Università
della
Svizzera
italiana

Faculty
of Informatics

Numerical Algorithms

Fall 2020

Assignment 2

September 17, 2020

Let n be an odd integer and consider the $n \times n$ matrix A with entries

- 6 on the main diagonal, i.e., $a_{i,i} = 6$, $i = 1, \dots, n$;
- -2 on the super- and subdiagonal, i.e., $a_{i,i+1} = a_{i+1,i} = -2$, $i = 1, \dots, n-1$;
- 1 on the antidiagonal, i.e. $a_{i,n+1-i} = 1$, $i = 1, \dots, (n-1)/2, (n+3)/2, \dots, n$.

For example, for $n = 7$,

$$A = \begin{pmatrix} 6 & -2 & 0 & 0 & 0 & 0 & 1 \\ -2 & 6 & -2 & 0 & 0 & 1 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 6 & -2 & 0 & 0 \\ 0 & 0 & 1 & -2 & 6 & -2 & 0 \\ 0 & 1 & 0 & 0 & -2 & 6 & -2 \\ 1 & 0 & 0 & 0 & 0 & -2 & 6 \end{pmatrix}.$$

Further let b be the vector of size n with entries

- 5 at the beginning and end, i.e., $b_1 = b_n = 5$;
- 2 in the middle, i.e., $b_{(n+1)/2} = 2$;
- 3 everywhere else, i.e. $b_i = 3$, $i = 2, \dots, (n-1)/2, (n+3)/2, \dots, n-1$.

For example, for $n = 7$,

$$b = (5, 3, 3, 2, 3, 3, 5).$$

Exercise 1 [5 points]

Solve the linear system $Ax = b$ with A and b as described above using the Gauss–Seidel Method with initial guess $x^{(0)} = (0, \dots, 0)$ for $n = 99999$. How many iterations does it take until the difference between $x^{(k)}$ and the next $x^{(k+1)}$ is below machine precision, i.e., what is the smallest k such that

$$\max_{i=1,\dots,n} \left| x_i^{(k+1)} - x_i^{(k)} \right| < 10^{-16},$$

and what is $x^{(k)}$ for this k?

Hand in the answer to this question and your commented source code. Make sure that your program *does not* store the matrix A *explicitly* as an $n \times n$ array and that each iteration is performed with $O(n)$ operations.

Exercise 2 [5 points]

Show that the matrix A above is symmetric positive-definite for any odd integer n, i.e., $A^T = A$ and $x^T A x > 0$ for all $x \neq 0$.

Hand in your proof, either on paper, or electronically (scanned or LaTeX'ed).

Solutions must be returned online or in class on September 24, 2020