ECSE-626

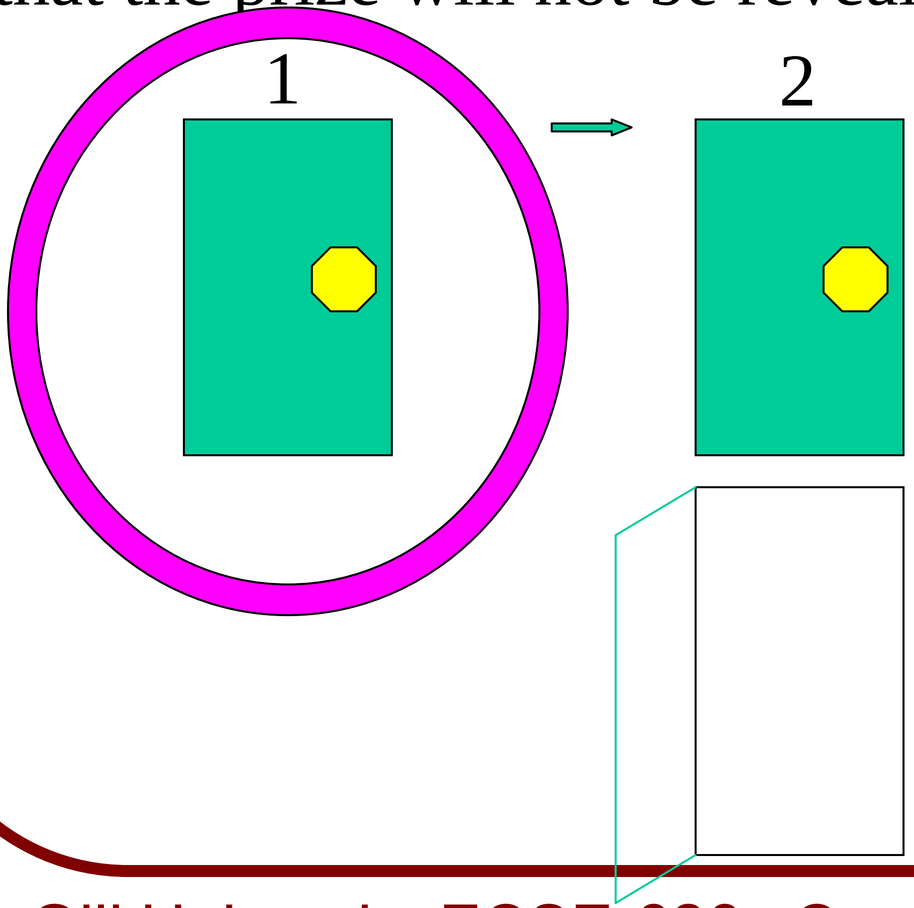
Statistical Computer Vision

Probabilistic Inference – Monty Hall

Game Show Problem

- Supposing you have a game show.
- Contestants told the following rules:
  - 3 doors, labelled 1, 2, 3.
  - 1 prize hidden behind one of them.
  - Select one door.
  - Initially, your chosen door will NOT be opened.
  - Instead, gameshow host will open one of other 2 doors. He will do so in a way as not to reveal the prize.

Example: You choose door 1, he will open either door 2, or door 3.
It is guaranteed that he will choose the one to open so that the prize will not be revealed.



- At this point, you are given fresh choice of door:
    - Stick with your first choice or
    - Switch to other closed door.
- All the doors will then be opened. You will receive whatever is behind your final choice of door.
- Suppose that you first choose door 1. Then the gameshow host opens door 3, revealing nothing behind the door.

Should you:

(a) stick with door 1.

(b) switch to door 2, or

Does it make a difference?

Formulate the problem as a Bayesian Inference Problem…

Solution

- Let H_i denote the hypothesis that the prize is behind door i.
- Let $D=n$ denote that the n^{th} door is opened. Bayesian inference requires us to estimate the posterior probabilities.
- We can then choose the action based on the resulting distribution. In this case, our action should maximize our chances of getting the prize.

Bayesian Inference

- We wish to compare the posterior probabilities of the hypotheses:

$$P(H_i \mid D=3) = \frac{P(D=3 \mid H_i)P(H_i)}{P(D=3)}$$

Prior Information

- Make the assumptions that the 3 hypotheses: H_1, H_2, H_3 are equiprobable *a priori*, i.e.:

$$P(H_1) = P(H_2) = P(H_3) = 1/3.$$

Likelihood Function

- We build distributions based on the general case.
- The data we receive after choosing door 1 is either $D=2$ or $D=3$.
- Let us examine each of these possibilities.

(1) The prize is behind door 1 – H_1.

(2) The prize is behind door 2 – H_2.

(3) The prize is behind door 3 – H_3.

Likelihood Function

(1) The prize is behind door 1 – H_1.

- The host has a free choice – he can choose door 2 or door 3.
- We assume that he can choose either at random.
- How do we express this possibility?

Likelihood Function

$$P(D = 2 \mid H_1) = 1/2$$

$$P(D = 3 \mid H_1) = 1/2$$

Likelihood Function

(2) The prize is behind door 2 – H_2.

- The choice of the host is forced to door 3.
- How do we express this possibility?

Likelihood Function

$$P(D = 2 \mid H_2) = 0$$

$$P(D = 3 \mid H_2) = 1$$

Likelihood Function

- Similarly for door 3…

(3) The prize is behind door 3 – H_3.

- The choice of the host is forced to door 2.
- How do we express this possibility?

$$P(D = 2 \mid H_3) = 1$$

$$P(D = 3 \mid H_3) = 0$$

Likelihood Function

This can be summarized as:

$$P(D=2\,|\,H_1)=1/2 \qquad P(D=3\,|\,H_1)=1/2$$

$$P(D=2\,|\,H_2)=0 \qquad P(D=3\,|\,H_2)=1$$

$$P(D=2\,|\,H_3)=1 \qquad P(D=3\,|\,H_3)=0$$

Normalization Constant

- What is the value of $P(D=3)$?
- In this case:

$$P(D=3)=\frac{1}{2}$$

Why?

Normalization Constant

- This can be deduced trivially as only 2 options are possible in this case: D=2 or D=3.
- We can also deduce this because the denominator is the normalizing constant for the posterior distribution.
- In this case, this implies:

Normalization Constant

$$\sum_{i=1}^{3} P(D=3 \mid H_i)P(H_i)$$

$$= P(D=3 \mid H_1)P(H_1) + P(D=3 \mid H_2)P(H_2) + P(D=3 \mid H_3)P(H_3)$$

$$= \frac{1}{2} * \frac{1}{3} + 1 * \frac{1}{3} + 0 = \frac{1}{2}$$

Posterior Probabilities

- Evaluating the posterior probabilities for the competing hypotheses gives:

$$P(H_1 \mid D = 3) = \frac{1}{3}$$

$$P(H_2 \mid D = 3) = \frac{2}{3}$$

$$P(H_3 \mid D = 3) = 0$$

So the contestant should *switch* to door 2 in order to have the biggest chance of getting the prize.

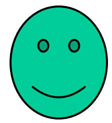
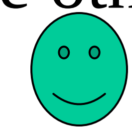
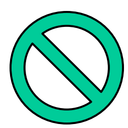
Intuition

- People find this solution surprising. The problem is referred to as the **Monty Hall paradox** – the solution is *counterintuitive*.
- They feel that the past can be ignored when assessing the probability. Thus, the first door choice and the host's reasoning about which door he opens are ignored.
- Because there are two doors to choose from, the two hypotheses should be equally likely once the 3^{rd} door is open – there is a 50-50 chance of choosing the right one.

Intuition

- However, here the choice was made based on reasoning and not randomly.
- To get a feel for this intuition, here is a frequentist way to see the solution. There are 3 possible scenarios, each with equal probability (1/3):

Intuition

1. The contestant picks empty door 1. The gamehost picks the other empty door. Switching will win the prize. 
2. The contestant picks empty door 2. The gamehost picks the other empty door. Switching will win the prize. 
3. The player picks the door with the prize. The gamehost picks either of the two empty doors. Switching will lose. 

Intuition

- In the first 2 scenarios, the player wins by switching. The third scenario is the only one where the player wins by staying.
- Since 2/3 of the scenarios win by switching and each scenario is equally likely, the odds of winning by switching are 2/3.

Intuition

- Another way to look at it: examine the experiment where the game is played with 1 million doors.
- The rules are now that you can choose 1 door, then the gameshow host opens 999,998 in such a way as to not reveal the prize. The door you chose as well as one other door remain closed.
- You may now stick or switch.

Intuition

- You are now confronted with door 1 and door 235,443 which have not been opened, door 1 being the door you chose initially. Where do you think the prize is?

New 3-door Game

- The main idea behind the inference task is unveiled by examining the case where an earthquake occurs and opens one of the doors before the gameshow host can get to it.
- Door 3 happens to be opened and it happens to not have the prize in it. (You still chose door 1initially).
- Should you stick with door 1 or switch to door 2, or does it make a difference?

New 3-door Game

- What is the difference between this game and the previous one? What has changed?
- The inference comes out differently, even though visually the scene looks the same.
- The nature of the data, and the probability of the data are both different now.

New 3-door Game

- In this case, the possible data outcomes are that any number of doors might have opened.
- These outcomes can be labelled:

  $\mathbf{d}$ = (0,0,0), (0,0,1),…,(1,1,1).
- Also, the prize can be visible should the earthquake open one or more doors.

New 3-door Game

- The probabilities of the outcomes is hard to determine explicitly as they depend on the reliability of door hinges, the properties of earthquakes, etc.
- Can determine the posterior probabilities without naming the exact probabilities $P(\boldsymbol{d}|H_i)$.
- All that we care about is the relative values of $P(D|H_i)$ for the value of D that actually occurred.

New 3-door Game

- Since $P(H_3|D) = 0$, $P(D|H_3) = 0$. We wish to compare $P(D|H_2)$ and $P(D|H_1)$.

  Difference here: In this case, they must be equal! Earthquakes are not sensitive to decisions the way gameshow hosts are!

  We don't know how sensitive door 3 hinge is to earthquakes, but we know that the door is just as likely to fall off its hinge whether or not there is a prize behind it!

New 3-door Game

$$P(H_1 \mid D) = \frac{1}{2}$$

$$P(H_2 \mid D) = \frac{1}{2}$$

$$P(H_3 \mid D) = 0$$

The two possible hypotheses are now equally likely.

Take-home Messages

- The key difference in the two cases is that the gameshow host opened a door based on the actions of the contestant and based on the prizes behind the door. The earthquake is not discriminating in how it opens doors.
- Things are not always immediately obvious.
- When solving inference problems, ALWAYS write out all the probabilities and evaluate them using laws of mathematics.

Take-home Messages

- Bayesian inference forces you to make all your assumptions explicit. It relies on formal mathematics. Don’t need to do guesswork.
- Inference is always performed based on the data that you received. Nothing is ignored.