Digital Signal Processing

Module 5: Linear Filters

Module Overview:

- Module 5.1–5.3: LTI systems and convolution; examples; stability
- Module 5.4: Convolution theorem
- Module 5.5–5.6: Ideal filters and their approximation
- Module 5.7–5.8: Realizable filters; implementation
- Module 5.9–5.10: Filter design

Digital Signal Processing

Module 5.1: Linear Filters

Overview:

- Linearity and time invariance
- Convolution

A generic signal processing device

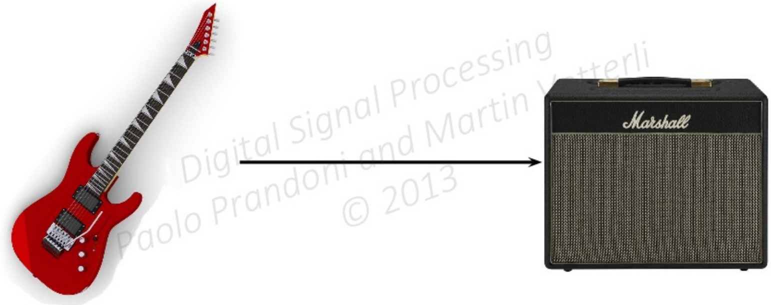
$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n]$$

$$y[n] = \mathcal{H}\{x[n]\}$$

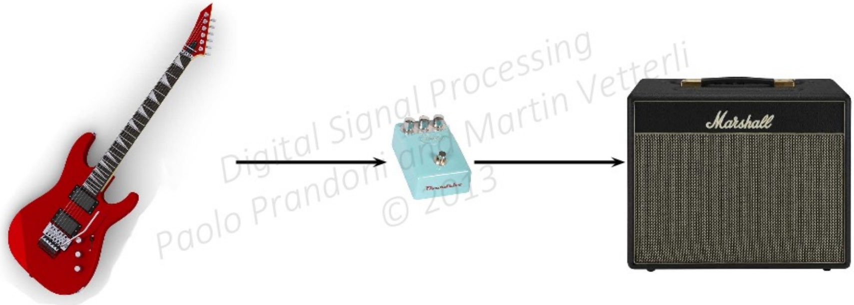
Linearity

$$\mathcal{H}\{\alpha\, x_1[n] + \beta\, x_2[n]\} = \alpha\, \mathcal{H}\{x_1[n]\} + \beta\, \mathcal{H}\{x_2[n]\}$$

Linearity



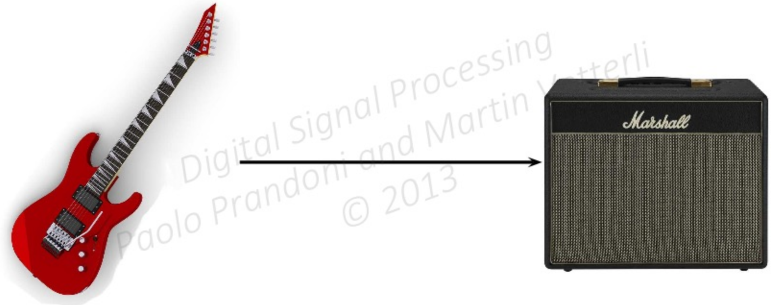
(Non) Linearity



Time invariance

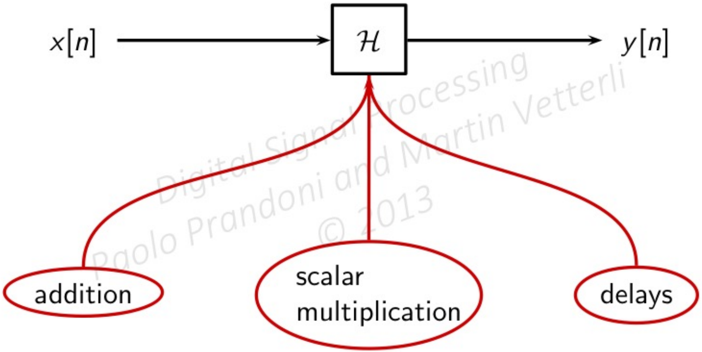
$$y[n] = \mathcal{H}\{x[n]\} \iff \mathcal{H}\{x[n-n_0]\} = y[n-n_0]$$

Time invariance



Time (in)variance

Linear, time-invariant systems



Linear, time-invariant systems

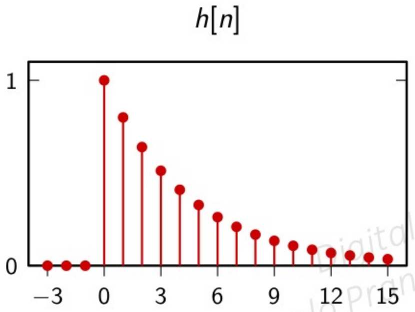
$$y[n] = H(x[n], x[n-1], x[n-2], \ldots, y[n-1], y[n-2], \ldots)$$

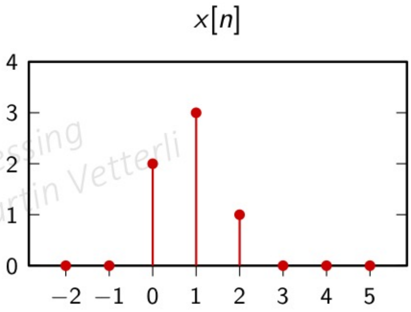
with $H(\cdot)$ a linear function of its arguments

$$h[n] = \mathcal{H}\{\delta[n]\}$$

Fundamental result: impulse response fully characterizes the LTI system!

Example





$$h[n] = \alpha^n \, u[n]$$

$$x[n] = \begin{cases} 2 & n = 0 \\ 3 & n = 1 \\ 1 & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

Example

- $x[n] = 2\delta[n] + 3\delta[n-1] + \delta[n-2]$
- we know the impulse response $h[n] = \mathcal{H}\{\delta[n]\}$;
- compute $y[n] = \mathcal{H}\{x[n]\}$ exploiting linearity and time-invariance

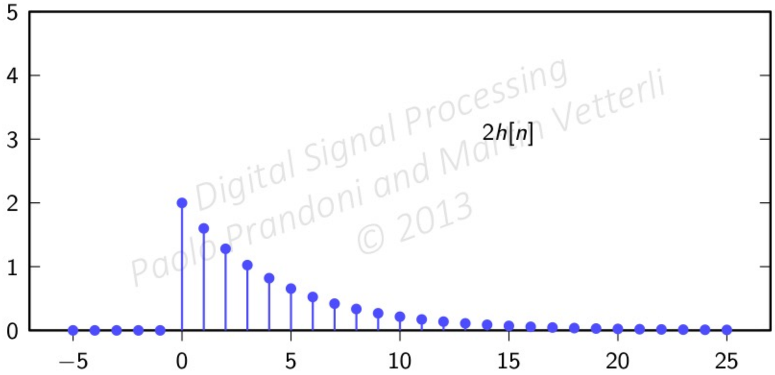
Example

$$y[n] = \mathcal{H}\{2\delta[n] + 3\delta[n-1] + \delta[n-2]\}$$

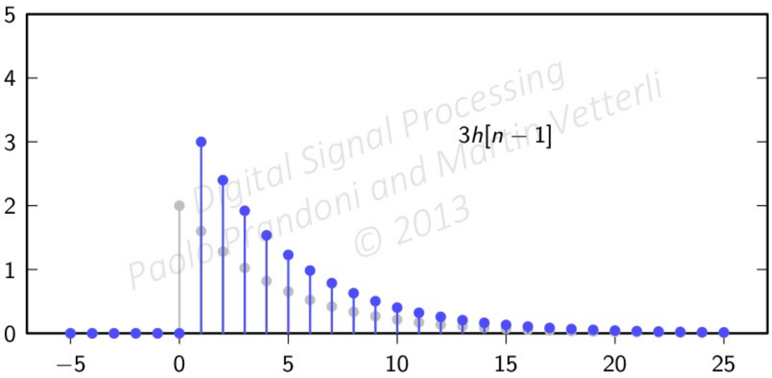
$$= 2\mathcal{H}\{\delta[n]\} + 3\mathcal{H}\{\delta[n-1]\} + \mathcal{H}\{\delta[n-2]\}$$

$$= 2h[n] + 3h[n-1] + h[n-2]$$

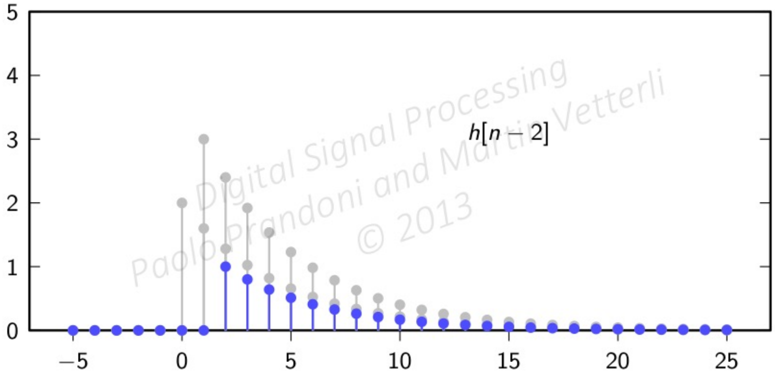
Example



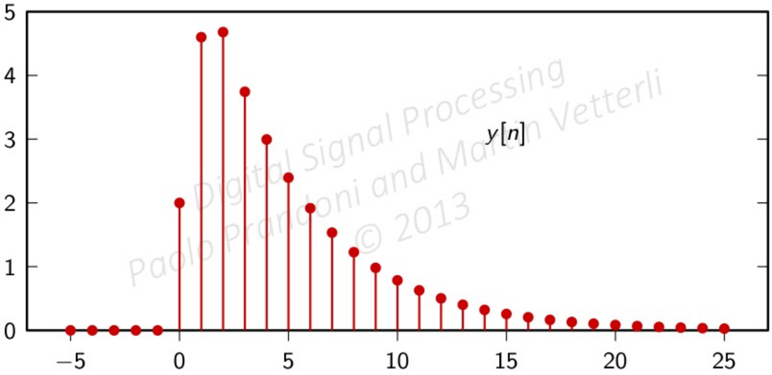
Example



Example



Example



Convolution

We can always write (remember Module 3.2):

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

by linearity and time invariance:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$= x[n] * h[n]$$

Performing the convolution algorithmically

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

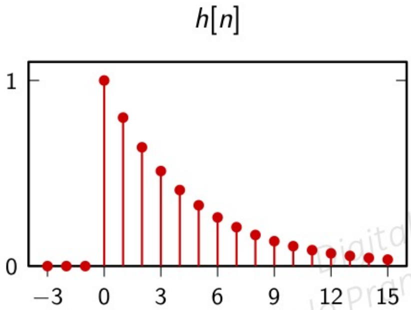
The recipe:

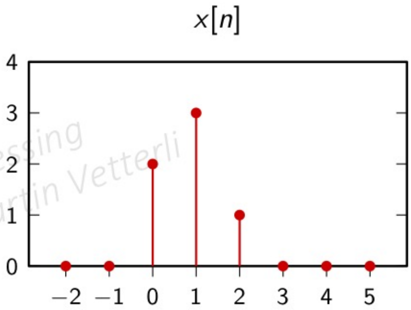
- time-reverse $h[m]$
- at each step n (from $-\infty$ to ∞):
  - center the time-reversed $h[m]$ in n (i.e. shift by $-n$)
  - compute the inner product

Ingredients:

- a sequence $x[m]$
- a second sequence $h[m]$

Same example, different perspective

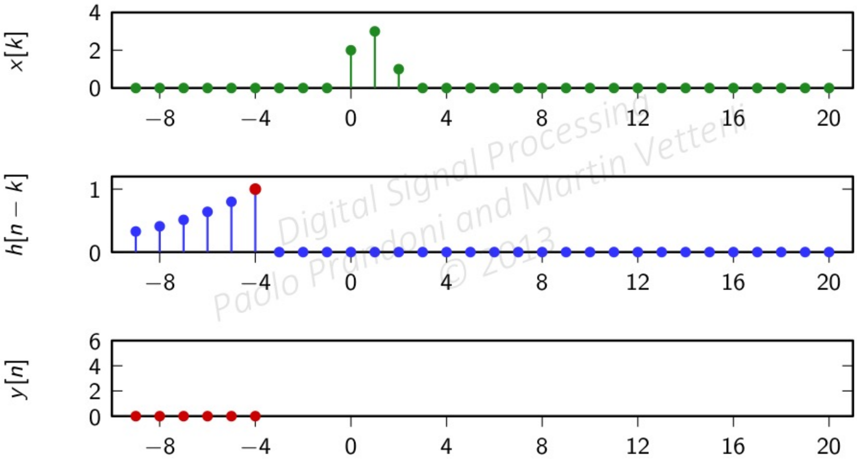




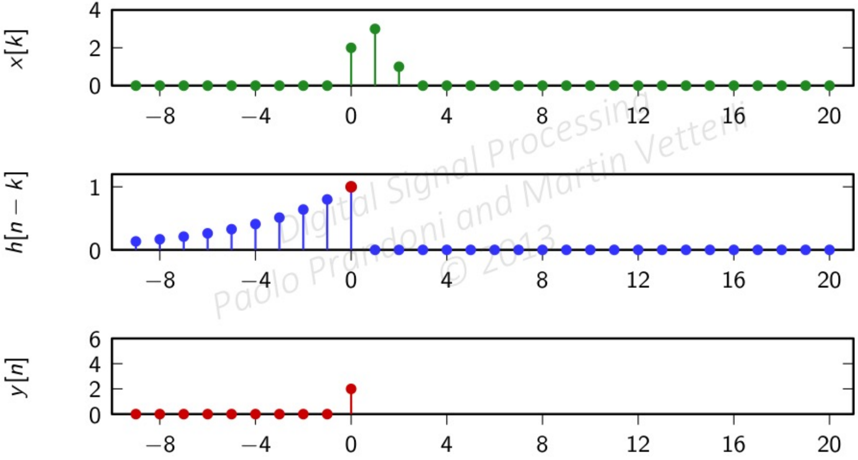
$$h[n] = \alpha^n \, u[n]$$

$$x[n] = \begin{cases} 2 & n = 0 \\ 3 & n = 1 \\ 1 & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

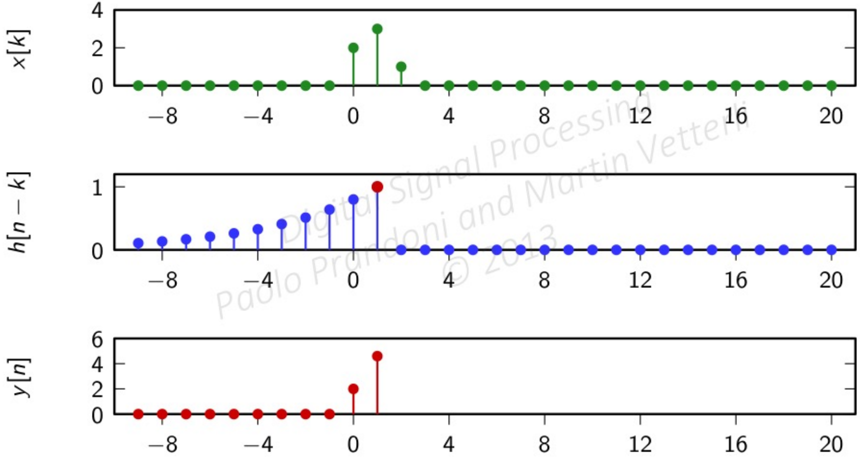
Convolution example



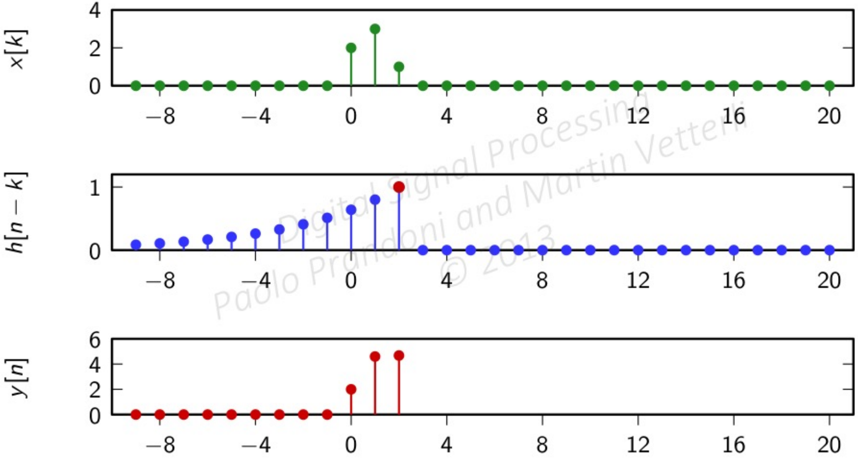
Convolution example



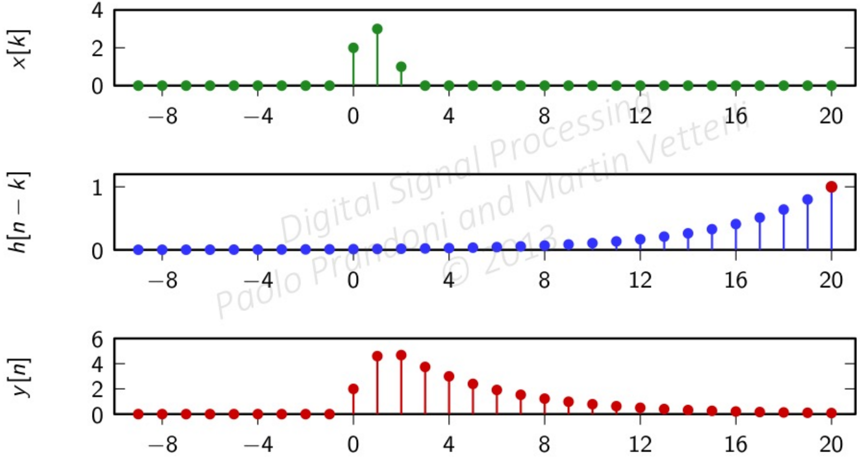
Convolution example



Convolution example

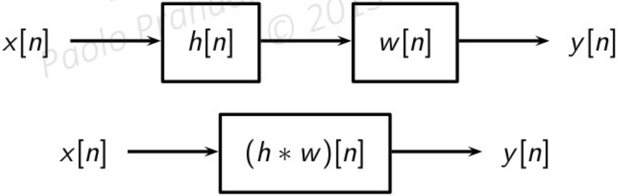


Convolution example



Convolution properties

- linearity and time invariance (by definition)
- commutativity: $(x * h)[n] = (h * x)[n]$
- associativity for absolutely- and square-summable sequences:
$((x * h) * w)[n] = (x * (h * w))[n]$



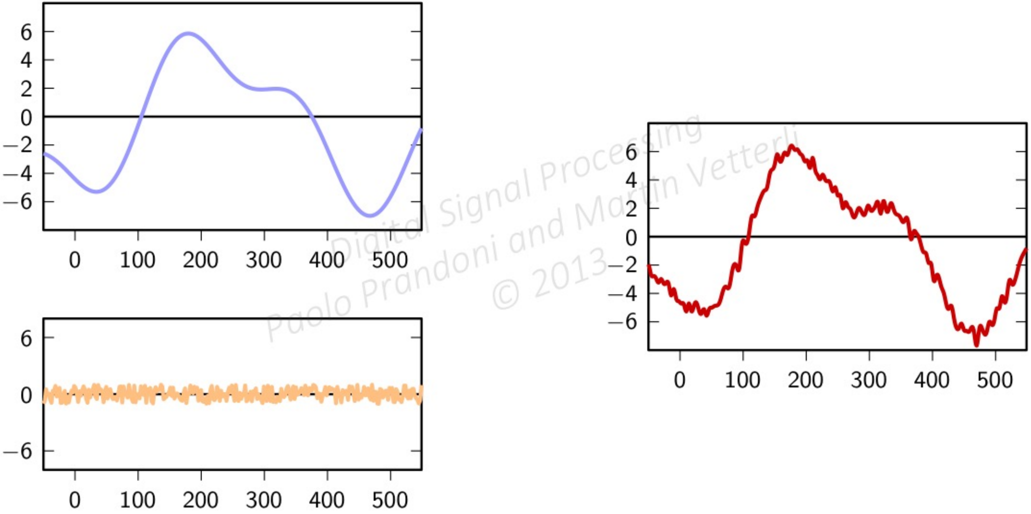
Digital Signal Processing

Module 5.2: Filtering by example

Overview:

- Moving average filter
- Leaky integrator

Typical filtering scenario: denoising

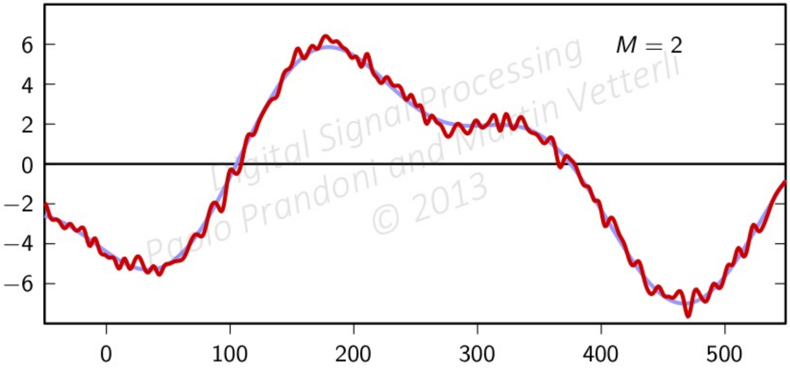


Denoising by Moving Average

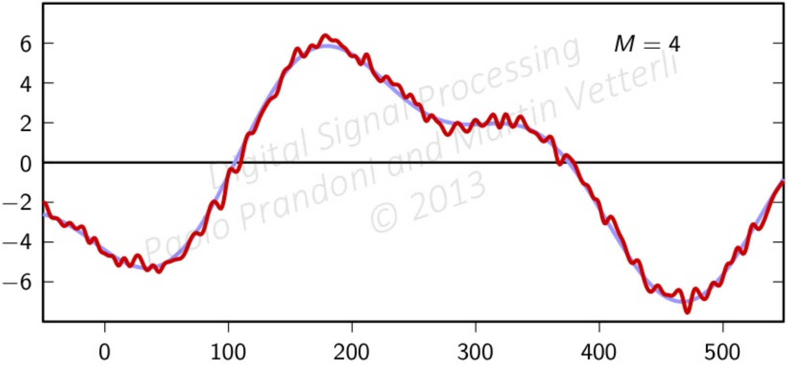
- idea: replace each sample by the local average
- for instance: $y[n] = (x[n] + x[n-1])/2$
- more generally:

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$$

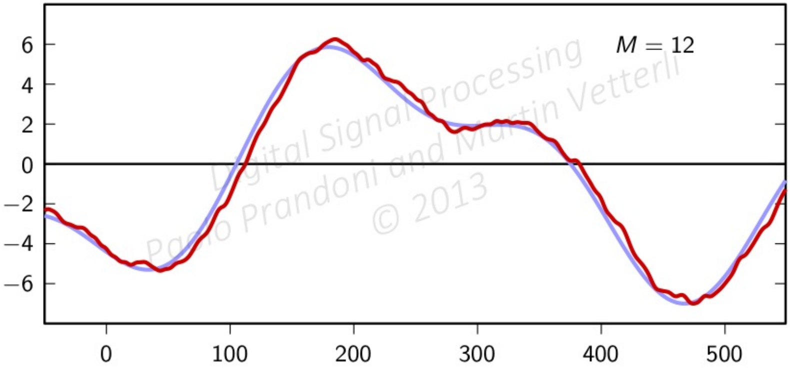
Denoising by Moving Average



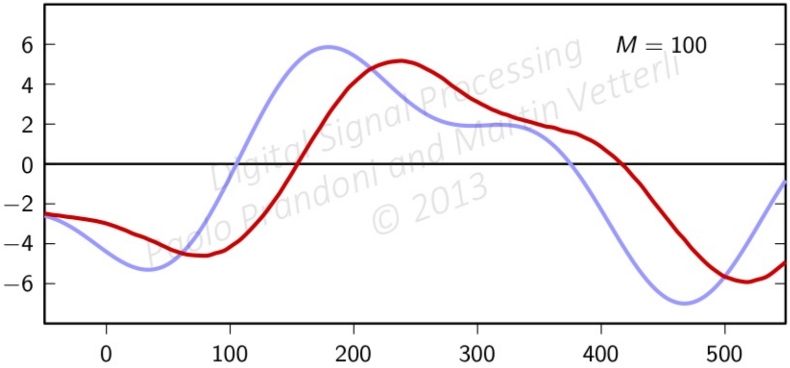
Denoising by Moving Average



Denoising by Moving Average



Denoising by Moving Average

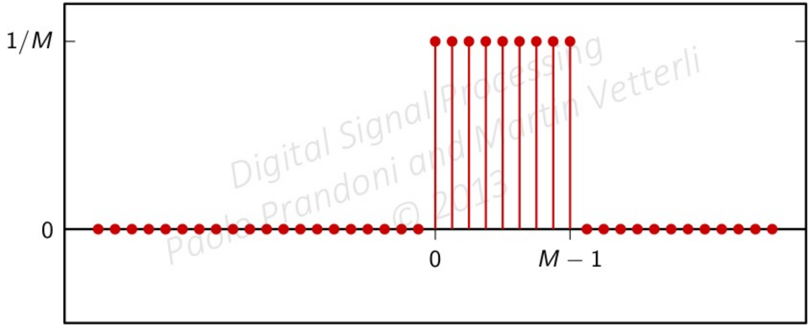


MA: impulse response

$$h[n] = \frac{1}{M} \sum_{k=0}^{M-1} \delta[n-k]$$

$$= \begin{cases} 1/M & \text{for } 0 \leq n < M \\ 0 & \text{otherwise} \end{cases}$$

MA: impulse response



MA: analysis

- smoothing effect proportional to M
- number of operations and storage also proportional to M

From the MA to a first-order recursion

$$y_M[n] = \underbrace{\boxed{\frac{1}{M}\left(x[n] + x[n-1] + x[n-2] + \ldots + x[n-M+1]\right)}}_{\text{moving average over } M \text{ points}}$$

From the MA to a first-order recursion

$$y_M[n] = \frac{1}{M}x[n] + \boxed{\frac{1}{M}\left(x[n-1] + x[n-2] + \ldots + x[n-M+1]\right)}$$

"almost" $y_{M-1}[n-1]$

i.e., moving average over $M-1$ points, delayed by one

From the MA to a first-order recursion

Formally:

$$y_M[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

$$y_M[n-1] = \frac{1}{M}\sum_{k=0}^{M-1} x[(n-1)-k]$$

From the MA to a first-order recursion

Formally:

$$y_M[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

$$y_M[n-1] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-(k+1)]$$

From the MA to a first-order recursion

Formally:

$$y_M[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$$

$$y_M[n-1] = \frac{1}{M} \sum_{k=0+1}^{M-1+1} x[n-k]$$

From the MA to a first-order recursion

Formally:

$$y_M[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

$$y_M[n-1] = \frac{1}{M}\sum_{k=1}^{M} x[n-k]$$

From the MA to a first-order recursion

Formally:

$$y_M[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

$$y_M[n-1] = \frac{1}{M}\sum_{k=1}^{M} x[n-k]$$

$$y_{M-1}[n] = \frac{1}{M-1}\sum_{k=0}^{M-2} x[n-k]$$

From the MA to a first-order recursion

Formally:

$$y_M[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

$$y_M[n-1] = \frac{1}{M}\sum_{k=1}^{M} x[n-k]$$

$$y_{M-1}[n-1] = \frac{1}{M-1}\sum_{k=1}^{M-1} x[n-k]$$

From the MA to a first-order recursion

$$y_M[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k] \qquad y_{M-1}[n-1] = \frac{1}{M-1}\sum_{k=1}^{M-1} x[n-k]$$

$$\sum_{k=0}^{M-1} x[n-k] = x[n] + \sum_{k=1}^{M-1} x[n-k]$$

$$My_M[n] = x[n] + (M-1)y_{M-1}[n-1]$$

From the MA to a first-order recursion

$$y_M[n] = \frac{M-1}{M} y_{M-1}[n-1] + \frac{1}{M} x[n]$$

$$y_M[n] = \lambda y_{M-1}[n-1] + (1-\lambda) x[n], \qquad \lambda = \frac{M-1}{M}$$

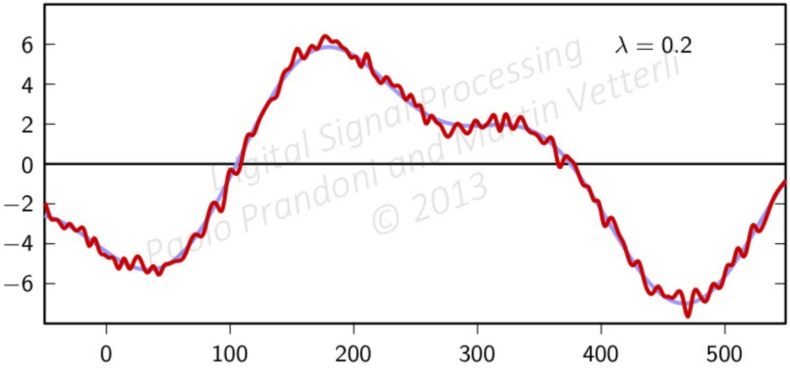
The Leaky Integrator

- when M is large, $y_{M-1}[n] \approx y_M[n]$ (and $\lambda \approx 1$)
- try the filter

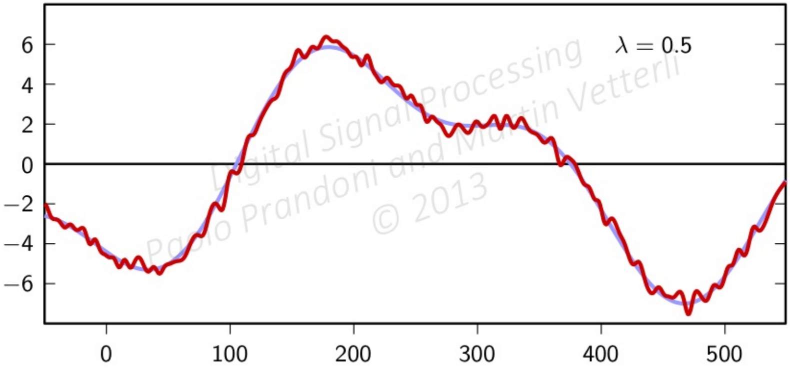
$$y[n] = \lambda y[n-1] + (1-\lambda)x[n]$$

- filter is now recursive, since it uses its previous output value

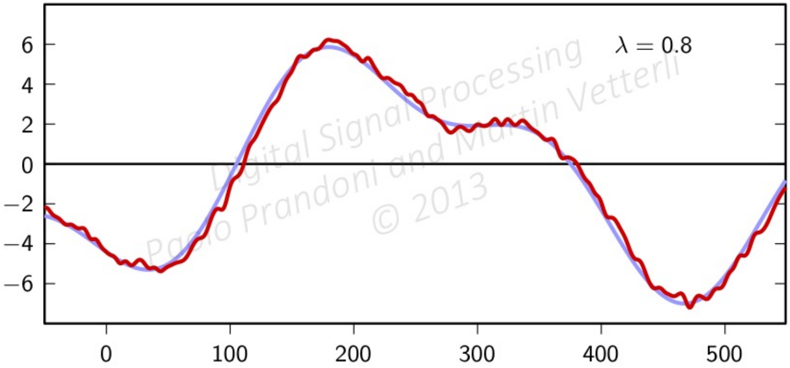
Denoising recursively with the Leaky Integrator



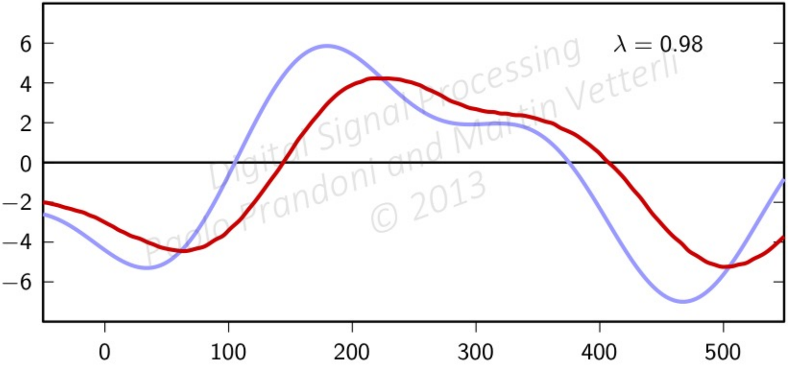
Denoising recursively with the Leaky Integrator



Denoising recursively with the Leaky Integrator



Denoising recursively with the Leaky Integrator



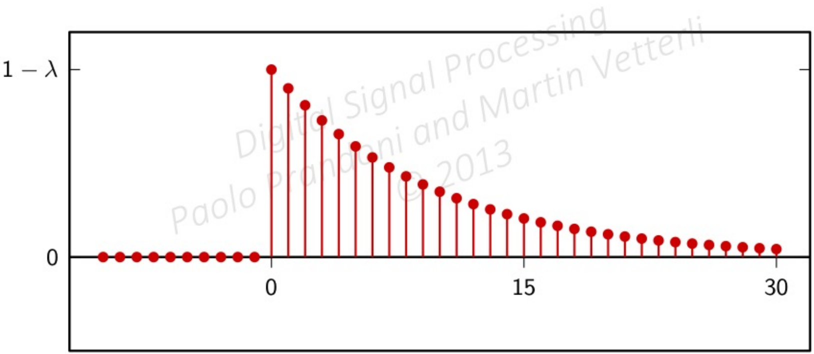
What about the impulse response?

$$y[n] = \lambda y[n-1] + (1-\lambda)\delta[n]$$

- $y[n] = 0$ for all $n < 0$
- $y[0] = \lambda y[-1] + (1-\lambda)\delta[0] = (1-\lambda)$
- $y[1] = \lambda y[0] + (1-\lambda)\delta[1] = \lambda(1-\lambda)$
- $y[2] = \lambda y[1] + (1-\lambda)\delta[2] = \lambda^2(1-\lambda)$
- $y[3] = \lambda y[2] + (1-\lambda)\delta[3] = \lambda^3(1-\lambda)$
- ...

Impulse response

$$h[n] = (1 - \lambda)\lambda^n \, u[n]$$



Leaky Integrator: why the name

Discrete-time integrator is a boundless accumulator:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

We can rewrite the integrator as

$$y[n] = y[n-1] + x[n]$$

Leaky Integrator: why the name

To prevent "explosion" pick $\lambda < 1$

$$y[n] = \lambda y[n-1] + (1-\lambda)x[n]$$

keep only a fraction λ of the accumulated value so far and forget ("leak") a fraction $1-\lambda$

add only a fraction $1-\lambda$ of the current value to the accumulator

Digital Signal Processing

Module 5.3: Filter Stability

Overview:

- Filter classification in the time domain
- Filter stability

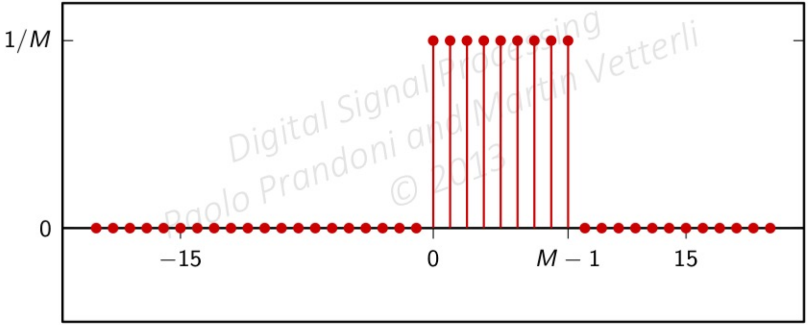
Filter types according to impulse response

- Finite Impulse Response (FIR)
- Infinite Impulse Response (IIR)
- causal
- noncausal

- impulse response has finite support
- only a finite number of samples are involved in the computation of each output sample

FIR (example)

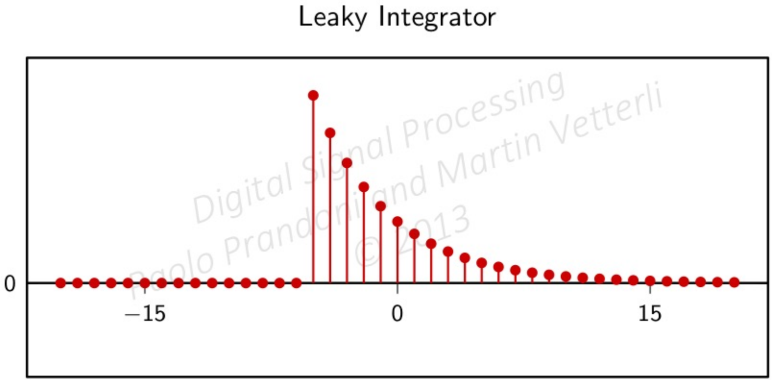
Moving Average filter



IIR

- impulse response has infinite support
- a potentially infinite number of samples are involved in the computation of each output sample
- surprisingly, in many cases the computation can still be performed in a finite amount of steps

IIR (example)


Leaky Integrator
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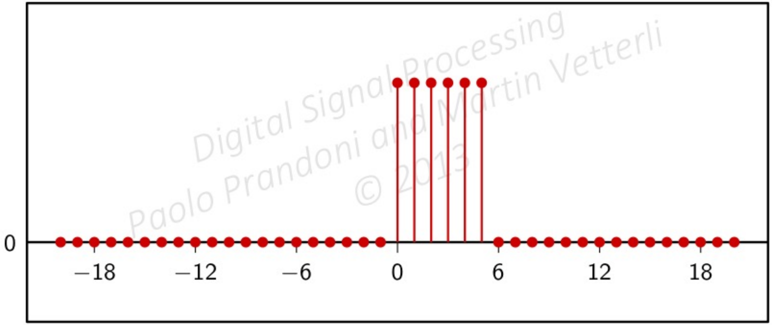
Causal vs Noncausal

- causal:
  - impulse response is zero for $n < 0$
  - only past samples (with respect to the present) are involved in the computation of each output sample
  - causal filters can work "on line" since they only need the past
- noncausal:
  - impulse response is nonzero for some (or all) $n < 0$
  - can still be implemented in a offline fashion (when all input data is available on storage, e.g. in Image Processing)

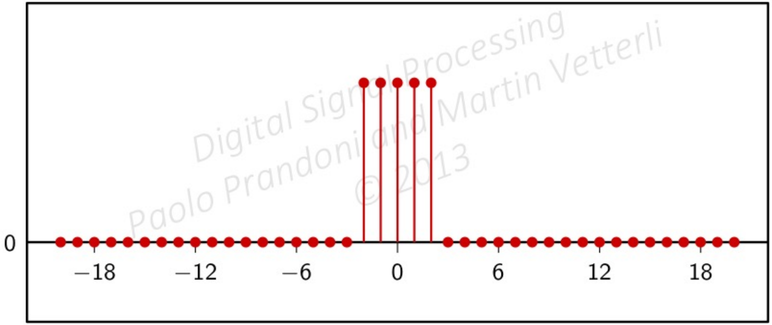
Causal example

Moving Average filter



Noncausal example

Zero-centered Moving Average filter



Stability

- key concept: avoid “explosions” if the input is nice
- a nice signal is a bounded signal: $|x[n]| < M$ for all n
- Bounded-Input Bounded-Output (BIBO) stability: if the input is nice the output should be nice

Fundamental Stability Theorem

A filter is BIBO stable if and only if its impulse response is absolutely summable

Proof ($\Rightarrow$)

Hypotheses:

- $|x[n]| < M$
- $\sum_n |h[n]| = L < \infty$

Thesis:

- $|y[n]|$ bounded

Proof:

$$
\begin{aligned}
|y[n]| &= \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \\
&\leq \sum_{k=-\infty}^{\infty} |h[k]x[n-k]| \\
&\leq M \sum_{k=-\infty}^{\infty} |h[k]| \\
&\leq ML
\end{aligned}
$$

Proof ($\Leftarrow$)

Hypotheses:

- $|x[n]| < M$
- $|y[n]| < P$

Thesis:

- $h[n]$ absolutely summable

Proof (by contradiction):

- assume $\sum_n |h[n]| = \infty$
- build $x[n] = \begin{cases} +1 & \text{if } h[-n] \geq 0 \\ -1 & \text{if } h[-n] < 0 \end{cases}$
- clearly, $x[n]$ is bounded
- however

$$y[0] = (x*h)[0] = \sum_{k=-\infty}^{\infty} h[k]x[-k] = \sum_{k=-\infty}^{\infty} |h[k]| = \infty$$

The good news

FIR filters are always stable

Let's check the Leaky Integrator:

$$\sum_{n=-\infty}^{\infty} |h[n]| = |1-\lambda| \sum_{n=0}^{\infty} |\lambda|^n$$

$$= \lim_{n\to\infty} |1-\lambda| \frac{1-|\lambda|^{n+1}}{1-|\lambda|}$$

$$< \infty \quad \text{for } |\lambda| < 1$$

stability is guaranteed for $|\lambda| < 1$

Checking the stability of IIRs

We will study indirect methods for filter stability later in this Module.

Digital Signal Processing

Module 5.4: Frequency Response

Overview:

- Eigensequences
- Convolution theorem
- Frequency and phase response

A remarkable result

$$e^{j\omega_0 n} \longrightarrow \boxed{\mathcal{H}} \longrightarrow ?$$

A remarkable result

$$
\begin{aligned}
y[n] &= e^{j\omega_0 n} * h[n] \\
&= h[n] * e^{j\omega_0 n} \\
&= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 (n-k)} \\
&= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \\
&= H(e^{j\omega_0})\, e^{j\omega_0 n}
\end{aligned}
$$

A remarkable result

$$e^{j\omega_0 n} \longrightarrow \boxed{\mathcal{H}} \longrightarrow H(e^{j\omega_0})\, e^{j\omega_0 n}$$

- complex exponentials are *eigensequences* of LTI systems, i.e., linear filters cannot change the frequency of sinusoids
- DTFT of impulse response determines the frequency characteristic of a filter

Magnitude and phase

If $H(e^{j\omega_0}) = Ae^{j\theta}$, then

$$\mathcal{H}\{e^{j\omega_0 n}\} = Ae^{j(\omega_0 n + \theta)}$$

amplitude:
amplification ($A > 1$)
or attenuation ($0 \leq A < 1$)

phase shift:
delay ($\theta < 0$)
or advancement ($\theta > 0$)

The convolution theorem

In general:

$$\text{DTFT}\{x[n] * h[n]\} = ?$$

Intuition: the DTFT reconstruction formula tells us how to build $x[n]$ from a set of complex exponential "basis" functions. By linearity...

The convolution theorem

$$
\begin{aligned}
\mathrm{DTFT}\{x[n] * h[n]\} &= \sum_{n=-\infty}^{\infty} (x*h)[n] e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]h[n-k] e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]h[n-k] e^{-j\omega (n-k)} e^{-j\omega k} \\
&= \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k} \sum_{n=-\infty}^{\infty} h[n-k] e^{-j\omega (n-k)} \\
&= H(e^{j\omega}) X(e^{j\omega})
\end{aligned}
$$

Frequency response

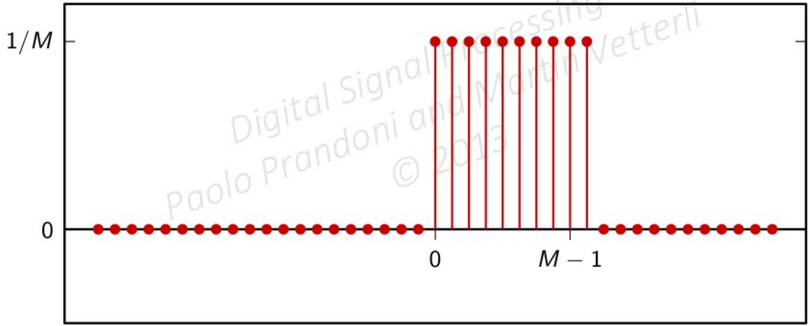
$$H(e^{j\omega}) = \text{DTFT}\{h[n]\}$$

Two effects:

- magnitude: amplification ($|H(e^{j\omega})| > 1$) or attenuation ($|H(e^{j\omega})| < 1$) of input frequencies
- phase: overall delay and shape changes

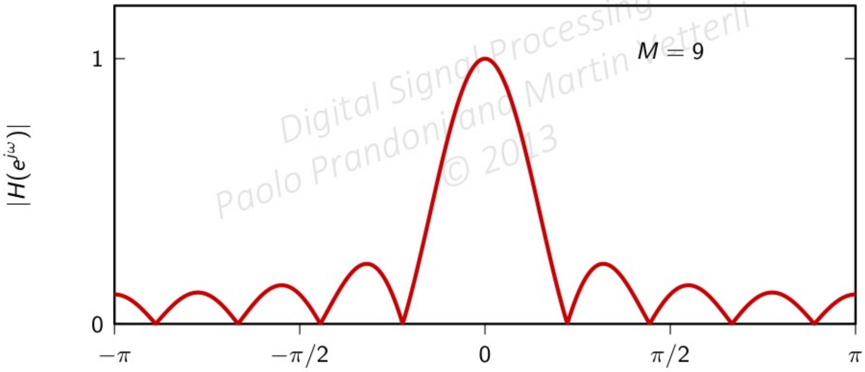
Moving Average revisited

$$h[n] = (u[n] - u[n-M])/M$$



Moving Average, magnitude response

$$|H(e^{j\omega})| = \frac{1}{M}\left|\frac{\sin\left(\frac{\omega}{2}M\right)}{\sin\left(\frac{\omega}{2}\right)}\right| \qquad \text{(see Module 4.7)}$$



Moving Average, magnitude response

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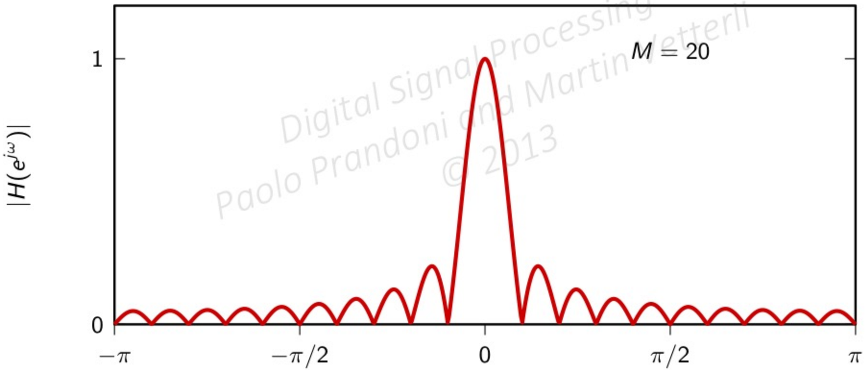
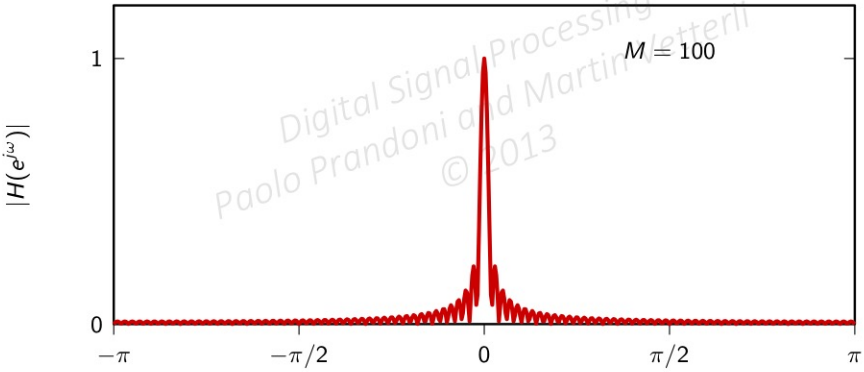
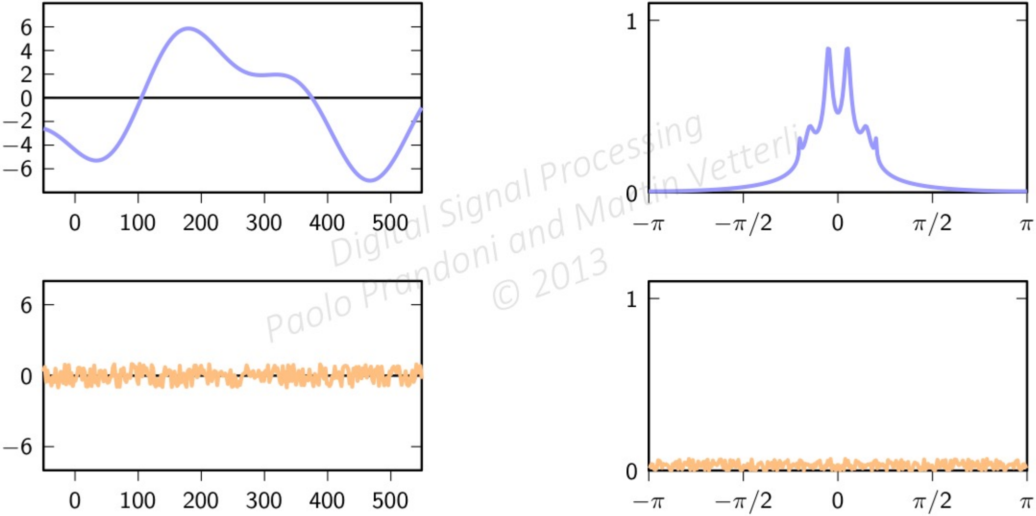


Moving Average, magnitude response

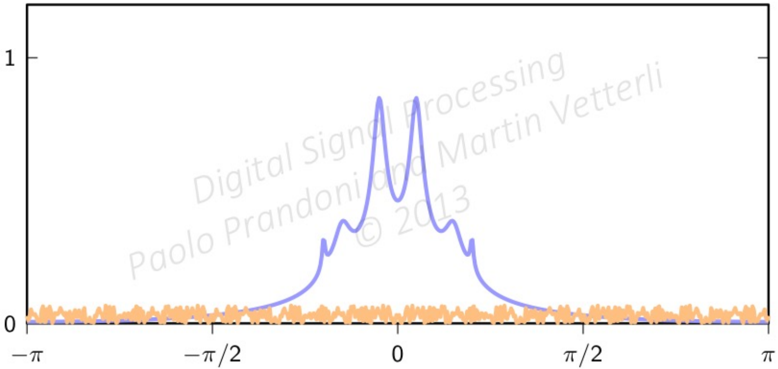
$$|H(e^{j\omega})| = \frac{1}{M}\left|\frac{\sin\left(\frac{\omega}{2}M\right)}{\sin\left(\frac{\omega}{2}\right)}\right| \qquad \text{(see Module 4.7)}$$



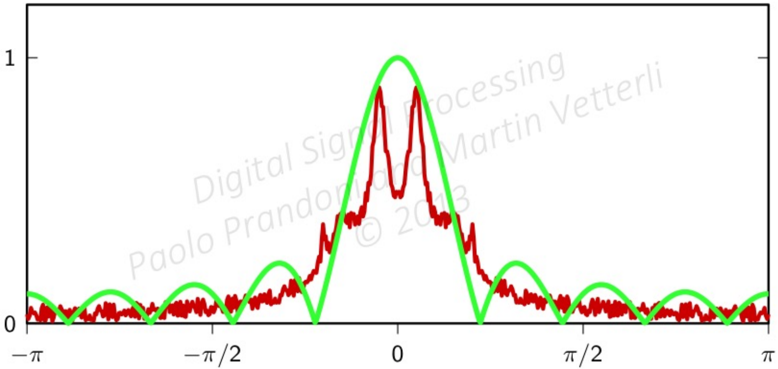
Denoising revisited



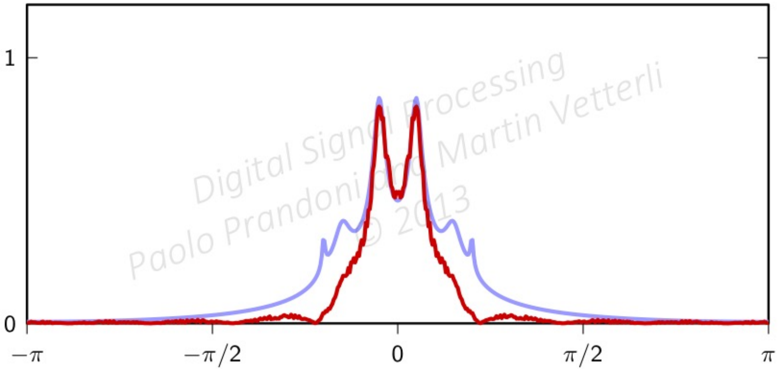
Denoising revisited



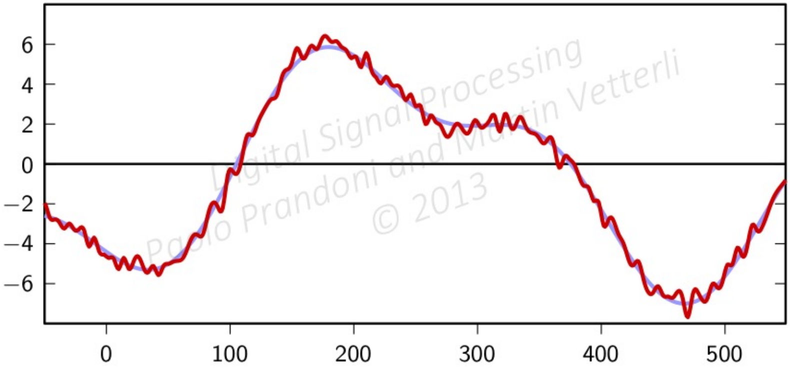
Denoising revisited



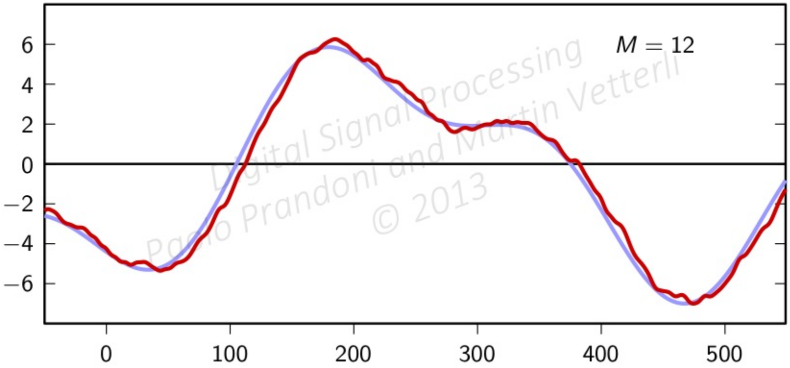
Denoising revisited



By the way, remember the time-domain analysis...



By the way, remember the time-domain analysis...

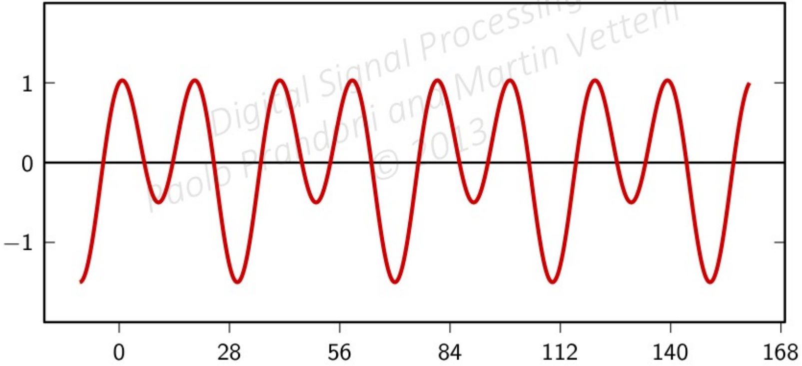


What about the phase?

Assume $|H(e^{j\omega})| = 1$

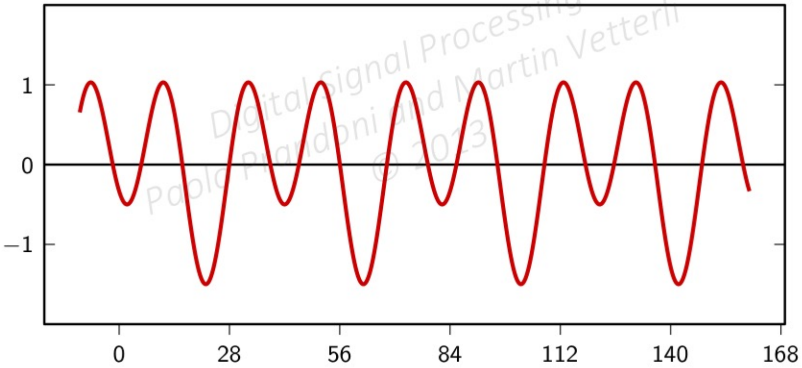
- zero phase: $\angle H(e^{j\omega}) = 0$
- linear phase: $\angle H(e^{j\omega}) = d\omega$
- nonlinear phase

$$x[n] = \frac{1}{2}\sin(\omega_0 n) + \cos(2\omega_0 n) \qquad \omega_0 = \frac{2\pi}{40}$$



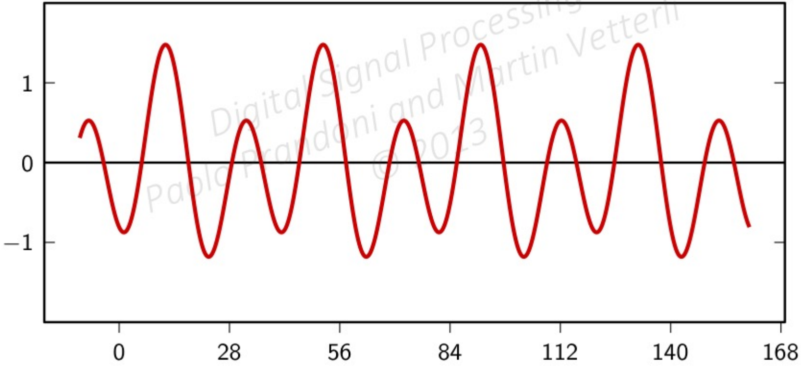
Phase and signal shape: linear phase

$$x[n] = \frac{1}{2}\sin(\omega_0 n + \theta_0) + \cos(2\omega_0 n + 2\theta_0) \qquad \theta_0 = \frac{8\pi}{5}$$



Phase and signal shape: nonlinear phase

$$x[n] = \frac{1}{2}\sin(\omega_0 n) + \cos(2\omega_0 n + 2\theta_0)$$



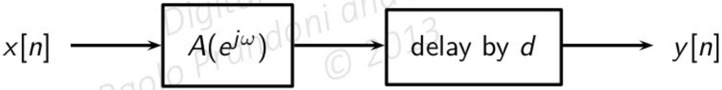
Linear phase

$$x[n] \longrightarrow \boxed{\mathcal{D}} \longrightarrow x[n-d]$$

- $y[n] = x[n-d]$
- $Y(e^{j\omega}) = e^{-j\omega d}\, X(e^{j\omega})$
- $H(e^{j\omega}) = e^{-j\omega d}$
- linear phase term

Linear phase

In general, if $H(e^{j\omega}) = A(e^{j\omega})e^{-j\omega d}$, with $A(e^{j\omega}) \in \mathbb{R}$

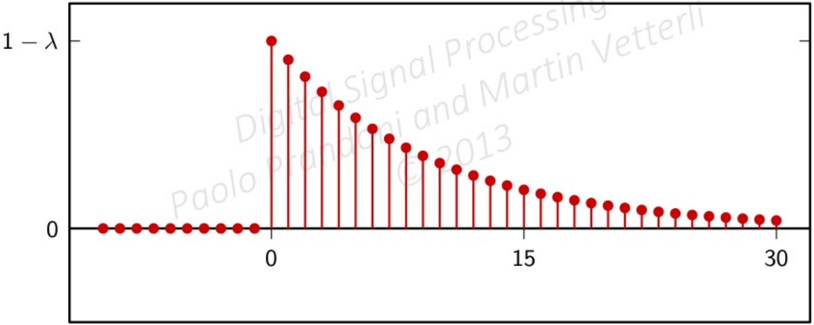


Moving Average is linear phase

$$H(e^{j\omega}) = \frac{1}{M}\,\frac{\sin\left(\frac{\omega}{2}M\right)}{\sin\left(\frac{\omega}{2}\right)}\,e^{-j\frac{M-1}{2}\omega}$$

Leaky integrator revisited

$$h[n] = (1 - \lambda)\lambda^n \, u[n]$$



Leaky integrator revisited

$$H(e^{j\omega}) = \frac{1-\lambda}{1-\lambda e^{-j\omega}} \qquad \text{(Module 4.4)}$$

Finding magnitude and phase require a little algebra...

Leaky integrator revisited

Recall from complex algebra:

$$\frac{1}{a + jb} = \frac{a - jb}{a^2 + b^2}$$

so that if $x = 1/(a + jb)$,

$$|x|^2 = \frac{1}{a^2 + b^2}$$

$$\angle x = \tan^{-1}\left[-\frac{b}{a}\right]$$

Leaky integrator revisited

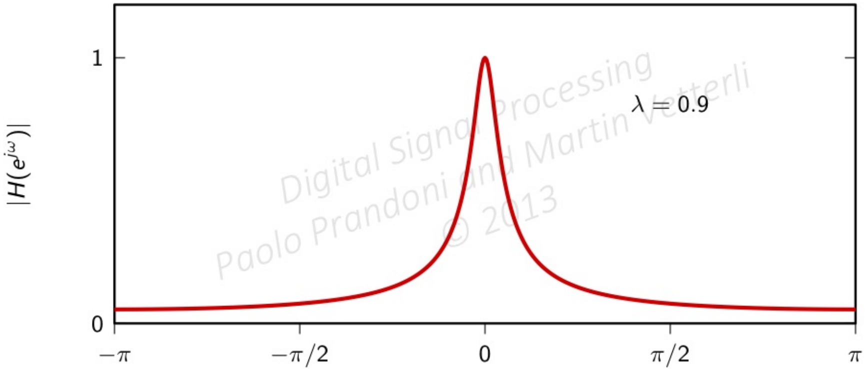
$$H(e^{j\omega}) = \frac{1-\lambda}{(1-\lambda\cos\omega) - j\lambda\sin\omega}$$

so that:

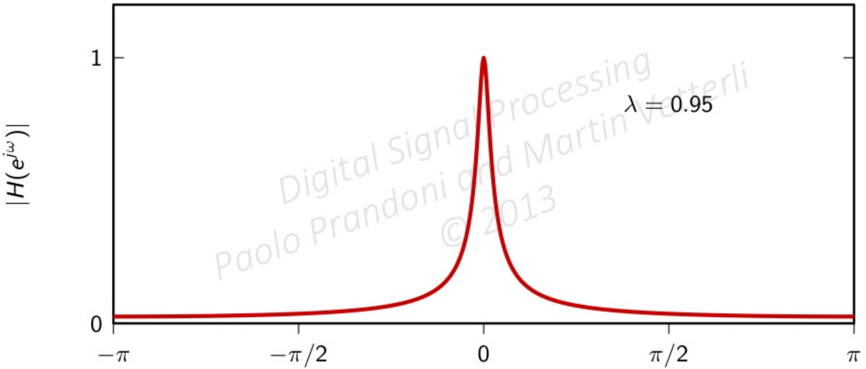
$$|H(e^{j\omega})|^2 = \frac{(1-\lambda)^2}{1-2\lambda\cos\omega+\lambda^2}$$

$$\angle H(e^{j\omega}) = \tan^{-1}\left[\frac{\lambda\sin\omega}{1-\lambda\cos\omega}\right]$$

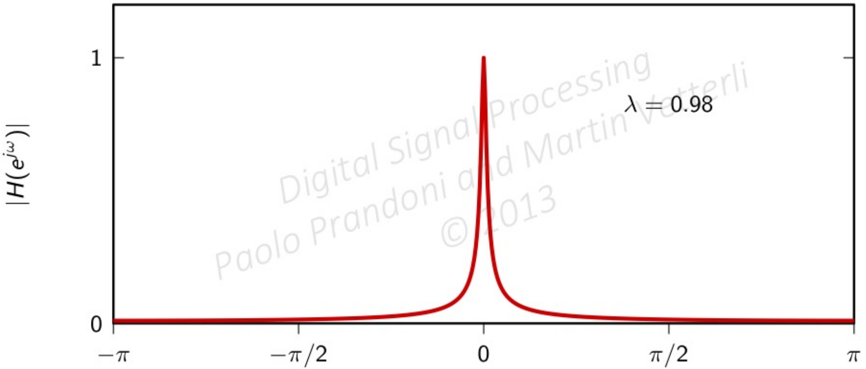
Leaky integrator, magnitude response



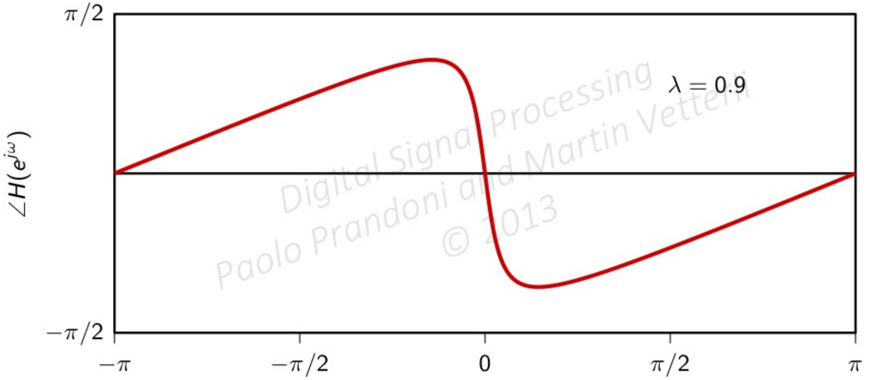
Leaky integrator, magnitude response



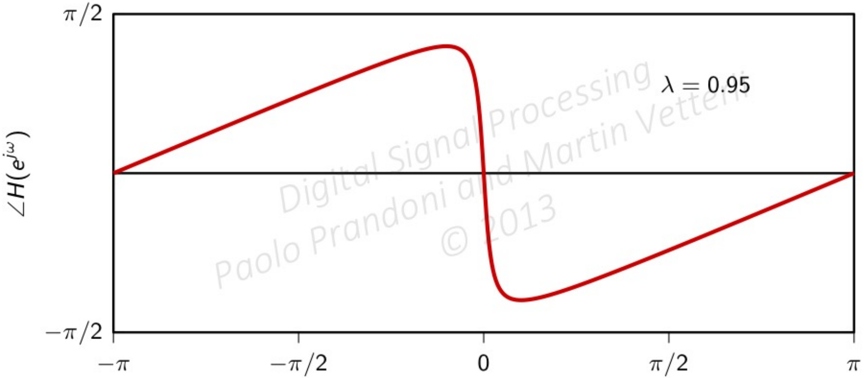
Leaky integrator, magnitude response



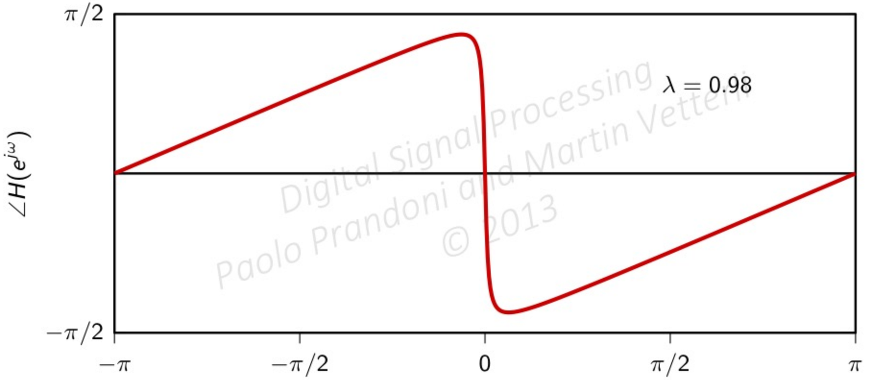
Leaky integrator, phase response



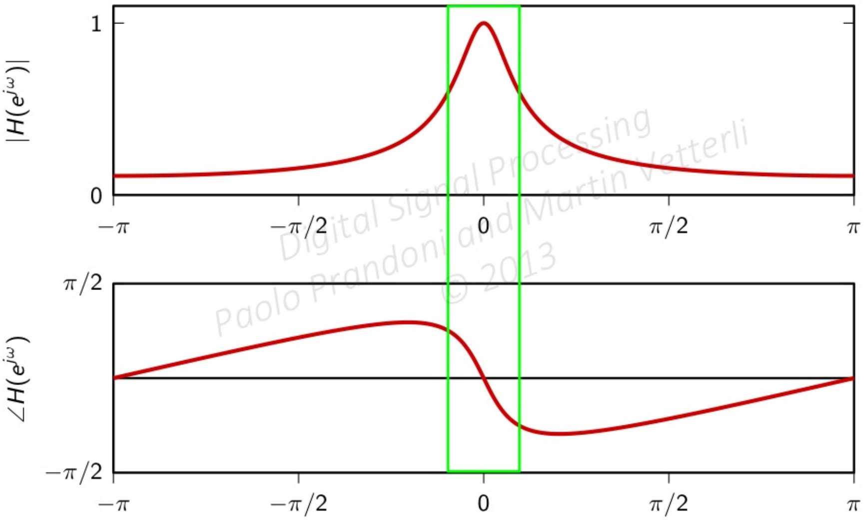
Leaky integrator, phase response



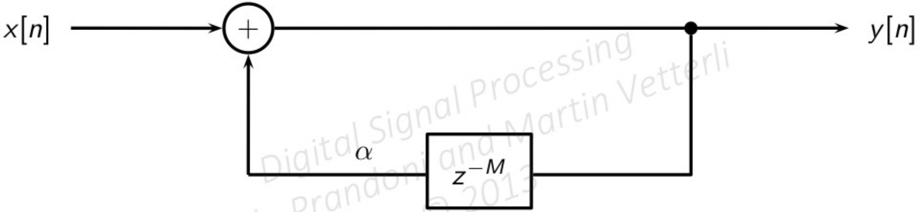
Leaky integrator, phase response



Phase is sufficiently linear where it matters



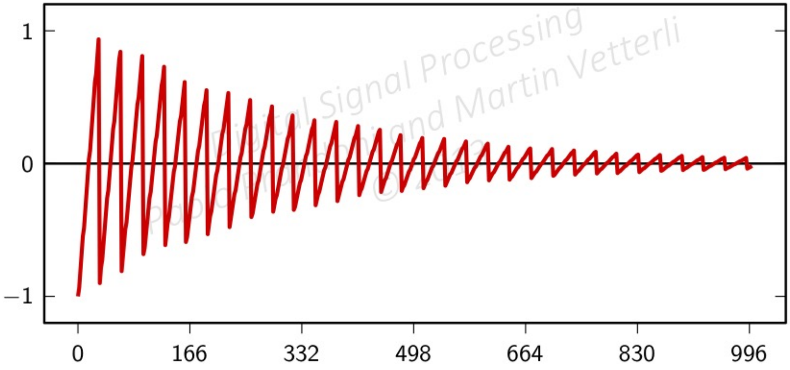
Karplus-Strong revisited, again!



$$y[n] = \alpha\, y[n - M] + x[n]$$

KS revisited

$$y[n] = \alpha^{\lfloor n/M \rfloor}\bar{x}[n \mod M]\,u[n]$$



Karplus-Strong revisited

$$y[n] = \underbrace{\bar{x}[0], \bar{x}[1], \ldots, \bar{x}[M-1]}_{1^{\text{st}} \text{ period}}, \underbrace{\alpha\bar{x}[0], \alpha\bar{x}[1], \ldots, \alpha\bar{x}[M-1]}_{2^{\text{nd}} \text{ period}}, \underbrace{\alpha^2\bar{x}[0], \alpha^2\bar{x}[1], \ldots}_{\ldots}$$

DTFT of KS signal, using the convolution theorem

key observation:

$$y[n] = \bar{x}[n] * w[n], \qquad w[n] = \begin{cases} \alpha^k & \text{for } n = kM \\ 0 & \text{otherwise} \end{cases}$$

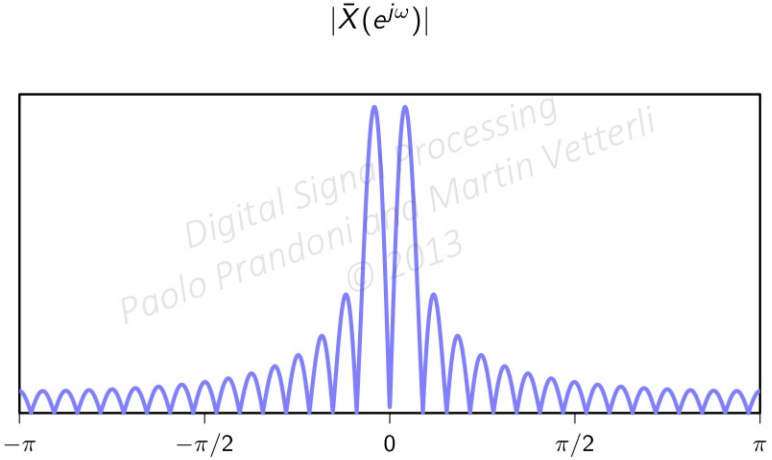
$$Y(e^{j\omega}) = \bar{X}(e^{j\omega})W(e^{j\omega})$$

DTFT of KS signal, using the convolution theorem

$$\bar{X}(e^{j\omega}) = e^{-j\omega}\left(\frac{M+1}{M-1}\right)\frac{1-e^{-j(M-1)\omega}}{(1-e^{-j\omega})^2} - \frac{1-e^{-j(M+1)\omega}}{(1-e^{-j\omega})^2}$$

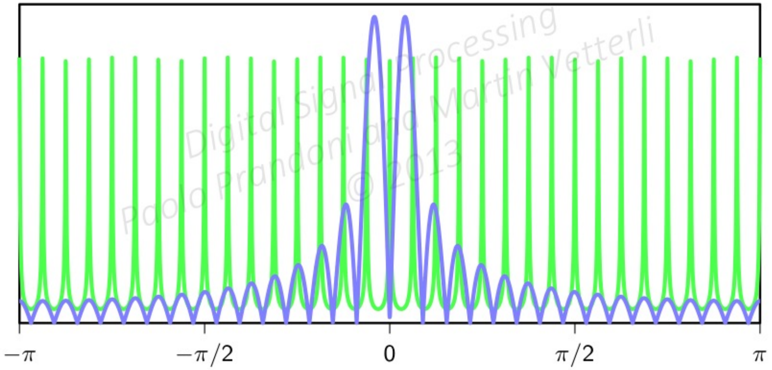
$$W(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega M}}$$

DTFT of KS

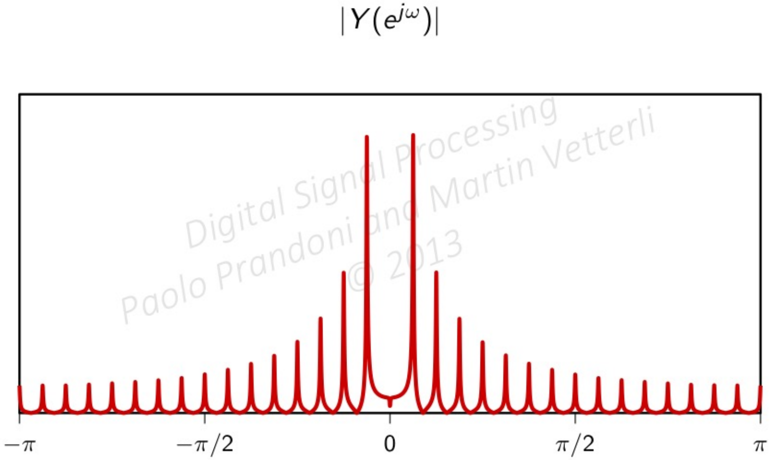


DTFT of KS

$|W(e^{j\omega})|$



DTFT of KS


$|Y(e^{j\omega})|$
Digital Signal Processing
Paolo Prandoni and Martin Vetterli
© 2013
$-\pi$
$-\pi/2$
0
$\pi/2$
π

Digital Signal Processing

Module 5.5: Ideal Filters

Overview:

- Filter classification in the frequency domain
- Ideal filters
- Demodulation revisited

Filter types according to magnitude response

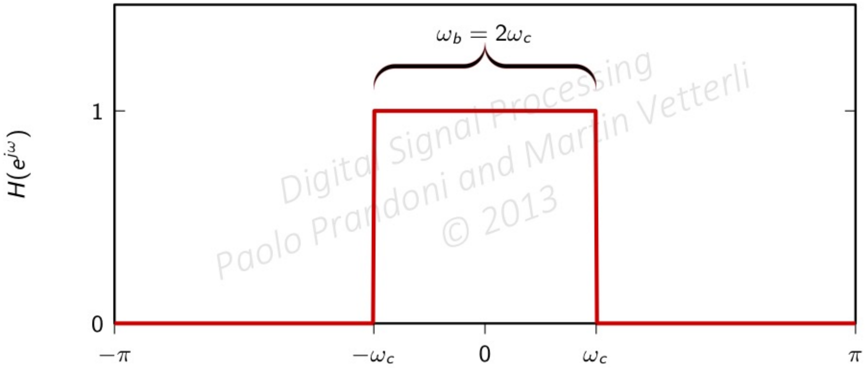
- Lowpass
- Highpass
- Bandpass
- Allpass

Moving Average and Leaky Integrator are lowpass filters

Filter types according to phase response

- Linear phase
- Nonlinear phase

What is the best lowpass we can think of?



Ideal lowpass filter

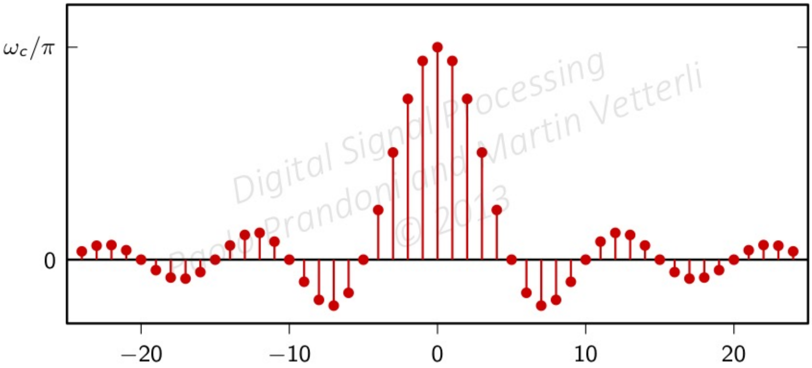
$$H(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases} \qquad (2\pi\text{-periodicity implicit})$$

- perfectly flat passband
- infinite attenuation in stopband
- zero-phase (no delay)

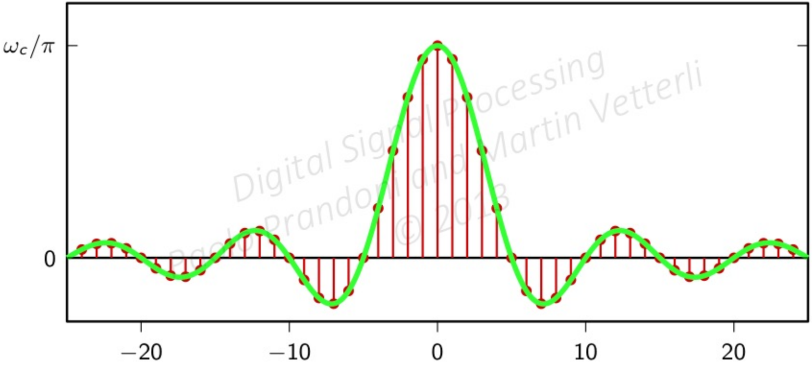
Ideal lowpass filter: impulse response

$$
\begin{aligned}
h[n] &= \mathrm{IDTFT}\left\{H(e^{j\omega})\right\} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega \\
&= \frac{1}{\pi n}\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j} \\
&= \frac{\sin \omega_c n}{\pi n}
\end{aligned}
$$

Ideal lowpass filter: impulse response



Ideal lowpass filter: impulse response



The bad news

- impulse response is infinite support, two-sided
  $\Longrightarrow$ cannot compute the output in a finite amount of time
  $\Longrightarrow$ that's why it's called "ideal"
- impulse response decays slowly in time
  $\Longrightarrow$ we need a lot of samples for a good approximation

Nevertheless...

The sinc-rect pair:

$$\mathrm{rect}(x) = \begin{cases} 1 & |x| \leq 1/2 \\ 0 & |x| > 1/2 \end{cases}$$

$$\mathrm{sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

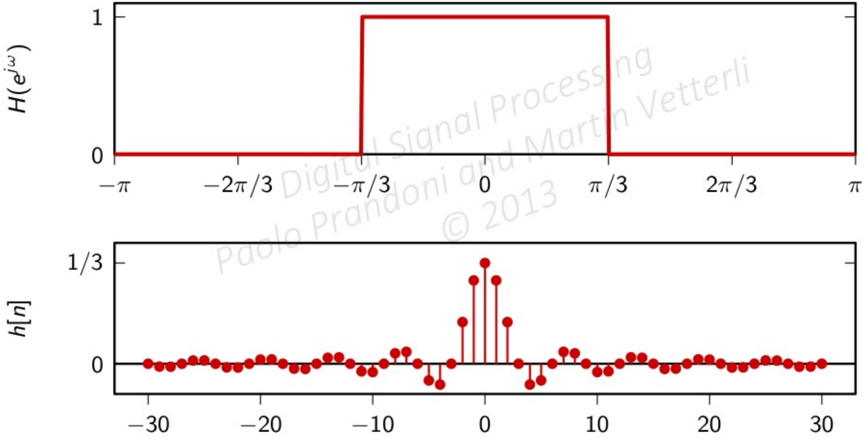
(note that $\mathrm{sinc}(x) = 0$ when x is a nonzero integer)

The ideal lowpass in canonical form

$$\text{rect}\left(\frac{\omega}{2\omega_c}\right) \stackrel{\text{DTFT}}{\longleftrightarrow} \frac{\omega_c}{\pi}\,\text{sinc}\left(\frac{\omega_c}{\pi}n\right)$$

Example

$$\omega_c = \pi/3:\ H(e^{j\omega}) = \mathrm{rect}(3\omega/2\pi),\ h[n] = (1/3)\mathrm{sinc}(n/3)$$



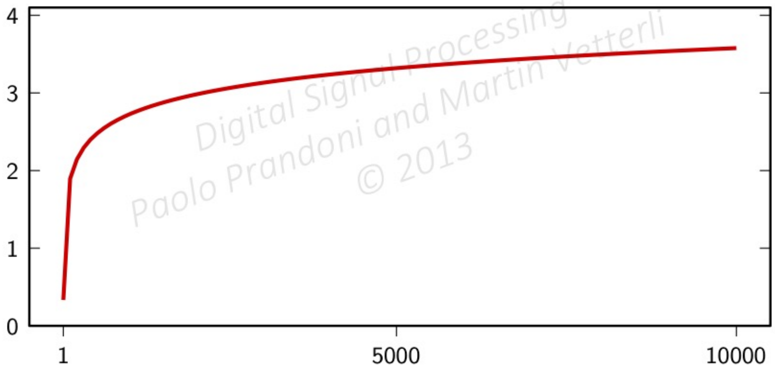
Little-known fact

- the sinc is not absolutely summable
- the ideal lowpass is not BIBO stable!
- example for $\omega_c = \pi/3$: $h[n] = (1/3)\,\text{sinc}(n/3)$
- take $x[n] = \text{sign}\{\text{sinc}(-n/3)\}$ and

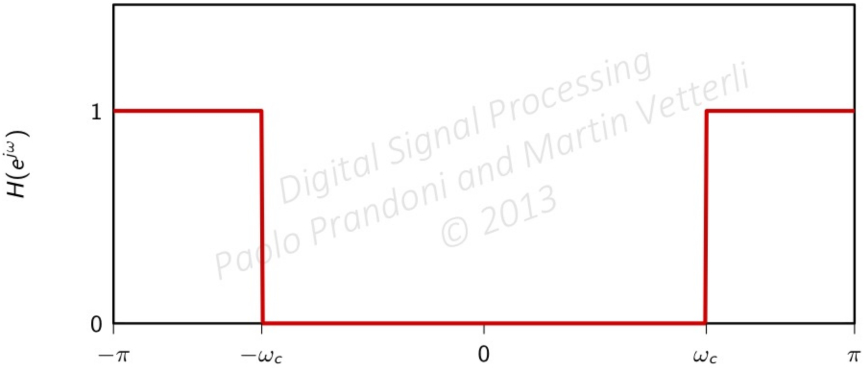
$$y[0] = (x * h)[0] = \frac{1}{3} \sum_{k=-\infty}^{\infty} |\,\text{sinc}(k/3)| = \infty$$

Divergence is however very slow...

$$s(n) = (1/3) \sum_{k=-n}^{n} |\operatorname{sinc}(k/3)|$$



Ideal highpass filter



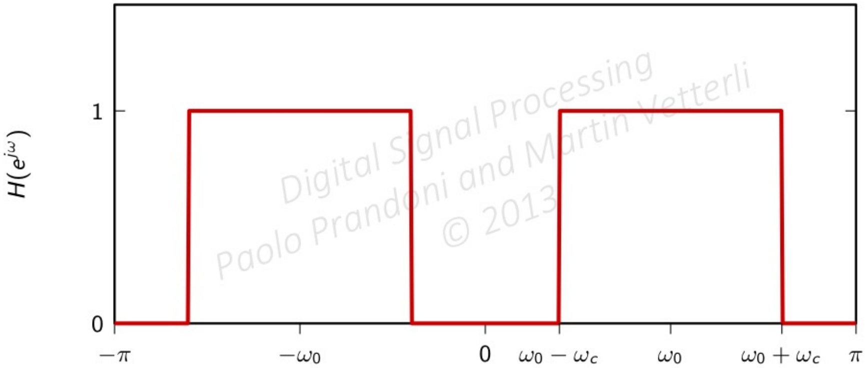
Ideal highpass filter

$$H_{hp}(e^{j\omega}) = \begin{cases} 1 & \text{for } \pi \geq |\omega| \geq \omega_c \\ 0 & \text{otherwise} \end{cases} \qquad (2\pi\text{-periodicity implicit})$$

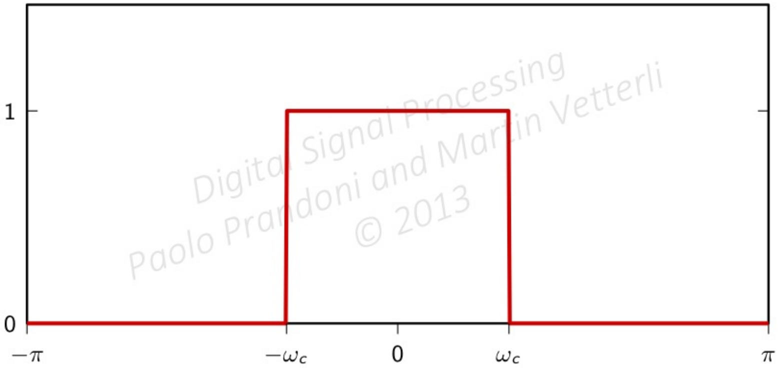
$$H_{hp}(e^{j\omega}) = 1 - H_{lp}(e^{j\omega})$$

$$h_{hp}[n] = \delta[n] - \frac{\omega_c}{\pi}\text{sinc}(\frac{\omega_c}{\pi}n)$$

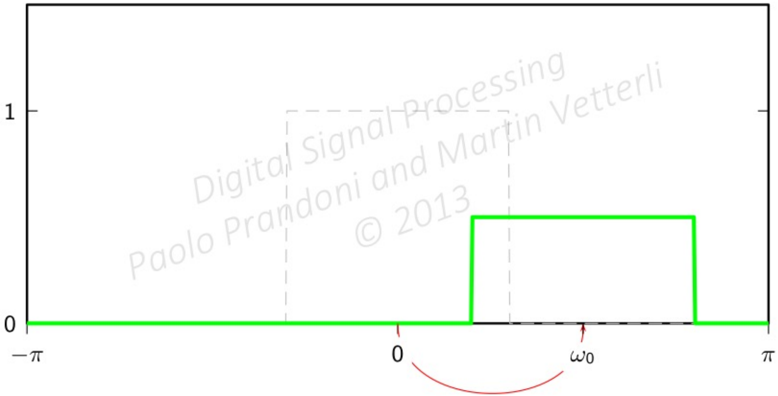
Ideal bandpass filter



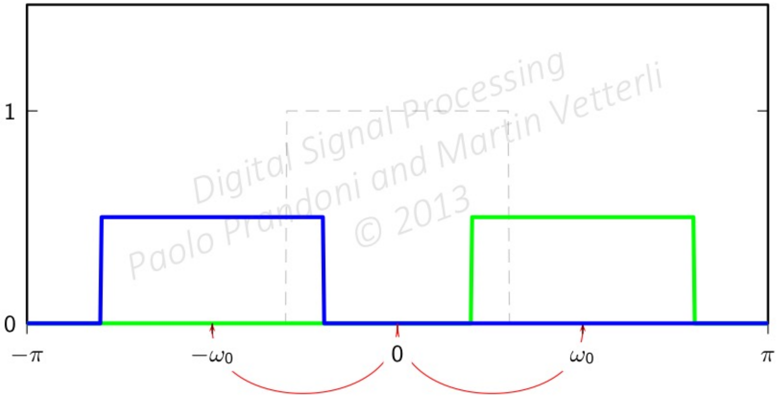
Ideal bandpass filter



Ideal bandpass filter



Ideal bandpass filter



Ideal bandpass filter

$$H_{bp}(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega \pm \omega_0| \leq \omega_c \\ 0 & \text{otherwise} \end{cases} \qquad (2\pi\text{-periodicity implicit})$$

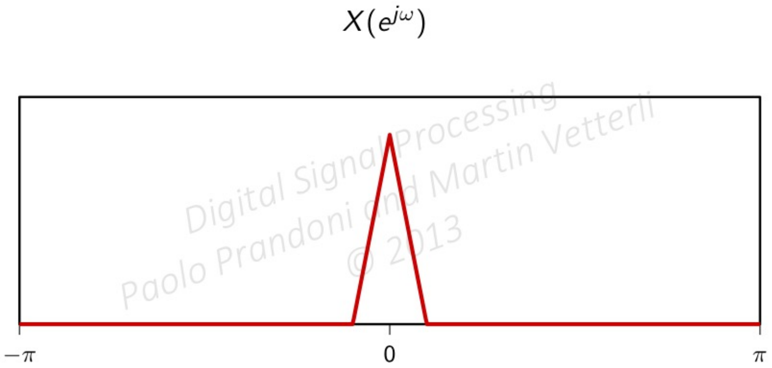
$$h_{bp}[n] = 2\cos(\omega_0 n)\frac{\omega_c}{\pi}\mathrm{sinc}(\frac{\omega_c}{\pi}n)$$

Demodulation revisited

remember Module 4.8:

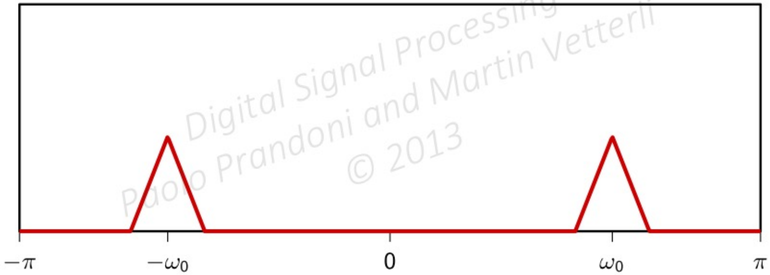
- apply sinusoidal modulation to $x[n]$: $y[n] = x[n] \cos \omega_0 n$
- demodulate by multiplying by the carrier $x'[n] = y[n] \cos \omega_0 n$
- demodulated signal contains unwanted high-frequency components

Demodulation revisited

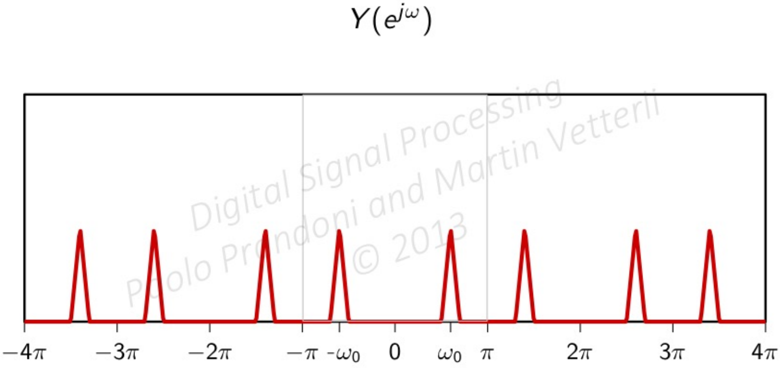


Demodulation revisited

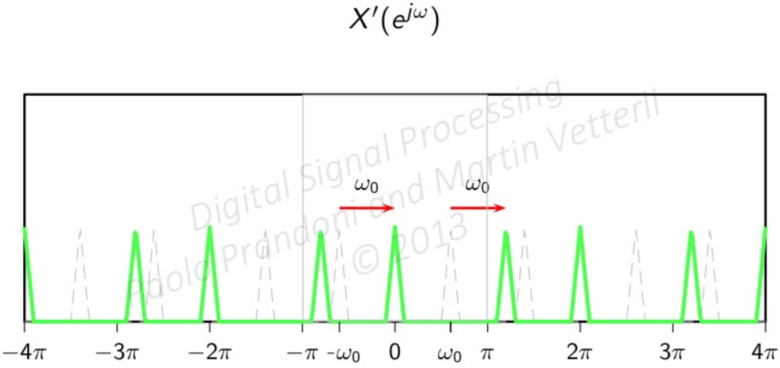
$Y(e^{j\omega})$



Demodulation revisited

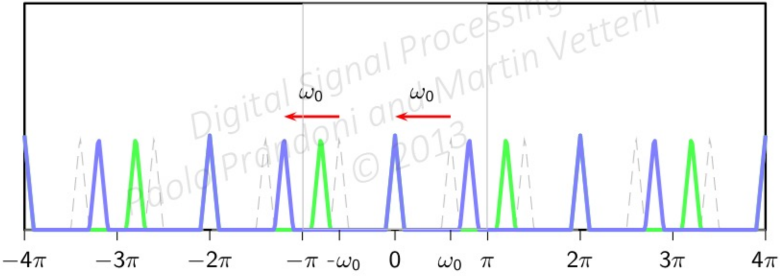


Demodulation revisited

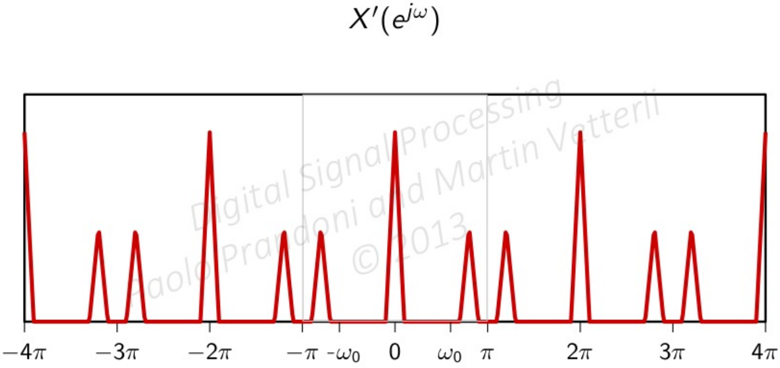


Demodulation revisited

$X'(e^{j\omega})$

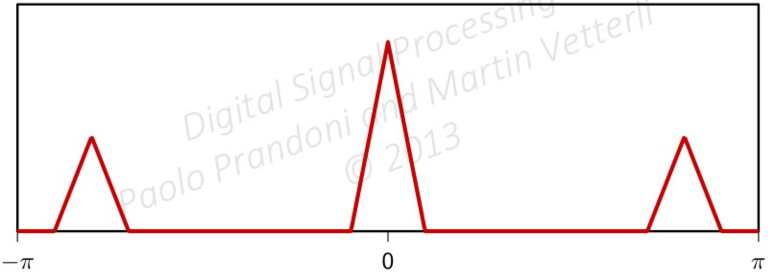


Demodulation revisited



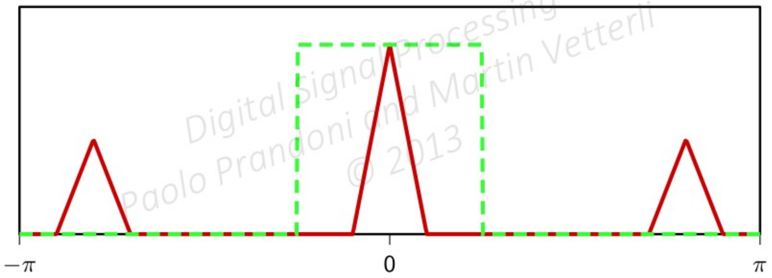
Solution: lowpass filtering

$X'(e^{j\omega})$

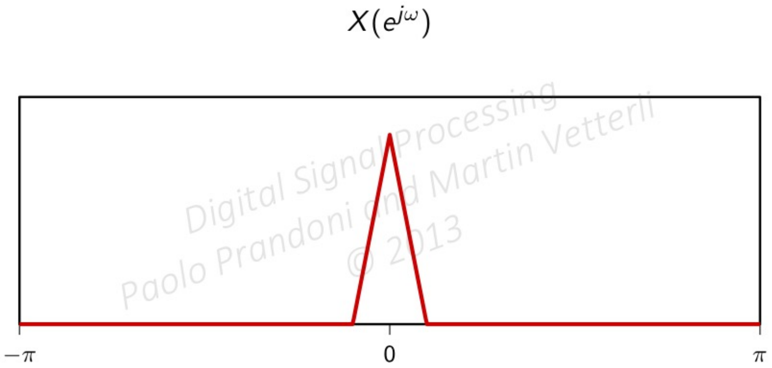


Solution: lowpass filtering

$X'(e^{j\omega})$



Solution: lowpass filtering



Digital Signal Processing

Module 5.6: Filter Design - Part I: Approximation of Ideal Filters

Overview:

- Impulse truncation
- Window method
- Frequency sampling

How can we approximate an ideal lowpass?

Idea #1:

- pick ω_c
- compute ideal impulse response $h[n]$
- truncate $h[n]$ to a finite-support $\hat{h}[n]$
- $\hat{h}[n]$ defines an FIR filter

Approximation by truncation

FIR approximation of length $M = 2N + 1$:

$$\hat{h}[n] = \begin{cases} \dfrac{\omega_c}{\pi} \operatorname{sinc}\left(\dfrac{\omega_c}{\pi} n\right) & |n| \leq N \\ \\ 0 & \text{otherwise} \end{cases}$$

Why it could be a good idea

$$\begin{aligned}
\text{MSE} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega}) - \hat{H}(e^{j\omega})|^2 d\omega \\
&= \|H(e^{j\omega}) - \hat{H}(e^{j\omega})\|^2 \\
&= \|h[n] - \hat{h}[n]\|^2 \\
&= \sum_{n=-\infty}^{\infty} |h[n] - \hat{h}[n]|^2
\end{aligned}$$

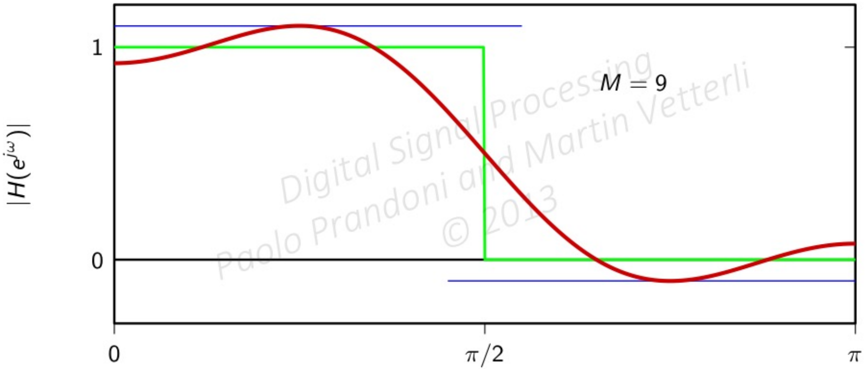
MSE is minimized by symmetric impulse truncation around zero

Why it could be a good idea

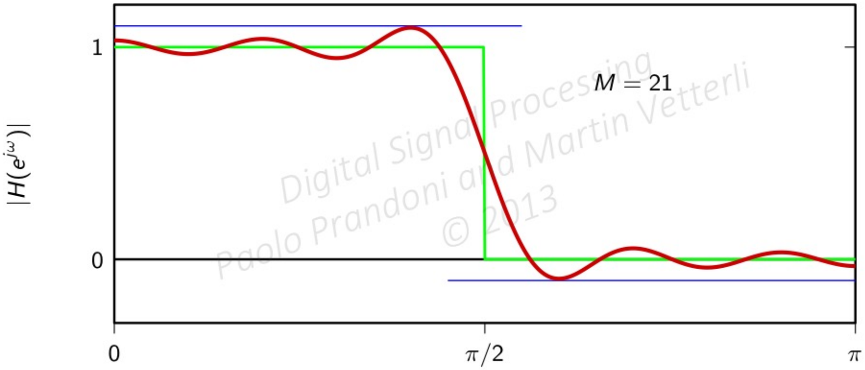
$$\begin{aligned}\text{MSE} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega}) - \hat{H}(e^{j\omega})|^2 d\omega \\ &= \|H(e^{j\omega}) - \hat{H}(e^{j\omega})\|^2 \\ &= \|h[n] - \hat{h}[n]\|^2 \\ &= \sum_{n=-\infty}^{\infty} |h[n] - \hat{h}[n]|^2\end{aligned}$$

MSE is minimized by symmetric impulse truncation around zero

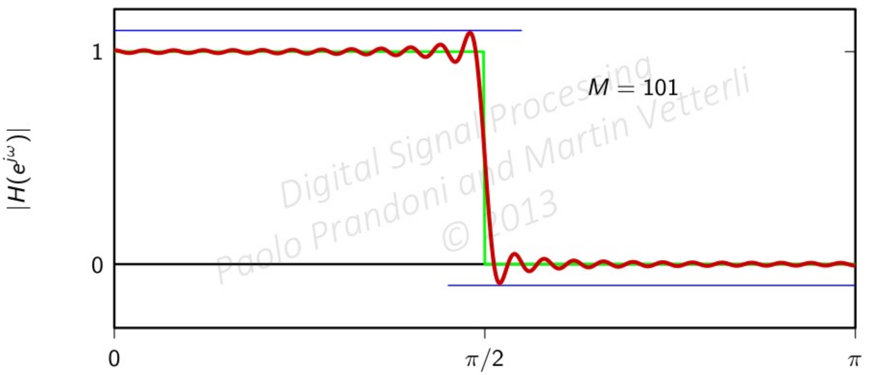
Why it's not such a good idea



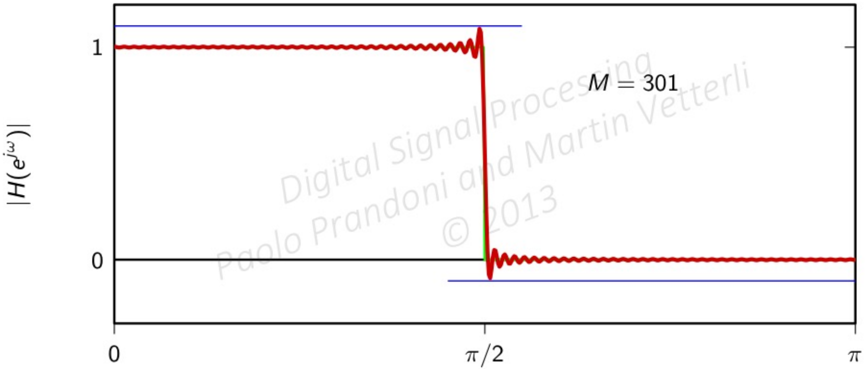
Why it's not such a good idea



Why it's not such a good idea



Why it's not such a good idea



The maximum error around the cutoff frequency
is around 9% of the height of the jump
regardless of N

Understanding the Gibbs phenomenon

$$\hat{h}[n] = h[n]\, w[n]$$

$$w[n] = \begin{cases} 1 & |n| \leq N \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{H}(e^{j\omega}) = ?$$

The modulation theorem

$$\mathrm{DTFT}\{(x * y)[n]\} = X(e^{j\omega})\, Y(e^{j\omega})$$

$$\mathrm{DTFT}\{x[n]\, y[n]\} = (X * Y)(e^{j\omega})$$

Convolution of DTFTs

in $\mathbb{C}^\infty$:

$$(x * y)[n] = \langle x^*[k], y[n-k] \rangle$$
$$= \sum_{k=-\infty}^{\infty} x[k]y[n-k]$$

in $L_2([-\pi, \pi])$:

$$(X * Y)(e^{j\omega}) = \langle X^*(e^{j\sigma}), Y(e^{j(\omega-\sigma)}) \rangle$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\sigma})Y(e^{j(\omega-\sigma)})d\sigma$$

Modulation theorem: proof

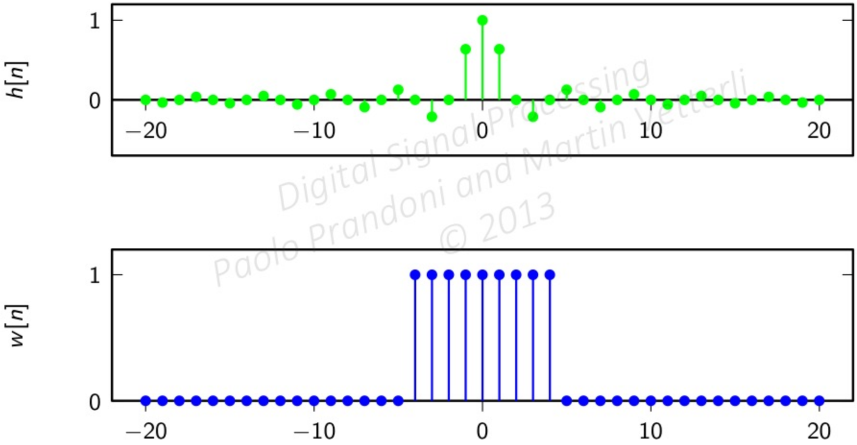
$$
\begin{aligned}
\text{IDTFT}\left\{(X*Y)(e^{j\omega})\right\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(X*Y)(e^{j\omega})e^{j\omega n}d\omega \\
&= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}X(e^{j\sigma})Y(e^{j(\omega-\sigma)})e^{j\omega n}d\sigma d\omega \\
&= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}X(e^{j\sigma})Y(e^{j(\omega-\sigma)})e^{j\sigma n}e^{j(\omega-\sigma)n}d\sigma d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\sigma})e^{j\sigma n}d\sigma\,\frac{1}{2\pi}\int_{-\pi}^{\pi}Y(e^{j(\omega-\sigma)})e^{j(\omega-\sigma)n}d\omega \\
&= x[n]\,y[n]
\end{aligned}
$$

Aside: sinusoidal modulation revisited

$$
\begin{aligned}
\text{DTFT}\left\{x[n]\cos\omega_c n\right\} &= X(e^{j\omega}) * \frac{1}{2}[\tilde{\delta}(\omega-\omega_c)+\tilde{\delta}(\omega+\omega_c)] \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} X(e^{j\sigma})\tilde{\delta}(\sigma-\omega+\omega_c)d\sigma + \frac{1}{4\pi}\int_{-\pi}^{\pi} X(e^{j\sigma})\tilde{\delta}(\sigma-\omega-\omega_c)d\sigma \\
&= \frac{1}{2}\left[X(e^{j(\omega-\omega_c)}) + X(e^{j(\omega+\omega_c)})\right]
\end{aligned}
$$

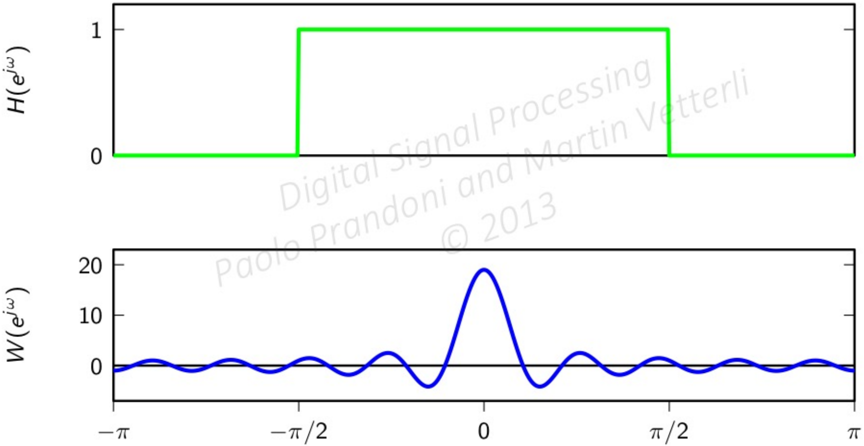
Understanding the Gibbs phenomenon

$$\hat{h}[n] = h[n]\, w[n]$$

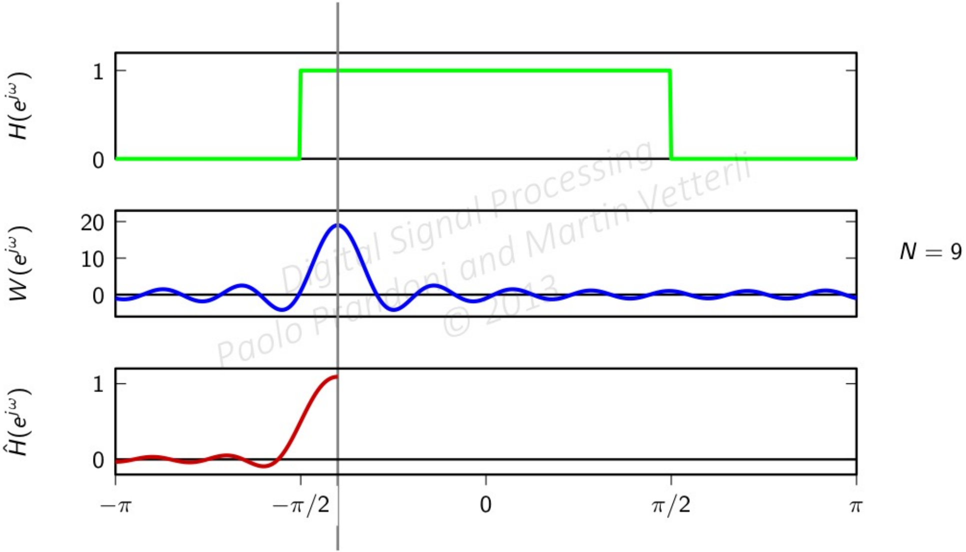


Understanding the Gibbs phenomenon

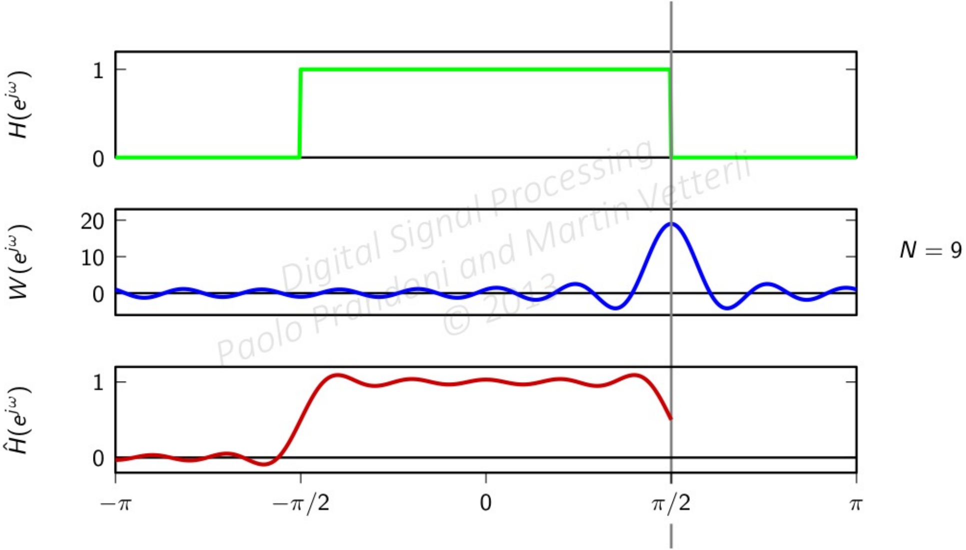
$$\hat{H}(e^{j\omega}) = (H * W)(e^{j\omega}), \qquad W(e^{j\omega}) = \sin(\omega(2N+1)/2)/\sin(\omega/2) \quad \text{(Module 4.7)}$$



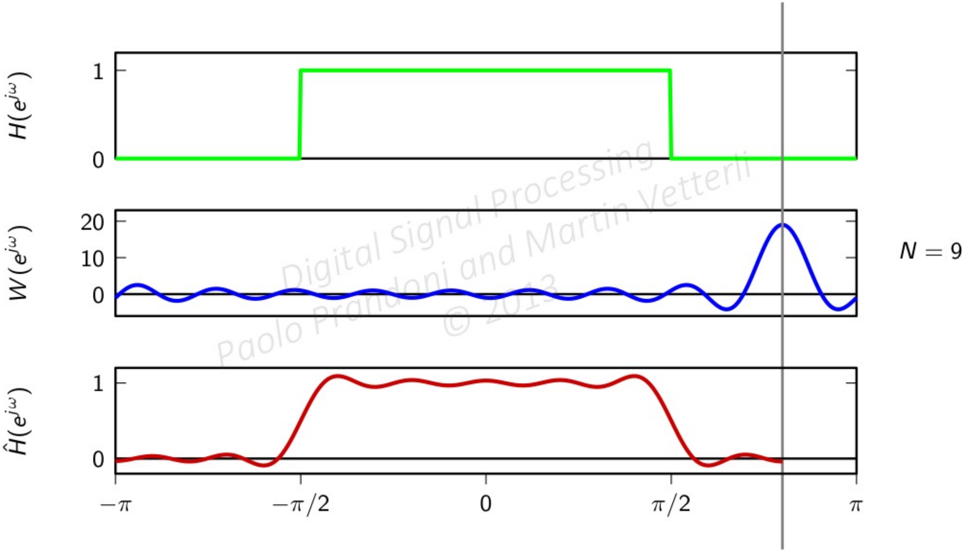
Implicit frequency-domain convolution



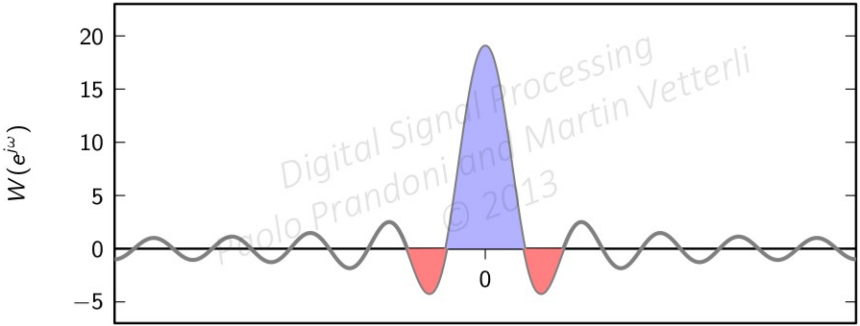
Implicit frequency-domain convolution



Implicit frequency-domain convolution



Mainlobe and sidelobes



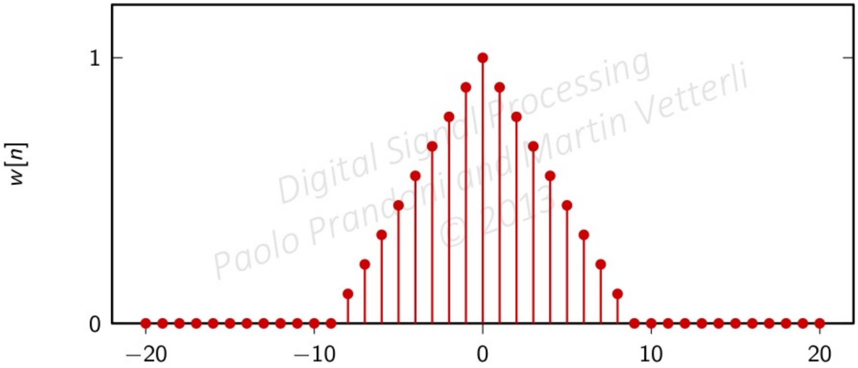
What if we change the window?

We want:

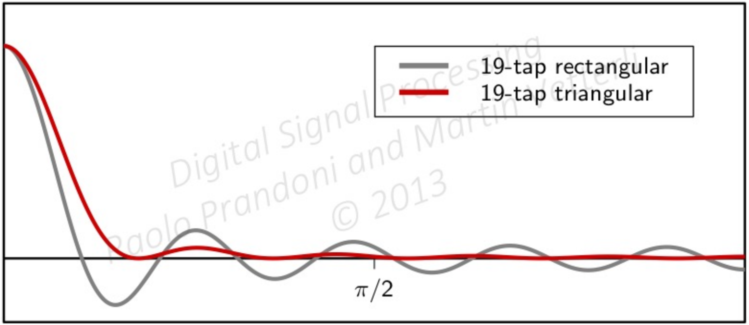
- narrow mainlobe so that transition is sharp
- small sidelobe so Gibbs error is small
- short window so FIR is efficient

very conflicting requirements!

Triangular window



Rectangular vs Triangular Window

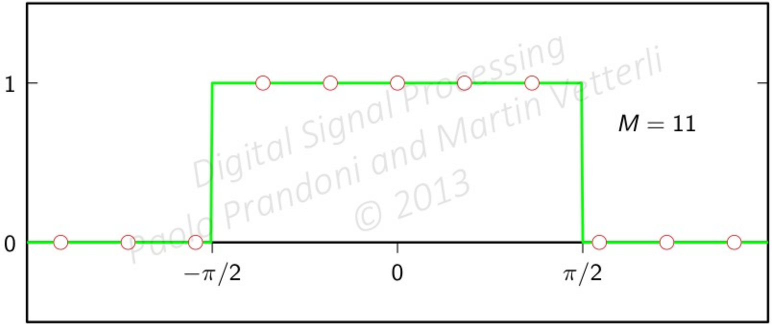


Frequency sampling

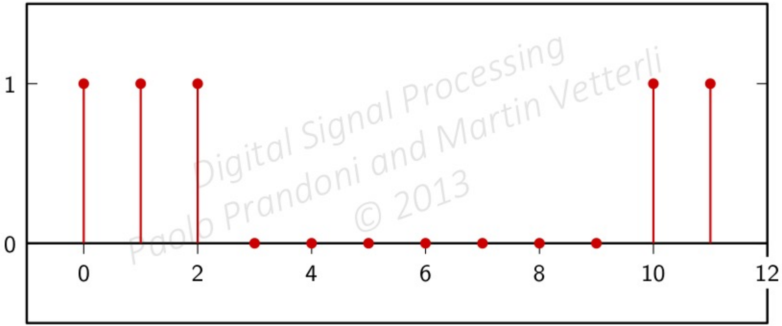
Idea #2:

- draw desired frequency response $H(e^{j\omega})$
- take M values at $\omega_k = (2\pi/M)k$
- compute IDFT of values
- use result as an M-tap impulse response $\hat{h}[n]$

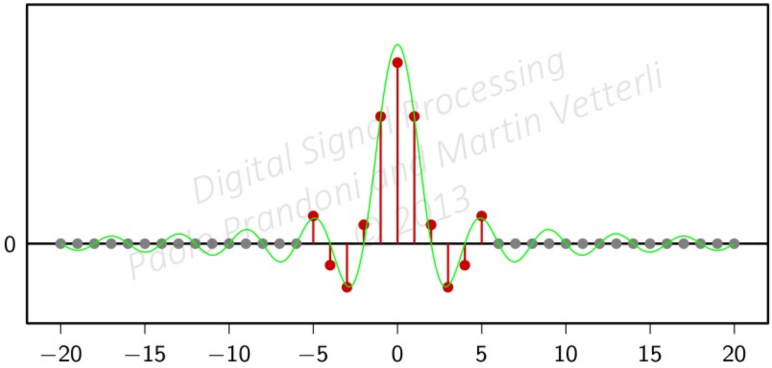
Frequency sampling: desired response



Frequency sampling: DFT samples



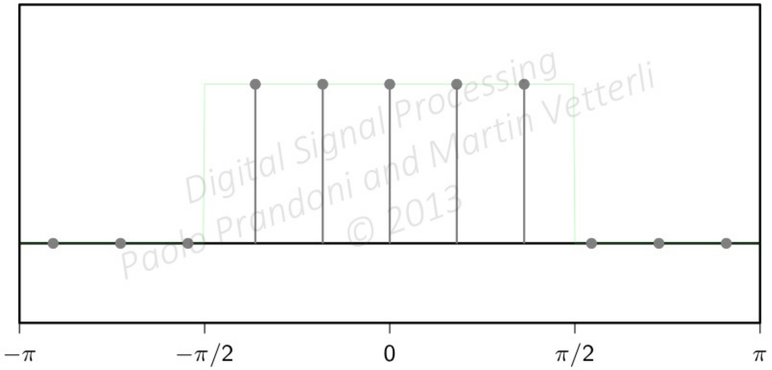
Frequency sampling: impulse response from IDFT



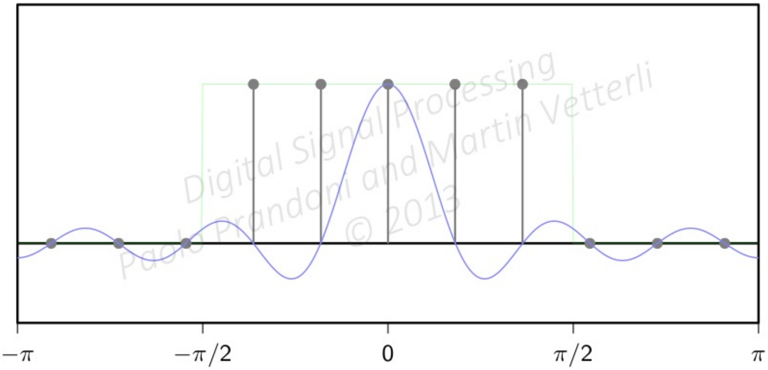
Why it's not such a good idea

- frequency response is DTFT of finite-support, whose DFT we know (see Module 4.7)
- frequency response is interpolation of frequency samples
- interpolator is transform of N-tap rectangular window
- again, no control over mainlobe and sidelobes

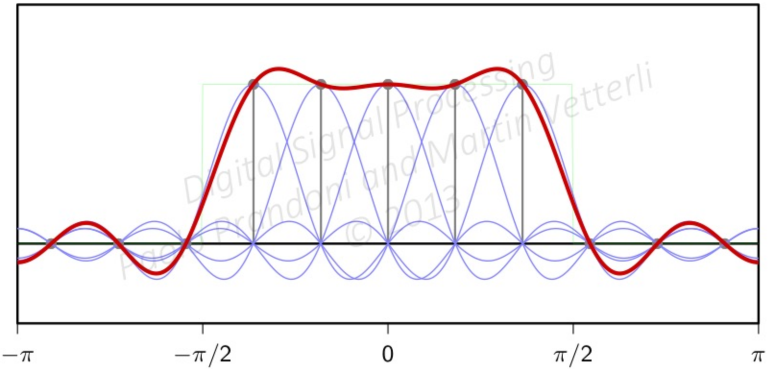
Frequency sampling: frequency response



Frequency sampling: frequency response



Frequency sampling: frequency response



Digital Signal Processing

Module 5.7: Realizable Filters

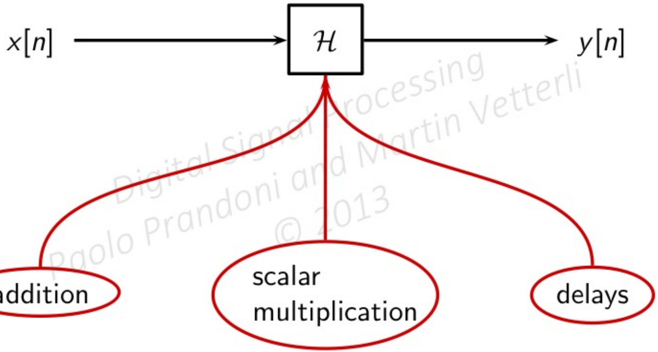
Overview:

- Constant-Coefficient Difference Equations
- The z-transform
- System transfer function
- Region of convergence and system stability

The most general LTI systems

- ideal filters cannot be implemented
- what is the most general, realizable LTI transformation?
  - linearity: only sums and multiplications
  - time-invariance: only multiplications by constants
  - realizability: only finite number of past and future samples

Linear, time-invariant systems



Constant-Coefficient Difference Equation

$$\sum_{k=0}^{N-1} a_k y[n-k] = \sum_{k=0}^{M-1} b_k x[n-k]$$

- uses M input and N output values
- how do we compute the frequency response?
- we need a new tool!

The z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \quad z \in \mathbb{C}$$

- think of it as a formal operator...
- ... or as the extension of the DTFT to the whole complex plane:

$$X(z)|_{z=e^{j\omega}} = \mathrm{DTFT}\{x[n]\}$$

Key properties

linearity:

$$\mathcal{Z}\{\alpha x[n] + \beta y[n]\} = \alpha X(z) + \beta Y(z)$$

time shift:

$$\mathcal{Z}\{x[n-N]\} = z^{-N} X(z)$$

Applying the z-transform to CCDE's

$$\sum_{k=0}^{N-1} a_k y[n-k] = \sum_{k=0}^{M-1} b_k x[n-k]$$

$$Y(z) \sum_{k=0}^{N-1} a_k z^{-k} = X(z) \sum_{k=0}^{M-1} b_k z^{-k}$$

$$Y(z) = H(z)X(z)$$

Rational Transfer Function

$$H(z) = \frac{\displaystyle\sum_{k=0}^{M-1} b_k z^{-k}}{\displaystyle\sum_{k=0}^{N-1} a_k z^{-k}}$$

by setting $z = e^{j\omega}$ we have the frequency response!

(and now the notation $X(e^{j\omega})$ should make more sense)

Leaky Integrator revisited

$$y[n] = (1-\lambda)x[n] + \lambda y[n-1]$$

$$Y(z) = (1-\lambda)X(z) + \lambda z^{-1} Y(z)$$

$$H(z) = \frac{(1-\lambda)}{1-\lambda z^{-1}}$$

$$H(e^{j\omega}) = \frac{(1-\lambda)}{1-\lambda e^{-j\omega}}$$

BTW, remember the delay block?

$$x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow x[n-1]$$

now the notation should make more sense!

Region of convergence (ROC)

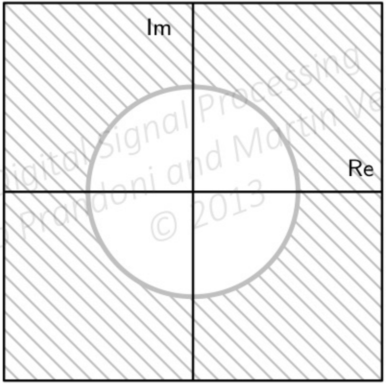
the z-transform is a power series, so convergence is always absolute

$$|X(z)| < \infty \Longleftrightarrow \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < \infty$$

Region of convergence (ROC)

- ROC is whole complex plane for finite-support sequences
- ROC has circular symmetry (depends only on $|z|$)
- ROC of causal sequences extends from a circle to infinity

ROC for causal systems



ROC for causal systems

Consider the transfer function for an LTI system:

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_{M-1} z^{-(M-1)}}{a_0 + a_1 z^{-1} + \ldots + a_{N-1} z^{-(N-1)}}$$

It can always be factored as:

$$H(z) = C \frac{\displaystyle\prod_{n=1}^{M-1} (1 - z_n z^{-1})}{\displaystyle\prod_{n=1}^{N-1} (1 - p_n z^{-1})}$$

ROC for causal systems

- z_n's: *zeros* of the transfer function
- p_n's: *poles* of the transfer function
- only trouble spots for ROC are the poles

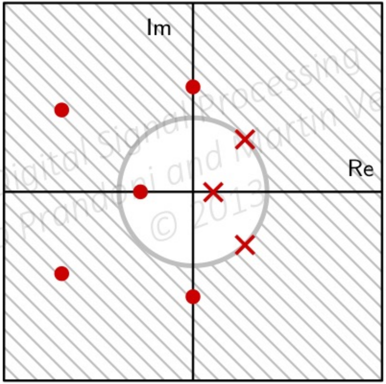
ROC for causal systems

We know:

- ROC extends outwards
- ROC cannot include poles

ROC extends outwards from a circle touching the largest-magnitude pole

ROC for causal systems



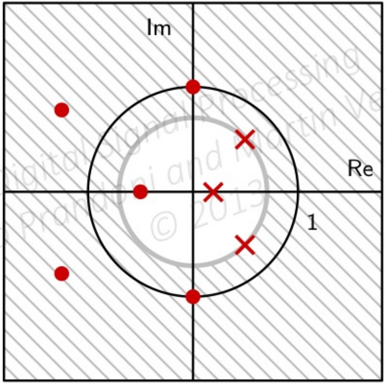
System stability criterion

Consider a filter with impulse response $h[n]$

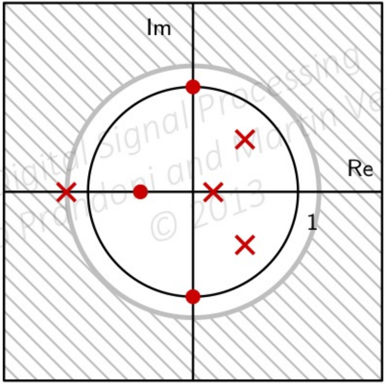
- BIBO stability $\iff \sum_{n=-\infty}^{\infty} |h[n]| < \infty$ (Module 5.3)
- $H(z)|_{|z|=1} < \infty \iff \sum_{n=-\infty}^{\infty} |h[n]| < \infty$ (absolute convergence of z-transform)

system is stable if and only if ROC includes the unit circle!

Stable system



Unstable system

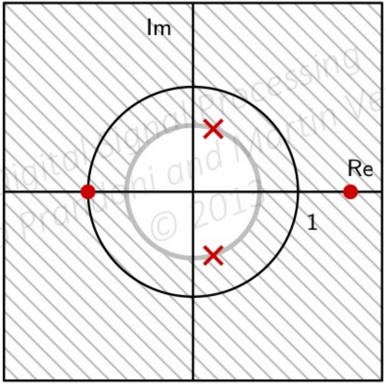


Estimating the frequency response from the pole-zero plot

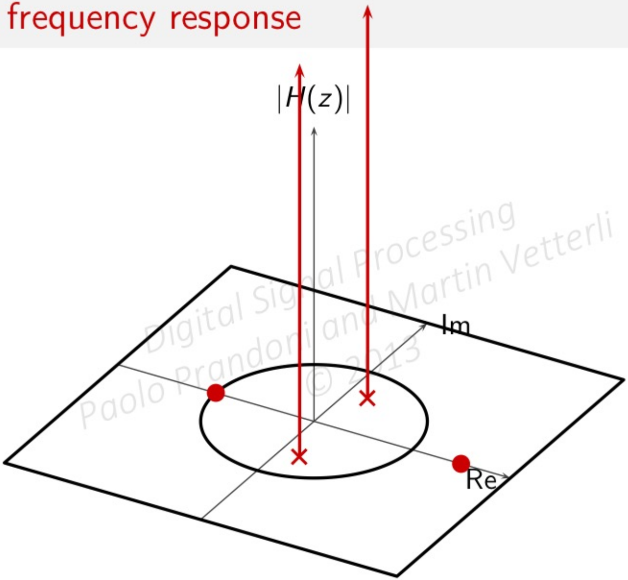
The "circus tent" method:

- magnitude of z transform is like a rubber sheet over the complex plane
- zeros glue the sheet to the ground
- poles are like ... poles, pushing it up
- frequency response (in magnitude) is sheet profile around the unit circle

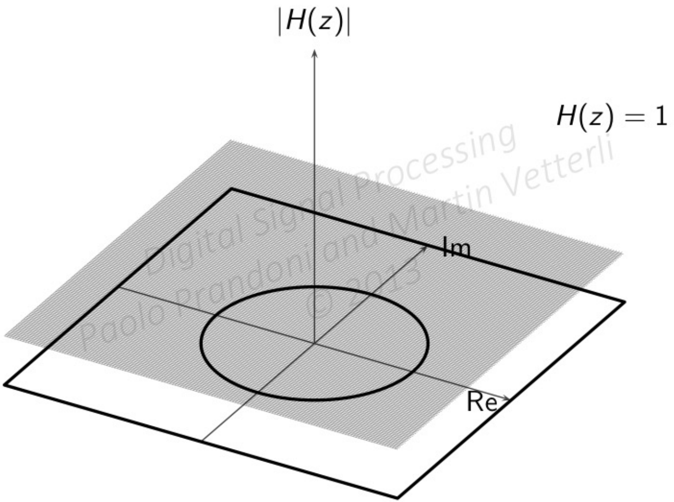
Estimating the frequency response



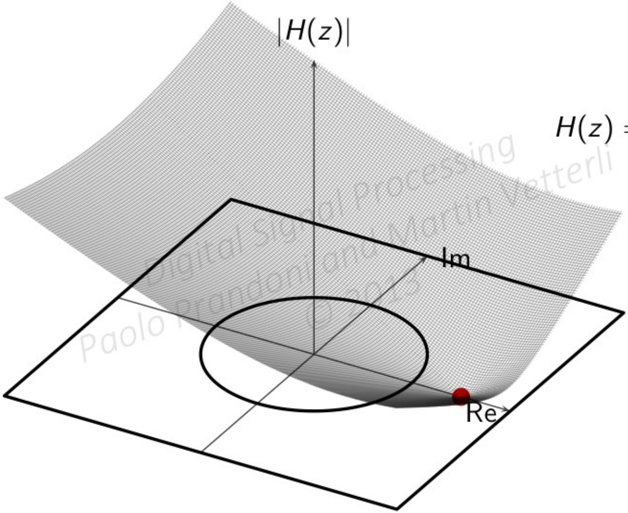
Estimating the frequency response



Estimating the frequency response

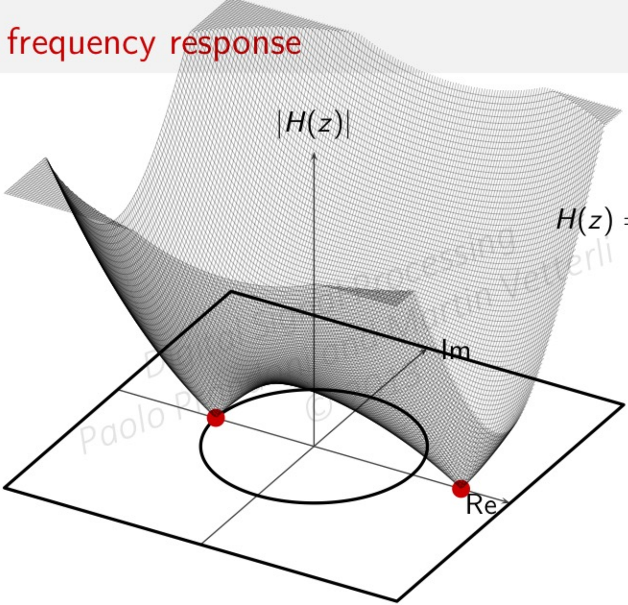


Estimating the frequency response



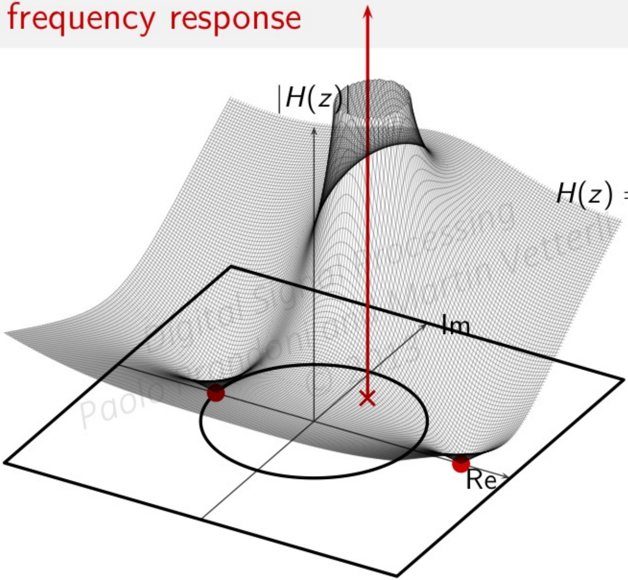
$$H(z) = (1 - z_0\, z^{-1})$$

Estimating the frequency response



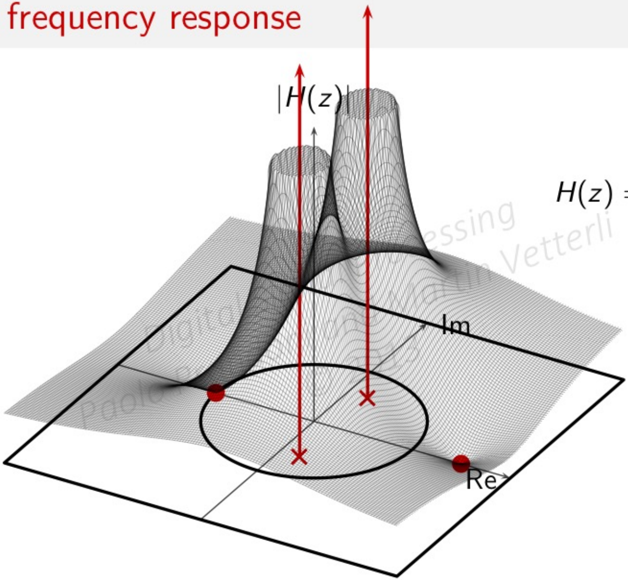
$$H(z) = (1 - z_0\, z^{-1})(1 + z_1\, z^{-1})$$

Estimating the frequency response



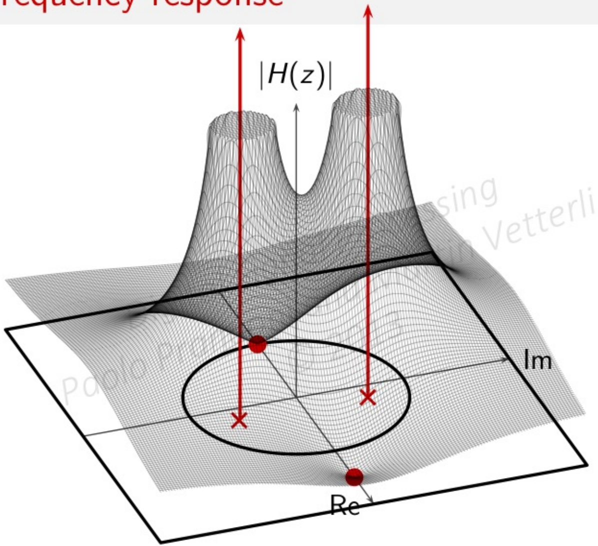
$$H(z) = \frac{(1 - z_0\, z^{-1})(1 + z_1\, z^{-1})}{(1 - p_0\, z^{-1})}$$

Estimating the frequency response

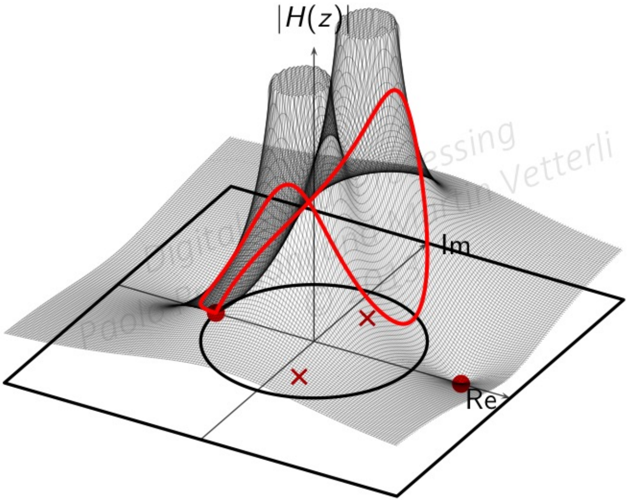


$$H(z) = \frac{(1 - z_0\, z^{-1})(1 + z_1\, z^{-1})}{(1 - p_0\, z^{-1})(1 - p_0^*\, z^{-1})}$$

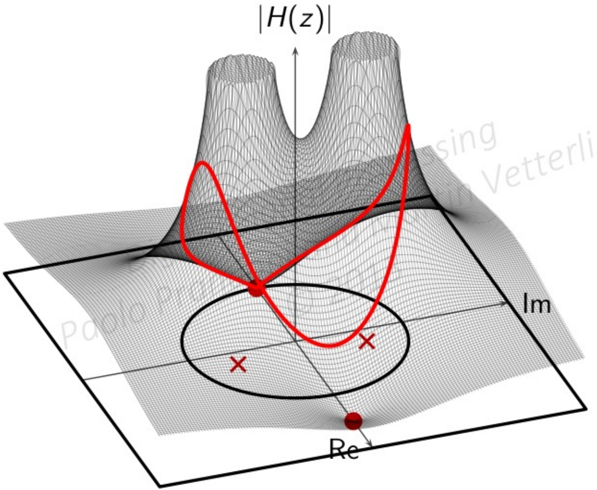
Estimating the frequency response



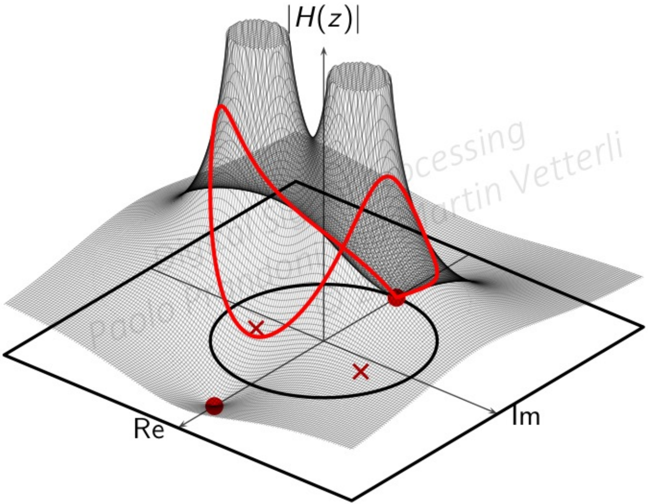
Estimating the frequency response



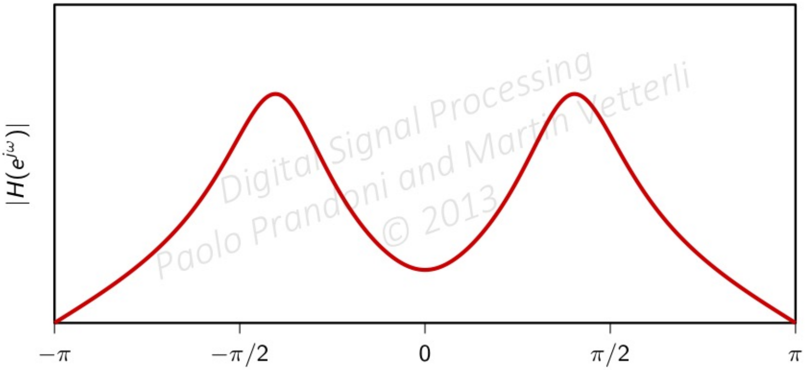
Estimating the frequency response



Estimating the frequency response



Estimating the frequency response



Digital Signal Processing

Module 5.8: Implementation of Digital Filters

Overview:

- Algorithms for CCDE's
- Block diagram
- Real-time processing

An old friend

```
double Leaky(double x) {
  static const double lambda = 0.95;
  static double y = 0;

  y = lambda * y + (1 - lambda) * x;
  return y;
}
```

Testing the code

```
int main() {
  int n;
  for (n = 0; n < 20; n++)
    printf("%.4f ", Leaky(n==0 ? 1.0 : 0.0));
}
```

```
0.0500 0.0475 0.0451 0.0429 0.0407 0.0387 0.0368 0.0349 0.0332 0.0315
0.0299 0.0284 0.0270 0.0257 0.0244 0.0232 0.0220 0.0209 0.0199 0.0189
```

Key points

- we need a "memory cell" to store previous output
- we need to initialize the storage before first use
- we need 2 multiplications and one addition per output sample

Some like it OOP

```
class Leaky:
  def __init__(self, lmb):
    self.lmb=lmb
    self.y=0

  def compute(self, x):
    res = []
    for v in x:
      self.y = self.lmb * self.y + (1 - self.lmb) * v
      res.append(self.y)
    return res
```

Testing the code

```
>>> from leaky import Leaky
>>> L=Leaky(0.95)
>>> print L.compute([0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0])
[0.0, 0.0, 0.0, 0.0, 0.050000000000000, 0.047500000000000,
 0.045125000000000, 0.042868750000000, 0.040725312500000,
 0.038689046875000, 0.036754594531250]
>>>
```

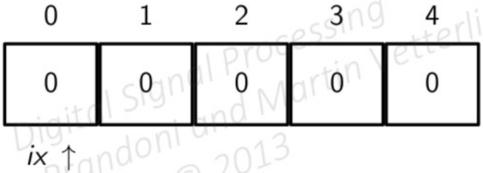
Another old friend

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$$

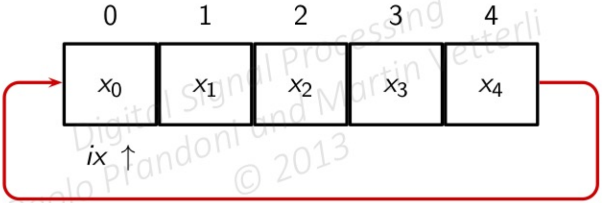
Another old friend

```
double MA(double x) {
  static const int M = 10;
  static double z[M];
  static int ix = -1;

  int n;
  double avg = 0;

  if (ix == -1) {
    for (n = 0; n < M; n++)
      z[n] = 0;
    ix = 0;
  }

  z[ix] = x;
  ix = (ix + 1) \% M;

  for (n = 0; n < M; n++)
    avg += z[n];
  return avg / M;
}
```

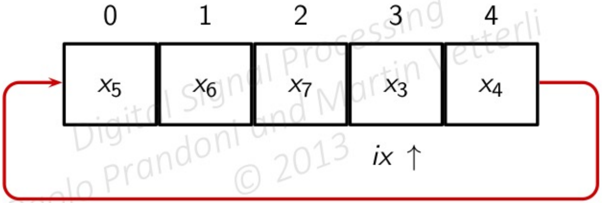
The circular buffer



The circular buffer



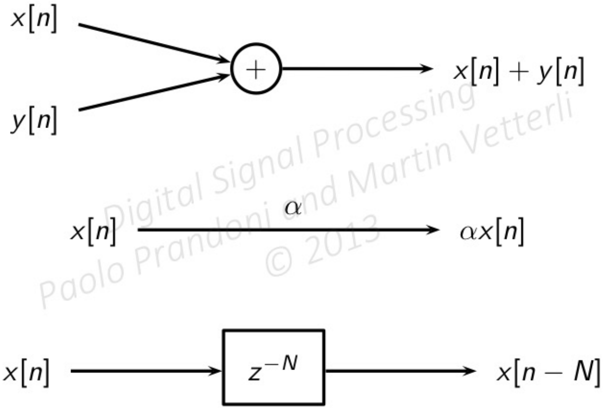
The circular buffer



Key points

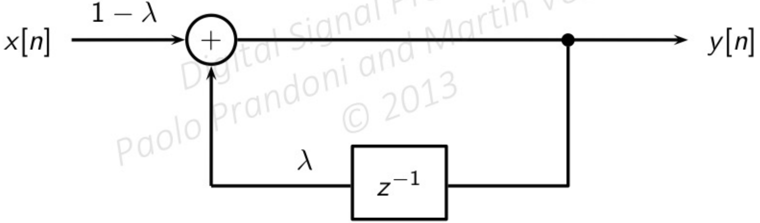
- we now need M memory cells to store previous input values
- we need to initialize the storage before first use
- we need 1 division and M additions per output sample

We can abstract from the implementation



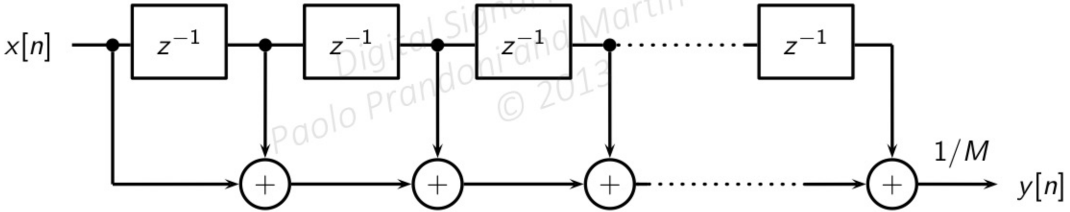
Leaky Integrator

$$y[n] = \lambda y[n-1] + (1-\lambda)x[n]$$



Moving Average

$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$



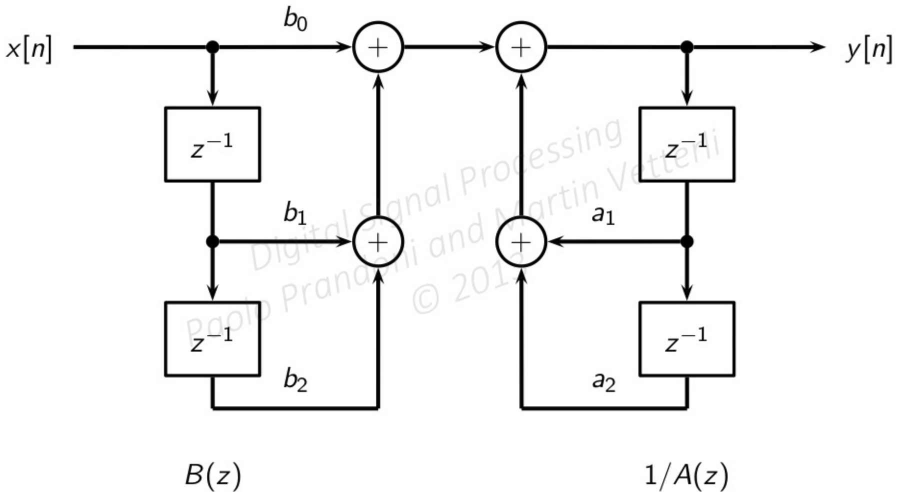
The second-order section

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{a_0 - a_1 z^{-1} - a_2 z^{-2}} = \frac{B(z)}{A(z)}$$

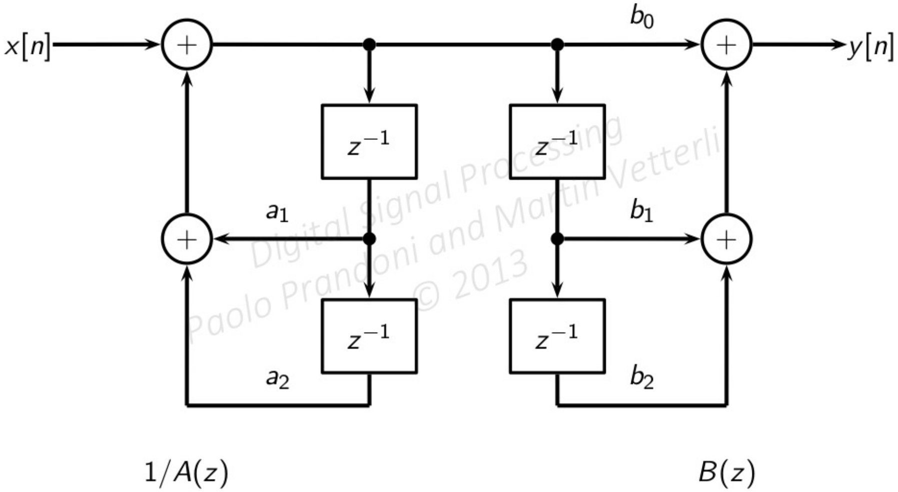
The second-order section

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{B(z)}{A(z)}$$

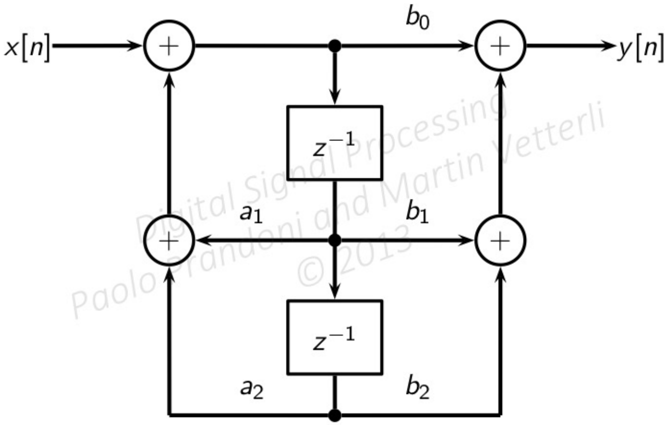
Second-order section, direct form I



Second-order section, direct form I, inverted order



Second-order section, direct form II



Real-time processing - speed

If input samples arrive every T seconds,

- each output sample must be computed in at most T seconds
- number of operations becomes important (IIR vs FIR)
- some common tricks:
  - circular buffers are size 2^K (mod operation faster)
  - exploit the parallelism some processor operations (fetch and compute)

Real-time processing - accuracy

Processing unit have finite precision arithmetic:

- overflow or underflow are a real problem
- more complex structures are more resilient (but less efficient)
- some common tricks:
  - use double precision in the accumulator
  - split the filter in low-order sections
  - use floating point

Digital Signal Processing

Module 5.9: Filter Design - Part II: Intuitive Filters

Overview:

- General problem
- "Intuitive" IIR design

Simple, useful filters

- many signal processing problems can be solved using simple filters
- we have seen simple lowpass filters already (Moving Average, Leaky Integrator)
- simple (low order) transfer functions allow for intuitive design and tuning

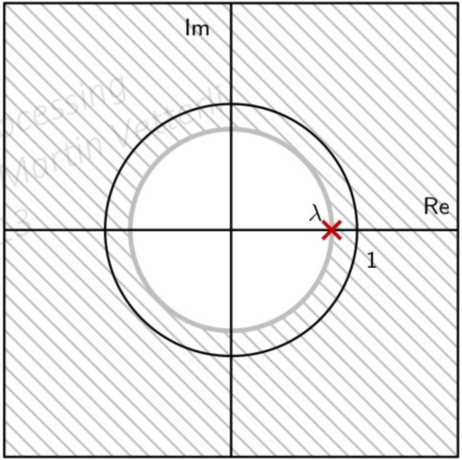
Simple lowpass

- let only low frequencies pass
- used to remove high frequency components (e.g. noise)
- useful in audio, communication, control systems
- we know a simple answer: leaky integrator

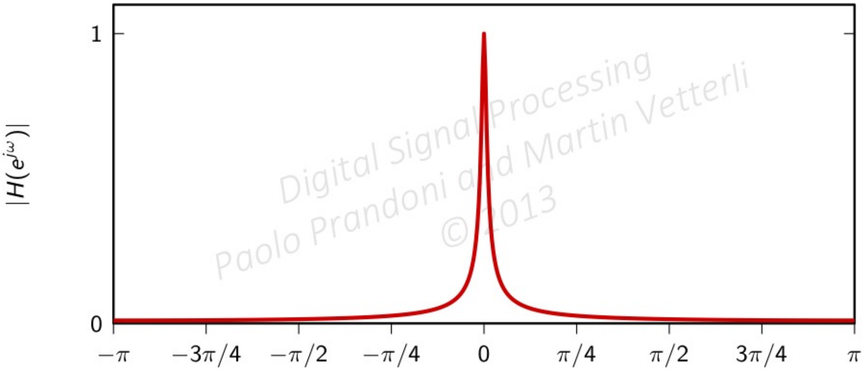
Leaky Integrator

$$H(z) = \frac{(1-\lambda)}{1-\lambda z^{-1}}$$

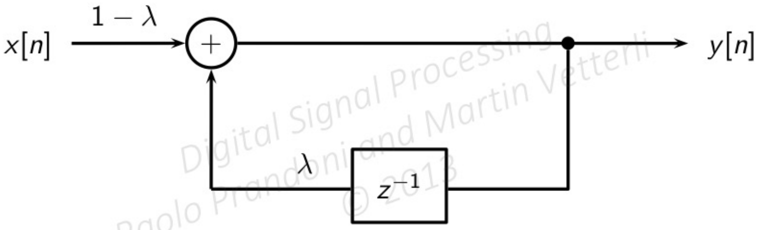
$$y[n] = (1-\lambda)x[n] + \lambda y[n-1]$$



Leaky Integrator, $\lambda = 0.98$



Leaky Integrator, filter structure



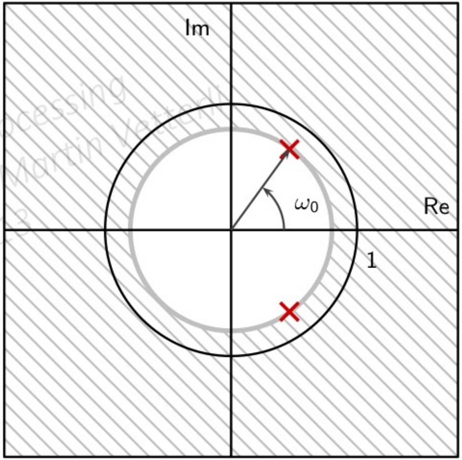
Resonator

- a resonator is a narrow bandpass filter
- used to detect the presence of a sinusoid of a given frequency
- useful in communication systems and telephony (DTMF)
- idea: shift the passband of the Leaky Integrator!

$$H(z) = \frac{G_0}{(1 - pz^{-1})(1 - p^* z^{-1})}$$

$$p = \lambda e^{j\omega_0}$$

$$y[n] = G_0 x[n] - a_1 y[n-1] - a_2 y[n-2]$$



$$H(z) = \frac{G_0}{(1 - pz^{-1})(1 - p^* z^{-1})}, \qquad p = \lambda e^{j\omega_0}$$

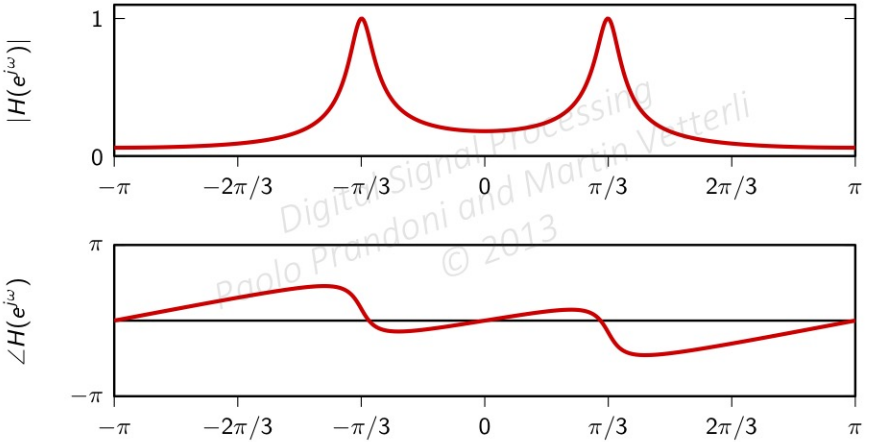
$$= \frac{G_0}{1 - 2\Re\{p\}\, z^{-1} + |p|^2\, z^{-2}}$$

$$= \frac{G_0}{1 - 2\lambda \cos\omega_0\, z^{-1} + |\lambda|^2\, z^{-2}}$$

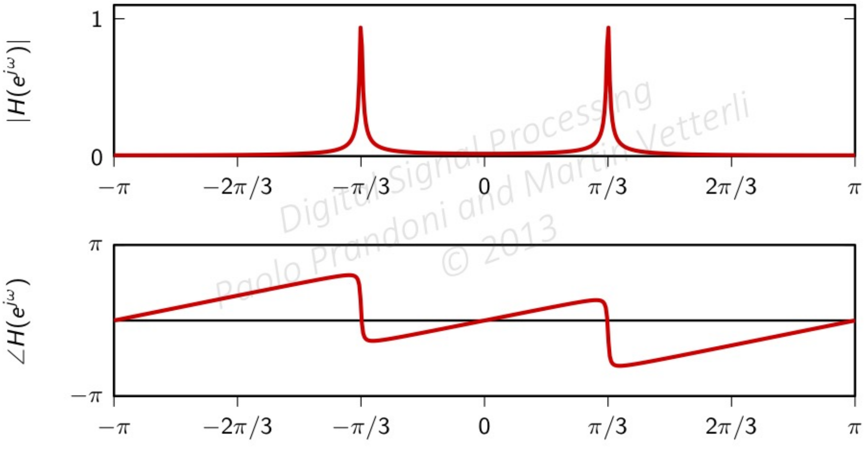
$$a_1 = 2\lambda\cos\omega_0$$

$$a_2 = -|\lambda|^2$$

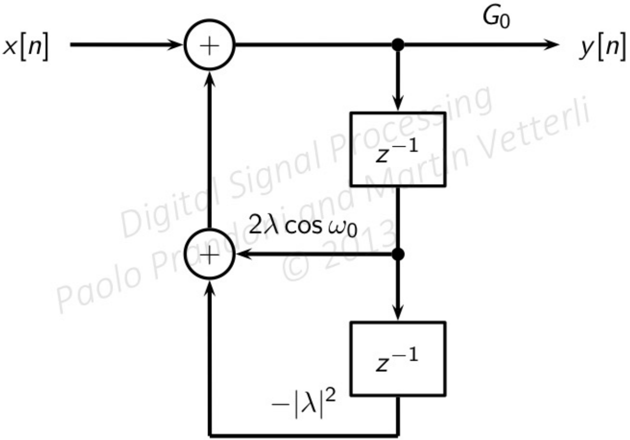
Resonator, $\lambda = 0.9, \omega_0 = \pi/3$



Resonator, $\lambda = 0.99, \omega_0 = \pi/3$



Resonator, filter structure



DC removal

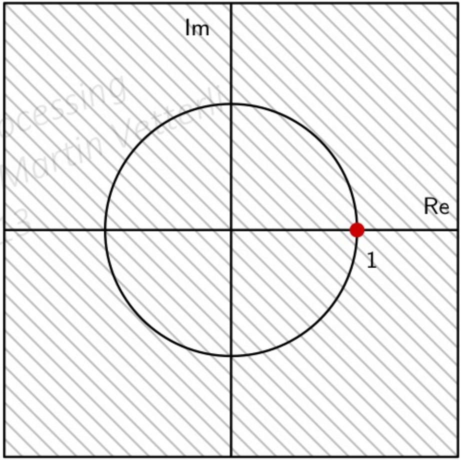
- a DC-balanced signal has zero sum: $\lim_{N\to\infty} \sum_{n=-N}^{N} x[n] = 0$

  i.e. there is no Direct Current component
- its DTFT value at zero is zero
- we want to remove the DC bias from a non zero-centered signal
- we want to kill the frequency component at $\omega = 0$

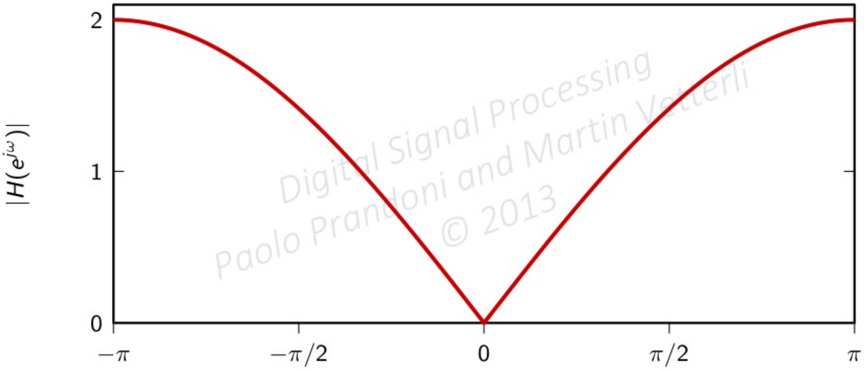
DC removal

$$H(z) = 1 - z^{-1}$$

$$y[n] = x[n] - x[n-1]$$



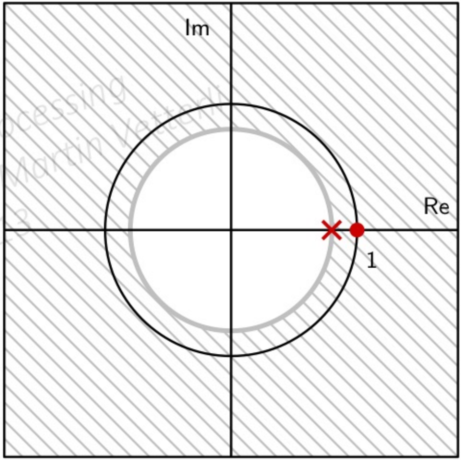
DC notch



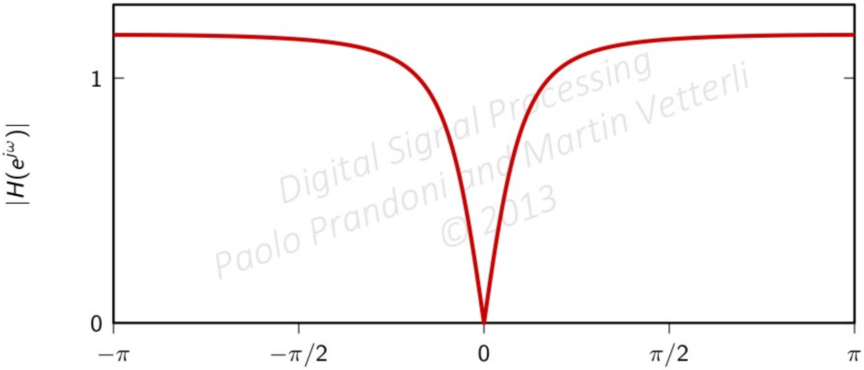
DC removal, improved

$$H(z) = \frac{1 - z^{-1}}{1 - \lambda z^{-1}}$$

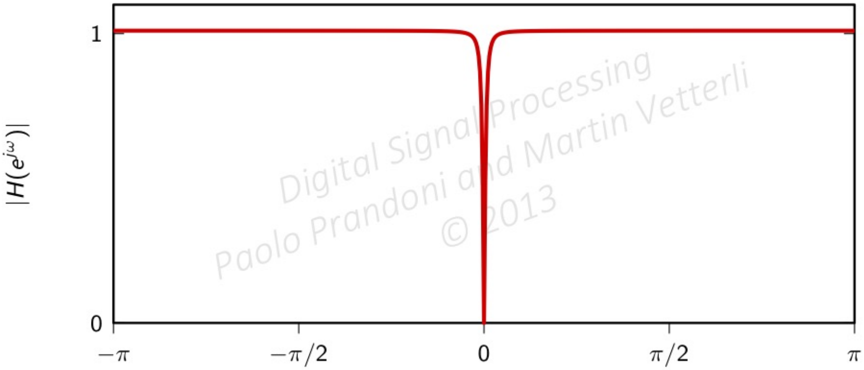
$$y[n] = \lambda y[n-1] + x[n] - x[n-1]$$



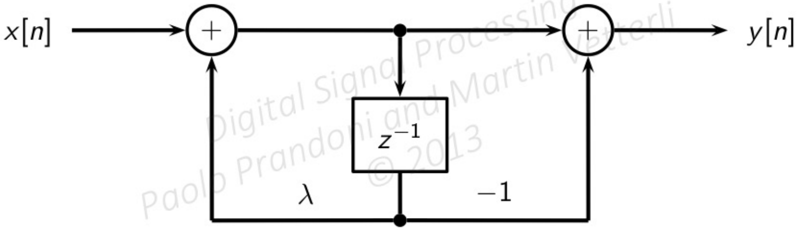
DC notch, $\lambda = 0.7$



DC notch, $\lambda = 0.98$



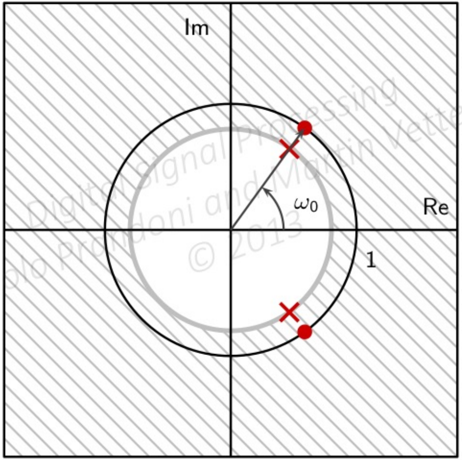
DC notch, filter structure



Hum removal

- similar to DC removal but we want to remove a specific nonzero frequency
- very useful for musicians: amplifiers for electric guitars pick up the hum from the electric mains (50Hz in Europe and 60Hz in North America)
- we need to tune the hum removal according to country

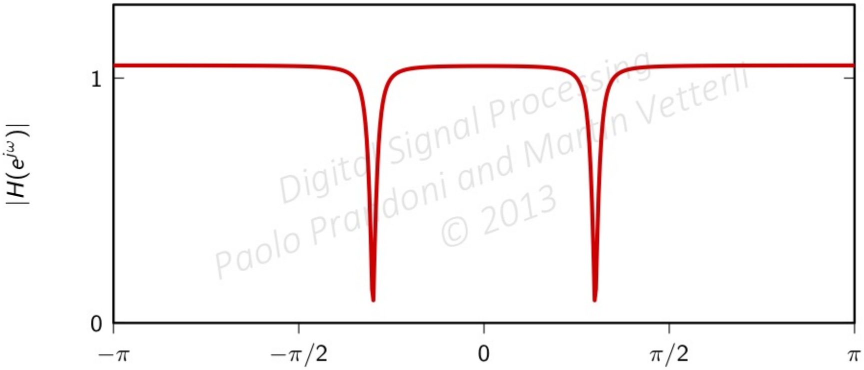
Hum removal



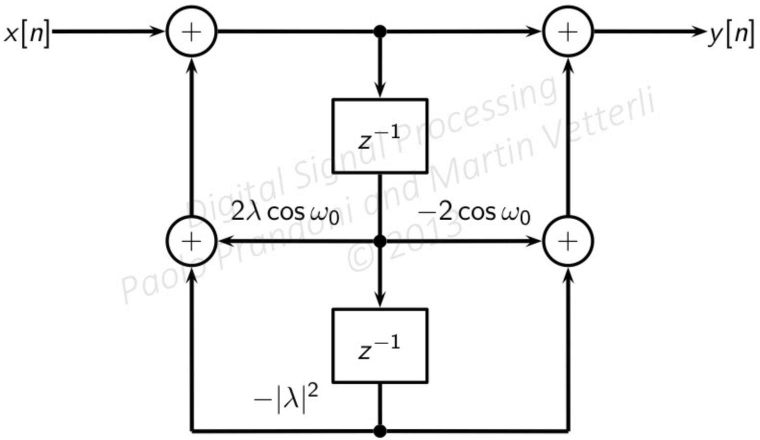
Hum removal

$$H(z) = \frac{(1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})}{(1 - \lambda e^{j\omega_0} z^{-1})(1 - \lambda e^{-j\omega_0} z^{-1})}$$

Hum removal, $\lambda = 0.95$



Hum removal, filter structure



Digital Signal Processing

Module 5.10: Filter Design - Part III: Design from Specs

Overview:

- Filter specifications
- IIR design
- FIR design

The filter design problem

You are given a set of requirements:

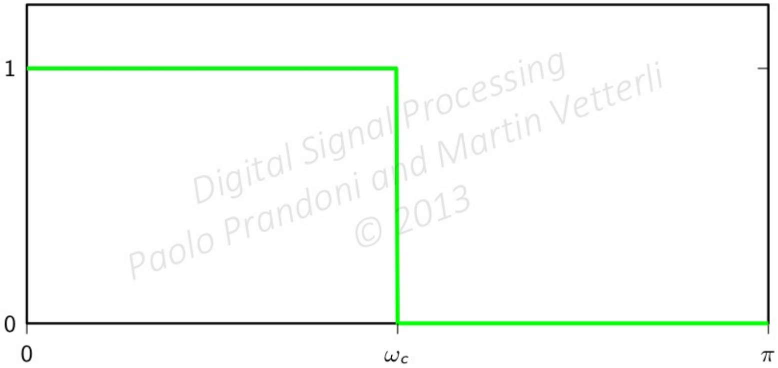
- frequency response: passband(s) and stopband(s)
- phase: overall delay, linearity
- some limit on computational resources and/or numerical precision

You must determine N, M, a_k's and b_k's in

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_{M-1} z^{-(M-1)}}{a_0 + a_1 z^{-1} + \ldots + a_{N-1} z^{-(N-1)}}$$

in order to best fulfill the requirements

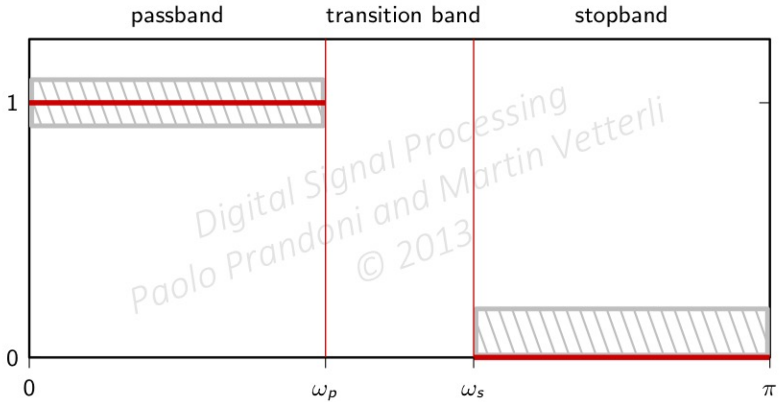
Example: lowpass specs



Practical limitations

- passband/stopband transitions cannot be infinitely sharp
  $\Rightarrow$ use *transition bands*
- magnitude response cannot be constant over an interval
  $\Rightarrow$ specify magnitude *tolerances* over bands
- in general:
  - smaller transition bands $\Rightarrow$ higher filter order
  - smaller error tolerances $\Rightarrow$ higher filter order
  - higher filter order $\Rightarrow$ more expensive, larger delay

Example: realistic lowpass specs

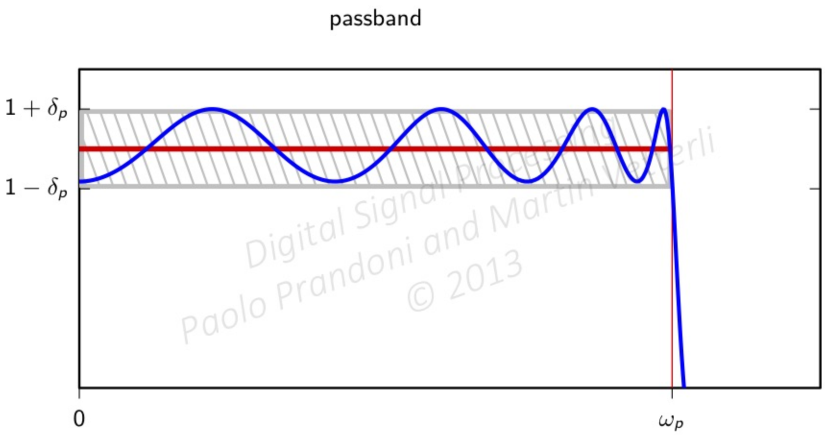


Why we can't have a flat response

$$H(z) = B(z)/A(z), \qquad \text{with } A \text{ and } B \text{ polynomials}$$

$$\begin{aligned}
H(e^{j\omega}) = c \text{ over an interval} \quad &\Rightarrow \quad B(z) - cA(z) = 0 \text{ over an interval} \\
&\Rightarrow \quad B(z) - cA(z) \text{ has an infinite number of roots} \\
&\Rightarrow \quad B(z) - cA(z) = 0 \text{ for all values of } z \\
&\Rightarrow \quad H(e^{j\omega}) = c \text{ over the entire } [-\pi, \pi] \text{ interval.}
\end{aligned}$$

Important case: equiripple error



The big questions

- IIR or FIR?
- how to determine the coefficients?
- how to evaluate the performance?

IIRs: pros and cons

Pros:

- computationally efficient
- strong attenuation easy
- good for audio

Cons:

- stability issues
- difficult to design for arbitrary response
- nonlinear phase

FIRs: pros and cons

Pros:

- always stable
- optimal design techniques exist
- can be designed with linear phase

Cons:

- computationally much more expensive
- may "sound" harsh

The design methods

- finding N, M, a_k's and b_k's from specs is a hard nonlinear problem
- established methods:
  - IIR: conversion of analog design
  - FIR: optimal minimax filter design

IIR: conversion of analog design

Filter design was an established art long before digital processing appeared

- lots of nice analog filters exist
- methods exist to "translate" the analog design into a rational transfer function
- most numerical packages (Matlab, etc.) provide ready-made routines
- design involves specifying some parameters and testing that the specs are fulfilled

Butterworth lowpass

Magnitude response:

- maximally flat
- monotonic over $[0, \pi]$

Design parameters:

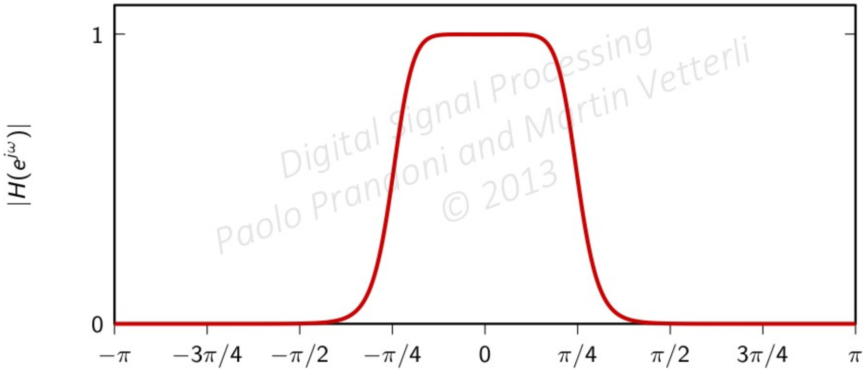
- order N
- cutoff frequency

Test values:

- width of transition band
- passband error

Butterworth lowpass example

$$N = 4, \omega_c = \pi/4$$



Chebyshev lowpass

Magnitude response:

- equiripple in passband
- monotonic in stopband

Design parameters:

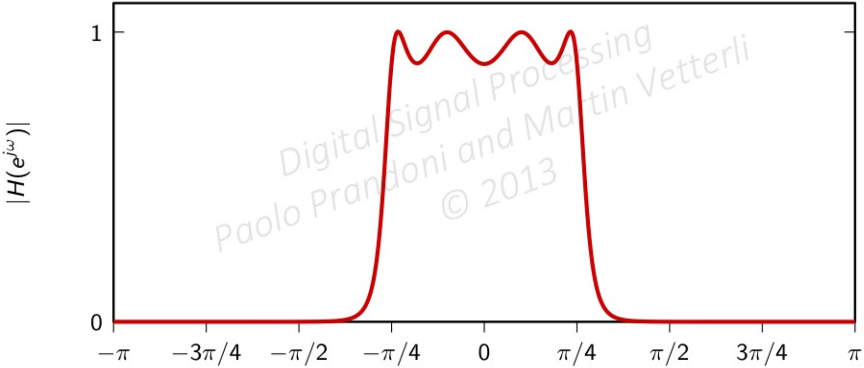
- order N
- passband max error
- cutoff frequency

Test values:

- width of transition band
- stopband error

Chebyshev lowpass example

$$N = 4, \omega_c = \pi/4, e_{\max} = 12\%$$



Elliptic lowpass

Magnitude response:

- equiripple in passband and stopband

Design parameters:

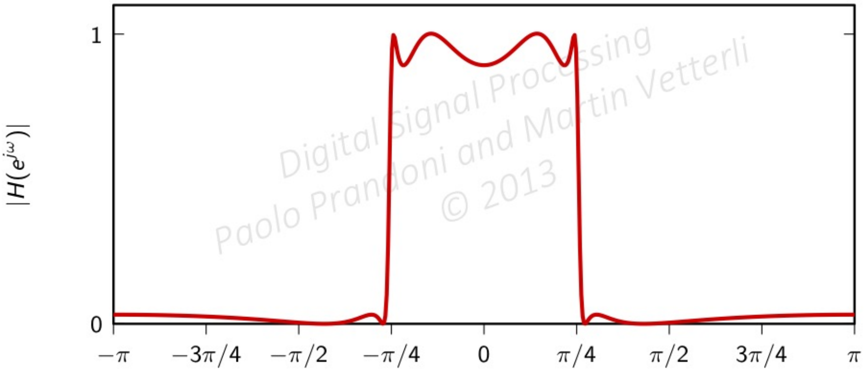
- order N
- cutoff frequency
- passband max error
- stopband min attenuation

Test value:

- width of transition band

Elliptic lowpass example

$$N = 4, \omega_c = \pi/4, e_{\text{max}} = 12\%, \text{att}_{\text{min}} = 0.03$$



FIR: optimal minimax design

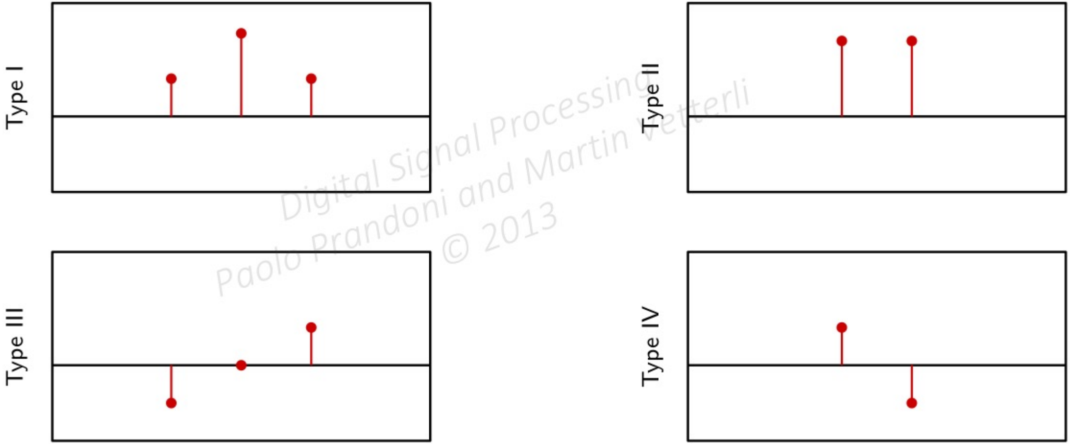
FIR filters are a digital signal processing “exclusivity”.
In the 70s Parks and McClellan developed an algorithm to design optimal FIR filters:

- linear phase
- equiripple error in passband and stopband

algorithm proceeds by **mini**mizing the **max**imum error in passband and stopband

Linear phase

Linear phase derives from a symmetric or antisymmetric impulse response



Linear phase (Type I)

$$h[C+n] = h[C-n]$$

$$h'[n] = h[n+C]$$
$$h'[n] = h'[-n]$$

$$H(z) = z^{-C}H'(z)$$

Linear phase (Type I)

$$H'(z) = \sum_{n=-M}^{M} h'[n]z^{-n}$$

$$= h'[0] + \sum_{n=1}^{M} h'[n](z^{n} + z^{-n})$$

$$H'(e^{j\omega}) = h'[0] + \sum_{n=1}^{M} h'[n](e^{j\omega n} + e^{-j\omega n})$$

$$= h'[0] + 2\sum_{n=1}^{M} h'[n]\cos\omega n \qquad \in \mathbb{R}$$

Linear phase (Type I)

$$H(e^{j\omega}) = \left[h[C] + 2\sum_{n=1}^{M} h[n+C]\cos n\omega \right] e^{-j\omega C}$$

Minimax lowpass

Magnitude response:

- equiripple in passband and stopband

Design parameters:

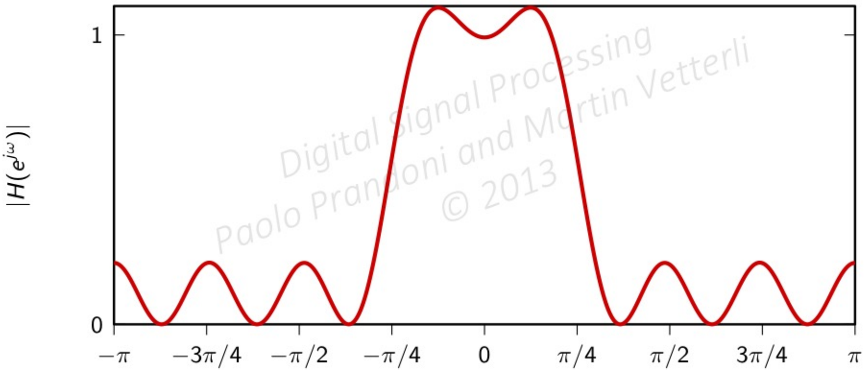
- order N (number of taps)
- passband edge ω_p
- stopband edge ω_s
- ratio of passband to stopband error δ_p/δ_s

Test value:

- passband max error
- stopband max error

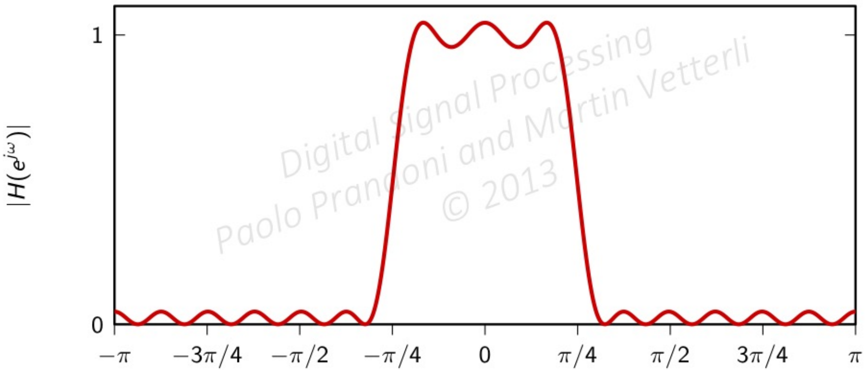
Minimax lowpass example

$$N = 9, \omega_s = 0.2\pi, \omega_p = 0.3\pi, \delta_p/\delta_s = 10$$



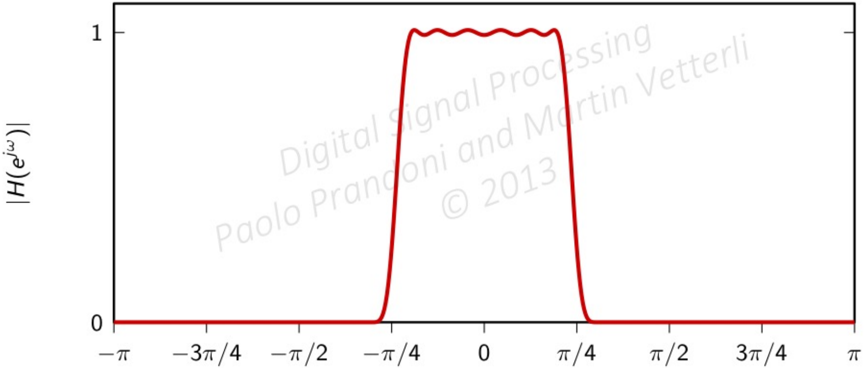
Minimax lowpass example

$$N = 19, \omega_s = 0.2\pi, \omega_p = 0.3\pi, \delta_p/\delta_s = 10$$



Minimax lowpass example

$$N = 51, \omega_s = 0.2\pi, \omega_p = 0.3\pi, \delta_p/\delta_s = 1$$



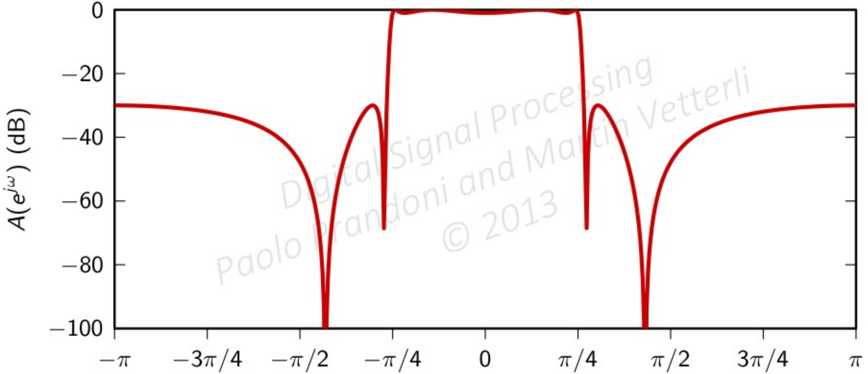
Magnitude response in decibels

- filter max passband magnitude G
- filter attenuation expressed in decibels as:

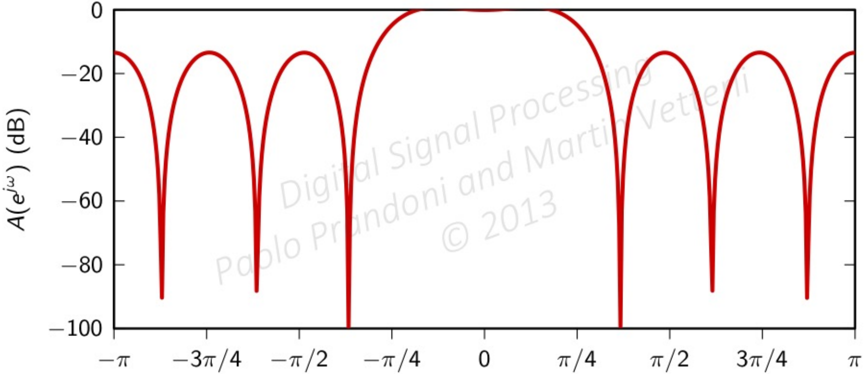
$$A_{\mathrm{dB}} = 20 \log_{10}(|H(e^{j\omega})|/G)$$

- useful to compare attenuations between filters

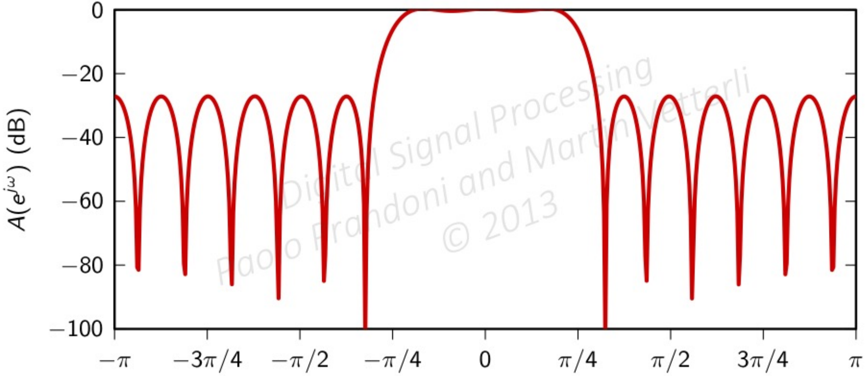
4th-order elliptic lowpass, $\omega_c = \pi/4$, log scale



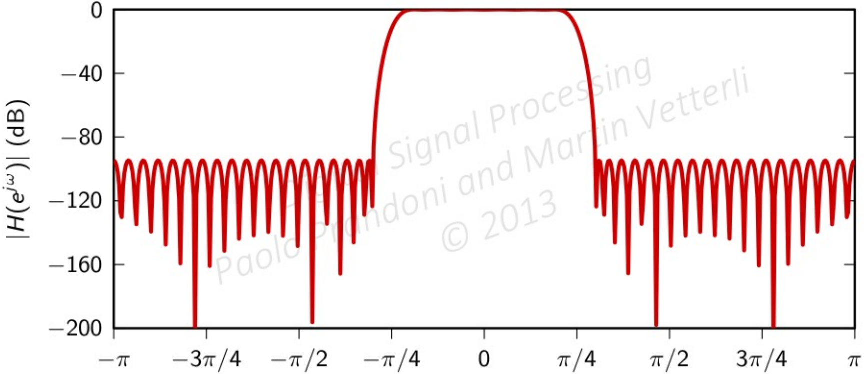
9-tap minimax lowpass, $\omega_c = \pi/4$, log scale



19-tap minimax lowpass, $\omega_c = \pi/4$, log scale



51-tap minimax lowpass, $\omega_c = \pi/4$, log scale



Life beyond lowpass

The IIR and FIR methods we just described can be used to design more general filter types than lowpass, with only minor modifications

- IIR bandpass and highpass can be obtain by modulating the lowpass response
- optimal FIR bandpass and highpass can be designed by the Parks-McClellan algorithm
- optimal FIR can also be designed with piecewise linear magnitude response
- the literature on filter design is vast: this is just the tip of the iceberg!

Digital Signal Processing

Module 5.11: Real-Time Processing

Summary:

- I/O and DMA
- multiple buffering
- implementation framework
- some guitar effects

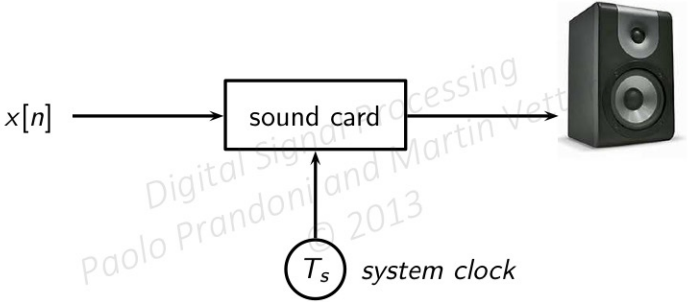
Real-time processing

Everything works in sync with a *system clock* of period T_s:

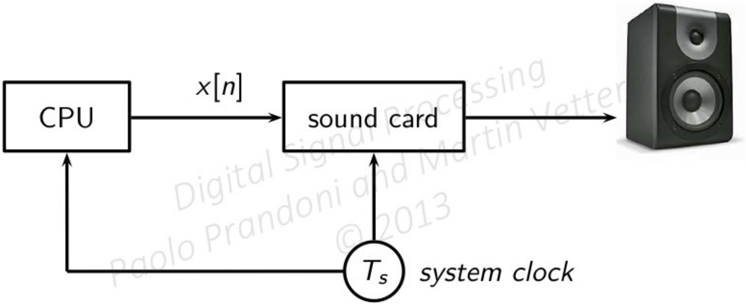
- "record" a value $x_i[n]$
- process the value in a causal filter
- "play" the output $x_o[n]$

everything needs to happen in at most T_s seconds!

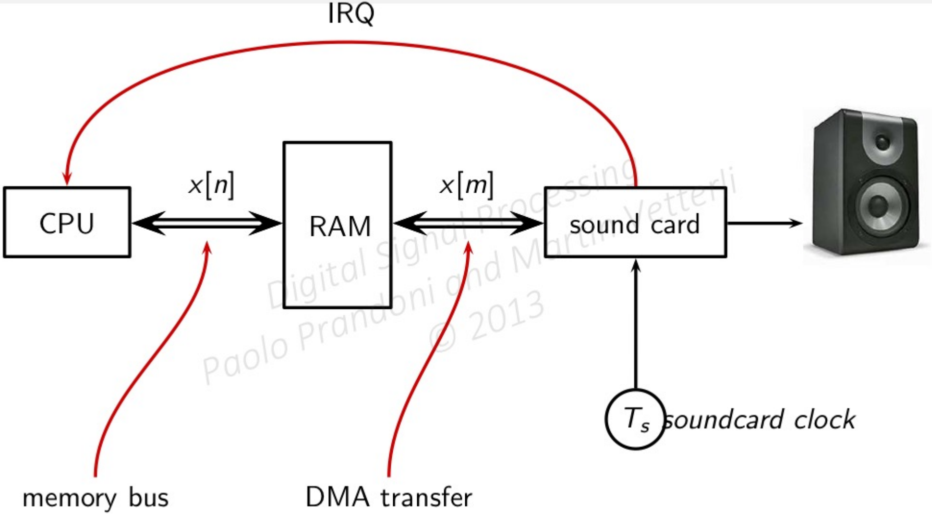
Playing a sound



On dedicated hardware...



On a PC...

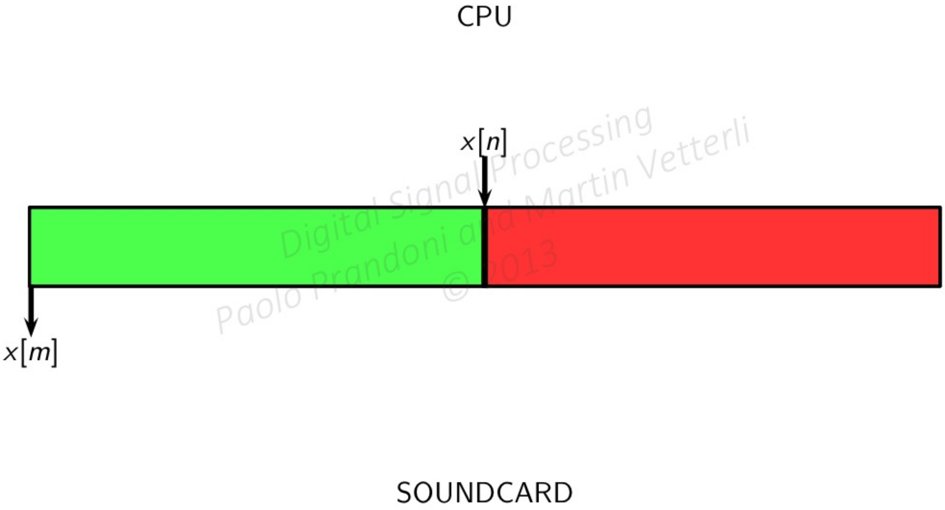

IRQ
CPU
$x[n]$
RAM
$x[m]$
sound card
T_s
soundcard clock
memory bus
DMA transfer

Buffering

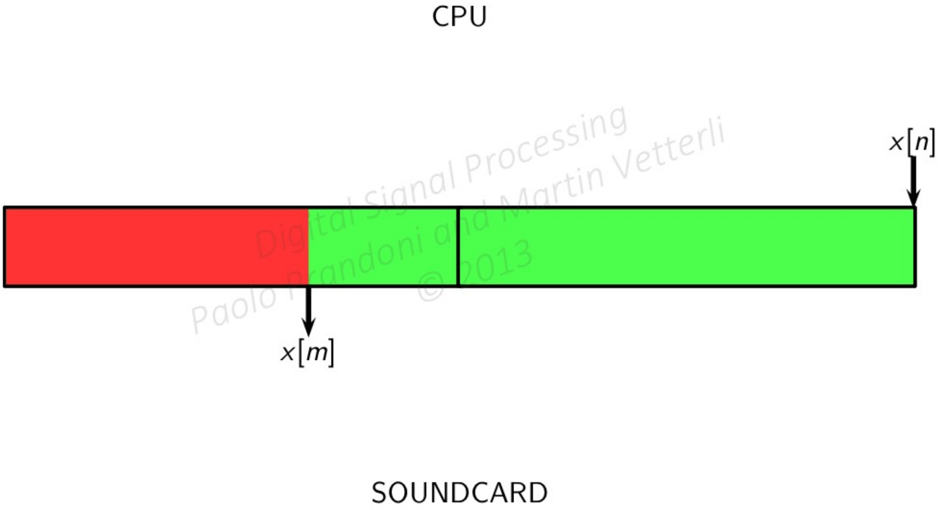
- interrupt for each sample would be too much overhead
- soundcard consumes sample in buffers
- soundcard notifies when buffer used up
- CPU can fill a buffer in less time than soundcard can empty it

buffering introduces delay!

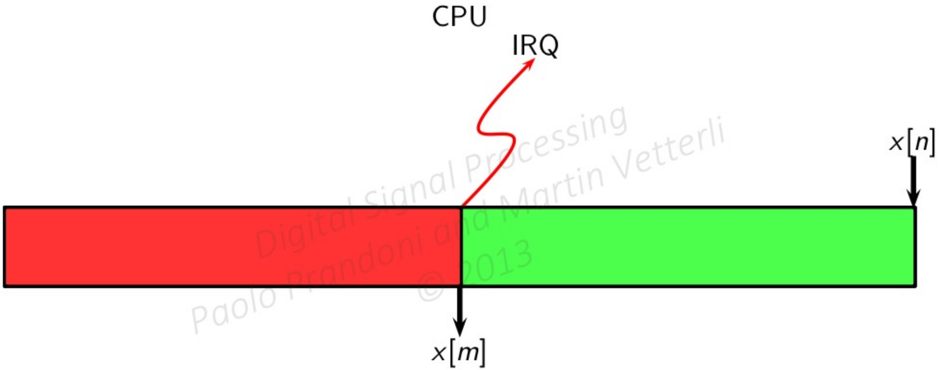
Example: double buffering (output)



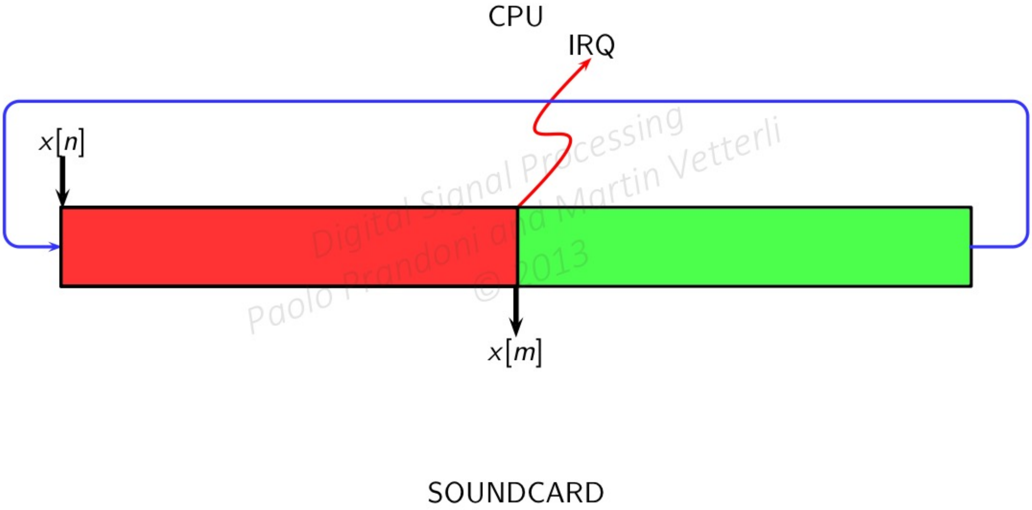
Example: double buffering (output)



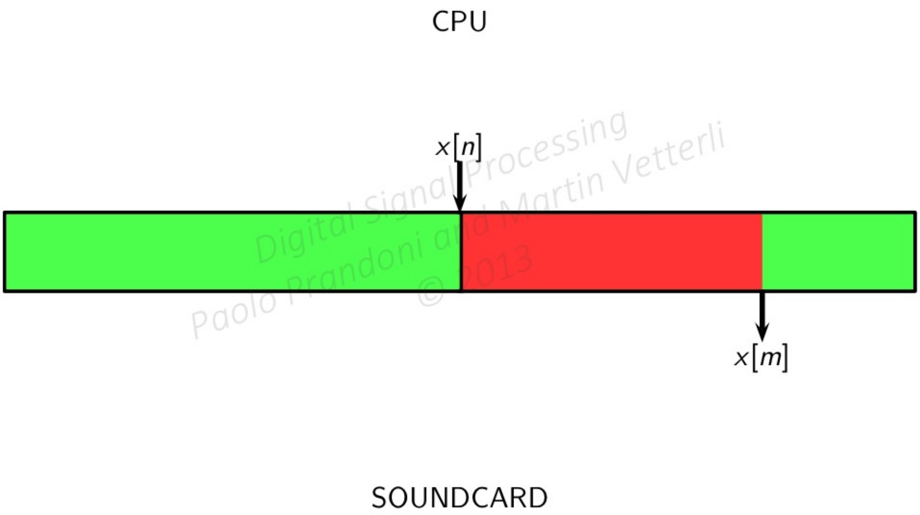
Example: double buffering (output)



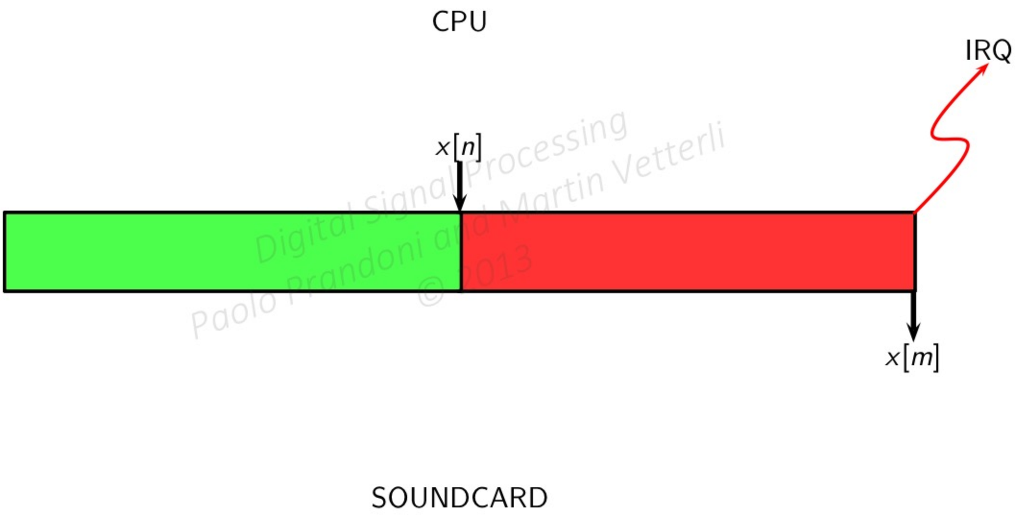
Example: double buffering (output)



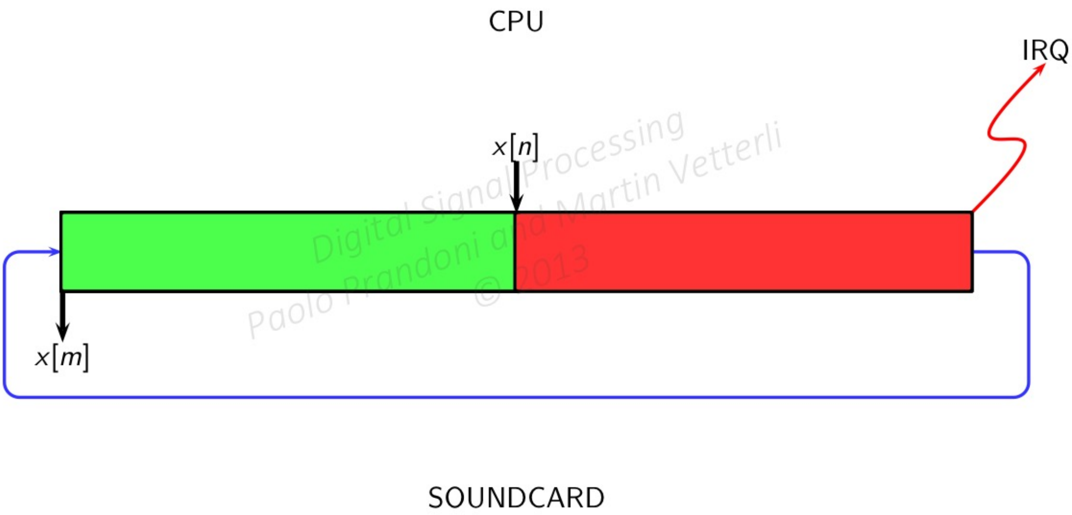
Example: double buffering (output)



Example: double buffering (output)



Example: double buffering (output)



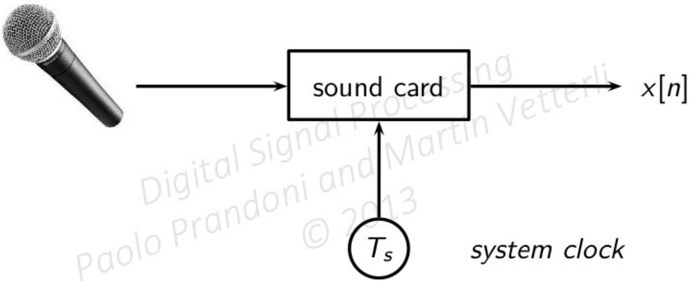
Example: double buffering

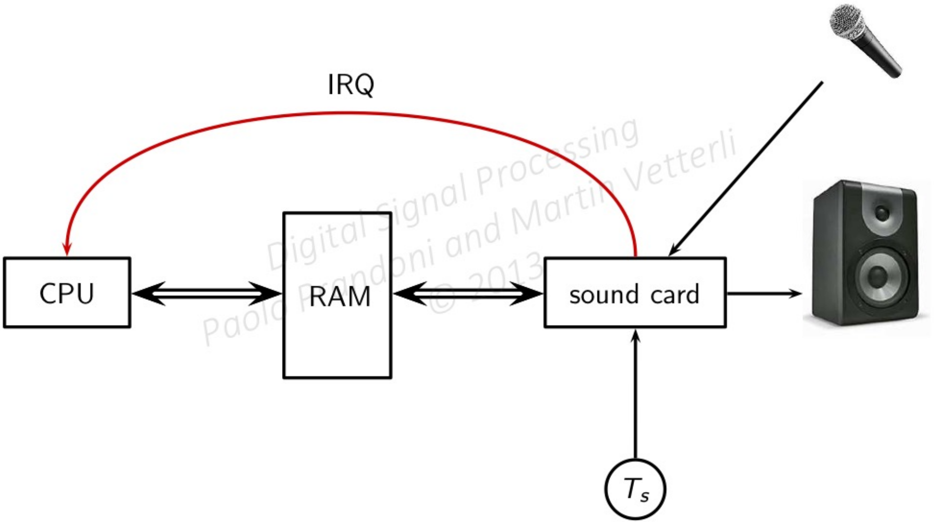
- double buffering introduces a delay $d = T_s \times \dfrac{L}{2}$ seconds
- if CPU doesn't fill the buffer fast enough: **underflow**

Multiple buffering

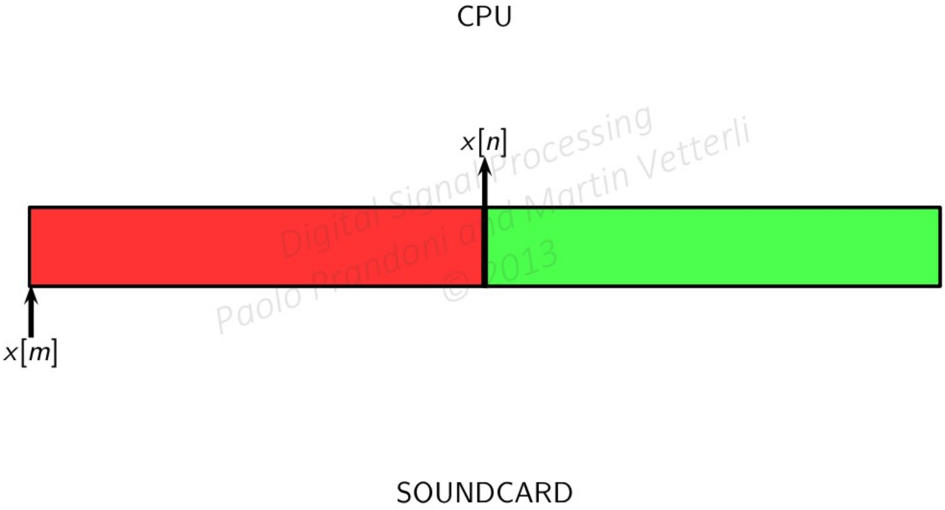
- call the CPU more often (balance load)
- keep reasonable underflow protection

What about the input?

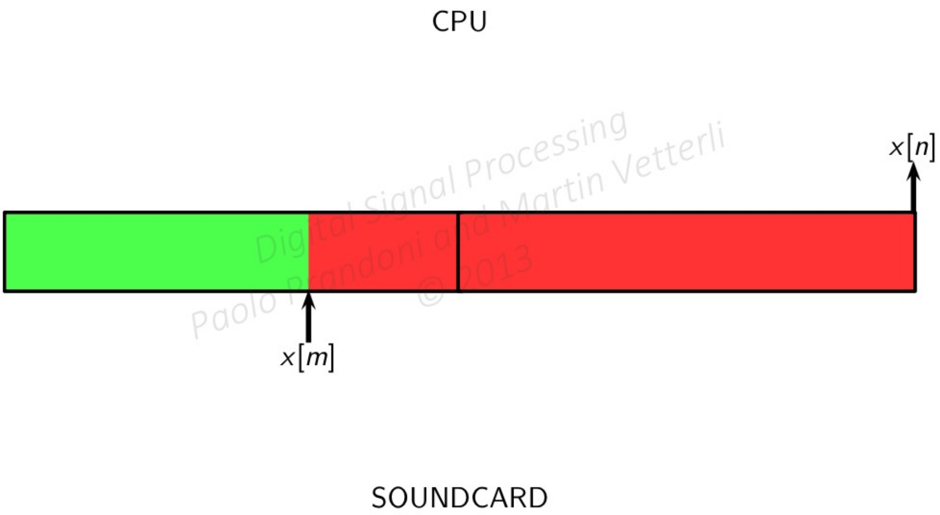



IRQ
CPU
RAM
sound card
T_s

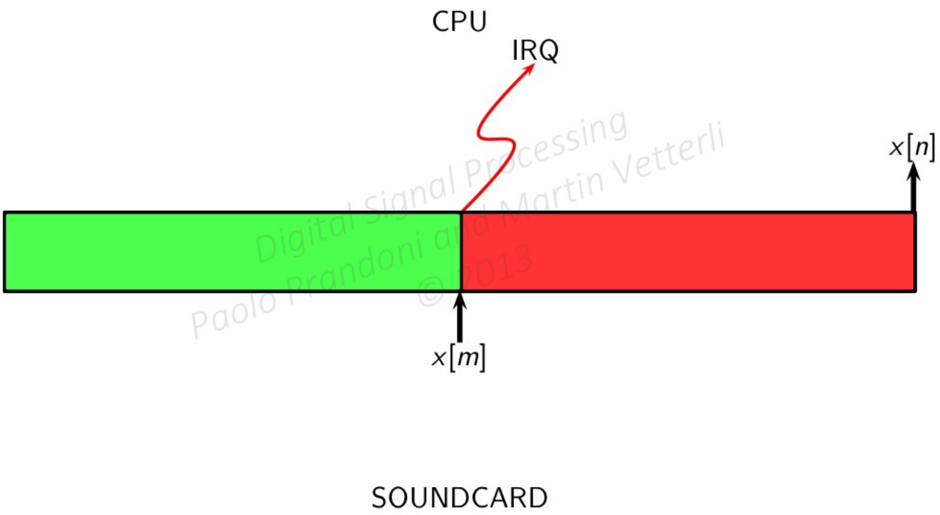
Example: double buffering (input)



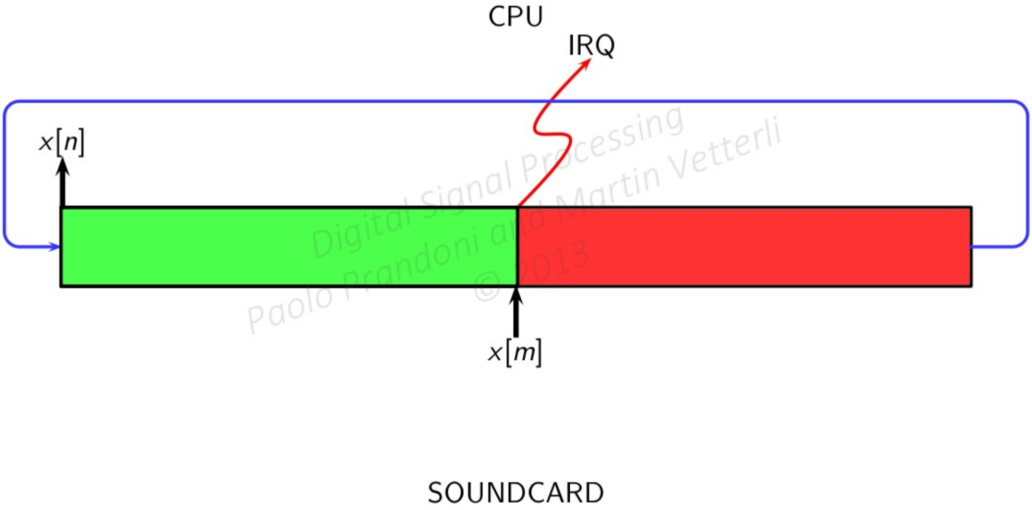
Example: double buffering (input)



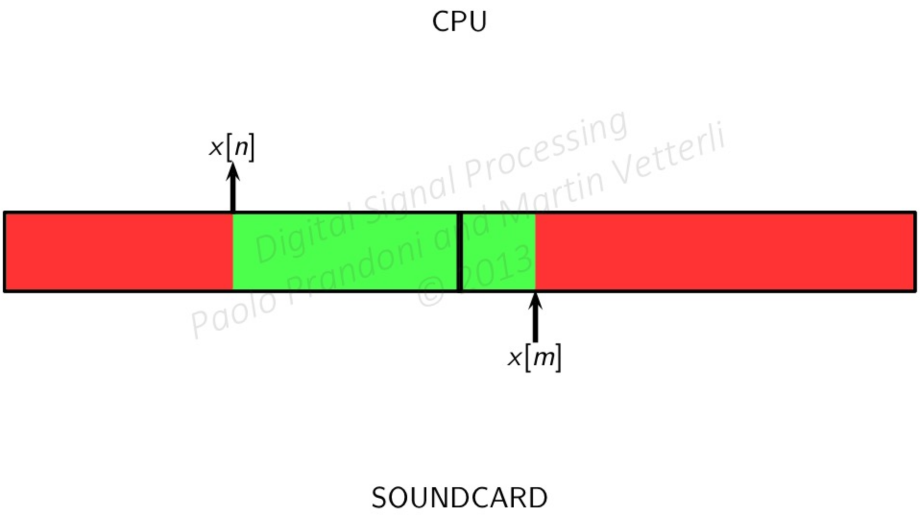
Example: double buffering (input)



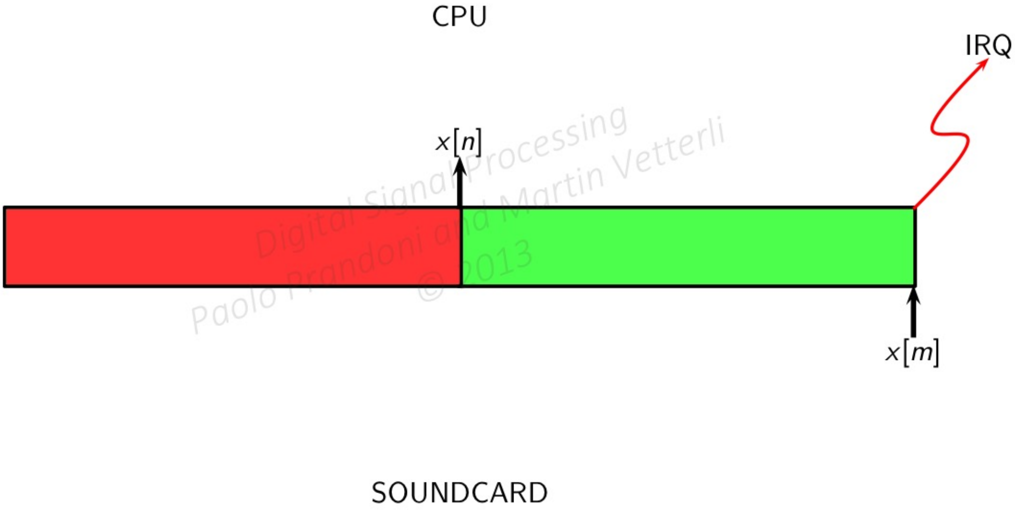
Example: double buffering (input)



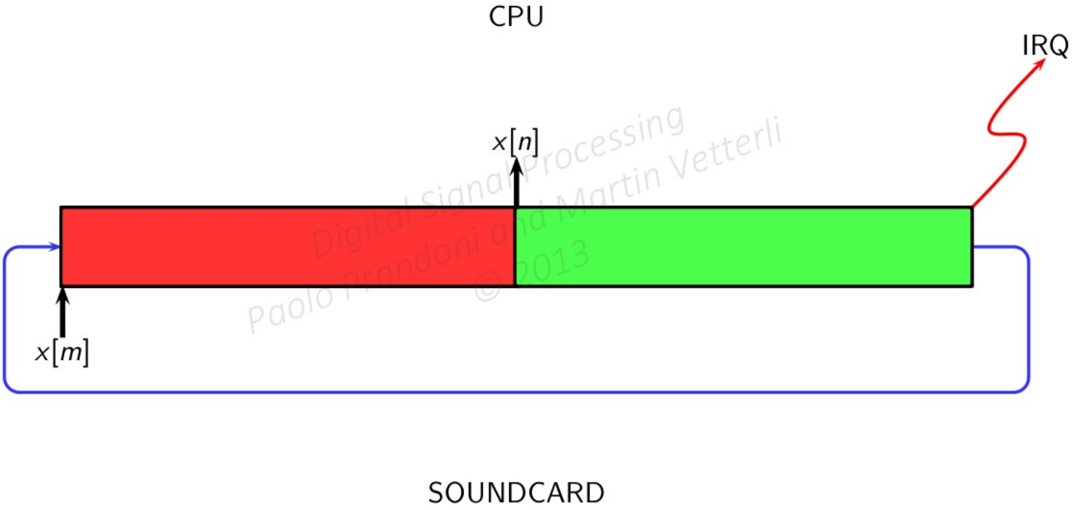
Example: double buffering (input)



Example: double buffering (input)



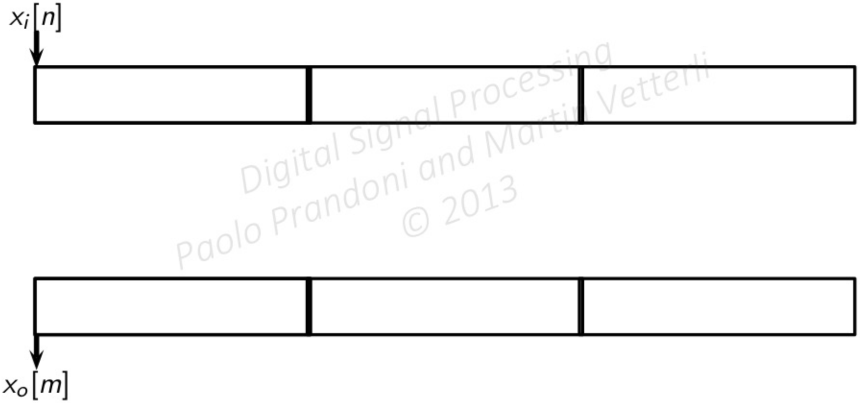
Example: double buffering (input)



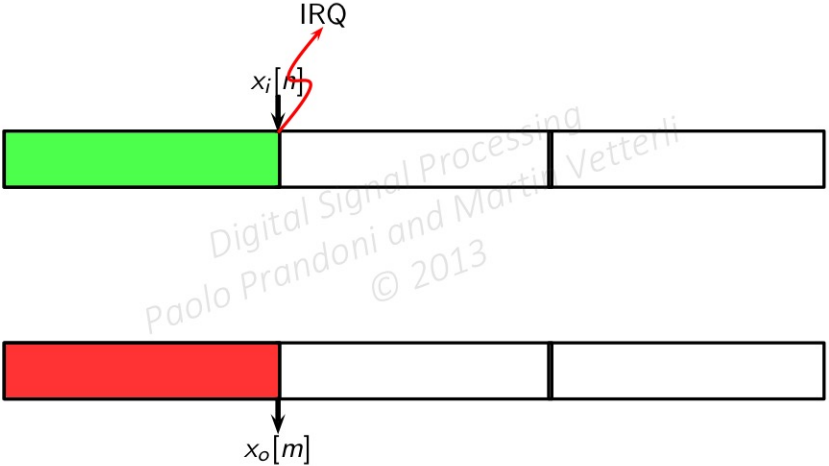
Putting it all together

- multiple input buffers and output buffers
- equal chunk sizes
- input IRQ drives processing

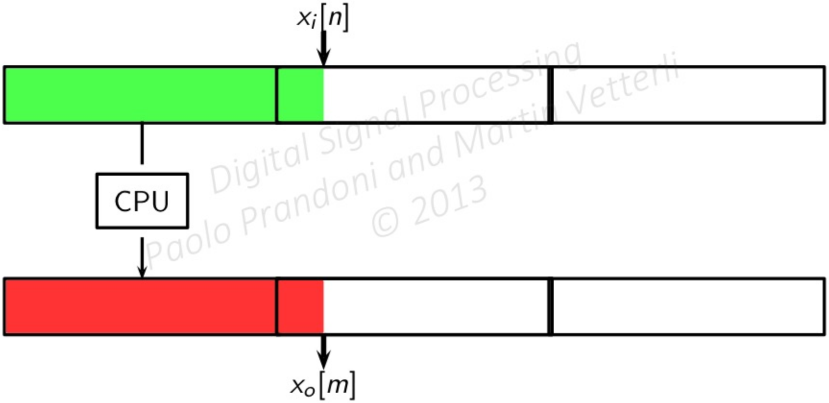
Real-time I/O processing with multiple buffering



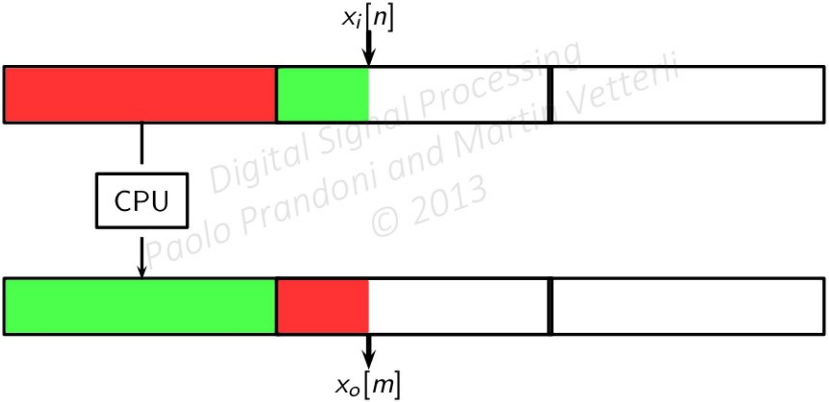
Real-time I/O processing with multiple buffering



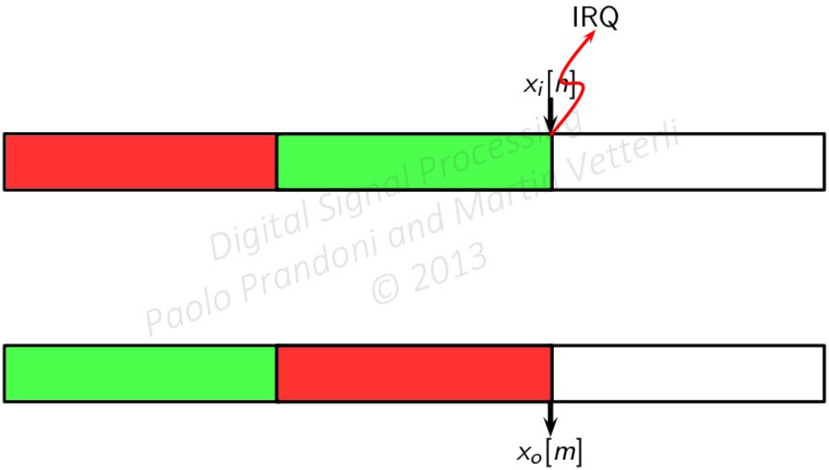
Real-time I/O processing with multiple buffering



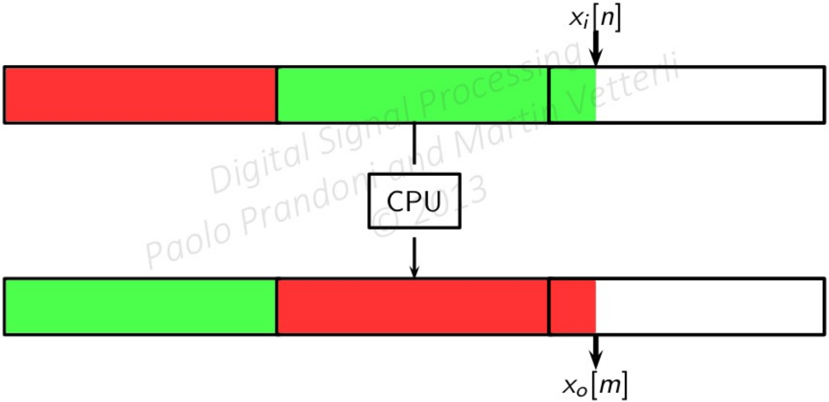
Real-time I/O processing with multiple buffering



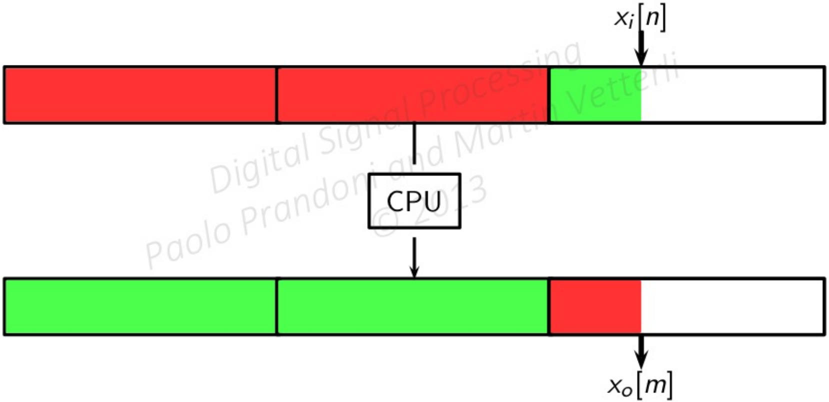
Real-time I/O processing with multiple buffering



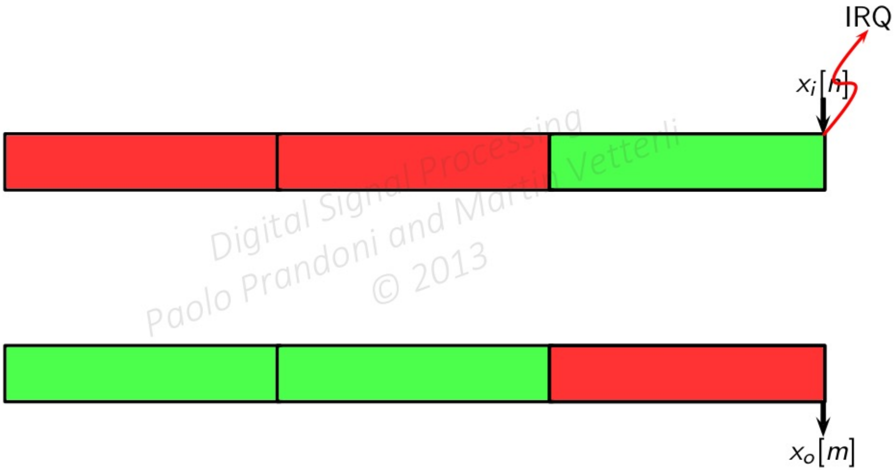
Real-time I/O processing with multiple buffering



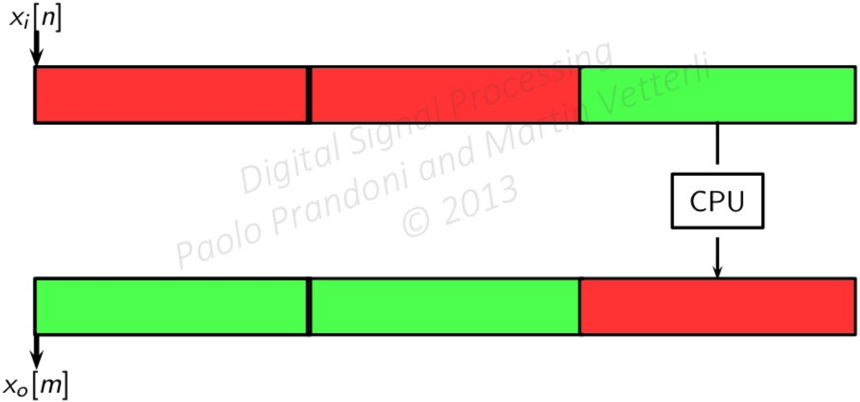
Real-time I/O processing with multiple buffering



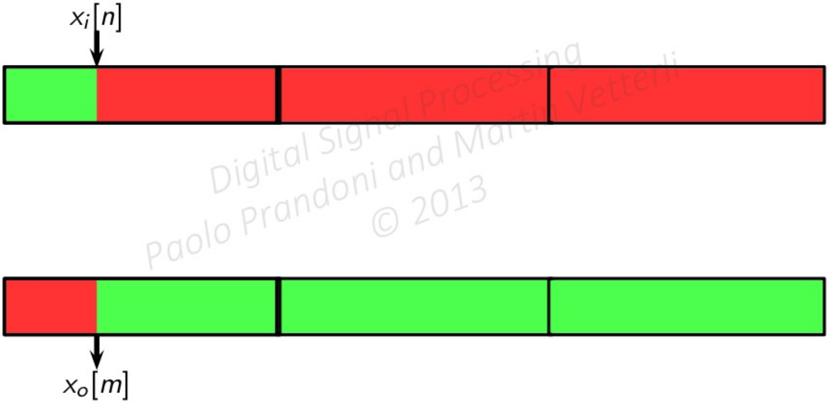
Real-time I/O processing with multiple buffering



Real-time I/O processing with multiple buffering



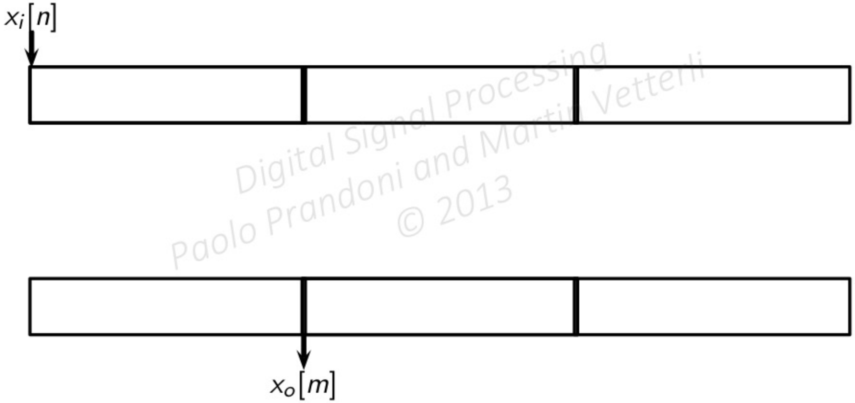
Real-time I/O processing with multiple buffering



Real-time I/O processing with multiple buffering

- total delay $d = T_s \times L$ seconds
- usually start output process first
- buffers can be collapsed

Less delay, more risk



Implementation

- low level:
  - study soundcard data sheet (each one is different)
  - write code to program soundcard via writes to IO ports
  - write an interrupt handler
  - write the code to handle the data
- high level:
  - choose a good API
  - write a callback function to handle the data

Callback prototype

```
int Callback( const void *input,
                        void *output,
              unsigned long samples,
                  ...);
```

Callback example

```
int Callback( const void *input,
                        void *output,
             unsigned long samples)
{
  float* pIn  = (float *)input;
  float* pOut = (float *)output;

  for (int n = 0; n < samples; n++)
    *pOut++ = Process(*pIn++);
}
```

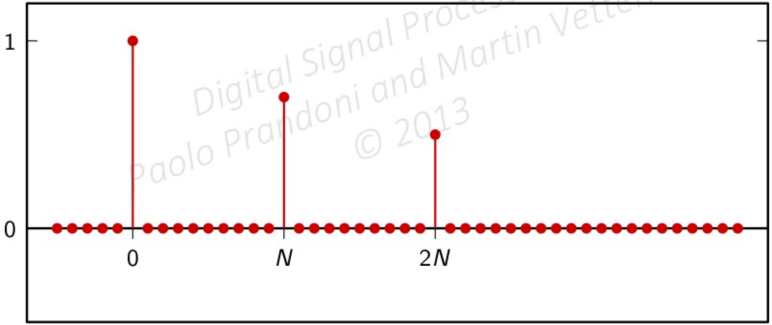
Processing gateway

```
// 10 sec @ 24KHz
enum {BUF_LEN = 0x10000};
enum {BUF_MASK = BUF_LEN -1};

float m_pY[BUF_LEN];
float m_pX[BUF_LEN];
int m_Ix;
int m_Iy;
```

```
float Process(float x)
{
  m_pX[m_Ix] = Sample;
  float y = Effect();
  m_pY[m_Iy] = y;
  m_Ix = (m_Ix + 1) & BUF_MASK;
  m_Iy = (m_Iy + 1) & BUF_MASK;

  return y;
}
```

Simple Echo

$$y[n] = \frac{a\,x[n] + b\,x[n-N] + c\,x[n-2N]}{a+b+c}$$

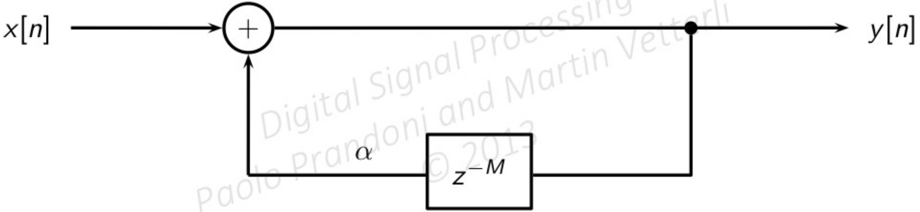


Simple Echo

```
float Echo()
{
  static float a = 1;
  static float b = 0.7f;
  static float c = 0.5f;
  static float norm = 1.0f / (a+b+c);
  static int N = (int)(0.3 * m_SR);

  return norm * (   a * m_pX[m_Ix]
                  + b * m_pX[(m_Ix + BUF_LEN - N)   & BUF_MASK]
                  + c * m_pX[(m_Ix + BUF_LEN - 2*N) & BUF_MASK]);
}
```

A better echo

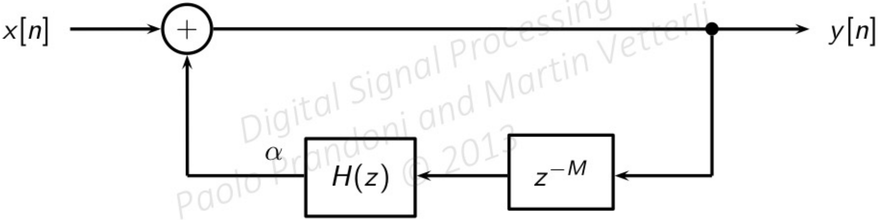
remember the KS algorithm? it's a sort of IIR echo



$$y[n] = \alpha\, y[n-M] + x[n]$$

A better echo

a natural echo has a lowpass characteristic

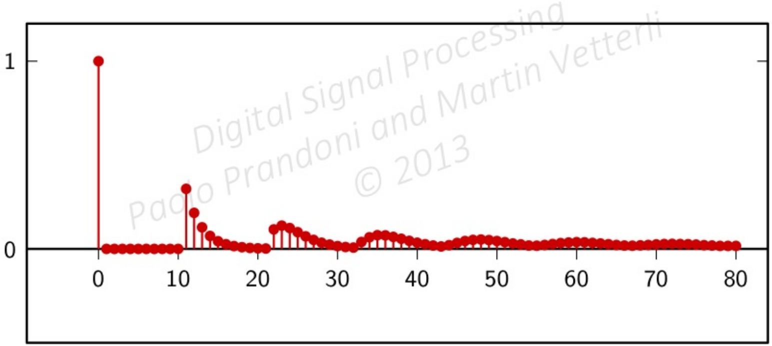


$$y[n] = \alpha(h * y)[n - M] + x[n]$$

A better echo

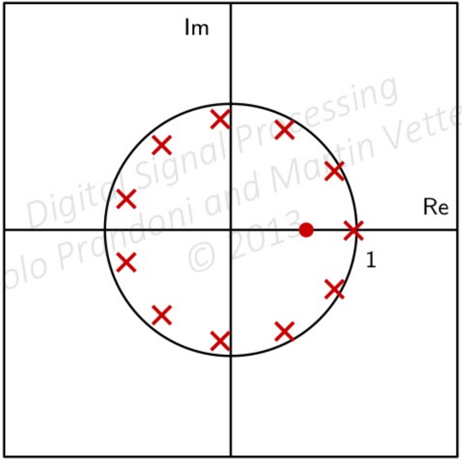
Choose for instance $H(z)$ = leaky integrator:

$$y[n] = x[n] - \lambda x[n-1] + \lambda y[n-1] - \alpha(1-\lambda)y[n-N]$$

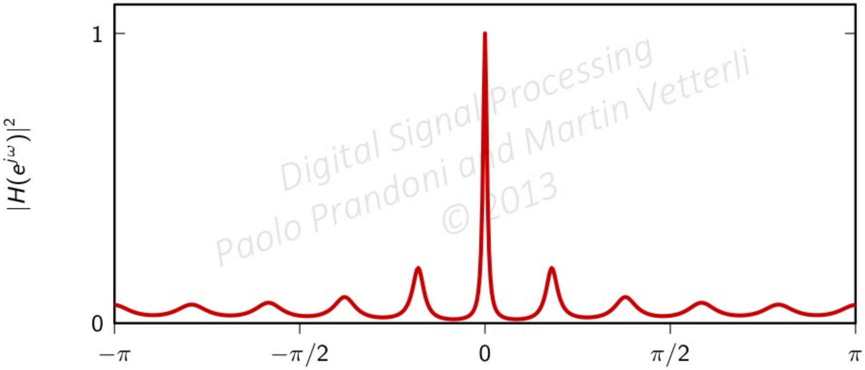


$N = 10, \lambda = 0.6, \alpha = 0.8$

A better echo



A better echo



“Natural” Echo

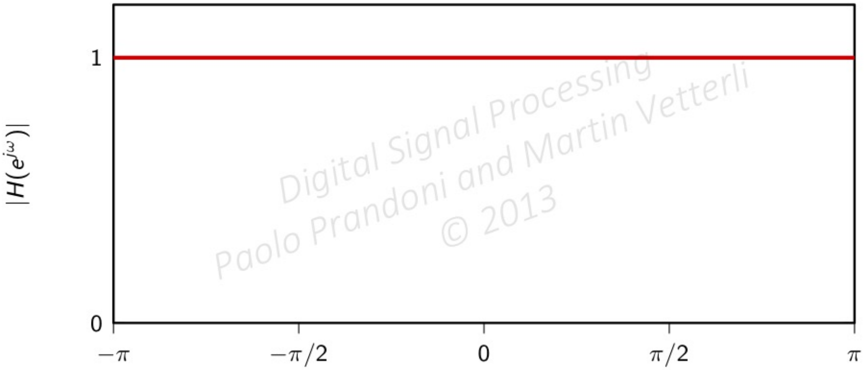
```
float NaturalEcho()
{
  static float a = 0.7;
  static float L = 0.6;
  static float norm = 1.0f / (1+a);
  static int N = (int)(0.3 * m_SR);

  return norm * (           m_pX[m_Ix]
                 - L *      m_pX[(m_Ix + BUF_LEN - 1) & BUF_MASK]
                 + L *      m_pY[(m_Iy + BUF_LEN - 1) & BUF_MASK]
                 + a*(1-L) * m_pY[(m_Iy + BUF_LEN - N) & BUF_MASK]);
}
```

Reverb

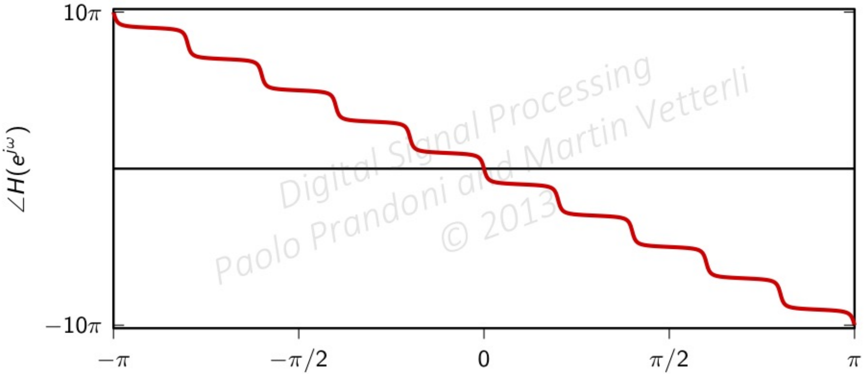
- reverb is given by the superposition of many many echos with different delays and magnitudes
- many ways to simulate, always rather costly
- a cheap alternative is to use an allpass filter

$$H(z) = \frac{-\alpha + z^{-N}}{1 - \alpha z^{-N}}$$

Reverb, magnitude response



Reverb, phase response



```
float Reverb()
{
  static float a = 0.8;
  static int N = (int)(0.006 * m_SR);

  return ( - a * m_pX[m_Ix]
           +     m_pX[(m_Ix + BUF_LEN - N) & BUF_MASK]
           + a * m_pY[(m_Iy + BUF_LEN - N) & BUF_MASK]);
}
```

Some non-LTI effects

- distortion: clip the signal

$$y[n] = \mathrm{trunc}(ax[n])/a$$

- tremolo: sinusoidal amplitude modulation

$$y[n] = (1 + \cos(\omega_0 n)/G)x[n]$$

- flanger: sinusoidal delay

$$y[n] = x[n] + x[n - \lfloor d(1 + \cos(\omega_0 n)) \rfloor]$$

```
float Fuzz()
{
  static float limit = 0.005;
  static float G = 5;

  float y = m_pX[m_Ix];
  if (y > limit)
    y = limit;
  if (y < -limit)
    y = -limit;
  return G*y;
}
```

Tremolo

```
float Tremolo()
{
  static double phi = 5 * 2*PI / m_SR;  // 5Hz LFO
  static double omega = 0;

  omega = omega + phi;
  return (1 + cos(omega)/4) * m_pX[m_Ix];
}
```

Flanger

```
float Flanger()
{
  static int N = (int)(0.002 * m_SR);   // max delay
  static double phi = 0.1 * 2*PI / m_SR;  // 0.1Hz LFO
  static double omega = 0;

  int d = (int)(N * (1 + cos(omega)));
  omega = omega + phi;
  return 0.5f + (m_pX[m_Ix] + m_pX[(m_Ix + BUF_LEN - d) & BUF_MASK]);
}
```

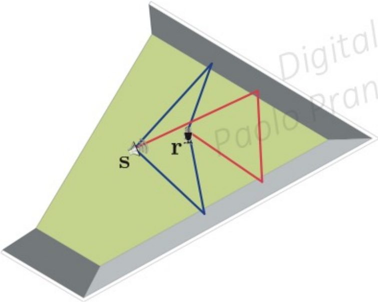
Digital Signal Processing

Module 5.12: Dereverberation and Echo Cancelation

Guest lecture by
Ivan Dokmanić

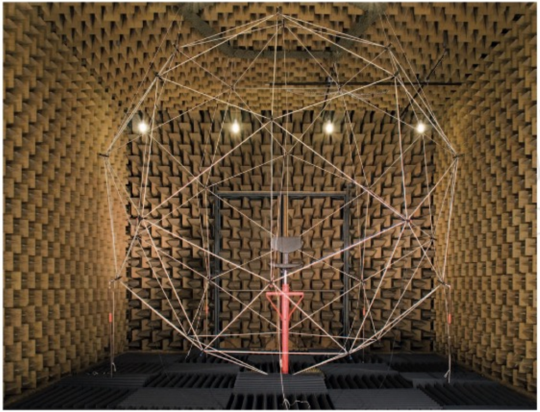
Sound Propagation

- Free space propagation, sound pressure $\sim \dfrac{1}{\text{distance}}$
- Every reflection attenuates the sound by $\alpha \in [0, 1]$



- Room is a linear filter, it acts by convolution
- $y_{\text{listener}}[n] = x_{\text{source}}[n] * h[n; \text{positions}]$
- We will supress the dependence on source and listener positions for simplicity

Anechoic Chamber



- Sound in this room Play

Small Classroom

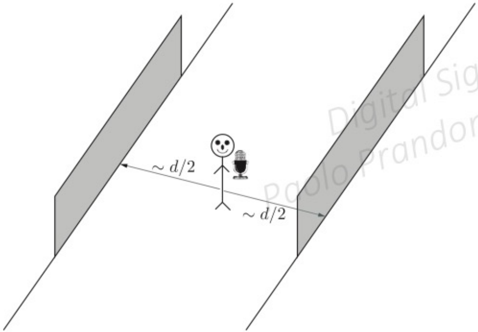
- Impulse response Play
- Sound in this room Play

Cathedral

- Impulse response Play
- Sound in this room Play

A Simple Room

- Consider (almost) colocated speaker and microphone between two walls at distance d



- The room impulse response is

$$h[n] = \sum_{k=0}^{\infty} \frac{\alpha^k}{B(k+\epsilon)T}\delta[n-kN]$$

where

$$N = \mathrm{round}(T \times F_s) = \mathrm{round}\left(\frac{d}{c}F_s\right)$$

and $c \approx 340\mathrm{m/s}$ is the speed of sound

Make it Easier

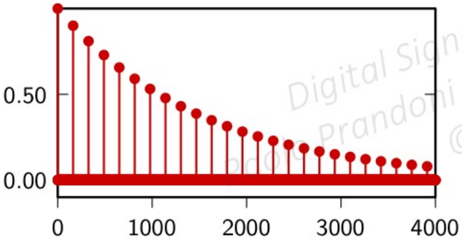
- $\frac{1}{k}$ term is difficult to handle in the z-domain, so we'll simplify
- Assume first that the dominant attenuation is due to reflections

$$h_a[n] = \sum_{k=0}^{\infty} \alpha^k \delta[n - kN]$$

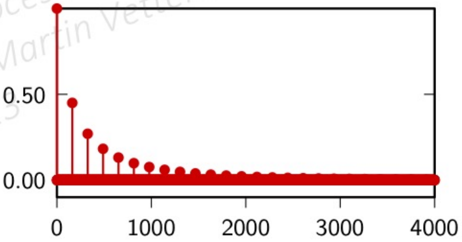
- This has a simple z-transform!

How do These Sound and Look?

- Simplified and realistic impulse responses for $F_s = 44100$ Hz, $d = 7$ m, $\alpha = 0.9$
- Original sound Play



Simplified room impulse response
Play



Realistic room impulse response
Play

Inverting the Room

- Reverberated sound is $y[n] = (h_a * x)[n]$
- We want to design a filter $h_i[n]$ such that $(h_i * y)[n] = x[n]$, that is

$$x = h_i * y = h_i * (h_a * x) = (h_i * h_a) * x.$$

  so that $(h_i * h_a)[n] = \delta[n]$, because $(\delta * x)[n] = x[n]$
- In the z-domain this becomes $H_i(z)H_a(z) = 1$ and we get that $H_i(z) = \dfrac{1}{H_a(z)}$

- Let's get rid of the room! The transfer function is computed as

$$
\begin{aligned}
H_a(z) &= \sum_{n=0}^{\infty} h_a[n] z^{-n} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \alpha^k \delta[n-kN] \right) z^{-n} \\
&= \sum_{n=0}^{\infty} \alpha^n z^{-nN} = \frac{1}{1-\alpha z^{-N}}
\end{aligned}
$$

Inverting the Room

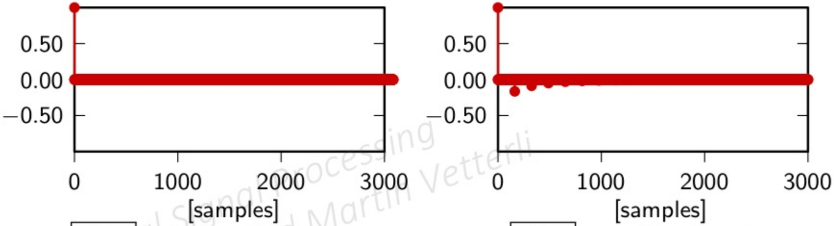
- The inverse is

$$H_i(z) = H_a(z)^{-1} = 1 - \alpha z^{-N} \leftrightarrow h_i[n] = \delta[n] - \alpha\delta[n - N]$$

- But this is a simple FIR filter!
- For motivated students: observation that exponentials are canceled by simple FIR filters is essential for the so-called Finite Rate of Innovation sampling

Equalized Room, Sounds and Plots

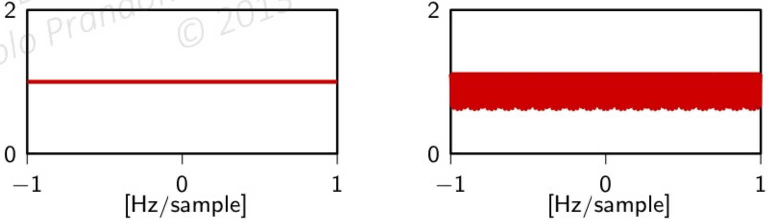
Equalized room impulse responses



Play equalized simple

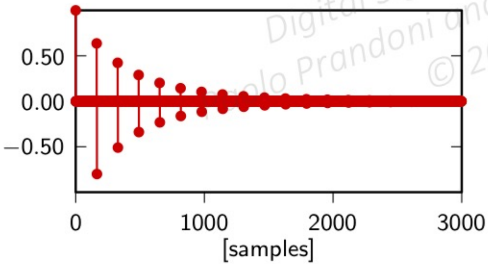
Play equalized real

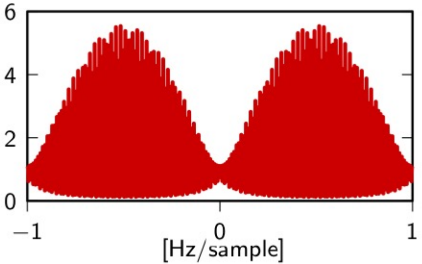
Magnitude of equalized room transfer functions



Equalized Room, Model Mismatch

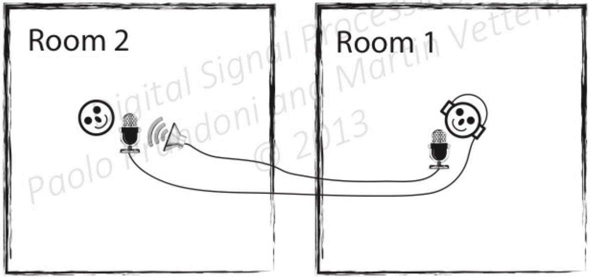
- Let's hear a different kind of model mismatch: 1% error in room size
- Play original sound
- Play "equalized", 1% error in d



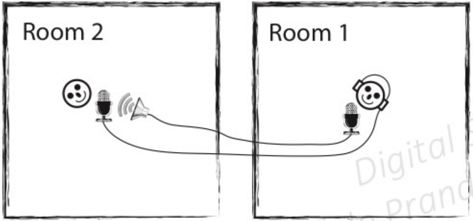


A Teleconference Call

- Consider now two people talking over a phone, as shown in the figure



A Teleconference Call

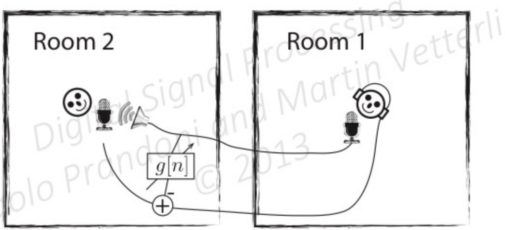


- Room 1 and Room 2 equal, $d = 7$ m, $\alpha = 0.8$, delay $= 0.3$ s

- Person in room one talks into the microphone
- Signal gets transmitted into another room with a delay and reproduced over a loudspeaker
- This gets convolved with the room 2, transmitted back, and reproduced over headphones, Play

Acoustic Echo Cancelation

- Must estimate and update the filter $g[n]$ that models Room 2, the loudspeaker and the microphone (not trivial!)



- Only subtracting the incoming sound (that is, setting $g[n] = \delta[n]$) is not enough, Play
- With ideal knowledge of $g[n]$, we get the expected result, only Room 1 remains Play

Is it Really This Simple?

- What if you are not halfway between the walls?