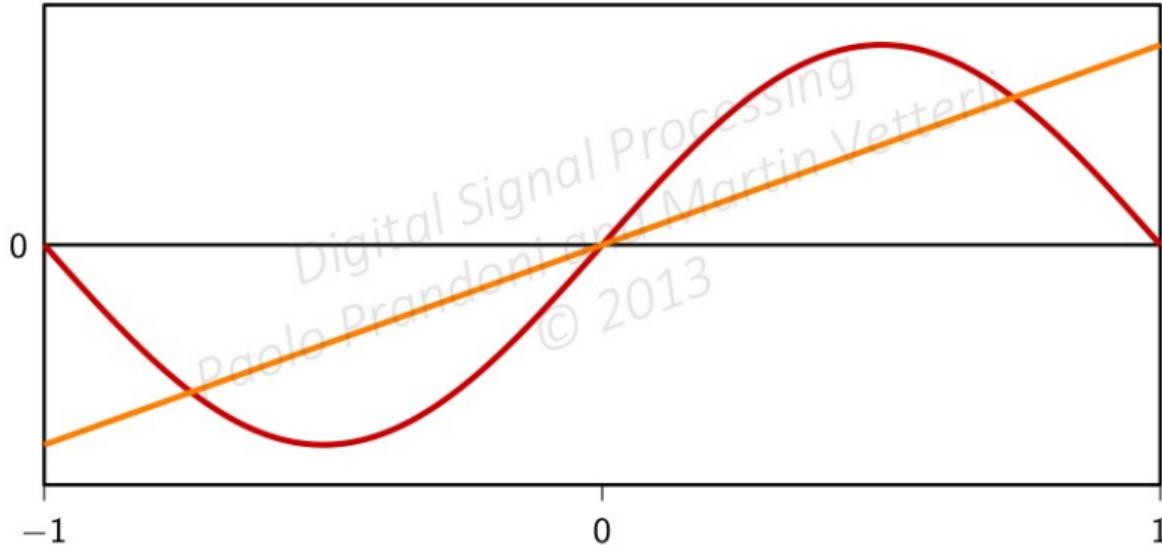
## Norm and distance in $L_2[-1, 1]$

$$\|\mathbf{x} - \mathbf{y}\|^2 = \int_{-1}^1 |x(t) - y(t)|^2 dt \text{ (MSE)}$$



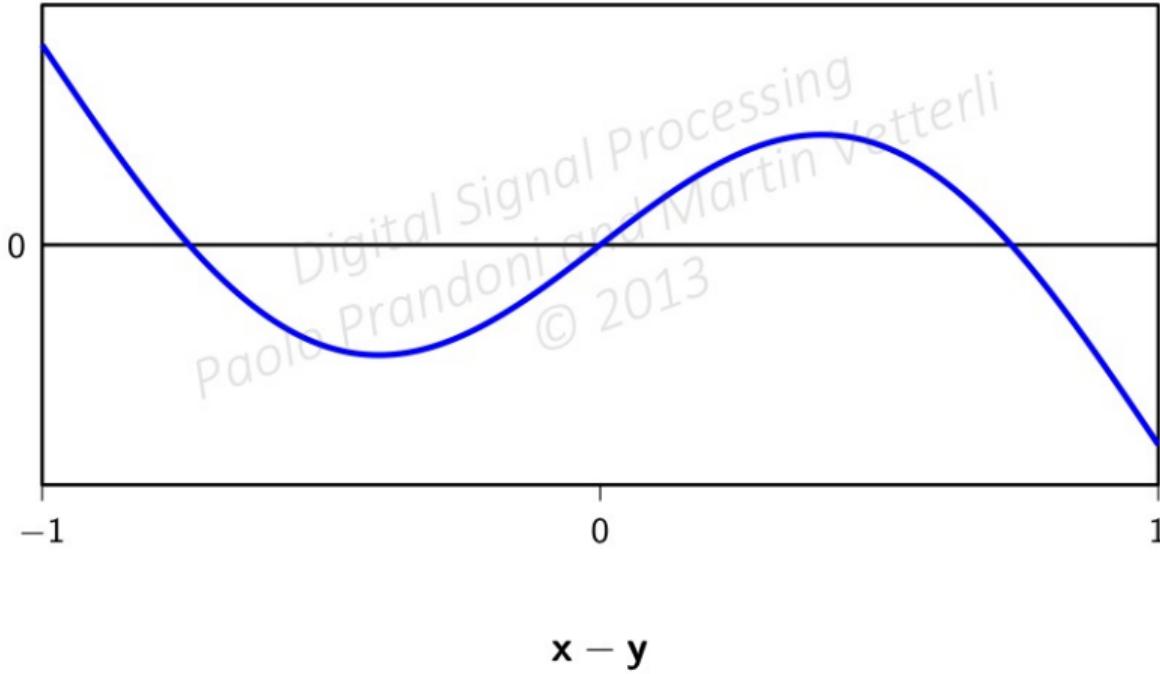
$$\mathbf{x} = \sin(\pi t)$$

$$\|\mathbf{x} - \mathbf{y}\|^2 = \int_{-1}^1 |x(t) - y(t)|^2 dt \text{ (MSE)}$$



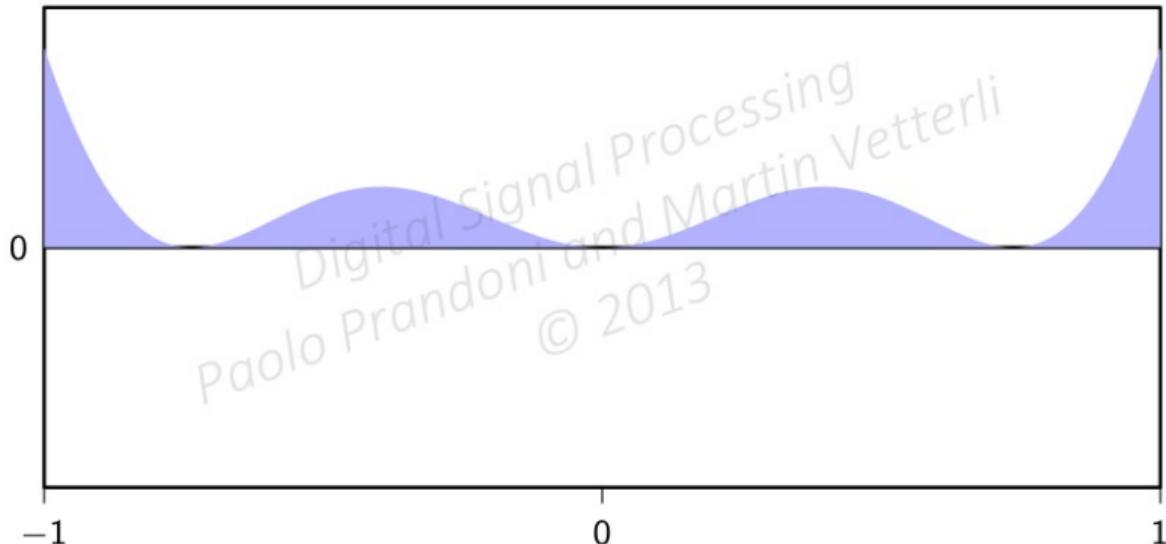
$$\mathbf{y} = t$$

$$\|\mathbf{x} - \mathbf{y}\|^2 = \int_{-1}^1 |x(t) - y(t)|^2 dt \text{ (MSE)}$$



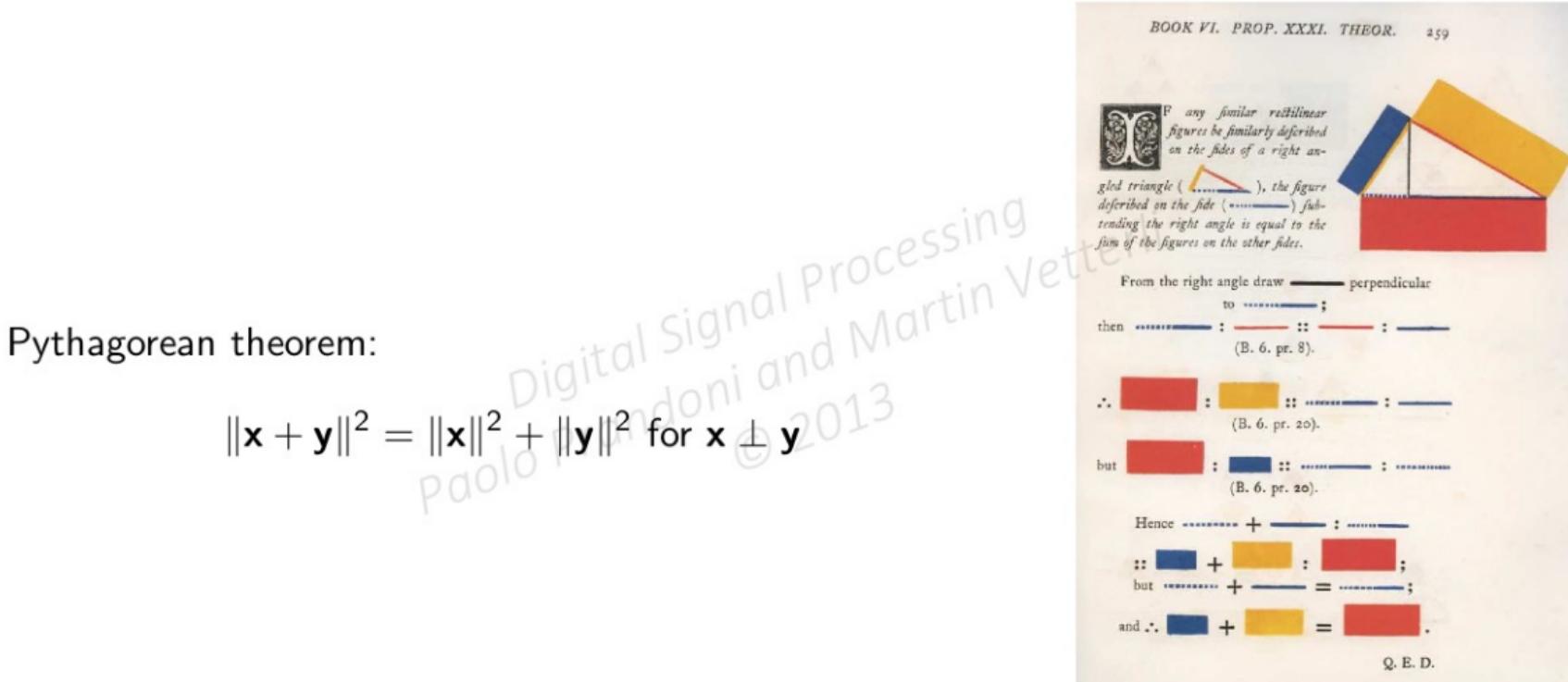
## Norm and distance in $L_2[-1, 1]$

$$\|\mathbf{x} - \mathbf{y}\|^2 = \int_{-1}^1 |x(t) - y(t)|^2 dt \text{ (MSE)}$$



$$\|\mathbf{x} - \mathbf{y}\| = \sqrt{5/3 - 4/\pi} \approx 0.6272$$

# A familiar result



Pythagorean theorem:

$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \text{ for } \mathbf{x} \perp \mathbf{y}$$

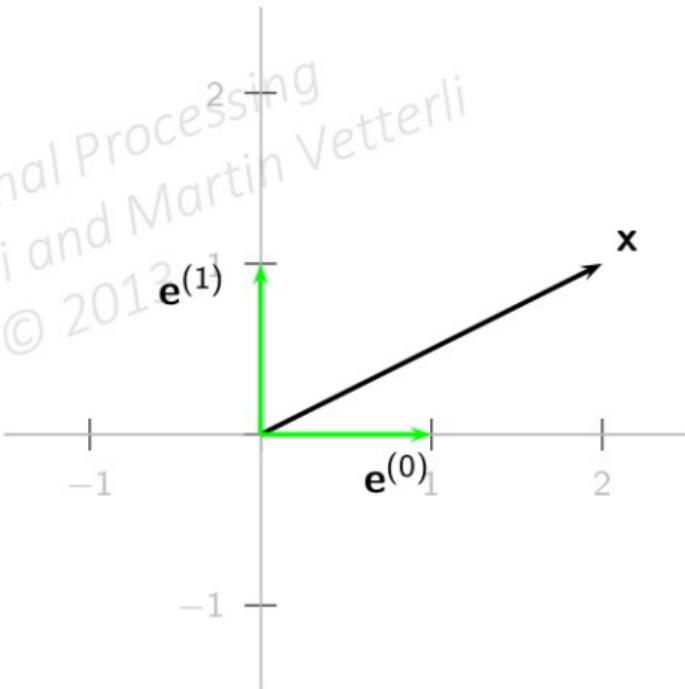
From Euclid's elements by Oliver Byrne (1810 - 1880)

$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \in \mathbb{R}^2$$

$$\mathbf{x} = 2\mathbf{e}^{(0)} + \mathbf{e}^{(1)}$$

$$\mathbf{x} = \mathbf{v}^{(0)} + \mathbf{v}^{(1)}$$

$$\mathbf{x} \neq \alpha_0 \mathbf{g}^{(0)} + \alpha_1 \mathbf{g}^{(1)} \quad \text{for any } \alpha_0, \alpha_1$$



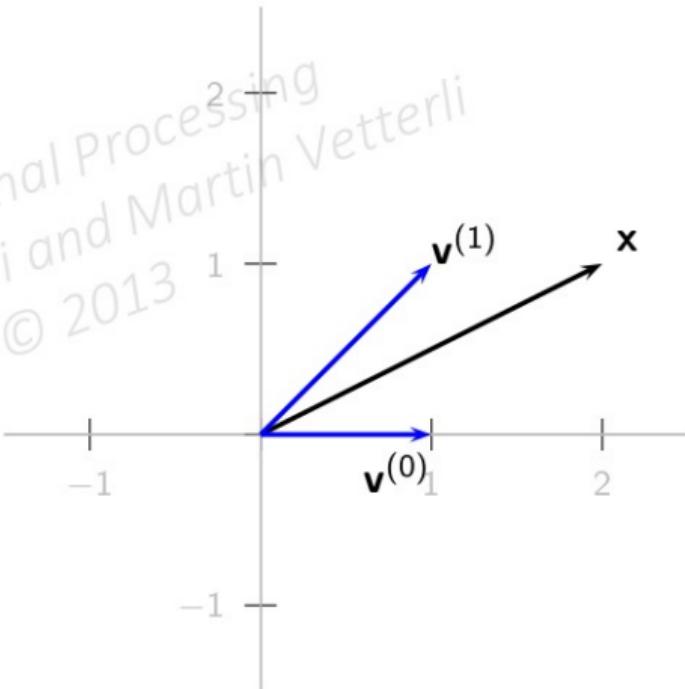
## Bases

$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \in \mathbb{R}^2$$

$$\mathbf{x} = 2\mathbf{e}^{(0)} + \mathbf{e}^{(1)}$$

$$\mathbf{x} = \mathbf{v}^{(0)} + \mathbf{v}^{(1)}$$

$$\mathbf{x} \neq \alpha_0 \mathbf{g}^{(0)} + \alpha_1 \mathbf{g}^{(1)} \quad \text{for any } \alpha_0, \alpha_1$$



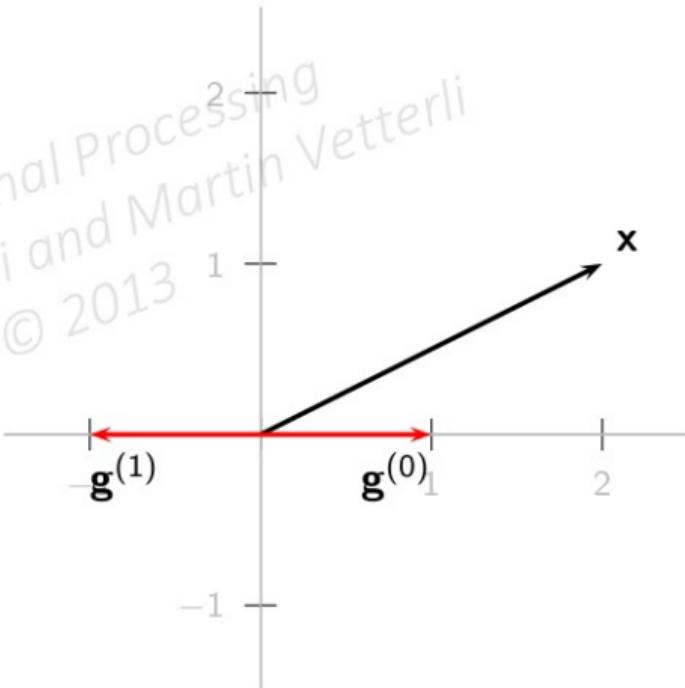
## Bases

$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \in \mathbb{R}^2$$

$$\mathbf{x} = 2\mathbf{e}^{(0)} + \mathbf{e}^{(1)}$$

$$\mathbf{x} = \mathbf{v}^{(0)} + \mathbf{v}^{(1)}$$

$$\mathbf{x} \neq \alpha_0 \mathbf{g}^{(0)} + \alpha_1 \mathbf{g}^{(1)} \quad \text{for any } \alpha_0, \alpha_1$$



- ▶ vector space  $H$
- ▶ set of  $K$  vectors from  $H$ :  $W = \{\mathbf{w}^{(k)}\}_{k=0,1,\dots,K-1}$

$W$  is a basis for  $H$  if:

- ▶ we can write for all  $\mathbf{x} \in H$ :

$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)}, \quad \alpha_k \in \mathbb{C}$$

- ▶ the coefficients  $\alpha_k$  are unique

Unique representation implies linear independence:

$$\sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)} = 0 \iff \alpha_k = 0, \quad k = 0, 1, \dots, K-1$$

Orthogonal basis:

$$\langle \mathbf{w}^{(k)}, \mathbf{w}^{(n)} \rangle = 0 \text{ for } k \neq n$$

Orthonormal basis:

$$\langle \mathbf{w}^{(k)}, \mathbf{w}^{(n)} \rangle = \delta[n - k]$$

We can always orthonormalize a basis via the Gram-Schmidt algorithm.

$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)}$$

how do we find the  $\alpha$ 's ?

Orthonormal bases are the best:

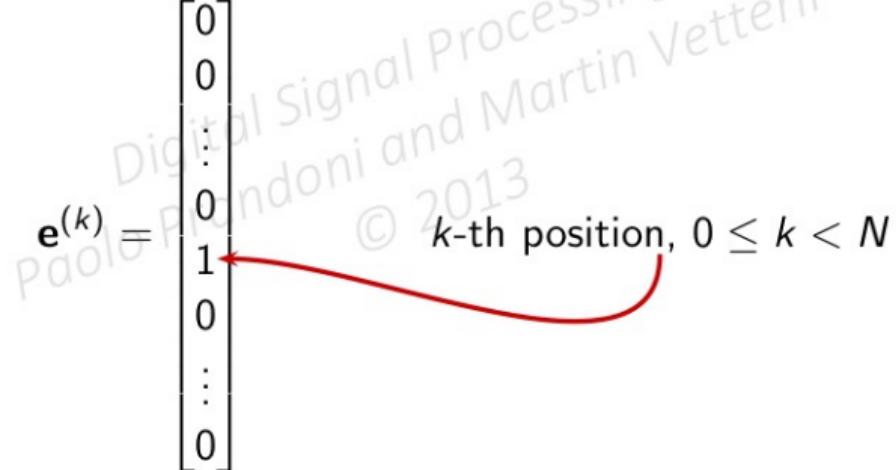
$$\alpha_k = \langle \mathbf{w}^{(k)}, \mathbf{x} \rangle$$

- ▶ a basis will contain  $N$  vectors
- ▶ canonical (orthonormal) basis:

$$\mathbf{e}^{(k)} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

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$k$ -th position,  $0 \leq k < N$



## Example: bases for sequences in $\ell_2(\mathbb{Z})$

- ▶ a basis will contain infinite vectors
- ▶ canonical (orthonormal) basis:

A diagram illustrating a unit vector  $\mathbf{e}^{(k)}$  in the space  $\ell_2(\mathbb{Z})$ . The vector is shown as a vertical column of numbers, with ellipses at the top and bottom indicating it continues indefinitely. The  $k$ -th position is highlighted with a red arrow pointing to the number 1. The text "k-th position,  $k \in \mathbb{Z}$ " is written next to the arrow. A watermark in the background reads "Digital Signal Processing Padavan and Martin Vetterli @ 2013".

$$\mathbf{e}^{(k)} = \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

k-th position,  $k \in \mathbb{Z}$

limiting operations must yield vector space elements

Example of an *incomplete* space: the set of rational numbers

$$x_n = \sum_{k=0}^n \frac{1}{k!} \in \mathbb{Q} \quad \text{but} \quad \lim_{n \rightarrow \infty} x_n = e \notin \mathbb{Q}$$



# Digital Signal Processing

Module 3.3: Hilbert Space and approximation

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- ▶ Norm conservation, Parseval
- ▶ Approximation by projection
- ▶ Examples

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$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)}$$

For an orthonormal basis:

$$\|\mathbf{x}\|^2 = \sum_{k=0}^{K-1} |\alpha_k|^2$$

## Example

- ▶ canonical basis  $E = \{\mathbf{e}^{(0)}, \mathbf{e}^{(1)}\}$

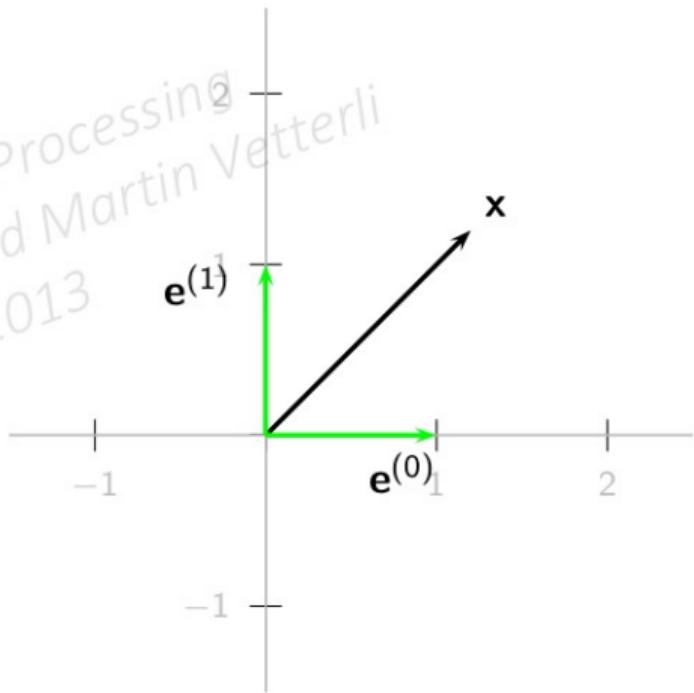
- ▶  $\mathbf{x} = \alpha_0 \mathbf{e}^{(0)} + \alpha_1 \mathbf{e}^{(1)}$

- ▶ new basis  $V = \{\mathbf{v}^{(0)}, \mathbf{v}^{(1)}\}$  with

$$\mathbf{v}^{(0)} = [\cos \theta \quad \sin \theta]^T$$

$$\mathbf{v}^{(1)} = [-\sin \theta \quad \cos \theta]^T$$

- ▶  $\mathbf{x} = \beta_0 \mathbf{v}^{(0)} + \beta_1 \mathbf{v}^{(1)}$



## Example

- ▶ canonical basis  $E = \{\mathbf{e}^{(0)}, \mathbf{e}^{(1)}\}$

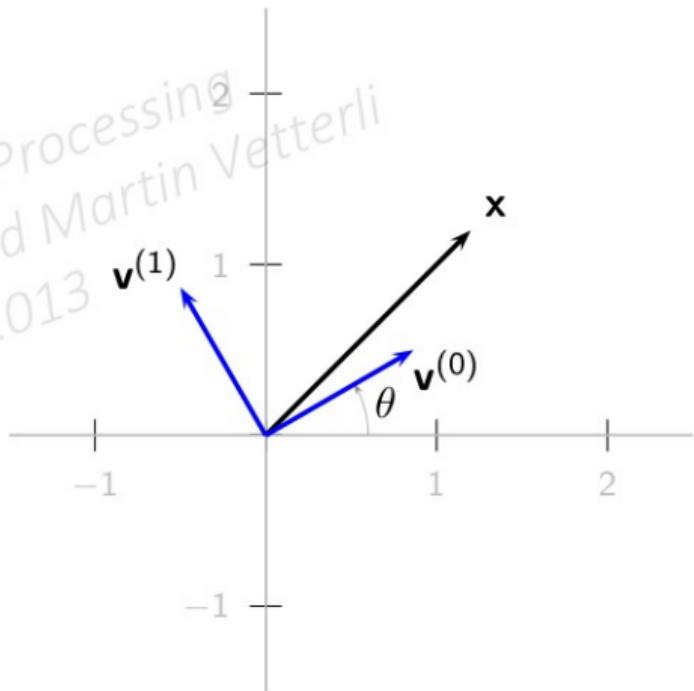
- ▶  $\mathbf{x} = \alpha_0 \mathbf{e}^{(0)} + \alpha_1 \mathbf{e}^{(1)}$

- ▶ new basis  $V = \{\mathbf{v}^{(0)}, \mathbf{v}^{(1)}\}$  with

$$\mathbf{v}^{(0)} = [\cos \theta \ \sin \theta]^T$$

$$\mathbf{v}^{(1)} = [-\sin \theta \ \cos \theta]^T$$

- ▶  $\mathbf{x} = \beta_0 \mathbf{v}^{(0)} + \beta_1 \mathbf{v}^{(1)}$



## Example

- ▶ new basis is orthonormal:

$$\beta_0 = \langle \mathbf{v}^{(0)}, \mathbf{x} \rangle$$

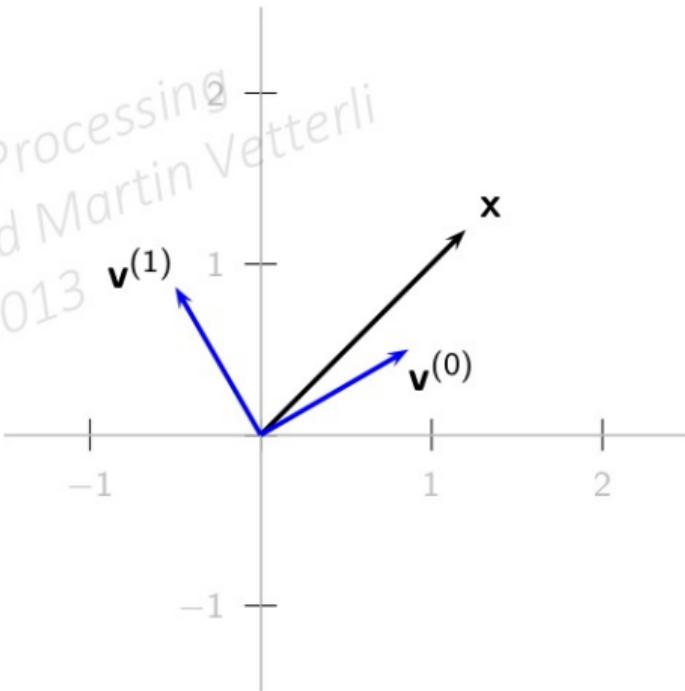
$$\beta_1 = \langle \mathbf{v}^{(1)}, \mathbf{x} \rangle$$

- ▶ in compact form:

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \mathbf{R}\boldsymbol{\alpha}$$

- ▶  $\mathbf{R}$ : rotation matrix

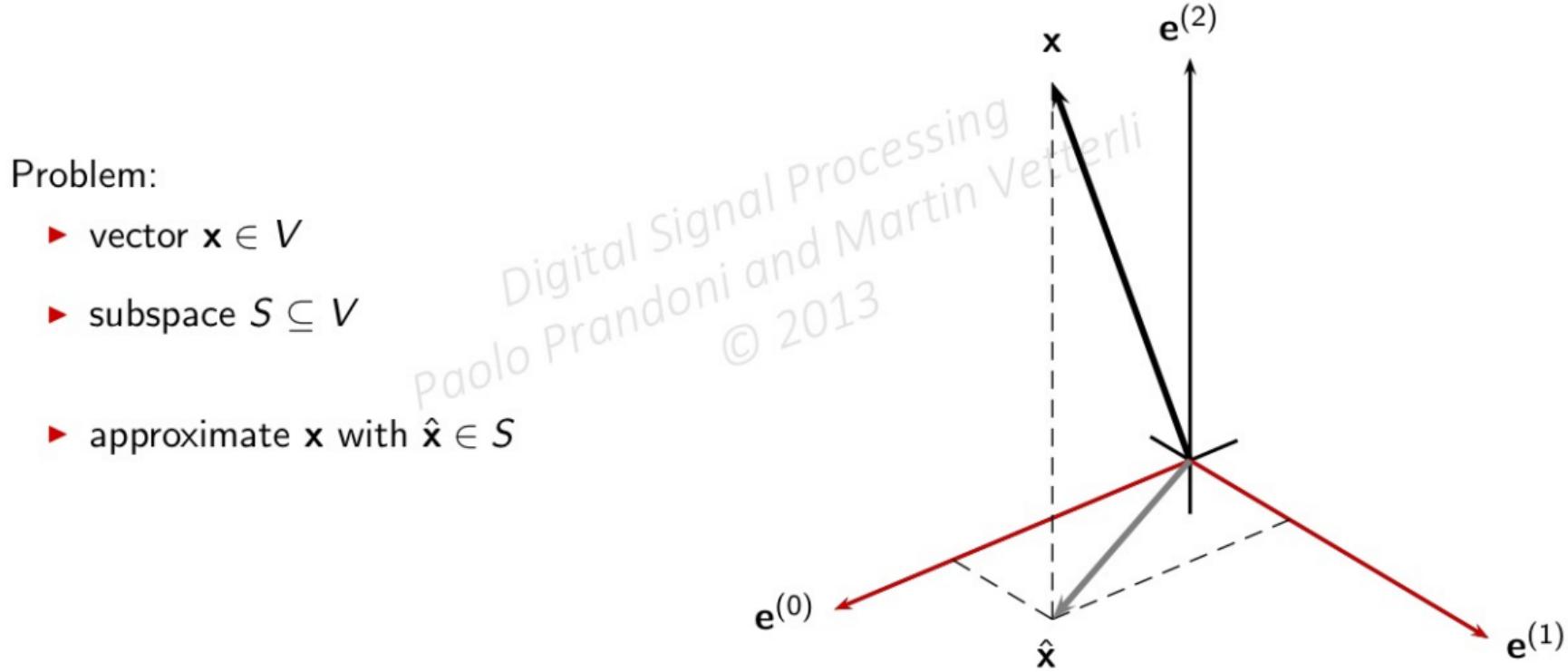
- ▶ key fact:  $\mathbf{R}^T \mathbf{R} = \mathbf{I}$



## Example

- ▶ square norm in canonical basis:  $\|\mathbf{x}\|^2 = \alpha_0^2 + \alpha_1^2$
- ▶ square norm in rotated basis:  $\|\mathbf{x}\|^2 = \beta_0^2 + \beta_1^2$
- ▶ let's verify Parseval:

$$\begin{aligned}\beta_0^2 + \beta_1^2 &= \boldsymbol{\beta}^T \boldsymbol{\beta} \\ &= (\mathbf{R}\boldsymbol{\alpha})^T (\mathbf{R}\boldsymbol{\alpha}) \\ &= \boldsymbol{\alpha}^T (\mathbf{R}^T \mathbf{R}) \boldsymbol{\alpha} \\ &= \boldsymbol{\alpha}^T \boldsymbol{\alpha} \\ &= \alpha_0^2 + \alpha_1^2\end{aligned}$$



- ▶  $\{\mathbf{s}^{(k)}\}_{k=0,1,\dots,K-1}$  orthonormal basis for  $S$
- ▶ orthogonal projection:

$$\hat{\mathbf{x}} = \sum_{k=0}^{K-1} \langle \mathbf{s}^{(k)}, \mathbf{x} \rangle \mathbf{s}^{(k)}$$

orthogonal projection is the “best” approximation over  $S$

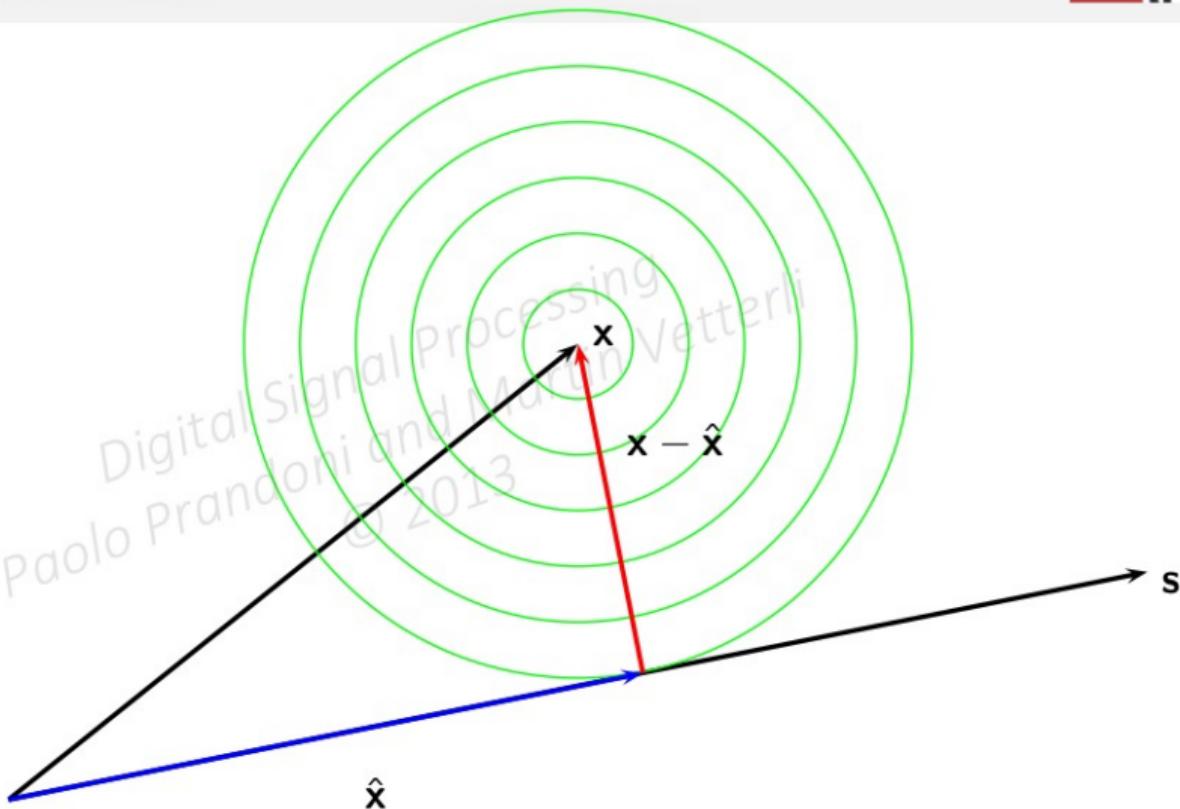
- ▶ orthogonal projection has minimum-norm error:

$$\arg \min_{\mathbf{y} \in S} \|\mathbf{x} - \mathbf{y}\| = \hat{\mathbf{x}}$$

- ▶ error is orthogonal to approximation:

$$\langle \mathbf{x} - \hat{\mathbf{x}}, \hat{\mathbf{x}} \rangle = 0$$

# Least Squares Approximation



- ▶ Hilbert space  $P_N[-1, 1] \subset L_2[-1, 1]$
- ▶ a self-evident, naive basis:  $\mathbf{s}^{(k)} = t^k, \quad k = 0, 1, \dots, N - 1$
- ▶ naive basis is not orthonormal

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goal: approximate  $\mathbf{x} = \sin t$  over  $P_3[-1, 1]$

- ▶ build orthonormal basis from naive basis
- ▶ project  $\mathbf{x}$  over the orthonormal basis
- ▶ compute approximation error
- ▶ compare error to Taylor approximation (well known but not optimal over the interval)

The Gram-Schmidt algorithm leads to an orthonormal basis for  $P_N([-1, 1])$   
(see appendix if interested in details)

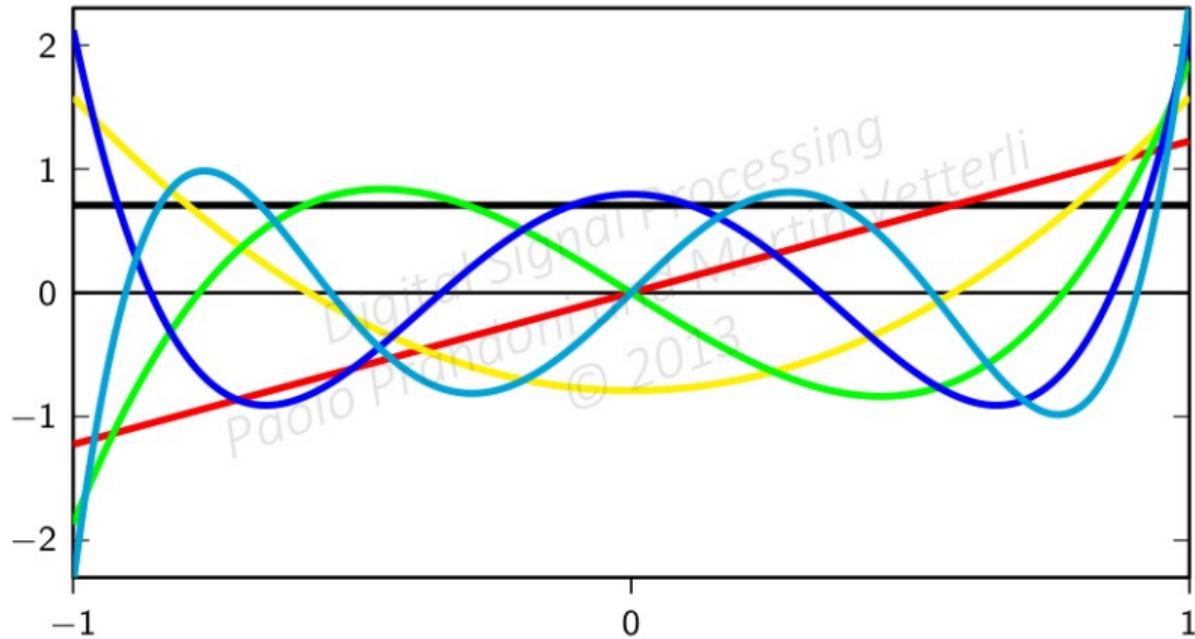
$$\mathbf{u}^{(0)} = \sqrt{1/2}$$

$$\mathbf{u}^{(1)} = \sqrt{3/2} t$$

$$\mathbf{u}^{(2)} = \sqrt{5/8}(3t^2 - 1)$$

$$\mathbf{u}^{(3)} = \dots$$

# Legendre Polynomials



$$\alpha_k = \langle \mathbf{u}^{(k)}, \mathbf{x} \rangle = \int_{-1}^1 u_k(t) \sin t \, dt$$

- ▶  $\alpha_0 = \langle \sqrt{1/2}, \sin t \rangle = 0$
- ▶  $\alpha_1 = \langle \sqrt{3/2} t, \sin t \rangle \approx 0.7377$
- ▶  $\alpha_2 = \langle \sqrt{5/8}(3t^2 - 1), \sin t \rangle = 0$

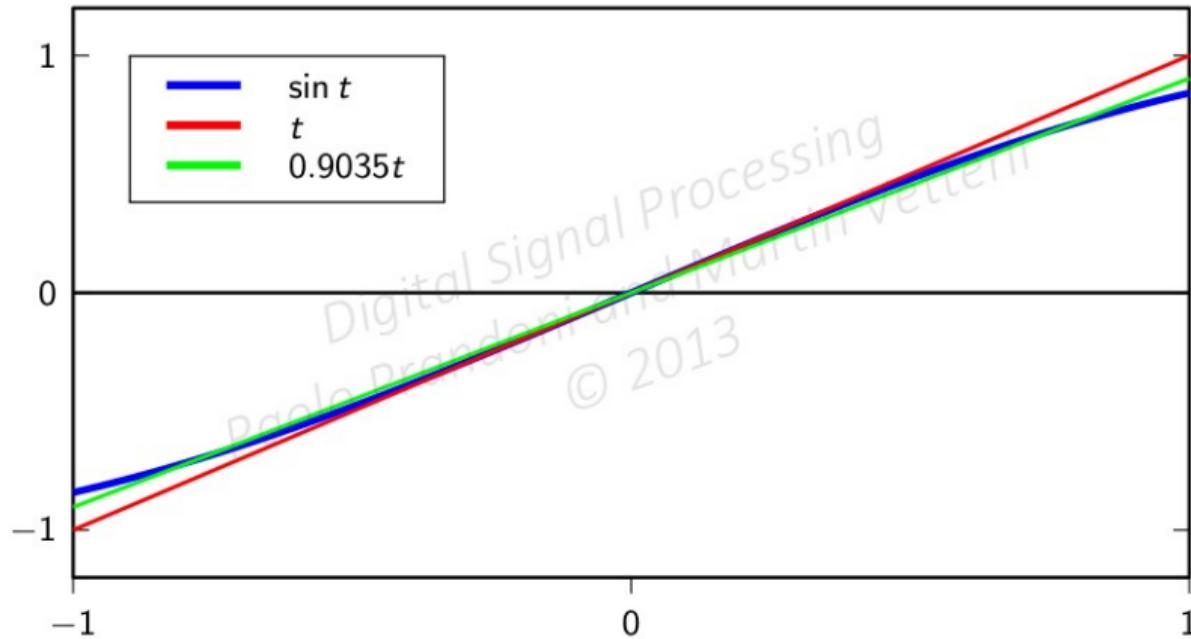
Using the orthogonal projection over  $P_3[-1, 1]$ :

$$\sin t \rightarrow \alpha_1 \mathbf{u}^{(1)} \approx 0.9035 t$$

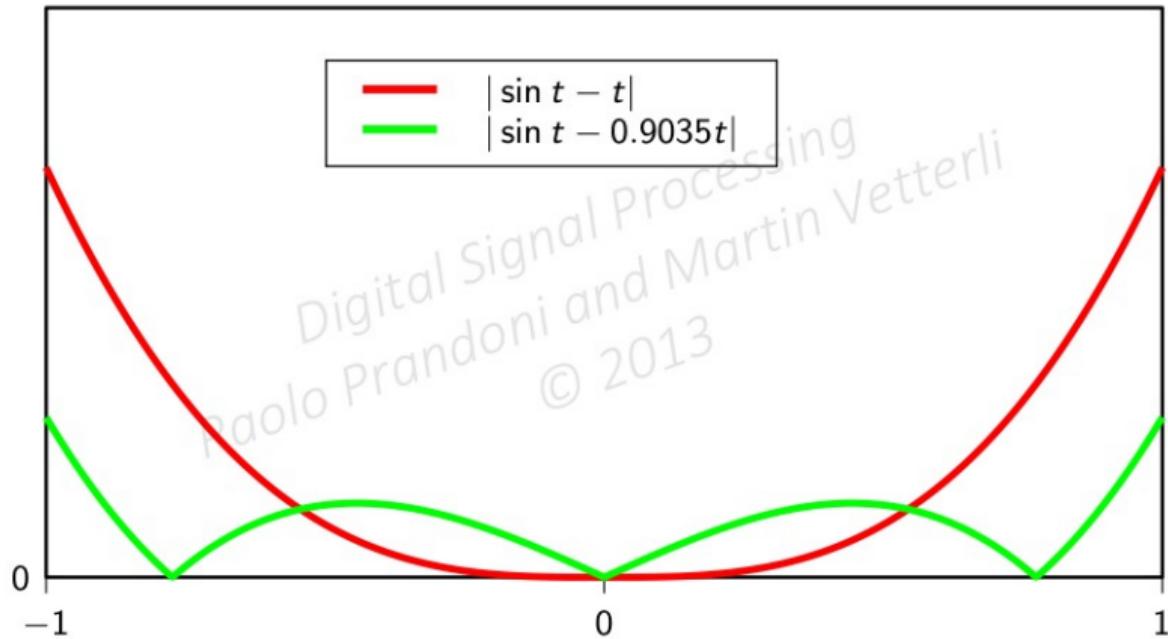
Using Taylor's series:

$$\sin t \approx t$$

# Sine approximation



## Approximation error



Orthogonal projection over  $P_3[-1, 1]$ :

$$\|\sin t - \alpha_1 \mathbf{u}^{(1)}\| \approx 0.0337$$

Taylor series:

$$\|\sin t - t\| \approx 0.0857$$

Why do we do all this?

- ▶ finite-length and periodic signals live in  $\mathbb{C}^N$
- ▶ infinite-length signals live in  $\ell_2(\mathbb{Z})$
- ▶ different bases are different observation tools for signals
- ▶ subspace projections are useful in filtering and compression

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# Appendix: orthonormalization of the naive polynomial basis

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Gram-Schmidt orthonormalization procedure:

$$\{\mathbf{s}^{(k)}\} \xrightarrow{\text{Gram-Schmidt}} \{\mathbf{u}^{(k)}\}$$

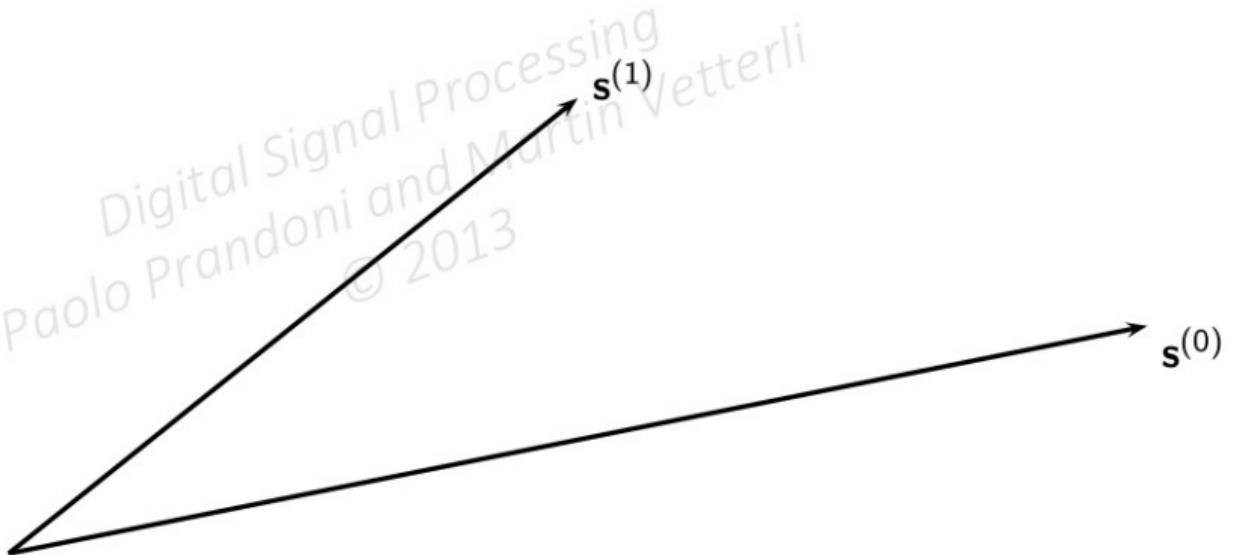
original set      orthonormal set

Algorithmic procedure: at each step  $k$

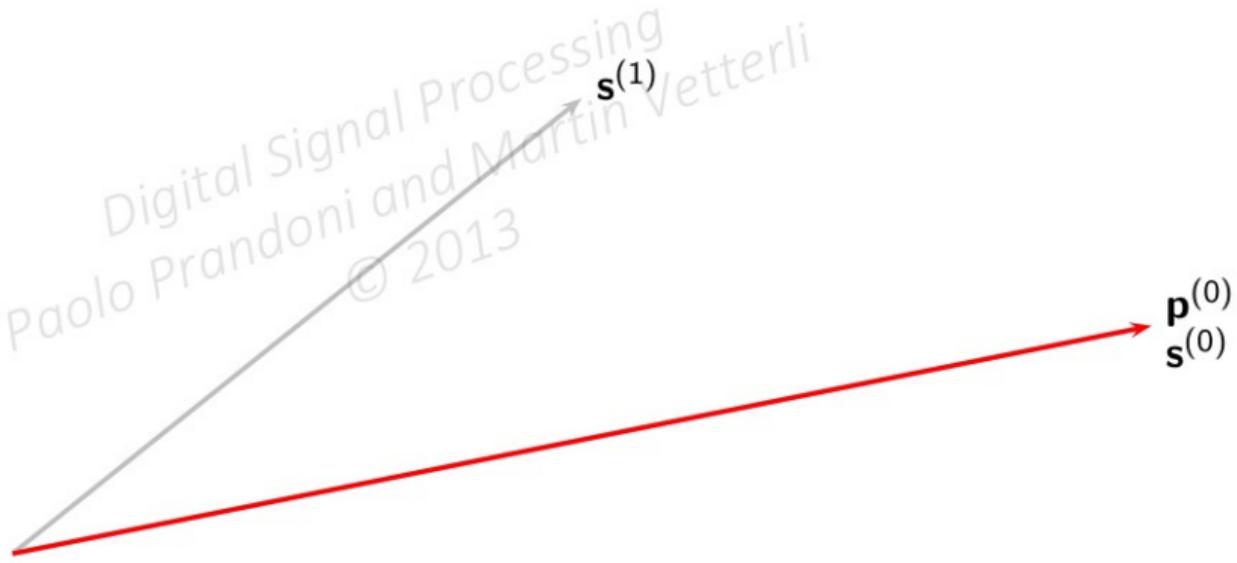
$$1. \quad \mathbf{p}^{(k)} = \mathbf{s}^{(k)} - \sum_{n=0}^{k-1} \langle \mathbf{u}^{(n)}, \mathbf{s}^{(k)} \rangle \mathbf{u}^{(n)}$$

$$2. \quad \mathbf{u}^{(k)} = \mathbf{p}^{(k)} / \|\mathbf{p}^{(k)}\|$$

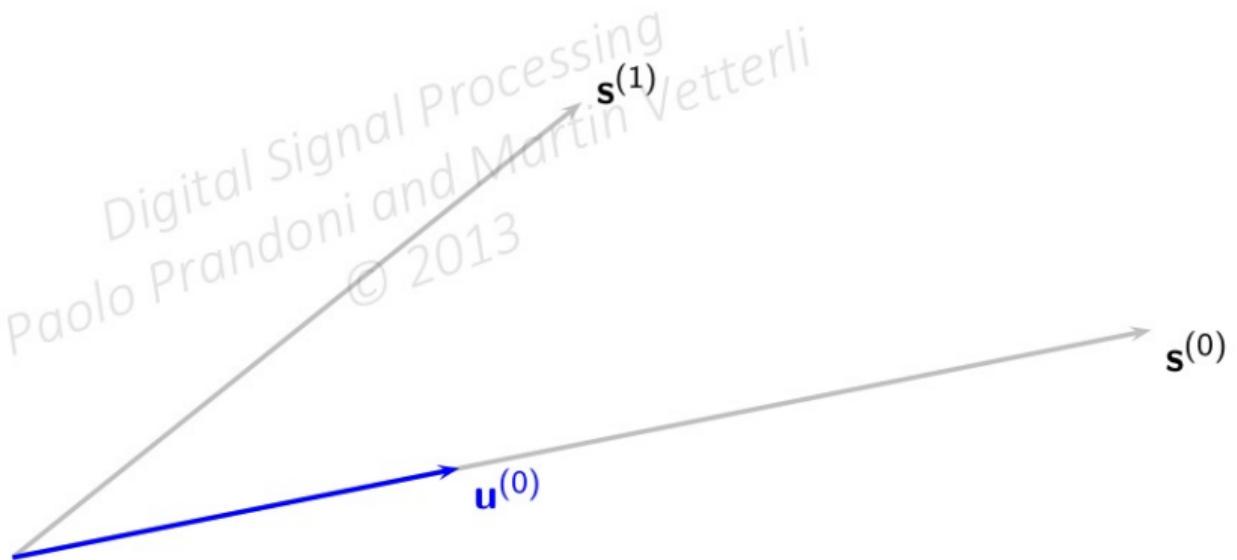
# Building an orthonormal basis



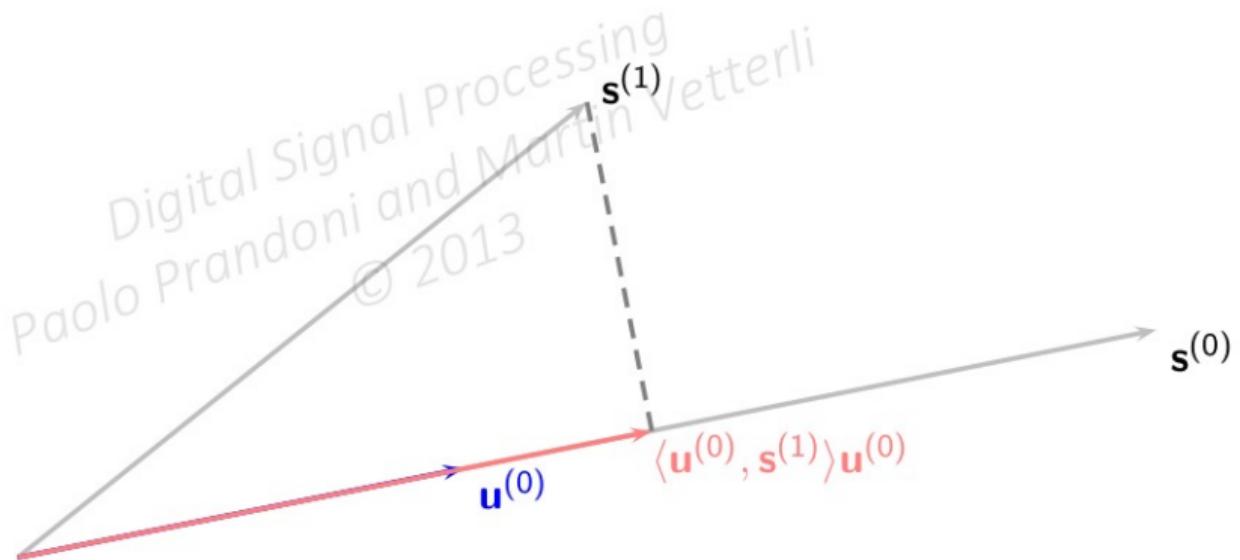
# Building an orthonormal basis



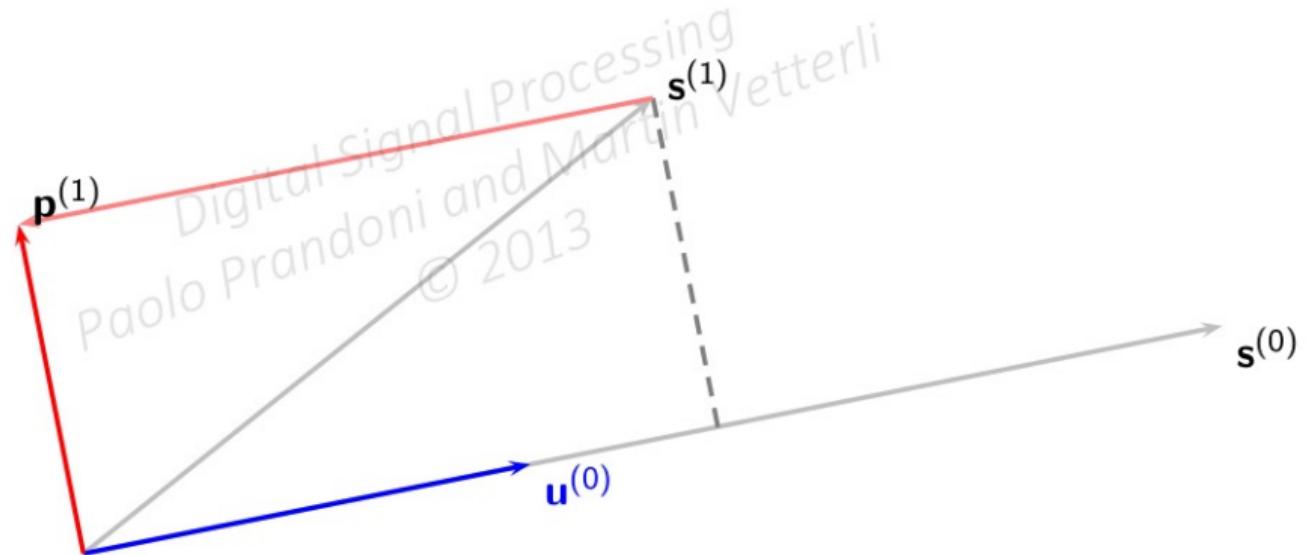
# Building an orthonormal basis



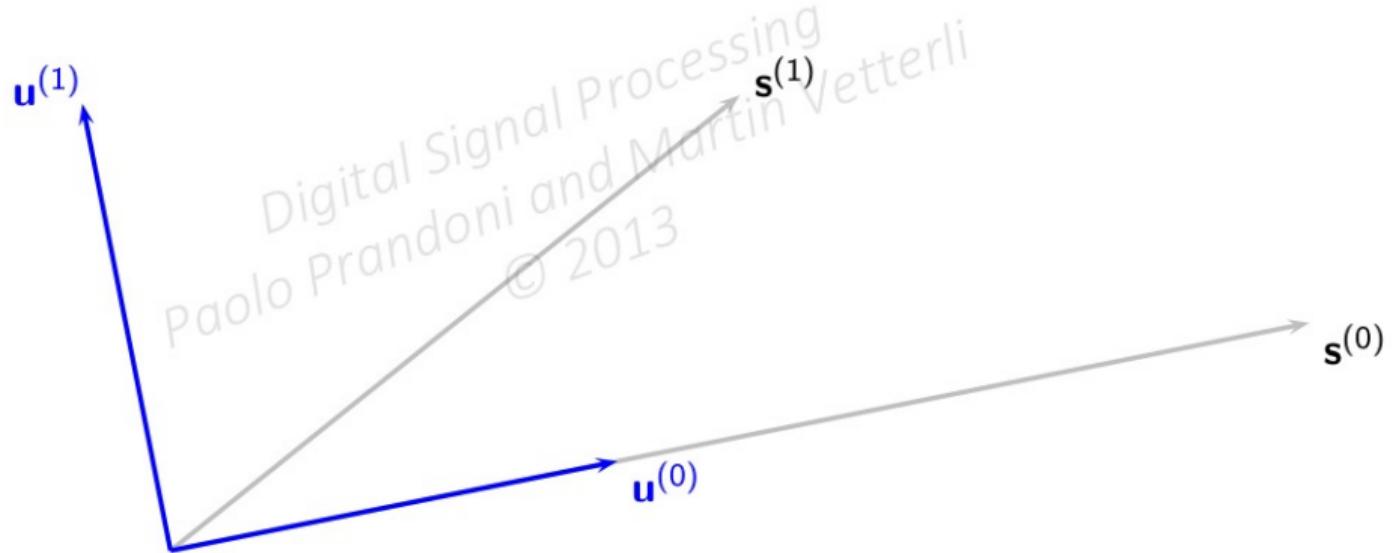
# Building an orthonormal basis



# Building an orthonormal basis



# Building an orthonormal basis



Gram-Schmidt orthonormalization of the naive basis:  $\{\mathbf{s}^{(k)}\} \rightarrow \{\mathbf{u}^{(k)}\}$

►  $\mathbf{s}^{(0)} = 1$

- $\mathbf{p}^{(0)} = \mathbf{s}^{(0)} = 1$
- $\|\mathbf{p}^{(0)}\|^2 = 2$
- $\mathbf{u}^{(0)} = \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2}$

►  $\mathbf{s}^{(1)} = t$

- $\langle \mathbf{u}^{(0)}, \mathbf{s}^{(1)} \rangle = \int_{-1}^1 t/\sqrt{2} = 0$
- $\mathbf{p}^{(1)} = \mathbf{s}^{(1)} = t$
- $\|\mathbf{p}^{(1)}\|^2 = 2/3$
- $\mathbf{u}^{(1)} = \sqrt{3/2} t$

►  $\mathbf{s}^{(2)} = t^2$

- $\langle \mathbf{u}^{(0)}, \mathbf{s}^{(2)} \rangle = \int_{-1}^1 t^2/\sqrt{2} = 2/3\sqrt{2}$
- $\langle \mathbf{u}^{(1)}, \mathbf{s}^{(2)} \rangle = \int_{-1}^1 t^3/\sqrt{2} = 0$
- $\mathbf{p}^{(2)} = \mathbf{s}^{(2)} - (2/3\sqrt{2})\mathbf{u}^{(0)} = t^2 - 1/3$
- $\|\mathbf{p}^{(2)}\|^2 = 8/45$
- $\mathbf{u}^{(2)} = \sqrt{5/8}(3t^2 - 1)$

$$\mathbf{u}^{(0)} = \sqrt{1/2}$$

$$\mathbf{u}^{(1)} = \sqrt{3/2} t$$

$$\mathbf{u}^{(2)} = \sqrt{5/8}(3t^2 - 1)$$

$$\mathbf{u}^{(3)} = \dots$$