Numerical analysis: Assignment 8

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Exercise 1

The integral $\int_a^b f(x)$ can be approximated using a Lagrange polynomial of degree 2 as follows:

$$\int_a^b f(x) \approx \int_{a=x_0}^{b=x3} P_2 \tag{1}$$

The polynomial P_2 is given by:

$$P(x) = \sum_{j=0}^{3} f(x_j) l_j(x) \tag{2}$$

with $l_j(x)$ defined as:

$$l_j(x) \prod_{j=0, m\neq j}^{n} \frac{x - x_m}{x_j - x_m} \tag{3}$$

Then by inserting Eq. 2 and Eq. 3 into Eq. 1:

$$\int_a^b f(x) \approx \int_{a=x_0}^{b=x3} P = \int_{a=x_0}^{b=x_3} \sum_{j=0}^{3} f(x_j) \prod_{j=0, m\neq j}^{n} \frac{x - x_m}{x_j - x_m} \tag{4}$$

By rearranging Eq. 4 we get :

$$\int_{a=x_1}^{b=x3} P = \int_{x_0}^{x_3} f_1 \frac{x - x_2}{x_1 - x_2} \frac{x - x_3}{x_1 - x_3} + f_2 \frac{x - x_1}{x_2 - x_1} \frac{x - x_3}{x_2 - x_3} + f_3 \frac{x - x_1}{x_3 - x_1} \frac{x - x_2}{x_3 - x_2} \tag{5}$$

By using the fact that the points are equidistant, Eq. 5 becomes:

$$= \int_{x_1 - h}^{x_1 + 2h} f_1 \frac{x^2 - x_2 x - x_3 x + x_2 x_3}{2h^2} + f_2 \frac{x^2 - x_3 x - x_1 x + x_1 x_3}{-h^2} + f_3 \frac{x^2 - x_2 x - x_1 x + x_1 x_2}{2h^2} = \tag{6}$$

$$= \frac{1}{h^2} \int_{x_1 - h}^{x_1 + 2h} f_1 \frac{x^2 - x_2 x - x_3 x + x_2 x_3}{2} - f_2 (x^2 - x_3 x - x_1 x + x_1 x_3) + f_3 \frac{x^2 - x_2 x - x_1 x + x_1 x_2}{2} = \tag{7}$$

By splitting the integrals:

$$= \frac{1}{h^2}\left(\frac{f_1}{2}\int_{x_1-h}^{x_1+2h} x^2 - x_2x - x_3x + x_2x_3 - f_2\int_{x_1-h}^{x_1+2h}(x^2 - x_3x - x_1x + x_1x_3) + \frac{f_3}{2}\int_{x_1-h}^{x_1+2h} x^2 - x_2x - x_1x + x_1x_2\right) = \tag{8}$$

Solving the three integrals gives the following relation :

$$\int_a^b f(x) \approx \frac{4}{3}h(2f_1 - f_2 + 2f_3) + \mathcal{O}(h^5) \tag{9}$$

The code has been implemented as required (i.e. dividing the interval (a,b) into 100 subpieces) in *exercise1.py* and outputs:

Numerical solution: 0.9460279856300005
Analytical solution: 0.946083070367183
Absolute error: 5.50847371825203e-5

The console output of my code also verifies the order $\mathcal{O}(n^5)$ of accuracy.

Exercise 2

The implementation can be found in *code/exercise2.py*. The output (for $j = 5$) is :

Iterations: 5
Numerical solution: 4.10879206104986
Analytical solution: 4.670774270471606
Absolute error: 0.5619822094217461

Bonus exercise

By the theorem of uniqueness of the interpolating polynomial:

$$\begin{aligned} c(x) &= q_1(x) + h_1(x) \\ c(x) &= q_2(x) + h_2(x) \end{aligned} \tag{10}$$

with:

$$\begin{aligned} h_1(x) &= \lambda(x - x_0)(x - x_1)(x - x_2) \\ \lambda &= \frac{f_3 - q_1(x_3)}{6h^3} \\ h_2(x) &= \mu(x - x_1)(x - x_2)(x - x_3) \\ \mu &= \frac{f_0 - q_2(x_0)}{-6h^3} \end{aligned} \tag{11}$$

We need to show that:

$$\int_a^{a+3h} c(x) = \frac{1}{2}\int_a^{a+3h} q_1(x) + \frac{1}{2}\int_a^{a+3h} q_2(x) \tag{12}$$

By inserting Eq. 11 it follows:

$$\int_a^{a+3h} 2c(x) = \int_a^{a+3h} q_1(x) + \int_a^{a+3h} q_2(x) \implies \int_a^{a+3h} h_1(x) + h_2(x)dx = 0 \tag{13}$$

Therefore it suffices to show Eq. 13 is equal to zero. However, I got lost into calculus.