Università
della
Svizzera
italiana

Faculty
of Informatics

Numerical Algorithms

Fall 2020

Assignment 8

October 29, 2020

Exercise 1 [5 points]

Let $h = (b-a)/4$ and $x_i = a+ih$ for $i = 0, \dots, 4$. Work out the weights c_1, c_2, and c_3 of the open Newton–Cotes Rule that approximates the integral of f over $[a, b]$ as

$$\int_a^b f(x)dx \approx h(c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)) = I$$

and is based on interpolating f at x_1, x_2, and x_3 with a quadratic polynomial, that is,

$$I = \int_a^b g(x)dx,$$

where $g \in \Pi_2$ and $g(x_i) = f(x_i)$ for $i = 1, 2, 3$.

What is the degree of precision for this rule? You can check this numerically.

Turn this rule into a composite rule and compute an approximation of $\int_0^1 \frac{\sin x}{x} dx$ by splitting the integration interval into $m = 100$ pieces of equal length.

Exercise 2 [5 points] — (only for students of the 6 ECTS course)

Let us consider a variant of Romberg integration that is based on the Composite Midpoint Rule instead of the Composite Trapezoid Rule. That is, for any $j \in \mathbb{N}$, we let $h_j = (b-a)/2^{j-1}$ and

$$R_{j,1} = h_j \sum_{i=1}^{2^{j-1}} f(a + (i - 1/2)h_j).$$

Note that we cannot use $R_{j-1,1}$ to simplify the computation of $R_{j,1}$ in this case. Moreover, we use Richardson extrapolation, as in the original Romberg integration scheme, to get the approximations $R_{j,k}$ for $2 \leq k \leq j$. You can assume again that the approximations $R_{j,k}$ have an approximation order of $O(h^{2k})$.

Use this modified Romberg integration scheme to approximate $\int_1^2 \exp(x)dx$ and compute all values $R_{j,k}$ for $1 \leq k \leq j \leq 5$.

Hand in your code and the output.

Bonus Exercise [3 points]

Let $c \in \Pi_3$ be a cubic polynomial and $q_1, q_2 \in \Pi_2$ be two quadratic polynomials that interpolate f at a, $a+h$, $a+2h$, and $a+3h$ for some a and $h > 0$, like this:

$$c(a + ih) = f(a + ih), \quad i = 0, 1, 2, 3,$$

$$q_1(a + ih) = f(a + ih), \quad i = 0, 1, 2.$$

$$q_2(a+ih) = f(a+ih), \quad i = 1, 2, 3.$$

Show that

$$\int_a^{a+3h} c(x)dx = \frac{1}{2}\int_a^{a+3h} q_1(x)dx + \frac{1}{2}\int_a^{a+3h} q_2(x)dx.$$

Hand in your proof.

Solutions must be returned online or in class on November 5, 2020