Fluid Transport

1 Analyse

1.1 Dell Operator ∇

1.1.1 The gradient

The *gradient* of a *scalar field* $f(x, y, z)$ is a vector field function that can be represented as:

$$\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

1.1.2 The divergence

The *divergence* of a *vector field* $\vec{v}(x, y, z) = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$ is a *scalar field* function that can be represented as:

$$\operatorname{div} \vec{v} = \nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

1.1.3 The Curl

The *Curl* of a vector field $\vec{v}(x, y, z) = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$ is a *vector field* function that can be represented as:

$$\operatorname{curl} \vec{v} = \nabla \times \vec{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

1.1.4 Laplace operator

The *Laplace operator* of either a vector or a scalar fields is a scalar operator that can be represented as:

$$\Delta = \nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

1.2 Derivative

1.2.1 Dependent and independent variables

The "dependent variable" represents the output or effect while the "independent variables" represent the inputs or causes.

1.2.2 Type of derivative

The derivative measures the sensitivity to change of a dependent variable which is determined by another quantity called independent variable. For a function $f(x_1, x_2, \cdots, x_n)$ depending on several variables,

- His partial derivative (denoted with ∂) with respect of one variable x_i keep the others held constant:

$$\frac{\partial f}{\partial x_i}$$

- And his total derivative with respect to x_i does not assume that the other arguments are constant while x_i varies. The total derivative adds in these indirect dependencies to find the overall dependency of f on x_i.

$$\frac{\mathrm{d}\, f}{\mathrm{d}\, x_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\frac{\mathrm{d}\, x_i}{\mathrm{d}\, x_j} = \frac{\partial f}{\partial x_i} + \sum_{j=1,\cdots,i-1,i+1,\cdots n} \frac{\partial f}{\partial x_j}\frac{\mathrm{d}\, x_i}{\mathrm{d}\, x_j}$$

1.2.3 Chain Rule

The chain rule is a formula for computing the derivative of the composition of two or more functions.

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

2 Tools

2.1 Conserved quantity

2.2 Flux

A flux is defined as the rate of flow of a property per unit area, which has the dimensions [quantity][time]^1[area]^1 and express as

$$\boxed{j = \lim_{A \to 0} \frac{I}{A} = \frac{dI}{dA}}$$

where $I = \lim_{\Delta t \to 0} \frac{\Delta q}{\Delta t} = \frac{dq}{dt}$ is the flow of quantity q per unit time t, and A is the area through which the quantity flows.

Inversely, the total amount q over an area a during a time $t_2 - t_1$ with a flux j is computed with

$$q = \int_{t_1}^{t_2} \iint_S \mathbf{j} \cdot \hat{\mathbf{n}} \, \mathrm{d}A \, \mathrm{d}t$$

2.3 Incompressible flow

2.4 Diffusion, Advection and Dispersion

2.5 Convection

2.6 Lagrangian vs Eulerian

2.7 Gauss theorem

3 Continuity equation

A *conservation law* states that a particular measurable property of an isolated physical system does not change as the system evolves over time. While it does not mention transport, the *continuity equation* is the correct mathematical expression for a *local* conservation law.

The continuity equations states that the amount q can only increased (decresed) with an inward (outward) flow or internal creation.

$$\boxed{\frac{dq}{dt} + \oiint_S \mathbf{j} \cdot d\mathbf{S} = \Sigma \quad \text{or} \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = \sigma}$$

Where S is any closed surface, j is the flux of q, ρ is q per unit volume and Σ is the source/sink term.

3.1 Material derivative

$$\frac{D}{Dt} \stackrel{\text{def}}{=} \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

4 Navier-Stock

4.1 Newton Second Law of Motion

The second law states that the net force $\mathbf{F}$ on an object is equal to the rate of change of its momentum $\mathbf{p}$

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\mathbf{v})}{\mathrm{d}t} = \frac{\partial\,(m\mathbf{v})}{\partial t} + \sum_i \frac{\partial\,(m\mathbf{v})}{\partial x_i}\frac{\partial x_i}{\partial t}$$

4.2 Derivation of the Convective Term

Momemtum change by convection:

$$\int \frac{\partial\,|m\overrightarrow{v}|}{\partial t}dt + \left[\frac{\partial m\overrightarrow{v}}{\partial x}dx\right]\cdot\overrightarrow{i} + \left[\frac{\partial m\overrightarrow{v}}{\partial y}dy\right]\cdot\overrightarrow{j} + \left[\frac{\partial m\overrightarrow{v}}{\partial z}dz\right]\cdot\overrightarrow{k}$$

Momentum change per time by convection:

$$\frac{\partial}{\partial t}\left\{\int \frac{\partial\,|m\overrightarrow{v}|}{\partial t}dt + \left[\frac{\partial m\overrightarrow{v}}{\partial x}dx\right]\cdot\overrightarrow{i} + \left[\frac{\partial m\overrightarrow{v}}{\partial y}dy\right]\cdot\overrightarrow{j} + \left[\frac{\partial m\overrightarrow{v}}{\partial z}dz\right]\cdot\overrightarrow{k}\right\}$$

Rewritten:

$$\frac{\partial\,|m\overrightarrow{v}|}{\partial t} + \left[\frac{\partial m\overrightarrow{v}}{\partial x}\frac{\partial x}{\partial t}\right]\cdot\overrightarrow{i} + \left[\frac{\partial m\overrightarrow{v}}{\partial y}\frac{\partial x}{\partial t}\right]\cdot\overrightarrow{j} + \left[\frac{\partial m\overrightarrow{v}}{\partial z}\frac{\partial x}{\partial t}\right]\cdot\overrightarrow{k}$$

For an infinitesimal volume, with uniform density:

$$\rho\left\{\frac{\partial\,|\vec{v}|}{\partial t} + \left[\frac{\partial\vec{v}}{\partial x}v_x\right]\cdot\vec{i} + \left[\frac{\partial\vec{v}}{\partial y}v_y\right]\cdot\vec{j} + \left[\frac{\partial\vec{v}}{\partial z}v_z\right]\cdot\vec{k}\right\}d\cancel{V}$$

4.3 Derivation of the Forcing Term

$$\sum\overrightarrow{F} = \overrightarrow{F_g} + \overrightarrow{F_p} + \overrightarrow{F_f} + \overrightarrow{F}$$

$$\overrightarrow{F_g} = m\overrightarrow{g}$$

$$\overrightarrow{F_p} = -\left\{\frac{\partial P}{\partial x} + \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\right\}d\cancel{V} = -\nabla P d\cancel{V}$$

$$\overrightarrow{F_f} = \left\{\sum_{j=x,y,z}\sum_{i=x,y,z}\frac{\partial\tau_{i,j}}{\partial j}\right\}d\cancel{V} = \nabla\cdot T d\cancel{V}$$

$$\tau_{i,j} = \begin{cases} \mu\left(\frac{\partial v_i}{dj} + \frac{\partial v_j}{di}\right) & j \neq i \\ -\frac{2}{3}\mu\nabla\cdot\overrightarrow{v} + 2\frac{\partial v_i}{di} & j = i \end{cases} \quad j, i = x, y, z$$

Which end up to:

$$\sum\overrightarrow{F} = m\overrightarrow{g} - \nabla P d\cancel{V} + \nabla\cdot T d\cancel{V} + \overrightarrow{F}$$

4.4 Finale expression

$$\rho\left\{\frac{\partial\,|\vec{v}|}{\partial t} + \left[\frac{\partial\vec{v}}{\partial x}v_x\right]\cdot\vec{i} + \left[\frac{\partial\vec{v}}{\partial y}v_y\right]\cdot\vec{j} + \left[\frac{\partial\vec{v}}{\partial z}v_z\right]\cdot\vec{k}\right\}d\cancel{V} = m\overrightarrow{g} - \nabla P d\cancel{V} + \nabla\cdot T d\cancel{V} + \overrightarrow{F}$$

$$\rho\left(\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p + \nabla\cdot\mathbf{T} + \mathbf{f}$$

5 Euler Equation

the Euler equations are a set of quasilinear hyperbolic equations governing adiabatic and inviscid flow.

$$\frac{D\mathbf{u}}{Dt} = -\nabla w + \mathbf{g}$$
$$\nabla\cdot\mathbf{u} = 0$$

5.1 The vorticity equation

6 Convectiondiffusion equation

The convectiondiffusion equation can be easily derived from the continuity equation (**??**) where the total flux is composed by a diffusive flux (Fick's law) (ref) and an advective flux **??**:

$$\frac{\partial c}{\partial t} = \nabla\cdot(D\nabla c) - \nabla\cdot(\vec{v}c) + R$$

Comments:

- Assumption: D and $\vec{v}$ only vary with space and time and not concentration otherwise the equation become non-linear
- Comment simplification: the diffusion coefficient is constant, there are no sources or sinks, and the velocity field describes an incompressible flow (i.e., it has zero divergence)

$$\frac{\partial c}{\partial t} = D\nabla^2 c - \vec{v}\cdot\nabla c.$$