MAST90105 Lab and Workshop 6 Solutions

The Lab and Workshop this week cover problems arising from Module 3, Section 4 (Normal Distribution and Central Limit Theorem), Module 4 and Module 5, Section 1.

1 Lab

1. Suppose $X_1 \stackrel{d}{=} N(\mu = 3, \sigma^2 = 4)$, $X_2 \stackrel{d}{=} N(3, 4)$, and X_1 and X_2 are independent.

   a. Let $Y = 5X_1 - 2X_2 + 6$. Name the distribution of Y and give the values of the associated parameters.

   - *The MGF of* Y *is*

   $$\begin{aligned} M_Y(t) &= E(e^{Yt}) = E(e^{5X_1t}) \cdot E(e^{-2X_2t}) \cdot e^{6t} = M_{X_1}(5t) \cdot M_{X_2}(-2t) \cdot e^{6t} \\ &= e^{3 \cdot 5t + (2 \cdot 5t)^2/2} \cdot e^{3 \cdot (-2t) + (2 \cdot (-2t))^2/2} \cdot e^{6t} = e^{15t + 116t^2/2}, \end{aligned}$$

   and we conclude that $Y = 5X_1 - 2X_2 + 6 \sim N(15, 116)$.

   b. Use the command `rnorm(n,mean,sd)` in R to generate a sample of size $n = 100$, $Y_1, \ldots, Y_{100}$, from this distribution. Use the command `mean()` to compute the sample mean and standardized second moment:

   $$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i, \quad \bar{Y}_S^2 = \frac{1}{n}\sum_{i=1}^{n} \frac{(Y_i - 15)^2}{116}.$$

   Repeat this procedure 1000 times to get 1000 sample means and sample second moments, $\bar{Y}_1, \ldots, \bar{Y}_{1000}$, and $\bar{Y}_{S,1}^2, \ldots, \bar{Y}_{S,1000}^2$.

   - ```
     simf = function(){
     Y = rnorm(100,mean=15,sd=sqrt(116));
     Y2 = (Y-15)*(Y-15)/116;
     c(mean(Y), mean(Y2))
     }
     set.seed(1234);
     M = matrix(nrow=1000, ncol=2);
     for(i in 1:1000) M[i,]=simf()
     ```

   c. Estimate the mean and variance of $\bar{Y}$ and $\bar{Y}_S^2$ from the data $\bar{Y}_1, \ldots, \bar{Y}_{1000}$, and $\bar{Y}_{S,1}^2, \ldots, \bar{Y}_{S,1000}^2$. How close are these estimates to theoretical values?

   - *The estimates are:* `m1=apply(M,2,mean); var1=apply(M,2,var)`
   - *m1* $= (15.0315,\ 0.999)$, *var1* $= (1.171,\ 0.0187)$
   - *Note that* $Z = \frac{(Y_1-15)^2}{116} \sim \chi^2(1)$, $Var(Z) = E(Z^2) - [E(Z)]^2 = 3 - 1 = 2$, *and*

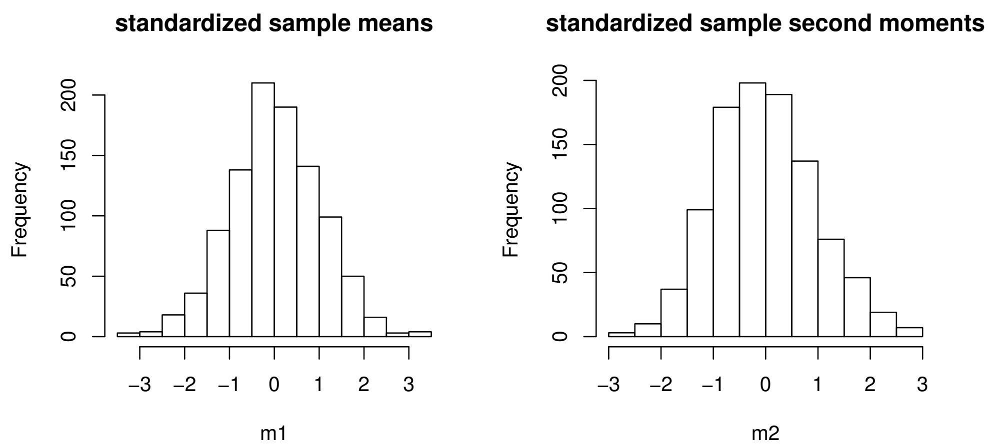
   $$\begin{aligned} E(\bar{Y}) &= E(Y_1) = 15, \quad E(\bar{Y}_S^2) = E\left(\frac{(Y_1 - 15)^2}{116}\right) = \frac{Var(Y_1)}{116} = 1, \\ Var(\bar{Y}) &= \frac{Var(Y_1)}{100} = 1.16, \quad Var(\bar{Y}_S^2) = \frac{Var\left(\frac{(Y_1-15)^2}{116}\right)}{100} = 0.02. \end{aligned}$$

   The exact values are therefore $(15, 1)$ *and* $(1.16, 0.02)$.
     ```

- *The mean estimates are quite accurate, the variance estimates are less accurate but still close to the true values*

d. Standardize the sample means and sample second moments, $\bar{Y}_1, \ldots, \bar{Y}_{1000}$, and $\bar{Y}^2_{S,1}, \ldots, \bar{Y}^2_{S,1000}$, so that they have zero mean and unit variance. Use the command `hist()` to construct the histograms of the standardized data. What distribution do the histograms resemble? What are the exact distributions?

```
m1 = (M[,1]-15)/sqrt(1.16); m2 = (M[,2] - 1)/sqrt(0.02);
par(mfrow=c(1,2)); #two plots/histograms in one row
hist(m1,main="standardized sample means");
hist(m2,main="standardized sample second moments");
```



- *The histograms resemble a bell shape typical for normal distribution. The exact distribution of m1 is $N(0,1)$, and m2 is a linearly transformed $\chi^2(100)$ distribution: $m2 = (Z-100)/\sqrt{2 \cdot 100} = (Z-100)/14.14$, where $Z \sim \chi^2(100)$.*

2. We would like to generate a random variable X which has the following cdf:

$$F(x) = \begin{cases} 0, & x < 0, \\ x^3, & 0 \le x < 1, \\ 1, & x \ge 1. \end{cases}$$

a. Use `runif(x)` function in R to generate random numbers from the uniform $U(0,1)$ distribution and write a function that generates random numbers from the cdf $F(x)$.

- *If $U \sim U(0,1)$, then $X = F^{-1}(U) = U^{1/3}$ has the cdf F.*

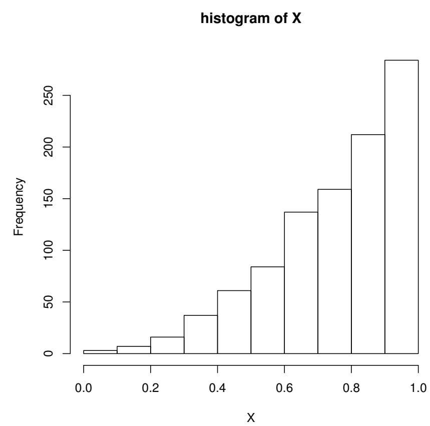
```
genf= function(n){
u = runif(n); x = u^(1/3); x
}
```

b. Use this function to generate a sample of size 1000 from this distribution and use this sample to estimate $\Pr(0.6 < X < 0.8)$. Compute the exact probability. How close is the estimated probability to the exact one?

- *The exact probability is* $pr1 = F(0.8) - F(0.6) = 0.296$
- ```
  set.seed(1234);
  X = genf(1000);
  pr2 = mean(X < 0.8 & X > 0.6);
  ```
- *We are lucky: for this simulation,* $pr2 = 0.296$*, which is the exact value! You can change seed to get a different outcome and check the variability of this estimate.*

c. Use the generated sample to construct the histogram of this distribution. Comment on the shape of the histogram.

- `hist(X, main = "histogram of X")`



- *The histogram resembles a quadratic function. Indeed, the density of* X *is* $f(x) = 3x^2$ *for* $0 < x < 1$.

2 Workshop

You can use R for numerical calculations as needed.

3. The serum zinc level X in micrograms per deciliter for males between ages 15 and 17 has a distribution which is approximately normal with $\mu = 90$ and $\sigma = 15$. Compute the conditional probability $P(X > 120|X > 105)$.

   - $P(X > 120|X > 105) = \frac{P(X>120)}{P(X>105)} = \frac{1-\Phi(2)}{1-\Phi(1)} = \frac{0.02275}{0.15865} = 0.1434.$

4. Let $Z_1, Z_2, \cdots, Z_7$ be a random sample from the standard normal distribution $N(0, 1)$. Let $W = Z_1^2 + Z_2^2 + \cdots + Z_7^2$. Name the distribution of W with associated parameter value. Justify your answer.

   - $W \stackrel{d}{=} \chi^2(7)$. *This is because a squared standard normal random variable has a* $\chi^2(1)$ *distribution, and the sum of independent chi-square random variables is chi-square with the degrees of freedom being the sum of the individual degrees of freedom of the random variables.*
  ```

5. If $X_1, X_2, \cdots, X_{16}$ is a random sample of size $n = 16$ from the normal distribution $N(50, 100)$.

    a. What is the distribution of $\frac{1}{100}\sum_{i=1}^{16}(X_i - 50)^2$?

    - $\chi^2(16)$

    b. What is the distribution of $\bar{X}$?

    - $N(50, \frac{100}{16} = 6.25)$.

6. Let $\bar{X}$ be the mean of a random sample of size 12 from the uniform distribution on the interval (0,1) (which has the mean 1/2 and variance 1/12). Approximate the probability $P(1/2 \le \bar{X} \le 2/3)$ using the central limit theorem.

    - *By CLT,* $\bar{X} \overset{d}{\approx} N(\frac{1}{2}, \frac{1}{144})$.
    - *So* $P(\frac{1}{2} \le \bar{X} \le \frac{2}{3}) \approx P(\frac{1/2-1/2}{1/12} \le Z \le \frac{2/3-1/2}{1/12})$
    $= P(0 \le Z \le 2) = \Phi(2) - \Phi(0) = 0.9772 - 0.4 = 0.4772.$

7. Let the distribution of Y be $Binomial(25, 1/2)$. Find the probability $P(10 \le Y \le 12)$ in two ways: using the binomial pmf formula, and using the normal approximation. Comment on any difference.

    - $P(10 \le Y \le 12) = \sum_{y=10}^{12}\binom{25}{y}0.5^{25} = 0.3852.$
    - *By CLT,* $Y \overset{d}{\approx} N(12.5, 6.25)$.
    So $P(10 \le Y \le 12) = P(9.5 \le Y \le 12.5) \approx P(\frac{9.5-12.5}{2.5} \le Z \le \frac{12.5-12.5}{2.5})$
    $= P(-1.2 \le Z \le 0) = \Phi(0) - \Phi(-1.2) = 0.5 - 0.1151 = 0.3849.$

8. The number X of flaws on a certain tape of length one yard follows a Poisson distribution with mean 0.3. We examine $n = 100$ such tapes and count the total number Y of flaws.

    a. Assuming independence, what is the distribution of Y?

    - $Y \overset{d}{=} \text{Poisson}(30)$. *By CLT* $Y \overset{d}{\approx} N(30, 30)$.

    b. Find the exact and approximate probabilities for $P(Y \le 25)$.

    - $P(Y \le 25) = P(Y \le 25.5) \approx P(Z \le \frac{25.5-30}{\sqrt{30}}) = P(Z \le -0.8216) = \Phi(-0.8216) = 0.2057.$

9. Let $Y = X_1 + X_2 + \cdots + X_{15}$ be the sum of a random sample of size 15 from the distribution whose pdf is $f(x) = (3/2)x^2,\ -1 < x < 1$. Approximate $P(-0.3 \le Y \le 1.5)$ using the central limit theorem.

    - $E(X_1) = \int_{-1}^{1}(3/2)x^3dx = 0$. $\text{Var}(X_1) = \int_{-1}^{1}(x-0)^2(3/2)x^2dx = \frac{3}{5}$.
    - *By CLT* $Y \overset{d}{\approx} N(15 \cdot 0, 15 \cdot \frac{3}{5}) = N(0, 9)$.

- *So* $P(-0.3 \leq Y \leq 1.5) \approx P(\frac{-0.3-0}{3} \leq Z \leq \frac{1.5-0}{3}) = P(-0.1 \leq Z \leq 0.5) = 0.2313$.

10. Let the joint pmf of X and Y be defined by

$$f(x,y) = \frac{x+y}{32}, \quad x = 1,2,\ y = 1,2,3,4.$$

a. Find the marginal pmf of X.

- $f_X(x) = \sum_{y=1}^{4} \frac{x+y}{32} = \frac{x+1}{32} + \frac{x+2}{32} + \frac{x+3}{32} + \frac{x+4}{32} = \frac{4x+10}{32}, \quad x = 1,2.$

b. Find the marginal pmf of Y.

- $f_Y(y) = \sum_{x=1}^{2} \frac{x+y}{32} = \frac{1+y}{32} + \frac{2+y}{32} = \frac{3+2y}{32}, \quad y = 1,2,3,4.$

c. Calculate $P(X > Y)$.

- $P(X > Y) = P(\{X = 2, Y = 1\}) = \frac{2+1}{32} = \frac{3}{32}$.

d. Calculate $P(Y = 2X)$.

- $P(Y = 2X) = P(\{X = 1, Y = 2\} \cup \{X = 2, Y = 4\}) = \frac{1+2}{32} + \frac{2+4}{32} = \frac{9}{32}$.

e. Calculate $P(X + Y = 3)$.

- $P(X + Y = 3) = P(\{X = 1, Y = 2\} \cup \{X = 2, Y = 1\}) = \frac{1+2}{32} + \frac{2+1}{32} = \frac{6}{32}$.

f. Calculate $P(X \leq 3 - Y)$.

- $P(X \leq 3 - Y) = P(X + Y \leq 3) = P(\{X = 1, Y = 1\} \cup \{X = 1, Y = 2\} \cup \{X = 2, Y = 1\}) = \frac{1+1}{32} + \frac{1+2}{32} + \frac{2+1}{32} = \frac{8}{32} = \frac{1}{4}$.

g. Are X and Y independent?

- *No, because* $f(x,y) \neq f_X(x)f_Y(y)$.

h. Find $E(X)$.

- $E(X) = \sum_{x=1}^{2} x\frac{4x+10}{32} = \frac{4+10}{32} + \frac{36}{32} = \frac{50}{32} = \frac{25}{16}$.

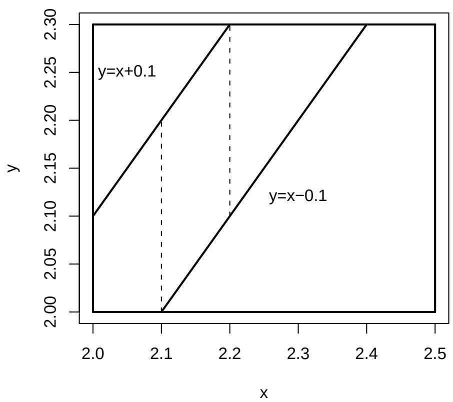
i. Find $E(X + Y)$.

- $E(X + Y) = \sum_{x=1}^{2}\sum_{y=1}^{4}(x+y)\frac{x+y}{32}$
$= \frac{(1+1)^2+(1+2)^2+(1+3)^2+(1+4)^2+(2+1)^2+(2+2)^2+(2+3)^2+(2+4)^2}{32} = \frac{140}{32}$.

11. Two construction companies make bids of X and Y (in \$100,000's) on a remodeling project. The joint pmf of X and Y is uniform on the space $2 < x < 2.5$, $2 < y < 2.3$. If X and Y are within 0.1 of each other, the companies will be asked to rebid; otherwise the lower bidder will be awarded the contract. What is the probability that they will be asked to rebid?

- $P(|X - Y| < 0.1) = P(X - 0.1 < Y < X + 0.1)$
$= \int_{2.0}^{2.1}\int_{2.0}^{x+0.1} \frac{1}{(2.5-2)(2.3-2)}dydx + \int_{2.1}^{2.2}\int_{x-0.1}^{x+0.1} \frac{1}{(2.5-2)(2.3-2)}dydx$
$+ \int_{2.2}^{2.4}\int_{x-0.1}^{2.3} \frac{1}{(2.5-2)(2.3-2)}dydx$
$= \frac{(x-1.9)^2}{0.3}\Big|_{2.0}^{2.1} + \frac{0.2\times0.1}{0.15} - \frac{(x-2.4)^2}{0.3}\Big|_{2.2}^{2.4} = \frac{11}{30}$.

- *In above calculation, note that $2 < X < 2.5$ and $2 < Y < 2.3$ are implicitly required.*
- *Because the joint pdf is uniform, this probability can also be obtained by calculating the proportion of the relevant area on the support of X and Y.*



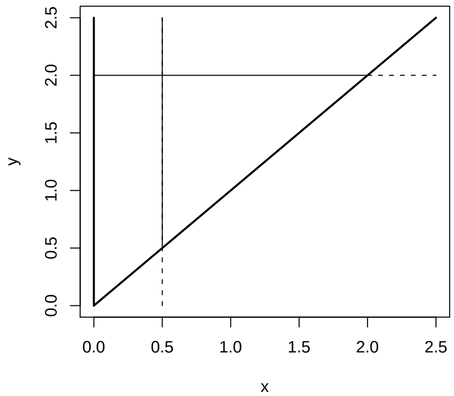
12. Let $f(x, y) = 2e^{-x-y}$, $0 \le x \le y < \infty$, be the joint pdf of X and Y.

   a. Find the marginal pdf $f_X(x)$ of X.
      - $f_X(x) = \int_x^\infty 2e^{-x-y}dy = 2e^{-2x}, \quad 0 \le x < \infty.$

   b. Find the marginal pdf $f_Y(y)$ of Y.
      - $f_Y(y) = \int_0^y 2e^{-x-y}dx = 2e^{-y}(1 - e^{-y}), \quad 0 \le y < \infty.$

   c. Compute $E(X)$ and $E(e^{-X-2Y})$.
      - $E(X) = \int_{-\infty}^\infty xf_X(x)dx = \int_0^\infty x \cdot 2e^{-2x}dx = \frac{1}{2}$. *Or* $E(X) = \int_{-\infty}^\infty \int_{-\infty}^\infty xf(x, y)dydx = \int_0^\infty \int_x^\infty x \cdot 2e^{-x-y}dydx = \frac{1}{2}$.
      - $E(e^{-X-2Y}) = \int_0^\infty \int_x^\infty e^{-x-2y} \cdot 2e^{-x-y}dydx$ $= \int_0^\infty \int_x^\infty 2e^{-2x-3y}dydx = \int_0^\infty \frac{2}{3}e^{-5x}dx = \frac{2}{15}$.

   d. Compute $P(X > \frac{1}{2})$.
      - $P(X > \frac{1}{2}) = \int_{1/2}^\infty f_X(x)dx = \int_{1/2}^\infty 2e^{-2x}dx = e^{-1}$.



e. Compute $P(X > \frac{1}{2}, Y > 2)$.

- $P(X > \frac{1}{2}, Y > 2) = \int_2^\infty \int_{1/2}^\infty f(x,y)dxdy = \int_2^\infty \int_{1/2}^y 2e^{-x-y}dxdy$
$= \int_2^\infty (2e^{-\frac{1}{2}-y} - 2e^{-2y})dy = [-2e^{-\frac{1}{2}-y} + e^{-2y}]_2^\infty = 2e^{-2.5} - e^{-4}$.

f. Compute $P(Y > 2|X > \frac{1}{2})$.

- $P(Y > 2|X > \frac{1}{2}) = \frac{P(Y>2, X>\frac{1}{2})}{P(X>\frac{1}{2})} = \frac{2e^{-2.5}-e^{-4}}{e^{-1}} = 2e^{-1.5} - e^{-3}$.

13. Let the joint pmf of X and Y be

$$f(x,y) = \frac{1}{4}, \quad (x,y) \in S = \{(0,0),(1,1),(1,-1),(2.0)\}.$$

a. Represent the joint pmf by a table.

	Y			
X	-1	*0*	*1*	$f_X(x)$
0		$\frac{1}{4}$		$\frac{1}{4}$
1	$\frac{1}{4}$		$\frac{1}{4}$	$\frac{1}{2}$
2		$\frac{1}{4}$		$\frac{1}{4}$
$f_Y(y)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	

b. Are X and Y independent?

- *No, because the space of X and Y is not rectangular.*

c. Calculate $\text{Cov}(X,Y)$ and ρ.

- *From the marginal pmf's in the table in (a),* $\mu_X = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1$.
$\mu_Y = (-1) \times \frac{1}{4} + 0 \times \frac{1}{2} + 1 \times \frac{1}{4} = 0$.

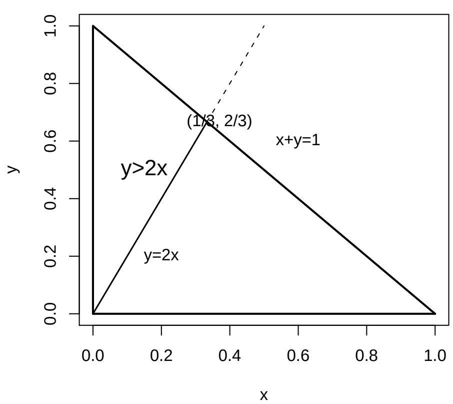
$\sigma_X^2 = (0-1)^2 \times \frac{1}{4} + (1-1)^2 \times \frac{1}{2} + (2-1)^2 \times \frac{1}{4} = \frac{1}{2}$.
$\sigma_Y^2 = (-1)^2 \times \frac{1}{4} + 0^2 \times \frac{1}{2} + 1^2 \times \frac{1}{4} = \frac{1}{2}$.

- $E(XY) = 0 \times 0 \times \frac{1}{4} + 1 \times (-1) \times \frac{1}{4} + 1 \times 1 \times \frac{1}{4} + 2 \times 0 \times \frac{1}{4} = 0$.
- *So* $\mathrm{Cov}(X,Y) = E(XY) - \mu_X\mu_Y = 0 - 1 \times 0 = 0$.
  And $\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y} = 0$.
- *This is another example of dependent variables having zero correlation coefficient.*

14. Consider continuous random variables X and Y which have the following joint pdf

$$f(x,y) = 24xy, \quad x > 0,\ y > 0,\ x + y < 1.$$

a. Sketch a graph of the support of X and Y.



b. Find the probability $P(Y > 2X)$.

- $P(Y > 2X) = \int_0^{1/3}\int_{2x}^{1-x} 24xydydx = \frac{7}{27}$.

c. Find the marginal pdf $f_1(x)$ of X.

- $f_1(x) = \int_0^{1-x} 24xydy = 12x(1-x)^2, \quad 0 < x < 1$.

d. Find the mean $E(X)$.

- $E(X) = \int_0^1 x12x(1-x)^2dx = \frac{2}{5}$.

e. Find the variance $\mathrm{Var}(X)$.

- $\mathrm{Var}(X) = \int_0^1 (x - \frac{2}{5})^2 12x(1-x)^2dx = \frac{1}{25}$.

f. Find the covariance $\mathrm{Cov}(X,Y)$.

- $\mathrm{Cov}(X,Y) = \int_0^1\int_0^{1-y} (x-\frac{2}{5})(y-\frac{2}{5})24xydxdy = -\frac{2}{75}$

g. Find the correlation coefficient ρ between X and Y.

- $\rho = \frac{-2/75}{\sqrt{1/25}\sqrt{1/25}} = -\frac{2}{3}$

h. Find the conditional pdf $h(y|x)$ of Y given $X = x$.

- $h(y|x) = \frac{24xy}{12x(1-x)^2} = \frac{2y}{(1-x)^2}, \quad 0 < y < 1-x.$

i. Find the condition probability $P(Y \leq \frac{1}{3}(1-X)|X = x)$.

- $P(Y \leq \frac{1}{3}(1-X)|X = x) = \int_0^{(1-x)/3} h(y|x)dy = \frac{1}{9}.$

j. Find the conditional mean $E(Y|X = x)$.

- $E(Y|x) = \int_{-\infty}^{\infty} y \cdot h(y|x)dy = \int_0^{1-x} y\frac{2y}{(1-x)^2}dy = \frac{2(1-x)}{3}.$

15. Show that $\mathrm{Cov}(aX + b, cX + d) = ac\mathrm{Var}(X)$, where X is a random variable and a, b, c are deterministic constants.

- $\mathrm{Cov}(aX + b, cX + d) = E[(aX + b - E[aX + b])(cX + d - E[cX + d])]$
$= E[(aX - E[aX])(cX - E[cX])] = E[ac(X - E[X])(X - E[X])]$
$= acE[(X - E[X])^2] = ac\mathrm{Var}(X).$

16. Let the pmf of X be $f_1(x) = \frac{1}{10}$, $x = 0, 1, 2, \cdots, 9$, and the conditional pmf of Y given $X = x$ be $h(y|x) = \frac{1}{10-x}$, $y = x, x+1, \cdots, 9$. Find

a. the joint pmf $f(x, y)$ of X and Y.

- $f(x, y) = f_1(x)h(y|x) = \frac{1}{10(10-x)}, \quad x = 0, 1, 2, \cdots, 9, y = x, x+1, \cdots, 9.$

b. The marginal pmf $f_2(y)$ of Y.

- $f_2(y) = \sum_{x=0}^{y} \frac{1}{10(10-x)}, \quad y = 0, 1, 2, \cdots, 9.$

c. $E(Y|x)$.

- $E(Y|x) = \sum_{y=x}^{9} y \cdot h(y|x) = \sum_{y=x}^{9} \frac{y}{10-x} = \frac{(x+9)(9-x+1)}{2(10-x)} = \frac{x+9}{2}.$

17. The marginal distribution of X is $U(0, 1)$. The conditional distribution of Y, given $X = x$, is $U(0, e^x)$.

a. Determine $h(y|x)$, the conditional pdf of Y, given $X = x$.

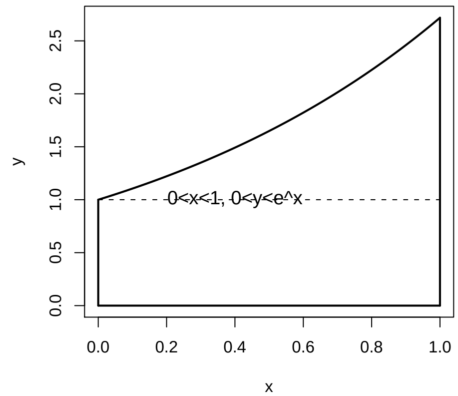
- $h(y|x) = \frac{1}{e^x} = e^{-x}, \quad 0 < y < e^x.$

b. Find $E(Y|x)$.

- $E(Y|x) = \frac{e^x}{2}, \quad 0 < x < 1.$

c. Find the joint pdf of X and Y. Sketch the region where $f(x, y) > 0$.

- $f(x, y) = f_1(x)h(y|x) = 1 \times e^{-x} = e^{-x}, \quad 0 < x < 1, 0 < y < e^x.$



d. Find $f_2(y)$, the marginal pdf of Y.

- $f_2(y) = \int_{-\infty}^{\infty} f(x,y)dx = \begin{cases} \int_0^1 e^{-x}dx = 1 - e^{-1}, & if\, 0 < y \le 1 \\ \int_{\ln(y)}^1 e^{-x}dx = y^{-1} - e^{-1}, & if\, 1 < y < e \end{cases}$

e. Find $g(x|y)$, the conditional pdf of X, given $Y = y$.

- $g(x|y) = \frac{f(x,y)}{f_2(y)} = \begin{cases} \frac{e^{-x}}{1-e^{-1}}, & 0 < x < 1, & if\, 0 < y \le 1 \\ \frac{e^{-x}}{y^{-1}-e^{-1}}, & \ln(y) < x < 1, & if\, 1 < y < e \end{cases}.$

18. An obstetrician does ultrasound examinations on her patients between their 16th and 25th weeks of pregnancy to check on the growth of the unborn child. Let X equal the widest diameter of the head and Y be the length of the femur, both in mm. Assume that X and Y have a bivariate normal distribution with $\mu_X = 60.6$, $\sigma_X = 11.2$, $\mu_Y = 46.8$, $\sigma_Y = 8.4$, $\rho = 0.94$

a. Find $P(40.5 < Y < 48.9)$

- *The marginal distribution of Y is $N(46.8, 8.4^2)$ so $P(40.5 < Y < 48.9) = P(Y \le 48.9) - P(Y \le 40.5) = 0.3721(4dp)$ can be found from the R commands*

```
pnorm(48.9, mean = 46.8, sd = 8.4) - pnorm(40.5, mean = 46.8,
    sd = 8.4)

## [1] 0.372079
```

b. Find $P(40.5 < Y < 48.9|X = 68.6)$

- *The conditional distribution of $Y|X = 68.6$ is $N(\mu_Y + \rho\sigma_Y z_X(68.6), (1-\rho^2)\sigma_Y^2)$ so the required probability is 0. 1084 (4dp) from the R commands*

```
(meanygivenx = 46.8 + 0.94 * 8.4 * (68.6 - 60.6)/11.2)

## [1] 52.44

(sigmaygivenx = (1 - 0.94^2)^0.5 * 8.4)

## [1] 2.865865

pnorm(40.5, mean = meanygivenx, sd = sigmaygivenx)

## [1] 1.548046e-05
```

19. Karl Pearson carried out a famous study on the resemblances between fathers and sons. He found that the distribution from his sample of 1078 pairs of fathers and sons had a mean and standard deviation of height for fathers of 69 and 2 (in.), whereas for sons it was 70 and 2 (in.). The correlation between fathers' and sons' heights was 0.5. Assuming that the distribution can of the pairs of heights can be modelled as bivariate normal with the sample means, sds and correlation, what is

    a. the expected height for the son of a father who is 74 in. tall,

    - *Let* (X, Y) *be the heights (in.) of the father and son, so that* (X, Y) *has a bivariate normal distribution with the parameters* $\mu_X = 69, \sigma_X = 2, \mu_Y = 70, \sigma_T = 2, \rho = 0.5$.
    - *Using the argument in Module 4,* $E(Y|X = 74) = \mu_Y + \rho\sigma_Y z_X(74) = 70 + 0.5 \times 2 \times \frac{74-69}{2} = 72.5$ *so the expected height is 72.5 in.*

    b. the chance that the son's height is more than 1 in. from the expected height in (a)?

    - *The conditional distribution of* $Y|X = 74$ *is* $N(\mu_Y + \rho\sigma_Y z_X(74), (1 - \rho^2)\sigma_Y^2)$ *so the required probability is (4dp) from the R commands*

```
(meanygivenx = 70 + 0.5 * 2 * (74 - 69)/2)

## [1] 72.5

(sigmaygivenx = (1 - 0.5^2)^0.5 * 2)

## [1] 1.732051

# Note that more than 1 inch different from 72.5 is
# < 71.5 or > 73.5.  Symmetry of the normal pdf
# about the mean implies the two probs have equal
# probability
2 * pnorm(71.5, mean = meanygivenx, sd = sigmaygivenx)

## [1] 0.5637029
```

c. the expected height of a father whose son is 72.5 in?

- *Using the argument in Module 4,* $E(X|Y = 72.5) = \mu_X + \rho\sigma_X z_Y(72.5) = 69 + 0.5 \times 2 \times \frac{72.5-70}{2} = 70.25$ *so the expected height is 70.25 in.*

d. the chance that both a father and son are above average height?

- *This one requires a subtle argument.*
- *From lectures the standardised* (X, Y) *satisfes:*

$$(z_X(X), z_Y(Y)) = (Z_1, \rho Z_1 + \sqrt{1-\rho^2}Z_2),$$

where (Z_1, Z_2) *have independent* $N(0, 1)$ *distributions.*

- *So the required probability is the same probability as*

$$P(z_X(X) \geq 0, z_Y(Y) \geq 0) = P(Z_1 \geq 0, \rho Z_1 + \sqrt{1-\rho^2}Z_2 \geq 0).$$

- *This last probability is the integral of the joint density of* (Z_1, Z_2) *over the set* $A = \{(z_1, z_2) : z_1 \geq 0, z_2 \geq \frac{-\rho z_1}{\sqrt{1-\rho^2}}\}$.
- *The set* A *is* $\frac{1}{3}$ *of the plane in the sense that if it is rotated through* 120° *three times it comes back to itself - draw a diagram and use some trigonometry to see this. Specifically,*

$$\mathbb{R}^2 = A \cup A_1 \cup A_2$$

where A_1 *is the rotation of* A *by* 120° *and* A_2 *is a rotation of* A *by* $240\circ$.

- *To see this note that* A *consists of the positive quadrant plus a wedge of the quadrant with non-negative* z_1 *and* z_2 *bounded by the line* $z_2 = -\frac{\rho}{\sqrt{1-\rho^2}}z_1 = -\frac{1}{\sqrt{3}}z_1$. *The angle that this line makes with the* z_1-*axis is* 30° *(since it's* $\tan = -\frac{1}{\sqrt{3}}$*) , so the angle of the boundary of* A *at the origin is* $90+30 = 120^\circ$.
- *Since the joint density of* (Z_1, Z_2) *is not affected by rotations, the probability is* $\frac{1}{3}$.
- *To see this in detail, if* ϕ *is the joint density of* (Z_1, Z_2) *then*

$$\begin{aligned} 1 &= \int_A \phi(x, y)\, dx\, dy + \int_{A_1} \phi(x, y)\, dx\, dy + \int_{A_2} \phi(x, y)\, dx\, dy \\ &= 3\int_A \phi(x, y)\, dxdy, \end{aligned} \tag{1}$$

and $3\int_A \phi(x, y)\, dxdy$ *is the required probability.*

- *The second equality in 1 follows because the rotational symmetry means that* $\phi(x, y) = \phi(x_1, y_1) = \phi(x_2, y_2)$ *where* (x_1, y_1) *is the rotation of* (x, y) *by* 120° *and* (x_2, y_2) *is the rotation of* (x, y) *by* 240° *so*

$$\int_A \phi(x, y)\, dx\, dy = \int_{A_1} \phi(x, y)\, dx\, dy = \int_{A_2} \phi(x, y)\, dx\, dy.$$

20. Scores in the two exams in a double weight course have means 65 and 60, standard deviations 18 and 20 and a correlation of 0.75. Assuming they can be modelled as bivariate normal, what is the expected score in the second exam for a student how is above average on the first exam?

    - *72 is the answer rounded to the nearest integer.*
    - *To see this the result of question 24 is used.*
    - *To see this, let Z_1, Z_2 be the random variables giving the scores in the first and second exams in statistical units.*
    - *From question 24 with $b = \infty, a = 0$, $E(Z_2|Z_1 > 0) = \frac{0.75[1-0]}{\sqrt{2\pi}(1/2}$ and the answer, z, is given in the following R code.*
    - *To get the expected second exam score, x for an above average student in the first exam note first that $Z_1 > 0$ is the same event as the first exam score being above average. So the required expected score, x is obtained by converting z from statistical units back to the original scale. This is done in the R code.*

```
(z = 0.75 * 2/sqrt(2 * pi))

## [1] 0.5984134

(x = 60 + 20 * z)

## [1] 71.96827
```

21. Suppose (X, Y) has a standard bivariate normal density with correlation ρ. For a, b, find $E(Y|a < X < b)$.

    - $E(Y|X = x) = \rho x$.
    - $f_X(x|a < X < b) = \frac{\phi(x)}{\Phi(b)-\Phi(a)}$ *for* $a < x < b$.
    - *So* $E(Y|a < X < b) = \int_a^b \rho x \frac{\phi(x)}{\Phi(b)-\Phi(a)}\, dx = \frac{\rho[e^{-a^2/2} - e^{-b^2/2}]}{\sqrt{2\pi}(\Phi(b)-\Phi(a)}$, *on changing variables in the integal from* x *to* $\frac{x^2}{2}$.

22. Suppose (X, Y) are random variables and a, b, c, d are constants. Show

$$Cov(aX + bY, cX + dY) = acVar(X) + bdVar(Y) + (bc + da)Cov(X, Y).$$

$$\begin{aligned}
Cov(aX+bY,cX+dY) &= E((aX+bY)(cX+dY) - E(aX+bY)E(cX+dY)\\
&= E((ad+bc)XY + acX^2 + bdY^2)\\
&- (aE(X)+bE(Y))(cE(X)+dE(Y))\\
&= (ad+bc)(E(XY) - E(X)E(Y))\\
&+ ac(E(X^2) - (E(X))^2) + bd(E(Y^2) - (E(Y))^2)\\
&= acVar(X) + bdVar(Y) + (bc+da)Cov(X,Y).
\end{aligned}$$

23. Suppose that $X \sim N(0,1)$ and that Z is an independent random variable which takes the values $1,-1$ each with probability $\frac{1}{2}$.

   a. Show that $X,-X$ have the same distribution.

$$\begin{aligned}
P(X \le x) &= \int_{-\infty}^{x} \phi(z)\,dz\\
&= \int_{-\infty}^{x} \phi(-z)\,dz\\
&= \int_{-x}^{-\infty} \phi(y)\,dy \quad (y=-z)\\
&= P(X \ge -x)\\
&= P(-X \le x),
\end{aligned}$$

   as required.

   b. What is the distribution of ZX?

   This is the standard normal because

$$\begin{aligned}
P(ZX \le x) &= P(X \le x, Z=1) + P(-X \le x, Z=-1)\\
&= P(X \le x)P(Z=1) + P(-X \le x)P(Z=-1)\\
&= P(X \le x)(P(Z=1) + P(Z=-1))\\
&= P(X \le x).
\end{aligned}$$

   c. Find $Cov(X,ZX), Corr(ZX,X)$.

   - *Since X, ZX both have variance 1, $Cov(X,ZX) = Corr(ZX,X)$.*
   - *Since X, ZX both mave mean 0 and Z is independent of X,*

$$Cov(X,ZX) = E(XZX) = E(X^2)E(Z) = 0,$$

     with the last step using $E(Z) = -1 \times 1/2 + 1 \times 1/2 = 0$.

   d. Are X and ZX independent? (Hint: consider $P(ZX \le -1 | X \le -1)$.)

They are not independent because the conditional probability would have to be $P(ZX \le -1)$ which is the same as $P(X \le -1)$. But

$$\begin{aligned} P(ZX \le -1 | X \le -1) &= \frac{P(X \le -1, Z = 1, X \le -1) + P(-X \le -1, Z = -1, X \le -1)}{P(X \le -1)} \\ &= \frac{P(X \le -1)}{2P(X \le -1)} \\ &= \frac{1}{2} = P(X \le 0) \ne P(X \le -1) \end{aligned}$$

e. Let $Y = ZX$. Is (X, Y) bivariate normal?

Although the marginal distributions are both normal, (X, Y) cannot be bivariate normal because they are uncorrelated but not independent.

24. Suppose U has a uniform pdf on [0,1]. Find the pdf of $-\ln(1-U)$.

- *For $x > 0$, $P(-\ln(1-U) \le x) = P(U \le 1-e^{-x})$ so $-\ln(1-U)$ has an exponential distribution with rate and scale $= 1$.*
- *Note that $p \to \ln(1-p)$ is the quantile function, so this is an example of the random number generation in Section 1 of Mondule 5.*

25. Suppose X has the Laplace pdf, f, given by $f(x) = \frac{1}{2}e^{-|x|}$ for all real x.

a. Find the cdf of X(hint: consider negative x first and compare with the exponential density).

- *For $x \le 0$, $P(X \le x) = \int_{-\infty}^{x} \frac{1}{2}e^{z}\,dz = \frac{1}{2}e^{-|x|}$.*
- *For $x > 0$, $P(X \le x) = \frac{1}{2} + \int_{0}^{x} \frac{1}{2}e^{-z}\,dz = 1 - \frac{1}{2}e^{-|x|}$.*
- *So the cdf, F_X, is given by*

$$F_X(x) = \begin{cases} \frac{1}{2}e^{-|x|} & x \le 0 \\ 1 - \frac{1}{2}e^{-|x|} & x > 0 \end{cases}.$$

b. Find the cdf of X^2.

- *For $x > 0$,*

$$\begin{aligned} P(X^2 \le x) &= P(-\sqrt{x} \le X \le \sqrt{x}) \\ &= F_X(\sqrt{x}) - F_X(-\sqrt{x}) \\ &= 1 - \frac{1}{2}e^{-\sqrt{x}} - \frac{1}{2}e^{-\sqrt{x}} \\ &= 1 - e^{-\sqrt{x}}. \end{aligned}$$

- *So the cdf, F_{X^2}, is given by*

$$F_{X^2}(x) = \begin{cases} 0 & x \le 0 \\ 1 - e^{-\sqrt{x}} & x > 0 \end{cases}.$$

c. Find the pdf of X^2.

Differentiating the pdf, f_{X^2}, is given by

$$f_{X^2}(x) = \begin{cases} 0 & x \leq 0 \\ \frac{e^{-\sqrt{x}}}{2\sqrt{x}} & x > 0 \end{cases}.$$

d. Find the cdf of X^4.

- *For $x > 0$,*

$$\begin{aligned} P(X^4 \leq x) &= P(-\sqrt[4]{x} \leq X \leq \sqrt[4]{x}) \\ &= F_X(\sqrt[4]{x}) - F_X(-\sqrt[4]{x}) \\ &= 1 - \frac{1}{2}e^{-\sqrt[4]{x}} - \frac{1}{2}e^{-\sqrt[4]{x}} \\ &= 1 - e^{-\sqrt[4]{x}}. \end{aligned}$$

- *So the cdf, F_{X^4}, is given by*

$$F_{X^4}(x) = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-\sqrt[4]{x}} & x > 0 \end{cases}.$$

e. Find the pdf of X^4.

Differentiating the pdf, f_{X^4}, is given by

$$f_{X^4}(x) = \begin{cases} 0 & x \leq 0 \\ \frac{e^{-\sqrt[4]{x}}}{4\sqrt[4]{x^3}} & x > 0 \end{cases}.$$