Lecture Summary: March 15, 2023

• Weighted least squares (WLS)

Motivation: Non-constant variance. Suppose that $Y_i = x_i'\beta + \epsilon_i$, where $\mathrm{var}(\epsilon_i) = \sigma^2 w_i$, $i = 1, \ldots, n$, where w_i are known constants. Then,

$$\frac{Y_i}{\sqrt{w_i}} = \left(\frac{x_i}{\sqrt{w_i}}\right)' \beta + \frac{\epsilon_i}{\sqrt{w_i}}.$$

If we define $\tilde{Y}_i = Y_i/\sqrt{w_i}$, $\tilde{x}_i = x_i/\sqrt{w_i}$, $\tilde{\epsilon}_i = \epsilon_i/\sqrt{w_i}$, then we have

$$\tilde{Y}_i = \tilde{x}_i'\beta + \tilde{\epsilon}_i, \quad i = 1, \ldots, n,$$

where $\mathrm{var}(\tilde{\epsilon}_i) = \mathrm{var}(\epsilon_i)/w_i = \sigma^2$. In other words, the regression of $\tilde{Y}$ on $\tilde{x}$ has constant variance.

Note that the LS is equivalent to finding β that minimizes $|Y - X\beta|^2 = (Y - X\beta)'(Y - X\beta)$, and the solution is $\hat{\beta} = (X'X)^{-1}X'Y$. Replacing X by $\tilde{X} = W^{-1/2}X$, and Y by $\tilde{Y} = W^{-1/2}Y$, where $W^{-1/2}$ is the diagonal matrix with $1/\sqrt{w_i}, i = 1, \ldots, n$ on the diagonal, leads to

$$\hat{\beta}_W = (\tilde{X}'\tilde{X})^{-1}\tilde{X}'\tilde{Y} = (X'W^{-1}X)^{-1}X'W^{-1}Y,$$

where W^{-1} is the diagonal matrix with $1/w_i, i = 1, \ldots, n$ on the diagonal.

The logistic regression model can be fitted by iteratively fitting WLS with updated weighting matrix in each iteration.

• Multiple logistic regression

$Y_i, x_i, i = 1, \ldots, n$, Y_i binary, $x_i' = (1, x_{i,1}, \ldots, x_{i,p-1})$. It is assumed that

$$\mathrm{logit}(p_i) = \beta_0 + \beta_1 x_{i,1} + \cdots + \beta_{p-1}x_{i,p-1} = x_i'\beta, \tag{1}$$

where $p_i = \mathrm{P}(Y_i = 1)$. The β coefficients can be estimated via maximum likelihood.

• Probit model

Replace the left side of (1) by $\Phi^{-1}(p_i)$, where $\Phi(x) = \mathrm{P}(Z \leq x)$ and Z is standard normal, and Φ^{-1} is the inverse function of Φ.

Motivation: Discrete response model in economics.