Università della Svizzera italiana
Institute of Computational Science ICS

Introduction to Differential Equations — Fall Semester 2018

Example of final exam

Date: January 14th, 2020

Note: unless otherwise noted you are *always* expected to depict/make clear *how* you found the solution.

Total points: 44.

Task A: Well-posedness of IVP (12 points)

Consider the IVP

$$\begin{cases} y'(t) = t\,y^2, \\ y(0) = 1, \end{cases}$$

1. State the local existence and uniqueness theorem. . . . . [__ / 2]
2. Verify the hypothesis of theorem on the above IVP. . . . . [__ / 4]
3. Solve the IVP by separation of variables. . . . . [__ / 4]
   Hint: write the problem in the form $\frac{dy}{dt} = t\,y^2$*, then integrate using the initial condition.*
4. Is the solution globally defined? If not, find the maximal interval. . . . . [__ / 2]

Task B: Contraction Theorem (8 points)

Consider the function

$$\psi(x) = 1 - ax,$$

for $x \in \mathbb{R}$ and $a \in \mathbb{R}$. Consider also the metric space $\mathcal{M} = (X, d)$, with $d(x, y) = |y - x|$, which is complete.

1. Is ψ continuous? . . . . [__ / 1]
2. Is ψ continuously differentiable? Compute ψ'. . . . . [__ / 1]
3. Find the fixed-point of ψ. What if $a = 1$? . . . . [__ / 2]
4. Find the Lipchitz constant of ψ. . . . . [__ / 2]
5. Find a condition on a such that the fixed-point iteration $x^{n+1} = \psi(x^n)$ is convergent. . . . . . . [__ / 2]

Task C: Absolute stability (10 points)

Consider the following 2-stage Runge-Kutta scheme:

$$\begin{aligned} k_1 &= f(t^n, u^n), \\ k_2 &= f\left(t^n + \alpha h, u^n + \alpha h k_1\right), \\ u^{n+1} &= u^n + h\left((1 - \tfrac{1}{2\alpha})k_1 + \tfrac{1}{2\alpha}k_2\right). \end{aligned}$$

1. Is the scheme explicit? . . . . [__ / 1]
2. Apply the scheme to the problem $y'(t) = \lambda y(t)$. . . . . [__ / 4]
3. Show that the stability region does not depend on α. . . . . [__ / 2]
4. Is the method unconditionally stable? . . . . [__ / 1]
5. Assuming $\lambda < 0$, find the stability region. . . . . [__ / 2]

Task D: System of ODEs (14 points)

Consider the second-order ODE (Van der Pol oscillator):

$$x'' - \mu(1 - x^2)x' + x = 0.$$

1. Convert the ODE to a first-order system of ODEs. . . . . [__ / 4]
2. Write down the backward Euler approximation of the system of ODEs. [__ / 4]
3. Write down the fixed-point iteration's method applied to the nonlinear problem. . . . . [__ / 3]
4. Write down the Newton's method applied to the nonlinear problem. . . . . [__ / 3]