

MAST90105: Lab and Workshop 3

1 Lab

1. An urn contains 17 balls marked LOSE and 3 balls marked WIN. You and an opponent take turns selecting at random a single ball from the urn without replacement. The person who selects the third WIN ball wins the game. It does not matter who selected the first two WIN balls.

- a. If you draw first, find the probability that you win the game on your second draw.
- b. If you draw first, find the probability that your opponent wins the game on his second draw.
- c. If you draw first, the probability that you win can be found from

$$P(\text{You win if you draw first}) = \sum_{k=1}^9 \frac{\binom{3}{2} \binom{17}{2k-2}}{\binom{20}{2k}} \times \frac{1}{20-2k} \quad (\text{Why?})$$

Note: You could win on your second, third, fourth, ..., or tenth draw, not on your first.

- d. If you draw second, the probability that you win can be found from

$$P(\text{You win if you draw second}) = \sum_{k=1}^9 \frac{\binom{3}{2} \binom{17}{2k-1}}{\binom{20}{2k+1}} \times \frac{1}{19-2k}. \quad (\text{Why?})$$

- e. Based on your results in (c) and (d), would you prefer to draw first or second? Why?
2. Suppose in a lot of 100 fuses there are 20 defective ones. A sample of 5 fuses are randomly selected from the lot without replacement. Let X be the number of defective fuses found in the sample.
 - a. Find the probability $P(X = 0)$,
 - b. The cumulative probability $P(X \leq 3)$,
 - c. The mean or expectation of X , $E(X)$,
 - d. The second moment of X , $E(X^2)$,
 - e. The variance of X , $Var(X)$,
 - f. A probability bargraph for the pmf of X .
 3. An urn contains n balls numbered from 1 to n . A random sample of n balls is selected from the urn, one at a time. A match occurs if ball numbered i is selected on the i th draw.

- If the draws are done *with replacement*, it can be shown that

$$P(\text{at least one match}) = 1 - \left(1 - \frac{1}{n}\right)^n.$$

- If the draws are done *without replacement*, it can be shown that

$$P(\text{at least one match}) = 1 - \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right) = 1 - \sum_{k=0}^n \frac{(-1)^k}{k!}.$$

- a. For each value of n given in the following table, find $P(\text{at least one match})$ and write down the results at appropriate places of the table.
- b. We can also use R to simulate the processes of drawing n balls with or without replacement from a set of n balls numbered from 1 to n . We can then simulate the probability of at least one match using the relevant relative frequencies.
 - Create a function `match.f` in R-Studio by writing a script and executing it as follows.

```
match.f <- function(n, simsize, rep = TRUE) {  
  freq = 0  
  for (i in 1:simsize) {  
    sam = sample(1:n, size = n, replace = rep)  
    freq = freq + (sum(sam == 1:n) >= 1)  
  }  
  freq/simsize  
}
```

- Note that `(sum(sam==1:n)>=1)` in `match.f` is for checking whether or not there is at least 1 match in `sam`.
- Simulate the drawing process 1000 times (`simsize=1000`) for each given n and `rep` (`rep=TRUE` indicates the “with replacement” procedure is used.) Execute the following and write down the results at appropriate places in the table that follows.

```
match.f(n = 1, simsize = 1000, rep = TRUE)  
match.f(n = 3, simsize = 1000, rep = TRUE)  
match.f(n = 10, simsize = 1000, rep = TRUE)  
match.f(n = 15, simsize = 1000, rep = TRUE)  
match.f(n = 100, simsize = 1000, rep = TRUE)  
match.f(n = 10000, simsize = 1000, rep = TRUE)  
match.f(n = 1, simsize = 1000, rep = FALSE)  
match.f(n = 3, simsize = 1000, rep = FALSE)  
match.f(n = 10, simsize = 1000, rep = FALSE)  
match.f(n = 15, simsize = 1000, rep = FALSE)  
match.f(n = 100, simsize = 1000, rep = FALSE)  
match.f(n = 10000, simsize = 1000, rep = FALSE)
```

n	$P(\text{at least one match})$			
	with replacement		without replacement	
	by Calculation	by R simulation	by Calculation	by R simulation
1				
3				
10				
15				
100				
∞				

2 Workshop

1. Let $f(x) = (1/4)^{|x|}(1/2)^{1-|x|}$ for $x = -1, 0, 1$ and $f(x) = 0$ for other x values. Find the mean, variance and moment generating function for a random variable with this pmf.
2. Let a random experiment be the cast of a pair of unbiased 6-sided dice and let X equal the smaller of the outcomes if they are different and the common value if they are equal.
 - a. With reasonable assumptions, find the pmf of X .
 - b. Let Y equal the range of the two outcomes (i.e., the absolute value of the difference of the largest and smallest outcomes). Determine the pmf of Y .
3. In a lot of 100 light bulbs, there are 5 bad bulbs. An inspector inspects 10 bulbs selected at random. Let X be the number of bad bulbs in the sample.
 - a. What probability distribution does X have?
 - b. Calculate the probability that at least one defective bulb will be found in the sample.
 - c. Find the mean of X , i.e. $E(X)$.
 - d. Find the variance of X , i.e. $\text{Var}(X)$. (Note that variance for Hypergeometric has not yet been discussed in class — it will be discussed in Module 4. The formula is given in the table in the textbook.)
 - e. Find the second moment of X , i.e. $E(X^2)$.

4. Given $E(X + 4) = 10$ and $E[(X + 4)^2] = 116$, determine
- $\text{Var}(X + 4)$.
 - $\mu = E(X)$.
 - $\sigma^2 = \text{Var}(X)$.
5. A box contains 4 coloured balls: 2 black and 2 white. Balls are randomly drawn successively without replacement. If X is the number of draws until the last black ball is obtained, what are the possible values of X ? Find the pmf $f(x)$ for X . (*Hint:* Define events $B_i = \{\text{the } i\text{-th draw is a black ball}\}$ and $W_j = \{\text{the } j\text{-th draw is a white ball}\}$. Then find how each outcome of X is related to B_i and W_j .)

6. Let X be the number of accidents in a factory per week having pmf

$$f(x) = \frac{1}{(x+1)(x+2)}, \quad x = 0, 1, 2, \dots$$

- Find the conditional probability of $X \geq 4$, given that $X \geq 1$. (*Hint:* Write $f(x) = \frac{1}{x+1} - \frac{1}{x+2}$.)
 - Does $E(X)$ exist? If yes, find it; if not, why?
7. Suppose that a school has 20 classes: 16 with 25 students in each, three with 100 students in each, and one with 300 students for a total of 1000 students.
- What is the average class size?
 - Select a student randomly out of the 1000 students. Let the random variable X equal the size of the class to which this student belongs. Find the pmf of X .
 - Find $E(X)$, $\text{Var}(X)$ and the mgf of X .
8. A certain type of mint has a label weight of 20.4 grams. Suppose that the probability is 0.90 that a mint weighs more than 20.7 grams. Let X equal the number of mints that weigh more than 20.7 grams in a sample of 8 mints selected at random.
- How is X distributed if we assume independence?
 - Find $P(X = 8)$ and $P(X \leq 7)$.
9. Define the pmf and give the values of μ and σ^2 when the moment-generating function (mgf) of X is defined by
- $M(t) = \frac{1}{3} + \frac{2}{3}e^t$.
 - $M(t) = (0.25 + 0.75e^t)^{12}$.
10. If the moment-generating function of X is

$$M(t) = \frac{2}{5}e^t + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t},$$

Find the mean, variance, and pmf of X .