Given a sequence of functions $\{t_n, t_{n-1}, t_{n-2}, \ldots, t_1, t_0\}$, a *recurrence equation* (RE) has the form:

$$F(t_n, t_{n-1}, t_{n-2}, \ldots, t_1, t_0) = 0.$$

A Special Case:
k^{th}-order Linear Recurrence Equation with Constant Coefficients (LRECC):

$$C_0\mathbf{t_n} + C_1\mathbf{t_{n-1}} + C_2\mathbf{t_{n-2}} + \ldots + C_k\mathbf{t_{n-k}} = f(n), \qquad \ldots\ldots\ldots\ldots\ldots (*)$$

where C_i = constant, $\forall i$, and $C_0 \neq 0$ and $C_k \neq 0$.

If $f(n) = 0 \Rightarrow (*)$ is a ***homogeneous LRE***; otherwise, (*) is an ***inhomogeneous LRE***.

Dfn: A ***solution*** of (*) is any function g(n) such that by assigning $\mathbf{t_n} = g(n)$, $\mathbf{t_{n-1}} = g(n-1)$, $\mathbf{t_{n-2}} = g(n-2)$, …, $\mathbf{t_{n-k}} = g(n-k)$, $C_0g(n) + C_1g(n-1) + C_2g(n-2) + \ldots + C_kg(n-k) = f(n)$.

A ***general solution*** of (*) is a solution of (*) with k unspecified constants such that any solution of (*) can be generated from the general solution by assigning specific value to these k constants. Once the k constants of a solution are specified, we have a ***particular solution*** of (*).

Remark:
In order to determine the k unspecified constants in the general solution, k initial conditions of (*) must be provided in order to set up a system of k linear equations in k unknowns (the constants). Standard method, such as Gaussian Eliminations, for solving systems of linear equations can then be used to solve for the k constants.

Solving a k-order LRE (*):

(1) If k initial conditions are not given, compute a general solution for (*) with k unspecified constants.
(2) If k initial conditions are given, compute a (particular) solution for (*) that will satisfy (*) and all of the k initial conditions.

Examples: Some REs.

1. $3\mathbf{t_n} + 5\mathbf{t_{n-1}} - 2\mathbf{t_{n-2}} = 0$, 2^{nd}-order HLRECC

2. $\mathbf{t_n} + 2\mathbf{t_{n-2}} + 2\mathbf{t_{n-4}} = 0$, 4^{th}-order HLRECC

3. $\mathbf{t_n} = \mathbf{t_{n-1}} + n - 1$, 1^{st}-order IHLRECC

4. $\mathbf{t_n} = n\mathbf{t_{n-3}} + 1$, 3^{rd}-order LRE

5. $\mathbf{t_n} + \mathbf{t_{n-1}}\mathbf{t_{n-2}} + 2\mathbf{t_{n-4}} = 1$, Non-Linear RE

I. Solving HLRECC:

Consider the following k^{th}-order HLRE with CC:

$$C_0\mathbf{t_n} + C_1\mathbf{t_{n-1}} + C_2\mathbf{t_{n-2}} + \ldots + C_k\mathbf{t_{n-k}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ (1)$$

Q: How can we find a solution of (1)?

Easy! Take $\mathbf{t_n} = 0$.

This is a trivial solution of (1).

Q: How can we find a nontrivial solution of (1)?

Let's try $\mathbf{t_n} = x^n \neq 0$.

Substituting $\mathbf{t_n} = x^n$ into (1) to get

$$C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_kx^{n-k} = 0.$$

Divide the above equation by $x^{n-k} \neq 0$ to get

$$C_0x^k + C_1x^{k-1} + C_2x^{k-2} + \ldots + C_k = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2)$$

Equation (2) is called the ***characteristic equation*** of (1).

Remarks:

1. The standard k^{th}-order HLRECC (1) can be transformed into an algebraic equation (2) of degree k with the same coefficients.
2. From the ***Fundamental Theorem of Algebra***, any polynomial equation of degree k has exactly k roots. Hence, the characteristic equation (2) must have exactly k roots, say $r_1, r_2, \ldots, r_k$.
3. The solution of a HLRECC (1) is completely characterized by the roots of its characteristic equation (2).
4. In order to solve (1), we need to solve for the k roots of (2).
5. The most difficult computational step in solving the k^{th}-order HLRECC (1) requires the computation of these k roots.
6. There is no simple algorithm for finding the k roots of a k^{th}-order algebraic equation when $k \geq 5$ (Galois Theorem).

Some Important Results:

Lemma 1: If r is a root of (2), then $\mathbf{t_n} = dr^n$ is a solution of (1) for any constant d.
Proof: Verify it by substituting $\mathbf{t_n} = dr^n$ back into (1).

Remark: The k roots of (2) generate k solutions $d_1r_1^n, d_2r_2^n, \ldots, d_kr_k^n$ of (1).

Lemma 2: If $S_1(n), S_2(n), \ldots, S_m(n)$ are m solutions of (1), $m \geq 1$, then any linear combination of these solutions $d_1S_1(n) + d_2S_2(n) + \ldots + d_mS_m(n)$ also forms a solution of (1), where each d_i, $1 \leq i \leq m$, is an arbitrary constant.
Proof: Use induction on m.

Corollary 3: $d_1r_1^n + d_2r_2^n + \ldots + d_kr_k^n$ is a solution of (1).

Q: Observe that the function $\mathbf{t_n} = d_1r_1^n + d_2r_2^n + \ldots + d_kr_k^n$ is a solution of (1) with exactly k constants. Will $\mathbf{t_n}$ be the general solution of (1)? If not, can we generate the general solution of (1) from the k solutions of (1)?

Need concept of linear independence of functions!

Defn: A set of functions $f_1(n), f_2(n), \ldots, f_m(n)$ is ***linearly independent*** if $h_1f_1(n) + h_2f_2(n) + \ldots + h_mf_m(n) = 0$, h_i = constant, $\forall i$, implies $h_1 = h_2 = \ldots = h_m = 0$; otherwise, they are ***linearly dependent***.

Remark:
If a set of functions $\{f_1(n), f_2(n), \ldots, f_m(n)\}$ is linearly dependent, then any function $f_i(n)$ in the set can be represented as a linear combination of the remaining functions such that $f_i(n) = h_1f_1(n) + \ldots + h_{i-1}f_{i-1}(n) + h_{i+1}f_{i+1}(n) + \ldots + h_mf_m(n)$, for some constants h_j, $1 \leq j \leq m$, $j \neq i$.

Examples:

1. $\{1, x, x^2, x^3\}$ is linearly independent since $h_1 + h_2x + h_3x^2 + h_4x^3 = 0$ implies $h_1 = h_2 = h_3 = h_4 = 0$. (HW: Prove it.)

2. $\{1, 2x-1, 2x+1, x^2\}$ is linearly dependent since $h_1 + h_2(2x-1) + h_3(2x+1) + h_4x^2 = 0$ if we choose $h_1 = 2, h_2 = 1, h_3 = -1, h_4 = 0$.

Theorem 4: If $\{S_1(n), S_2(n), \ldots, S_k\}$ is a set of k linearly independent solutions of (1), then $t_n = d_1S_1(n) + d_2S_2(n) + \ldots + d_kS_k(n)$ forms the general solution of (1), where d_i = constant, $\forall i$.

Remark: If we can extract k linearly independent solutions of (1) from $\{r_1, r_2, \ldots, r_k\}$, then we can form the general solution of (1) by taking a linear combination of them.

Q: How do we extract k linearly independent solutions from $\{r_1, r_2, \ldots, r_k\}$?

Theorem 5: Let $r_1, r_2, \ldots, r_q$ be the q distinct roots of (2) such that r_i has multiplicity $m_i, \forall\, i$, where $m_i \geq 1$, and $m_1 + m_2 + \ldots m_t = k$. Then $\{r_i^{\,n}, nr_i^{\,n}, n^2 r_i^{\,n}, \ldots, n^{m_i - 1} r_i^{\,n} \mid 1 \leq i \leq q\}$ forms a set of k linearly independent solutions of (1).

Corollary 6: The general solution of (1) is given by

$$
\begin{aligned}
t_n = {} & d_{11} r_1^{\,n} + d_{12} n r_1^{\,n} + d_{13} n^2 r_1^{\,n} + \ldots + d_{1m_1} n^{m_1 - 1} r_1^{\,n} \\
& + d_{21} r_2^{\,n} + d_{12} n r_2^{\,n} + d_{13} n^2 r_2^{\,n} + \ldots + d_{1m_2} n^{m_2 - 1} r_2^{\,n} \\
& + \ldots \\
& + d_{q1} r_t^{\,n} + d_{q2} n r_t^{\,n} + d_{q3} n^2 r_t^{\,n} + \ldots + d_{qm_t} n^{m_t - 1} r_q^{\,n}.
\end{aligned}
$$

Method for solving a k^{th}-order HLRE with CC:

1. Generate the corresponding characteristic equation.
2. Solve for the k roots of the characteristic equation.
3. Extract the set of k linearly independent solutions from the roots.
4. Form the general solution by taking a linear combination of the k linearly independent solutions.
5. Determine the k constants using k initial conditions if given.
6. Verify the correctness of the solution.

Examples: Solving HLRECC.

1. If the roots of a 3^{rd}-order HLRE with CC are 3, 5, ½, then the general solution is
   $t_n = d_1 3^n + d_2 5^n + d_3 (1/2)^n$.

2. If the roots of a 10^{th}-order HLRE with CC are 1(3), 3(2), 7(4), 8(1), then the general solution is: $t_n = d_1 + d_2 n + d_3 n^2 +$
   $d_4 3^n + d_5 n 3^n +$
   $d_6 7^n + d_7 n 7^n + d_8 n^2 7^n + d_9 n^3 7^n +$
   $d_{10} 8^n$

3. Given $t_0 = 1$, $t_1 = 2$, $3t_n + 5t_{n-1} = 2t_{n-2}$, $\forall n \geq 2$.
   Characteristic equation:
   $3x^2 + 5x - 2 = 0$
   $(x + 2)(3x - 1) = 0$
   Roots are: –2, 1/3
   General solution: $t_n = d_1(-2)^n + d_2(1/3)^n$
   Determining constants:
   $t_0 = 1$: $d_1(-2)^0 + d_2(1/3)^0 = d_1 + d_2 = 1$,
   $t_1 = 2$: $d_1(-2)^1 + d_2(1/3)^1 = -2d_1 + (1/3)d_2 = 2$.
   On solving, $d_1 = -5/7$, $d_2 = 12/7$.
   Hence, the solution is $t_n = (-5/7)(-2)^n + (12/7)(1/3)^n$.

Verification of Solution:

1. Verification using ***Strong Induction***:

Define P(n): $t_n = (-5/7)(-2)^n + (12/7)(1/3)^n$ is the solution to

$t_0 = 1$, $t_1 = 2$, $3t_n + 5t_{n-1} = 2t_{n-2}$, $\forall n \geq 2$.

(i) ***Basis step***:

When n = 0, $t_0 = (-5/7)(-2)^0 + (12/7)(1/3)^0$
$= (-5/7) + (12/7)$
$= 1.$

When n = 1, $t_1 = (-5/7)(-2)^1 + (12/7)(1/3)^1$
$= (10/7) + (12/21)$
$= 2.$

Hence, P(0) and P(1) are both true.

(ii) ***Inductive step***:

Assume that P(k) is true for all k, $2 \leq k < n$. Need to prove that

$t_{k+1} = (-5/7)(-2)^{k+1} + (12/7)(1/3)^{k+1}$.

t_{k+1}

$= (-5t_k + 2t_{k-1})/3$ (From RR)

$= \{[(-5)(-5/7)(-2)^k + (-5)(12/7)(1/3)^k] +$
$[(2)(-5/7)(-2)^{k-1} + (2)(12/7)(1/3)^{k-1}]\}/3$ (I.H.)

$= \{[(-25/14)(-2)^{k+1} + (-180/7)(1/3)^{k+1}] +$
$[(-5/14)(-2)^{k+1} + (216/7)(1/3)^{k+1}]\}/3$

$= [(-30/14)(-2)^{k+1} + (36/7)(1/3)^{k+1}]/3$

$= (-5/7)(-2)^{k+1} + (12/7)(1/3)^{k+1}$.

Hence, $P(2) \wedge P(3) \wedge \ldots \wedge P(k) \rightarrow P(k+1)$. By strong induction, P(n) is true for all $n \geq 2$.

2. Using ***Backward Substitutions***:

(i) ***Verifying the initial conditions***:

n = 0: $t_0 = (-5/7)(-2)^0 + (12/7)(1/3)^0 = 1$.

n = 1: $t_1 = (-5/7)(-2)^1 + (12/7)(1/3)^1 = 2$.

Observe that this is the same as in the basis step of an induction proof.

(ii) ***Verifying the general recurrence***:

Substitute t_n, t_{n-1}, t_{n-2}, … back into the given recurrence equation to verify that

$C_0\mathbf{t_n} + C_1\mathbf{t_{n-1}} + C_2\mathbf{t_{n-2}} + \ldots + C_k\mathbf{t_{n-k}} = f(n) = 0$.

Hence, we need to verify that

$$\begin{aligned}
&3t_n + 5t_{n-1} - 2t_{n-2} \\
&= 3[(-5/7)(-2)^n + (12/7)(1/3)^n] \\
&\quad + 5[(-5/7)(-2)^{n-1} + (12/7)(1/3)^{n-1}] \\
&\quad - 2[(-5/7)(-2)^{n-2} + (12/7)(1/3)^{n-2}] \\
&= 0. \qquad \text{(H.W.)}
\end{aligned}$$

4. Given $t_0 = 0$, $t_1 = 1$, $t_n = t_{n-1} + t_{n-2}$, $\forall n \geq 2$.

Characteristic equation:

$x^2 - x - 1 = 0$

Roots are: $(1+\sqrt{5})/2$, $(1-\sqrt{5})/2$

General solution: $t_n = d_1[(1+\sqrt{5})/2]^n + d_2[(1-\sqrt{5})/2]^n$

Determining constants:

$d_1 = 1/\sqrt{5}$,

$d_2 = -1/\sqrt{5}$,

The solution:

$t_n = (1/\sqrt{5})[(1+\sqrt{5})/2]^n + (-1/\sqrt{5})[(1-\sqrt{5})/2]^n$

Verification: HW

(1) Use backward substitutions.

(2) Use induction.

5. Given $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, $t_n = 5t_{n-1} - 8t_{n-2} + 4t_{n-3}$, $\forall n \geq 3$.

Characteristic equation:

$x^3 - 5x^2 + 8x - 4 = 0$

$(x-1)(x-2)^2 = 0$

Roots are: 1, 2(2)

General solution: $t_n = d_1 + d_2 2^n + d_3 n 2^n$

Determining constants:

$t_0 = 0$:	d_1	+	d_2			=	0,
$t_1 = 1$:	d_1	+	$2d_2$	+	$2d_3$	=	1,
$t_2 = 2$:	d_1	+	$4d_2$	+	$8d_3$	=	2.

On solving,

$d_1 = -2$, $d_2 = 2$, $d_3 = -(1/2)$

The solution:

$$\begin{aligned} t_n &= (-2) + 2*2^n - (1/2)n2^n \\ &= 2^{n+1} - n2^{n-1} - 2 \end{aligned}$$

Verification: HW

II. Solving IHLRECC:

Consider the following k^{th}-order IHLRE with CC:

$$C_0\mathbf{t_n} + C_1\mathbf{t_{n-1}} + C_2\mathbf{t_{n-2}} + \ldots + C_k\mathbf{t_{n-k}} = f(n) \ \ldots\ldots\ldots\ldots(*)$$

The corresponding HLRECC of (*) is given by

$$C_0\mathbf{t_n} + C_1\mathbf{t_{n-1}} + C_2\mathbf{t_{n-2}} + \ldots + C_k\mathbf{t_{n-k}} = 0 \ \ldots\ldots\ldots\ldots(**)$$

Theorem 7: Let p(n) be any particular solution of (*) and h(n) be the general solution of (**). Then the general solution of (*) is given by t_n = h(n) + p(n).

Method for Solving a k^{th}-order IHLRECC:

1. Compute the general solution h(n) for the corresponding HLRECC (**) using previous method.
2. Compute a particular solution p(n) for the given IHLRECC (*).
3. Form the general solution t_n = h(n) + p(n) for (*).
4. Determine the k constants using k initial conditions if given.
5. Verify the correctness of the solution.

Q: How do we compute a particular solution p(n) for (*)?

1. Educated “guess”.
2. From existing books/tables.
3. Use computer to analyze the behavior of t_n. Conjecture a solution and then verify using backward substitutions.

Some Suggestions for Possible p(n):

If f(n) =	**Then try: p(n) =**
$B_0 + B_1n + B_2n^2 + \ldots + B_mn^m$, B_i’s are constants	$E_0 + E_1n + E_2n^2 + \ldots + E_mn^m$, E_i’s are constants
Bs^n, B and s are constants	Es^n, E and s are constants
$B_0 + B_1n + B_2n^2 + \ldots + B_mn^m + Bs^n$, B_i’s and s are constants	$E_0 + E_1n + E_2n^2 + \ldots + E_mn^m + Es^n$, E_i’s and s are constants
$(B_0 + B_1n + B_2n^2 + \ldots + B_mn^m)s^n$, B_i’s and s are constants	$(E_0 + E_1n + E_2n^2 + \ldots + E_mn^m)s^n$, E_i’s and s are constants

Remark:

In order to verify that p(n) is indeed a (particular) solution of (*), one must verify the correctness of p(n) by substituting t_n = p(n) back into the given IHLRECC (*). In the process, the constant(s) in p(n) must also be determined.

Examples: Solving IHLRECC.

1. Given $t_0 = 1$, $t_n - 2t_{n-1} = 1$, $\forall n \geq 1$.

   Solving corresponding Homogeneous LRE:

   For $t_n - 2t_{n-1} = 0$, $x - 2 = 0$, $x = 2$.
   Hence, $h(n) = d2^n$.

   Computing p(n):

   Try $p(n) = E$ = constant and verify (by substituting p(n) back into the given IHLRE).
   $t_n - 2t_{n-1} = 1$: $E - 2E = 1$, $E = -1$.
   Hence, $p(n) = -1$.

   Computing general solution t_n:

   $t_n = d2^n - 1$.

   Determining constant in t_n:

   $t_0 = 1$: $d2^0 - 1 = 1$, $d = 2$.

   Hence, $t_n = 2^{n+1} - 1$.

   Verification: HW

2. Given $t_0 = 1$, $t_n - 2t_{n-1} = n$, $\forall n \geq 1$.

   From previous example, we know that $h(n) = d2^n$.

   Computing p(n):

   Since $f(n) = n$, try $p(n) = E_1 n + E_0$, E_i = constant.
   From $t_n - 2t_{n-1} = n$, we have

   $E_1 n + E_0 - 2[E_1(n - 1) + E_0] = n$,
   $E_1 n + E_0 - 2E_1 n + 2E_1 - 2E_0 = n$,
   $(E_1 n - 2E_1 n - n) + (E_0 + 2E_1 - 2E_0) = 0$

   Hence, we obtain

   $-E_1 - 1 = 0$,
   $2E_1 - E_0 = 0$.

   On solving, $E_1 = -1$, $E_0 = -2$.
   Hence, $p(n) = -n - 2$.

   Computing general solution t_n:

   $t_n = d2^n - n - 2$.

   Determining constant in t_n:

   $t_0 = 1$: $d2^0 - 0 - 2 = 1$, $d = 3$.

   Hence, $t_n = 3 * 2^n - n - 2$.

   Verification: HW

3. Given $t_1 = 8$, $t_n - t_{n-1} = n - 1$, $\forall n \geq 2$.

Solving corresponding Homogeneous LRE:

$t_n - t_{n-1} = 0$ implies that $x - 1 = 0$, $h(n) = d$ = constant.

Computing p(n):

Since $f(n) = n - 1$, try $p(n) = E_1n + E_0$, E_1 and E_0 are constant.

From $t_n - t_{n-1} = n - 1$, we have

$E_1n + E_0 - [E_1(n - 1) + E_0] = n - 1$,

$E_1n + E_0 - E_1n + E_1 - E_0 = n - 1$,

$E_1 = n - 1$, which is a contradiction since E_1 must be a constant.

Q: What is wrong?

Wrong guess of p(n); it doesn't work. Try again!

Motivated by Theorem 5, let's try $p(n) = (E_1n + E_0)*n = E_1n^2 + E_0n$ again.

Verification:

$E_1n^2 + E_0n - [E_1(n - 1)^2 + E_0(n - 1)] = n - 1$,

$E_1n^2 + E_0n - E_1n^2 + 2E_1n - E_1 - E_0n + E_0 = n - 1$,

$2E_1n - (E_1 - E_0) = n - 1$.

Hence, $E_1 = 1/2$, $E_0 = -1/2$.

$p(n) = n^2/2 - n/2$.

Computing general solution t_n:

$t_n = d + n^2/2 - n/2$

Determining constant in t_n:

$t_1 = 8$: $1/2 - 1/2 + d = 8$, $d = 8$.

Hence, $t_n = n^2/2 - n/2 + 8$.

Verification: HW

Some Questions:

1. What happened to our selection of p(n) in the above example? Why did it fail?
2. What if it fails again after multiplied by n?
3. Can we determine p(n) without guessing?

A Useful Theorem for Determining p(n):

Theorem 8: Given k^{th}-order IHLRECC $C_0\mathbf{t_n} + C_1\mathbf{t_{n-1}} + C_2\mathbf{t_{n-2}} + \ldots + C_k\mathbf{t_{n-k}} = f(n)$, where $f(n) = (B_0 + B_1n + B_2n^2 + \ldots + B_qn^q)*Bs^n$, B_i's and s are constants. If s is a root of multiplicity m of the characteristic equation of the corresponding HLRECC $C_0\mathbf{t_n} + C_1\mathbf{t_{n-1}} + C_2\mathbf{t_{n-2}} + \ldots + C_k\mathbf{t_{n-k}} = 0$, then $p(n) = (E_0 + E_1n + E_2n^2 + \ldots + E_qn^q)*n^ms^n$ is a solution of (*) for some constants E_i's.

4. Consider the previous recurrence equation

   $t_1 = 8$, $t_n - t_{n-1} = n - 1$, $\forall n \geq 2$.

   The characteristic equation of the corresponding HLRECC is $x - 1 = 0$ with root $x = 1$. Since $f(n) = n - 1 = (n - 1)1^n$, 1 is a root of multiplicity 1 of the characteristic equation of the corresponding HLRECC. Hence, from Theorem 8, $p(n) = (E_1n + E_0)*n*1^n = E_1n^2 + E_0n$. On solving, as in previous example, $p(n) = n^2/2 - n/2$.

Another Very Useful Theorem:

Theorem 9: $C_0\mathbf{t_n} + C_1\mathbf{t_{n-1}} + C_2\mathbf{t_{n-2}} + \ldots + C_k\mathbf{t_{n-k}} = b_1^nP_1(n) + b_2^nP_2(n) + \ldots$, where b_i = constant, $\mathbf{b_i \neq b_j}$, $\forall\ i \neq j$, and $P_i(n)$ is a polynomial function of degree e_i, $\forall i$. The general solution $\mathbf{t_n}$ is given by taking a linear combination of the linearly independent solutions generated by the roots of the following characteristic equation:

$(C_0x^k + C_1x^{k-1} + \ldots + C_k)(x - b_1)^{e1+1}(x - b_2)^{e2+1} \ldots = 0.$

5. Consider the previous recurrence equation $t_1 = 8$, $t_n - t_{n-1} = n - 1$, $\forall n \geq 2$.

   Applying the above theorem, $f(n) = n - 1= (n - 1)*1^n$,

   $b_1 = 1$, $P_1(n) = n - 1$, $e_1 = 1$.

   Characteristic equation of IHLRECC:

   $(x - 1)(x - 1)^2 = 0$, $x = 1(3)$.

   Hence, $t_n = d_1 + d_2n + d_3n^2$.

   Observe that there are two extraneous constants in the general solution that we must remove.

   From $t_1 = 8$, we have $t_2 = t_1 + 2 - 1= 9$ and $t_3 = t_2 + 3 - 1= 11$.

   From $t_1 = 8$, $d_1 + d_2 + d_3 = 8$.

   From $t_2 = 9$, $d_1 + 2d_2 + 4d_3 = 9$.

   From $t_3 = 11$, $d_1 + 3d_2 + 9d_3 = 11$.

   On solving, $d_1 = 8$, $d_2 = -1/2$, and $d_3 = 1/2$.

   Hence, $t_n = n^2/2 - n/2 + 8$.

6. Given $t_0 = 4$, $t_n - 2t_{n-1} = 3^n$, $\forall n \geq 1$.

Applying the above theorem, $f(n) = 3^n$, we have

$b_1 = 3$, $P_1(n) = 1$, $e_1 = 0$.

Characteristic equation of IHLRECC:

$(x - 2)(x - 3) = 0$, $x = 2(1), 3(1)$.

Hence, $t_n = d_1 2^n + d_2 3^n$

Observe that there is again an extraneous constant in the general solution that we must remove.

$t_0 = 4$, $t_n - 2t_{n-1} = 3^n$ implies $t_1 - 2t_0 = 3^1$, $t_1 = 11$.

$t_0 = 4$: $d_1 + d_2 = 4$;

$t_1 = 11$: $2d_1 + 3d_2 = 11$.

On solving, $d_1 = 1$, $d_2 = 3$.

Hence, $t_n = 2^n + 3^{n+1}$. (Verify!)

7. Given $t_0 = 0$, $t_n - 2t_{n-1} = n + 2^n$, $\forall n \geq 1$.

Applying the above theorem, $f(n) = n + 2^n$, we have

$b_1 = 1$, $P_1(n) = n$, $e_1 = 1$,

$b_2 = 2$, $P_2(n) = 1$, $e_2 = 0$.

Characteristic equation of IHLRECC:

$(x - 1)^2(x - 2)^2 = 0$,

$x = 1(2), 2(2)$.

Hence, $t_n = d_1 + d_2 n + d_3 2^n + d_4 n 2^n$

Observe that there are 4 constants in t_n. Hence, we need 4 initial conditions for computing the constants. Since the given equation is a 1st-order equation, there are three extraneous constants in t_n. To order to solve for the 4 constants, we need to generate 3 more initial conditions from the given initial condition and the recurrence equation as follow:

$$
\begin{array}{lrcrcrcrcl}
t_0 = 0, & d_1 & & & + & d_3 & & & = & 0, \\
t_1 = 2t_0 + 1 + 2^1 = 3, & d_1 & + & d_2 & + & 2d_3 & + & 2d_4 & = & 3, \\
t_2 = 2t_1 + 2 + 2^2 = 12, & d_1 & + & 2d_2 & + & 4d_3 & + & 8d_4 & = & 12, \\
t_3 = 2t_2 + 3 + 2^3 = 35, & d_1 & + & 3d_2 & + & 8d_3 & + & 24d_4 & = & 35.
\end{array}
$$

On solving, $d_1 = -2$, $d_2 = -1$, $d_3 = 2$, $d_4 = 1$. (Verify!)

Hence, $t_n = n2^n + 2^{n+1} - n - 2$. (Verify!)

8. Given $t_n - 2t_{n-1} = n3^n + 5*3^n$.

Observe that $f(n) = n3^n + 5*3^n = 3^n(n + 5)$.

Applying the above theorem, we have

$b_1 = 3$, $P_1(n) = n + 5$, $e_1 = 1$.

Characteristic equation of IHLRECC:

$(x - 2)(x - 3)^2 = 0$.

$x = 2, 3(2)$.

Hence, $t_n = d_1 2^n + d_2 3^n + d_3 n3^n$.

Since there is no initial condition given, you will not be able to determine all three of the constants in a_n. However, you must eliminate the two extraneous constants in t_n. By substituting $t_n = d_1 2^n+d_2 3^n+d_3 n3^n$ back into the recurrence equation, we have

$$d_1 2^n + d_2 3^n + d_3 n3^n - 2[d_1 2^{n-1} + d_2 3^{n-1} + d_3(n-1)3^{n-1}] = n3^n + 5*3^n,$$

$$d_1 2^n + d_2 3^n + d_3 n3^n - d_1 2^n - \frac{2}{3}d_2 3^n - \frac{2}{3}d_3 n3^n + \frac{2}{3}d_3 3^n = n3^n + 5*3^n.$$

Hence,

$$d_3 n3^n - \frac{2}{3}d_3 n3^n = n3^n,$$

$$d_2 3^n - \frac{2}{3}d_2 3^n + \frac{2}{3}d_3 3^n = 5*3^n.$$

On solving, $d_2 = 13$, $d_3 = 1$.

Hence, $t_n = d_1 2^n + 13*3^n + n3^n$. (Verify!)

Divide-and-Conquer Algorithms and Recurrence Equations
General recurrence equation from divide-and-conquer:

$$T(n) = \text{constant}, \quad \text{for } n \le n_0,$$
$$T(n) = aT(n/c) + bn^k, \quad \text{for } n = c^k > n_0.$$

It can be solved by converting it into a LRE using the following ***domain transformation*** method.

Substituting $n = c^m$ into the general recurrence to get

$$T(c^m) = aT(c^{m-1}) + bc^{km}$$

Let $t_m = T(c^m)$ to obtain the LRE

$$t_m = t_{m-1} + bc^{km}$$

Remark: The same approach can be used to solve more general REs from DAC algorithms.

Examples: Solving REs using domain transformation.

1. $T(2) = 1,$
   $T(n) = 2T(n/2) + 2, \quad \text{for } n = 2^m > 2.$

   Substituting $n = 2^m$ into the general recurrence to get
   $T(2^m) = 2T(2^{m-1}) + 2$

   Let $t_m = T(2^m)$ to obtain the LRE
   $t_m = 2t_{m-1} + 2,$
   $t_1 = T(2^1) = 1.$

   Characteristic equation: $(x-2)(x-1) = 0$
   Roots: 1, 2
   Solution: $t_m = d_1 + d_2 2^m$

   $t_1 = 1$ implies $t_2 = 2t_1 + 2 = 4.$
   $t_1 = 1$: $d_1 + 2d_2 = 1,$
   $t_2 = 4$: $d_1 + 4d_2 = 4.$
   On solving, $d_1 = -2$, $d_2 = 3/2.$
   Hence, $t_m = (3/2)2^m - 2.$

   Recall that $t_m = T(2^m) = T(n)$ and $n = 2^m$, we have
   $T(n) = (3/2)n - 2.$

2. $T(1) = 0,$
$T(n) = 2T(n/2) + n - 1, \quad \text{for } n = 2^m > 1.$

Substituting $n = 2^m$ into the general recurrence to get
$T(2^m) = 2T(2^{m-1}) + 2^m - 1$

Let $t_m = T(2^m)$ to obtain the LRE
$t_m = 2t_{m-1} + 2^m - 1,$
$t_0 = T(2^0) = 0.$

Characteristic equation: $(x-2)(x-2)(x-1) = 0$
Roots: 1, 2(2)
Solution: $t_m = d_1 + d_2 2^m + d_3 m2^m$

$t_0 = 0$ implies $t_1 = 2t_0 + 2^1 - 1 = 1,$
$t_1 = 1$ implies $t_2 = 2t_1 + 2^2 - 1 = 5.$
$t_0 = 0$: $d_1 + d_2 = 0,$
$t_1 = 1$: $d_1 + 2d_2 + 2d_3 = 1,$
$t_2 = 5$: $d_1 + 4d_2 + 8d_3 = 5.$

On solving, $d_1 = 1$, $d_2 = -1$, $d_3 = 1$.

Hence, $t_m = m2^m - 2^m + 1.$

Recall that $t_m = T(2^m) = T(n)$ and $n = 2^m$, we have

$T(n) = n\lg n - n + 1.$

3. Consider the following recurrence equation:

$$T(1) = 0,$$
$$T(2) = 1,$$
$$T(n) = 5T(\frac{n}{2}) - 6T(\frac{n}{2^2}) + n, n = 2^k > 1.$$

Substituting $n = 2^k$ into the given recurrence equation to get

$T(2^k) = 5T(2^{k-1}) - 6T(2^{k-2}) + 2^k.$

Let $t_k = T(2^k),$ we have

$t_k = 5t_{k-1} - 6t_{k-2} + 2^k,$

$t_0 = 0,\ t_1 = 1.$

Solving the corresponding LRECC:

$t_k - 5t_{k-1} + 6t_{k-2} = 2^k,$

$t_0 = 0,\ t_1 = 1.$

Characteristic equation for the inhomogeneous equation:

$(x^2 - 5x + 6)(x - 2) = 0,$

$(x - 3)(x - 2)^2 = 0,$

$x = 2(2),\ 3.$

$t_k = d_1 2^k + d_2 k 2^k + d_3 3^k$

$t_0 = 0,\ t_1 = 1, t_2 = 5t_1 - 6t_0 + 2^2 = 9.$

$t_0 = 0 : d_1 + d_3 = 0,$

$t_1 = 1 : 2d_1 + 2d_2 + 3d_3 = 1,$

$t_2 = 9 : 4d_1 + 8d_2 + 9d_3 = 9.$

On solving, $\mathrm{d}_1 = -5, \mathrm{d}_2 = -2, \mathrm{d}_3 = 5.$

$t_k = 5 * 3^k - 2k2^k - 5 * 2^k,$

$T(n) = 5n^{\lg 3} - 2n\lg n - 5n.$