

Numerical Algorithms: Assignment 13

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$$\Phi_E = \frac{Q}{\varepsilon_0} \quad (1)$$

Exercise 1

The function is odd (i.e. $f(-x) = -f(x)$) therefore all the cosine terms a_k are zero. The Fourier of expansion of $f(x)$ is then:

$$f(x) = \sum_{k=1}^{\infty} b_k \sin(kx) \quad (2)$$

To compute the coefficients b_k we need to solve:

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx \quad \text{for } k \geq 1 \quad (3)$$

To solve this integral I have used an "iterative" generalisation of integral by parts:

$$\int u v dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots \quad (4)$$

with u', u'', \dots and v_1, v_2, \dots are successive derivatives of u and integrals of v . Choosing $u = f(x)$ and $v = \sin(kx)$ yields:

$$b_k = \frac{1}{\pi} \left[x^3 \left(\frac{-\cos(kx)}{k} \right) - 3x^2 \left(\frac{-\sin(kx)}{k^2} \right) + 6x \left(\frac{\cos(kx)}{k^3} \right) - 6 \left(\frac{\sin(kx)}{k^4} \right) \right]_{-\pi}^{\pi} = \quad (5)$$

We notice that $\sin(k\pi) = \sin(-k\pi) = 0$.

$$\begin{aligned} &= \frac{1}{\pi} \left[x^3 \left(\frac{-\cos(kx)}{k} \right) + 6x \left(\frac{\cos(kx)}{k^3} \right) \right]_{-\pi}^{\pi} = \\ &= \frac{1}{\pi} \left(\pi^3 \left(\frac{-\cos(k\pi)}{k} \right) - (-\pi)^3 \left(\frac{-\cos(k(-\pi))}{k} \right) + 6\pi \left(\frac{\cos(\pi x)}{k^3} \right) - 6(-\pi) \left(\frac{\cos(-\pi x)}{k^3} \right) \right) = \end{aligned} \quad (6)$$

By rearranging the equation and recalling that cosine is an even function (i.e. $f(x) = f(-x)$) it follows

$$= \frac{1}{\pi} \left(-2\pi^3 \frac{\cos(k\pi)}{k} + 12\pi \left(\frac{\cos(\pi x)}{k^3} \right) \right) = 2\cos(k\pi) \left(-\frac{\pi^2}{k} + \frac{6}{k^3} \right) = \quad (7)$$

We notice that $\cos(k\pi) = -1^k$

$$= 2(-1)^k \left(\frac{6}{k^3} - \frac{\pi^2}{k} \right) \quad (8)$$

Therefore the Fourier expansion of $f(x)$ is

$$f(x) = \sum_{k=1}^{\infty} 2(-1)^k \left(\frac{6}{k^3} - \frac{\pi^2}{k} \right) \sin(kx) \quad (9)$$

Exercise 2

The code has been implemented in *exercise2.py* using Python 3. It was not specified what the code should return as output. I thought it was interesting (and easier to visualise) to plot the real and imaginary components of the Fourier coefficients (Fig.1). The imaginary part is close to zero, but not zero. The same results are reported in Lst. 1

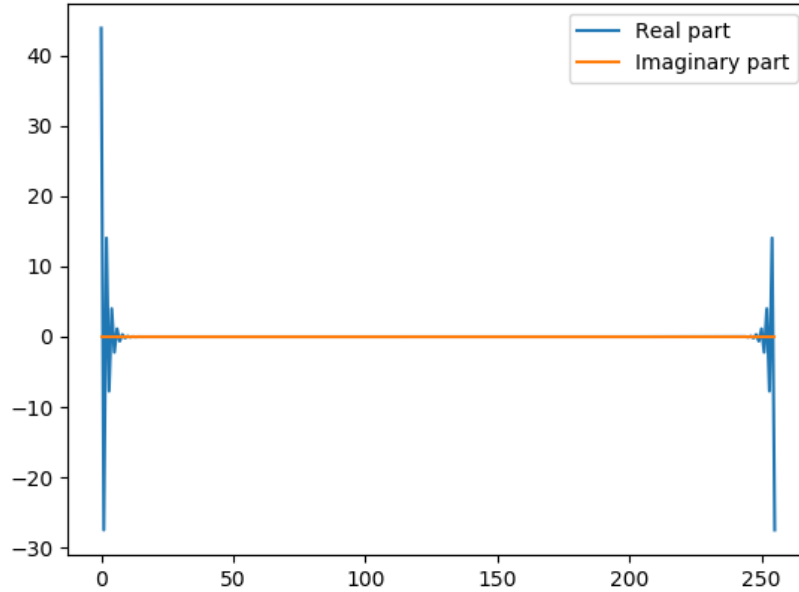


Figure 1: The real and imaginary components

Listing 1: Results in vector form

```
[ 4.39487644e+01+0.00000000e+00j -2.75285556e+01-1.69309011e-15j
 1.40598970e+01+9.71445147e-16j -7.75257338e+00-9.36750677e-16j
 4.00104568e+00+2.08166817e-16j -2.21830159e+00+2.63677968e-16j
 1.12639365e+00+3.15719673e-16j -6.42237697e-01+1.73472348e-18j
 3.11321361e-01+1.38777878e-17j -1.90623565e-01+2.02095285e-16j
```

8.19431099e-02-1.30971622e-16j	-5.99608818e-02+1.79543880e-16j
1.83897270e-02-1.39862080e-17j	-2.13634396e-02-2.25514052e-17j
1.45856134e-03-3.38000027e-16j	-9.42091295e-03+7.60520390e-16j
-2.56092214e-03-8.13151629e-20j	-5.34887253e-03+2.88363897e-16j
-3.13347079e-03-1.33289105e-16j	-3.70108669e-03+2.01986865e-16j
-2.87526258e-03-2.50450702e-17j	-2.86687432e-03+1.30619257e-16j
-2.48310609e-03-5.80048162e-17j	-2.34984465e-03+1.20996962e-16j
-2.12488230e-03-1.49077799e-18j	-1.98412550e-03+2.77189838e-16j
-1.82834327e-03-8.05833265e-17j	-1.70607737e-03+1.54444599e-16j
-1.58781412e-03-5.91330641e-17j	-1.48630091e-03+4.79081835e-17j
-1.39223577e-03-4.80706444e-16j	-1.30848070e-03+6.44470312e-16j
-1.23173460e-03-5.42101086e-20j	-1.16225162e-03-1.03036476e-15j
-1.09860957e-03-4.23092957e-18j	-1.04046339e-03+7.89468588e-17j
-9.87045902e-04-8.45000068e-18j	-9.37943036e-04-5.23805175e-17j
-8.92660153e-04-2.10064171e-16j	-8.50836904e-04+2.09738910e-16j
-8.12118981e-04-7.96888597e-18j	-7.76215219e-04+2.32073475e-16j
-7.42857690e-04+6.74509277e-17j	-7.11815092e-04+2.18548053e-16j
-6.82878945e-04+1.58022467e-17j	-6.55865332e-04-3.17413739e-16j
-6.30609199e-04+5.03747434e-17j	-6.06963330e-04+7.04799175e-17j
-5.84795422e-04+6.09863722e-20j	-5.63986664e-04+3.39416266e-16j
-5.44429930e-04+2.94482863e-16j	-5.26028532e-04-5.56805578e-17j
-5.08694970e-04+1.31866089e-17j	-4.92349930e-04+3.72254040e-17j
-4.76921374e-04-3.02966745e-17j	-4.62343758e-04-2.72043266e-16j
-4.48557340e-04-2.37694386e-17j	-4.35507581e-04-3.28552222e-16j
-4.23144607e-04-1.07183549e-16j	-4.11422741e-04+1.89715051e-16j
-4.00300088e-04-3.93543366e-16j	-3.89738167e-04-1.17886234e-16j
-3.79701582e-04-8.40355469e-16j	-3.70157735e-04-4.95560464e-16j
-3.61076559e-04+0.00000000e+00j	-3.52430295e-04+7.25418445e-16j
-3.44193280e-04+4.34833046e-16j	-3.36341759e-04+3.31111955e-16j
-3.28853725e-04-2.69995987e-17j	-3.21708761e-04+1.65246811e-16j
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-3.02149298e-04-9.20386000e-18j	-2.96199875e-04+3.40032906e-17j
-2.90511055e-04-1.26401033e-16j	-2.85069562e-04+2.00238589e-16j
-2.79863002e-04-8.53809211e-18j	-2.74879791e-04-4.22161221e-18j
-2.70109098e-04+1.74624312e-17j	-2.65540786e-04+2.34079249e-16j
-2.61165361e-04+3.38813179e-20j	-2.56973928e-04-2.39900059e-16j
-2.52958143e-04-2.00983978e-17j	-2.49110177e-04+1.30565047e-16j
-2.45422682e-04-2.96935870e-17j	-2.41888753e-04+2.55505794e-16j
-2.38501901e-04-2.28224557e-17j	-2.35256024e-04+2.24375640e-16j
-2.32145383e-04-5.74627151e-18j	-2.29164576e-04+1.53401055e-16j
-2.26308518e-04+1.71032893e-17j	-2.23572419e-04+5.08219768e-17j
-2.20951768e-04-5.72289340e-17j	-2.18442314e-04+2.24768663e-16j
-2.16040051e-04-7.56314025e-16j	-2.13741203e-04+1.64460573e-15j
-2.11542211e-04+5.42101086e-20j	-2.09439720e-04+1.83704506e-17j
-2.07430567e-04-3.20921302e-16j	-2.05511771e-04+1.05719876e-16j
-2.03680524e-04+2.73693286e-17j	-2.01934178e-04+1.40471944e-16j
-2.00270240e-04-1.12079400e-16j	-1.98686366e-04+4.05030827e-16j
-1.97180345e-04-6.50521303e-19j	-1.95750103e-04-1.76182853e-18j

-1.94393688e-04-1.00126071e-16j	-1.93109270e-04-9.28890211e-17j
-1.91895131e-04+3.88144378e-17j	-1.90749664e-04-1.47993597e-17j
-1.89671366e-04+5.05780313e-17j	-1.88658833e-04+9.36256010e-16j
-1.87710759e-04+1.35525272e-19j	-1.86825929e-04-2.59740959e-16j
-1.86003217e-04-8.07866144e-17j	-1.85241585e-04-2.79724161e-17j
-1.84540076e-04-1.62630326e-17j	-1.83897814e-04+1.66533454e-16j
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-1.82318923e-04-5.20417043e-17j	-1.81906433e-04-2.00360561e-16j
-1.81549947e-04-8.32667268e-17j	-1.81249032e-04+1.73472348e-16j
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-1.80676403e-04+3.60822483e-16j	-1.80594796e-04+4.16333634e-16j
-1.80567604e-04+0.00000000e+00j	-1.80594796e-04+1.94289029e-16j
-1.80676403e-04-1.38777878e-16j	-1.80812524e-04+1.73472348e-16j
-1.81003324e-04-1.38777878e-17j	-1.81249032e-04-6.93889390e-17j
-1.81549947e-04+3.81639165e-17j	-1.81906433e-04+2.09901541e-16j
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-1.83314004e-04-1.03216047e-16j	-1.83897814e-04+8.58688121e-17j
-1.84540076e-04-6.25584654e-17j	-1.85241585e-04+1.05818132e-16j
-1.86003217e-04+1.00126071e-16j	-1.86825929e-04+5.32404253e-16j
-1.87710759e-04+2.71050543e-20j	-1.88658833e-04+6.44415890e-16j
-1.89671366e-04-1.36975392e-16j	-1.90749664e-04+1.87892236e-16j
-1.91895131e-04-3.93565389e-17j	-1.93109270e-04+1.82877801e-16j
-1.94393688e-04-4.84638371e-17j	-1.95750103e-04+2.57823277e-16j
-1.97180345e-04+2.43945489e-19j	-1.98686366e-04+2.92328011e-17j
-2.00270240e-04-2.30121911e-17j	-2.01934178e-04+1.88813808e-16j
-2.03680524e-04-1.36328259e-16j	-2.05511771e-04-6.02951933e-17j
-2.07430567e-04-7.25627715e-16j	-2.09439720e-04+5.24557552e-16j
-2.11542211e-04+5.42101086e-20j	-2.13741203e-04+2.60269508e-16j
-2.16040051e-04-2.08386199e-16j	-2.18442314e-04+2.03467478e-16j
-2.20951768e-04+1.74624312e-17j	-2.23572419e-04+1.45269539e-16j
-2.26308518e-04-1.39753660e-16j	-2.29164576e-04+3.61906685e-16j
-2.32145383e-04+5.96311195e-19j	-2.35256024e-04-2.00468982e-16j
-2.38501901e-04-1.48007149e-16j	-2.41888753e-04+1.19451974e-16j
-2.45422682e-04-6.54587062e-17j	-2.49110177e-04+3.13402190e-16j
-2.52958143e-04-3.51457687e-16j	-2.56973928e-04+9.70665876e-16j
-2.61165361e-04-6.77626358e-21j	-2.65540786e-04+2.85151948e-16j
-2.70109098e-04-2.11676922e-16j	-2.74879791e-04-4.71018081e-17j
-2.79863002e-04-3.09675246e-17j	-2.85069562e-04-1.93350517e-16j
-2.90511055e-04-1.08481204e-16j	-2.96199875e-04+2.02654327e-16j
-3.02149298e-04+2.95275685e-18j	-3.08373555e-04-2.94213506e-16j
-3.14887911e-04+7.94279735e-17j	-3.21708761e-04+2.12740795e-16j
-3.28853725e-04-1.89323722e-16j	-3.36341759e-04+3.95441990e-16j
-3.44193280e-04-7.69467917e-16j	-3.52430295e-04-5.59151409e-16j
-3.61076559e-04+0.00000000e+00j	-3.70157735e-04-1.11399650e-15j
-3.79701582e-04-4.90344197e-16j	-3.89738167e-04+1.58327399e-17j
-4.00300088e-04-1.11778279e-16j	-4.11422741e-04+2.90422187e-17j
-4.23144607e-04+5.09151505e-17j	-4.35507581e-04-1.27007508e-16j
-4.48557340e-04+1.61427539e-17j	-4.62343758e-04+2.17531613e-16j

$-4.76921374e-04+1.01745598e-17j$	$-4.92349930e-04+1.58849171e-16j$
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$-5.44429930e-04-2.33869185e-16j$	$-5.63986664e-04+6.95420826e-16j$
$-5.84795422e-04+2.03287907e-20j$	$-6.06963330e-04+3.21797981e-16j$
$-6.30609199e-04+1.01115405e-16j$	$-6.55865332e-04-4.94667241e-17j$
$-6.82878945e-04-7.38612730e-18j$	$-7.11815092e-04-1.80424794e-16j$
$-7.42857690e-04+9.45966395e-17j$	$-7.76215219e-04+1.60082451e-16j$
$-8.12118981e-04-6.88468380e-18j$	$-8.50836904e-04-2.89820793e-16j$
$-8.92660153e-04-8.00412254e-17j$	$-9.37943036e-04+2.99809006e-16j$
$-9.87045902e-04-2.48654992e-17j$	$-1.04046339e-03+1.20685254e-16j$
$-1.09860957e-03-8.12909378e-16j$	$-1.16225162e-03+8.50102383e-16j$
$-1.23173460e-03-5.42101086e-20j$	$-1.30848070e-03+6.40702498e-16j$
$-1.39223577e-03+4.22516128e-16j$	$-1.48630091e-03-3.60378638e-16j$
$-1.58781412e-03+1.91293921e-17j$	$-1.70607737e-03-6.68275114e-17j$
$-1.82834327e-03+1.14952535e-16j$	$-1.98412550e-03+1.68241072e-16j$
$-2.12488230e-03-5.85469173e-18j$	$-2.34984465e-03-3.03928974e-16j$
$-2.48310609e-03+8.70072243e-17j$	$-2.86687432e-03-1.47966491e-16j$
$-2.87526258e-03+4.25007252e-17j$	$-3.70108669e-03+2.06161043e-16j$
$-3.13347079e-03+3.86518074e-17j$	$-5.34887253e-03+9.07199392e-16j$
$-2.56092214e-03-1.89735380e-19j$	$-9.42091295e-03+1.74807272e-16j$
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$1.83897270e-02+2.45029691e-17j$	$-5.99608818e-02-6.59194921e-17j$
$8.19431099e-02-5.89805982e-17j$	$-1.90623565e-01+3.50414142e-16j$
$3.11321361e-01+8.67361738e-17j$	$-6.42237697e-01-1.72604986e-16j$
$1.12639365e+00+2.77555756e-16j$	$-2.21830159e+00-1.21430643e-15j$
$4.00104568e+00+1.51267887e-15j$	$-7.75257338e+00-2.08166817e-15j$
$1.40598970e+01+5.13478149e-15j$	$-2.75285556e+01-1.12410081e-14j$]