Introduction to
Partial Differential Equations

Lecture Notes

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Preliminary version – Do not distribute

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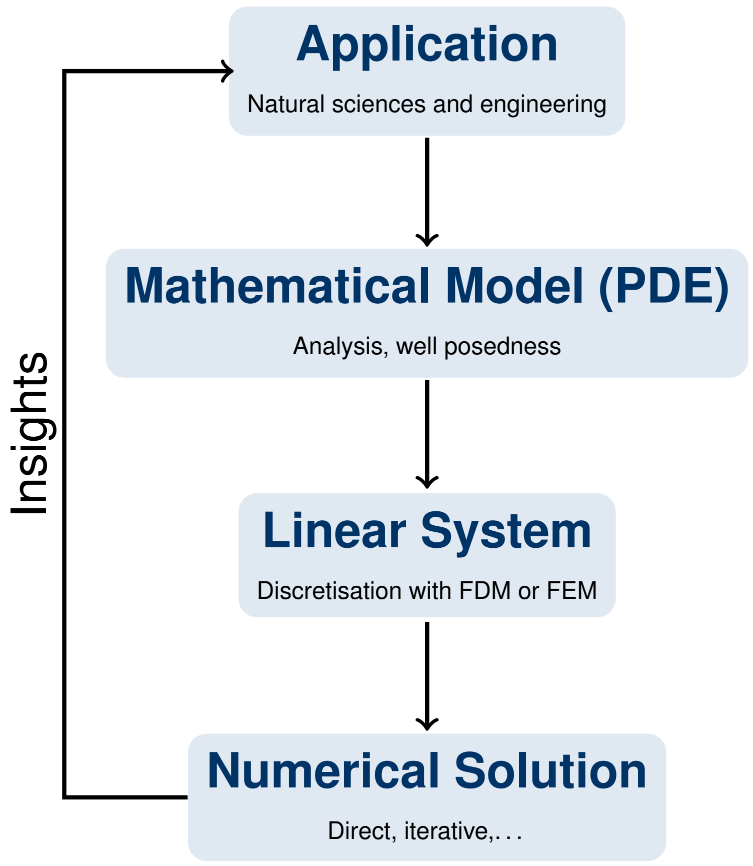
Contents

Habt Euch vorher wohl präpariert,
Paragraphos wohl einstudiert,
Damit Ihr nachher besser seht,
Daß er nichts sagt, als was im Buche steht;
Doch Euch des Schreibens ja befleißt,
Als diktiert, Euch der Heilig Geist!

–J. W. v. Goethe, Faust I



Application

Natural sciences and engineering

Mathematical Model (PDE)

Analysis, well posedness

Linear System

Discretisation with FDM or FEM

Numerical Solution

Direct, iterative,. . .

I. Partial Differential Equations

1 Preliminaries

In what follows, let $\Omega \subset \mathbb{R}^d$ for $d \in \{1,2,3,4\}$ denote a connected, open set with Lipschitz continuous boundary $\Gamma := \partial D$. We call Ω a *domain*. In this course, we focus on linear second order partial differential equations of the form

$$(\mathcal{L}u)(\boldsymbol{x}) := -\sum_{i,j=1}^{d} a_{i,j}(\boldsymbol{x}) u_{x_i x_j}(\boldsymbol{x}) + \sum_{i=1}^{d} b_i(\boldsymbol{x}) u_{x_i}(\boldsymbol{x}) + c(\boldsymbol{x})u(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{1.1}$$

where we use the abbreviations

$$u_{x_i} := \frac{\partial u}{\partial x_i} \quad \text{and} \quad u_{x_i x_j} := \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Often, equation (1.1) is rewritten in vector notation according to

$$-\operatorname{div}\big(\boldsymbol{A}(\boldsymbol{x})\nabla u(\boldsymbol{x})\big) + \boldsymbol{b}^{\intercal}(\boldsymbol{x})\nabla u(\boldsymbol{x}) + c(\boldsymbol{x})u(\boldsymbol{x}) = f(\boldsymbol{x})$$

with

$$\boldsymbol{A}(\boldsymbol{x}) := \begin{bmatrix} a_{1,1}(\boldsymbol{x}) & \cdots & a_{1,d}(\boldsymbol{x}) \\ \vdots & \ddots & \vdots \\ a_{d,1}(\boldsymbol{x}) & \cdots & a_{d,d}(\boldsymbol{x}) \end{bmatrix} \quad \text{and} \quad \boldsymbol{b}(\boldsymbol{x}) := \begin{bmatrix} \tilde{b}_1(\boldsymbol{x}) \\ \vdots \\ \tilde{b}_d(\boldsymbol{x}) \end{bmatrix}.$$

In this context, for a given given scalar field $v\colon \mathbb{R}^d \to \mathbb{R}$ and a vector field $\boldsymbol{f}\colon \mathbb{R}^d \to \mathbb{R}^d$, we set

$$\nabla v := \big[v_{x_1}, \ldots, v_{x_d}\big]^{\intercal} \quad \text{and} \quad \operatorname{div} \boldsymbol{f} := \sum_{i=1}^{d} \frac{\partial f_i}{\partial x_i}.$$

(1.2) **Definition.**

Multi indices We call $\boldsymbol{\alpha} = [\alpha_1, \dots, \alpha_d]^\intercal \in \mathbb{N}$ a *multi index* and set

$$\partial^{\boldsymbol{\alpha}} := \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}, \quad \text{where } |\boldsymbol{\alpha}| := \alpha_1 + \ldots + \alpha_d.$$

Moreover, we set

$$\boldsymbol{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$$

and

$$\boldsymbol{\alpha}! := \alpha_1! \cdots \alpha_d!, \quad \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} := \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d}.$$

Classical Function Spaces Let $\Omega \subset \mathbb{R}^d$ denote an open set. For $s \in \mathbb{N}$, we define the spaces

$$C^s(\Omega) := \{v \colon \Omega \to \mathbb{R} : \partial^{\boldsymbol{\alpha}} v \text{ is continuous for all } |\boldsymbol{\alpha}| \leqslant s\}.$$

If Ω is bounded, we can further define the norms

$$\|v\|_{C^s(\overline{\Omega})} := \max_{|\boldsymbol{\alpha}| \leqslant s, \boldsymbol{x} \in \overline{\Omega}} |\partial^{\boldsymbol{\alpha}} v(\boldsymbol{x})|$$

with the corresponding spaces

$$C^s(\overline{\Omega}) := \{v \colon \overline{\Omega} \to \mathbb{R} : \partial^{\boldsymbol{\alpha}} v \text{ is continuous for all } |\boldsymbol{\alpha}| \leqslant s \text{ and } \|v\|_{C^s(\overline{\Omega})} < \infty\}.$$

These spaces are complete with respect to the norm $\|\cdot\|_{C^s(\overline{\Omega})}$ and hence *Banach spaces.*

Space of Test Functions We further define

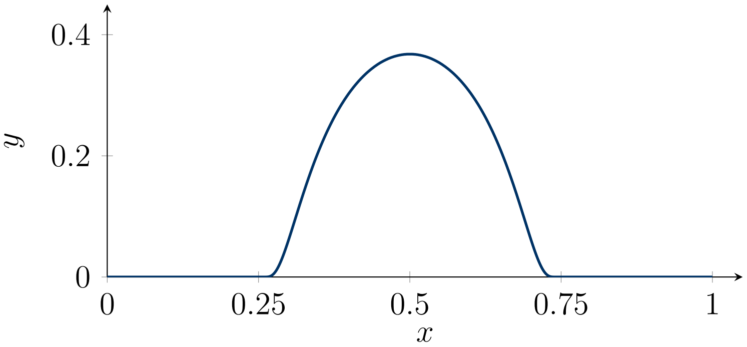
$$C^\infty(\Omega) := \bigcap_{s \in \mathbb{N}} C^s(\Omega).$$

Then, the *space of test functions* is defined as

$$C_0^\infty(\Omega) := \{v \in C^\infty(\Omega) : \operatorname{supp}(v) \subset \Omega\},$$

where $\operatorname{supp}(v) := \overline{\{\boldsymbol{x} \in \Omega : v(\boldsymbol{x}) \neq 0\}}$ is the *support* of the function $v \colon \Omega \to \mathbb{R}$.

(1.3) **Example.**



The function

$$\varphi(x) := \begin{cases} e^{\frac{-1}{1-(4x-2)^2}}, & 0.25 < x < 0.75, \\ 0, & \text{otherwise}, \end{cases}$$

is in $C_0^\infty(0,1)$, where $\operatorname{supp}(\varphi) = [0.25, 0.75]$.

In what follows, we make the assumption $u \in C^2(\Omega)$. Therefore, Schwartz's theorem yields $u_{x_i x_j} = u_{x_j x_i}$. Hence, without loss of generality the matrix $\boldsymbol{A}(\boldsymbol{x})$ is symmetric, i.e. $a_{i,j}(\boldsymbol{x}) = a_{j,i}(\boldsymbol{x})$, and has only real eigen values. Depending on the eigen values of $\boldsymbol{A}$, we distinguish three different types of partial differential equations that have to be supplemented by appropriate boundary- and/or initial- conditions in order to obtain a *well posed problem*, i.e. the problem exhibits a unique solution which depends continuously on the given data.

(1.4) **Definition.** The differential operator $\mathcal{L}$ from (1.1) is called…

- … *elliptic* at $\boldsymbol{x} \in \Omega$, iff all eigen values of $\boldsymbol{A}(\boldsymbol{x})$ are positive.
- … *parabolic* at $\boldsymbol{x} \in \Omega$, iff $d-1$ eigen values of $\boldsymbol{A}(\boldsymbol{x})$ are positive, one eigen value is zero and $\operatorname{rank}[\boldsymbol{A}(\boldsymbol{x}), \boldsymbol{b}(\boldsymbol{x})] = d$.
- … *hyperbolic* at $\boldsymbol{x} \in \Omega$, iff $d-1$ eigen values of $\boldsymbol{A}(\boldsymbol{x})$ are positive and one eigen value is negative.

The differential operator $\mathcal{L}$ is called *elliptic/parabolic/hyperbolic*, iff it is *elliptic/parabolic/hyperbolic* for every $\boldsymbol{x} \in \Omega$. Accordingly, equation (1.1) is called *elliptic/parabolic/hyperbolic*, iff the underlying differential operator exhibits this property.

Next, we shall consider three different examples, which are prototypes for the three different types of partial differential equations.

2 The Plateau Problem

We consider a soap film that is supported by a wireframe. This wireframe shall be represented by a smooth and closed curve in $\mathbb{R}^3$. We assume that its parallel projection onto the (x,y)-plane has no double points. Then, the shape of the soap film can be described by the graph of a function $u\colon \Omega \to \mathbb{R}$, while the wireframe is the graph of a function $g\colon \Gamma \to \mathbb{R}$. Due to surface tension, the soap film minimises its surface area

$$\int_\Omega \sqrt{1 + u_x^2 + u_y^2}\, \mathrm{d}x\, \mathrm{d}y \to \min.$$

To solve this nonlinear variational problem approximately, we employ the first order Taylor expansion $\sqrt{1+z} = 1 + \frac{z}{2} + \mathcal{O}(z^2)$. Hence, for small values of u_x and u_y, we may replace the integrand by a quadratic expression. We arrive at the minimisation problem

$$F(u) := \frac{1}{2} \int_\Omega u_x^2 + u_y^2\, \mathrm{d}x\, \mathrm{d}y \to \min. \tag{2.1}$$

The values of u at Γ are prescribed by the position of the wireframe, i.e. $u|_\Gamma = g$.
Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$, $u|_\Gamma = g$ be a solution to (2.1). Then, for any $v \in C^1(\Omega) \cap C(\overline{\Omega})$ with $v|_\Gamma = 0$, the Gâteaux derivative satisfies

$$0 = \lim_{\epsilon \to 0} \frac{F(u + \epsilon v) - F(u)}{\epsilon} = \int_\Omega u_x v_x + u_y v_y \,\mathrm{d}x \,\mathrm{d}y = \int_\Omega \langle \nabla u, \nabla v \rangle \,\mathrm{d}\boldsymbol{x}, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ denotes the dot product in $\mathbb{R}^d$.
On the other hand, we have for a continuously differentiable vector field $\boldsymbol{f}$ by the *divergence theorem*

$$\int_\Omega \operatorname{div} \boldsymbol{f} \,\mathrm{d}\boldsymbol{x} = \int_\Gamma \langle \boldsymbol{f}, \boldsymbol{n} \rangle \,\mathrm{d}\sigma,$$

where $\boldsymbol{n}$ denotes the outward pointing normal vector. Hence, for $\boldsymbol{f} := \nabla u v$, the multivariate product rule yields

$$\int_\Omega \Delta u v \,\mathrm{d}\boldsymbol{x} + \int_\Omega \langle \nabla u, \nabla v \rangle \,\mathrm{d}\boldsymbol{x} = \int_\Omega \operatorname{div} \boldsymbol{f} \,\mathrm{d}\boldsymbol{x} = \int_\Gamma \langle \boldsymbol{f}, \boldsymbol{n} \rangle \,\mathrm{d}\sigma = \int_\Gamma v \langle \nabla u, \boldsymbol{n} \rangle \,\mathrm{d}\sigma = 0,$$

since $v|_\Gamma = 0$. This can be rewritten as

$$\int_\Omega -\Delta u v \,\mathrm{d}\boldsymbol{x} = \int_\Omega \langle \nabla u, \nabla v \rangle \,\mathrm{d}\boldsymbol{x}.$$

Inserting the latter into (2.2), leads to

$$\int_\Omega -\Delta u v \,\mathrm{d}\boldsymbol{x} = 0.$$

Since this equation holds for any *test function* $v \in C^1(\Omega) \cap C(\overline{\Omega})$ with $v|_\Gamma = 0$, the *fundamental lemma of calculus of variations* yields *Laplace's equation*

$$\Delta u(\boldsymbol{x}) = 0 \quad \text{for } \boldsymbol{x} \in \Omega.$$

Summary. The first order approximation of the Plateau problem leads to Laplace's equation

$$\Delta u(\boldsymbol{x}) = 0 \quad \text{for } \boldsymbol{x} \in \Omega.$$

This equation is of the form (1.1) with

$$\boldsymbol{A}(\boldsymbol{x}) := \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \in \mathbb{R}^{d \times d}, \quad \boldsymbol{b}(\boldsymbol{x}) = \boldsymbol{0} \in \mathbb{R}^d, \quad c(\boldsymbol{x}) = 0$$

and thus elliptic.
In order to obtain a well posed problem, the equation has to be supplemented by boundary conditions. Let g, κ be continuous functions. Common boundary conditions are ...

- ... **Dirichlet conditions**

$$u = g \quad \text{on } \Gamma.$$

- ... **Neumann conditions**

$$\langle \boldsymbol{n}, \boldsymbol{A}\nabla u\rangle = g \quad \text{on } \Gamma.$$

- ... **Robin conditions**

$$\langle \boldsymbol{n}, \boldsymbol{A}\nabla u\rangle + \kappa u = g \quad \text{on } \Gamma.$$

3 The Heat Equation

Let $u\colon [0,\infty) \times \Omega \to \mathbb{R}$ denote the distribution of temperature in an object. Moreover, we denote by $f \in C\big([0,\infty)\times\Omega\big)$ a heat source inside the object. We consider the balance of heat in a control volume $V \subset \Omega$. The energy principle states that the rate of change in the total energy in V is comprised of the inflow of heat via the surface ∂V and the heat injection f. Hence, there holds for the energy $E(t,\boldsymbol{x})$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V E \,\mathrm{d}\boldsymbol{x} = -\int_{\partial V}\langle \boldsymbol{j}, \boldsymbol{n}\rangle \,\mathrm{d}\sigma + \int_V f \,\mathrm{d}\boldsymbol{x},$$

where the vector field $\boldsymbol{j}(t,\boldsymbol{x})$ denotes the heat flux. Now, the application of the divergence theorem yields

$$\int_V \frac{\partial E}{\partial t} + \operatorname{div}\boldsymbol{j} - f \,\mathrm{d}\boldsymbol{x} = 0 \quad \text{for } t > 0$$

and hence, since $V \subset \Omega$ can be chosen arbitrarily

$$\frac{\partial E}{\partial t}(t,\boldsymbol{x}) + \operatorname{div}\boldsymbol{j}(t,\boldsymbol{x}) = f(t,\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Omega,\ t > 0.$$

In accordance with *Fourier's law*, we assume that

$$\boldsymbol{j} = -\kappa\nabla u,$$

where κ is a material dependent diffusion constant. Making further the assumption that the energy depends linearly on the temperature, i.e. $E = E_0 + \lambda u$ for some $\lambda \in \mathbb{R}$, we obtain the *heat equation*

$$\frac{\partial u}{\partial t} - \frac{\kappa}{\lambda}\Delta u = \frac{f}{\lambda} \quad \text{for } \boldsymbol{x} \in \Omega,\ t > 0.$$

Summary. Based on the conservation of energy, we have derived the heat equation

$$\frac{\partial u}{\partial t} - \kappa \Delta u = f \quad \text{for } \boldsymbol{x} \in \Omega,\ t > 0.$$

This equation is of the form (1.1) with

$$\boldsymbol{A}(\boldsymbol{x}) := \kappa \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \in \mathbb{R}^{\tilde{d}\times\tilde{d}},\ \boldsymbol{b}(\boldsymbol{x}) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\tilde{d}},\ c(\boldsymbol{x}) = 0,\ \tilde{d} := d + 1$$

and thus parabolic. The additional dimension denotes the time.

In order to obtain a well posed problem, the equation has to be supplemented by boundary conditions and an initial condition at $t = 0$.

4 The Wave Equation

The motion of molecules in an ideal gas is described by three laws. In what follows, we denote the velocity by $\boldsymbol{v}$, the density by ρ, and the pressure by p.

Due to the conservation of mass, the change of mass, i.e. $\int_V \frac{\partial \rho}{\partial t}\,\mathrm{d}\boldsymbol{x}$ in a volume $V \subset \Omega$ equals the flow through its surface, i.e. $-\int_{\partial V} \rho\langle \boldsymbol{v}, \boldsymbol{n}\rangle\,\mathrm{d}\sigma$. As in the derivation of the heat equation, we arrive at the *continuity equation*

$$\frac{\partial \rho}{\partial t} = -\rho_0 \operatorname{div} \boldsymbol{v},$$

where ρ is approximated by the fixed density ρ_0.

By *Newton's third law*, the pressure gradient induces an accelaration of the molecules, i.e.

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\nabla p.$$

Finally, the *ideal gas law* states that the pressure is proportional to the density for constant temperature, i.e.

$$p = c^2 \rho, \quad c > 0.$$

Combining these three laws yields

$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 \rho}{\partial t^2} = c^2 \frac{\partial}{\partial t}\big(-\rho_0 \operatorname{div} \boldsymbol{v}\big) = -c^2 \operatorname{div}\left(\rho_0 \frac{\partial \boldsymbol{v}}{\partial t}\right) = c^2 \operatorname{div}(\nabla p) = c^2 \Delta p.$$

Summary. Based on the conservation of mass, Newton's third law and the ideal gas law, we have derived the *wave equation*

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0 \quad \text{for } \boldsymbol{x} \in \Omega,\ t > 0.$$

This equation is of the form (1.1) with

$$\boldsymbol{A}(\boldsymbol{x}) := \begin{bmatrix} c^2 & & & \\ & \ddots & & \\ & & c^2 & \\ & & & -1 \end{bmatrix} \in \mathbb{R}^{\tilde{d}\times\tilde{d}},\ \boldsymbol{b}(\boldsymbol{x}) = \boldsymbol{0} \in \mathbb{R}^{\tilde{d}},\ c(\boldsymbol{x}) = 0,\ \tilde{d} := d+1$$

and thus hyperbolic. The additional dimension denotes the time.

In order to obtain a sensible problem, the equation has to be supplemented by boundary conditions and two initial conditions at $t = 0$, one for u and one for $\partial u/\partial t$.

5 Analytical Solutions in One Dimension

Poisson's Equation

We consider the one dimensional *Poisson's equation* with Dirichlet boundary conditions, i.e.

$$\begin{aligned} -u''(x) &= f(x) \quad \text{for } x \in \Omega := (0,1), \\ u(0) &= u_0,\ u(1) = u_1. \end{aligned}$$

The application of the *Fundamental theorem of calculus* yields

$$u'(x) = -\int_0^x f(s)\,\mathrm{d}s + \alpha$$

and hence

$$u(x) = \int_0^x u'(y)\,\mathrm{d}y + \beta = -\int_0^x \int_0^y f(s)\,\mathrm{d}s\,\mathrm{d}y + \alpha x + \beta$$

for some constants $\alpha, \beta \in \mathbb{R}$.

By considering $x = 0$ and $x = 1$, the constants can be determined according to

$$\alpha = u_1 - u_0 + \int_0^1 \int_0^y f(s)\,\mathrm{d}s\,\mathrm{d}y, \quad \beta = u_0.$$

By differentiation it can be seen that α and β and therefore $u(x)$ are uniquely determined. Moreover, the solution depends continuously on the data. Hence, the one dimensional Dirichlet problem is well posed.

The Heat Equation

We consider the one dimensional heat equation with an initial condition and Dirichlet boundary conditions, i.e.

$$\begin{aligned} u_t(t,x) &= u_{xx}(t,x) \quad \text{for } x \in \Omega := (0,1), \\ u(t,0) &= u(t,1) = 0 \\ u(0,x) &= f(x). \end{aligned}$$

Let the initial values be given by a Fourier series

$$f(x) = \sum_{k=1}^{\infty} a_k \sin(k\pi x) \quad \text{for } x \in \Omega$$

with coefficients $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R}$. The functions

$$e^{-(k\pi)^2 t} \sin(k\pi x) \quad \text{for } k \in \mathbb{Z}$$

satisfy the homogenous heat equation $u_t = u_{xx}$ with homogenous Dirichlet boundary conditions. Therefore, the function

$$u(t,x) = \sum_{k=1}^{\infty} a_k e^{-(k\pi)^2 t} \sin(k\pi x)$$

solves the initial boundary value problem at hand.

In the case $\Omega = \mathbb{R}$, the boundary condition drops out and we face a pure initial value problem. It is called the *Cauchy problem* for the one dimensional heat equation. Given a bounded and continuous initial condition f, we can represent the solution by means of Fourier integrals instead of Fourier series. It holds

$$u(t,x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{\frac{-\xi^2}{4t}} f(x-\xi) \, \mathrm{d}\xi.$$

We remark that the solution in (t,x) depends on the knowledge of f in the entire domain. This means that the propagation of data is performed with infinite speed. The solution is unique and depends continuously on the initial condition. Hence, the Cauchy problem for the one dimensional heat equation is well posed.

The Wave Equation

We consider the pure initial value problem for the the one dimensional wave equation, which reads

$$\begin{aligned} u_{tt}(t,x) &= u_{xx}(t,x) \quad \text{for } \Omega := \mathbb{R}, \\ u(0,x) &= f(x), \;\; u_t(0,x) = g(x). \end{aligned}$$

To solve this equation, we consider the change of variables

$$\xi = x + t, \qquad \eta = x - t.$$

By the chain rule, we obtain

$$u_x = u_\xi \frac{\partial \xi}{\partial x} + u_\eta \frac{\partial \eta}{\partial x},$$

et cetera, we obtain

$$\begin{aligned} u_x &= u_\xi + u_\eta, \quad & u_{xx} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \\ u_t &= u_\xi - u_\eta, \quad & u_{tt} &= u_{\xi\xi} - 2u_{xi\eta} + u_{\eta\eta}. \end{aligned}$$

Therefore, the wave equation in the new coordinates reads

$$u_{\xi\eta} = 0.$$

Its general solution is

$$u(\xi,\eta) = \varphi(\xi) + \psi(\eta) = \varphi(x+t) + \psi(x-t),$$

where the functions φ and ψ have to be determined from the initial conditions. It holds

$$\varphi(x) + \psi(x) = f(x), \quad \varphi'(x) - \psi'(x) = g(x).$$

By differentiating the first equation, we obtain two equations for φ' and ψ', which read

$$\begin{aligned}\varphi' &= \frac{1}{2}(f'+g), \quad \varphi(\xi) = \frac{1}{2}f(\xi) + \frac{1}{2}\int_{x_0}^{\xi} g(s)\,\mathrm{d}s\\ \psi' &= \frac{1}{2}(f'-g), \quad \psi(\eta) = \frac{1}{2}f(\eta) - \frac{1}{2}\int_{x_0}^{\eta} g(s)\,\mathrm{d}s.\end{aligned}$$

Hence, we obtain

$$u(t,x) = \varphi(\xi) + \psi(\eta) = \frac{1}{2}\big(f(x+t) + f(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,\mathrm{d}s.$$

Note that the solution $u(t,x)$ only depends on the initial values between $x-t$ and $x+t$. This corresponds to the fact that the underlying physical system propagates changes in the data only with finite velocity. Further, we remark that the solution is unique and depends continuously on the data. Therefore, the initial value problem for the one dimensional wave equation is well posed.

6 The Maximum Principle

An important tool into show the uniqueness and continuous dependency of the data of the solution to elliptic and parabolic partial differential equations is the maximum principle. Its discrete analogue also is also used in the analysis of finite difference methods. We consider here the version for the elliptic case.

In what follows, let

$$(\mathcal{L}u)(\boldsymbol{x}) := -\sum_{i,j=1}^{d} a_{i,j}(\boldsymbol{x})u_{x_i x_j}(\boldsymbol{x}) \tag{6.1}$$

be an elliptic differential operator and let Ω be a bounded domain.

(6.2) **Theorem (Maximum principle).** For $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, let

$$\mathcal{L}u = f \leqslant 0 \quad \text{in } \Omega.$$

Then u attains its maximum at the boundary Γ.

Proof. Without loss of generality, we assume that the matrix $\boldsymbol{A}(\boldsymbol{x})$ is diagonal, i.e. $a_{i,j}(\boldsymbol{x}) = 0$ if $i \neq j$, see e.g. [Braess].
First, we carry out the proof under the stronger assumption $f < 0$. Suppose that u attains its maximum at $\boldsymbol{x}_0 \in \Omega$, i.e.

$$u(\boldsymbol{x}_0) = \sup_{\boldsymbol{x}\in\Omega} u(\boldsymbol{x}) > \max_{\boldsymbol{x}\in\Gamma} u(\boldsymbol{x}).$$

Since $\boldsymbol{x}_0$ is a maximum, the gradient of u vanishes at $\boldsymbol{x}_0$ and the Hessian is negative definite, i.e.

$$\nabla u(\boldsymbol{x}_0) = \boldsymbol{0} \quad \text{and} \quad u_{x_i x_i} \leqslant 0.$$

Due to the ellipticity of $\mathcal{L}$, there further holds $a_{i,i}(\boldsymbol{x}) > 0$ and hence

$$(\mathcal{L}u)(\boldsymbol{x}_0) = -\sum_{i=1}^{d} a_{i,i}(\boldsymbol{x}_0) u_{x_i x_i}(\boldsymbol{x}_0) \geqslant 0.$$

This is a contradiction to the assumption $(\mathcal{L}u)(\boldsymbol{x}_0) = f(\boldsymbol{x}_0) < 0$.
Now, let $f \leqslant 0$ and again suppose there exists an $\boldsymbol{x}_0 \in \Omega$ such that $u(\boldsymbol{x}_0) > \max_{\boldsymbol{x}\in\Gamma} u(\boldsymbol{x})$. We introduce the auxiliary function $h(\boldsymbol{x}) := \|\boldsymbol{x} - \boldsymbol{x}_0\|_2^2$, which is bounded on Γ. Hence, for some $\delta > 0$ sufficiently small, the function

$$w = u + \delta h$$

still attains its maximum at a point $\tilde{\boldsymbol{x}}_0 \in \Omega$. Since $h_{x_i x_j} = 2\delta_{i,j}$, we obtain

$$(\mathcal{L}w)(\boldsymbol{x}) = (\mathcal{L}u)(\boldsymbol{x}) + \delta(\mathcal{L}h)(\boldsymbol{x}) = f(\boldsymbol{x}) - 2\delta \sum_{i=1}^{d} a_{i,i}(\boldsymbol{x}) < 0$$

for all $x \in \Omega$. This yields a contradiction as in the first part of the proof. □

Several simple but important consequences can be derived from the maximum principle.

(6.3) **Corollary.** Let the conditions of Theorem (6.2) be satisfied.

(a) **Minimum principle** If $\mathcal{L}u \geqslant 0$ in Ω, then u attains its minimum at the boundary Γ.

(b) **Comparison principle** Assume $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and

$$\begin{aligned} \mathcal{L}u \leqslant \mathcal{L}v \quad & \text{in } \Omega, \\ u \leqslant v \quad & \text{on } \Gamma. \end{aligned}$$

Then, there holds $u \leqslant v$ in Ω.

(c) **Continuous dependency on the boundary data** The solution to the Dirichlet problem

$$\mathcal{L}u = f \text{ in } \Omega, \quad u = g \text{ on } \Gamma$$

depends continuously on the boundary data: If $\mathcal{L}u_1 = \mathcal{L}u_2 = f$ and $u_1|_\Gamma = g_1$, $u_2|_\Gamma = g_2$, then

$$\sup_{\boldsymbol{x}\in\Omega} |u_1(\boldsymbol{x}) - u_2(\boldsymbol{x})| = \max_{\boldsymbol{z}\in\Gamma} |g_1(\boldsymbol{z}) - g_2(\boldsymbol{z})|.$$

(d) **Uniqueness of the solution** The solution to the Dirichlet problem is unique.

(e) **Helmholtz terms** If $(\mathcal{L} + c)u \leqslant 0$ for $c(\boldsymbol{x}) \geqslant 0$, then

$$\sup_{x\in\Omega} u(\boldsymbol{x}) \leqslant \max\left\{0, \max_{\boldsymbol{z}\in\Gamma} u(\boldsymbol{z})\right\}.$$

Proof. (a) Apply the maximum principle to $v := -u$.

(b) There holds $\mathcal{L}w := \mathcal{L}v - \mathcal{L}u \geqslant 0$ in Ω and $w := v - u \geqslant 0$ on Γ.

Hence, the minimum principle yields $w(\boldsymbol{x}) \geqslant 0$ for $\boldsymbol{x} \in \Omega$.

(c) We have $\mathcal{L}w = 0$ for $w := u_1 - u_2$. Hence, the maximum principle yields

$$w(\boldsymbol{x}) \leqslant \max_{\boldsymbol{z}\in\Gamma} w(\boldsymbol{z}) \leqslant \max_{\boldsymbol{z}\in\Gamma} |w(\boldsymbol{z})| \quad \text{for } \boldsymbol{x} \in \Omega.$$

On the other hand, the minimum principle gives us

$$w(\boldsymbol{x}) \geqslant \min_{\boldsymbol{z}\in\Gamma} w(\boldsymbol{z}) \geqslant -\max_{\boldsymbol{z}\in\Gamma} |w(\boldsymbol{z})| \quad \text{for } \boldsymbol{x} \in \Omega.$$

Consequently, there holds

$$|w(\boldsymbol{x})| \leqslant \max_{\boldsymbol{z}\in\Gamma} |w(\boldsymbol{z})| \quad \text{for all } \boldsymbol{x} \in \Omega.$$

From this, the assertion is obtained by the continuity of w.

(d) This follows directly from the continuous dependency on the boundary data.

(e) If $u(\boldsymbol{x}) \leqslant 0$ for all $\boldsymbol{x} \in \Omega$, there is nothing to show. Thus, suppose there exists $\boldsymbol{x}_0 \in \Omega$ with $u(\boldsymbol{x}_0) > 0$. Then, it holds $(\mathcal{L}u)(\boldsymbol{x}_0) \leqslant (\mathcal{L}u)(\boldsymbol{x}_0) + c(\boldsymbol{x}_0)u(\boldsymbol{x}_0) \leqslant 0$ and the maximum principle yields the assertion. □

Under stronger assumptions on the differential operator $\mathcal{L}$ we can also show the continuous dependence of the solution on the right hand side.

(6.4) **Definition.** An elliptic operator of the form (6.1) is called *uniformly elliptic* if there exists a constant $\alpha > 0$ such that

$$\boldsymbol{\xi}^\mathsf{T} \boldsymbol{A}(\boldsymbol{x}) \boldsymbol{\xi} \geqslant \alpha \|\boldsymbol{\xi}\|_2^2$$

for all $\boldsymbol{\xi} \in \mathbb{R}^d$, $\boldsymbol{x} \in \Omega$. The largest such α is called *constant of ellipticity*.

(6.5) **Corollary (Continuous dependence on the right hand side).** Let the second order differential operator $\mathcal{L}$ is uniformly elliptic in Ω. Then there exists a constant $c > 0$ which only depends on Ω and α such that

$$|u(\boldsymbol{x})| \leqslant \max_{\boldsymbol{z} \in \Gamma} |u(\boldsymbol{z})| + c \sup_{\boldsymbol{z} \in \Omega} |(\mathcal{L}u)(\boldsymbol{z})|$$

for every $\boldsymbol{x} \in \Omega$ and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

Proof. Since Ω is bounded, there is $R > 0$ such that $\Omega \subset B_R(\boldsymbol{0}) := \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\|_2 < R\}$. Let

$$w(\boldsymbol{x}) := R^2 - \|\boldsymbol{x}\|_2^2.$$

Since $w_{x_i x_j} = -2\delta_{i,j}$, we have

$$\mathcal{L}w \geqslant 2\alpha \quad \text{and} \quad 0 \leqslant w(\boldsymbol{x}) \leqslant R^2 \text{ for all } \boldsymbol{x} \in \Omega.$$

Therefore, the function

$$v(\boldsymbol{x}) := \max_{\boldsymbol{z} \in \Gamma} |u(\boldsymbol{z})| + w(\boldsymbol{x}) \frac{1}{2\alpha} \sup_{\boldsymbol{z} \in \Omega} |(\mathcal{L}u)(\boldsymbol{z})|$$

satisfies $\mathcal{L}v \geqslant |\mathcal{L}u|$ in Ω and $v \geqslant |u|$ on Γ.

Now, the comparison principle yields $-v(\boldsymbol{x}) \leqslant u(\boldsymbol{x}) \leqslant v(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$. Since $w \leqslant R^2$, we arrive at the assertion with $c := R^2/(2\alpha)$. □

II. The Finite Difference Method

1 The Poisson Problem

In what follows, we restrict ourselves to Poisson's equation with Dirichlet boundary conditions. Let $f \in C^0(\Omega)$ and $g \in C^0(\Gamma)$. We want to compute $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that

$$-\Delta u = f \text{ in } \Omega, \quad u = g \text{ on } \Gamma.$$

(1.1) **Definition.** A solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to Poisson's equation is called *classical solution*. In the case $f \equiv 0$, the solution u is called *harmonic*.

In order to compute the classical solution numerically, we discretise the partial derivatives occurring in the Laplacian.

(1.2) **Definition.** For $v \in C^0(\mathbb{R}^d)$, a direction $\boldsymbol{e}_i \in \mathbb{R}^d$, i.e. $e_{i,j} = \delta_{i,j}$ and $h > 0$, we define the *forward difference*

$$(\partial_i^+ v)(\boldsymbol{x}) := \frac{v(\boldsymbol{x} + h\boldsymbol{e}_i) - v(\boldsymbol{x})}{h},$$

the *backward difference*

$$(\partial_i^- v)(\boldsymbol{x}) := \frac{v(\boldsymbol{x}) - v(\boldsymbol{x} - h\boldsymbol{e}_i)}{h}$$

and the *central difference*

$$(\partial_i^\bullet v)(\boldsymbol{x}) := \frac{v(\boldsymbol{x} + h\boldsymbol{e}_i) - v(\boldsymbol{x} - h\boldsymbol{e}_i)}{2h}.$$

For $v \in C^1(\mathbb{R}^d)$, obviously the limit $h \to 0$ exists and there holds

$$\lim_{h\to 0}(\partial_i^+ v)(\boldsymbol{x}) = \lim_{h\to 0}(\partial_i^- v)(\boldsymbol{x}) = \lim_{h\to 0}(\partial_i^\bullet v)(\boldsymbol{x}) = v_{x_i}(\boldsymbol{x}).$$

The next lemma tells us how good this approximation to the actual derivative is. The corresponding approximation order is called *consistency* of the difference operator.

(1.3) **Lemma.** For $\boldsymbol{x}_0 \in \Omega$ let $\{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x} - \boldsymbol{x}_0\| \leqslant h\} =: \overline{B_h(\boldsymbol{x}_0)} \subset \overline{\Omega}$ and suppose $v \in C^4(\overline{\Omega})$. Then, it holds

$$\begin{aligned} v_{x_i}(\boldsymbol{x}_0) &= \partial_i^{\pm}(\boldsymbol{x}_0) + \mathcal{O}(h), \\ v_{x_i}(\boldsymbol{x}_0) &= \partial_i^{\bullet}(\boldsymbol{x}_0) + \mathcal{O}(h^2) \end{aligned}$$

and

$$\begin{aligned} v_{x_i x_i}(\boldsymbol{x}_0) &= (\partial_i^- \partial_i^+ v)(\boldsymbol{x}_0) + \mathcal{O}(h^2) \\ &= \frac{v(\boldsymbol{x}_0 + h\boldsymbol{e}_i) - 2v(\boldsymbol{x}_0) + v(\boldsymbol{x}_0 - h\boldsymbol{e}_i)}{h^2} + \mathcal{O}(h^2). \end{aligned}$$

Proof. Since we only consider directional derivatives, it is sufficient prove the assertions for $d = 1$. A *Taylor expansion* of v yields

$$v(x_0 \pm h) = v(x_0) \pm hv'(x_0) + \frac{h^2}{2} v''(\xi), \quad \xi \in (x_0, x_0 \pm h).$$

This directly yields the claim for $\partial_i^{\pm}$. For the central difference, it holds

$$\begin{aligned} v(x_0 + h) &= v(x_0) + hv'(x_0) + \frac{h^2}{2} v''(x_0) + \frac{h^3}{6} v'''(\xi_1), \quad \xi_1 \in (x_0, x_0 + h), \\ v(x_0 - h) &= v(x_0) - hv'(x_0) + \frac{h^2}{2} v''(x_0) - \frac{h^3}{6} v'''(\xi_2), \quad \xi_2 \in (x_0 - h, x_0). \end{aligned}$$

Subtracting these two equations yields

$$v(x + h) - v(x - h) = 2hv'(x_0) + \frac{h^3}{6}\big(v'''(\xi_1) + v'''(\xi_2)\big).$$

The assertion is hence obtained by dividing by $2h$. Finally, the claim on the second order derivative is obtained by adding up the following three equations

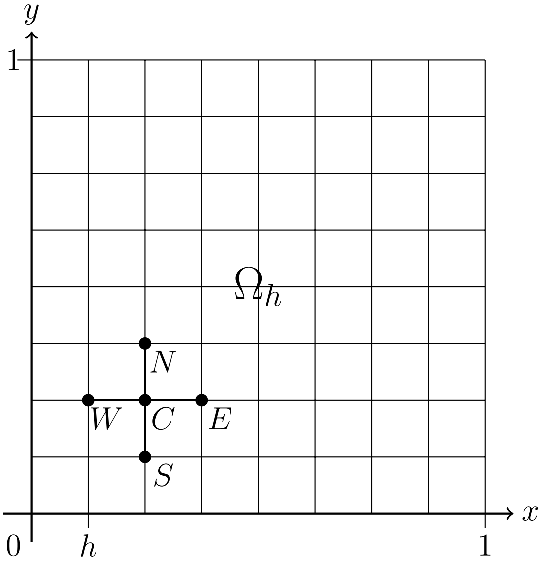
$$\begin{aligned} v(x_0 + h) &= v(x_0) + hv'(x_0) + \frac{h^2}{2} v''(x_0) + \frac{h^3}{6} v'''(x_0) + \frac{h^4}{24} v^{(4)}(\xi_1), \\ &\qquad \xi_1 \in (x_0, x_0 + h), \\ -2v(x_0) &= -2v(x_0) \\ v(x_0 - h) &= v(x_0) - hv'(x_0) + \frac{h^2}{2} v''(x_0) - \frac{h^3}{6} v'''(x_0) + \frac{h^4}{24} v^{(4)}(\xi_2), \\ &\qquad \xi_2 \in (x_0 - h, x_0), \end{aligned}$$

which yields

$$v(x_0 + h) - 2v(x_0) + v(x_0 - h) = h^2 v''(x_0) + \frac{h^4}{24}\big(v^{(4)}(\xi_1) + v^{(4)}(\xi_2)\big).$$

Division by h^2 yields the assertion. □

The discretisation by finite differences is based on a *mesh* with *meshwidth* h for Ω. For the sake of simplicity, we shall assume in what follows that the domain Ω is given by the hypercube, i.e. $\Omega := (0,1)^d \subset \mathbb{R}^d$. In principle, it is also possible to consider the finite difference method on more general domains. Nevertheless, this leads to a cumbersome discussion of corner cases. We refer to [Braess,Hackbusch] for a more general discussion of this topic.



For $n \in \mathbb{N}^*$ let $h = 1/n$. A mesh with meshwidth h for Ω is defined by

$$\Omega_h := \{\boldsymbol{x} \in \Omega : \boldsymbol{x} = h\boldsymbol{k} \text{ for } \boldsymbol{k} \in \mathbb{N}^d\},$$
$$\Gamma_h := \{\boldsymbol{x} \in \Gamma : \boldsymbol{x} = h\boldsymbol{k} \text{ for } \boldsymbol{k} \in \mathbb{N}^d\}.$$

As in the continuous case, we set $\overline{\Omega}_h := \Omega_h \cup \Gamma_h$.

The previous definition of a mesh is also valid for the slightly more general case that Ω is comprised of cubes of edge length h.

For each mesh point $\boldsymbol{x} \in \Gamma_h$, we have $u(\boldsymbol{x}) = g(\boldsymbol{x})$. For each point $\boldsymbol{x} \in \Omega_h$, we obtain an equation for $u(\boldsymbol{x})$ by approximating Poisson's equation by means of finite differences. For each $\boldsymbol{x} \in \Omega_h$, we set

$$(1.4) \qquad (\Delta_h u)(\boldsymbol{x}) := \sum_{i=1}^{d} (\partial_i^- \partial_i^+ u)(\boldsymbol{x}) = (\Delta u)(\boldsymbol{x}) + \mathcal{O}(h^2).$$

For $d = 2$, we can express the equation for $u(x,y)$ in terms of the *standard five-point stencil*

$$\begin{bmatrix} \alpha_{NW} & \alpha_N & \alpha_{NE} \\ \alpha_W & \alpha_C & \alpha_E \\ \alpha_{SW} & \alpha_S & \alpha_{SE} \end{bmatrix}_* = \frac{1}{h^2} \begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_* .$$

It holds

$$\alpha_C u_C + \alpha_E u_E + \alpha_S u_S + \alpha_W u + \alpha_N u_N = f(x,y) \quad \text{for } (x,y) \in \Omega_h$$

with $u_C := u(x,y), u_E = u(x+h,y), u_S = u(x,y-h), u_W = u(x-h,y), u_N = u(x,y+h)$.

(1.5) Example.

Poisson's Equation in One Dimension For $d = 1$, we have

$$-u'' = f \text{ in } (0,1), \quad u(0) = \alpha, \; u(1) = \beta.$$

Setting $u_i := u(x_i)$ and $f_i := f(x_i)$ for $x_i = hi$, $i = 1, \ldots, n-1$ and $h = 1/n$, we end up with the linear system of equations

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}}_{\boldsymbol{L}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{bmatrix}}_{\boldsymbol{u}} = \underbrace{\begin{bmatrix} f_1 + \alpha/h^2 \\ f_2 \\ \vdots \\ f_{n-2} \\ f_{n-1} + \beta/h^2 \end{bmatrix}}_{\boldsymbol{f}}.$$

Poisson's equation in d Dimensions Using the matrix $\boldsymbol{L} \in \mathbb{R}^{(n-1)\times(n-1)}$ and the identity matrix $\boldsymbol{I} \in \mathbb{R}^{(n-1)\times(n-1)}$, the finite difference approximation of the d-dimensional Laplacian can be written as

$$\boldsymbol{L}^{(d)} := \sum_{i=1}^{d} \underbrace{\boldsymbol{I} \otimes \cdots \otimes \boldsymbol{I}}_{(i-1)\text{-times}} \otimes \boldsymbol{L} \otimes \underbrace{\boldsymbol{I} \otimes \cdots \otimes \boldsymbol{I}}_{(d-i)\text{-times}}. \tag{1.6}$$

Herein, for two matrices $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{B} \in \mathbb{R}^{m'\times n'}$, $(\boldsymbol{A} \otimes \boldsymbol{B}) \in \mathbb{R}^{(mm')\times(nn')}$ denotes their *Kronecker product*, which is defined according to

$$\boldsymbol{A} \otimes \boldsymbol{B} := \begin{bmatrix} a_{1,1}\boldsymbol{B} & \cdots & a_{1,n}\boldsymbol{B} \\ \vdots & \ddots & \vdots \\ a_{m,1}\boldsymbol{B} & \cdots & a_{m,n}\boldsymbol{B} \end{bmatrix}.$$

The Kronecker product ist associative but not commutative.

We will also make use of the *column-wise vectorisation* of a matrix, which is defined as follows. Let $\boldsymbol{A} := [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n] \in \mathbb{R}^{m\times n}$ with columns $\boldsymbol{a}_i \in \mathbb{R}^{m\times 1}$ for $i = 1, \ldots, n$. Then, we set

$$\operatorname{vec}(\boldsymbol{A}) := \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_n \end{bmatrix} \in \mathbb{R}^{nm}.$$

Now, for $d = 2$ we obtain for the Poisson problem $\Delta u = f$ in Ω, $u = 0$ on Γ the linear system of equations

$$(\boldsymbol{L} \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{L}) \operatorname{vec}(\boldsymbol{U}) = \operatorname{vec}(\boldsymbol{F}),$$

where we set

$$\boldsymbol{U} := [u(jh, ih)]_{i,j=1}^{n-1} \quad \text{and} \quad \boldsymbol{F} := [f(jh, ih)]_{i,j=1}^{n-1}.$$

Neumann Boundary Conditions Often, instead of Dirichlet conditions, which fix the function values at Γ, information on the flux are available and shall be incorporated. This leads to Neumann boundary conditions

$$\frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}) = g(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Gamma.$$

To sustain the overall accuracy $\mathcal{O}(h^2)$ from the discretization of the Laplacian, we have to employ central differences for the discretisation of the gradient. We show the procedure here for $d = 1$.

At the point $x_0 = 1$ and $x_n = 1$, we obtain the equations

$$\frac{u_{n+1} - u_{n-1}}{2h} = g(x_n) \quad \text{and} \quad \frac{u_{-1} - u_1}{2h} = g(x_0),$$

respectively, where we set $x_{-1} := -h$ and $x_{n+1} = 1 + h$. Hence, there holds

$$u_{n+1} = u_{n-1} + 2hg(x_n) \quad \text{and} \quad u_{-1} = u_1 + 2hg(x_0).$$

Now we can insert these expressions into the discretisation of the Laplacian and end up with the equations

$$\Delta_h u(x_0) = \frac{2hg(x_0) - 2u_0 + 2u_1}{h^2} \quad \text{and} \quad \Delta_h u(x_n) = \frac{2hg(x_n) - 2u_n + 2u_{n-1}}{h^2}.$$

△

Next, we are concerned with the solvability and the rate of convergence of the finite difference method. To that end, we start with a discrete version of the maximum principle for the problem at hand. For a more general version, we refer to [Braess].

(1.7) **Theorem (Discrete maximum principle).** Let u_h be the solution to the discrete problem

$$(1.8) \qquad -\Delta_h u_h = f \quad \text{in } \Omega_h \text{ with } f \leqslant 0.$$

Then, there holds

$$\max_{\boldsymbol{x} \in \Omega_h} u_h(\boldsymbol{x}) \leqslant \max_{\boldsymbol{z} \in \Gamma_h} u_h(\boldsymbol{z}).$$

Proof. For a proof of this theorem, see [Braess]. □

As a consequence, the properties derived in Corollary (I.6.3) directly transfer to u_h. This accounts particularly for the comparison principle and the continuous dependence on the data. Moreover, there holds the following

(1.9) **Corollary.** The solution $\boldsymbol{u} \in \mathbb{R}^{(n-1)^d}$ to

$$\boldsymbol{L}^{(d)} \boldsymbol{u} = \boldsymbol{f},$$

cf. (1.6), is unique.

Proof. The solution of the homogenous system $\boldsymbol{L}^{(d)}\boldsymbol{u} = \boldsymbol{0}$ corresponds to the solution of (1.8) with $f \equiv 0$ and homogenous Dirichlet data. Hence,

$$\min_{\boldsymbol{x}\in\Omega_h} u_h(\boldsymbol{x}) = \max_{\boldsymbol{x}\in\Omega_h} u_h(\boldsymbol{x}) = 0.$$

Therefore, the homogenous system has only the trivial solution and the matrix $\boldsymbol{L}^{(d)}$ is nonsingular. □

2 Convergence of the Finite Difference Method

On Ω_h and $\overline{\Omega}_h$, we introduce the sup norm according to

$$\|v_h\|_{\Omega_h} := \max_{\boldsymbol{x}\in\Omega_h} |v_h(\boldsymbol{x})| \quad \text{and} \quad \|v_h\|_{\overline{\Omega}_h} := \max_{\boldsymbol{x}\in\overline{\Omega}_h} |v_h(\boldsymbol{x})|,$$

respectively.

(2.1) **Definition.** Let $\mathcal{L}_h$ denote the finite difference approximation of the second order differential operator $\mathcal{L}$. The corresponding finite difference method is called …

- … *convergent* of order p, iff

$$\|u - u_h\|_{\overline{\Omega}_h} = \mathcal{O}(h^p).$$

- … *consistent* of order p, iff

$$\|\mathcal{L}_h u - \mathcal{L}u\|_{\Omega_h} = \mathcal{O}(h^p).$$

- … *stable*, iff there exists $C_s > 0$ such that for all $v_h \colon \Omega_h \to \mathbb{R}$ with $v_h|_{\Gamma_h} = 0$ holds

$$\|v_h\|_{\overline{\Omega}_h} \leqslant C_s \|\mathcal{L}_h v_h\|_{\Omega_h}.$$

Note that there holds $\|\Delta_h u - \Delta u\|_{\Omega_h} = \mathcal{O}(h^2)$. Thus, the discretisation of the Laplacian is consistent of order 2.

The stability of the discretisation translates to $\|(\boldsymbol{L}^{(d)})^{-1}\|_\infty \leqslant C_s$ independently of the mesh width $h > 0$: Let $\boldsymbol{v}, \boldsymbol{w}$ contain the values of a grid function $v_h \colon \Omega_h \to \mathbb{R}$ with $v_h|_{\Gamma_h} = 0$ and $\Delta_h v_h$, respectively. Hence, there holds $\boldsymbol{w} = \boldsymbol{L}^{(d)}\boldsymbol{v}$. Now, the stability condition translates to

$$\left\|\left(\boldsymbol{L}^{(d)}\right)^{-1}\boldsymbol{w}\right\|_\infty = \|\boldsymbol{v}\|_\infty = \|v_h\|_{\overline{\Omega}_h} \leqslant C_s\|\Delta_h v_h\|_{\Omega_h} = C_s\|\boldsymbol{w}\|_\infty.$$

A similar consideration also applies for general finite difference approximations $\mathcal{L}_h$.

The next theorem tells us under which conditions a finite difference method is convergent.

(2.2) **Theorem.** If a finite difference method is stable and consistent of order p, then it is also convergent of order p.

Proof. There holds

$$\|u - u_h\|_{\overline{\Omega}_h} \leqslant C_s \|\mathcal{L}_h(u - u_h)\|_{\Omega_h} = C_s \|\mathcal{L}_h u - \mathcal{L} u\|_{\Omega_h} = \mathcal{O}(h^p),$$

since $(\mathcal{L}_h u_h)(\boldsymbol{x}) = f(\boldsymbol{x}) = (\mathcal{L}u)(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega_h$. □

The theorem tells us that, in order to show convergence of the finite element discretisation (1.4) of the Laplacian, it remains to show the stability, which also holds on more general domains.

(2.3) **Theorem.** Assume $\Omega \subset B_R(\mathbf{0})$ and let $v_h \colon \Omega_h \to \mathbb{R}$, $v_h|_{\Gamma_h} = 0$. Then, it holds

$$\|v_h\|_{\overline{\Omega}_h} \leqslant \frac{R^2}{2d} \|\Delta_h v_h\|_{\Omega_h}.$$

Proof. Let $w(\boldsymbol{x}) := (R^2 - \|\boldsymbol{x}\|_2^2)/(2d)$. It holds

$$-\Delta_h w = -\Delta w = 1 \text{ in } \Omega_h, \quad w \geqslant 0 \text{ on } \Gamma_h.$$

Now, consider the solution u_h to $-\Delta_h u_h = 1$ in Ω_h, $u_h|_{\Gamma_h} = 0$. The discrete comparison principle gives us $u_h(\boldsymbol{x}) \leqslant w(\boldsymbol{x})$ for all $\boldsymbol{x} \in \overline{\Omega}_h$. Together with the minimum principle, we arrive at

$$0 \leqslant u_h(\boldsymbol{x}) \leqslant \frac{1}{2d}(R^2 - \|\boldsymbol{x}\|_2^2) \quad \text{for all } \boldsymbol{x} \in \overline{\Omega}_h. \tag{2.4}$$

From this, the stability is derived as follows: Let $v_h \colon \Omega_h \to \mathbb{R}$, $v_h|_{\Gamma_h} = 0$. It holds

$$-\big(\Delta_h(-u_h)\big)(\boldsymbol{x}) = -1 \leqslant -\frac{(\Delta_h v_h)(\boldsymbol{x})}{\|\Delta_h v_h\|_{\Omega_h}} \leqslant 1 = -(\Delta_h u_h)(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in \Omega_h.$$

The discrete comparison principle yields

$$-u_h(\boldsymbol{x}) \leqslant \frac{v_h(\boldsymbol{x})}{\|\Delta_h v_h\|_{\Omega_h}} \leqslant u_h(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in \overline{\Omega}_h$$

and hence due to (2.4) we arrive at

$$\frac{\|v_h\|_{\overline{\Omega}_h}}{\|\Delta_h v_h\|_{\Omega_h}} \leqslant \|u_h\|_{\overline{\Omega}_h} \leqslant \frac{R^2}{2d}.$$

□

III. Variational Formulation

1 Sobolev Spaces

In what follows, let $\Omega \subset \mathbb{R}^d$ denote a domain with piecewise smooth boundary. The function space $L^2(\Omega)$ consists of all equivalence classes of square integrable functions on Ω. Two functions $f, g\colon \Omega \to \mathbb{R}$ are identified, if $f(\boldsymbol{x}) = g(\boldsymbol{x})$ for almost every $\boldsymbol{x} \in \Omega$. Endowed with the inner product

$$(f, g)_{L^2(\Omega)} := \int_\Omega f(\boldsymbol{x}) g(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x},$$

$L^2(\Omega)$ becomes a *Hilbert space* with the induced norm

$$\|f\|_{L^2(\Omega)} := \sqrt{(f, f)_{L^2(\Omega)}}.$$

Note that two functions $f, g\colon \Omega \to \mathbb{R}$ are identified, iff $\|f - g\|_{L^2(\Omega)} = 0$.

(1.1) **Definition.** The function $u \in L^2(\Omega)$ posseses the *weak derivative* $v = \partial^{\boldsymbol{\alpha}} u$ in $L^2(\Omega)$, iff $v \in L^2(\Omega)$ and

$$(v, \varphi)_{L^2(\Omega)} = (-1)^{|\boldsymbol{\alpha}|}(u, \partial^{\boldsymbol{\alpha}}\varphi) \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

(1.2) **Example.** Let $\Omega = (0, 1)$ and consider the function $u(x) = |x - 0.5|$. For $\varphi \in C_0^\infty(0, 1)$, integration by parts yields

$$\begin{aligned} -\int_0^1 u\varphi'(x)\,\mathrm{d}x &= -\int_0^{0.5} (0.5 - x)\varphi'(x)\,\mathrm{d}x - \int_{0.5}^1 (x - 0.5)\varphi'(x)\,\mathrm{d}x \\ &= -[u(x)\varphi(x)]_0^1 + \int_0^{0.5} -1 \cdot \varphi(x)\,\mathrm{d}x + \int_{0.5}^1 1 \cdot \varphi(x)\,\mathrm{d}x \\ &= \int v(x)\varphi(x)\,\mathrm{d}x, \end{aligned}$$

where v is the *heavyside function*, which is defined as

$$v(x) = \begin{cases} 1, & x \geqslant 0, \\ -1, & x < 0. \end{cases}$$

Hence, the weak derivative of u is given by $u' = v$. △

(1.3) **Remark.** Let $u \in C^1(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$, the divergence theorem yields

$$(u_{x_i}, \varphi)_{L^2(\Omega)} + (u, \varphi_{x_i})_{L^2(\Omega)} = \int_\Omega \frac{\partial}{\partial x_i}(u\varphi)\,\mathrm{d}\boldsymbol{x} = \int_\Gamma u\varphi n_i \,\mathrm{d}\sigma,$$

where $\boldsymbol{n} = [n_1, \dots, n_d]^\intercal \in \mathbb{R}^d$ is the outward pointing normal vector. Consequently, it holds

$$(u_{x_i}, \varphi)_{L^2(\Omega)} = -(u, \varphi_{x_i})_{L^2(\Omega)} \quad \text{for all } \varphi \in C_0^\infty(\Omega),$$

i.e. the weak derivative coincides with the classical derivative. △

(1.4) **Definition.** Let $m \in \mathbb{N}$. The *Sobolev space* $H^m(\Omega)$ is defined according to

$$H^m(\Omega) := \{v \in L^2(\Omega) : \partial^{\boldsymbol{\alpha}} v \in L^2(\Omega) \text{ for all } |\boldsymbol{\alpha}| \leqslant m\}.$$

(1.5) **Theorem.** The Sobolev space $H^m(\Omega)$ endowed with the inner product

$$(v, w)_{H^m(\Omega)} := \sum_{|\boldsymbol{\alpha}| \leqslant m} (\partial^{\boldsymbol{\alpha}} v, \partial^{\boldsymbol{\alpha}} w)_{L^2(\Omega)}$$

and the corresponding norm $\|v\|_{H^m(\Omega)} = \sqrt{(v, v)_{H^m(\Omega)}}$ is a Hilbert space.

Proof. Let $\{v_n\}_{n\in\mathbb{N}}$ denote a Cauchy sequence in $H^m(\Omega)$. Then, due to the definition of the $\|\cdot\|_{H^m(\Omega)}$-norm, the sequence $\{\partial^{\boldsymbol{\alpha}} v_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$ for every $|\boldsymbol{\alpha}| \leqslant m$. The completeness of $L^2(\Omega)$ implies that there exist the limits

$$\big\|\partial^{\boldsymbol{\alpha}} v_n - v^{(\boldsymbol{\alpha})}\big\|_{L^2(\Omega)} \to 0, \quad n \to \infty$$

for certain functions $v^{(\boldsymbol{\alpha})} \in L^2(\Omega)$. It remains to show $\partial^{\boldsymbol{\alpha}} v^{(\boldsymbol{0})} = v^{(\boldsymbol{\alpha})}$. To that end, let $\{w_n\}_{n\in\mathbb{N}}$ denote a Cauchy sequence in $L^2(\Omega)$ with limit $w \in L^2(\Omega)$. Due to the *Cauchy-Schwarz inequality*, it holds

$$\big|(w_n - w, \varphi)_{L^2(\Omega)}\big| \leqslant \|w - w_n\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)}$$

and hence $(w_n, \varphi)_{L^2(\Omega)} \to (w, \varphi)_{L^2(\Omega)}$ for every $\varphi \in C_0^\infty(\Omega)$. Consequently, we have

$$\begin{aligned}(v^{(\boldsymbol{\alpha})}, \varphi)_{L^2(\Omega)} &= \lim\nolimits_{n\to\infty} (\partial^{\boldsymbol{\alpha}} v_n, \varphi)_{L^2(\Omega)} \\ &= \lim\nolimits_{n\to\infty} (-1)^{|\boldsymbol{\alpha}|} (v_n, \partial^{\boldsymbol{\alpha}}\varphi)_{L^2(\Omega)} = (-1)^{|\boldsymbol{\alpha}|} (v^{(\boldsymbol{0})}, \partial^{\boldsymbol{\alpha}}\varphi)_{L^2(\Omega)}.\end{aligned}$$

By the definition of the weak derivative, we arrive at $\partial^{\boldsymbol{\alpha}} v^{(\boldsymbol{0})} = v^{(\boldsymbol{\alpha})}$. □

(1.6) **Remark.** The smoothness of the functions in $H^m(\Omega)$ in the classical sense is dependent on the spatial dimension: For $d = 1$, there holds $H^1(\Omega) \subset C^0(\Omega)$. For $d = 2$, functions in $H^1(\Omega)$ may exhibit point singularities. For example, it holds

$$v(r, \varphi) = \log\left(\log \frac{2}{r}\right) \in H^1\big(B_1(\boldsymbol{0})\big).$$

More general, there holds

$$\|\boldsymbol{x}\|_2^{-\beta} \in H^1\big(B_1(\boldsymbol{0})\big) \quad \text{for } \beta < \frac{d-2}{2} \text{ and } d \geqslant 3.$$

△

Note that it is also possible to introduce Sobolev spaces without the recourse to the concept of weak derivatives.

(1.7) **Theorem.** Let $\Omega \subset \mathbb{R}^d$ denote a domain and let $m \geqslant 0$. Then $C^\infty(\Omega) \cap H^m(\Omega)$ is dense in $H^m(\Omega)$.

Proof. See for example [Alt,Wloka]. □

The theorem tells us that $H^m(\Omega)$ is the completion of $C^\infty(\Omega) \cap H^m(\Omega)$ with respect to the $\|\cdot\|_{H^m(\Omega)}$-norm, i.e.

$$H^m(\Omega) = \overline{C^\infty(\Omega) \cap H^m(\Omega)}^{\|\cdot\|_{H^m(\Omega)}}.$$

Based on this fact, we introduce a corresponding generalisation for functions with zero boundary values.

(1.8) **Definition.** We define the Sobolev spaces $H_0^m(\Omega)$ for $m \in \mathbb{N}$ as the completion of $C_0^\infty(\Omega)$ with respect to the $\|\cdot\|_{H^m(\Omega)}$-norm, i.e.

$$H_0^m(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^m(\Omega)}}.$$

Note that $H_0^m(\Omega)$ is a closed subspace of $H^m(\Omega)$ and hence also a Hilbert space. Moreover, there holds $H_0^0(\Omega) = H^0(\Omega) = L^2(\Omega)$ such that we arrive at the following scheme

$$\begin{array}{ccccccccccc}
L^2(\Omega) & = & H^0(\Omega) & \supset & H^1(\Omega) & \supset & H^2(\Omega) & \supset & \dots & & \\
 & & \| & & \cup & & \cup & & & & \\
 & & H_0^0(\Omega) & \supset & H_0^1(\Omega) & \supset & H_0^2(\Omega) & \supset & \dots & \supset & C_0^\infty(\Omega).
\end{array}$$

In particular, $C_0^\infty(\Omega)$ is a dense subset of $L^2(\Omega)$.
The functional

$$|v|_{H^m(\Omega)} := \sqrt{\sum_{|\boldsymbol{\alpha}|=m} \|\partial^{\boldsymbol{\alpha}} v\|_{L^2(\Omega)}^2}$$

defines a *seminorm* on $H^m(\Omega)$, i.e. $|\cdot|_{H^m(\Omega)}$ satisfies all properties of a norm except for that it is not point-separating. Obviously, there holds $|v|_{H^m(\Omega)} = 0$, $m \geqslant 1$, for any constant function $v \in H^m(\Omega)$. However , the next theorem tells us that $|\cdot|_{H^m(\Omega)}$ is an equivalent norm on $H_0^m(\Omega)$.

(1.9) **Theorem (Poincaré inequality).** Let $\Omega \subset [0,s]^d$ for some $s > 0$. Then, it holds

$$\|v\|_{L^2(\Omega)} \leqslant s|v|_{H^1(\Omega)} \quad \text{for every } v \in H_0^1(\Omega).$$

Proof. Since $C_0^\infty(\Omega)$ is a dense subset of $H_0^1(\Omega)$, it is sufficient to show the result for $v \in C_0^\infty(\Omega)$. We set $v(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in [0,s]^d \setminus \Omega$. Then, the fundamental theorem of calculus yields

$$v(\boldsymbol{x}) = v(0, x_2, \dots, x_d) + \int_0^{x_1} v_{x_1}(y, x_2, \dots, x_d)\,\mathrm{d}y.$$

Since $v(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in [0,s]^d \setminus \Omega$, the first term vanishes. Now, the application of the Cauchy-Schwarz inequality yields

$$\begin{aligned}\big(v(\boldsymbol{x})\big)^2 &= \bigg(\int_0^{x_1} v_{x_1}(y, x_2, \dots, x_d)\,\mathrm{d}y\bigg)^2 \leqslant \int_0^{x_1} 1^2\,\mathrm{d}y \int_0^{x_1} \big(v_{x_1}(y, x_2, \dots, x_d)\big)^2\,\mathrm{d}y \\ &\leqslant s \int_0^{s} \big(v_{x_1}(y, x_2, \dots, x_d)\big)^2\,\mathrm{d}y.\end{aligned}$$

Since the right hand side is independent of x_1, integration with respect to x_1 yields

$$\int_0^s \big(v(\boldsymbol{x})\big)^2\,\mathrm{d}x_1 \leqslant s^2 \int_0^s \big(v_{x_1}(y, x_2, \dots, x_d)\big)^2\,\mathrm{d}y = s^2 \int_0^s \big(v_{x_1}(\boldsymbol{x})\big)^2\,\mathrm{d}x_1.$$

Finally, integrating with respect to the other coordinates yields

$$\begin{aligned}\|v\|_{L^2(\Omega)}^2 &= \int_\Omega \big(v(\boldsymbol{x})\big)^2\,\mathrm{d}\boldsymbol{x} = \int_{[0,s]^d} \big(v(\boldsymbol{x})\big)^2\,\mathrm{d}\boldsymbol{x} \\ &\leqslant s^2 \int_{[0,s]^d} \big(v_{x_1}(\boldsymbol{x})\big)^2\,\mathrm{d}\boldsymbol{x} = s^2 \int_\Omega \big(v_{x_1}(\boldsymbol{x})\big)^2\,\mathrm{d}\boldsymbol{x} \leqslant s^2 |v|_{H^1(\Omega)}^2.\end{aligned}$$ □

(1.10) **Remark.** The Poincaré inequality also holds under the weaker assumption of homogenous boundary conditions on part $\Gamma_D \subset \Gamma$ of the boundary with $|\Gamma_D| > 0$. △

(1.11) **Corollary.** Let $\Omega \subset [0,s]^d$ for some $s > 0$. Then, it holds

$$|v|_{H^m(\Omega)} \leqslant \|v\|_{H^m(\Omega)} \leqslant (1+s)^m |v|_{H^m(\Omega)} \quad \text{for every } v \in H_0^m(\Omega),\ m \geqslant 0.$$

Proof. The proof is by induction on m. See for example [Braess]. □

The boundary Γ of a domain $\Omega \subset \mathbb{R}^d$ is a null set with respect to the d-dimensional Lebesgue measure. Hence, functions $v \in L^2(\Omega)$ do not exhibit boundary values. The next theorem tells us that the situation is different for functions in $H^1(\Omega)$.

(1.12) **Theorem (Trace theorem).** Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with piecewise smooth boundary. In addition, let Ω satisfy the *cone condition*, i.e. there exist an angle $\alpha_0 > 0$ such that the interior angle at every corner of Ω is bigger than α_0. Then, there exists a continuous linear mapping

$$\gamma \colon H^1(\Omega) \to L^2(\Gamma), \quad \|\gamma(v)\|_{L^2(\Gamma)} \leqslant c\|v\|_{H^1(\Omega)}, \quad c > 0,$$

such that $\gamma(v) = v|_\Gamma$ for every $v \in C^1(\overline{\Omega})$.

Proof. See for example [Braess]. □

2 Variational Formulation of Dirichlet Problems

We consider the Dirichlet problem

$$\begin{aligned}-\operatorname{div}\big(\boldsymbol{A}(\boldsymbol{x})\nabla u(\boldsymbol{x})\big) + c(\boldsymbol{x})u(\boldsymbol{x}) &= f(\boldsymbol{x}), & \boldsymbol{x} &\in \Omega, \\ u(\boldsymbol{x}) &= 0, & \boldsymbol{x} &\in \Gamma.\end{aligned} \tag{2.1}$$

In addition to the classical solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ we will derive in this section also *weak solutions*, which are obtained from the *variational formulation*.

(2.2) **Theorem (Characterisation theorem).** Let V denote a vector space and denote by

$$a\colon V \times V \to \mathbb{R}$$

a symmetric and positive bilinear form, i.e. $a(v,w) = a(w,v)$ and $a(v,v) > 0$ for all $v \in V \setminus \{0\}$. Moreover, let

$$\ell\colon V \to \mathbb{R}$$

be a linear functional. Then, the functional

$$J(v) := \frac{1}{2}a(v,v) - \ell(v)$$

attains its minimum at $u \in V$, iff

$$a(u,v) = \ell(v) \quad \text{for all } v \in V. \tag{2.3}$$

In addition, there exists at most one solution to (2.3).

Proof. Let $u, v \in V$ and $t \in \mathbb{R}$. It holds

$$\begin{aligned} J(u+tv) &= \frac{1}{2}a(u+tv, u+tv) - \ell(u+tv) \\ &= J(u) + t[a(u,v) - \ell(v)] + \frac{1}{2}t^2 a(v,v). \end{aligned} \tag{2.4}$$

Hence, if $u \in V$ satisfies (2.3), there holds

$$J(u+tv) = J(u) + \frac{1}{2}t^2 a(v,v) > J(u) \quad \text{for every } v \neq 0,\ t > 0.$$

Therefore, u is the unique minimum of J. Vice versa, if u is a minimum of J then the derivative of the function $t \mapsto J(u+tv)$ must vanish at $t=0$. The derivative is according to (2.4) given by $a(u,v) - \ell(v)$, which directly yields (2.3). □

The next theorem tells us that every classical solution to (2.1) is also a minimum in the sense of the characterisation theorem.

(2.5) **Theorem.** Every classical solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to (2.1) is also a solution to the variational problem

$$J(v) := \int_\Omega \frac{1}{2}\big(\langle \boldsymbol{A}\nabla v, \nabla v\rangle + cv^2\big) - fv \,\mathrm{d}\boldsymbol{x} \to \min$$

among all functions $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with $v|_\Gamma = 0$.

Proof. The divergence theorem yields

$$\int_\Omega \operatorname{div}(\boldsymbol{A}\nabla u)v + \langle \boldsymbol{A}\nabla u, \nabla v\rangle \,\mathrm{d}\boldsymbol{x} = \int_\Omega \operatorname{div}\big((\boldsymbol{A}\nabla u)v\big) \,\mathrm{d}\boldsymbol{x} = \int_\Gamma \langle \boldsymbol{A}\nabla u, \boldsymbol{n}\rangle v \,\mathrm{d}\sigma.$$

Hence, for functions v with $v|_\Gamma = 0$, we obtain

$$\int_\Omega \operatorname{div}(\boldsymbol{A}\nabla u) v \,\mathrm{d}\boldsymbol{x} = -\int_\Omega \langle \boldsymbol{A}\nabla u, \nabla v\rangle \,\mathrm{d}\boldsymbol{x}.$$

Next, we define

$$a(u,v) := \int_\Omega \langle \boldsymbol{A}\nabla u, \nabla v\rangle + cuv \,\mathrm{d}\boldsymbol{x}, \quad \ell(v) := (f,v)_{L^2(\Omega)}.$$

Then, it holds for every $v \in C^1(\Omega) \cap C^0(\overline{\Omega})$ with $v|_\Gamma = 0$ that

$$\begin{aligned} a(u,v) - \ell(v) &= \int_\Omega \langle \boldsymbol{A}\nabla u, \nabla v\rangle + cuv - fv \,\mathrm{d}\boldsymbol{x} \\ &= \int_\Omega \big(-\operatorname{div}(\boldsymbol{A}\nabla u) + cu - f\big) v \,\mathrm{d}\boldsymbol{x} = 0, \end{aligned}$$

if u is a classical solution to (2.1). The minimal property is now implied by the characterisation theorem. □

In a similar fashion, one shows that every solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to the variational problem is also a classical solution to (2.1). However, a minimum of J does not necessarily exist in $C^2(\Omega) \cap C^0(\overline{\Omega})$.

(2.6) **Definition.** Let H denote a Hilbert space with norm $\|\cdot\|_H$. A bilinear form $a\colon H \times H \to \mathbb{R}$ is *continuous*, iff there exists $c_S > 0$ such that

$$|a(v,w)| \leqslant c_S \|v\|_H \|w\|_H \quad \text{for all } v, w \in H.$$

The bilinear form is called *elliptic*, iff there exists $c_E > 0$ such that

$$a(v,v) \geqslant c_E \|v\|_H^2 \quad \text{for all } v \in H.$$

The *energy norm*

$$\|v\|_a := \sqrt{a(v,v)} \tag{2.7}$$

induced by a continuous and elliptic bilinear form a on a Hilbert space H is equivalent to the Hilbert space norm. Obviously, there holds

$$\sqrt{c_E}\|v\|_H \leqslant \|v\|_a \leqslant \sqrt{c_S}\|v\|_H \quad \text{for all } v \in H.$$

(2.8) **Theorem (Lax-Milgram).** Let $V \subset H$ be a closed subspace of the Hilbert space H with dual space V'. Moreover, let $a\colon H \times H \to \mathbb{R}$ be a continuous bilinear form that is elliptic on V. Then, for every $\ell \in V'$, the variational problem

$$J(v) := \frac{1}{2} a(v,v) - \ell(v) \to \min$$

exhibits a unique solution $u \in V$.

Proof. See for example [Braess]. □

(2.9) **Remark.** In the particular case $V = H$ and $a(v,w) = (v,w)_H$, the Lax-Milgram theorem yields the Riesz representation theorem: For every $\ell \in H'$ there exists a $u \in H$ such that

$$(u,v)_H = \ell(v) \quad \text{for all } v \in H.$$

△

In view of the Lax-Milgram theorem, we can now specify the notion of *weak solution*.

(2.10) **Definition.** A function $u \in H_0^1(\Omega)$ is called *weak solution* to (2.1), iff there holds

$$a(u,v) = \ell(v) \quad \text{for all } v \in H_0^1(\Omega),$$

where a and ℓ are given as in the characterisation theorem.

The next theorem specifies under which conditions (2.1) exhibits a unique weak solution.

(2.11) **Theorem.** Let $f \in L^2(\Omega)$ and

$$0 \leqslant c(\boldsymbol{x}) \leqslant \overline{c} < \infty, \qquad 0 < \underline{\alpha}\|\boldsymbol{\xi}\|_2^2 \leqslant \boldsymbol{\xi}^\intercal \boldsymbol{A}(\boldsymbol{x})\boldsymbol{\xi} \leqslant \overline{\alpha}\|\boldsymbol{\xi}\|_2^2 < \infty$$

for all $\boldsymbol{x} \in \Omega$, $\boldsymbol{\xi} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\}$. Then, (2.1) exhibits a unique weak solution $u \in H_0^1(\Omega)$, which solves the minimisation problem

$$J(v) = \frac{1}{2}a(v,v) - \ell(v) \to \min.$$

Proof. Since

$$\begin{aligned} a(v,w) = \int_\Omega \langle \boldsymbol{A}\nabla v, \nabla w\rangle + cvw \,\mathrm{d}\boldsymbol{x} &\leqslant \int_\Omega \overline{\alpha}\|\nabla v\|_2\|\nabla w\|_2 + \overline{c}|v||w| \,\mathrm{d}\boldsymbol{x} \\ &\leqslant \overline{\alpha}\sqrt{\int_\Omega \|\nabla v\|_2^2 \,\mathrm{d}\boldsymbol{x}}\sqrt{\int_\Omega \|\nabla w\|_2^2 \,\mathrm{d}\boldsymbol{x}} + \overline{c}\sqrt{\int_\Omega v^2 \,\mathrm{d}\boldsymbol{x}}\sqrt{\int_\Omega v^2 \,\mathrm{d}\boldsymbol{x}} \\ &\leqslant \max\{\overline{\alpha},\overline{c}\}\|v\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)}, \end{aligned}$$

the bilinear form a is continuous on $H^1(\Omega)$. The ellipticity of a follows from

$$a(v,v) = \int_\Omega \langle \boldsymbol{A}\nabla v, \nabla v\rangle + cv^2 \,\mathrm{d}\boldsymbol{x} \geqslant \underline{\alpha}\int_\Omega \|\nabla v\|_2^2 \,\mathrm{d}\boldsymbol{x} = \underline{\alpha}|v|_{H^1(\Omega)}^2,$$

since by the Poincaré inequality $|\cdot|_{H^1(\Omega)}$ is an equivalent norm on $H_0^1(\Omega)$. Finally, since

$$|\ell(v)| = \left|\int_\Omega fv \,\mathrm{d}\boldsymbol{x}\right| \leqslant \|f\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)},$$

also the linear form is continuous. Therefore, the unique solvability follows from the Lax-Milgram theorem. □

(2.12) **Remark.**

(a) The dual space of $H^{-1}(\Omega) := [H_0^1(\Omega)]'$ can be defined as completion of $L^2(\Omega)$ with respect to the *dual norm*

$$\|f\|_{H^{-1}(\Omega)} := \sup_{0\neq v\in H^1(\Omega)} \frac{|(f,v)_{L^2(\Omega)}|}{\|v\|_{H^1(\Omega)}}, \quad f\in L^2(\Omega).$$

Hence, we can extend the linear form to $H^{-1}(\Omega)$ according to

$$|\ell(v)| = \|v\|_{H^1(\Omega)} \frac{|(f,v)_{L^2(\Omega)}|}{\|v\|_{H^1}(\Omega)} \leqslant \|v\|_{H^1(\Omega)}\|f\|_{H^{-1}(\Omega)}.$$

Since $H^{-1}(\Omega) \supset L^2(\Omega)$, this implies that the right hand side f in (2.1) does not even have to be square integrable.

(b) The case of inhomogenous Dirichlet conditions $u = g \not\equiv 0$ at Γ is usually reduced to the homogenous case as follows: Determine $u_g \in H^1(\Omega)$ with $u_g = g$ on Γ in the sense of the trace theorem. Now, letting $u = u_0 + u_g$ leads to the variational problem:

$$\begin{aligned}&\text{find } u_0\in H_0^1(\Omega) \text{ such that}\\ &\qquad a(u_0,v) = \ell(v) - a(u_g,v) \quad \text{for all } v\in H_0^1(\Omega).\end{aligned}$$

△

3 Variational Formulation of Neumann Problems

We consider the Neumann problem

$$\begin{aligned} -\operatorname{div}\big(\boldsymbol{A}(\boldsymbol{x})\nabla u(\boldsymbol{x})\big) + c(\boldsymbol{x})u(\boldsymbol{x}) &= f(\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega,\\ \langle \boldsymbol{A}(\boldsymbol{x})\nabla u(\boldsymbol{x}), \boldsymbol{n}\rangle &= g(\boldsymbol{x}), \quad \boldsymbol{x}\in\Gamma. \end{aligned} \tag{3.1}$$

with a uniformly elliptic differential operator and a bounded, positive reaction term, i.e.

$$0 < \underline{\alpha}\|\boldsymbol{\xi}\|_2^2 \leqslant \boldsymbol{\xi}^\intercal \boldsymbol{A}(\boldsymbol{x})\boldsymbol{\xi} \leqslant \overline{\alpha}\|\boldsymbol{\xi}\|_2^2 < \infty, \quad 0 < \underline{c} \leqslant c(\boldsymbol{x}) \leqslant \overline{c} < \infty.$$

Multiplying (3.1) by $\varphi \in C^\infty \cap H^1(\Omega)$ and integration yield

$$\begin{aligned}&\int_\Omega \big(-\operatorname{div}(\boldsymbol{A}\nabla u) + cu\big)\varphi \,\mathrm{d}\boldsymbol{x}\\ &\quad = \int_\Omega \langle \boldsymbol{A}\nabla u, \nabla\varphi\rangle + cu\varphi\,\mathrm{d}\boldsymbol{x} - \int_\Gamma \langle \boldsymbol{A}\nabla u, \boldsymbol{n}\rangle\varphi\,\mathrm{d}\sigma = \int_\Omega f\varphi\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned} a(v,w) &= \int_\Omega \langle \boldsymbol{A}\nabla v, \nabla w\rangle + cvw\,\mathrm{d}\boldsymbol{x},\\ \ell(v) &= \int_\Gamma gv\,\mathrm{d}\sigma + \int_\Omega fv\,\mathrm{d}\boldsymbol{x} \qquad\qquad \text{for } v,w\in H^1(\Omega).\end{aligned}$$

The continuity of the bilinear form a follows as for the Dirichlet problem. In addition, there holds

$$a(v,v) \geqslant \underline{\alpha}|v|^2_{H^1(\Omega)} + \underline{c}\|v\|^2_{L^2(\Omega)} \geqslant \min\{\underline{\alpha},\underline{c}\}\|v\|^2_{H^1(\Omega)},$$

which shows the ellipticity of a on $H^1(\Omega)$. Moreover, for $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$, we obtain

$$|\ell(v)| \leqslant \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)}\|\gamma(v)\|_{L^2(\Gamma)} \leqslant C\|v\|_{H^1(\Omega)}, \quad C > 0,$$

by the trace theorem. Hence, the linear form is continuous.

(3.2) **Theorem.** Let $\Omega \subset \mathbb{R}^d$ satisfy the cone condition and let $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$. Then, (3.1) exhibits a unique weak solution $u \in H^1(\Omega)$, which solves the minimisation problem

$$J(v) = \frac{1}{2}a(v,v) - \ell(v) \to \min.$$

In addition, it holds $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, iff (3.1) has a classical solution.

Proof. The first part of the proof follows from the Lax-Milgram theorem. The second part follows by the fundamental lemma of calculus of variations. □

If $c \equiv 0$ in (3.1), then obviously $u + \eta$, $\eta \in \mathbb{R}$, is a solution to (3.1). In particular, the bilinear form a is not elliptic anymore. Let $V := \big\{v \in H^1(\Omega) : (v,1)_{L^2(\Omega)} = 0\big\} \subset H^1(\Omega)$ be the subspace of functions with vanishing *mean value*

$$\overline{v} := \frac{1}{|\Omega|}(v,1)_{L^2(\Omega)} = 0.$$

Since there holds a Poincaré inequality of the form

$$\|v\|_{L^2(\Omega)} \leqslant C(|\overline{v}| + |v|_{H^1(\Omega)}), \quad C > 0,$$

in $H^1(\Omega)$, the bilinear form a is elliptic on V. Consequently, the Lax-Milgram theorem indicates the unique solvability of (3.1) in V in the case $c \equiv 0$.

(3.3) **Remark.** For $v \equiv 1$, the variational formulation yields the *compatibility condition*

$$\int_\Omega f \,\mathrm{d}\boldsymbol{x} + \int_\Gamma g \,\mathrm{d}\sigma = 0,$$

which is necessary to guarantee the existence of a solution. However, there holds

$$(f+\eta, v)_{L^2(\Omega)} = \int_\Omega fv \,\mathrm{d}\boldsymbol{x} + \eta\int_\Omega v \,\mathrm{d}\boldsymbol{x} = (f,v)_{L^2(\Omega)} \quad \text{for every } v \in V,\ \eta \in \mathbb{R}.$$

To enforce the compatibility condition, we set $\tilde{f} = f - \eta$ and make the ansatz

$$0 = \int_\Omega \tilde{f} \,\mathrm{d}\boldsymbol{x} + \int_\Gamma g \,\mathrm{d}\sigma = \int_\Omega f \,\mathrm{d}\boldsymbol{x} - \eta|\Omega| + \int_\Gamma g \,\mathrm{d}\sigma.$$

This yields

$$\eta := \frac{1}{|\Omega|}\bigg(\int_\Omega f \,\mathrm{d}\boldsymbol{x} + \int_\Gamma g \,\mathrm{d}\sigma\bigg).$$

△

4 Galerkin Methods

Let $a\colon V \times V \to \mathbb{R}$ denote a continuous, elliptic bilinear form on the Hilbert space V. In addition let $\ell \in V'$ denote a continuous linear form. Our goal is to compute the solution $u \in V$ to the variational problem

$$\begin{aligned} &\text{Find } u \in V \text{ such that} \\ &\qquad a(u, v) = \ell(v) \quad \text{for all } v \in V. \end{aligned} \tag{4.1}$$

To that end, we restrict the variational problem to a finite dimensional subspace $V_h \subset V$. This yields the *Galerkin method*

$$\begin{aligned} &\text{Find } u_h \in V_h \text{ such that} \\ &\qquad a(u_h, v_h) = \ell(v_h) \quad \text{for all } v_h \in V_h. \end{aligned} \tag{4.2}$$

Since V_h is a finite dimensional subspace of V, it is particularly closed, i.e. it is itself a Hilbert space. Hence, the existence and the uniqueness of the solution $u_h \in V_h$ are guaranteed by the Lax-Milgram theorem. In addition, there holds

$$c_E \|u_h\|_V^2 \leqslant a(u_h, u_h) = \ell(u_h) \leqslant \|\ell\|_{V'} \|u_h\|_V,$$

where $c_E > 0$ is the constant of ellipticity. From this, we directly infer the stability estimate

$$\|u_h\|_V \leqslant \frac{1}{c_E} \|\ell\|_{V'}.$$

(4.3) **Remark.** If the bilinear form a from (4.1) is additionally symmetric, then, due to the Characterisation theorem (2.2), the variational problem (4.1) is equivalent to the minimisation problem

$$J(v) = \frac{1}{2} a(v, v) - \ell(v) \to \min, \quad v \in V.$$

Replacing V by V_h, we end up with the finite dimensional minimisation problem

$$J(v_h) = \frac{1}{2} a(v_h, v_h) - \ell(v_h) \to \min, \quad v_h \in V_h.$$

The latter procedure is called *Ritz-Galerkin method*. Herein, the solution $u_h \in V_h$ is also described by the Characterisation theorem (2.2). △

The Galerkin method leads to a linear system of equations in a straightforward manner: Let $\{\phi_1, \ldots, \phi_n\}$ denote a basis of V_h. Then, due to linearity, (4.2) is equivalent to

$$\begin{aligned} &\text{Find } u_h \in V_h \text{ such that} \\ &\qquad a(u_h, \phi_i) = \ell(\phi_i) \quad \text{for all } i = 1, \ldots, n. \end{aligned}$$

Hence, making the ansatz

$$u_h = \sum_{i=1}^{n} u_i \phi_i$$

results in the linear system

$$\sum_{j=1}^{n} a(\phi_j, \phi_i) u_j = \ell(\phi_i) \quad \text{for all } i = 1, \dots, n.$$

In matrix notation, this system reads

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f},$$

where $\boldsymbol{A} := [a(\phi_j, \phi_i)]_{i,j=1}^{n}$, $\boldsymbol{u} := [u_i]_{i=1}^{n}$, $\boldsymbol{f} := [\ell(\phi_i)]_{i=1}^{n}$.

The matrix $\boldsymbol{A}$ is referred to as *stiffness matrix.* In case of an elliptic, continuous and symmetric bilinear form a, the matrix $\boldsymbol{A}$ is symmetric and positive definite. There holds

$$\boldsymbol{v}^{\intercal}\boldsymbol{A}\boldsymbol{v} = \sum_{i,j=1}^{n} a(\phi_j, \phi_i) v_i v_j = a\left(\sum_{j=1}^{n} v_j \phi_j, \sum_{i=1}^{n} v_i \phi_i\right) \geqslant c_E \left\| \sum_{i=1}^{n} v_i \phi_i \right\|_V^2.$$

The latter norm only becomes zero, iff $\boldsymbol{v} = \boldsymbol{0}$.

The next lemma quantifies the quality of the approximation by the Galerkin method.

(4.4) **Theorem (Céa's lemma).** Let $a \colon V \times V \to \mathbb{R}$ denote an elliptic and continuous bilinear form. Moreover, let $u \in V$ and $u_h \in V_h \subset V$ denote the solutions to (4.1) and (4.2), respectively. Then, it holds

$$\|u - u_h\|_V \leqslant \frac{c_S}{c_E} \inf_{v_h \in V_h} \|u - v_h\|_V,$$

where $c_E > 0$ is the constant of ellipticity and is $c_S > 0$ the constant of continuity associated to a.

Proof. By definition, there holds

$$a(u, v_h) = \ell(v_h) = a(u_h, v_h) \quad \text{for all } v_h \in V_h \subset V$$

and hence

$$a(u - u_h, v_h) = 0 \quad \text{for all } v_h \in V_h. \tag{4.5}$$

This property is called *Galerkin orthogonality* .

Now, suppose $v_h \in V_h$. Then (4.5) yields for $v_h - u_h \in V_h$ that

$$a(u - u_h, v_h - u_h) = 0$$

and consequently

$$\begin{aligned} c_E \|u - u_h\|_V^2 &\leqslant a(u - u_h, u - u_h) \\ &= a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h) \\ &\leqslant c_S \|u - u_h\|_V \|u - v_h\|_V. \end{aligned}$$

Simplifying this inequality and taking into account that it holds for any $v_h \in V_h$ yields the assertion. □

(4.6) **Corollary.** Let $a\colon V\times V\to\mathbb{R}$ denote an elliptic, continuous and symmetric bilinear form. Moreover, let $u\in V$ and $u_h\in V_h\subset V$ denote the solutions to (4.1) and (4.2), respectively. Then, it holds

$$\|u-u_h\|_V\leqslant\sqrt{\frac{c_S}{c_E}}\inf_{v_h\in V_h}\|u-v_h\|_V,$$

where $c_E>0$ is the constant of ellipticity and is $c_S>0$ the constant of continuity associated to a.

Proof. Due to the Galerkin orthogonality (4.5), the Cauchy-Schwarz inequality for the energy norm, cf. (2.7), reads

$$\begin{aligned}\|u-u_h\|_a^2&=a(u-u_h,u-u_h)=a(u-u_h,u-v_h)\\&\leqslant\|u-u_h\|_a\|u-v_h\|_a&&\text{for all } v_h\in V_h.\end{aligned}$$

This yields Céa's lemma in the energy norm

$$\|u-u_h\|_a\leqslant\|u-v_h\|_a\quad\text{for all } v\in V_h.$$

The claim is now obtained via the inequality

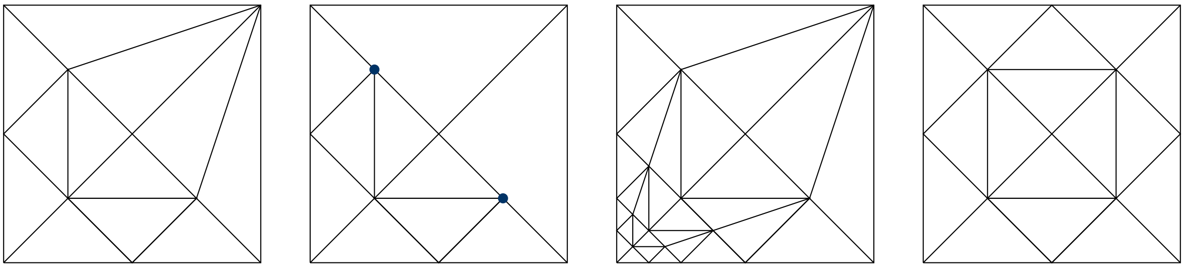
$$c_E\|u-u_h\|_V^2\leqslant\|u-u_h\|_a^2\leqslant\|u-v_h\|_a^2\leqslant c_S\|u-v_h\|_V^2.$$

□

IV. The Finite Element Method

Céa's lemma tells us that the quality of the approximation u_h to the solution u of ((III.4.1)) is governed by the approximation quality of the subspace V_h. Hence, the next step is to construct suitable approximation spaces V_h in a systematic fashion. For the sake of simplicity, we assume in what follows that $\Omega \subset \mathbb{R}^d$ is a polygonal domain.

1 Meshing



admissible non-admissible shape-regular uniform

(1.1) **Definition.** A partition $\mathcal{T} = \{T_1, \dots, T_m\}$ of Ω into simplices is called *admissible*, iff the following two conditions are satisfied:

(a) It holds $\overline{\Omega} = \cup_{i=1}^m T_i$.

(b) If $T_i \cap T_j \neq \varnothing$ for $i \neq j$, then $T_i \cap T_j$ is either a point, an edge or a face of T_i as well as of T_j.

We write $\mathcal{T}_h$ instead of $\mathcal{T}$, iff $\operatorname{diam}(T_i) \leqslant 2h$ for $i = 1, \dots, m$.
A family $\{\mathcal{T}_h\}_{h>0}$ of triangulations is *shape-regular* , iff there exists $\kappa > 0$ such that

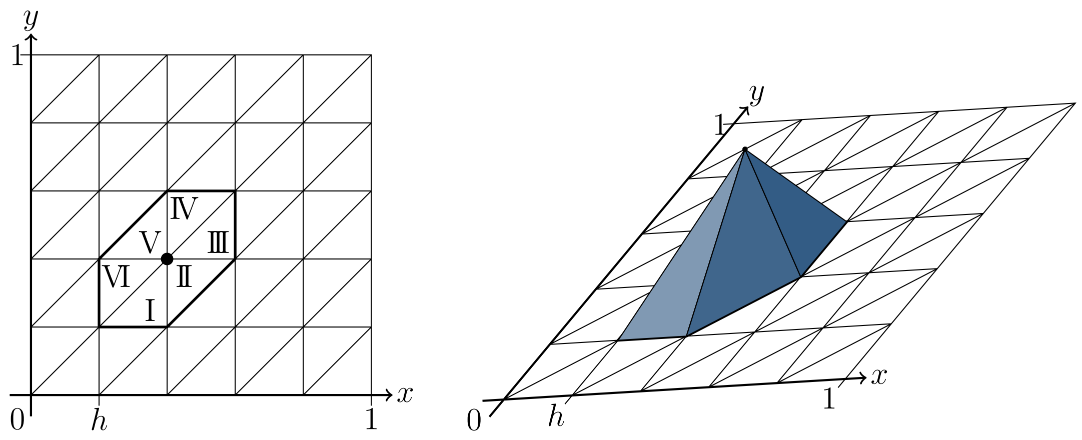
$$\rho_T \geqslant h_T/\kappa \qquad (1.2)$$

for all $T \in \mathcal{T}_h$, where ρ_T is the radius of the incircle of T and h_T is the diameter of T. The family is called *uniform*, iff there exists $h > 0$ such that $h_T = h$ independently of $T \in \mathcal{T}_h$ in ((1.2)).

(1.3) **Example (Courant 1943).** We consider the Poisson's equation

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma,$$

where $\Omega := (0,1)^2$ is the unit square, which is triangulated with congruent triangles of mesh size $h > 0$.



The finite dimensional approximation space in the Galerkin method is given by

$$V_h := \{v \in C^0(\overline{\Omega}) : v|_T \text{ is linear for every } T \in \mathcal{T}_h \text{ and } v|_\Gamma = 0\} \subset H_0^1(\Omega).$$

For every $T \in \mathcal{T}_h$, a function $v \in V_h$ satisfies $v|_T(x,y) = a + bx + cy$ and is hence uniquely determined by its values at the vertices of T. Globally, $v \in V_h$ is determined by its values at the grid points $(x_i, y_i) \in \overline{\Omega}$. A suitable basis of V_h is given by the hat functions

$$\phi_i(x_j, y_j) = \delta_{ij}.$$

Thus, if $h = 1/n$ for $n \in \mathbb{N}^*$ then $\dim V_h = (n-1)^2$. In the sequel, we compute the entries of the stiffness matrix $\boldsymbol{A}$. Let ϕ_Z denote the basis function. supported on the elements I–Ⅵ with maximum in (x_i, y_i). Analogously, we define$\phi_N, \phi_S, \phi_W, \phi_E$ via

$$\phi_N(x_i, y_i + h) = 1,\ \phi_S(x_i, y_i - h) = 1,\ \phi_W(x_i - h, y_i) = 1,\ \phi_E(x_i + h, y_i) = 1.$$

The derivatives of the basis function ϕ_Z are given by

	I	Ⅱ	Ⅲ	Ⅳ	V	Ⅵ	else
$\partial_x \phi_Z$	0	$-1/h$	$-1/h$	0	$1/h$	$1/h$	0
$\partial_y \phi_Z$	$1/h$	$1/h$	0	$-1/h$	$-1/h$	0	0

Exploiting symmetry, we obtain

$$\begin{aligned} a(\phi_Z, \phi_Z) &= \int_{\text{I–Ⅵ}} \langle \nabla \phi_Z, \nabla \phi_Z \rangle \,\mathrm{d}\boldsymbol{x} = 2 \int_{\text{I–Ⅲ}} (\partial_x \phi_Z)^2 + (\partial_y \phi_Z)^2 \,\mathrm{d}\boldsymbol{x} \\ &= 2 \int_{\text{Ⅱ–Ⅲ}} (\partial_x \phi_Z)^2 \,\mathrm{d}\boldsymbol{x} + 2 \int_{\text{I–Ⅱ}} (\partial_y \phi_Z)^2 \,\mathrm{d}\boldsymbol{x} \\ &= \frac{2}{h^2} \int_{\text{Ⅱ–Ⅲ}} 1 \,\mathrm{d}\boldsymbol{x} + \frac{2}{h^2} \int_{\text{I–Ⅱ}} 1 \,\mathrm{d}\boldsymbol{x} = 4. \end{aligned}$$

Moreover, we have

$$\begin{aligned} a(\phi_Z, \phi_S) = a(\phi_S, \phi_Z) &= \int_{\mathrm{I}-\mathrm{II}} \langle \nabla\phi_Z, \nabla\phi_S \rangle \,\mathrm{d}\boldsymbol{x} \\ &= \int_{\mathrm{I}-\mathrm{II}} \partial_y\phi_Z \partial_y\phi_S \,\mathrm{d}\boldsymbol{x} = \int_{\mathrm{I}-\mathrm{II}} \frac{1}{h}\Big(-\frac{1}{h}\Big) \,\mathrm{d}\boldsymbol{x} = -1. \end{aligned}$$

Accordingly, there holds due to symmetry

$$a(\phi_Z, \phi_S) = a(\phi_Z, \phi_N) = a(\phi_Z, \phi_W) = a(\phi_Z, \phi_E) = -1.$$

Finally, we infer

$$a(\phi_Z, \phi_{SW}) = \int_{\mathrm{I}+\mathrm{VI}} \langle \nabla\phi_Z, \nabla\phi_{SW} \rangle \,\mathrm{d}\boldsymbol{x} = 0 = a(\phi_Z, \phi_{NE}).$$

Consequently, as in the finite difference method, we end up with the standard five-point stencil

$$\begin{bmatrix} \alpha_{NW} & \alpha_N & \alpha_{NE} \\ \alpha_W & \alpha_C & \alpha_E \\ \alpha_{SW} & \alpha_S & \alpha_{SE} \end{bmatrix}_* = \begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_*$$

for the particular choice of the mesh and the basis functions. However, we remark that the scaling of the stencil is different here. △

As in the example, we typically choose *piecewise* polynomial basis functions. Conseqently, the functions $v \in V_h$ are piecewise polynomials, as well. Here and in what follows, we say that a function satisfies a given property piecewise, iff this property holds for all elements $T \in \mathcal{T}$. Hence, $v \in V_h$ satisfies $v|_T \in \mathcal{P}_m(T)$ for all $T \in \mathcal{T}_h$, where

$$\mathcal{P}_m(T) := \Big\{ v\colon T \to \mathbb{R} : v(\boldsymbol{x}) = \sum_{0 \leqslant |\boldsymbol{\alpha}| \leqslant m} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \text{ with } c_{\boldsymbol{\alpha}} \in \mathbb{R} \text{ for } 0 \leqslant |\boldsymbol{\alpha}| \leqslant m \Big\}.$$

The next theorem gives a sufficient criterion for $V_h \subset H^k(\Omega)$ for $k \geqslant 1$.

(1.4) **Theorem.** Let $k \geqslant 1$ and assume $\Omega \subset \mathbb{R}^d$ is bounded. Suppose $v\colon \overline{\Omega} \to \mathbb{R}$ satisfies $V|_T \in C^\infty(T)$ for all $T \in \mathcal{T}$. Then, it holds $v \in H^k(\Omega)$, iff $v \in C^{k-1}(\overline{\Omega})$.

Proof. For a proof of this theorem, see [Braess]. □

2 The P1-Element

Although it is possible to construct finite elements also on quadrilaterals and hexahedrons and for higher order polynomials, we restrict ourselves in this lecture to the situation of finite elements of lowest order on triangles and tetrahedrons and refer to [Braess] for a more comprehensive discussion.

For historical reasons, we start by the definition of finite elements as introduced by Ciarlet.

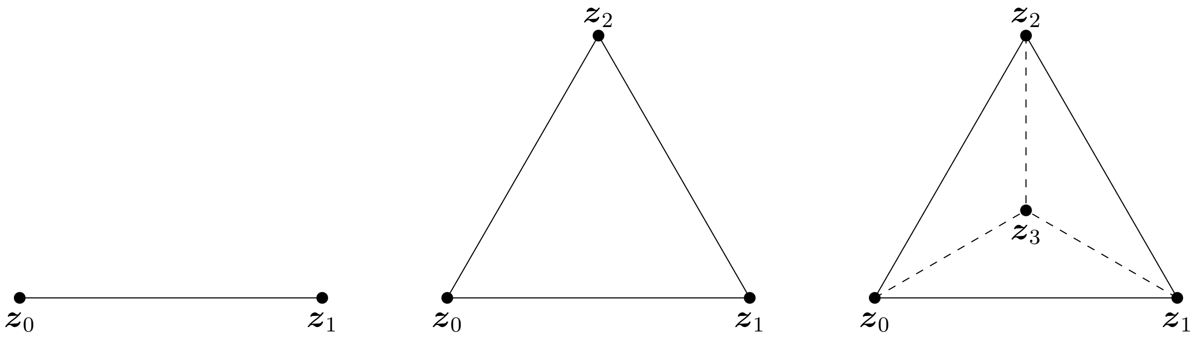
(2.1) **Definition.** A *finite element* is a triple (T, Π, Σ) with the following properties:

(a) $T \subset \mathbb{R}^d$ is closed set called *element*.

(b) $\Pi \subset C^0(T)$ is an s-dimensional subspace. The basis funcions $\{v_1, \dots v_s\} \subset \Pi$ are called *shape functions*.

(c) Σ is a set of s bounded, linear functionals $\chi_1, \dots, \chi_s$ on Π and every $p \in \Pi$ is uniquely determined by the values $\chi_i(p)$, $i = 1, \dots, s$. (These functionals correspond to generalised interpolation conditions).

The dimension s corresponds the *number of local degrees of freedom*.



(2.2) **Definition.** Let $z_0, \dots, z_d \in \mathbb{R}^d$ be $(d+1)$ non-collinear vertices, i.e. the vectors $z_1 - z_0, \dots, z_d - z_0$ are linearly independent. We set

$$T := \operatorname{conv}\{z_0, \dots, z_d\} \subset \mathbb{R}^d,$$

i.e. T is the convex hull of the vertices. Moreover, we define $\Pi := \mathcal{P}_1(T)$ and

$$\Sigma := \{\chi_i \colon \Pi \to \mathbb{R} : \chi_i(v) := v(z_i),\ i = 0, \dots, d\}.$$

The triplet (T, Π, Σ) is called *P1-element*.

The P1-element satisfies the definition of a finite element.

(2.3) **Theorem.** The P1-element is a finite element.

Proof. Obviously, T is a closed set. Moreover, it holds

$$\mathcal{P}_1(T) = \begin{cases} \operatorname{span}\{1, x_1\}, & d = 1, \\ \operatorname{span}\{1, x_1, x_2\}, & d = 2, \\ \operatorname{span}\{1, x_1, x_2, x_3\}, & d = 3. \end{cases}$$

Therefore, the dimension of $\mathcal{P}_1(T)$ is $s = d + 1$. In order to show the uniqueness of the representation, it suffices to show that from $\chi_i(q) = 0$ for $i = 0, \dots, d$ already follows $q \equiv 0$. Let

$$q = c_0 + \sum_{i=1}^{d} c_i x_i \quad \text{for } c_0, \dots, c_d \in \mathbb{R}.$$

Then, the coefficients are determined by the linear system

$$\begin{bmatrix} 1 & z_{0,1} & \cdots & z_{0,d} \\ 1 & z_{1,1} & \cdots & z_{1,d} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_{d,1} & \cdots & z_{d,d} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_d \end{bmatrix} = \mathbf{0}.$$

The non-collinearity of the points implies that the matrix is regular. Hence,

$$c_0 = c_1 = \ldots = c_d = 0$$

is the only solution, which shows $q \equiv 0$. □

(2.4) Remark. If $\Omega \subset \mathbb{R}^d$ is a bounded *Lipschitz domain*, i.e. Γ can locally be parametrised by a Lipschitz continuous function, then the Sobolev imbedding theorem guarantees $H^2(\Omega) \subset C^0(\overline{\Omega})$ for $d \leqslant 3$. Hence, the point evaluations in the definition of the P1-element are well defined on $H^2(\Omega)$. △

(2.5) Theorem (Nodal Basis). Let (T, Π, Σ) be a finite element. There exists a basis $\{\phi_1, \ldots, \phi_s\} \subset \Pi$ such that

$$\chi_i(\phi_j) = \delta_{ij} \quad \text{for } i, j = 1, \ldots, s.$$

Proof. Let $\{\psi_1, \ldots, \psi_s\}$ denote a basis of Π and define the matrix

$$\boldsymbol{A} := [\chi_i(\psi_j)]_{i,j=1}^s \in \mathbb{R}^{s \times s}.$$

This matrix is regular, since the ψ_j form a basis of Π and the functionals χ_i are linear. Now, let $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_s\} \subset \mathbb{R}^s$ denote the canonical basis. Then, the functions

$$\phi_j := \sum_{i=1}^s c_{j,i}\psi_i, \quad \text{where } \boldsymbol{c}_j := \boldsymbol{A}^{-1}\boldsymbol{e}_j,$$

satisfy

$$\chi_\ell(\phi_j) = \sum_{i=1}^s c_{j,i}\chi_\ell(\psi_i) = \sum_{i=1}^s c_{j,i}a_{\ell,i} = [a_{\ell,i}]_{\ell=1}^s \boldsymbol{A}^{-1}\boldsymbol{e}_j = \boldsymbol{e}_\ell^\intercal \boldsymbol{e}_j = \delta_{\ell j}$$

and form a basis of Π. The latter is verified by observing that $\psi_j = \sum_{i=1}^s \chi_i(\psi_j)\phi_i$ for $i = 1, \ldots, s$. □

(2.6) Example. Let $\widehat{T} := \operatorname{conv}\{[0,0]^\intercal, [1,0]^\intercal, [0,1]^\intercal\} \subset \mathbb{R}^2$, then the nodal basis of the P1-element is given by

$$\phi_1(x, y) = 1 - x - y, \ \phi_2(x, y) = x, \ \phi_3(x, y) = y.$$

Let $\widehat{T} := \operatorname{conv}\{[0,0,0]^\intercal, [1,0,0]^\intercal, [0,1,0]^\intercal, [0,0,1]^\intercal\} \subset \mathbb{R}^3$, then the nodal basis of the P1-element is given by

$$\phi_1(x, y, z) = 1 - x - y - z, \ \phi_2(x, y, z) = x, \ \phi_3(x, y, z) = y, \ \phi_4(x, y, z) = z.$$

△

(2.7) **Definition.** Given a finite element (T, Π, Σ) and a function $v \in H^2(T)$ the *nodal interpolant* $\mathcal{I}_T v \in \Pi$ is defined as

$$\mathcal{I}_T v := \sum_{i=1}^{s} \chi_i(v)\phi_i,$$

where $\{\phi_1, \dots, \phi_s\} \subset \Pi$ denotes the nodal basis. In particular, there holds

$$\chi_i(\mathcal{I}_T v) = \chi_i(v) \quad \text{for } i = 1, \dots, s.$$

(2.8) **Remark.** The nodal interpolant is unique: Assume $\mathcal{I}_T v = \sum_{i=1}^{s} c_i\phi_i$, then it holds

$$\chi_i(\mathcal{I}_T v) = \chi_i(v) = c_i$$

by the definition of the nodal interpolant. △

3 Affine Families

Although it is possible to compute the finite element stiffness matrix and the corresponding right hand side directly on the actual triangulation as in Example (1.3), it is much more convenient to perform the calculations on a reference configuration $(\widehat{T}, \widehat{\Pi}, \widehat{\Sigma})$. This accounts particularly, when numerical quadrature has to be employed, e.g. in the case of a non trivial diffusion coefficient $\boldsymbol{A}(\boldsymbol{x})$ in the variational formulation.

(3.1) **Definition.** Let $\mathcal{T}_h$ denote a triangulation for $\Omega \subset \mathbb{R}^d$. An *affine family* is a familiy of finite elements

$$(T, \Pi_T, \Sigma_T)_{T \in \mathcal{T}_h}$$

such that each finite element (T, Π_T, Σ_T) is obtained by an affine transformation of a common *reference element* $(\widehat{T}, \widehat{\Pi}, \widehat{\Sigma})$, i.e. for each $T \in \mathcal{T}_h$ there exists an affine diffeomorphism Φ_T such that

$$T = \Phi_T(\widehat{T}), \; \Pi_T = \left\{\hat{p} \circ \Phi_T^{-1} : \hat{p} \in \widehat{\Pi}\right\}, \; \Sigma_T = \left\{\hat{\chi} \circ \Phi_T^{-1} : \hat{\chi} \in \widehat{\Sigma}\right\}.$$

The next theorem tells us under which conditions there exists an affine diffeomorphism that can be used to construct an affine family and establishes norm “equivalences” for the Sobolev spaces defined on T and $\widehat{T}$, respectively.

(3.2) **Theorem.** Let $\widehat{T} := \operatorname{conv}\{\mathbf{0}, \boldsymbol{e}_1, \dots \boldsymbol{e}_d\}$, with the canonical basis $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_d\} \subset \mathbb{R}^d$, denote the unit simplex. Assume $T = \operatorname{conv}\{\boldsymbol{z}_0, \dots, \boldsymbol{z}_d\} \subset \mathbb{R}^d$ is a non-degenerate simplex. Then, there exists a unique affine diffeomorphism $\Phi_T \colon \widehat{T} \to T$ with

$$\Phi_T(\mathbf{0}) = \boldsymbol{z}_0 \quad \text{and} \quad \Phi_T(\boldsymbol{e}_i) = \boldsymbol{z}_i \text{ for } i = 1, \dots, d.$$

In addition, there holds for every $v \in H^m(T)$ and $\hat{v} := v \circ \Phi_T \in H^m(\widehat{T})$ that

$$\begin{aligned} |v|_{H^k(T)} &\leqslant c\rho_T^{-k} |\det \boldsymbol{B}|^{\frac{1}{2}} |\hat{v}|_{H^k(\widehat{T})}, \\ |\hat{v}|_{H^k(\widehat{T})} &\leqslant c' h_T^k |\det \boldsymbol{B}|^{-\frac{1}{2}} |v|_{H^k(T)}, \end{aligned}$$

for $0 \leqslant k \leqslant m$ and constants $c, c' > 0$. Herein, the H^k-semi-norm is defined via

$$|v|_{H^k(\Omega)} := \left(\sum_{|\boldsymbol{\alpha}|=k} \|\partial^{\boldsymbol{\alpha}} v\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$

The constants are independent of ρ_T, h_T, cf. Definition (1.1), and of the Jacobian $\boldsymbol{B} \in \mathbb{R}^{d \times d}$ of Φ_T.

Proof. Let

$$\boldsymbol{B} := [\boldsymbol{z}_1 - \boldsymbol{z}_0, \dots, \boldsymbol{z}_d - \boldsymbol{z}_0] \in \mathbb{R}^{d \times d}.$$

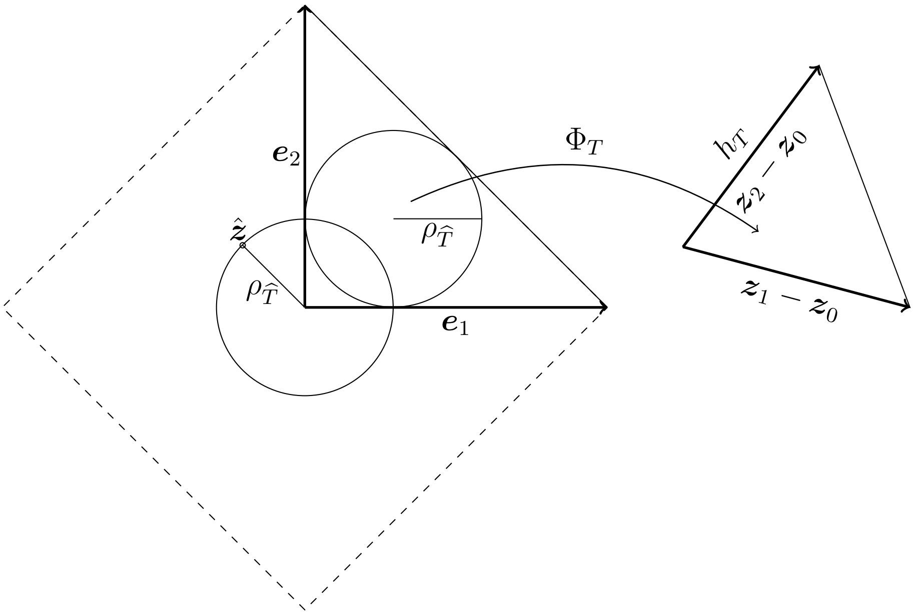
Due to the non-degeneracy of T, $\boldsymbol{B}$ is a regular matrix. Moreover, the transformation

$$\Phi_T \colon \widehat{T} \to \mathbb{R}^d, \ \hat{\boldsymbol{x}} \mapsto \boldsymbol{B}\hat{\boldsymbol{x}} + \boldsymbol{z}_0$$

satisfies $\Phi_T(\mathbf{0}) = \boldsymbol{z}_0$ and $\Phi_T(\boldsymbol{e}_i) = \boldsymbol{z}_i$ for $i = 1, \dots, d$. Since the image of a convex set under an affine transformation is convex, we have $\Phi_T(\widehat{T}) = T$.
The inverse of Φ_T is given by

$$\Phi_T^{-1} \colon T \to \widehat{T}, \ \boldsymbol{x} \mapsto \boldsymbol{B}^{-1}(\boldsymbol{x} - \boldsymbol{z}_0).$$

Hence, Φ_T is an affine diffeomorphism with Jacobian $\boldsymbol{B}$.



Now, let $\hat{\boldsymbol{z}} \in \mathbb{R}^d$ with $\|\hat{\boldsymbol{z}}\|_2 = \rho_{\widehat{T}}$. Then, there exist $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}} \in \widehat{T}$ with $\hat{\boldsymbol{z}} = \hat{\boldsymbol{y}} - \hat{\boldsymbol{x}}$ and it holds $\boldsymbol{B}\hat{\boldsymbol{z}} = \boldsymbol{B}(\hat{\boldsymbol{y}} - \hat{\boldsymbol{x}})$ and $\|\boldsymbol{B}(\hat{\boldsymbol{y}} - \hat{\boldsymbol{x}})\|_2 \leqslant h_T$. From this, we deduce

$$\|\boldsymbol{B}\|_2 = \frac{1}{\rho_{\widehat{T}}} \sup_{\|\hat{\boldsymbol{z}}\|_2 = \rho_{\widehat{T}}} \|\boldsymbol{B}\hat{\boldsymbol{z}}\|_2 = \frac{1}{\rho_{\widehat{T}}} \sup_{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}} \in \widehat{T}} \|\boldsymbol{B}(\hat{\boldsymbol{y}} - \hat{\boldsymbol{x}})\|_2 = \frac{h_T}{\rho_{\widehat{T}}}.$$

By interchanging the roles of T and $\widehat{T}$, we find analogously

$$\|\boldsymbol{B}^{-1}\|_2 \leqslant \frac{h_{\widehat{T}}}{\rho_T}.$$

Next, let $v \in C^\infty(T) \cap H^k(T)$. By the chain rule, it holds

$$\nabla v = \nabla(\hat{v} \circ \Phi_T^{-1}) = \boldsymbol{B}^{-\intercal} \widehat{\nabla} \hat{v} \circ \Phi_T^{-1} \quad \text{or} \quad \widehat{\nabla v} = \boldsymbol{B}^{-\intercal} \widehat{\nabla} \hat{v}.$$

Multiplication with $\boldsymbol{B}^\intercal$ then gives

$$\widehat{\nabla} \hat{v} = \boldsymbol{B}^\intercal \widehat{\nabla v}.$$

With the previous estimate on the norm of $\boldsymbol{B}$, we obtain

$$\|\widehat{\nabla v}\|_2 \leqslant c \rho_T^{-1} \|\widehat{\nabla} \hat{v}\|_2 \quad \text{and} \quad \|\widehat{\nabla} \hat{v}\|_2 \leqslant c' h_T \|\widehat{\nabla v}\|_2$$

for some constants $c, c' > 0$. More generally, since the Jacobian of Φ_T is constant, we have for any multi index $\boldsymbol{\alpha} \in \mathbb{N}^d$ that

$$|\widehat{\partial^{\boldsymbol{\alpha}} v}| \leqslant c \rho_T^{-|\boldsymbol{\alpha}|} \max_{|\boldsymbol{\beta}| = |\boldsymbol{\alpha}|} |\hat{\partial}^{\boldsymbol{\beta}} \hat{v}| \quad \text{and} \quad |\hat{\partial}^{\boldsymbol{\alpha}} \hat{v}| \leqslant c' h_T^{|\boldsymbol{\alpha}|} \max_{|\boldsymbol{\beta}| = |\boldsymbol{\alpha}|} |\widehat{\partial^{\boldsymbol{\beta}} v}|$$

for some other constants $c, c' > 0$ which depend on k, d, where $\boldsymbol{\beta} \in \mathbb{N}^d$. From these expressions, it is easy to derive

$$\sum_{|\boldsymbol{\alpha}| = k} (\widehat{\partial^{\boldsymbol{\alpha}} v})^2 \leqslant c \rho_T^{-2k} \sum_{|\boldsymbol{\alpha}| = k} (\hat{\partial}^{\boldsymbol{\alpha}} \hat{v})^2 \quad \text{and} \quad \sum_{|\boldsymbol{\alpha}| = k} (\hat{\partial}^{\boldsymbol{\alpha}} \hat{v})^2 \leqslant c' h_T^{2k} \sum_{|\boldsymbol{\alpha}| = k} (\widehat{\partial^{\boldsymbol{\alpha}} v})^2$$

for some other constants $c, c' > 0$ which depend on k, d. For a given function $w \in L^1(T)$, the transformation formula yields with $\hat{w} := w \circ \Phi_T \in L^1(\widehat{T})$ that

$$\int_{\widehat{T}} \hat{w} \,\mathrm{d}\hat{\boldsymbol{x}} = \int_{\Phi_T(\widehat{T})} \hat{w} \circ \Phi_T^{-1} |\det \boldsymbol{B}^{-1}| \,\mathrm{d}\boldsymbol{x} = |\det \boldsymbol{B}^{-1}| \int_T w \,\mathrm{d}\boldsymbol{x}.$$

Hence, applying the transformation formula to $w = (\partial^{\boldsymbol{\alpha}} v)^2$ gives

$$\begin{aligned} |v|^2_{H^k(T)} &= \sum_{|\boldsymbol{\alpha}|=k} \int_T (\partial^{\boldsymbol{\alpha}} v)^2 \,\mathrm{d}\boldsymbol{x} = |\det \boldsymbol{B}| \sum_{|\boldsymbol{\alpha}|=k} \int_{\widehat{T}} (\widehat{\partial^{\boldsymbol{\alpha}} v})^2 \,\mathrm{d}\hat{\boldsymbol{x}} \\ &\leqslant |\det \boldsymbol{B}| c\rho_T^{-2k} \sum_{|\boldsymbol{\alpha}|=m} \int_{\widehat{T}} (\hat{\partial}^{\boldsymbol{\alpha}} \hat{v})^2 \,\mathrm{d}\hat{\boldsymbol{x}} = c\rho_T^{-2k} |\det \boldsymbol{B}| |\hat{v}|_{H^k(\widehat{T})}. \end{aligned}$$

The second inequality is derived in a similar fashion. The assertion is then obtained by a density argument. □

4 Approximation Properties

In this section, we consider the global approximation error induced by the interpolation of a function $v \in H^2(\Omega)$ in the finite element space V_h.

(4.1) **Theorem (Bramble-Hilbert Lemma).** Let $F\colon H^m(T) \to \mathbb{R}$ denote a bounded and sublinear functional, i.e.

$$|F(v)| \leqslant c\|v\|_{H^m(\Omega)} \text{ for } c > 0 \quad \text{and} \quad |F(v+w)| \leqslant \big(|F(v)| + |F(w)|\big),$$

and assume $\mathcal{P}_{m-1}(T) \subset \ker(F)$. Then, there exists $C > 0$ such that

$$|F(v)| \leqslant C|v|_{H^m(T)} \quad \text{for all } v \in H^m(T).$$

Proof. For a proof of the Bramble-Hilbert lemma, see [Bartels]. □

(4.2) **Corollary.** Let (T, Π, Σ) be a finite element with $\mathcal{P}_{m-1}(T) \subseteq \Pi$. Then, there exists $C > 0$ such that

$$\|v - \mathcal{I}_T v\|_{H^m(T)} \leqslant C|v|_{H^m(T)} \quad \text{for all } v \in H^m(T).$$

Proof. We set $F(v) = \|v - \mathcal{I}_T v\|_{H^m(T)}$ and note that F is sublinear. Moreover, with the definition of the interpolant $\mathcal{I}_T v = \sum_{i=1}^s \chi_i(v)\phi_i$ and the continuity of the functionals χ_i, i.e. $|\chi_i(v)| \leqslant c\|v\|_{H^m(T)}$ for $i = 1, \dots, s$, we have

$$\begin{aligned} |F(v)| &\leqslant \|v\|_{H^m(T)} + \|\mathcal{I}_T v\|_{H^m(T)} \leqslant \|v\|_{H^m(T)} + \sum_{i=1}^s |\chi_i(v)| \|\phi_i\|_{H^m(T)} \\ &\leqslant \|v\|_{H^m(T)} + c\|v\|_{H^m} \sum_{i=1}^s \|\phi_i\|_{H^m(T)} \leqslant C\|v\|_{H^m} \end{aligned}$$

for some $C > 0$. In addition, there holds $F(v) = 0$ for all $v \in \Pi$. Hence, the conditions of the Bramble-Hilbert lemma are satisfied, which implies the assertion. □

(4.3) **Definition.** Let $\mathcal{T}_h$ be a triangulation for $\Omega \subset \mathbb{R}^d$ and let $(T, \Pi_T, \Sigma_T)_{T\in\mathcal{T}_h}$ denote an affine family. The *global interpolant* $\mathcal{I}\colon H^m(\Omega) \to L^\infty(\Omega)$ is defined via

$$(\mathcal{I}v)|_T = \mathcal{I}_T(v|_T)$$

for all $T \in \mathcal{T}_h$. The affine family is called a C^k-*element*, iff $\mathcal{I}v \in C^k(\overline{\Omega})$ for all $v \in C^k(\overline{\Omega}) \cap H^m(\Omega)$.

In practice, for a given triangulation $\mathcal{T}_h$, one usually considers the *finite element spaces*

$$V_h = \{v \in C^0(\overline{\Omega}) : v|_T \in \mathcal{P}_m(T),\ T \in \mathcal{T}_h\} \subset H^1(\Omega), \tag{4.4}$$

which are based on a C^0-element. It is also possible to construct finite element spaces with higher global smoothness, e.g. C^1-elements for an H^2-conforming discretisation. However, they are much more difficult to construct.

In order to estimate the global approximation error of a function $v \in H^m$ by its interpolant $\mathcal{I}v \in V_h$, we introduce the mesh dependent norm

$$\|v\|_{m,h} := \left(\sum_{T\in\mathcal{T}_h} \|v\|^2_{H^m(T)}\right)^{\frac{1}{2}}.$$

This norm is a bit more general than the usual H^m-norm, since it does not require global smoothness of a function. However, it holds

$$\|v\|_{m,h} = \|v\|_{H^m(\Omega)} \quad \text{for all } v \in H^m.$$

We remark that, according to Theorem (1.4), this equality holds for the C^0-element for $m = 0, 1$.

(4.5) **Theorem.** Let $(T, \Pi_T, \Sigma_T)_{T\in\mathcal{T}_h}$ denote an affine family with $\mathcal{P}_{m-1}(\widehat{T}) \subset \widehat{T}$, where $\mathcal{T}_h$ denotes a shape-regular triangulation for $\Omega \subset \mathbb{R}^d$. Moreover, let $m \geqslant 2$ and $0 \leqslant k \leqslant m$. Then, it holds

$$\|v - \mathcal{I}v\|_{k,h} \leqslant ch^{m-k}|v|_{H^m(\Omega)} \quad \text{for all } v \in H^m(\Omega)$$

with a constant $C > 0$, which only depends on Ω, κ and m.

Proof. From Theorem (3.2) we obtain for $T \in \mathcal{T}$ that

$$\begin{aligned}|v - \mathcal{I}_T v|_{H^\ell(T)} &\leqslant c\left(\frac{h_T}{\kappa}\right)^{-\ell} |\det \boldsymbol{B}|^{\frac{1}{2}} |\hat{v} - \mathcal{I}_{\widehat{T}}\hat{v}|_{H^\ell(\widehat{T})} \\ &\leqslant Ch_T^{-\ell} |\det \boldsymbol{B}|^{\frac{1}{2}} |\hat{v} - \mathcal{I}_{\widehat{T}}\hat{v}|_{H^\ell(\widehat{T})}\end{aligned}$$

for all $0 \leqslant \ell \leqslant m$ and some constants $c, C > 0$.

Moreover, due to Corollary (4.2), there holds

$$|\hat{v} - \mathcal{I}_{\widehat{T}}\hat{v}|_{H^\ell(\widehat{T})} \leqslant \|\hat{v} - \mathcal{I}_{\widehat{T}}\hat{v}\|_{H^m(\widehat{T})} \leqslant C|\hat{v}|_{H^m(\widehat{T})} \quad \text{for some } C > 0$$

and consequently

$$|v - \mathcal{I}_T v|_{H^\ell(T)} \leqslant ch_T^{-\ell} |\det \boldsymbol{B}|^{\frac{1}{2}} |\hat{v}|_{H^m(\widehat{T})}$$

for some other constant $c > 0$. Now, mapping the norm back to T results in

$$|v - \mathcal{I}_T v|_{H^\ell(T)} \leqslant ch_T^{-\ell} |\det \boldsymbol{B}|^{\frac{1}{2}} |\hat{v}|_{H^m(\widehat{T})} \leqslant ch_T^{-\ell} |\det \boldsymbol{B}|^{\frac{1}{2}} c' h_T^m |\det \boldsymbol{B}|^{-\frac{1}{2}} |v|_{H^m(T)} \\ \leqslant Ch_T^{m-\ell} |v|_{H^m(T)}$$

for some constant $C > 0$. Thus, by summation we obtain

$$\|v - \mathcal{I}_T v\|^2_{H^k(T)} = \sum_{\ell=0}^{k} |v - \mathcal{I}_T v|^2_{H^\ell(T)} \\ \leqslant C^2 |v|^2_{H^m(T)} \sum_{\ell=0}^{k} h_T^{2(m-\ell)} \leqslant ch_T^{2(m-k)} |v|_{H^m(T)}$$

for some constant $c > 0$. The assertion is obtained from this expression by taking the maximum over all diameters h_T and summation over all $T \in \mathcal{T}_h$. □

Consequently, for $v \in H^2(\Omega)$ and the P1-element, we obtain on a uniform triangulation the approximation estimates

$$\|v - \mathcal{I}v\|_{H^1(\Omega)} \leqslant ch|v|_{H^2(\Omega)} \quad \text{and} \quad \|v - \mathcal{I}v\|_{L^2(\Omega)} \leqslant ch^2|v|_{H^2(\Omega)}.$$

For functions $v \in V_h$, cf. ((4.4)), we can estimate a given Sobolev norm by a weaker one by introducing negative powers of $h > 0$.

(4.6) **Theorem (Inverse Estimate).** Let $\mathcal{T}_h$ denote a uniform triangulation of $\Omega \subset \mathbb{R}^d$. Moreover, let the finite element space V_h be given by ((4.4)), where the piecewise polynomial degree is $s \geqslant 0$. Then, there exists $c > 0$, which only depends on s, t, κ, such that

$$\|v\|_{m,h} \leqslant ch^{t-m} \|v\|_{t,h} \quad \text{for all } v \in V_h.$$

and $0 \leqslant t \leqslant m$

Proof. For a proof of this theorem, see [Braess]. □

From the theorem, we infer that, in the case of piecewise linear finite elements, there holds

$$\|v\|_{H^1(\Omega)} \leqslant ch^{-1} \|v\|_{L^2(\Omega)} \quad \text{for all } v \in V_h.$$

V. Error Estimates for Elliptic Problems

In the last paragraph, we have seen that we can derive approximation results based on interpolation, if the underlying function is in $H^2(\Omega)$, i.e.

$$\|v - \mathcal{I}v\|_{H^1(\Omega)} \leqslant ch|v|_{H^2(\Omega)} \quad \text{for all } v \in H^2(\Omega).$$

Hence, if the solution u to the elliptic variational problem

$$\begin{aligned}&\text{Find } u \in V \text{ such that}\\ &\qquad a(u,v) = \ell(v) \quad \text{for all } v \in V\end{aligned}$$

is contained in $H^2(\Omega)$, Céa's lemma yields

$$\|u - u_h\|_{H^1(\Omega)} \leqslant \frac{c_S}{c_E} ch|u|_{H^2(\Omega)}.$$

In this chapter, we shall investigate when this is the case for solutions to elliptic problems.

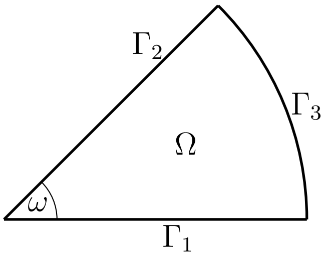
(1.1) **Definition.** Let $H^1_0(\Omega) \subset V \subset H^1(\Omega)$ and $a\colon V \times V \to \mathbb{R}$ an elliptic bilinear form. The variational problem

$$\begin{aligned}&\text{Find } u \in V \text{ such that}\\ &\qquad a(u,v) = \ell(v) \quad \text{for all } v \in V\end{aligned}$$

is called $H^s(\Omega)$-*regular* for $s \geqslant 2$, iff there exists $c > 0$ such that

$$\|u\|_{H^s(\Omega)} \leqslant c\|f\|_{H^{s-2}(\Omega)} \quad \text{for all } f \in H^{s-2}(\Omega).$$

(1.2) **Example.**



We consider the domain

$$\Omega := \{(r\cos\alpha, r\sin\alpha) : 0 < r < 1,\ 0 < \alpha < \omega\}$$

with boundaries

$$\begin{aligned}
\Gamma_1 &:= \{(r,0) : 0 \leqslant r \leqslant 1\},\\
\Gamma_2 &:= \{(r\cos\omega, r\sin\omega) : 0 \leqslant r \leqslant 1\},\\
\Gamma_3 &:= \{(\cos\alpha, \sin\alpha) : 0 < \alpha < \omega\}.
\end{aligned}$$

The two dimensional Laplacian in polar coordinates is given by

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \alpha^2},$$

where $x(r,\alpha) = r\cos\alpha$ and $y(r,\alpha) = r\sin\alpha$.
The function

$$u(x,y) = \hat{u}(r,\alpha) = \left(r^2 - r^{\frac{\pi}{\omega}}\right)\sin\left(\frac{\pi}{\omega}\alpha\right)$$

satisfies

$$\Delta u = \left(4 - \frac{\pi^2}{\omega^2}\right)\sin\left(\frac{\pi}{\omega}\alpha\right).$$

Hence, u is the unique solution to the Dirichlet problem

$$-\Delta u = \left(\frac{\pi^2}{\omega^2} - 4\right)\sin\left(\frac{\pi}{\omega}\alpha\right) \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma.$$

The function $r^{\pi/\omega}\sin(\pi\alpha/\omega)$ is in $H^2(\Omega)$, if $\pi/\omega \geqslant 1$, i.e. $\omega \leqslant \pi$. Consequently, the same accounts for u. On the other hand, the right hand side is always in $L^2(\Omega)$. Hence, the problem is for $\omega > \pi$ not $H^2(\Omega)$-regular. △

(1.3) **Theorem.** Let $\Omega \subset \mathbb{R}^d$ denote a convex and polygonal domain and let

$$a(v,w) := \int_\Omega \langle \boldsymbol{A}\nabla v, \nabla w\rangle \,\mathrm{d}\boldsymbol{x}, \quad v,w \in H_0^1(\Omega)$$

denote an elliptic bilinear form with Lipschitz continuous coefficients $a_{i,j}(\boldsymbol{x})$. Then, the variational problem

Find $u \in H_0^1(\Omega)$ such that
$$a(u,v) = \ell(v) \quad \text{for all } v \in H_0^1(\Omega)$$

is $H^2(\Omega)$-regular.

Proof. For a proof of this theorem, see [Grisvard]. □

Based on the preceding regularity result, we can now state the convergence result for the finite element method in the elliptic case.

(1.4) **Theorem.** Let $\Omega \subset \mathbb{R}^d$ denote a convex and polygonal domain and let $\mathcal{T}_h$ denote a uniform triangulation for Ω. Then, if $f \in L^2(\Omega)$, the piecewise linear Galerkin approximation $u_h \in V_h$ to the elliptic variational problem

$$\begin{aligned} &\text{Find } u \in H_0^1(\Omega) \text{ such that} \\ &\qquad a(u,v) = \ell(v) \quad \text{for all } v \in H_0^1(\Omega) \end{aligned}$$

satisfies the error estimate

$$\|u - u_h\|_{H^1(\Omega)} \leqslant ch\|f\|_{L^2(\Omega)} \quad \text{for some } c > 0.$$

Proof. According to Theorem (1.3), the underlying variational problem is $H^2(\Omega)$-regular. Hence, we have $\|u\|_{H^2(\Omega)} \leqslant c\|f\|_{L^2(\Omega)}$ for some constant $c > 0$. Consequently, we obtain by Theorem (IV.4.5) that

$$\|u - \mathcal{I}u\|_{H^1(\Omega)} \leqslant ch|u|_{H^2(\Omega)} \leqslant ch\|u\|_{H^2(\Omega)} \leqslant Ch\|f\|_{L^2(\Omega)}$$

for some constants $c, C > 0$. Finally, observing $\mathcal{I}u \in V_h$, Céa's lemma yields

$$\|u - u_h\|_{H^1(\Omega)} \leqslant \frac{c_S}{c_E} \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)} \leqslant \frac{c_S}{c_E}\|u - \mathcal{I}u\|_{H^1(\Omega)} \leqslant \frac{c_S}{c_E} Ch\|f\|_{L^2(\Omega)}.$$

□

(1.5) **Remark.** In the case that *P2-elements*, i.e. piecewise quadratic finite elements are considered, Theorem (IV.4.5) provides a higher approximation order. This results in an overall higher convergence rate given that the problem at hand is $H^3(\Omega)$-regular. In general, a $H^3(\Omega)$-regular solution is only obtained on a smooth, curved domain, which cannot be decomposed into simplices. △

The rate of convergence of the finite element is increased, if the error is measured with respect to $L^2(\Omega)$.

(1.6) **Theorem (Aubin-Nitsche Lemma).** Let $\Omega \subset \mathbb{R}^d$ denote a convex and polygonal domain and let $\mathcal{T}_h$ denote a uniform triangulation for Ω. Then, if $f \in L^2(\Omega)$, the piecewise linear Galerkin approximation $u_h \in V_h$ to the elliptic variational problem

$$\begin{aligned} &\text{Find } u \in H_0^1(\Omega) \text{ such that} \\ &\qquad a(u,v) = \ell(v) \quad \text{for all } v \in H_0^1(\Omega) \end{aligned}$$

satisfies the error estimate

$$\|u - u_h\|_{L^2(\Omega)} \leqslant ch^2\|f\|_{L^2(\Omega)} \quad \text{for some } c > 0.$$

Proof. We consider the *dual problem*

$$\begin{aligned}&\text{Find } \phi_g \in H_0^1(\Omega) \text{ such that}\\ &\qquad a(w,\phi_g) = (g,w)_{L^2(\Omega)} \quad \text{for all } w \in H_0^1(\Omega),\end{aligned}$$

which obviously exhibits a unique weak solution due to the Lax-Milgram lemma. It holds for $w := u - u_h$ that

$$\begin{aligned}(g, u-u_h)_{L^2(\Omega)} &= a(u-u_h, \phi_g)\\ &= a(u-u_h, \phi_g - v_h) \leqslant c_S \|u-u_h\|_{H^1(\Omega)} \|\phi_g - v_h\|_{H^1(\Omega)}\end{aligned}$$

for all $v_h \in V_h$ due to the Galerkin orthogonality $a(u-u_h, v_h) = 0$.
By the Riesz representation theorem, there holds

$$\|w\|_{L^2(\Omega)} = \sup_{0\neq g\in L^2(\Omega)} \frac{(g,w)_{L^2(\Omega)}}{\|g\|_{L^2(\Omega)}}.$$

Consequently, we infer

$$\begin{aligned}\|u-u_h\|_{L^2(\Omega)} &= \sup_{0\neq g\in L^2(\Omega)} \frac{(g,u-u_h)_{L^2(\Omega)}}{\|g\|_{L^2(\Omega)}}\\ &\leqslant c_S \|u-u_h\|_{H^1(\Omega)} \sup_{0\neq g\in L^2(\Omega)} \frac{\|\phi_g - v_h\|_{H^1(\Omega)}}{\|g\|_{L^2(\Omega)}}.\end{aligned}$$

Since the choice of $v_h \in V_h$ in the previous estimate is arbitrary, employing Theorem (1.4) for $\phi_g \in H_0^1(\Omega)$ results in

$$\|\phi_g - \phi_{g,h}\|_{H^1(\Omega)} \leqslant ch\|g\|_{L^2(\Omega)}$$

for some constant $c > 0$, where $\phi_{g,h} \in V_h$ is the Galerkin approximation to ϕ_g. Inserting this estimate in the preceding one, results in

$$\|u-u_h\|_{L^2(\Omega)} \leqslant Ch\|u-u_h\|_{H^1(\Omega)}$$

for some constant $C > 0$. The assertion is now obtained by another application of Theorem (1.4). □

For the sake of completeness, we also state an error bound on the finite element approximation with respect to the $L^\infty(\Omega)$.

(1.7) **Theorem.** For the discretisation of an $H^2(\Omega)$-regular, elliptic variational problem by piecewise linear finite elements on a uniform triangulation $\mathcal{T}_h$, there holds the error estimate

$$\|u-u_h\|_{L^\infty(\Omega)} \leqslant ch^2|\log h|^{\frac{3}{2}} \max_{|\boldsymbol{\alpha}|\leqslant 2} \|\partial^{\boldsymbol{\alpha}} u\|_{L^\infty(\Omega)}$$

for some constant $c > 0$.

Proof. For a proof of this theorem, see [Ciarlet]. □

VI. Implementation

For the numerical realisation of the finite element method the following steps have to be implemented

(a) Mesh generation

(b) Assembly of stiffness matrix and right hand side

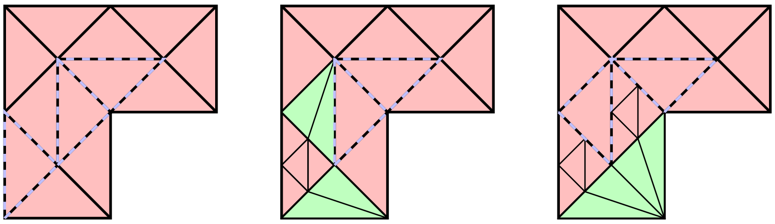
(c) Solution of the linear system

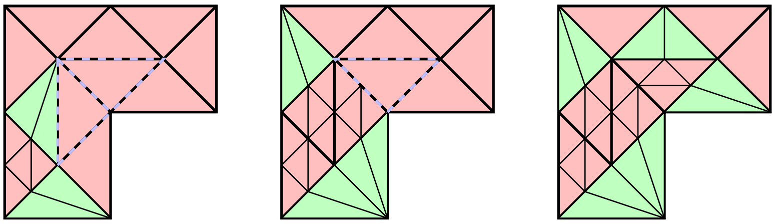
(d) A-posteriori error analysis (if the solution is insufficent, mark elements to be refined and go back to 1).

(e) Visualisation of the solution and quantities of interest.

Mesh generation: For the generation of the initial triangulation, there exist several possibilities, for example they might manually be provided or by an automatic meshing tool. Based on this initial triangulation, one might either consider uniform or shape regular refinements. Next, we introduce an algorithm that is suitable for adaptive mesh refinement. It starts from all elements being colored "red".

```
Red-green refinement
  input:     admissible partition and marked elements
  output:    admissible, refined partition
    while there are marked elements do
      ① choose a marked element and refine it uniformly
      ② color all "red" elements with a hanging node "green" and bisect them
      ③ color all "green" elements with a hanging node "red",
         undo the bisection and mark them
```





Note that in this procedure, bisected "green" elements are still considered as a single element. The same procedure also works in three spatial dimensions but is more involved. Finally, we remark that a shape-regular partition stays shape-regular since all angles are at most bisected.

Assembly: Given an affine family $(T, \Pi_T, \Sigma_T)_{T\in\mathcal{T}_h}$ of P1- or P2-elements, it is convenient to assemble the stiffness matrix with respect to the nodal basis $\{\hat{\phi}_i\}_{i=1}^s$ for $\Pi_{\widehat{T}}$. Then, obviously $\phi_i := \hat{\phi}_i \circ \Phi_T^{-1}$ for $i = 1, \dots, s$ is the nodal basis for Π_T, where

$$\Phi_T\colon \widehat{T} \to T, \quad \hat{\boldsymbol{x}} \mapsto \boldsymbol{B}_T\hat{\boldsymbol{x}} + \boldsymbol{z}_{0,T}.$$

In particular, we can also introduce a nodal basis $\{\psi_i\}_{i=1}^N$ for V_h with respect to the nodes $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\}$ of the mesh such that $\psi_i(\boldsymbol{x}_j) = \delta_{ij}$. In particular, we have $\psi_i|_T = \phi_j$ for some $1 \leqslant j \leqslant s$, Now, we may consider an element based assembly. Let

$$a(\psi_j, \psi_i) = \int_\Omega \langle \boldsymbol{A}\nabla\psi_j, \nabla\psi_i\rangle \,\mathrm{d}\boldsymbol{x}.$$

It holds

$$\begin{aligned} a(\psi_j, \psi_i) &= \sum_{T\in\mathcal{T}_h} \int_T \langle \boldsymbol{A}\nabla\psi_j, \nabla\psi_i\rangle \,\mathrm{d}\boldsymbol{x} \\ &= \sum_{T\in\mathcal{T}_h} \int_{\widehat{T}} \langle (\boldsymbol{A}\circ\Phi_T)\boldsymbol{B}_T^{-\intercal}\hat{\nabla}\hat{\psi}_j, \boldsymbol{B}_T^{-\intercal}\hat{\nabla}\hat{\psi}_i\rangle |\det \boldsymbol{B}_T| \,\mathrm{d}\hat{\boldsymbol{x}} =: \sum_{T\in\mathcal{T}_h} a_T(\hat{\psi}_j, \hat{\psi}_i). \end{aligned}$$

From this, we derive the element stiffness matrix for $T \in \mathcal{T}_h$, which is given by

$$\boldsymbol{A}_T := \begin{bmatrix} a_T(\hat{\phi}_1, \hat{\phi}_1) & \cdots & a_T(\hat{\phi}_s, \hat{\phi}_1) \\ \vdots & \ddots & \vdots \\ a_T(\hat{\phi}_1, \hat{\phi}_s) & \cdots & a_T(\hat{\phi}_s, \hat{\phi}_s) \end{bmatrix}. \tag{0.1}$$

In order to compute the element stiffness matrix ((0.1)) numerically, one usually employs numerical quadrature. Common quadrature rules for $d = 2$ are denoted in the subsequent table.

quadrature points	weights	exactness
$[1/3, 1/3]$	$1/2$	$\mathcal{P}_1$
$[1/2, 1/2], [1/2, 0], [0, 1/2]$	$1/6$	$\mathcal{P}_2$
$[1/3, 1/3]$	$-27/96$	
$[1/5, 1/5], [1/5, 3/5], [3/5, 1/5]$	$25/96$	$\mathcal{P}_3$
$[1/3, 1/3]$	$9/80$	
$\left[\frac{6+\sqrt{15}}{21}, \frac{6+\sqrt{15}}{21}\right], \left[\frac{9-2\sqrt{15}}{21}, \frac{6+\sqrt{15}}{21}\right], \left[\frac{6+\sqrt{15}}{21}, \frac{9-2\sqrt{15}}{21}\right]$	$\frac{155+\sqrt{15}}{2400}$	
$\left[\frac{6-\sqrt{15}}{21}, \frac{6-\sqrt{15}}{21}\right], \left[\frac{9+2\sqrt{15}}{21}, \frac{6-\sqrt{15}}{21}\right], \left[\frac{6-\sqrt{15}}{21}, \frac{9+2\sqrt{15}}{21}\right]$	$\frac{155-\sqrt{15}}{2400}$	$\mathcal{P}_5$

A very elegant way to assemble the right hand side is based on the finite element mass matrix

$$\boldsymbol{M} := [(\psi_i, \psi_j)_{L^2(\Omega)}]_{i,j=1}^N,$$

which can be assembled in a similar fashion as the stiffness matrix. Then, assuming

$$f(\boldsymbol{x}) \approx \sum_{i=1}^N f(\boldsymbol{x}_i)\psi_i(\boldsymbol{x})$$

yields

$$\ell(\psi_j) = \int_\Omega f\psi_i \,\mathrm{d}\boldsymbol{x} \approx \sum_{i=1}^N f(\boldsymbol{x}_i) \int_\Omega \psi_i\psi_j \,\mathrm{d}\boldsymbol{x}.$$

Consequently, we obtain

$$\begin{bmatrix} \ell(\psi_1) \\ \vdots \\ \ell(\psi_N) \end{bmatrix} \approx \boldsymbol{M} \begin{bmatrix} f(\boldsymbol{x}_1) \\ \vdots \\ f(\boldsymbol{x}_N) \end{bmatrix}.$$

On the other hand, if f is only in $L^2(\Omega)$ and, hence, does not allow for pointwise evaluations, the right hand side has to be assembled in an element based fashion, as before.

Dirichlet boundary conditions: The numerical treatment of homogenous Dirichlet problems is straightforward for $V_h \subset H_0^1(\Omega)$. For the case of Dirichlet boundary conditions $g \not\equiv 0$, we may also consider the interpolation of the Dirichlet data in the boundary nodes:

$$g_h(\boldsymbol{x}) := \sum_{i=1}^{N_D} g(\boldsymbol{x}_i^D)\phi_i^D(\boldsymbol{x})\big|_\Gamma \approx g \quad \text{on } \Gamma.$$

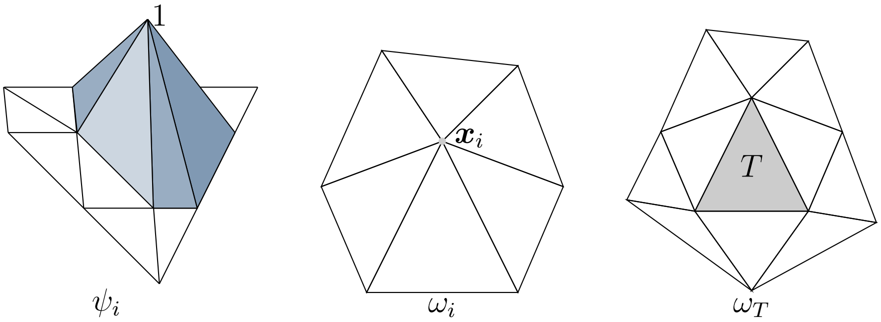
Obviously, this function satisfies $g_h \in H^1(\Omega)$. Hence, it suffices to find $u_h \in H_0^1(\Omega)$ such that

$$a(u_h, v_h) = \ell(v_h) - a(g_h, v_h) \quad \text{for all } v_h \in V_h.$$

However, we remark that the extension g_h of the boundary values is not uniformly stable in $H^1(\Omega)$ with respect to the mesh width $h > 0$. In the case of a non-linear partial differential equation, this effects the rate of convergence for numerical solution methods.

Remaining steps: The numerical solution of the emerging linear system can either be performed by a sparse direct solver or an iterative solver. In particular, the iterative multigrid method is of linear cost in terms of degrees of freedom, independently on the mesh size $h > 0$. In the sequel, we shall consider suitable adaptive error estimators.

VII. Error Control and Adaptivity



The interpolation operator in Theorem (IV.4.5) requires functions in $H^2(\Omega)$. In the sequel, we consider an approximation method that also works for $H^1(\Omega)$ functions. It is based on an idea von Philippe Clément. To that end, let $\mathcal{T}_h$ denote a shape-regular triangulation of $\Omega \subset \mathbb{R}^2$. For each node $\boldsymbol{x}_i$, we define the union of the adjacent elements

$$\omega_i := \operatorname{supp} \psi_i,$$

where $\{\psi_i\}_i$ are the nodal basis functions of finite element space based on the P1-element. Accordingly, we define the patch ω_T to be the union of all elements that share a lower dimensional face with T, i.e.

$$\omega_T := \bigcup_{\boldsymbol{x}_i \in T} \omega_i.$$

For shape-regular triangulations, there obviously holds

$$|\omega_T| \leqslant C|T| \leqslant C h_T^2,$$

where the constant $C > 0$ depends on κ.

(1.1) **Definition.** The *Clément operator* is given according to

$$C_h \colon H^1(\Omega) \to V_h, (C_h v)(\boldsymbol{x}) := \sum_{i=1}^{N} (Q_i v)\psi_i(\boldsymbol{x}),$$

where $Q_i \colon L^2(\omega_i) \to \mathcal{P}_0(\omega_i)$ is the L^2-projection, i.e. $(v - Q_i v, 1)_{L^2(\omega_i)} = 0$.

In particular, there holds

$$\|v - Q_i v\|_{L^2(\omega_i)} \leqslant C \operatorname{diam}(\omega_i) \|v\|_{H^1(\omega_i)} \quad \text{for some } C > 0,$$

which can be easily derived as in the proof of the Poincaré inequality.

(1.2) **Theorem (Clément).** Let $\mathcal{T}_h$ denote a shape-regular triangulation of $\Omega \subset \mathbb{R}^2$. Then, the Clément operator C_h is well defined, linear and has the following properties: There exists $C > 0$ which depends on $\mathcal{T}_h$ such that for $k = 0, 1$ there holds

$$\|v - C_h v\|_{H^k(T)} \leqslant C h_T^{1-k} \|v\|_{H^1(\omega_T)},$$
$$\|v - C_h v\|_{L^2(e)} \leqslant C h_T^{1/2} \|v\|_{H^1(\omega_T)}$$

for all $T \in \mathcal{T}_h$, $v \in H^1(\Omega)$, where e denotes an arbitrary edge of T.

Proof. For a proof of this theorem, see [P. Clément. Approximation by finite element functions using local regularization]. □

In the sequel, we restrict ourselves to the homogenous Dirichlet problem

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma.$$

If we insert the Galerkin solution $u_h \in V_h$ into this equation, we obtain a residual. Moreover, the derivatives at the element boundaries of u_h exhibit jumps. Hence, we consider the area based residuals

$$R_T := R_T(u_h) := f + \Delta u_h \quad \text{for } T \in \mathcal{T}_h$$

and edge based jumps

$$R_e := R_e(u_h) := \left[\!\!\left[\frac{\partial u_h}{\partial \boldsymbol{n}} \right]\!\!\right] := \frac{\partial u_h}{\partial \boldsymbol{n}}\bigg|_{T_1} - \frac{\partial u_h}{\partial \boldsymbol{n}}\bigg|_{T_2} \quad \text{for } e = T_1 \cap T_2 \in \mathcal{E},$$

where $\mathcal{E}$ denotes the set of interior edges in $\mathcal{T}_h$. Moreover, we will also refer to the set of edges of T by $\mathcal{E}_T$. Based on these quantities, we define the local error estimators

$$\eta_T^2 := h_T^2 \|R_T\|_{L^2(T)}^2 + \frac{1}{2} \sum_{e \in \mathcal{E}_T} h_e \|R_e\|_{L^2(e)}^2.$$

From these, we obtain the global error estimator

$$\eta^2 := \sum_{T \in \mathcal{T}_h} \eta_T^2 = \sum_{T \in \mathcal{T}_h} h_T^2 \|R_T\|_{L^2(T)}^2 + \sum_{e \in \mathcal{E}} h_e \|R_e\|_{L^2(e)}^2.$$

(1.3) **Theorem.** Let $\mathcal{T}_h$ denote a shape-regular triangulation. Then, there exists $c > 0$ depending on Ω and κ, such that

$$\|u - u_h\|_{H^1(\Omega)} \leqslant c\eta.$$

Proof. For $v \in H_0^1(\Omega)$, there holds by the divergence theorem

$$\begin{aligned}\ell(v) &:= \big(\nabla(u-u_h), \nabla v\big)_{L^2(\Omega)} \\ &= (f,v)_{L^2(\Omega)} - \sum_{T\in\mathcal{T}_h} (\nabla u_h, \nabla v)_{L^2(\Omega)} \\ &= (f,v)_{L^2(\Omega)} - \sum_{T\in\mathcal{T}_h} \Big[-(\Delta u_h, v)_{L^2(\Omega)} + \sum_{e\in\mathcal{E}_T} \Big(\frac{\partial u_h}{\partial \boldsymbol{n}}, v\Big)_{L^2(e)} \Big] \\ &= \sum_{T\in\mathcal{T}_h} (\Delta u_h + f, v)_{L^2(T)} + \sum_{e\in\mathcal{E}} \Big(\Big[\!\Big[\frac{\partial u_h}{\partial \boldsymbol{n}} \Big]\!\Big], v \Big)_{L^2(e)} \\ &= \sum_{T\in\mathcal{T}_h} \Big[(R_T, v)_{L^2(T)} + \frac{1}{2} \sum_{e\in\mathcal{E}_T} (R_e, v)_{L^2(e)} \Big].\end{aligned}$$

Now, let $v_h := C_h v$. Due to Galerkin orthogonality, it holds

$$\big(\nabla(u-u_h), \nabla v_h\big)_{L^2(\Omega)} = 0 \quad \text{for all } v_h \in V_h$$

and hence

$$\ell(v) = \ell(v-v_h) \leqslant \sum_{T\in\mathcal{T}_h} \Big[\|R_T\|_{L^2(T)} \|v-v_h\|_{L^2(T)} \frac{1}{2} \sum_{e\in\mathcal{E}_T} \|R_e\|_{L^2(e)} \|v-v_h\|_{L^2(e)} \Big].$$

Since $\cup_{T\in\mathcal{T}_h} \omega_T$ is only a finite covering of Ω, Theorem (1.2) yields

$$\begin{aligned}\ell(v) &\leqslant c \sum_{T\in\mathcal{T}_h} \Big[h_T \|R_T\|_{L^2(T)} + \frac{1}{2} \sum_{e\in\mathcal{E}_T} h^{\frac{1}{2}} \|R_e\|_{L^2(e)} \Big] \|v\|_{H^1(\omega_T)} \\ &\leqslant c \sum_{T\in\mathcal{T}_h} \eta_T \|v\|_{H^1(\omega_T)} \leqslant c \Big(\sum_{T\in\mathcal{T}_h} \eta_T^2 \Big)^{\frac{1}{2}} \Big(\sum_{T\in\mathcal{T}_h} \|v\|_{H^1(\omega_T)}^2 \Big)^{\frac{1}{2}} \leqslant C\eta \|v\|_{H^1(\Omega)}\end{aligned}$$

for some constants $c, C > 0$. The proof is completed the duality argument

$$\begin{aligned}|u-u_h|_{H^1(\Omega)} &= \sup_{v\in H_0^1(\Omega), \|v\|_{H_0^1(\Omega)}=1} \big(\nabla(u-u_h), \nabla v\big)_{L^2(\Omega)} \\ &= \sup_{v\in H_0^1(\Omega), \|v\|_{H_0^1(\Omega)}=1} \ell(v).\end{aligned}$$

□

(1.4) Remark. In practice, the area based residual is hard to compute if f cannot be integrated exactly. Hence, one applies the splitting $f = f_h + f - f_h$, such that $\Delta u_h + f_h$ can be calculated exactly. In this case, one ends up with the bound

$$\|u-u_h\|_{H^1(\Omega)} \leqslant c \Big[\eta + \Big(\sum_{T\in\mathcal{T}_h} h_T^2 \|f-f_h\|_{L^2(T)}^2 \Big)^{\frac{1}{2}} \Big].$$

In this estimate, the newly added term is called *data oscillation*.

In particular, one can show the lower bound

$$\eta^2 \leqslant c' \left[\|u - u_h\|^2_{H^1(\Omega)} + \sum_{T \in \mathcal{T}_h} h_T^2 \|f - f_h\|^2_{L^2(T)} \right]$$

for some $c' > 0$. Therefore, the error estimator η is equivalent to the true error up to the data oscillation and the error is actually localised in the estimators η_T. In practice, one computes η_T for all elements and marks a certain fraction of the elements that provide the largest error contributions for refinement. △

VIII. The Heat Equation

In this section, we consider the heat equation

$$(0.1) \qquad \frac{\partial u}{\partial t}(t,\boldsymbol{x}) - \Delta u(t,\boldsymbol{x}) = f(t,\boldsymbol{x}), \quad (t,\boldsymbol{x}) \in [0,T] \times \Omega$$

for a timepoint $T > 0$ and a domain $\Omega \subset \mathbb{R}^d$ as an example for parabolic equations. For the sake of simplicity, we restrict ourselves to homogenous Dirichlet conditions, i.e.

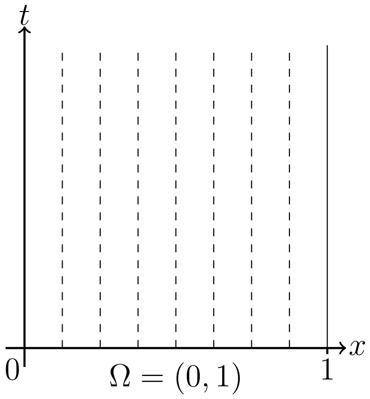
$$u(t,\boldsymbol{x}) = 0 \quad \text{for all } [0,T] \times \Gamma.$$

This corresponds to fixing the temperature for all times to zero degrees at the boundary. To end up with a well posed problem, we also prescribe the initial temperature distribution

$$u(0,\boldsymbol{x}) = g(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in \Omega.$$

In the stationary case $\partial u/\partial t \equiv 0$, equation ((0.1)) reduces to the well known Poisson's equation.

1 The Method of Lines



In Order to compute a solution to the heat equation, we first apply a semi-discretisation with respect to the spatial variable. To that end, we consider the variational formulation

Find $u(t) \in H_0^1(\Omega)$ such that

$$\frac{\partial}{\partial t}\big(u(t), v\big)_{L^2(\Omega)} + \big(\nabla u(t), \nabla v\big)_{L^2(\Omega)} = \big(f(t), v\big)_{L^2(\Omega)} \quad \text{for all } v \in H_0^1(\Omega).$$

The restriction of the variational formulation to the finite element space $V_h \subset H_0^1(\Omega)$ hence reads

Find $u_h(t) \in V_h$ such that

$$\frac{\partial}{\partial t}\big(u_h(t), v_h\big)_{L^2(\Omega)} + \big(\nabla u_h(t), \nabla v_h\big)_{L^2(\Omega)} = \big(f(t), v_h\big)_{L^2(\Omega)} \quad \text{for all } v_h \in V_h.$$

This translates to the linear system

$$\boldsymbol{M}\frac{\partial}{\partial t}\boldsymbol{u}(t) + \boldsymbol{A}\boldsymbol{u}(t) = \boldsymbol{f}(t), \quad \boldsymbol{u}(0) = \boldsymbol{g}. \tag{1.1}$$

Herein,

$$\boldsymbol{M} = [(\psi_i, \psi_j)_{L^2(\Omega)}]_{i,j}$$

denotes the finite element mass matrix, while

$$\boldsymbol{A} = [(\nabla\psi_j, \nabla\psi_i)_{L^2(\Omega)}]_{i,j}$$

is the finite element stiffness matrix. Note that these matrices are independent of the time variable t. This is in contrast to the coefficients of the discrete solution $\boldsymbol{u}(t) = [u_i(t)]_i$ and the coefficients of the right hand side $\boldsymbol{f}(t) = \big[\big(f(t), \phi_i\big)_{L^2(\Omega)}\big]_i$, which both depend on t.

The finite element solution is of the form

$$u_h(t, \boldsymbol{x}) = \sum_i u_i(t)\psi_i(\boldsymbol{x}).$$

In view of the vector valued function $t \mapsto \boldsymbol{u}(t)$, the semi-discretisation ((1.1)) is referred to as *method of lines*. For each $t \geqslant 0$, we obtain a vector that represents the function $u_h(t, \boldsymbol{x})$ with respect to the triangulation $\mathcal{T}_h$.

2 The θ-Scheme

Next, we introduce an appropriate discretisation for the time. To that and, we consider a subdivision of $[0, T]$ into M sub-intervals $[t_i, t_{i+1}]$, $i = 0, \dots, M-1$ and set $k_i := t_{i+1} - t_i$ to the length of each sub-interval.

Then, the *forward Euler method* yields

$$\boldsymbol{M}\frac{\boldsymbol{u}_{i+1} - \boldsymbol{u}_i}{k_i} + \boldsymbol{A}\boldsymbol{u}_i = \boldsymbol{f}(t_i) \quad \text{or} \quad \boldsymbol{M}\boldsymbol{u}_{i+1} = (\boldsymbol{M} - k_i\boldsymbol{A})\boldsymbol{u}_i + k_i\boldsymbol{f}(t_i), \tag{2.1}$$

while the *backward Euler method* results in

$$\boldsymbol{M}\frac{\boldsymbol{u}_{i+1} - \boldsymbol{u}_i}{k_i} + \boldsymbol{A}\boldsymbol{u}_{i+1} = \boldsymbol{f}(t_{i+1}) \quad \text{or} \quad (\boldsymbol{M} + k_i\boldsymbol{A})\boldsymbol{u}_{i+1} = \boldsymbol{M}\boldsymbol{u}_i + k_i\boldsymbol{f}(t_{i+1}). \tag{2.2}$$

In both cases, the initial value is given by $\boldsymbol{u}_0 = \boldsymbol{g}$.

The idea of the *θ-scheme* is now to combine the two equations ((2.1)) and ((2.2)) with respect to the parameter $\theta \in [0, 1]$. We obtain

$$\text{(2.3)}\qquad \boldsymbol{M}\frac{\boldsymbol{u}_{i+1}-\boldsymbol{u}_i}{k_i}+(1-\theta)\boldsymbol{A}\boldsymbol{u}_i+\theta\boldsymbol{A}\boldsymbol{u}_{i+1}=(1-\theta)\boldsymbol{f}(t_i)+\theta\boldsymbol{f}(t_{i+1})$$

or

$$(\boldsymbol{M}+k_i\theta\boldsymbol{A})\boldsymbol{u}_{i+1}=\big(\boldsymbol{M}-k_i(1-\theta)\boldsymbol{A}\big)\boldsymbol{u}_i+k_i(1-\theta)\boldsymbol{f}(t_i)+k_i\theta\boldsymbol{f}(t_{i+1}),$$

respectively. In particular, it holds

$$\theta=\begin{cases}0, & \text{forward Euler method}\\ 1/2, & \text{trapezoidal rule}\\ 1, & \text{backward Euler method}\end{cases}$$

In the context of parabolic equations, the trapezoidal rule is also called *Crank-Nicolson method*. In view of the local truncation error, the θ-scheme is consistent of order one, for $\theta=0.5$, it is even consistent of order two.

(2.4) **Theorem.** The θ-scheme for the heat equation is stable for $1/2\le\theta\le 1$, i.e.

$$\|u_{h,M}\|^2_{L^2(\Omega)}+\sum_{i=0}^{M-1}\Big[k_i|u_{h,i+\theta}|^2_{H^1(\Omega)}+(2\theta-1)\|u_{h,i+1}-u_{h,i}\|^2_{L^2(\Omega)}\Big]$$
$$\le\|u_{h,0}\|^2_{L^2(\Omega)}+c\sum_{i=0}^{M-1}k_i\|f_{i+\theta}\|^2_{L^2(\Omega)},$$

where we set

$$u_{h,i+\theta}:=(1-\theta)u_{h,i}+\theta u_{h,i+1},\qquad f_{i+\theta}:=(1-\theta)f(t_i)+\theta f(t_{i+1}).$$

Proof. In view of ((2.3)), inserting $u_{h,i+\theta}$ as a test function into the corresponding variational formulation yields

$$\text{(2.5)}\qquad (u_{h,i+1}-u_{h,i},u_{h,i+\theta})_{L^2(\Omega)}+k_i(\nabla u_{h,i+\theta},\nabla u_{h,i+\theta})_{L^2(\Omega)}=k_i(f_{i+\theta},u_{h,i+\theta})_{L^2(\Omega)}.$$

On the other hand, there holds

$$\begin{aligned}&(u_{h,i+1}-u_{h,i},u_{h,i+\theta})_{L^2(\Omega)}\\ &\quad=\big(u_{h,i+1}-u_{h,i},(1-\theta)u_{h,i}+\theta u_{h,i+1}\big)_{L^2(\Omega)}\\ &\quad=\left(u_{h,i+1}-u_{h,i},\frac{1}{2}u_{h,i+1}+\frac{1}{2}u_{h,i}+\Big(\theta-\frac{1}{2}\Big)(u_{h,i+1}-u_{h,i})\right)_{L^2(\Omega)}\\ &\quad=\frac{1}{2}\|u_{h,i+1}\|^2_{L^2(\Omega)}-\frac{1}{2}\|u_{h,i}\|^2_{L^2(\Omega)}+\Big(\theta-\frac{1}{2}\Big)\|u_{h,i+1}-u_{h,i}\|^2_{L^2(\Omega)}.\end{aligned}$$

The combination of ((2.5)), the Cauchy-Schwarz inequality and the Poincaré inequality yields

$$\begin{aligned}&\|u_{h,i+1}\|^2_{L^2(\Omega)}-\|u_{h,i}\|^2_{L^2(\Omega)}+(2\theta-1)\|u_{h,i+1}-u_{h,i}\|^2_{L^2(\Omega)}+2k_i|u_{h,i+\theta}|^2_{H^1(\Omega)}\\ &\quad=2k_i(f_{i+\theta},u_{h,i+\theta})_{L^2(\Omega)}\\ &\quad\le 2k_ic\|f_{i+\theta}\|_{L^2(\Omega)}|u_{h,i+\theta}|_{H^1(\Omega)}\\ &\quad\le k_ic^2\|f_{i+\theta}\|^2_{L^2(\Omega)}+k_i|u_{h,i+\theta}|^2_{H^1(\Omega)}\end{aligned}$$

for some constant $c > 0$. From this, we infer

$$\|u_{h,i+1}\|^2_{L^2(\Omega)} - \|u_{h,i}\|^2_{L^2(\Omega)} + (2\theta - 1)\|u_{h,i+1} - u_{h,i}\|^2_{L^2(\Omega)} \\ + k_i|u_{h,i+\theta}|^2_{H^1(\Omega)} \leq k_i c^2 \|f_{i+\theta}\|^2_{L^2(\Omega)}.$$

Now, summation with respect to i yields the assertion. □

Consequently, the θ-scheme is stable for all $\theta \in [0.5, 1]$. This means that errors will not be amplified exponentially. For $\theta < 0.5$, the method is only stable if $k_i \sim h^2$. This is the so called *CFL-condition* for parabolic problems. Herein, CFL stands for Courant, Friedrichs and Lewy. Given that $\theta > 0.5$, high frequency contributions to the solution are exponentially dampened. Accordingly, local perturbations in the data $\boldsymbol{f}(t_i)$ and $\boldsymbol{g}$ are substantially reduced.

(2.6) **Remark.** The Crank-Nicolson method is the simplest second order method. Hence, it is rather popular. However, since errors do not get dampened exponentially, it might produce unphysical oscillations. Hence, it is preferable to choose $\theta = 1/2 + \epsilon$ for a small quantity $\epsilon > 0$.

3 Error Analysis

In the sequel, we consider the error analysis for the backward Euler method, i.e. $\theta = 1$. For the sake of simplicity, we slightly modify the right hand side according to

$$\overline{f}(t_{i+1}) := \frac{1}{k_i}\int_{t_i}^{t_{i+1}} f(t)\,\mathrm{d}t = f(t_{i+1}) + \mathcal{O}(k_i)$$

In order to estimate the error, we introduce the discrete semi-norm

$$\|u\|_{h,\infty} := \max_{i=1}^{M} \|u(t_i)\|_{L^2(\Omega)}.$$

For functions $u_h \in V_h \times \{t_1, \ldots, t_M\}$, which are discrete with respect to time, this is even a norm.

(3.1) **Theorem.** Let Ω denote a convex, polygonal domain and $\mathcal{T}_h$ a shape-regular mesh, which is fixed with respect to time.
Assume that the continuous solution to the heat equation satisfies $u \in H^1(0, T) \otimes H^2(\Omega)$.
Then the backward Euler method together with a piecewise linear finite element discretisation satisfies the error estimate

$$\|u - u_h\|_{h,\infty} \leq c\left[\sqrt{T}h^2 k^{-1/2}\|\Delta u\|_{h,\infty} + \left(\sum_{i=0}^{M-1} k_i^2 \int_{t_i}^{t_{i+1}} \left|\frac{\partial u}{\partial t}\right|^2_{H^1(\Omega)} \mathrm{d}t\right)^{1/2}\right],$$

given that $\min_{i=1}^{M}\{k_i\} \geq k$.

Proof. Let $P_h u_i$ denote the Galerkin projection of $u_i := u(t_i)$ onto V_h. It is given via the Galerkin method accoring to

Find $P_h u_i \in V_h$ such that

$$(\nabla P_h u_i, \nabla v_h)_{L^2(\Omega)} = (\nabla u_i, \nabla v_h)_{L^2(\Omega)} \quad \text{for all } v_h \in V_h.$$

We split the error $u_i - u_{h,i}$ into $\xi_i = (P_h - I)u_i$ and $\eta_i = u_{h,i} - P_h u_i \in V_h$. According to Theorem (V.1.6), see also Theorem (V.1.4), with respect to ξ_i there holds the error estimate

$$(3.2) \qquad \|\xi_i\|_{L^2(\Omega)} = \|(P_h - I)u_i\|_{L^2(\Omega)} \leq ch^2 |u_i|_{H_2(\Omega)}.$$

From this, we derive

$$\|(P_h - I)u\|_{h,\infty} \leq ch^2 \|\Delta u\|_{h,\infty}.$$

For the error contribution η_{i+1}, we employ the identity

$$(3.3) \qquad \|\eta_{i+1}\|^2_{L^2(\Omega)} - \|\eta_i\|^2_{L^2(\Omega)} = 2(\eta_{i+1} - \eta_i, \eta_{i+1})_{L^2(\Omega)} - \|\eta_{i+1} - \eta_i\|^2_{L^2(\Omega)}.$$

We compute a bound for the term $(\eta_{i+1} - \eta_i, \eta_{i+1})_{L^2(\Omega)}$. Remind that the Galerkin formulation for the backward Euler method reads at time t_{i+1}

Find $u_{h,i+1} \in V_h$ such that

$$(u_{h,i+1} - u_{h,i}, v_h)_{L^2(\Omega)} + k_i(\nabla u_{h,i+1}, \nabla v_h)_{L^2(\Omega)} = k_i(f_{i+1}, v_h)_{L^2(\Omega)}$$

for all $v_h \in V_h$. Hence, it holds due to $\eta_{i+1} \in V_h$ and the definition of the Galerkin projection that

$$\begin{aligned}
&(\eta_{i+1} - \eta_i, \eta_{i+1})_{L^2(\Omega)} \\
&\quad = k_i(f_{i+1}, \eta_{i+1})_{L^2(\Omega)} - k_i(\nabla u_{h,i+1}, \nabla \eta_{i+1})_{L^2(\Omega)} - (P_h u_{i+1} - P_h u_i, \eta_{i+1})_{L^2(\Omega)} \\
&\quad = k_i(f_{i+1}, \eta_{i+1})_{L^2(\Omega)} - k_i\big(\nabla(\eta_{i+1} + u_{i+1}), \nabla \eta_{i+1}\big)_{L^2(\Omega)} - (u_{i+1} - u_i, \eta_{i+1})_{L^2(\Omega)} \\
&\qquad - (P_h u_{i+1} - u_{i+1} - P_h u_i + u_i, \eta_{i+1})_{L^2(\Omega)} \\
&\quad = k_i(f_{i+1}, \eta_{i+1})_{L^2(\Omega)} - k_i\big(\nabla(\eta_{i+1} + u_{i+1}), \nabla \eta_{i+1}\big)_{L^2(\Omega)} - (u_{i+1} - u_i, \eta_{i+1})_{L^2(\Omega)} \\
&\qquad - (\xi_{i+1} - \xi_i, \eta_{i+1})_{L^2(\Omega)}.
\end{aligned}$$

The first three terms on the right hand side of the previous identity can be bounded by using the heat equation according to

$$\begin{aligned}
E_1 &:= k_i(f_{i+1}, \eta_{i+1})_{L^2(\Omega)} - k_i\big(\nabla(\eta_{i+1} + u_{i+1}), \nabla \eta_{i+1}\big)_{L^2(\Omega)} - (u_{i+1} - u_i, \eta_{i+1})_{L^2(\Omega)} \\
&= \int_{t_i}^{t_{i+1}} \left(f - \frac{\partial u}{\partial t}, \eta_{i+1}\right)_{L^2(\Omega)} \mathrm{d}t - k_i(\nabla u_{i+1}, \nabla \eta_{i+1})_{L^2(\Omega)} - k_i \|\nabla \eta_{i+1}\|^2_{L^2(\Omega)} \\
&= \int_{t_i}^{t_{i+1}} \big(\nabla(u - u_{i+1}), \nabla \eta_{i+1}\big)_{L^2(\Omega)} \mathrm{d}t - k_i \|\nabla \eta_{i+1}\|^2_{L^2(\Omega)}.
\end{aligned}$$

Now, the fundamental theorem of calculus yields for a differentiable function g that

$$\int_x^y \{g(z) - g(x)\} \,\mathrm{d}z = \int_x^y \int_x^z g'(t) \,\mathrm{d}t \,\mathrm{d}z.$$

Since

$$\{(t, z) : x \leq z \leq y,\ x \leq t \leq z\} = \{(t, z) : x \leq t \leq y,\ t \leq z \leq y\},$$

we end up with the identity

$$\int_x^y \{g(z) - g(x)\}\,\mathrm{d}z = \int_x^y \int_t^y g'(t)\,\mathrm{d}z\,\mathrm{d}t = \int_x^y g'(t)(y-t)\,\mathrm{d}t.$$

Inserting this into the representation of E_1 gives

$$\begin{aligned} E_1 &= \int_{t_i}^{t_{i+1}} (t_i - t)\left(\nabla \frac{\partial u}{\partial t}, \nabla \eta_{i+1}\right)_{L^2(\Omega)} \mathrm{d}t - k_i \|\nabla \eta_{i+1}\|^2_{L^2(\Omega)} \\ &\leq \int_{t_i}^{t_{i+1}} (t - t_i)\left|\frac{\partial u}{\partial t}\right|_{H^1(\Omega)} \|\nabla \eta_{i+1}\|_{L^2(\Omega)}\,\mathrm{d}t - k_i \|\nabla \eta_{i+1}\|^2_{L^2(\Omega)} \\ &\leq \left(\frac{k_i^2}{2}\int_{t_i}^{t_{i+1}} \left|\frac{\partial u}{\partial t}\right|^2_{H^1(\Omega)} \mathrm{d}t\right)^{1/2} \left(2\int_{t_i}^{t_{i+1}} \|\nabla \eta_{i+1}\|^2_{L^2(\Omega)}\,\mathrm{d}t\right)^{1/2} - k_i \|\nabla \eta_{i+1}\|^2_{L^2(\Omega)}. \end{aligned}$$

Now, estimating the geometric average by the arithmetic one, i.e. $\sqrt{ab} \leq (a+b)/2$ for all $a, b > 0$, we arrive at

$$E_1 \leq \frac{k_i^2}{4}\int_{t_i}^{t_{i+1}} \left|\frac{\partial u}{\partial t}\right|^2_{H^1(\Omega)} \mathrm{d}t.$$

Additionally, it holds

$$\begin{aligned} E_2 &:= (\xi_{i+1} - \xi_i, \eta_{i+1})_{L^2(\Omega)} \\ &= (\xi_{i+1}, \eta_{i+1})_{L^2(\Omega)} - (\xi_i, \eta_i)_{L^2(\Omega)} + (\xi_i, \eta_i - \eta_{i+1})_{L^2(\Omega)} \\ &\leqslant (\xi_{i+1}, \eta_{i+1})_{L^2(\Omega)} - (\xi_i, \eta_i)_{L^2(\Omega)} + \|\eta_i - \eta_{i+1}\|_{L^2(\Omega)} \|\xi_i\|_{L^2(\Omega)} \\ &\leq (\xi_{i+1}, \eta_{i+1})_{L^2(\Omega)} - (\xi_i, \eta_i)_{L^2(\Omega)} + \frac{1}{2}\|\eta_i - \eta_{i+1}\|^2_{L^2(\Omega)} + ch^4 |u_i|^2_{H^2(\Omega)}, \end{aligned}$$

where we employed ((3.2)) end the estimate on the geometric and arithmetic average in the last step.
Inserting the equality $(\eta_{i+1} - \eta_i, \eta_{i+1})_{L^2(\Omega)} = E_1 + E_2$ into ((3.3)) yields

$$\begin{aligned} \|\eta_{i+1}\|^2_{L^2(\Omega)} - \|\eta_i\|^2_{L^2(\Omega)} &= 2E_1 + 2E_2 - \|\eta_{i+1} - \eta_i\|^2_{L^2(\Omega)} \\ &\leq 2(\xi_{i+1}, \eta_{i+1})_{L^2(\Omega)} - 2(\xi_i, \eta_i)_{L^2(\Omega)} + \frac{k_i^2}{2}\int_{t_i}^{t_{i+1}} \left|\frac{\partial u}{\partial t}\right|^2_{H^1(\Omega)} \mathrm{d}t + ch^4 |u_i|^2_{H^2(\Omega)}. \end{aligned}$$

Now, summation with respect to i results in

$$\begin{aligned}
\|\eta_M\|^2_{L^2(\Omega)} &\leq \|\eta_0\|^2_{L^2(\Omega)} + 2 \underbrace{(\xi_M, \eta_M)_{L^2(\Omega)}}_{\leq(\sqrt{2}\|\xi_M\|_{L^2(\Omega)})(\|\eta_M\|_{L^2(\Omega)}/\sqrt{2})} -2 \underbrace{(\xi_0, \eta_0)_{L^2(\Omega)}}_{\leq(\sqrt{2}\|\xi_0\|_{L^2(\Omega)})(\|\eta_0\|_{L^2(\Omega)}/\sqrt{2})} \\
&\quad + \sum_{i=0}^{M-1} \frac{k_i^2}{2} \int_{t_i}^{t_{i+1}} \left|\frac{\partial u}{\partial t}\right|^2_{H^1(\Omega)} \mathrm{d}t + ch^4 \sum_{i=0}^{M-1} |u_i|^2_{H^2(\Omega)} \\
&\leq \|\eta_0\|^2_{L^2(\Omega)} + 2\|\xi_M\|^2_{L^2(\Omega)} + \frac{1}{2}\|\eta_M\|^2_{L^2(\Omega)} + 2\|\xi_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|\eta_0\|^2_{L^2(\Omega)} \\
&\quad + \sum_{i=0}^{M-1} \frac{k_i^2}{2} \int_{t_i}^{t_{i+1}} \left|\frac{\partial u}{\partial t}\right|^2_{H^1(\Omega)} \mathrm{d}t + ch^4 \sum_{i=0}^{M-1} k_i |k_i^{-1/2} u_i|^2_{H^2(\Omega)} \\
&\leq \frac{3}{2}\|\eta_0\|^2_{L^2(\Omega)} + 2\|\xi_M\|^2_{L^2(\Omega)} + \frac{1}{2}\|\eta_M\|^2_{L^2(\Omega)} + 2\|\xi_0\|^2_{L^2(\Omega)} \\
&\quad + \sum_{i=0}^{M-1} \frac{k_i^2}{2} \int_{t_i}^{t_{i+1}} \left|\frac{\partial u}{\partial t}\right|^2_{H^1(\Omega)} \mathrm{d}t + ch^4 T \|k^{-1/2}\Delta u\|^2_{h,\infty}.
\end{aligned}$$

Projecting the initial value $u_0 = g$ via a suitable projection $Q_h \colon H^1_0(\Omega) \to V_h$ yields

$$\begin{aligned}
\|\eta_0\|_{L^2(\Omega)} = \|Q_h u_0 - P_h u_0\|_{L^2(\Omega)} &\leq \|(I - Q_h)u_0\|_{L^2(\Omega)} + \|(I - P_h)u_0\|_{L^2(\Omega)} \\
&\leq ch^2 |u_0|_{H^2(\Omega)}.
\end{aligned}$$

Thus, we arrive at

$$\begin{aligned}
\frac{1}{2}\|\eta_M\|^2_{L^2(\Omega)} \leq 2\|\xi_M\|^2_{L^2(\Omega)} &+ 2\|\xi_0\|^2_{L^2(\Omega)} + ch^4(T+1)\|k^{-1/2}\Delta u\|^2_{h,\infty} \\
&+ \sum_{i=0}^{M-1} \frac{k_i^2}{2} \int_{t_i}^{t_{i+1}} \left|\frac{\partial u}{\partial t}\right|^2_{H^1(\Omega)} \mathrm{d}t.
\end{aligned}$$

Since $\|u - u_h\|_{h,\infty} \leq \|\xi\|_{h,\infty} + \|\eta\|_{h,\infty}$, we finally obtain the assertion. □

(3.4) **Remark.** In the case of uniform time steps $k \leq ck_i$, the error estimate from Theorem (3.1), indicates that

$$\|u - u_h\|_{h,\infty} = \mathcal{O}(h^2 k^{-1/2} + k).$$

In view of the factor $k^{-1/2}$ is not optimal. However, under the condition $h \leq ck^{3/4}$ we obtain the optimal order of convergence $\mathcal{O}(k)$. Employing the Crank-Nicolson method, the discretisation error becomes $\mathcal{O}(k^2 + h^2)$. This implies quadratic convergence of order $\mathcal{O}(k^2)$, if $h \sim k$ is chosen.

IX. Appendix

1 Basics

We start be recalling the basic definitions on vector spaces and norms. In what follows, let always always $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Moreover, we set $\mathbb{N} := \{0, 1, 2, \dots\}$ and $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

(1.1) **Definition.** A $\mathbb{K}$-*vector space* (or vector space, linear space) is a set V together with two operations

$$\begin{array}{lll} +\colon V \times V \to V, & (x, y) \mapsto x + y & \text{(addition)} \\ \cdot\colon \mathbb{K} \times V \to V, & (\alpha, x) \mapsto \alpha \cdot x & \text{(scalar multiplication)} \end{array}$$

which satisfy the following eight conditions:

$$\begin{array}{llll} \text{(A1)} & \forall x, y, z \in V: & x + (y + z) = (x + y) + z & \text{(associativity)} \\ \text{(A2)} & \forall x, y \in V: & x + y = y + x & \text{(commutativity)} \\ \text{(A3)} & \exists! 0 \in V\ \forall x \in V: & x + 0 = x & \text{(identity element)} \\ \text{(A4)} & \forall x \in V\ \exists (-x) \in V: & x + (-x) = 0 & \text{(inverse element)} \\ & & & \\ \text{(S1)} & \forall \alpha, \beta \in \mathbb{K}, x \in V: & (\alpha\beta) \cdot x = a\alpha \cdot (\beta \cdot x) & \text{(compatibility)} \\ \text{(S2)} & \forall x \in V, 1 \in \mathbb{K}: & 1 \cdot x = x & \text{(identity element)} \\ \text{(S3)} & \forall \alpha \in \mathbb{K}, x, y \in V: & \alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y & \text{(distributivity I)} \\ \text{(S4)} & \forall \alpha, \beta \in \mathbb{K}, x \in V: & (\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x & \text{(distributivity II)} \end{array}$$

The elements $x \in V$ are referred to as *vectors*, while the elements $\alpha \in \mathbb{K}$ are called *scalars*.

(1.2) **Example.**

- For $d \in \mathbb{N}^*$, the spaces $\mathbb{R}^d$ and $\mathbb{C}^d$ with the usual operations form an $\mathbb{R}$-vector space and a $\mathbb{C}$-vector space, respectively.

- The set $C^k([a, b])$, $k \in \mathbb{N}$ of k-times continuously differentiable functions on the interval $[a, b]$ with the operations

  $$(f + g)(x) := f(x) + g(x), \quad (\alpha \cdot f)(x) = \alpha f(x),$$

  where $f, g \in C([a, b]), \alpha \in \mathbb{R}$, forms an $\mathbb{R}$-vector space.

- For $1 \leqslant p < \infty$, let

  $$\ell^p(\mathbb{K}) := \left\{ (x_n)_{n \in \mathbb{N}} : \sum_{n \in \mathbb{N}} |x_n|^p < \infty \right\}$$

  denote the set of $\mathbb{K}$-valued, *p-summable* sequences. Together with the operations

  $$(x_n)_{n \in \mathbb{N}} + (y_n)_{n \in \mathbb{N}} := (x_n + y_n)_{n \in \mathbb{N}}, \quad \alpha \cdot (x_n)_{n \in \mathbb{N}} := (\alpha x_n)_{n \in \mathbb{N}},$$

  where $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \in \ell^p(\mathbb{K}), \alpha \in \mathbb{K}$, $\ell^p(\mathbb{K})$ becomes a $\mathbb{K}$-vector space.

△

(1.3) **Definition.** Let V denote a $\mathbb{K}$-vector space. A subset $U \subset V$ endowed with the addition and scalar multiplication of V is called a *subspace*, iff it satisfies the conditions of a vector space from Definition (1.1).

(1.4) **Example.**

- The space spanned by the vector $[1, 0]^\intercal$ forms a subspace of $\mathbb{R}^2$.
- The space $C^k([a, b])$ is a subspace of $C^\ell([a, b])$ for $0 \leqslant \ell \leqslant k$.

△

(1.5) **Definition.** Let V denote a $\mathbb{K}$-vector space. A mapping

$$\|\cdot\| \colon V \to [0, \infty)$$

is called *norm* on V, iff

$$\begin{array}{llll} \text{(N1)} & \forall x \in V \setminus \{0\}: & \|x\| > 0 & \text{(definiteness)} \\ \text{(N2)} & \forall x \in V\ \forall \alpha \in \mathbb{K}: & \|\alpha x\| = |\alpha| \cdot \|x\| & \text{(homogeneity)} \\ \text{(N3)} & \forall x, y \in V: & \|x + y\| \leq \|x\| + \|y\| & \text{(triangle inequality)} \end{array}$$

The tuple $(V, \|\cdot\|)$ is called *normed vector space*.

Based on the distance concept which is introduced by a norm, we can now consider convergence in normed vector spaces.

(1.6) **Definition.** Let $(V, \|\cdot\|)$ denote a normed vector space. A sequence $(x_n)_{n \in \mathbb{N}} \subset V$ is called *convergent* towards $x^\star \in V$, iff for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon)$ such that

$$\|x_n - x^\star\| < \varepsilon \text{ for all } n \geqslant n_0,$$

i.e. iff all x_n except for finitely many are contained in every vicinitiy of $x^\star$.
A sequence $(x_n)_{n \in \mathbb{N}} \subset V$ is called a *Cauchy sequence*, iff for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon)$ such that

$$\|x_n - x_m\| < \varepsilon \text{ for all } m, n \geqslant n_0,$$

i.e. iff any x_n, x_m become asymptotically arbitrarily close to each other.

It is straightforward to verify that every convergent sequence is a Cauchy sequence. The converse is a special property and is only true in certain spaces.

(1.7) **Definition.** A normed vector space $(V, \|\cdot\|)$ is called *complete* if every Cauchy sequence $(x_n)_{n \in \mathbb{N}} \subset V$ has a limit $x^\star \in V$. A complete normed vector space is called a *Banach space*.

(1.8) **Example.**

- For $d \in \mathbb{N}^*$, the vector spaces $\mathbb{R}^d$ and $\mathbb{C}^d$ endowed with the norm

$$\|x\|_2 := \sqrt{\sum_{k=1}^{d} |x_k|^2}$$

are Banach spaces.

- The vector spaces $\ell^p(\mathbb{K})$, $1 \leqslant p < \infty$ endowed with the norms

$$\|(x_n)_{n\in\mathbb{N}}\|_{\ell^p} := \left(\sum_{n\in\mathbb{N}} |x_n|^p\right)^{\frac{1}{p}}$$

are Banach spaces.

- The vector spaces $C^k([a,b])$, $k \in \mathbb{N}$, endowed with the norms

$$\|f\| := \max_{x\in[a,b]} |f(x)| + \max_{x\in[a,b]} \left|f'(x)\right| + \cdots + \max_{x\in[a,b]} \left|f^{(k)}(x)\right|$$

are Banach spaces.

- The vector space $C([a,b])$ endowed with the norm

$$\|x\|_{L^p} := \left(\int_a^b |x(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty,$$

where the integral has to be understood in the Riemannian sense, is not complete.

△

(1.9) **Remark.** More generally, one can show that every finite dimensional normed $\mathbb{K}$-vector space is complete. This is a consequence of the completeness of $\mathbb{K}$. △

(1.10) **Theorem.**

(a) Let $(V, \|\cdot\|)$ denote a Banach space and $U \subset V$ a closed subspace, i.e. for every convergent sequence in U, the limit is also contained in U, then U is complete.

(b) Let $(V, \|\cdot\|)$ denote a normed vector space. If $U \subset V$ a complete subspace, then U is closed.

Proof.

(a) Let $(x_n)_{n\in\mathbb{N}}$ denote a Cauchy sequence in U. Since V is complete, the limit $x := \lim_{n\to\infty} x_n$ exists. From the closedness of U we infer $x \in U$.

(b) Let $(x_n)_{n\in\mathbb{N}} \subset U$ be a convergent sequence with limit $x \in V$. Since $(x_n)_{n\in\mathbb{N}}$ is convergent, it is particular a Cauchy sequence in U. Due to the completeness, the sequence must have a limit point in U. From the uniqueness of the limit, we infer $x \in U$, which implies that U is closed. □

(1.11) **Example.**

- For $d \in \mathbb{N}^*$, the subspace corresponding to the set $U := \{x \in \mathbb{R}^d : x_d = 0\} \subset \mathbb{R}^d$ is a complete subspace of $\mathbb{R}^d$.

△

(1.12) Theorem. Let $(V, \|\cdot\|_V)$ denote a normed vector space. Then, there exists a Banach space $(\overline{V}, \|\cdot\|_{\overline{V}})$ called *completion* of V and an injective mapping $J\colon V \to \overline{V}$ such that

$$J(x+y) = J(x) + J(y), \quad J(\alpha \cdot x) = \alpha \cdot J(x), \quad \text{and } \|x\|_V = \|J(x)\|_{\overline{V}}.$$

The completion is uniquely determined up to isometry.

Proof. We consider the vector space of all Cauchy sequences on V, which we denote by

$$\tilde{V} := \{\tilde{x} = (x_n)_{n\in\mathbb{N}} \subset V : (x_n)_{n\in\mathbb{N}} \text{ is a Cauchy sequence}\}.$$

On $\tilde{V}$ we introduce the *equivalence relation*

$$(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}} \quad :\Leftrightarrow \quad \|x_n - y_n\|_V \text{ is a null sequence in } \mathbb{R}$$

and define the *equivalence classes* $[\tilde{x}] := \{y \in \tilde{V} : x \sim y\}$. With the addition and scalar multiplication for Cauchy sequences, the set $\overline{V} := \{[\tilde{x}] : \tilde{x} \in \tilde{V}\}$ becomes a vector space. Since $|\|x_n\|_V - \|x_m\|_V| \leqslant \|x_n - x_m\|_V$ for any Cauchy sequence $(x_n)_{n\in\mathbb{N}}$, the limit $\lim_{n\to\infty} \|x_n\|_V$ exists in $\mathbb{R}$. We define

$$\|[\tilde{x}]\|_{\overline{V}} := \lim_{n\to\infty} \|x_n\|_V.$$

The mapping $J\colon V \to \overline{V}$ is given by $J(x) = [(x)_{n\in\mathbb{N}}]$, i.e. the equivalence class which contains the constant sequence with value x. The linearity of the mapping J is consequence of the fact that $\overline{V}$ is a vector space. Moreover J is injective, since $x \neq y$ obviously implies $J(x) \neq J(y)$. Furher, it holds $\|J(x)\|_{\overline{V}} = \lim_{n\to\infty} \|x\|_V = \|x\|_V$.
It remains to show that that $(\overline{V}, \|\cdot\|_{\overline{V}})$ is a Banach space. To that end, let $\big([\tilde{x}]_k\big)_{k\in\mathbb{N}} \subset \overline{V}$ be a Cauchy sequence. We denote the n-th element of some representer of $[\tilde{x}]_k$ by $x_{k,n}$. For each k we can now choose n_k such that

$$\text{(1.13)} \qquad \|x_{k,m} - x_{k,n_k}\|_V \leqslant k^{-1} \quad \text{if } m > n_k.$$

We show that the sequence

$$\text{(1.14)} \qquad \tilde{x}^\star := \big(x_{1,n_1}, x_{2,n_2}, \ldots, x_{k,n_k}, \ldots\big) \subset V$$

is a Cauchy sequence and that $\big([\tilde{x}]_k\big)_{k\in\mathbb{N}}$ converges towards $[\tilde{x}^\star]$. We have

$$\|[\tilde{x}]_k - J(x_{k,n_k})\|_{\overline{V}} = \lim_{m\to\infty} \|x_{k,m} - x_{k,n_k}\|_V \leqslant k^{-1}$$

due to (1.13). Note that the limit exists, since the sum of two Cauchy sequences forms a Cauchy sequence. Further, we derive

$$\text{(1.15)} \qquad \begin{aligned} \|x_{k,n_k} - x_{m,n_m}\|_V &= \|J(x_{k,n_k}) - J(x_{m,n_m})\|_{\overline{V}} \\ &\leqslant \|[\tilde{x}]_k - J(x_{k,n_k})\|_{\overline{V}} + \|[\tilde{x}]_m - J(x_{m,n_m})\|_{\overline{V}} + \|[\tilde{x}]_k - [\tilde{x}]_m\|_{\overline{V}} \\ &\leqslant k^{-1} + m^{-1} + \|[\tilde{x}]_k - [\tilde{x}]_m\|_{\overline{V}}. \end{aligned}$$

This implies that (1.14) is a Cauchy sequence. We find

$$\begin{aligned}\|[\tilde{x}^\star] - [\tilde{x}]_k\|_{\overline{V}} &\leqslant \|[\tilde{x}^\star] - J(x_{k,n_k})\|_{\overline{V}} + \|J(x_{k,n_k}) - [\tilde{x}]_k\|_{\overline{V}} \\ &\leqslant \|[\tilde{x}^\star] - J(x_{k,n_k})\|_{\overline{V}} + k^{-1}.\end{aligned}$$

The first term on the right hand side can be bounded with the help of (1.15) in accordance with

$$\begin{aligned}\|[\tilde{x}^\star] - J(x_{k,n_k})\|_{\overline{V}} &= \lim_{m\to\infty} \|x_{m,n_m} - x_{k,n_k}\|_V \\ &\leqslant \lim_{m\to\infty} \left(m^{-1} + \|[\tilde{x}]_k - [\tilde{x}]_m\|_{\overline{V}}\right) + k^{-1} \\ &= \lim_{m\to\infty} \|[\tilde{x}]_k - [\tilde{x}]_m\|_{\overline{V}} + k^{-1}.\end{aligned}$$

Again the limit exists and is bounded by some ε_k. The sequence $(\varepsilon_k)k \in \mathbb{N}$ converges to 0, since $[\tilde{x}]_k$ is a Cauchy sequence. This proves

$$\lim_{k\to\infty} \|[\tilde{x}^\star] - [\tilde{x}]_k\|_{\overline{V}} = 0,$$

which implies the completeness of $\overline{V}$. □

(1.16) **Example.** The space $C([a,b])$ is *dense* in

$$L^1([a,b]) := \left\{ f\colon [a,b] \to \mathbb{R} : f \text{ is Lebesgue measurable and } \int_a^b |f(t)|\,\mathrm{d}t < \infty \right\}$$

(later!). Hence, the completion of $C([a,b])$ with respect to the integral norm is $L^1([a,b])$.

2 Linear operators

(2.1) **Definition.** Let $(V, \|\cdot\|_V), (W, \|\cdot\|_W)$ denote two normed $\mathbb{K}$-vector spaces. A *linear operator* is a mapping $T\colon V \to W$ which satisfies

$$\begin{array}{llll} \text{(L1)} & \forall x, y \in V: & T(x+y) = T(x) + T(y) & \text{(linearity I)} \\ \text{(L2)} & \forall x \in V,\ \alpha \in \mathbb{K}: & T(\alpha x) = \alpha T(x) & \text{(linearity II)} \\ \text{(L3)} & \exists C > 0 \forall x \in V: & \|T(x)\|_W \leqslant C\|x\|_V & \text{(continuity)} \end{array}$$

The space of all continuous operators from V to W is denoted by $\mathscr{L}(V;W)$.
The smallest constant $C > 0$ such that (L3) holds, i.e.

$$C = \|T\|_{\mathscr{L}(V;W)} := \sup_{\|x\|_V \leqslant 1} \|T(x)\|_W,$$

is called the *operator norm* of T.
The space $V' := \mathscr{L}(V;\mathbb{K})$ is called *dual space* of V.

(2.2) **Remark.**

- To emphasise the linearity the parentheses around the argument of a linear operator are usually omitted, i.e. one writes Tx instead of $T(x)$.

- The postulated *Lipschitz continuity* (L3) is equivalent to the usual ε-δ-continuity of a linear operator. In fact, a linear operator is already continuous everywhere, if it is continuous in a single point $x_0 \in X$.
- The space $(\mathscr{L}(V;W), \|\cdot\|_{\mathscr{L}(V;W)})$ is a normed vector space. It is a Banach space if W is a Banach space.
- If V is finite dimensional, then every linear operator on V is continuous.

△

(2.3) **Example.**

- The linear operators in $\mathscr{L}(\mathbb{R}^{d_1}, \mathbb{R}^{d_2})$ coincide with the $(d_2 \times d_1)$-matrices, i.e. $\mathscr{L}(\mathbb{R}^{d_1}, \mathbb{R}^{d_2}) = \mathbb{R}^{d_2 \times d_1}$.
- The operator

$$T\colon C([a,b]) \to \mathbb{R}, \quad f \mapsto \int_a^b f(t)\,\mathrm{d}t$$

is linear and continuous, i.e. $T \in \big(C([a,b])\big)'$.

△

(2.4) **Definition.** Let V, W denote two Banach spaces. An operator $T \in \mathscr{L}(V;W)$ is *compact*, iff for every bounded sequence $(x_n)_{n\in\mathbb{N}} \subset V$ the sequence $(Tx_n)_{n\in\mathbb{N}} \subset W$ exhibits a convergent subsequence or equivalently, if $\overline{T(\{x \in V : \|x\|_V < 1\})}$ is compact. The set of all compact operators is denoted by $\mathscr{K}(V,W)$. It is a closed subset of $\mathscr{L}(V;W)$.

(2.5) **Remark.** Every operator $T\colon V \to W$ with a finite range, i.e. $\dim\big(T(V)\big) < \infty$ is compact. Consequently, if T is the limit of finite range operators, it is compact as well.

(2.6) **Example.** We consider the integral operator

$$T\colon C([a,b]) \to C([a,b]), \quad (Tx)(s) := \int_a^b k(s,t)x(t)\,\mathrm{d}t, \quad k \in C([a,b]^2).$$

The Arzelà-Ascoli theorem states: Let $M \subset C([a,b])$ be bounded and equicontinuous, i.e. for every $\varepsilon > 0$ and all $x \in M$ exists a $\delta > 0$ such that $|s,t| < \delta \Rightarrow |x(s) - x(t)| < \varepsilon$. Then $\overline{M}$ is compact.

We verify the conditions of the Arzelà-Ascoli theorem to show that the closure of

$$M := T(\{x \in C([a,b]) : \|x\| \leqslant 1\})$$

is compact.

Obviously, T is bounded. Hence, M is bounded as well. Moreover, let $\varepsilon > 0$. Since k is continuous, there exists $\delta(\varepsilon) > 0$ such that

$$\|(s,t)-(s',t')\| < \delta \quad \Rightarrow \quad |k(s,t)-k(s',t')| < \frac{\varepsilon}{b-a}.$$

This implies for all $|s-s'| < \delta$ that

$$|(Tx)(s)-(Tx)(s')| \leqslant \int_a^b |k(s,t)-k(s',t)||x(t)|\,\mathrm{d}t < \frac{\varepsilon}{b-a}\|x\|(b-a) < \varepsilon.$$

Hence, the Arzelà-Ascoli theorem applies and T is compact.

$\triangle$

3 Banach fixed-point theorem

(3.1) **Theorem (Banach fixed-point theorem).** Let $(V, \|\cdot\|)$ denote a Banach space. Moreover, let $\Phi \colon V \to V$ be a contraction of V, i.e. there exists $0 \leqslant L < 1$ such that

$$\|\Phi(x) - \Phi(y)\| \leqslant L\|x-y\| \quad \text{for all } x, y, \in V.$$

Then, there exists a unique fixed point $x^\star \in V$ of Φ and for every initial value $x_0 \in V$ the sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n := \Phi(x_{n-1})$ converges to $x^\star$.
In addition, there hold the following estimates

$$\begin{array}{lll} \text{(i)} & \|x^\star - x_n\| \leq L\|x^\star - x_{n-1}\| & \text{(monotonicity)} \\ \text{(ii)} & \|x^\star - x_n\| \leq \dfrac{L^n}{1-L}\|x_1 - x_0\| & \text{(a-piori bound)} \\ \text{(iii)} & \|x^\star - x_n\| \leq \dfrac{L}{1-L}\|x_n - x_{n-1}\| & \text{(a-posteriori bound)} \end{array}$$

Proof. Let $x_0 \in V$. We start by proving that the sequence $(x_n)_{n\in\mathbb{N}}$ forms a Cauchy sequence. Since $\Phi(z) \in V$ for every $z \in V$, there holds $x_n \in V$ for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$ we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\Phi(x_n) - \Phi(x_{n-1})\| \leq L\,\|x_n - x_{n-1}\| \\ &= L\,\|\Phi(x_{n-1}) - \Phi(x_{n-2})\| \leq L^2\|x_{n-1} - x_{n-2}\| \\ &= L^2\|\Phi(x_{n-2}) - \Phi(x_{n-3})\| \leq L^3\|x_{n-2} - x_{n-3}\| \\ &= \ldots \leq L^n\|x_1 - x_0\|. \end{aligned}$$

Moreover, we obtain by the triangle inequality

$$\|x_n - x_m\| \leq \sum_{k=m}^{n-1} \|x_{k+1} - x_k\|.$$

Combining these two estimates yields

$$\begin{aligned} \|x_n - x_m\| &\leq \sum_{k=m}^{n-1} L^k\|x_1 - x_0\| = \|x_1 - x_0\|\sum_{k=m}^{n-1} L^k = \|x_1 - x_0\|L^m\sum_{k=0}^{n-m-1} L^k \\ &= \|x_1 - x_0\|L^m\frac{1-L^{n-m}}{1-L} \leq \|x_1 - x_0\|\frac{L^m}{1-L} \xrightarrow{m\to\infty} 0. \end{aligned} \tag{3.2}$$

Hence, for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|x_n - x_m\| \leq \varepsilon \quad \text{for all } m, n > n_0.$$

Consequently $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since V is complete, its limit $x^\star$ is contained in V.

By assumption, Φ is Lipschitz continuous. In particular, it is also continuous on V. Hence, it follows

$$x^\star = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \Phi(x_n) = \Phi\Big(\lim_{n\to\infty} x_n\Big) = \Phi(x^\star).$$

Therefore, $x^\star \in V$ is a fixed point of Φ.
Let $\tilde{x}^\star$ denote a further fixed point if Φ. It holds

$$0 \leq \|\tilde{x}^\star - x^\star\| = \|\Phi(\tilde{x}^\star) - \Phi(x^\star)\| \leq L\|\tilde{x}^\star - x^\star\|,$$

which implies $\|\tilde{x}^\star - x^\star\| = 0$. This shows the uniqueness of the fixed point.
The monotonicity of the iteration follows from

$$0 \leq \|x - x_n\| = \|x - \Phi(x_{n-1})\| \leq L\|x - x_{n-1}\|.$$

Next, to derive the a-posteriori estimate, we employ again the triangle inequality and obtain

$$\|x - x_i\| \leq L\|x - x_n + x_n - x_{n-1}\| \leq L\|x - x_n\| + L\|x_i - x_{n-1}\|.$$

From this, the a-priori estimate can be inferred by employing (3.2). □

(3.3) **Example.** We want to compute the solution $x \in C([a,b])$ of the *Fredholm integral equation*

$$x(s) - \lambda \int_a^b k(s,t)x(t)\,\mathrm{d}t = g(s), \quad s \in [a,b], \tag{3.4}$$

we assume $g \in C([a,b])$ and $k \in C([a,b]^2)$. We shall choose the parameter $\lambda \in \mathbb{R}$ such that the equation exhibits a unique solution. Remind that $C([a,b])$ is a Banach space. We define the operator (

$$(T_\lambda x)(s) := \lambda \int_a^b k(s,t)x(t)\,\mathrm{d}t + g(s), \quad s \in [a,b]$$

and rewrite (3.4) as fixed point equation $x = T_\lambda x$. In view of Banach's fixed point theorem, we have to determine λ such that T_λ becomes a contraction. It holds

$$\begin{aligned}\max_{a\leqslant s\leqslant b} |(T_\lambda x)(s) - (T_\lambda y)(s)| &= \max_{a\leqslant s\leqslant b} \left|\lambda \int_a^b k(s,t)x(t)\,\mathrm{d}t - \lambda \int_a^b k(s,t)y(t)\,\mathrm{d}t\right| \\ &= \max_{a\leqslant s\leqslant b} \left|\lambda \int_a^b k(s,t)[x(t) - y(t)]\,\mathrm{d}t\right| \\ &\leqslant \lambda(b-a)\Big(\max_{a\leqslant s,t\leqslant b} |k(s,t)|\Big)\Big(\max_{a\leqslant t\leqslant b} |x(t) - y(t)|\Big)\end{aligned}$$

We have to guarantee $L := \lambda(b-a)\big(\max_{a\leqslant s,t\leqslant b} |k(s,t)|\big) < 1$. This is true if

$$|\lambda| < \Big((b-a) \max_{a\leqslant s,t\leqslant b} |k(s,t)|\Big)^{-1}.$$

△

Exercises

Exercise. Show that $\ell^p(\mathbb{K})$ is a vector space for $1 \leqslant p < \infty$.

Exercise. Show that every convergent sequence is a Cauchy sequence.

Exercise. Show that $\ell^p(\mathbb{K})$ is a Banach space for $1 \leqslant p < \infty$.

Exercise. Show that $C([0,1])$ equipped with the integral norm

$$\|f\| = \int_0^1 |f| \, \mathrm{d}x \text{ (in the Riemannian sense)}$$

is not complete.

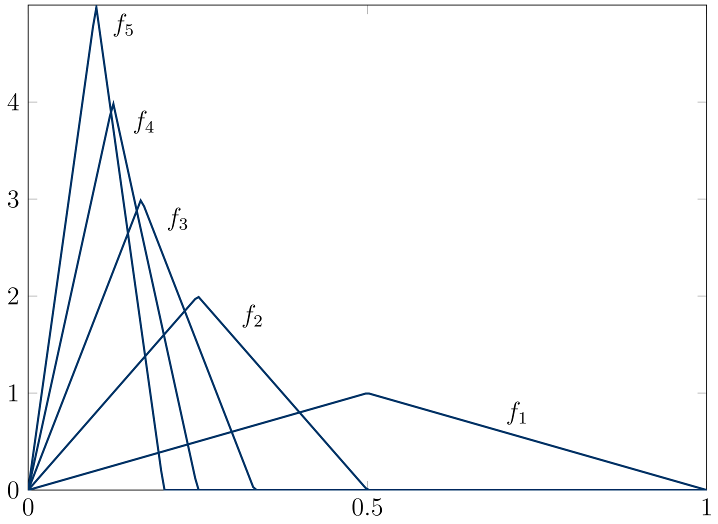


Figure IX.1: Sequence $\{f_n\} \subset C([0,1])$.

As already indicated, the spacse $C([0,1])$ is no Banach space with respect to the integral norm induced by the Riemann integral. Consider for example the sequence

$$f_n(x) := \begin{cases} 2n^2x, & 0 \leqslant x < \frac{1}{2n}, \\ 2n - 2n^2x, & \frac{1}{2n} \leqslant x < \frac{1}{n}, \\ 0, & \text{else.} \end{cases}$$

It holds that $f_n \to 0$ but

$$\int_0^1 |f_n| \,\mathrm{d}x = \frac{1}{2} \text{ for all } n \in \mathbb{N}^*, \text{ while } \int_0^1 0 \,\mathrm{d}x = 0.$$

In particular, the limiting procedure is not interchangeable with the integration. In the sequel, we construct a more general concept of the integral, which exhibits this property.

1 Basics

In what follows, let Ω always denote a non-empty set, unless stated otherwise.
In order to define measures later on, we consider special subsets $\mathcal{F} \subset 2^\Omega$.

(1.1) **Definition.** A set $\mathcal{F} \subset 2^\Omega$ is called *σ-field* on Ω, iff

(i) $\Omega \in \mathcal{F}$,

(ii) $A \in \mathcal{F} \Rightarrow A^\complement \in \mathcal{F}$,

(iii) $A_i \in \mathcal{F}$ for all $i \in \mathbb{N} \Rightarrow \bigcup_{i\in\mathbb{N}} A_i \in \mathcal{F}$.

Sets $A \in \mathcal{F}$ are called *measurable* and the tuple $(\Omega, \mathcal{F})$ is a *measurable space*.

Note that the power set 2^Ω is always a *σ-field* on Ω. The next theorem tells us, that a given set $\mathcal{A} \subset 2^\Omega$ can always be augmented to a σ-field.

(1.2) **Theorem (Generated σ-field).** Let $\mathcal{A} \subset 2^\Omega$. There exists a smallest σ-field

$$\sigma(\mathcal{A}) := \bigcap_{\substack{\mathcal{F} \text{ is a } \sigma\text{-field} \\ \mathcal{F} \supset \mathcal{A}}} \mathcal{F}$$

such that $\mathcal{A} \subset \sigma(\mathcal{A})$. $\sigma(\mathcal{A})$ is called the σ-field *generated* by $\mathcal{A}$.

Proof. We verify the conditions of Definition (1.1) for $\sigma(\mathcal{A})$. Formally, $\sigma(\mathcal{A})$ is the intersection $\cap\mathcal{C}$ of the class $\mathcal{C}$ of all σ-fields which contain $\mathcal{A}$. This intersection is defined as

$$\cap\mathcal{C} := \{A : A \in \mathcal{F} \text{ for all } \mathcal{F} \in \mathcal{C}\}.$$

Since $\mathcal{A} \subset 2^\Omega$ and 2^Ω is a σ-field, the intersection is non-empty. Moreover, from $\Omega \in \mathcal{F}$ for all $\mathcal{F} \in \mathcal{C}$, we infer $\Omega \in \sigma(\mathcal{A})$. Now, let $A \in \sigma(\mathcal{A})$. Thus, $A \in \mathcal{F}$ for all $\mathcal{F} \in \mathcal{C}$ and consequently $A^\complement \in \mathcal{F}$ for all $\mathcal{F} \in \mathcal{C}$. This implies $A^\complement \in \sigma(\mathcal{A})$. Now assume $A_i \in \sigma(\mathcal{A})$ for every $i \in \mathbb{N}$. Then, since $A_i \in \mathcal{F}$ for every $i \in \mathbb{N}$ and all $\mathcal{F} \in \mathcal{C}$, it holds $\bigcup_{i\in\mathbb{N}} A_i =: A \in \mathcal{F}$ for all $\mathcal{F} \in \mathcal{C}$. The latter implies $A \in \sigma(\mathcal{A})$. □

(1.3) **Example.** Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_1, \mathcal{F}_1)$ denote two measurable spaces. The *product σ-field* is defined as

$$\mathcal{F}_1 \otimes \mathcal{F}_2 := \sigma(\{A_1 \times A_2 : A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\}).$$

Note that $\{A_1 \times A_2 : A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\}$ is no σ-field in general.

A very important σ-field, which we shall consider is the one that is generated by a *topology*.

(1.4) **Definition.** A set $\tau \subset 2^\Omega$ is called a *topology* on Ω, iff

(i) $\varnothing, \Omega \in \tau$,

(ii) $A, B \in \tau \Rightarrow A \cap B \in \tau$,

(iii) $\mathcal{A} \subset \tau \Rightarrow \left(\bigcup_{A\in\mathcal{A}} A\right) \in \tau$.

Sets $A \in \tau$ are called *open* and the tuple (Ω, τ) is a *topological space*. The σ-field which is generated by τ, i.e. $\mathcal{B}(\Omega) := \sigma(\tau)$, is called the *Borel σ-field*.

(1.5) **Example.** In the case $\Omega = \mathbb{R}^d$ for $d = 1, 2, \ldots$, the canonical topology is generated by the Euclidean norm. We define the ball

$$B_r(\boldsymbol{x}) := \left\{\boldsymbol{y} \in \mathbb{R}^d : \|\boldsymbol{x} - \boldsymbol{y}\|_2 < r\right\}$$

with radius $r > 0$ and center $\boldsymbol{x} \in \mathbb{R}^d$ and the union of all these balls

$$\mathcal{G} = \left\{B_r(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^d, r > 0\right\} \subset 2^{\mathbb{R}^d}.$$

It holds

$$\tau = \left\{ \Big(\bigcup_{A \in \mathcal{A}} A \Big) : \mathcal{A} \subset \mathcal{G} \right\}$$

and $\mathcal{B}(\mathbb{R}^d) = \sigma(\tau)$. We remark that the same σ-field is already generated by the set

$$\mathcal{G} = \big\{ (-\boldsymbol{\infty}, \boldsymbol{b}) : \boldsymbol{b} \in \mathbb{Q}^d \big\} \quad \text{with } (-\boldsymbol{\infty}, \boldsymbol{b}) := (-\infty, b_1) \times \cdots \times (-\infty, b_d).$$

△

(1.6) **Definition.** Let $\mathcal{F}$ denote a σ-field on Ω. A function $\mu\colon \mathcal{F} \to [0,\infty]$ with $\mu(\varnothing) = 0$ is called a *measure*, iff

$$\mu\Big(\dot{\bigcup}_{i \in \mathbb{N}} A_i \Big) = \sum_{i \in \mathbb{N}} \mu(A_i)$$

for every sequence $\{A_i\}_{i \in \mathbb{N}} \subset \mathcal{F}$ of disjoint sets, i.e. $A_i \cap A_j = \varnothing$ for $i \neq j$. This property is called *σ-additivity*. A measure μ is called *σ-finite*, iff there exists a sequence $(A_i)_{i \in \mathbb{N}} \subset \mathcal{F}$ such that $\Omega = \bigcup_{i \in \mathbb{N}} A_i$ and $\mu(A_i) < \infty$ for all $i \in \mathbb{N}$. It is called *finite*, iff $\mu(\Omega) < \infty$. The triplet $(\Omega, \mathcal{F}, \mu)$ is called a *measure space*.

Note that the definition directly implies for $A \subset B \subset \mathcal{F}$ that

$$\mu(B) = \mu\big(A \dot{\cup} (B \setminus A)\big) = \mu(A) + \mu(B \setminus A) \geqslant \mu(A).$$

This property is called *monotonicity* of the measure μ.

(1.7) **Example.** The function $\lambda\colon \mathcal{B}(\mathbb{R}^d) \to [0,\infty]$ with

$$\lambda\big((\boldsymbol{a}, \boldsymbol{b})\big) = \prod_{i=1}^{d} (b_i - a_i) \quad \text{for all } \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d \text{ with } \boldsymbol{a} < \boldsymbol{b}$$

is a measure, the so called *Lebesgue-Borel measure*. The *measure extension theorem* guarantees that it is uniquely defined by the previous property. △

(1.8) **Definition.** Let Ω denote a set, $\mathcal{F} \subset 2^{\Omega}$ a σ-field and $\mu\colon \mathcal{F} \to [0,\infty]$ a measure. The triplet $(\Omega, \mathcal{F}, \mu)$ is called a *measure space*.

We close this paragraph by introducing the concept of *outer measures* and the idea of null-sets.

(1.9) **Definition.** A function $\mu^*\colon 2^\Omega \to [0,\infty]$ is called an *outer measure*, iff

(i) $\mu^*(\varnothing) = 0$,

(ii) μ^* is monotone,

(iii) μ^* is *σ-subadditive*, i.e. $A \subset \bigcup_{i\in\mathbb{N}} A_i \Rightarrow \mu^*(A) \leqslant \sum_{i\in\mathbb{N}} \mu(A_i)$, $A, A_i \in 2^\Omega$.

A set $A \in 2^\Omega$ is *μ^*-measurable*, iff

$$\mu^*(A\cap E) + \mu^*(A^\complement \cap E) = \mu^*(E) \quad \text{for every } E \in 2^\Omega.$$

We set $\mathcal{M}(\mu^*) := \{A \in 2^\Omega : A \text{ is } \mu^*\text{-measurable}\}$. Note that $\mathcal{M}(\mu^*)$ is a σ-field and that μ^* a measure on $\mathcal{M}(\mu^*)$.

(1.10) **Example.** Let $\mathcal{F} \subset 2^\Omega$ be a σ-field and let μ be a measure on $\mathcal{F}$. The function $\mu^*\colon 2^\Omega \to [0,\infty]$ defined via

$$\mu^*(B) := \inf\{\mu(A) : A \supset B \text{ and } A \in \mathcal{F}\}$$

is an outer measure with $\mu(A) = \mu^*(A)$ for all $A \in \mathcal{F}$. Note that $\inf\varnothing = \infty$ and $\sup\varnothing = -\infty$. △

Proof. We verify the conditions for an outer measure from the previous definition. It holds $\{\varnothing\} \in \mathcal{F}$ and hence $\mu^*(\varnothing) = 0$.
Next, we define for $E \in 2^\Omega$ the set

$$\mathcal{U}(E) := \{A \in \mathcal{F} : A \supset E\}.$$

Then, for $A \subset B \subset \Omega$ we have $\mathcal{U}(B) \subset \mathcal{U}(A)$ and therefore $\mu^*(A) \leqslant \mu^*(B)$.
Let $A_i \subset \Omega$ for $i \in \mathbb{N}$ and $A \subset \bigcup_{i\in\mathbb{N}} A_i$. Assume without loss of generality that $\mu^*(A_i) < \infty$, which implies $\mathcal{U}(A_i) \neq \varnothing$ for every $i \in \mathbb{N}$. Now, let $\varepsilon > 0$. We choose for every $i \in \mathbb{N}$ a set $C_i \in \mathcal{U}(A_i)$ with

$$\mu(C_i) \leqslant \mu^*(A_i) + \varepsilon 2^{-i}.$$

Note that such a set always exists due to the definition of μ^*. It holds $\left(\bigcup_{i\in\mathbb{N}} C_i\right) \in \mathcal{U}(A)$. We infer

$$\mu^*(A) \leqslant \mu\left(\bigcup_{i\in\mathbb{N}} C_i\right) \leqslant \sum_{i\in\mathbb{N}} \mu(C_i) \leqslant \sum_{i\in\mathbb{N}} \mu^*(A_i) + \epsilon$$

which implies the σ-subadditivity for $\varepsilon \to 0$. Finally, for $A \in \mathcal{F}$, we have $\mu^*(A) \leqslant \mu(A)$, since $A \in \mathcal{U}(A)$. On the other hand, it follows $\mu(A) \leqslant \mu^*(A)$ due to the monotonicity of μ, since $A \subset B$ for every $B \in \mathcal{U}(A)$. □

(1.11) **Remark.** Due to the σ-subadditivity of an outer measure μ^*, a set $A \in 2^\Omega$ is already measurable, iff

$$\mu^*(E) \geqslant \mu^*(A\cap E) + \mu^*(A^\complement \cap E)$$

for every $E \in 2^\Omega$ with $\mu^*(E) \leqslant \infty$. △

(1.12) **Definition.** Let $(\Omega, \mathcal{F}, \mu)$ denote a measure space. A set $A \in \mathcal{F}$ is called a *μ-null set*, iff $\mu(A) = 0$. We set $\mathcal{N}_\mu := \{N \subset \Omega : N \subset A, A \in \mathcal{F} \text{ and } \mu(A) = 0\} \subset 2^\Omega$. Let $P(\omega)$ be a property of points $\omega \in \Omega$. We say that P holds *μ-almost everywhere* or for *μ-almost every* $\omega \in \Omega$, iff $\{\omega \in \Omega : \neg P(\omega)\}$ is a μ-null set. A measure space is called *complete*, iff $\mathcal{N}_\mu \subset \mathcal{F}$.

The next theorem tells us that every measure space can be augmented to a complete measure space in a canonical way.

(1.13) **Theorem.** Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. Then, $\big(\Omega, \mathcal{M}(\mu^*), \mu^*\big)$, where $\mu^*(B) = \inf\{\mu(A) : A \supset B \text{ and } A \in \mathcal{F}\}$ for $B \in \mathcal{M}(\mu^*)$, is a complete measure space. Moreover, it holds

$$\mathcal{M}(\mu^*) = \sigma(\mathcal{F} \cup \mathcal{N}_\mu) = \{A \cup N : A \in \mathcal{F}, N \in \mathcal{N}_\mu\}$$

and $\mu^*(A \cup N) = \mu(A)$ for any $A \in \mathcal{F}, N \in \mathcal{N}_\mu$.
Therefore, the space $\big(\Omega, \mathcal{M}(\mu^*), \mu^*\big)$ is called the *completion* of $(\Omega, \mathcal{F}, \mu)$. The measure μ^* is called an *outer measure*.

Proof. At first, we show that $\mathcal{N}_{\mu^*} \subset \mathcal{M}(\mu^*)$. Let $N \in 2^\Omega$ with $N \subset A$, $A \in \mathcal{M}(\mu^*)$ and $\mu^*(A) = 0$. Then it holds $\mu^*(N) = 0$ and we obtain for any $E \in 2^\Omega$ with $\mu^*(E) < \infty$ that

$$\mu^*(E) \geqslant \mu^*(N^\complement \cap E) = \mu^*(N^\complement \cap E) + \mu^*(N) \geqslant \mu^*(N \cap E) + \mu^*(N \cap E),$$

which shows the measurability of N, i.e. $N \in \mathcal{M}(\mu^*)$.
Next, we show $\mathcal{F} \cup \mathcal{N}_\mu \subset \mathcal{M}(\mu^*)$, which implies $\sigma(\mathcal{F} \cup \mathcal{N}_\mu) \subset \mathcal{M}(\mu^*)$. Let $A \in \mathcal{F}$ and $E \in 2^\Omega$ with $\mu^*(E) < \infty$. For $\varepsilon > 0$ choose $(A_i)_{i \in \mathbb{N}} \subset \mathcal{F}$ with $E \subset \bigcup_{i \in \mathbb{N}} A_i$ and $\sum_{i \in \mathbb{N}} \mu(A_i) \leqslant \mu^*(E) + \varepsilon$. Then $(A \cap E) \subset \bigcup_{i \in \mathbb{N}} (A \cap A_i)$ and $(A^\complement \cap E) \subset \bigcup_{i \in \mathbb{N}} (A^\complement \cap A_i)$. Hence, we obtain

$$\mu^*(A \cap E) + \mu^*(A^\complement \cap E) \leqslant \sum_{i \in \mathbb{N}} \mu(A \cap A_i) + \mu(A^\complement \cap A_i) = \sum_{i \in \mathbb{N}} \mu(A_i),$$

and therefore $\mu^*(A \cap E) + \mu^*(A^\complement \cap E) \leqslant \mu^*(E)$ for $\varepsilon \to 0$. Observing $\mathcal{N}_\mu = \mathcal{N}_{\mu^*}$, this shows $\mathcal{F} \cup \mathcal{N}_\mu \subset \mathcal{M}(\mu^*)$.
Finally, we show that every $B \in \mathcal{M}(\mu^*)$ can be written as $B = A \cup N$ with $A \in \mathcal{F}, N \in \mathcal{N}_\mu$ and $\mu^*(B) = \mu(A)$. Since μ and hence μ^* are σ-finite, we can always represent B as a countable union of disjoint sets of finite outer measure. Moreover, if $\mu^*(B) = \infty$ and $B \subset A, A \in \mathcal{F}$, then obviously $\mu(A) = \infty$. Hence, assume without loss of generality that $\mu^*(B) < \infty$. Then, for every $i \in \mathbb{N}$ there exists a set $A_i \in \mathcal{F}$ such that $B \subset A_i$ and

$$\mu(A_i) \leqslant \mu^*(B) + 2^{-i}.$$

For $A := \bigcap_{i \in \mathbb{N}} A_i = \big(\bigcup_{i \in \mathbb{N}} A_i^\complement\big)^\complement$, we have $B \subset A$ and $A \in \mathcal{F}$. It holds

$$\mu^*(B) \leqslant \mu(A) \leqslant \mu(A_i) \leqslant \mu^*(B) + 2^{-i}.$$

Since this inequality holds for every $i \in \mathbb{N}$, we infer $\mu^*(B) = \mu(A)$. Next, consider the set $B \setminus A =: C \in \mathcal{M}(\mu^*)$. It holds

$$\mu^*(C) = \mu^*(B \setminus A) = \mu^*(B) - \mu^*(A) = \mu^*(B) - \mu(A) = 0,$$

which implies $C \in \mathcal{N}_\mu$. □

(1.14) **Example.** The Lebesgue-Borel measure λ on $\mathcal{B}(\mathbb{R}^d)$ can uniquely be extended to a measure λ^* on

$$\mathcal{B}^*(\mathbb{R}^d) := \sigma\big(\mathcal{B}(\mathbb{R}^d) \cup \mathcal{N}_\lambda\big).$$

$\mathcal{B}^*(\mathbb{R}^d)$ is called the σ-field of *Lebesgue measurable* sets. △

2 Measurable mappings

(2.1) **Definition.** Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ denote two measurable spaces. A mapping $f\colon \Omega \to \Omega'$ is called *$\mathcal{F}/\mathcal{F}'$-measurable* or simply *measurable*, iff

$$f^{-1}(A') := \{\omega \in \Omega : X(\omega) \in A'\} \in \mathcal{F} \quad \text{for all } A' \in \mathcal{F}'.$$

(2.2) **Example.** Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ denote two measurable spaces and $f\colon \Omega \to \Omega'$ a mapping.

(i) The identity mapping $\mathrm{id}\colon \Omega \to \Omega$ is $\mathcal{F}/\mathcal{F}$-measurable.

(ii) The *characteristic function* of $A \subset \Omega$ defined via

$$\mathbb{1}_A\colon \Omega \to \{0,1\}, \quad \mathbb{1}_A(\omega) := \begin{cases} 1, & \omega \in A, \\ 0, & \text{else,} \end{cases}$$

is $\mathcal{F}/2^{\{0,1\}}$-measurable, iff $A \in \mathcal{F}$.

(iii) If $\mathcal{F} = 2^\Omega$, then all mappings $f\colon \Omega \to \Omega'$ are $\mathcal{F}/\mathcal{F}'$-measurable.

(iv) If $\mathcal{F}' = \sigma(\mathcal{A}')$ and $f^{-1}(A') \in \mathcal{F}$ for all $A' \in \mathcal{A}'$, then f is $\mathcal{F}/\mathcal{F}'$-measurable.

(v) If (Ω, τ) and (Ω', τ') are topological spaces and f is continuous, i.e. $f^{-1}(O') \in \tau$ for all $O' \in \tau'$, then f is also $\mathcal{B}(\tau)/\mathcal{B}(\tau')$-measurable.

(vi) If $f\colon \Omega_0 \to \Omega_1$ is $\mathcal{F}_0/\mathcal{F}_1$-measurable and $g\colon \Omega_1 \to \Omega_2$ is $\mathcal{F}_1/\mathcal{F}_2$-measurable, then the composition $g \circ f\colon \Omega_0 \to \Omega_2$ is $\mathcal{F}_0/\mathcal{F}_2$-measurable.

(vii) For $x, y \in [-\infty, \infty]$ let $x \vee y := \max\{x, y\}$ and $x \wedge y := \min\{x, y\}$. If $f\colon \Omega \to \mathbb{R}$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable, then due to (v) and (vi) the functions

$$f^+ := f \vee 0, \quad f^- := -f \vee 0, \quad |f| := f^+ + f^-$$

are also $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable.

(viii) If f, g are $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable, so are αf for $\alpha \in \mathbb{R}$, $f + g$ and $f \cdot g$.

(ix) Let $f_0, f_1, f_2, \dots$ be a sequence of $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable functions, then

$$\inf_{i\in\mathbb{N}} f_i, \quad \sup_{i\in\mathbb{N}} f_i, \quad \liminf_{i\in\mathbb{N}} f_i, \quad \limsup_{i\in\mathbb{N}} f_i$$

are also $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable.

△

Measurable mappings induce corresponding σ-fields in a straightforward manner.

(2.3) **Theorem.** Let $f\colon \Omega \to \Omega'$, where $\Omega \neq \varnothing$ and $(\Omega', \mathcal{F}')$ is a measurable space. Then,

$$\sigma(f) := \{f^{-1}(A') : A' \in \mathcal{F}'\}$$

is a σ-field. $\sigma(f)$ is called the σ-field *generated* by f. It is also the smallest σ-field $\mathcal{F}$ on Ω such that f is $\mathcal{F}/\mathcal{F}'$-measurable.

Proof. Since $f^{-1}(\Omega') = \{\omega \in \Omega : f(\omega) \in \Omega'\} = \Omega$, it holds $\Omega \in \sigma(f)$. Now, let $A \in \sigma(f)$. There exists $A' \in \mathcal{F}'$ with $A' = f(A)$ and $A'^{\complement} \in \mathcal{F}'$. Therefore, we obtain

$$A^{\complement} = \left(f^{-1}(A')\right)^{\complement} = f^{-1}(A'^{\complement}) \in \sigma(f).$$

Next, let $\{A_i\}_{i\in\mathbb{N}} \in \sigma(f)$ be a sequence of disjoint sets. Then, $A_i' := f(A_i)$ also forms a sequence of disjoint sets and $\dot\bigcup_{i\in\mathbb{N}} A_i' =: A' \in \mathcal{F}'$.

$$\dot\bigcup_{i\in\mathbb{N}} A_i = \dot\bigcup_{i\in\mathbb{N}} f^{-1}(A_i') = f^{-1}\left(\dot\bigcup_{i\in\mathbb{N}} A_i'\right) = f^{-1}(A') \in \mathcal{F}.$$

Finally, assume there exists a σ-field $\mathcal{F} \subsetneq \sigma(f)$ such that f is $\mathcal{F}/\mathcal{F}'$-measurable. Hence, there must be a set $A \neq \varnothing$, which is contained in $\sigma(f)$ but not in $\mathcal{F}$. In particular, there must exist a non empty set $A' \in \mathcal{F}'$ with $A = f^{-1}(A')$. Thus, since $A \notin \mathcal{F}$, f cannot be $\mathcal{F}/\mathcal{F}'$-measurable. □

In what follows, we consider the approximation of measurable functions by simpler ones.

(2.4) **Definition.** Let $(\Omega, \mathcal{F})$ be a measurable space. A mapping $f\colon \Omega \to \mathbb{R}$ is called a *simple function*, iff there exist disjoint sets $A_1, \dots, A_n \in \mathcal{F}$ and numbers $\alpha_1, \dots, \alpha_i \in \mathbb{R}$ such that

$$f = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}.$$

Every simple function $f\colon \Omega \to \mathbb{R}$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable. In turn, every measurable map $f\colon \Omega \to \mathbb{R}$ which assumes only finitely many values is a simple function.

The following theorem characterises simple functions.

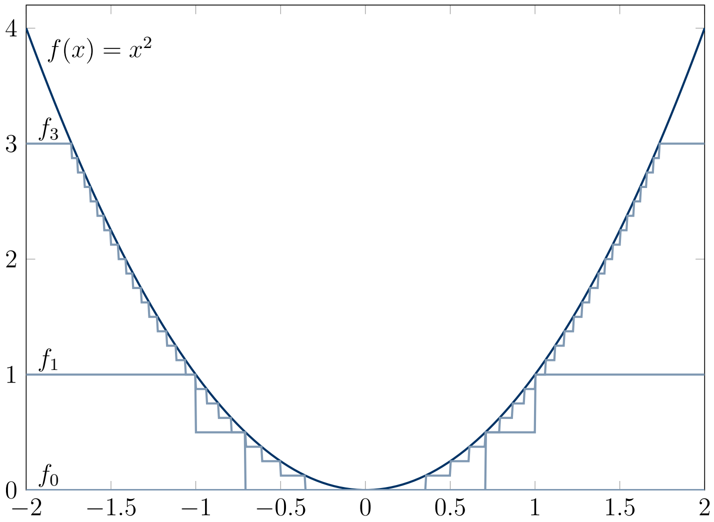


Figure IX.2: Approximation of $f(x) = x^2$ by $f_i := (2^{-i}\lfloor 2^i f \rfloor) \wedge i$.

(2.5) **Theorem.** Let $(\Omega, \mathcal{F})$ be a measurable space and $f\colon \Omega \to [0,\infty]$ a measurable mapping. Then, there exists an *isotone sequence* of simple functions $f_i \nearrow f$, i.e.

$$f_i(\omega) \leqslant f_j(\omega),\ i \leqslant j, \text{ and } \lim_{i\to\infty} f_i(\omega) = f(\omega) \quad \text{for every } \omega \in \Omega.$$

In particular, there exist sets $A_i \in \mathcal{F}$ and numbers $\alpha_i \geqslant 0$, $i \in \mathbb{N}$, such that $f = \sum_{i\in\mathbb{N}} \alpha_i \mathbb{1}_{A_i}$.

Proof. For $i \in \mathbb{N}$, we define the function $f_i := (2^{-i}\lfloor 2^i f \rfloor) \wedge i$, cp. Figure IX.2. This function is measurable and assumes at most $i2^i + 1$ different values and it holds $f_i(\omega) \leqslant f_j(\omega)$, $i \leqslant j$, and $\lim_{i\to\infty} f_i(\omega) = f(\omega)$ for every $\omega \in \Omega$.

Next, define the sets $A_{i,j} := \{\omega \in \Omega : f_i(\omega) - f_{i-1}(\omega) = j2^{-i}\}$ and the coefficients $\alpha_{i,j} = j2^{-i}$. We obtain $f_i - f_{i-1} = \sum_{j=1}^{2^i} \alpha_{i,j} \mathbb{1}_{A_{i,j}}$. Hence, expanding f in a telescoping sum yields

$$f = f_0 + \sum_{i=1}^{\infty} (f_i - f_{i-1}) = 0 + \sum_{i=1}^{\infty} \sum_{j=1}^{2^i} \alpha_{i,j} \mathbb{1}_{A_{i,j}} = \sum_{n\in\mathbb{N}} \alpha_n \mathbb{1}_{A_n}$$

for an appropriate enumeration $(i,j) \mapsto n$. □

3 The measure integral

The construction of the measure integral is performed in four steps.

At first, we define the integral for characteristic functions.

(3.1) **Definition (Step 1).** Let $f = \mathbb{1}_A$ for some $A \in \mathcal{F}$. We set

$$\int f \, \mathrm{d}\mu := \mu(A).$$

Next, we consider linear combinations of characteristic functions with positive coefficients.

(3.2) **Definition (Step 2).** Let $f = \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i}$ with $\alpha_i \geqslant 0$. We set

$$\int f \, \mathrm{d}\mu := \sum_{i=1}^n \alpha_i \mu(A_i).$$

Although the representation of a simple function $f \geqslant 0$ is not unique, the integral is well defined, i.e. it is independent of the particular representation of f.

(3.3) **Lemma.** Let $0 \leqslant f = \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i} = \sum_{i=1}^m \beta_i \mathbb{1}_{B_i}$, where $A_1, \dots, A_n$ and $B_1, \dots, B_m$ are two families of disjoint subsets. Then

$$\sum_{i=1}^n \alpha_i \mu(A_i) = \sum_{i=1}^m \beta_i \mu(B_i).$$

Proof. It holds $\Omega = \dot{\bigcup}_{i=1}^n A_i = \dot{\bigcup}_{i=1}^m B_i$ and therefore

$$\sum_{i=1}^n \alpha_i \mu(A_i) = \sum_{i=1}^n \sum_{j=1}^m \alpha_i \mu(A_i \cap B_j) = \sum_{j=1}^m \sum_{i=1}^n \beta_j \mu(A_i \cap B_j) = \sum_{j=1}^m \beta_i \mu(B_j),$$

since $\alpha_i = \beta_j$ whenever $A_i \cap B_j \neq \varnothing$. □

From the definition of the integral for non negative simple functions, we can already infer its main properties.

(3.4) **Lemma.** Let $f, g \geqslant 0$ denote two simple functions and let $\alpha > 0$. It holds

(i) $$\int f + g \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu,$$

(ii) $$\int \alpha f \, \mathrm{d}\mu = \alpha \int f \, \mathrm{d}\mu,$$

(iii) $$f \leqslant g \text{ implies } \int f \, \mathrm{d}\mu \leqslant \int g \, \mathrm{d}\mu.$$

Now, we shall extend the integral to non-negative measurable functions.

(3.5) **Definition (Step 3).** Let $f \colon \Omega \to [0, \infty]$ be measurable. We set

$$\int f \, \mathrm{d}\mu := \sup_{i \in \mathbb{N}} \int f_i \, \mathrm{d}\mu,$$

where $\{f_i\}_{i \in \mathbb{N}}$ is an arbitrary isotone sequence of non-negative simple functions with $f_i \nearrow f$.

The following lemma guarantees that the integral for this type of functions is well defined.

(3.6) **Lemma.** Let $(f_i)_{i \in \mathbb{N}}$ and $(g_i)_{i \in \mathbb{N}}$ be two isotone sequences of non negative simple functions with $\sup_{i \in \mathbb{N}} f_i = \sup_{i \in \mathbb{N}} g_i$. Then, it holds

$$\sup_{i \in \mathbb{N}} \int f_i \, \mathrm{d}\mu = \sup_{i \in \mathbb{N}} \int g_i \, \mathrm{d}\mu.$$

Proof. Let $g_i = \sum_{m=1}^{n^{(i)}} \alpha_m^{(i)} \mathbb{1}_{A_m^{(i)}}$ and define for $\alpha \in \mathbb{R}$ the sets $B_{j,i}^\alpha := \{f_j \geqslant \alpha g_i\} \in \mathcal{F}$. For $\alpha \in (0,1)$, we obtain due to $\sup_{j\in\mathbb{N}} f_j \geqslant g_i$ that

$$\Omega = \bigcup_{j\in\mathbb{N}} B_{j,i}^\alpha \quad \text{and hence} \quad \Big(\bigcup_{j\in\mathbb{N}} B_{j,i}^\alpha\Big) \cap A_m^{(i)} = A_m^{(i)}.$$

Moreover, it holds $f_j \geqslant \alpha g_i \mathbb{1}_{B_{j,i}^\alpha}$, where the latter is obviously a simple function. Hence, we infer from (iii) in Lemma (3.4) that

$$\int f_j \,\mathrm{d}\mu \geqslant \int g_i \mathbb{1}_{B_{j,i}^\alpha} \,\mathrm{d}\mu = \alpha \sum_{m=1}^{n^{(i)}} \alpha_m^{(i)} \mu(A_m^{(i)} \cap B_{j,i}^\alpha).$$

This implies

$$\sup_{j\in\mathbb{N}} \int f_j \,\mathrm{d}\mu \geqslant \alpha \sup_{j\in\mathbb{N}} \sum_{m=1}^{n^{(i)}} \alpha_m^{(i)} \mu(A_m^{(i)} \cap B_{j,i}^\alpha) = \alpha \sum_{m=1}^{n^{(i)}} \alpha_m^{(i)} \mu(A_m^{(i)}) = \alpha \int g_i \,\mathrm{d}\mu.$$

Consequently, for $\alpha \to 1$ we arrive at

$$\sup_{j\in\mathbb{N}} \int f_j \,\mathrm{d}\mu \geqslant \sup_{i\in\mathbb{N}} \int g_i \,\mathrm{d}\mu.$$

Since we can proceed analogously to show the reverse estimate, this completes the proof.□

The integral is linear and monotone.

(3.7) Theorem. Let $f, g \geqslant 0$ denote two measurable functions and let $\alpha \in \mathbb{R}$. It holds

(i) $$\int f + g \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu + \int g \,\mathrm{d}\mu,$$

(ii) $$\int \alpha f \,\mathrm{d}\mu = \alpha \int f \,\mathrm{d}\mu,$$

(iii) $$f \leqslant g \text{ implies } \int f \,\mathrm{d}\mu \leqslant \int g \,\mathrm{d}\mu.$$

We remark that $\int f^+ \,\mathrm{d}\mu \leqslant \int |f| \,\mathrm{d}\mu$ as well as $\int f^- \,\mathrm{d}\mu \leqslant \int |f| \,\mathrm{d}\mu$ and $f^-, f^+, |f| \geqslant 0$. This gives rise to the following

(3.8) Definition (Step 4). An $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mapping $f\colon \Omega \to \mathbb{R}$ is called *integrable*, iff $\int |f| \,\mathrm{d}\mu < \infty$. We then set

$$\int f \,\mathrm{d}\mu := \int f^+ \,\mathrm{d}\mu - \int f^- \,\mathrm{d}\mu \quad \text{and} \quad \int_A f \,\mathrm{d}\mu := \int f \mathbb{1}_A \,\mathrm{d}\mu \quad \text{for } A \in \mathcal{F}. \tag{3.9}$$

If there only holds $\int f^- \,\mathrm{d}\mu < \infty$ or $\int f^+ \,\mathrm{d}\mu < \infty$, this definition is still sensible. Then the values $-\infty$ and ∞ are possible. For the (outer) Lebesgue measure $\mu = \lambda$, the integral (3.9) is called *Lebesgue integral*.

The most important properties of the general intgral are given in the following theorem.

(3.10) **Theorem.**

- Let $\{f_n\}_{n\in\mathbb{N}}$ denote an isotone sequence of measurable functions with pointwise limit f. Then
$$\lim_{n\to\infty}\int f_n\,\mathrm{d}\mu = \int f\,\mathrm{d}\mu. \quad (\textbf{Monotone convergence theorem})$$
- Let $\{f_n\}_{n\in\mathbb{N}}$ denote an sequence of measurable functions with pointwise limit f and let g be an integrable *majorant*, i.e. $|f_n| \leqslant g$ for all $n \in \mathbb{N}$, then
$$\lim_{n\to\infty}\int f_n\,\mathrm{d}\mu = \int f\,\mathrm{d}\mu. \quad (\textbf{Dominated convergence theorem})$$
- Let $\{f_n\}_{n\in\mathbb{N}}$ denote an sequence of non-negative, measurable functions, then
$$\int \liminf_{n\to\infty} f_n\,\mathrm{d}\mu \leqslant \liminf_{n\to\infty}\int f_n\,\mathrm{d}\mu. \quad (\textbf{Fatou's lemma})$$

Finally, we can introduce the important *L^p-spaces*. They play a crucial rule in the theory of partial differential equations.

(3.11) **Definition.** Further, we define for $1 \leqslant p < \infty$ the *$\mathcal{L}^p$-spaces*

$$\mathcal{L}^p\big(\mathbb{R}^d\big) := \Big\{ f\colon \mathbb{R}^d \to \mathbb{R} : f \text{ is } \mathcal{B}^*(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})\text{-measurable and } \|f\|_{L^p} < \infty \Big\}$$

with the *semi-norm*

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}^d} |f|^p\,\mathrm{d}\lambda^\star \right)^{\frac{1}{p}}.$$

Introducing the equivalence relation

$$f \sim g \quad :\Leftrightarrow \quad f - g = 0 \ \lambda^*\text{-almost everywhere},$$

the quotient spaces $L^p\big(\mathbb{R}^d\big) := \mathcal{L}^p\big(\mathbb{R}^d\big)/\!\sim$ are Banach spaces. The spaces $L^p(\Omega)$ for $\Omega \subset \mathbb{R}^d$ are defined analogously.

(3.12) **Theorem.** For $1/p + 1/q = 1$ there holds *Hölder's inequality*

$$\|fg\|_{L^1} \leqslant \|f\|_{L^p}\|g\|_{L^q}.$$

Exercises

Exercise. Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ denote two measurable spaces. Show that

$$\{A_1 \times A_2 : A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\}$$

is no σ-field in general.

Exercise. Compute the Lebesgue integral of the *Dirichlet function*

$$f(x) := \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{else.} \end{cases}$$

Exercise. Show that $\|f\|_{L^p} = 0 \Leftrightarrow f = 0$ almost everywhere.

Exercise. Let $\alpha, \beta > 0$ and $r \in (0, 1)$. Show that $\alpha^r \beta^{1-r} \leqslant r\alpha + (1-r)\beta$. Use this fact to prove Hölder's inequality.

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