Solution to 1:

We see that in the first graph, the data seems to create some sort of a horizontal ellipse, almost a disk. This is because the two components of the vector are independent (as their covariance is equal to zero) and so the data are not located around an oblique line, but rather around a horizontal line. The ellipse looks almost like a disk because the variances of the two components are not very different to each other. Recall that the contour lines of a bivariate normal distribution are ellipses, and in the case where the components are independent, the axes of the ellipse are parallel to the main axes, whence the shape of the scatterplot. In the other three examples, the two components of the normal vector are not independent as the contour lines of their distribution become more and more elongated ellipses. From case 2 to 4, the ellipses become increasingly more tight around an oblique line because the correlation between the two components increases. Recall that we can compute the correlation matrices by taking:

$$R = diag(\sigma_{11}, \sigma_{22})^{-1/2} \Sigma diag(\sigma_{11}, \sigma_{22})^{-1/2}.$$

We deduce that from the first to the fourth case, $\rho_{12} = 0, 0.3162278, 0.6324555, 0.8944272$

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library(MASS)
par(mfrow=c(2,2))
mu=c(1,2)

sigma=matrix(c(1,0,0,2),nrow=2,byrow=T)
N=200
X <- mvrnorm(N, mu = mu, Sigma = sigma )
plot(X,pch='*',col=2,xlab='X1',ylab='X2')
dim(X)
title("Scatterplot 1")
D=diag(diag(1/sqrt(sigma)))
R1=D%*%sigma%*%D

sigma=matrix(c(5,1,1,2),nrow=2,byrow=T)
N=200
X <- mvrnorm(N, mu = mu, Sigma = sigma )
dim(X)
plot(X,pch='*',col=2,xlab='X1',ylab='X2')
title("Scatterplot 2")
D=diag(diag(1/sqrt(sigma)))
R2=D%*%sigma%*%D

sigma=matrix(c(5,2,2,2),nrow=2,byrow=T)
N=200
X <- mvrnorm(N, mu = mu, Sigma = sigma )
dim(X)
plot(X,pch='*',col=2,xlab='X1',ylab='X2')
title("Scatterplot 3")
D=diag(diag(1/sqrt(sigma)))
R3=D%*%sigma%*%D

sigma=matrix(c(5,2,2,1),nrow=2,byrow=T)
N=200
X <- mvrnorm(N, mu = mu, Sigma = sigma )
dim(X)
plot(X,pch='*',col=2,xlab='X1',ylab='X2')
title("Scatterplot 4")
D=diag(diag(1/sqrt(sigma)))
R4=D%*%sigma%*%D
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2. $X \sim N_p(\mu, \Sigma)$, Σ invertible

$$Y = (X-\mu)^T \Sigma^{-1} (X-\mu)$$

$$= (X-\mu)^T \Sigma^{-1/2} \Sigma^{-1/2}(X-\mu)$$

$$= Z^T Z, \quad \text{where } Z = \Sigma^{-1/2}(X-\mu) \sim N_p(0, I_p)$$

$$= \sum_{j=1}^{p} z_j^2 \sim \chi^2_p$$

$Z = \begin{pmatrix} z_1 \\ \vdots \\ z_p \end{pmatrix}$ where each $z_j \sim N(0,1)$ and the z_j's are independent since $\Sigma_Z = I_p = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

(covs are all zero).

3. when $p = 1$, $Y = MM^T$ where $M \sim 1 \times n$ matrix whose columns are independent and have a $N(0, \sigma^2)$ distribution

Thus $M = (M_1, \ldots, M_n)$ where $M_1, \ldots, M_n \sim N(0, \sigma^2)$

since $p = 1$, $Y = MM^T \sim W_1(\Sigma, n)$ by definition of a Wishart. Moreover,

$$Y = M.M^T = (M_1, \ldots, M_n)\begin{pmatrix} M_1 \\ \vdots \\ M_n \end{pmatrix} = M_1^2 + \cdots + M_n^2$$

where the M_j's are independent $N(0, \sigma^2)$.

Thus $M_j = \sigma Z_j$ where $Z_j \sim N(0,1)$ and the Z_j's are independent.

$$\text{thus } Y = \sigma^2 . \underbrace{\sum_{j=1}^{n} Z_j^2}_{\chi^2_n} = \text{[illegible]}$$

4. By $B' = (BMM'B^T) = (BM)(BM)' = vv'$ where $BM = v$.

- Let $B = \begin{pmatrix} B_1^T \\ \vdots \\ B_q^T \end{pmatrix} = \begin{pmatrix} B_{11} & \cdots & B_{1p} \\ \vdots & & \\ B_{q1} & \cdots & B_{qp} \end{pmatrix}$ where $B_j = (B_{j1}, \dots, B_{jp})^T$

then let $M = (M_1, \dots, M_n) = \begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & & \\ M_{p1} & \cdots & M_{pn} \end{pmatrix}$

$$\begin{array}{ccc} \shortparallel & & \shortparallel \\ M_1 & & M_n \end{array}$$

- In this notation,

$$BM = \begin{pmatrix} B_1^T M_1, & \cdots, & B_1^T M_n \\ \vdots & & \\ B_q^T M_1, & \cdots, & B_q^T M_n \end{pmatrix}$$

where $B_j^T M_k$ is a scalar since $\underset{\text{dimensions}}{(1\times p)\cdot(p\times 1)} = (1\times 1)$

$$= (\underbrace{BM_1}_{}, \dots, \underbrace{BM_n}_{})$$, where $BM_j \sim N_q(0, B\Sigma B^T)$

$(q\times p)\cdot(p\times 1)$
$= (q\times 1)$ matrix
$= q$-vector

- and M_j and M_k indep $\Leftarrow$ $\mathrm{cov}(M_j, M_k) = 0_{p\times p}$

in the normal case, we have $\Leftrightarrow$

Thus BM_j and BM_k are indep since

$$\mathrm{cov}(BM_j, BM_k) = B\underbrace{\mathrm{cov}(M_j, M_k)}_{=0_{p\times p}}B^T = 0_{q\times q}$$

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- Thus $V = BM = (BM_1, \dots, BM_n)$ is a $q\times n$ matrix whose columns are independent $N_q(0, B\Sigma B^T)$

$\Rightarrow ByB^T = vV^T \sim W_q(B\Sigma B^T, n)$

5. $M = \Pi \Pi'$ where columns of Π are indep $N_p(0, \Sigma)$.

Let $a = p \times 1$ vector. Then from 4, we have

$$a^T M a \sim W_1(a^T \Sigma a, n) = a^T \Sigma a \cdot \chi^2_n,$$

from 3.

Thus, if $\overbrace{a^T \Sigma a}^{= \text{number (not matrix)}} \neq 0$, we can write

$$\frac{a^T M a}{a^T \Sigma a} \sim \chi^2_n$$