Università della Svizzera italiana

Faculty of Informatics

Numerical Algorithms

Fall 2020

Assignment 9 (only for students of the 6 ECTS course) November 5, 2020

Exercise 1 [5 points]

Let $A \in \mathbb{R}^{m \times n}$ with $m > n$ have full rank n and let $b \in \mathbb{R}^m$. Show that the least squares solution $x^* \in \mathbb{R}^n$ of the linear system $Ax = b$ (that is, the vector x^* that solves the normal equations $A^T Ax^* = A^T b$) satisfies

$$x^* = \arg\min_{x \in \mathbb{R}^n} \|b - Ax\|.$$

Hand in your proof.

Exercise 2 [5 points]

Fit the $m = 4$ data points $P_i = (x_i, y_i) \in \mathbb{R}^2$,

$$P_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad P_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad P_4 = \begin{pmatrix} 2 \\ -2 \end{pmatrix},$$

with the parametric circle model $F\colon [0, 2\pi) \to \mathbb{R}^2$,

$$F(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + c_3 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

with $n = 3$ model parameters c_1, c_2, c_3, using the initial parameter values

$$t_1 = 1, \qquad t_2 = 2, \qquad t_3 = 3, \qquad t_4 = 4.$$

That is, write a program that iterates the following steps:

1. Consider the $2m$ equations $x_i = x(t_i)$, $i = 1, \ldots, m$ and $y_i = y(t_i)$, $i = 1, \ldots, m$, expressed in matrix/vector notation as $Ac = b$, where $c = (c_1, \ldots, c_n)$.
2. Find the least squares solution of this linear system, which in turns gives the best fitting parametric circle model.
3. Compute the root mean squared error (RMSE) for this best fit.
4. For each $i = 1, \ldots, m$, replace the current parameter value t_i with the parameter t_i^* that minimizes the distance between $F(t)$ and P_i,
$$t_i^* = \arg\min_{t \in [0, 2\pi)} \|F(t) - P_i\|.$$

How many iterations does it take before the RMSE of the best fit does not improve any further, that is, what is the smallest k, such that the difference between the RMSE at iteration k and the RMSE at iteration $k + 1$ is smaller than 10^{-8}?

Hand in your code, and print the smallest k as well as the optimal parameter values and model parameters after these k iterations.

Solutions must be returned online or in class on November 12, 2020