ECSE-626
Statistical Computer Vision

Ill-posed Problems and Regularization

The Vision Problem

The ***Vision Problem*** is that of inference about the ***state*** of the world from a set of samples or data measurements.

Traditionally, these are samples of the flux of photons present in the environment.

The Vision Problem

In biological vision, samples of the photon flux are acquired through the eye onto the retina.

In artificial vision, these measurements are generated by sensors such as cameras.

Samples of the world can also be acquired from laser rangefinders, medical imaging devices, etc.

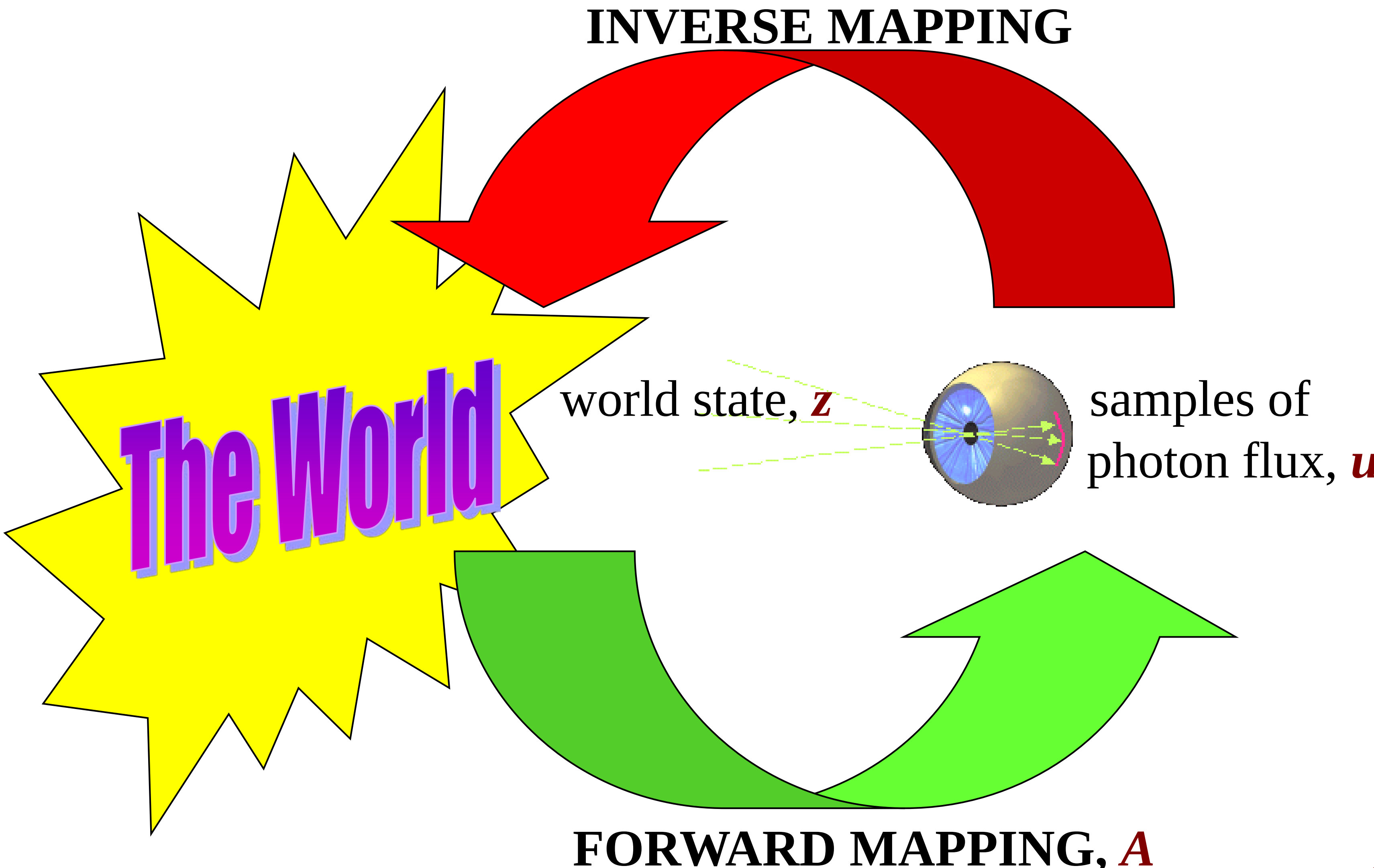
Vision as an Inverse Problem

The ***Vision Problem*** is part of a class of mathematical problems known as ***Inverse Problems***.

Inverse problems typically have a known forward mapping, $\boldsymbol{A}$, as well as measured data, $\boldsymbol{u}$, provided by the forward mapping:

$$\boldsymbol{u} = \boldsymbol{A}\boldsymbol{z}$$

The inverse problem is to somehow determine the value of $\boldsymbol{z}$ given $\boldsymbol{A}$ and $\boldsymbol{u}$.


INVERSE MAPPING
The World
world state, z
samples of
photon flux, u
FORWARD MAPPING, A

There are three main aspects to solving the vision problem:

- **Defining the "state" of the world**
- **Specifying the forward mapping**
- **Solving for the inverse mapping**

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“State” of the World

- Defining the "state" of the world is essentially a philosophical problem.
- In practice, intelligent systems use abstract descriptions of the world, in terms of quantities such as objects and surfaces.

“State” of the World

- The inverse computer vision problem would typically be:

  Given a 2D image of a scene as acquired by a camera, infer the underlying 3D objects that generated the image.

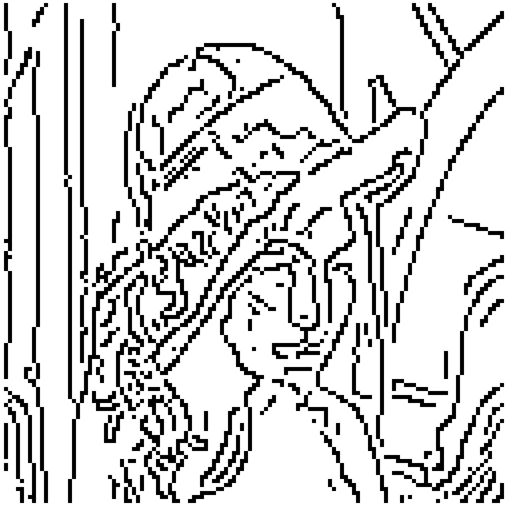
- ***In general, there is no natural definition of abstract descriptions of the state of the world, leading to endless arguments (c.f. edge detection).***

Inverse Problems in Computer Vision

- Edge detection
- Shape-from-shading
- Structure-from-stereo
- Optical flow computation
- Surface reconstruction
- Object recognition
- Scene understanding
- Etc.

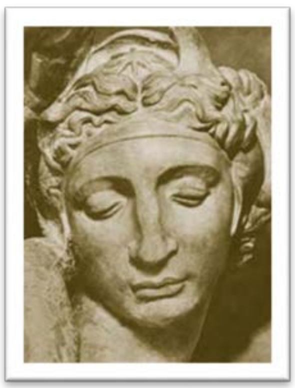
Edge Detection

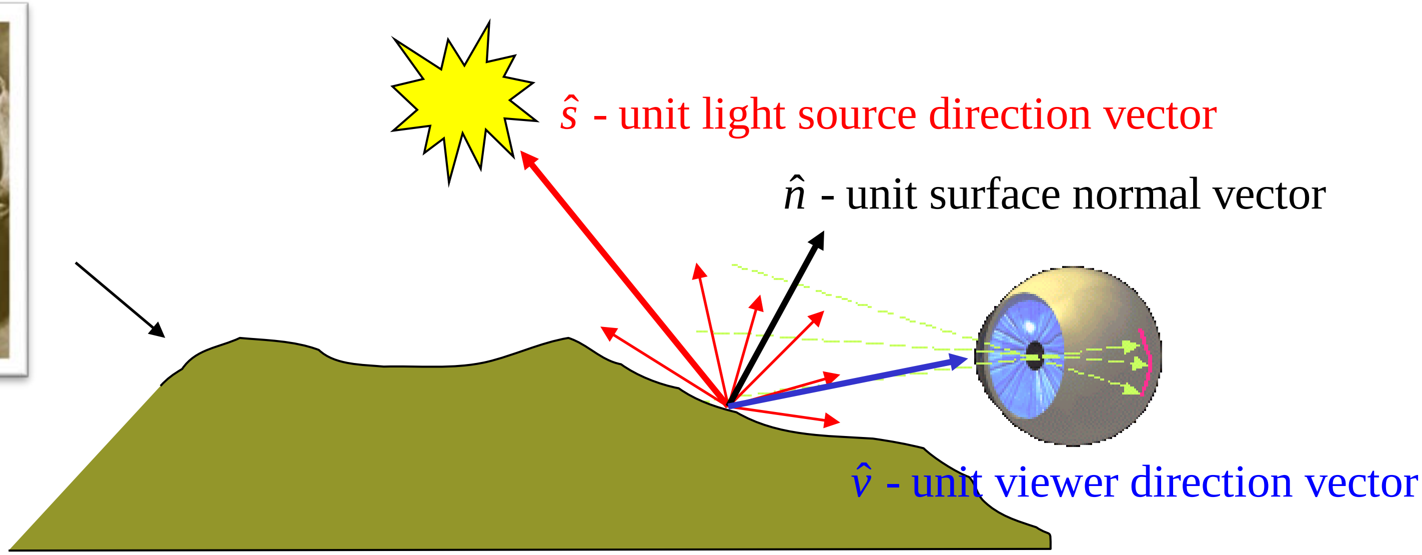
• Find edges in the scene (which characterize boundaries) based on intensity changes in the image.



Shape-from-Shading

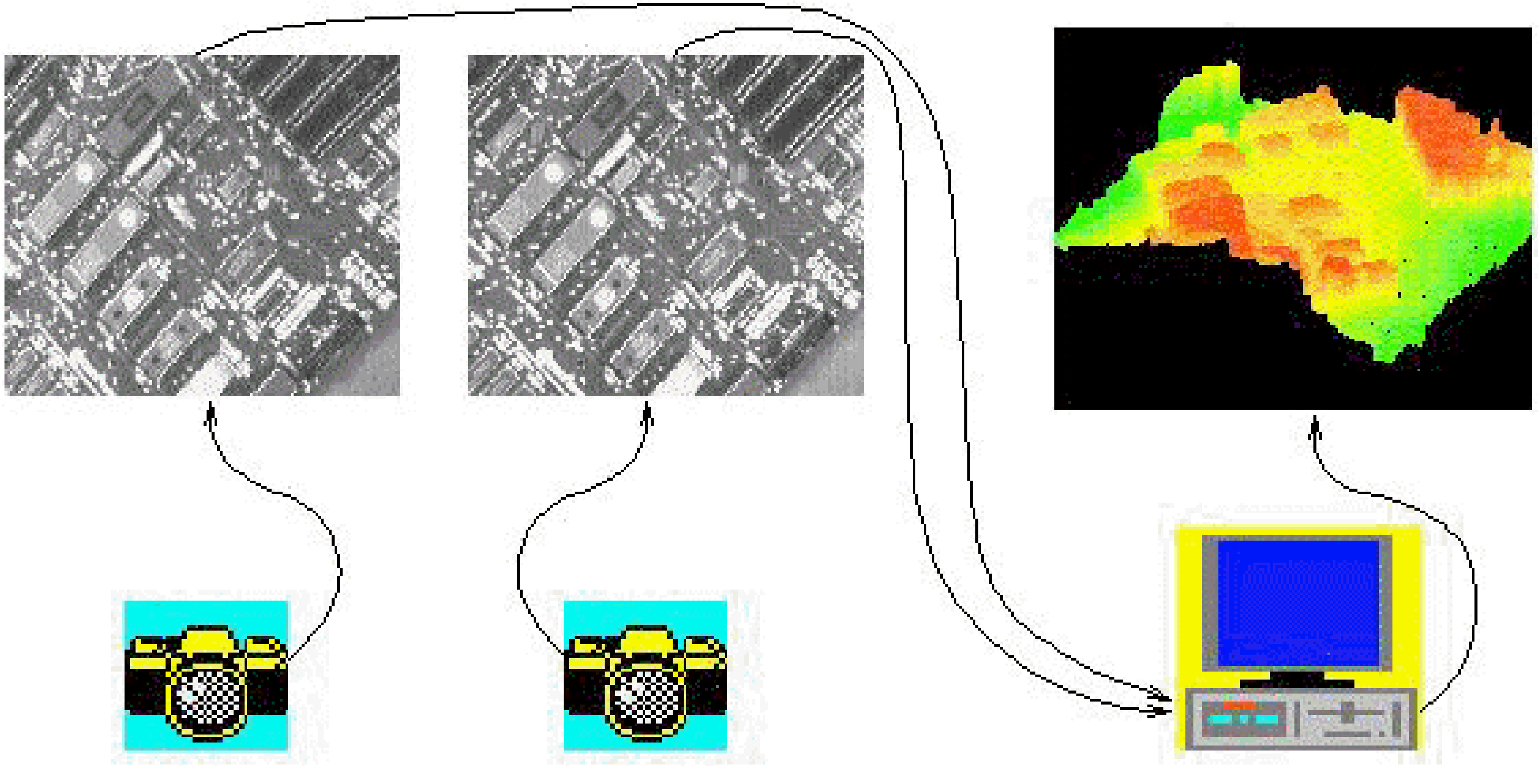
- Extract information about the 3-D shape of a surface from measurements of the light reflecting off of the surface.





Structure-from-Stereo

- Estimate depth from a pair of stereo images.



http://www.cmis.csiro.au/iap/RecentProjects/stereoEE02.htm

Optical Flow

- Compute the vector displacement field mapping points on one image onto another image.

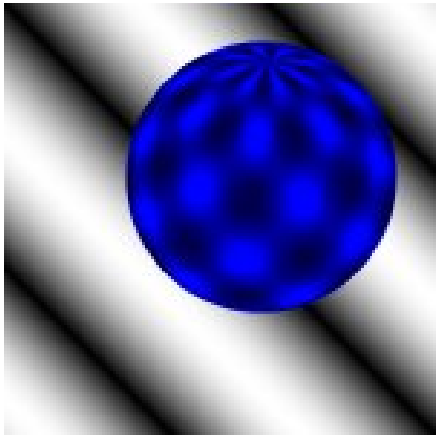


Image 1

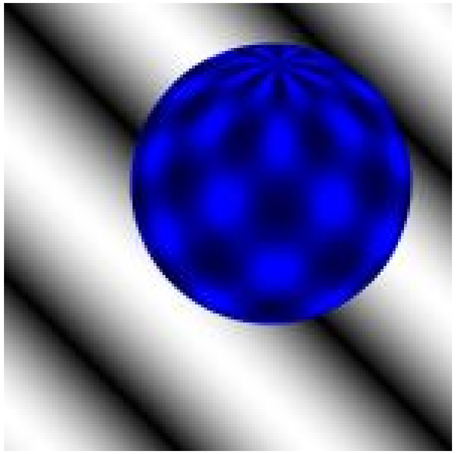
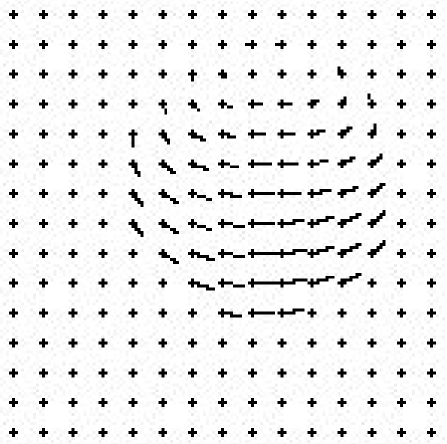


Image 2

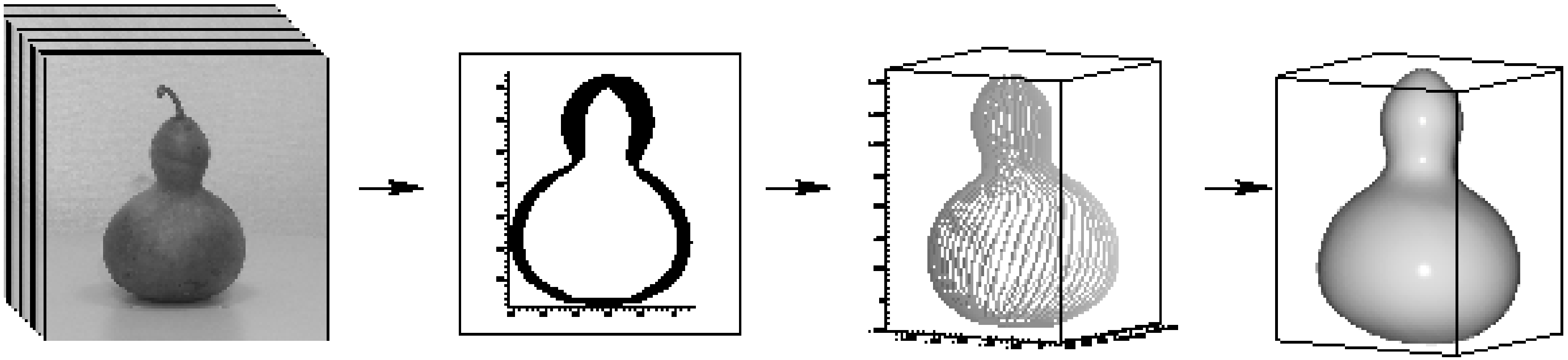


Flow field

http://www.cs.otago.ac.nz/research/vision/Research/OpticalFlow/opticalflow.html

Surface Reconstruction

- Given a sequence of images of an object, build a 3D model of the object that gave rise to the images (one interpretation):



http://homepages.inf.ed.ac.uk/rbf/CVonline/LOCAL_COPIES/BOYER/node1.html#SECTION00010000000000000000

Object Recognition

- Which 3D object, in a database of known objects, gave rise to this 2D image?

What is this?

Database of objects

There are three main aspects to solving the vision problem:

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Forward problem

- Specifying the forward problem is presumably possible in principle, using the quantum mechanical world description.
- Getting an exact solution for the forward problem for vision problems would be enormously complicated.
- Solving for the inverse problem would likewise be complicated.

Forward problem

- Computer graphics: *Given a 3D object in a scene and known environmental conditions (e.g. lighting position, color, etc.), determine the appearance of 2D images of the scene.*

https://www.videogamer.com/news/game-listing-indicates-call-of-duty-ww2-will-run-in-4k-with-hdr-on-xbox-one-x

Forward problem

- In practice, in computer vision, simpler models are used for the forward mapping (e.g. geometrical optics, simple shading models) from abstract world descriptions.

- ***These models are approximate at best, so that any solution of the inverse mapping is also only approximate at best, and might even be completely wrong.***

There are three main aspects to solving the vision problem:

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Solution of Inverse Problems

What is meant by the "*solution*" to an inverse problem?

We want to find the "solution" z to some problem given "data" u.

Typically this is done by applying a solution operator to the data:

$$z = R\,u$$

The challenge lies in finding an appropriate solution operator.

Difficulties with Solving Inverse Problems

1. ***Existence***: The solution process might not provide any answer for some data values.
2. ***Uniqueness***: The solution process might provide multiple solutions for a given input.
3. ***Stability***: Even if the answer exists and is unique it might not be stable with respect to small perturbations in the input.

To define the problem more completely let us specify that ***u,z*** are elements of metric spaces ***U, F***, respectively, with *metric* functions:

$$\rho_U(u_1, u_2),\ \rho_F(z_1, z_2)$$

Example:

$U = \mathbb{Z}_{+}^{N}$ (N-pixel image taking on positive integral values)

$F = L_2(\mathbb{R}^2)$ (Finite energy 2-D signal)

$$\rho_U(u_1, u_2) = \sum_{i=1}^{\infty} |u_1^i - u_2^i|$$

$$\rho_F(z_1, z_2) = \iint (z_1(x,y) - z_2(x,y))^2 \, dxdy$$

(note: F is a Hilbert Space, but U is not)

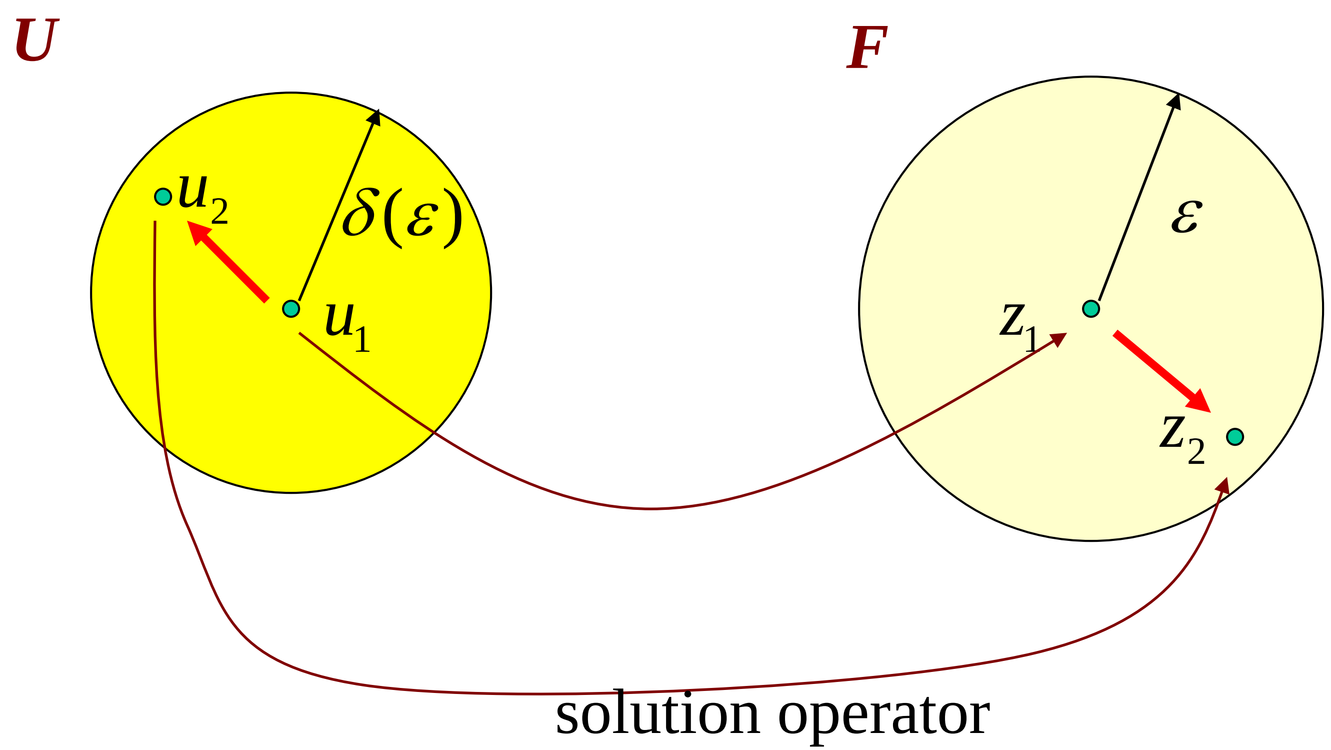
Stability of a Solution Operator

The problem of determining $z = Ru$ in F from $u \in U$ is said to be *stable on the spaces* (F, U) if, for every positive number ε, there exists a positive number $\delta(\varepsilon)$ such that

$$\rho_{\mathrm{U}}(u_1, u_2) < \delta(\varepsilon) \text{ implies } \rho_{\mathrm{F}}(z_1, z_2) \leq \varepsilon$$

where $z_1 = Ru_1$ and $z_2 = Ru_2$

Stability means that if the data is perturbed slightly then the resulting perturbation of the solution will be small.



Well-Posed Inverse Problems

An inverse problem is said to be

Well-Posed in the sense of Hadamard

if

- For every $u \in U$ there exists a solution $z \in F$
- The solution is unique
- The solution is stable on (U, F)

(Hadamard's notion of well-posedness was for initial-value and boundary-value problems)

Jacques Salomon Hadamard

Born: 8 Dec 1865 in Versailles, France
Died: 17 Oct 1963 in Paris, France

(image taken from http://www-groups.dcs.st-and.ac.uk/~history/PictDisplay/Hadamard.html)

Ill-Posed Inverse Problems

An inverse problem that does not satisfy all the conditions for well-posedness is said to be ***ill-posed***.

Most vision problems are ill-posed.

Ill-posed problems, by definition, do not have a solution process that yields a unique, stable, solution.

Ill-Posed Inverse Problems

Infer identity of person from single camera image.

Examples:

Solution non-unique!

http://arstechnica.com/business/2015/06/facebooks-facial-recognition-will-one-day-find-you-even-while-facing-away/

Solution to Ill-Posed Inverse Problems

So, what are we to do if we are faced with an ill-posed problem (like vision)?

In this course, we will examine various strategies for finding solutions to ill-posed vision problems.

We will then examine probabilistic inference solutions in greater depth.

Regularization

- What do we do with an *ill-posed* problem that we can't solve?

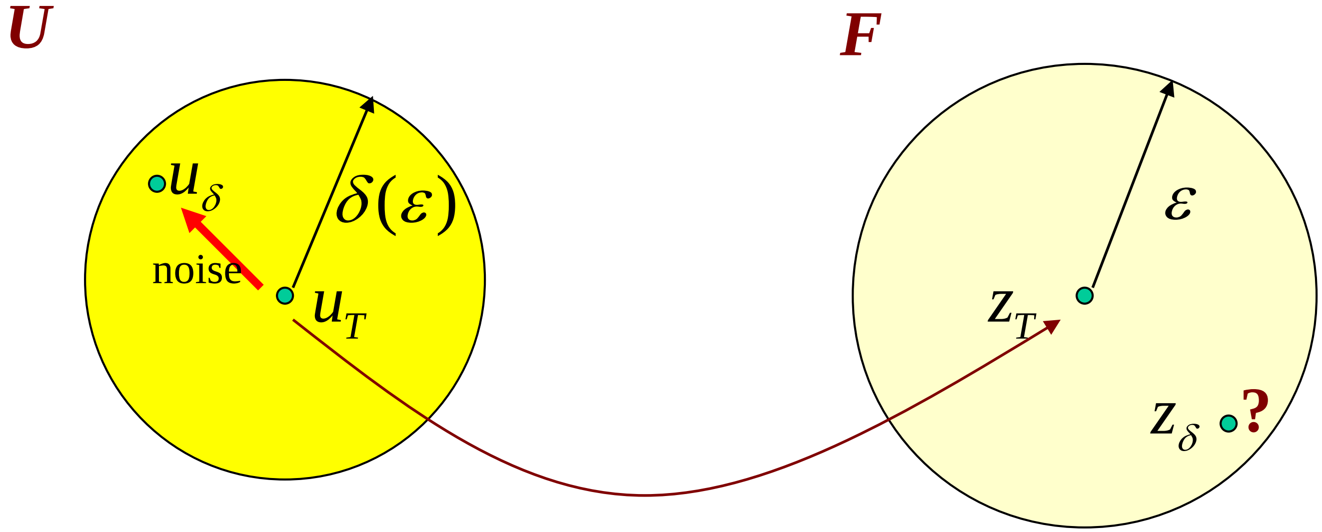
Replace it with a problem that we *can* solve!

Regularization

A.N. Tikhonov and V.Y. Arsenin, **Solutions of Ill-Posed Problems**, V.H. Winston, Washington, 1977

Tikhonov proposed handling ill-posed problems through ***regularization*** of the problem, which entails ***replacing the ill-posed problem with a well-posed one*** whose solution is in some sense close to the desired "true" solution.

Suppose that u_T is the "true", or noise-free data generated by the "true" solution z_T.
Let u_δ be such that $\rho_U(u_\delta, u_T) \le \delta$. That is, we know that the noise level in the data is less than or equal to δ.



In general, the approximate solution z_δ cannot be obtained merely by

$$z_\delta = A^{-1} u_\delta$$

even if A^{-1} exists, due to the ill-posedness.

Instead, we use another operator, R, which has parameters that depend on δ, which can provide a *stable*, *approximate*, solution:

$$z_\delta = R(u_\delta, \delta)$$

Definition:

An operator $R(u,\alpha)$, depending on a parameter α, is called a *regularizing* operator for the equation

$$Az = u$$

in a neighborhood of $u = u_T$ if:

1) There exists a positive number δ_1 such that the operator $R(u,\alpha)$ is defined for every $\alpha > 0$ and every $u \in U$ for which $\rho_U(u,u_T) \le \delta < \delta_1$.

2) There exists a function $\alpha = \alpha(\delta)$ of δ such that for every $\varepsilon > 0$ there exists a number $\delta(\varepsilon) \le \delta$ such that the inclusion $u_\delta \in U$ and the inequality $\rho_U(u_\delta,u_T) \le \delta(\varepsilon)$ imply that $\rho_F(z_\alpha,z_T) \le \varepsilon$ where $z_\alpha = R(u_\delta,\alpha(\delta))$.

z_α is called the Regularized Solution of $Az = u$.

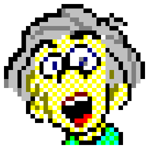
α is called the Regularization parameter.
It is required that α be consistent with the accuracy (noise level) δ of the data u_δ.

The second condition in the definition is essentially a stability condition.

In the regularization approach, the task of finding a solution to the inverse problem that is stable under small changes in the data reduces to:

- Finding a regularizing operator on the problem spaces
- Estimating the noise level in the data
- Determining the value of the regularizing parameter α
- Applying the regularizing operator

Well, that's all well and good, you say, but how can we find a suitable regularizing operator???



Tikhonov developed a ***Variational*** approach, meaning that Calculus of Variations is used to solve an optimization problem.

To see how regularization can be cast in terms of an optimization problem, let us begin by considering what is to be optimized.

In the calculus of variations, typically it is a ***functional*** that is optimized.

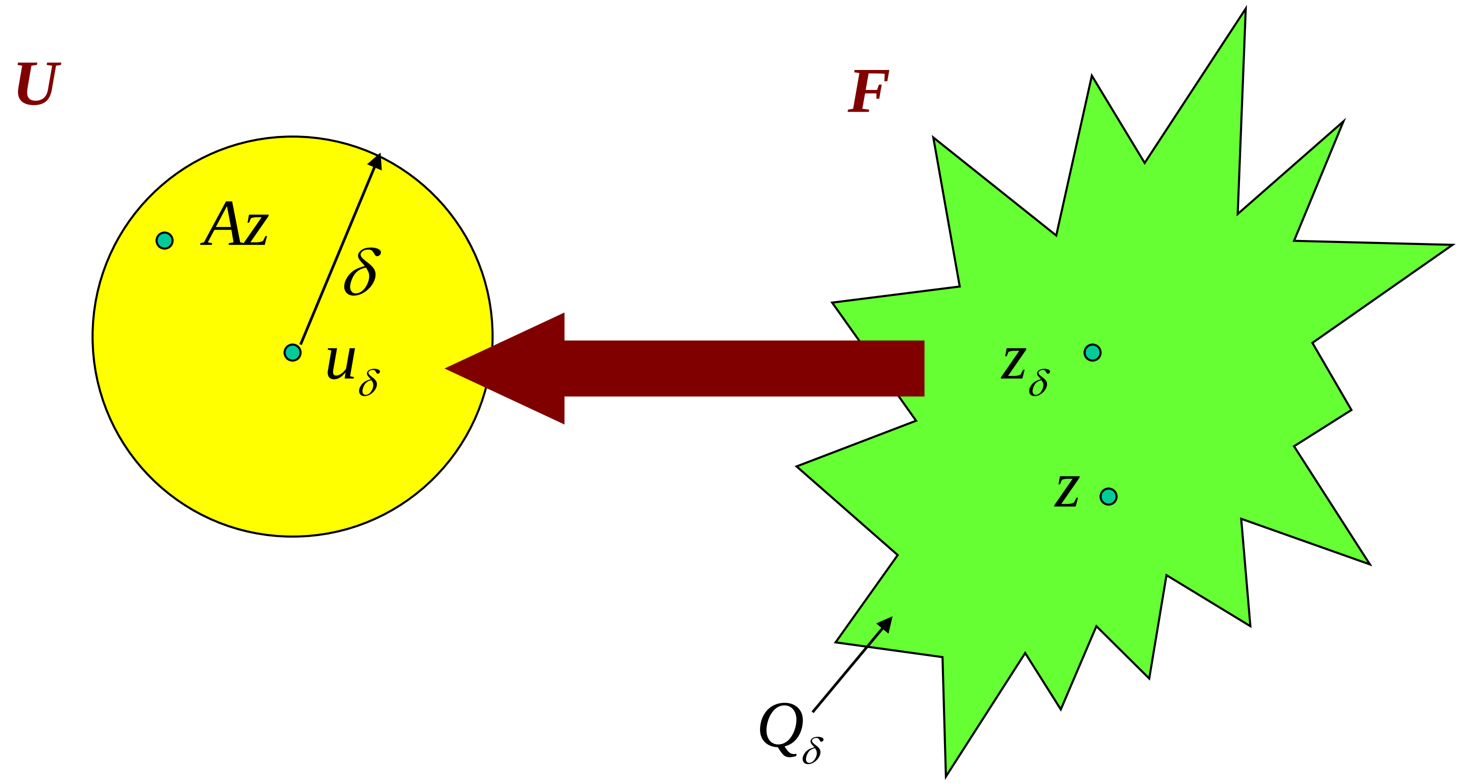
A functional is a mapping from a space of functions to a scalar.

Let $Q_\delta \subset F$ be the set defined by all values z such that

$$\rho_U(Az, u_\delta) \leq \delta$$

That is, the set of all values which produce data that are within a distance δ of u_δ.

We would say then that Q_δ is the set of all possible solutions that are consistent with noisy data, with a noise level δ.


U
F
Az
δ
u_δ
z_δ
z
Q_δ

We could obtain an approximate solution by just taking any element $z_\delta \in Q_\delta$. But this solution will, in general, not be continuous wrt δ.

It may also be quite difficult to determine the set Q_δ.

Instead, Tikhonov proposed a variational principle.

Constrained optimization problem:

Constrain the solution space so that the problem is well-posed.

Let $\Omega[z]$ be a continuous non-negative functional defined on a subset F_1 of F : Stabilizing Functionals or Stabilizers.

The regularizing operation corresponds to finding the value z which minimizes $\Omega[z]$ subject to the constraint that $\rho_U(Az, u_\delta) = \delta$.

We therefore have that the regularizing operation corresponds to finding the value z which minimizes $\Omega[z]$ subject to the constraint that $\rho_U(Az, u_\delta) = \delta$.

Thus, we need to solve a constrained optimization problem.

Lagrange Multipliers

Constrained optimization problems can be solved using the technique of Lagrange Multipliers.

The method of Lagrange Multipliers involves adding an auxiliary variable to the optimization problem, which incorporates the constraint.

Lagrange Multipliers

Suppose we want to minimize $f(x)$ subject to the constraint $g(x)=0$.

In the Lagrange multiplier approach
we do an unconstrained minimization of $f(x)+\mu g(x)$
by taking the gradient wrt to x and setting to zero.

Lagrange Multipliers

The value of the auxiliary variable (the "Lagrange Multiplier") μ is taken to be the value which causes the solution to satisfy the constraint.

In the case of the regularization problem, the Lagrange Multiplier formulation then becomes the unconstrained minimization of the following functional:

$$M^{\alpha}(z,u_{\delta}) = \Omega[z] + \alpha(\rho_U(Az,u_{\delta}) - \delta)$$

where α is the Lagrange Multiplier, which is chosen to make $\rho_U(Az_{\alpha},u_{\delta}) = \delta$ where z_{α} is a solution $z \in F_1$ for which $M^{\alpha}(z,u_{\delta})$ is a minimum.

Note that α depends on δ, A, and u_δ.

The quantity δ does not depend on z or α, so we can remove it from the functional. i.e.

$$M^\alpha(z,u_\delta)=\Omega[z]+\alpha(\rho_U(Az,u_\delta)-\delta)$$

becomes

$$M^\alpha(z,u_\delta)=\Omega[z]+\alpha\rho_U(Az,u_\delta)$$

In summary, regularization is a way to handle ill-posed problems by solving a related well-posed problem.

The variational approach to regularization involves solving a constrained optimization problem.