2025 / Spring Semester

Topics on Quantum Computing

Lecture Note – Quantum Phase Estimation

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Quantum Phase Estimation (QPE)

Estimating phase of an eigenvalue of a unitary operator

- The Quantum Phase Estimation (QPE) algorithm is one of the most important tools in quantum computing - maybe the most important.
  - https://arxiv.org/abs/quant-ph/9511026
- The QPE finds an eigenvalue of a unitary operator, given its eigenstate
  - We are given a unitary operator U and one of its eigenstate (eigenvector) $|\psi\rangle$.
  - a unitary operator is a square matrix
  - its eigenvalues has a magnitude of 1.
  - eigenvectors can be always represented in a normalized way.
  - The master equation then becomes $U|\psi\rangle = e^{i\phi}|\psi\rangle$, where ϕ is the phase of an eigenvalue.

Quantum Phase Estimation (QPE)

Estimating phase of an eigenvalue of a unitary operator

$$\boxed{U|\psi\rangle = e^{i\phi}|\psi\rangle}$$

- **The core of QPE algorithm** is to secure a quantum circuit that conducts the following transformation (note that $|0\rangle$ here = $|0\rangle^{\otimes n}$) using $|\boldsymbol{\psi}\rangle$

$$|0\rangle \rightarrow |\boldsymbol{\theta}\rangle \qquad (\phi = 2\boldsymbol{\pi\theta},\ 0 \leq \boldsymbol{\theta} < 1)$$

  - We could then get $|\boldsymbol{\theta}\rangle$ directly measuring the 2nd register: **Estimation register**
  - How can we express a scalar value $\boldsymbol{\theta}$ with a quantum state $|\boldsymbol{\theta}\rangle$? → Binary string

$$0.15625 = 1 \times 10^{-1} + 5 \times 10^{-2} + 6 \times 10^{-3} + 2 \times 10^{-4} + 5 \times 10^{-5}$$ Decimal fraction

$$0.00101 = 0 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3} + 0 \times 2^{-4} + 1 \times 2^{-5}$$ Binary fraction

  - (e.g.) 0.3125 (decimal) = 0.0101 (binary), and we can encode this with $|0101\rangle$.

Quantum Phase Estimation (QPE)

Estimating phase of an eigenvalue of a unitary operator

❑ **Mission: Finding the operator that makes $|0\rangle \rightarrow |\boldsymbol{\theta}\rangle$ with $|\boldsymbol{\psi}\rangle$**

$$\mathrm{QFT}|\theta\rangle = \frac{1}{\sqrt{2^n}} \sum_{k=0} e^{2\pi i\theta k}|k\rangle$$

- If we have the state in the right-hand side, $\boldsymbol{\theta}$ can be estimated with inverse QFT!
- Recall that U and $|\boldsymbol{\psi}\rangle$ are given (see the slide #2). Then, the factor $e^{2\boldsymbol{\pi} i \boldsymbol{\theta} k}$ can be obtained by applying U k times to $|\boldsymbol{\psi}\rangle$ as follows,

$$U^k|\psi\rangle = e^{2\pi i\theta k}|\psi\rangle$$

- Now, if we have an operator that makes $|\boldsymbol{\psi}\rangle \otimes |k\rangle$ → $(U^k|\boldsymbol{\psi}\rangle) \otimes |k\rangle$, then the right-hand side state in the first equation of this slide can be obtained (see more details in the next slide).

Quantum Phase Estimation (QPE)

Estimating phase of an eigenvalue of a unitary operator

❑ **Mission: Finding the operator that makes $|0\rangle \rightarrow |\theta\rangle$ with $|\psi\rangle$** (con't)

- With an operator that makes the transformation: $|\psi\rangle \otimes |k\rangle \rightarrow (U^k|\psi\rangle) \otimes |k\rangle$, we get the right-hand side state in the first equation of slide 4, as follows:

$$\frac{1}{\sqrt{2^n}} \sum_{k=0} |\psi\rangle|k\rangle \rightarrow \frac{1}{\sqrt{2^n}} \sum_{k=0} U^k|\psi\rangle|k\rangle = |\psi\rangle \frac{1}{\sqrt{2^n}} \sum_{k=0} e^{2\pi i\theta k}|k\rangle$$

Quantum Phase Estimation (QPE)

Estimating phase of an eigenvalue of a unitary operator

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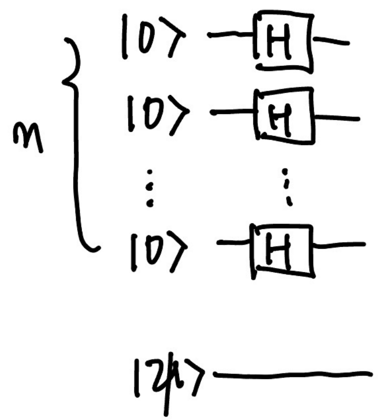
$$\boxed{\frac{1}{\sqrt{2^n}}\sum_{k=0}|\psi\rangle|k\rangle} \rightarrow \frac{1}{\sqrt{2^n}}\sum_{k=0} U^k|\psi\rangle|k\rangle = |\psi\rangle\frac{1}{\sqrt{2^n}}\sum_{k=0} e^{2\pi i\theta k}|k\rangle$$

- The left-most superposition state can be generated

$$\frac{1}{\sqrt{2^n}}\sum_{k=0}|\psi\rangle \otimes |k\rangle = |\psi\rangle \otimes \frac{1}{\sqrt{2^n}}(|0\rangle + |1\rangle + \cdots + |n-1\rangle)$$



$$\frac{1}{\sqrt{2^n}}\sum_{k=0}^{n-1}|k\rangle$$

Basis Rep.

Quantum Phase Estimation (QPE)

Estimating phase of an eigenvalue of a unitary operator

❑ **Mission: Finding the operator that makes $|0\rangle \rightarrow |\theta\rangle$ with $|\psi\rangle$** (con't)

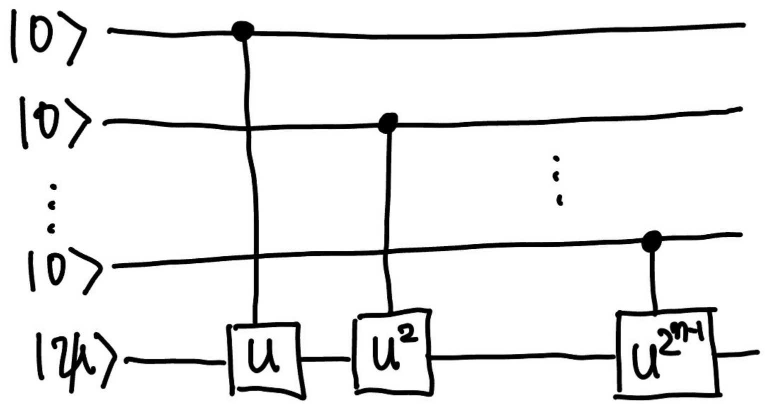
- With an operator that makes the transformation: $|\boldsymbol{\psi}\rangle \otimes |k\rangle \rightarrow (U^k|\boldsymbol{\psi}\rangle) \otimes |k\rangle$, we get the right-hand side state in the first equation of slide 4, as follows:

$$\frac{1}{\sqrt{2^n}}\sum_{k=0}|\psi\rangle|\boldsymbol{k}\rangle \rightarrow \boxed{\frac{1}{\sqrt{2^n}}\sum_{k=0}U^k|\psi\rangle|\boldsymbol{k}\rangle} = |\psi\rangle\frac{1}{\sqrt{2^n}}\sum_{k=0}e^{2\pi i\theta k}|\boldsymbol{k}\rangle$$

- Transformation with a controlled sequence

$$\sum_{k=0} U^k|\boldsymbol{\psi}\rangle\otimes|k\rangle = |\boldsymbol{\psi}\rangle\otimes|00\rangle + U|\boldsymbol{\psi}\rangle\otimes|01\rangle + U^2|\boldsymbol{\psi}\rangle\otimes|10\rangle + U^3|\boldsymbol{\psi}\rangle\otimes|11\rangle$$

$(n = 2)$

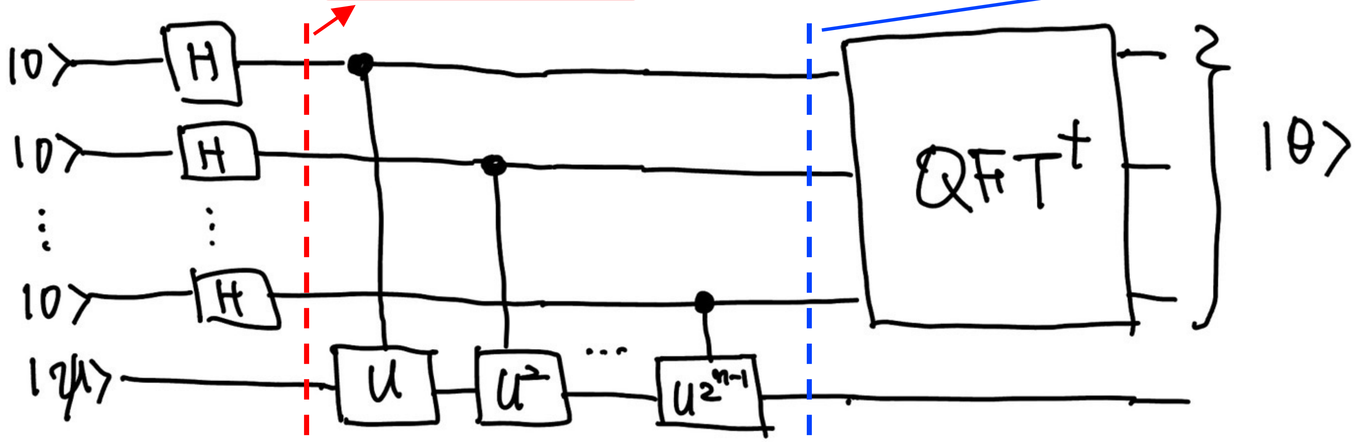


Quantum Phase Estimation (QPE)

Estimating phase of an eigenvalue of a unitary operator

- **Mission: Finding the operator that makes $|0\rangle \rightarrow |\boldsymbol{\theta}\rangle$ with $|\boldsymbol{\psi}\rangle$** (con't)
  - With an operator that makes the transformation: $|\boldsymbol{\psi}\rangle \otimes |k\rangle \rightarrow (U^k|\boldsymbol{\psi}\rangle) \otimes |k\rangle$, we get the right-hand side state in the first equation of slide 4, as follows:

$$\boxed{\frac{1}{\sqrt{2^n}} \sum_{k=0} |\psi\rangle |k\rangle} \rightarrow \frac{1}{\sqrt{2^n}} \sum_{k=0} U^k |\psi\rangle |k\rangle = \boxed{|\psi\rangle \frac{1}{\sqrt{2^n}} \sum_{k=0} e^{2\pi i\theta k} |k\rangle}$$



- Estimation qubits
  - Qubits that are used to drive $|\boldsymbol{\theta}\rangle$
  - In this case?

Quantum Phase Estimation (QPE)

Estimating phase of an eigenvalue of a unitary operator

❑ Summary: Process

1. Start with the state $|\boldsymbol{\psi}\rangle \otimes |0\rangle$

2. Apply a Hadamard gate to all estimation qubits to implement the following transformation

$$|\boldsymbol{\psi}\rangle \otimes |0\rangle \;\rightarrow\; \frac{1}{\sqrt{2^n}} \sum_{k=0} |\boldsymbol{\psi}\rangle \otimes |k\rangle$$

3. Apply a controlled sequence operation to get a state of $|\psi\rangle \dfrac{1}{\sqrt{2^n}} \sum_{k=0} e^{2\pi i\theta k} |\boldsymbol{k}\rangle$

4. Conduct the inverse QFT to get $|\boldsymbol{\theta}\rangle$, and we estimate $\boldsymbol{\theta}$

Quantum Phase Estimation (QPE)

Coding Practice

- Problem description
  - Let's say, $U = R_\phi(2\boldsymbol{\pi}/5)$, where $R_\phi(\phi) = e^{i\phi/2} R_z(\phi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$ $\boxed{U|\psi\rangle = e^{i\phi}|\psi\rangle}$
  - $|\boldsymbol{\psi}\rangle = e^{2\boldsymbol{\pi} i\boldsymbol{\theta}}|1\rangle$ with $\boldsymbol{\theta} = 1/5 = 0.2$
  - The mission is to find $|\boldsymbol{\theta}\rangle$ and $\boldsymbol{\theta}$ with QPE
  - Total circuit size = 5 qubits, so 4 qubits are needed to get $|\boldsymbol{\theta}\rangle$ ($|\boldsymbol{\psi}\rangle$: 1-qubit state)
- Useful functionalities
  - `qml.ControlledQubitUnitary`
  - `qml.probs`
  - `qml.counts` with setting `shots` in device plugin (`qml.device(...)`)

Quantum Phase Estimation (QPE)

Coding Practice

- ❑ Define U and prepare the emulator plugin with 2048 shots

```
import pennylane as qml
import numpy as np
import matplotlib.pyplot as plt

def myU(ind, w):
    return qml.PhaseShift(2 * np.pi * ind / 5, wires=w)

dev = qml.device("default.qubit", wires=5, shots=2048)
```

https://docs.pennylane.ai/en/stable/code/api/pennylane.PhaseShift.html

***class* PhaseShift(*phi, wires, id=None*)**

Bases: `pennylane.operation.Operation`

Arbitrary single qubit local phase shift

$$R_\phi(\phi) = e^{i\phi/2} R_z(\phi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$$

- ❑ We then create a single subroutine named as `circuit_QPE` (next slide)

Quantum Phase Estimation (QPE)

Coding Practice

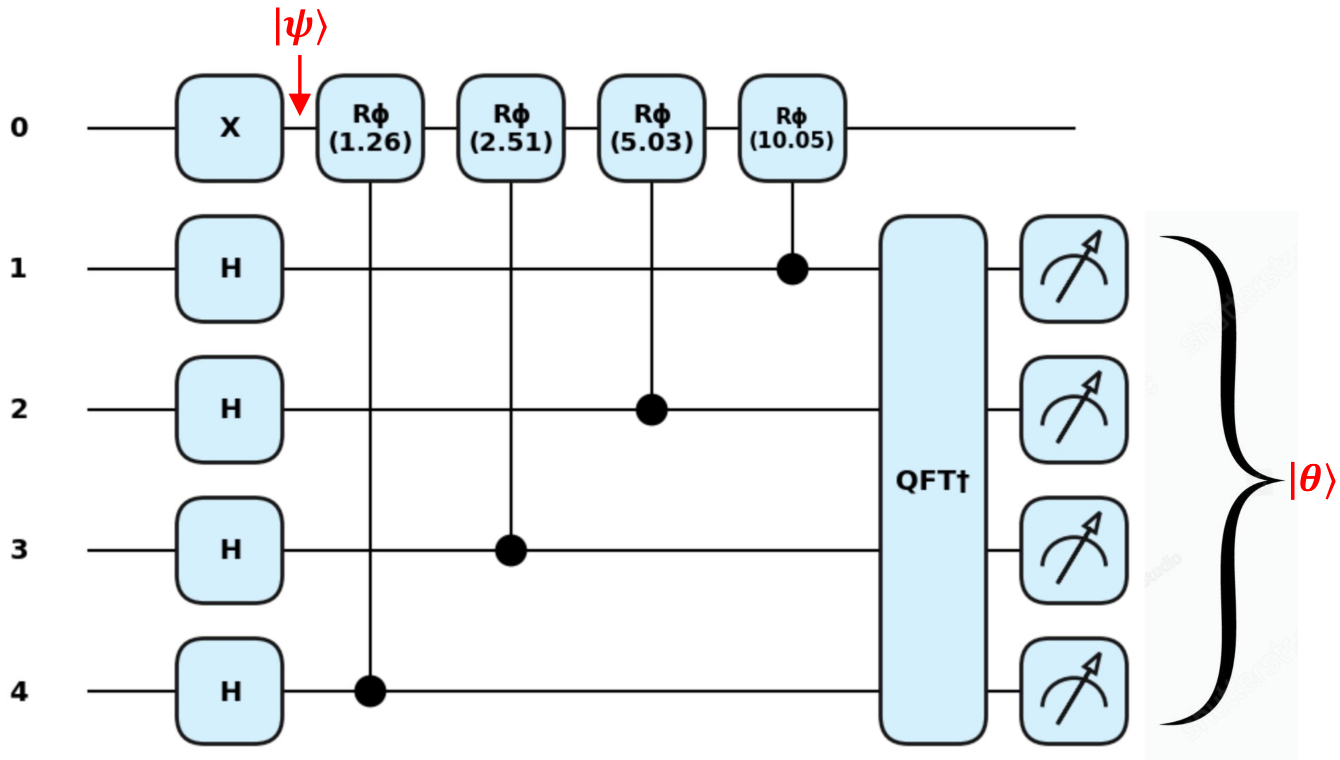
- The subroutine `circuit_QPE()`

```
@qml.qnode(dev)
def circuit_QPE():
    qml.PauliX(wires=0)                     # wire 0 is for eigenvector
                                              (should be flipped since |ψ⟩ = e^{2πiθ}|1⟩)

    for w in range(1,5):
        qml.Hadamard(wires=w)               # Hadamard operations

    for w in range(1,5):
        qml.ControlledQubitUnitary(myU(2**(w-1),0), control_wires=5-w, wires=0)
        # A sequence of controlled unitaries
    qml.adjoint(qml.QFT)(wires=range(1,5))  # Inverse QFT
    return qml.counts(wires=range(1,5))     # Get sampling counts for |θ⟩
```

Quantum Phase Estimation (QPE)

Coding Practice

- ❑ The subroutine `circuit_QPE()`



Quantum Phase Estimation (QPE)

Coding Practice

- ❑ Check the results

```
results = circuit_QPE()
results
```

```
...   {'0000': 9,
       '0001': 20,
       '0010': 53,
       '0011': 1782,
       '0100': 120,
       '0101': 16,
       '0110': 10,
       '0111': 8,
       '1000': 7,
       '1001': 4,
       '1010': 1,
       '1011': 5,
       '1100': 2,
       '1101': 2,
       '1110': 1,
       '1111': 8}
```

- ❑ 0011 is observed with a probability of 1782/2048
  - This corresponds to 0.0011 (binary fraction)
  - 0.0011 (bin) = $2^{-3} + 2^{-4}$ = 0.1875 (dec)
  - Recall the given $\boldsymbol{\theta}$ = 0.2

    → The inaccuracy ~ 0.0125 with 4 qubits (why 4 qubits, not 5)?

    → How can we reduce this inaccuracy?

Quantum Phase Estimation (QPE)

Misson

- Write a code that estimate the eigenvalue of U, where
  - U = 2-qubit controlled phase gate with phase ϕ = $\boldsymbol{\pi}$/3 ($\boldsymbol{\theta}$ = 1/6) (https://docs.pennylane.ai/en/stable/code/api/pennylane.ControlledPhaseShift.html)
  - 4-qubit estimation register (what should $|\boldsymbol{\psi}\rangle$ be to estimate $\boldsymbol{\theta}$)?
  - Use 2048 shots and what do you get for $\boldsymbol{\theta}$?
- Do phase estimation for the remaining three eigenvalues, which should be 0. What can you learn from the results?
- Increase the size of the estimation register to 6, 8, 10-qubit. How does $\boldsymbol{\theta}$ change & what can you learn from the results?