MAST90085 Week 3 Lab

- Recall that if $Z_1, \dots, Z_n$ are independent $N(0,1)$ then

$$X = \sum_{k=1}^{n} Z_k^2 \sim \chi_n^2$$

is a chi square with n degrees of freedom.

- Recall that if M is an $p \times n$ matrix whose columns are independent and all have a $N_p(0, \Sigma)$ distribution, then

$$\mathcal{Y} = MM^T \sim W_p(\Sigma, n)\,. \tag{1}$$

Problems

1. In R, generate a sample $X_1, \dots, X_n$ of size $n = 200$ from a $N_2(\mu, \Sigma)$ distribution and draw the scatterplot of the pairs (X_{1i}, X_{2i}), for $i = 1, \dots, n$, using a red $*$ as the symbol to represent the data points. Do this for the following four different matrices Σ:

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \Sigma = \begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix}, \Sigma = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}, \Sigma = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

and plot the four scatterplots on a single graph window showing 4 scatterplots presented as the first two scatterplots on the first row and the last two scatterplots on the second row. Label the axes as $X1$ and $X2$ and add the title `Scatterplot 1`,..., `Scatterplot 4` to the first,..., fourth scatterplot. What do you notice about the cloud of points when you compare the four cases? Explain what is going on. Hint: install and use the package `MASS`.

2. Show that if $X \sim N_p(\mu, \Sigma)$ and Σ is invertible, then

$$Y = (X - \mu)^T \Sigma^{-1} (X - \mu) \sim \chi_p^2\,. \tag{2}$$

3. Let $p = 1$ and take $\Sigma = \sigma^2$, a number. Show that $\mathcal{Y}$ defined above is $W_1(\sigma^2, n)$, is also equal to σ^2 times a χ_n^2.

4. Show that if $\mathcal{Y}$ is defined at (1) and B is a $q \times p$ matrix then

$$B\mathcal{Y}B^T \sim W_q(B\Sigma B^T, n)\,.$$

5. Show thatif $\mathcal{Y}$ is defined at (1) and a is a $p \times 1$ vector such that $a^T \Sigma a \neq 0$, then

$$a^T \mathcal{Y} a / a^T \Sigma a \sim \chi_n^2\,.$$