MAST90105 Lab and Workshop 8 Solutions

1 Lab

In this lab, you will learn how to use R to find a minimum of function and to generate random variables from a given CDF which does not have its inverse in closed form.

1. Function `nlm()` in R can be used to find a minimum of a nonlinear function.

   a. Define a function $f(x_1, x_2) = \exp(x_1^2 + x_2^2) - 2(\exp(x_1) + \exp(x_2))$ in R and use `nlm()` to find the minimum of this function.

```r
func1 = function(x) {
    x1 = x[1]
    x2 = x[2]
    out = exp(x1^2 + x2^2) - 2 * (exp(x1) + exp(x2))
    out
}
mle1 = nlm(f = func1, p = c(1, 1))
mle1
```

```
## $minimum
## [1] -5.397726
##
## $estimate
## [1] 0.7235762 0.7235762
##
## $gradient
## [1] 1.646683e-06 1.646683e-06
##
## $code
## [1] 1
##
## $iterations
## [1] 7
```

   b. As you have learned, to generate a random variable Z from a continuous distribution with given CDF $F_Z(z)$, one can follow these steps:

      i. Generate $U \sim U(0, 1)$
      ii. Compute $Z = F_Z^{-1}(U)$

   The quantile function $F_Z^{-1}(q)$ might not be available in closed form and can be computed numerically. For a strictly increasing CDF $F_Z(z)$ there exists a unique solution to the equation: $F_Z(z_q) = q$. It implies the function $G_Z(z) = (F_Z(z) - q)^2$ attains its minimum (which is zero) at $z = z_q$. The point z_q at which this minimum is attained can be found using `nlm()` function in R, and one can follow these steps to generate a random variable Z in this case:

i. Generate $U \sim U(0,1)$
ii. Compute $Z = \text{argmin}_z (F_Z(z) - U)^2$

Use this approach to generate a sample of size $N = 1000$ from a continuous random variable with the PDF:

$$f_Z(z) = \begin{cases} 0.5 + z^2 + z^5, & 0 < z < 1, \\ 0, & \text{otherwise.} \end{cases}$$

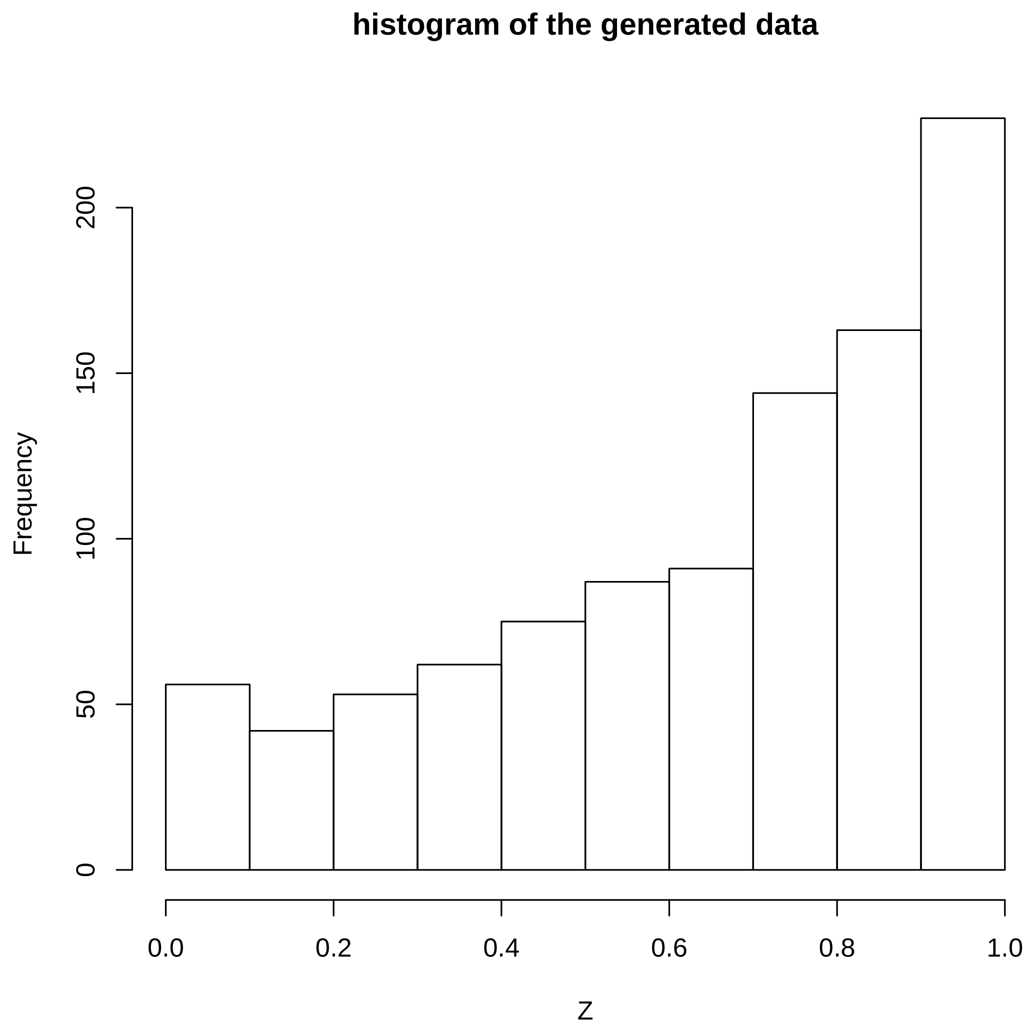
Draw a histogram of the generated data.

The CDF is

$$F_Z(z) = \int_0^z (0.5 + x^2 + x^4) dx = \frac{z}{2} + \frac{z^3}{3} + \frac{z^6}{6}.$$

The R code:

```r
func2 = function(z, q) {
    if (z < 0 || z > 1)
        return(1e+100)  #to force nlm() search a solution in (0,1) interval
    diff = (z/2 + z^3/3 + z^6/6 - q)^2
    diff
}
genf = function(N) {
    U = runif(N)
    Z = rep(0, N)
    for (i in 1:N) {
        Ui = U[i]
        mle2 = nlm(f = func2, p = 0.5, q = Ui)
        Z[i] = mle2$estimate
    }
    Z
}
set.seed(1234)
Z = genf(1000)
hist(Z, main = "histogram of the generated data")
```



2. A similar approach can be used to generate a random pair (X, Y) from a continuous bivariate distribution with given joint CDF $F_{X,Y}(x, y)$. Define

$$F_{Y|X}(y|x) = \Pr(Y \leq y|X = x) = \frac{\frac{\partial}{\partial x} F_{X,Y}(x, y)}{f_X(x)},$$

where $F_X(x)$ and $f_X(x)$ are the marginal CDF and PDF of X, respectively. Assuming $F_X(x)$ and $F_{Y|X}(y|x)$ are strictly increasing and continuously differentiable functions of x and y, respectively, one can follow these steps to generate a random pair (X, Y):

   a. Generate independent $U_X \sim U(0, 1)$ and $U_Y \sim U(0, 1)$
   b. Compute $X = F_X^{-1}(U_X)$
   c. Compute $Y = F_{Y|X}^{-1}(U_Y|X)$

The inverse CDFs F_X^{-1} and $F_{Y|X}^{-1}$ can be computed numerically using `nlm()` function if they are not available in closed form.

a. **Bonus:** Show that the joint CDF of (X, Y) generated using the algorithm above is $F_{X,Y}$.
b. Use this algorithm to generate a sample of size $N = 1000$ from a continuous bivariate random variable (X, Y) with the joint PDF:

$$f_{X,Y}(x, y) = \begin{cases} x + y, & 0 < x, y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Draw a scatter plot of the generated data.

We find the joint CDF first:

$$F_{X,Y}(x, y) = \int_0^x \int_0^y f_{X,Y}(x^*, y^*)dx^*dy^* = 0.5(xy^2 + x^2y), \quad 0 < x, y < 1.$$

It implies that $F_X(x) = F_{X,Y}(x, 1) = 0.5(x + x^2)$*,* $f_X(x) = F'_X(x) = 0.5 + x$*,* $0 < x < 1$*, and*

$$F_{Y|X}(y|x) = \frac{\frac{\partial}{\partial x} F_{X,Y}(x, y)}{f_X(x)} = \frac{0.5y^2 + xy}{0.5 + x}.$$

The inverse CDFs can be computed in closed form in this case by solving quadratic equations. Note that the range of random variables X *and* Y *is* $(0, 1)$*, and therefore a positive root should be selected:*

$$0.5(x + x^2) = q \quad \Rightarrow \quad F_X^{-1}(q) = -0.5 + \sqrt{0.25 + 2q},$$

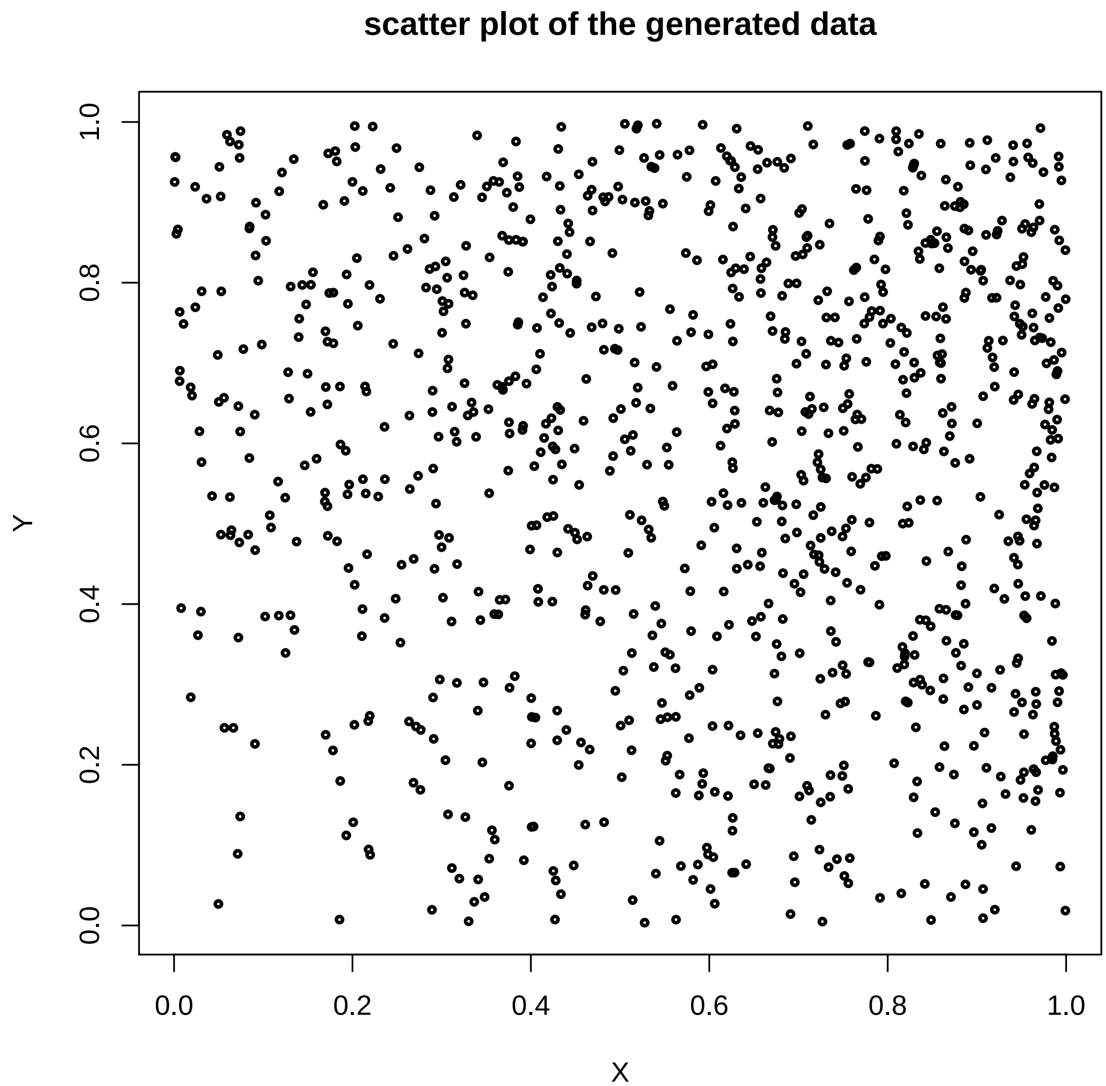
$$\frac{0.5y^2 + xy}{0.5 + x} = q \quad \Rightarrow \quad F_{Y|X}^{-1}(q|x) = -x + \sqrt{x^2 + q(1 + 2x)}.$$

R code:

```
genXY = function(N) {
    Ux = runif(N)
    Uy = runif(N)
    X = -0.5 + sqrt(0.25 + 2 * Ux)
    Y = -X + sqrt(X^2 + Uy * (1 + 2 * X))
    cbind(X, Y)
}

set.seed(1234)
Z = genXY(1000)

plot(Z, cex = 0.5, lwd = 2, main = "scatter plot of the generated data")
```



2 Workshop

1. a. A random sample $X_1, \ldots, X_n$ of size n is taken from a Poisson distribution with mean $\lambda > 0$.
   i. Show the maximum likelihood estimator of λ is $\hat{\lambda} = \bar{X}$.
   ii. Suppose with $n = 40$ we observe 5 zeros, 7 ones, 12 twos, 9 threes, 5 fours, 1 five, and 1 six. What is the maximum likelihood estimate of λ.
   - $$L(\lambda) = \prod_{i=1}^{n} \frac{\exp(-\lambda)\lambda^{x_i}}{x_i!} = e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i} \frac{1}{\prod_{i=1}^{n} x_i!}$$

so that

$$\log L(\lambda) = \ell(\lambda) = -n\lambda + \ln\lambda \sum_{i=1}^{n} x_i - \ln \prod_{i=1}^{n} x_i!$$

and

$$\ell'(\lambda) = -n + \frac{\sum_{i=1}^{n} x_i}{\lambda}$$

so that setting $\ell'(\lambda) = 0$ *yields* $\hat{\lambda} = \bar{X}$.

- *Since* $\ell''(\lambda) = -\frac{\sum_{i=1}^{n} x_i}{\lambda^2} < 0$, *so* $\bar{X}$ *is the mle.*
- $\bar{x} = (7 + 24 + 27 + 20 + 5 + 6)/40 = 2.225$.

b. Find the maximum likelihood estimator, $\hat{\theta}$, if $X_1, \cdots, X_n$ is a random sample from the following probability density function:

$$f(x;\theta) = (1/2)\exp(-|x-\theta|), -\infty < x < \infty, 0 < \theta < \infty$$

This involves minimizing $\sum_{i=1}^{n} |x_i - \theta|$, *which is difficult. Try* $n = 5$ *and a sample* $6.1, -1.1, 3.2, 0.7, 1.7$. *Then deduce the MLE.*

- $$L(\theta) = \left(\frac{1}{2}\right)^n \prod_{i=1}^{n} \exp(-|x_i - \theta|)$$

$$\log L(\theta) = \ell(\theta) = -n\log 2 - \sum_{i=1}^{n} |x_i - \theta|$$

$$\ell'(\theta) = \sum_{i=1}^{n} signum(x_i - \theta)$$

where $signum(x) = 1$ *if* $x > 0$, $signum(x) = 0$ *if* $x = 0$, $signum(x) < 0$ *if* $x < 0$. *Note that* $\ell(\theta)$ *is piecewise linear and nondifferentiable when* θ *equals any* x_i, *so* $\ell'(\theta)$ *is not defined at those points. If* n *is odd,* $\ell'(\theta) > 0$ *for* $\theta < x_{[n/2]+1}$ *and* $\ell'(\theta) < 0$ *for* $\theta > x_{[n/2]+1}$ $\hat{\theta}$ *must equal the middle value,* $x_{[n/2]+1}$. *If* n *is even, a similar argument shows that* $\hat{\theta}$ *can be taken anywhere between the middle two ordered values, for example at the average of the middle two values. So* $\hat{\theta}$ *can be taken to be the median in either case.*

2. Let $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, $\theta \in \Omega = \{\theta : 0 < \theta < \infty\}$ and let $X_1, \ldots, X_n$ denote a random sample from this distribution. Note that

$$\int_0^1 x\theta\, x^{\theta-1} dx = \frac{\theta}{\theta+1}$$

a. Sketch the p.d.f. of X for $\theta = 1/2$ and $\theta = 2$.

- *Sketch above.*

b. Show that $\hat{\theta} = -n/\ln\left(\prod_{i=1}^{n} X_i\right)$ is the maximum likelihood estimator of θ.

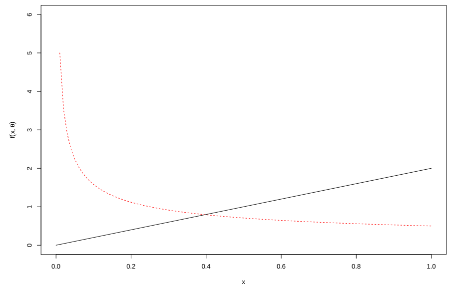


Figure 1: Sketch for Question 2(a)

- *The likelihood is*
$$L(\theta) = \theta^n \prod_{i=1}^{n} x_i^{\theta-1}.$$
Hence the log-likelihood is
$$\ell(\theta) = n \ln\theta + (\theta - 1)\sum_{i=1}^{n} \ln(x_i),$$
and the derivative is
$$\ell'(\theta) = \frac{n}{\theta} + \sum_{i=1}^{n} \ln(x_i),$$
so that setting $\ell'(\theta) = 0$ yields
$$\hat{\theta} = \frac{-n}{\sum_{i=1}^{n} \ln(X_i)} = \frac{-n}{\ln(\prod_{i=1}^{n} X_i)}$$
- *Differentiating again shows*
$$\ell''(\theta) = -\frac{n}{\theta^2} < 0$$
so $\hat{\theta}$ is the mle.

c. For each of the following three sets of observations from this distribution compute the maximum likelihood estimates and the methods of moments estimates.

X	Y	Z
0.0256	0.9960	0.4698
0.3051	0.3125	0.3675
0.0278	0.4374	0.5991
0.8971	0.7464	0.9513
0.0739	0.8278	0.6049
0.3191	0.9518	0.9917
0.7379	0.9924	0.1551
0.3671	0.7112	0.0710
0.9763	0.2228	0.2110
0.0102	0.8609	0.2154

($\sum_{i=1}^n \ln(x_i) = -18.2063$, $\sum_{i=1}^n \ln(y_i) = -4.5246$, $\sum_{i=1}^n \ln(z_i) = -10.42968$, $\sum_{i=1}^n x_i = 3.7401$, $\sum_{i=1}^n y_i = 7.0592$, $\sum_{i=1}^n z_i = 4.6368$.)

- *Substituting gives* $\hat{\theta}_X = 0.549$, $\hat{\theta}_Y = 2.210$, $\hat{\theta}_Z = 0.959$. *To find the method of moments estimators solve* $\bar{x} = \theta/(\theta + 1)$ *which yields* $\tilde{\theta} = \bar{x}/(1 - \bar{x})$ *and hence* $\tilde{\theta}_X = 0.598$, $\tilde{\theta}_Y = 2.400$ *and* $\tilde{\theta}_Z = 0.865$.

3. Let $X_1, \ldots, X_n$ be a random sample from the exponential distribution whose p.d.f. is $f(x;\theta) = (1/\theta)\exp(-x/\theta)$, $0 < x < \infty$, $0 < \theta < \infty$.

   a. Show that $\bar{X}$ is an unbiased estimator of θ.

   - *Recall that for a random sample* $X_1, \cdots, X_n$,

$$E(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = E(X_1)$$

   So $\bar{X}$ *is unbiased for* $E(X_1) = \theta$.

   b. Show that the variance of $\bar{X}$ is θ^2/n. What is a good estimate of θ if a random sample of size 5 yielded the values 3.5, 8.1, 0.9, 4.4 and 0.5?

   - *Recall that*

$$Var(\bar{X}) = \frac{1}{n^2}\sum_{i=1}^{n} \text{Var}(X_i) = \frac{\text{Var}(X_1)}{n}$$

and $\sigma^2 = Var(X_i) = \theta^2$. *Hence* $Var(\bar{X}) = \sigma^2/n = \theta^2/n$ *as required. A good estimator of* θ *is* $\hat{\theta} = \bar{x} = 3.48$. *This is a good estimator as it is unbiased and its variance tends to zero as* $n \to \infty$, *and we showed it is the maximum likelihood estimator in class.*

4. Let $X_1, \ldots, X_n$ be a random sample from a distribution having finite variance σ^2. Show that

$$S^2 = \sum_{i=1}^{n} \frac{\left(X_i - \bar{X}\right)^2}{n-1}$$

is an unbiased estimator of σ^2. HINT: Write

$$S^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right)$$

and compute $E(S^2)$.

- $$\begin{aligned} S^2 &= \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n-1} \\ &= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i^2 - 2X_i\bar{X} + \bar{X}^2\right) \\ &= \frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 - 2n\bar{X}^2 + n\bar{X}^2\right) \\ &= \frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right) \end{aligned}$$

as required. Now in general, $E(X^2) = \sigma^2 + \mu^2$, *so that* $E(X_i^2) = \sigma^2 + \mu^2$ *and* $E(\bar{X}^2) = \sigma^2/n + \mu^2$ *and hence*

$$\begin{aligned} E(S^2) &= \frac{1}{n-1}\left\{n(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)\right\} \\ &= \frac{(n-1)\sigma^2}{n-1} = \sigma^2 \end{aligned}$$

5. Let $X_1, \ldots, X_n$ be a random sample of size n from the distribution with p.d.f. $f(x;\theta) = (1/\theta)x^{(1-\theta)/\theta}$, $0 < x < 1$.

   a. Show the mean of X is $E(X) = 1/(1+\theta)$.

   - $$E(X) = \int_0^1 x \times \frac{1}{\theta}x^{(1-\theta)/\theta}\,dx = \left[\frac{\theta}{\theta(\theta+1)}x^{(\theta+1)/\theta}\right]_0^1 = \frac{1}{\theta+1}$$

b. Show the maximum likelihood estimator of θ is

$$\hat{\theta} = -\frac{1}{n}\sum_{i=1}^{n} \ln X_i$$

- *The log likelihood is*

$$\ell(\theta) = \ln L(\theta) = -n\log(\theta) + \sum_{i=1}^{n} \frac{1-\theta}{\theta}\ln(x_i),$$

so

$$\ell'(\theta) = \ln L(\theta) = \frac{-n}{\theta} - \frac{1}{\theta^2}\sum_{i=1}^{n}\ln(x_i).$$

Equating this to zero gives the required expression for $\hat{\theta}$*. Writing* $t = \sum_{i=1}^{n}\ln(x_i)$ *and differentiating gives*

$$\ell''(\hat{\theta}) = \frac{n}{\hat{\theta}^2} + \frac{2t}{\hat{\theta}^3} = -\frac{n^3}{t^2} < 0,$$

so $\hat{\theta}$ *is the MLE.*

c. Is the MLE unbiased? (You can do this using integration by parts combined with rules about expectation of the sample mean.)

- *Yes, the MLE is unbiased because*

$$E(\hat{\theta}) = E(-\ln(X_1)),$$

being the sample average of the logarithms of the sample values and

$$E(-ln(X_1)) = \int_0^1 -ln(x)f(x;\theta)\,dx = \left[-\ln(x)x^{1/\theta}\right]_0^1 + \int_0^1 \frac{1}{x}x^{1/\theta}\,dx = \theta.$$

d. Show the method of moments estimator of θ is

$$\widetilde{\theta} = \frac{1-\bar{X}}{\bar{X}}.$$

- *Equating* $\bar{X}$ *to* $\frac{1}{\theta+1}$ *is the same as* $\theta = \frac{1}{\bar{X}} - 1 = \frac{1-\bar{X}}{\bar{X}}$*, as required.*