Disjoint Sets and Union-Find Operations

Consider a finite set $S = \{x_1, x_2, \ldots, x_n\}$, $n \geq 1$.

Observations:

1. A trivial partition P of S can be obtained by simply defining $S_i = \{x_i\}$, for all i, $1 \leq i \leq n\}$.
2. Given a partition P of S with k sets, $k \geq 2$, a new partition of S can be formed by taking some unions of the sets in P.
3. One can perform at most n-1 union operations on the sets in P before S is re-generated.

Questions:

For any given partition P of S,

1. How do we represent/naming a set?
2. What kind of data structure should we use to implement a set so as to support the following two basic operations:
   (i) find(x), $x \in S$: Return the (unique) set containing x.
   (ii) union(x,y): Return the union of the two sets containing with representative x, and y, respectively.

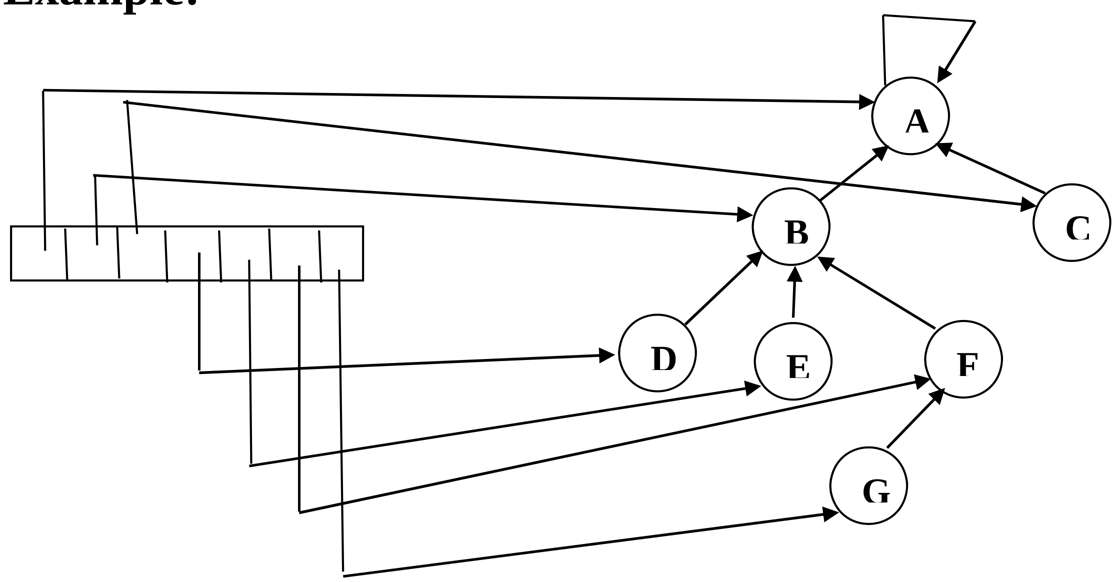
A Simple Approach:

- Given any set S_i. Use an element $x \in S_i$ as the representative/name of the set.
- Use a tree to implement a set with the root of the tree being the representative of the set.

Implementing a Set using a Tree Structure:

Each set S_i will be implemented as a (rooted) tree T_i such that each element $x \in S_i$ will be represented by a node with a parent pointer pointing to its parent in T_i.

Example:



Remark: The array of pointers will be used to access the nodes of the tree.

Set Union and Trees Merging:

Given two sets S_i and S_j with representatives x_i and x_j, the operation union(x_i,x_j) is equivalent to merging two trees with roots x_i and x_j together. To merge two trees together, one can simply set the parent pointer of x_i to point at x_j with x_j being the representative of the new set. Observe that, independent of the sizes of the two sets, union(x_i,x_j) can now be executed in O(1) time.

Union and Find Operations in Disjoint Sets:

If each element $x \in S$ is being used to represent a data object, then one of the most important and relevant operations in a partition of S will be the *find* operation, find(x), which will return the representative of the unique set that contains x.

Basic find Operation:

Approach:

- Using the array of pointers to locate the element x, and the tree containing x, among the trees in a partition.
- Follow the parent pointer of x to the root of the tree that contains x.
- Return the root of the tree that contains x.

Complexity of find Operations:

Recall that if one performs O(n) union operations on the sets of a given partition using the above tree structures and trees merging algorithm, a tree with height O(n) can be formed. Hence, in order to find the root of the tree that contains x, one may have to follow the parent pointers of a leaf to the root, resulting in $T(n) = O(n)$.

Amortized Analysis of Disjoint Set Operations:

Given a sequence of O(n) union operations intermixed with a sequence of O(m) find operations, where $m >> n$, what is a good data structure and its complexity T(m,n) in performing these O(n) union and O(m) find operations?

Simple Approach:

Using the above implementation, we have T(m,n) = O(mn).

Questions:

Can we do it better?

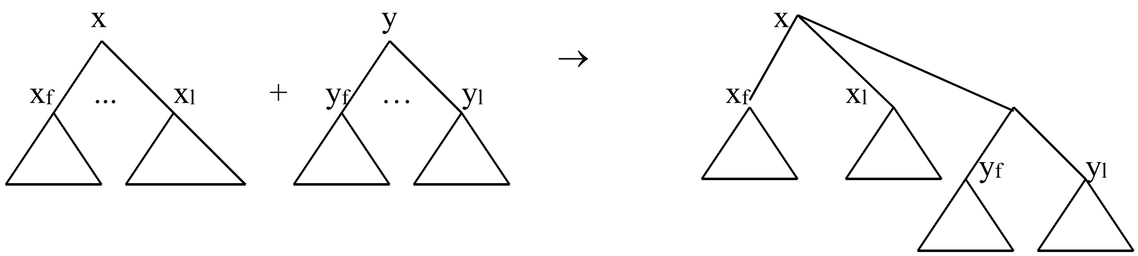
Two Possible Approaches:

1. Minimize the height of the resulting tree during union operation.
2. Modify the structure of a tree during find operations so that subsequent find operations performed on the same tree can be speeded up.

Union-by-Height Heuristic:

For each node x in a tree T, recall that the height of x, h(x), is the length of a longest path from x to a leaf node in T. In performing union(x_i,x_j) operation, the parent pointer of x_i is set to point at x_j iff $h(x_i) \leq h(x_j)$. Otherwise, we will set the parent pointer of x_j to point at x_i instead. Observe that, by using this union-by-height heuristic, the height of the resulting tree will increase by 1 iff both trees have the same height.

Example: Performing union(x,y) using union-by-height with $h(x) \geq h(y)$.



Other Union Heuristics:

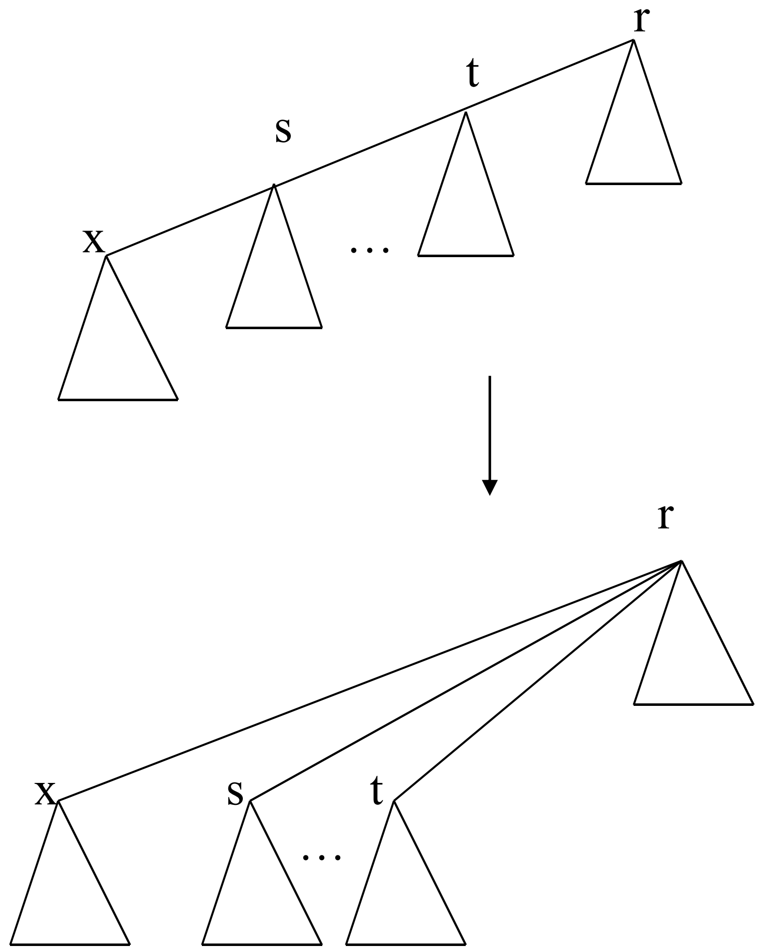
1. Union-by-Rank Heuristic:
   For each node x in S, define rank(x) as followed:
   Initially, when x is in a tree by itself, rank(x) = 0.
   When performing union(x_i,x_j) operation, the parent pointer of x_i is set to point at x_j iff $rank(x_i) \leq rank(x_j)$. And if $rank(x_i) = rank(x_j)$, then $rank(x_j) = rank(x_j) + 1$. If $rank(x_i) > rank(x_j)$, then we will set the parent pointer of x_j to point at x_i instead. Observe that, by using this union-by-rank heuristic, the rank of the resulting tree will increase by 1 iff both trees have the same rank. Also, the rank of a root can only be changed during union operation. Hence, the $rank(x) \geq h(x)$, $\forall\ x \in S$.

2. Union-by-Weight/Size Heuristic:
   For each node x in S, define w(x) be the number of nodes, including x, in the tree rooted at x. When performing union(x_i,x_j) operation, the parent pointer of x_i is set to point at x_j iff $w(x_i) \leq w(x_j)$. Otherwise, we will set the parent pointer of x_j to point at x_i instead. Observe that, by using this union-by-weight heuristic, the weight of the resulting tree will always be increased by the size of the additional tree.

Path Compression Heuristic:

In performing find(x) operation, after the tree T that contains x and the root r of T is identified, every node on the path from x to r will be made a new child of r.

Example: Performing find(x) using path compression.



Theorem (Tarjan): By using union-by-rank and path compression heuristics, $T(m,n) = O(m\alpha(m,n))$, where $\alpha(m,n)$ is the inverse Ackerman's function $A(m,n)$.

Review: Ackermann's Function

Define the **Ackermann's function** $A : N \times N \to N$ by

$$
\begin{aligned}
A(0, n) &= n + 1,\\
A(m, 0) &= A(m-1, 1), \text{ if } m > 0,\\
A(m, n) &= A(m-1, A(m, n-1)), \text{ if } m, n > 0.
\end{aligned}
$$

Example:

$$
\begin{aligned}
A(0, 0) &= 1,\\
A(0, 1) &= 2,\\
A(0, 2) &= 3,\\
A(0, 3) &= 4,\\
&\ldots
\end{aligned}
$$

$$
\begin{aligned}
&A(1, 0) = A(0, 1) = 2,\\
&A(1, 1) = A(0, A(1, 0)) = A(0, 2) = 3,\\
&A(1, 2) = A(0, A(1, 1)) = A(0, 3) = 4,\\
&A(1, 3) = 5,\\
&\ldots
\end{aligned}
$$

$$
\begin{aligned}
&A(2, 0) = A(1, 1) = 3,\\
&A(2, 1) = A(1, A(2, 0)) = A(1, 3) = 5,\\
&A(2, 2) = A(1, A(2, 1)) = A(1, 5) = 7,\\
&A(2, 3) = 9,\\
&\ldots
\end{aligned}
$$

$A(3,0) = A(2, 1) = 5,$
$A(3, 1) = A(2, A(3,0)) = A(2, 5) = 13,$
$A(3, 2) = A(2, A(3, 1)) = A(2, 13) = 29,$
$A(3, 3) = A(2, A(3, 2)) = A(2, 29) = 61,$
…

$A(4,0) = A(3,1) = 13,$
$A(4,1) = A(3,A(4,0)) = A(3,13),$
$A(4,2) = ?$

Remark: $A(m,n)$ is an extremely fast growing function and, hence, its inverse function $\alpha(m,n)$, which often appears in data structures analyses and counting, grows extremely slow. For all practical purpose, $\alpha(m,n)$ can be treated as a constant.