K5

Beta-spectroscopy and Fermi theory of beta decay

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Abstract

In this laboratory exercise we will measure the beta-decay energy spectrum of ^{137}Cs and analyze it in terms of the Fermi theory of beta decay using a Kurie plot. Quantities such as the end point energy and the ft-value will be extracted. The beta decay under investigation will be classified according to selection rules. Combined with spin-parity assignments based on the nuclear shell model we will employ these selection rules to produce a decay scheme.

Prerequisites

- Before the laboratory excercise you must perform the tasks presented in the "Laboratory session preparation task" document:
  - Solve the homework problems
  - Perform the excercise calibration task. *Needed for admission to the lab!*
- This laboratory excercise illustrates a number of concepts from introductory nuclear physics. In particular, we will encounter:
  - beta decay and selection rules
  - the nuclear shell model
  - gamma emission and inner conversion electrons
  - Fermi theory of beta decay (see also appendix B)

  Most of these topics will be encountered and discussed in some detail during the lab session.

1 Introduction

The study of nuclear spectroscopy (excitation energies, spin and parity, life times, etc.) can often be performed utilizing radioactive decay. Much information can be extracted by studying the properties of the decay particles (α, β, γ-radiation, etc). Modern nuclear physics involves the study of exotic, radioactive isotopes. These isotopes are produced at radioactive-beam accelerator facilities, involving advanced experimental techniques such as ionization, acceleration and isotope separation of very short-lived, radioactive isotopes. Theoretically, the study of these isotopes challenges the traditional nuclear models and provides stringent tests of our knowledge of nuclear forces and quantum many-body systems.

In this laboratory excercise we will study the β-decay of the long-lived ^{137}Cs isotope. We will measure the energy spectrum of the β-particle and we will employ theoretical knowledge to determine a decay scheme and energy spectrum.

Please study Section 2.6 and Section 7.7 about β-decay in the textbook Nuclear and Particle Physics by B.R. Martin.

2 Experiment

2.1 Experimental setup

We are interested in studying β-particles (electrons) with an energy up to $\approx$ 1 MeV. For this purpose we will use a Si-detector for detecting charged particles. The experimental equipment and setup is illustrated in Fig. 1 and described shortly below. It will also be described and demonstrated during the lab session.

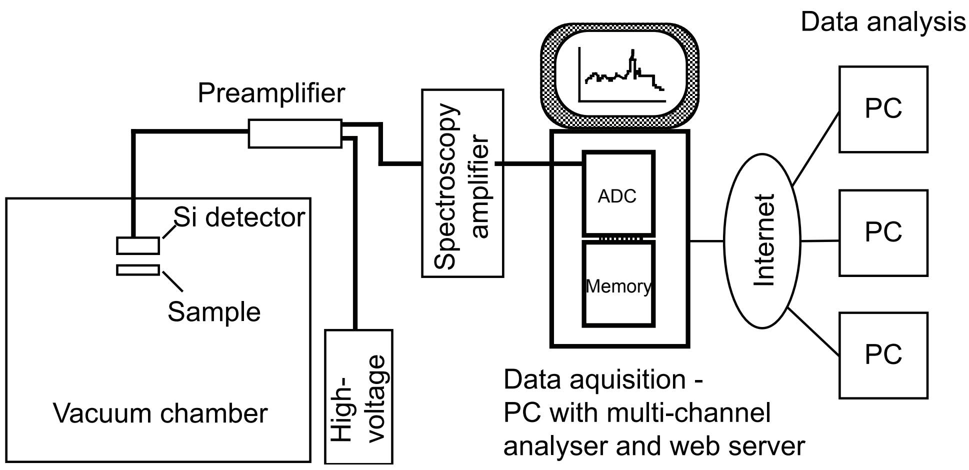

Figure 1: Experimental setup for beta-decay spectroscopy.

The basic idea of our semiconductor detector is to utilize the electron band structure of such materials. By creating a so called *pn* junction with a large reverse voltage we will have a strong electric potential field but no electric current. The depletion volume is characterized by a lack of free charge carriers. A charged, energetic particle entering this volume excites a large number of electron-hole pairs through ionization of the Si atoms. The end result is a current pulse with an amplitude proportional to the kinetic energy loss of the incoming particle. The Li-doping of the Si detector in combination with the high voltage (≈ 500 V) guarantees a large depletion volume. The majority of electrons with an energy of $\lesssim 1.5$ MeV will come to a complete halt within a depth of about 2 mm, and this can be seen as the design criterion for our detector. Note that high-energy photons will typically not interact very strongly with our detector due to the small active volume. However, X-ray photons have small enough energy that they might get photoabsorbed and will then show up as a low-energy ($\lesssim 100$ keV) peak.

The detector signal is amplified (to about 1–10 Volts) and re-shaped (to 2–4 μs) in a pre-amplifier and a linear spectroscopy amplifier. This analog signal can then be fed into an ADC (analog to digital converter) and converted to a binary number N. The number of channels of the ADC should optimally correspond to the detector resolution. One can often assume a linear relation between the signal amplitude (particle kinetic energy) and the channel number N. These numbers obtained from many decay events are stored in memory and by collecting data (statistics) we build a histogram representing the energy spectrum. However, this energy spectrum is a function of the channel number and the setup therefore requires an energy calibration.

2.2 Measurements

The following measurements will be performed during the lab session

1. **Energy calibration**. A calibration of channel number versus electron energy will be performed utilizing the decay of ^{207}Bi.

   The decay of ^{207}Bi is a multistep process that involves the emission of conversion electrons (see the sections on gamma decay and on internal conversion in the course text book). These conversion electrons are emitted with discrete energies and in this particular decay there are four different peaks in the energy spectrum (see Fig 2). By determining the channel numbers of the conversion electron peaks in the spectrum it is possible to perform the energy calibration by finding a linear relation between channel numbers and energies (see Appendix A.1.3).

2. **Beta spectrum of ^{137}Cs**. Measure the electron energy spectrum from the beta decay of ^{137}Cs. Perform the analysis excercises of Sec. 3.

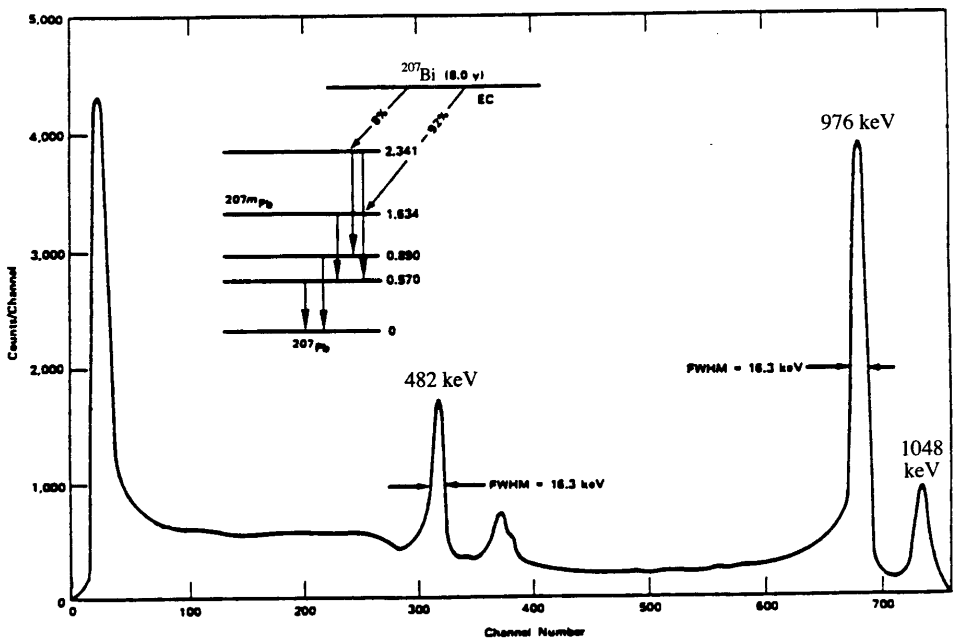

Figure 2: Decay scheme of ^{207}Bi and the corresponding electron energy spectrum. The peaks correspond to conversion electrons (please see text on gamma deexcitation).

3 Analysis

Use the energy-calibrated data from the ^{137}Cs measurement and perform a data analysis as described below.

3.1 Decay scheme

Study the measured electron energy spectrum and try to explain the different features.
In particular, focus on the shape of the spectrum, peak(s) and end points.

Sketch a simple decay scheme that would correspond to the observed features.

3.2 Kurie plot

Find the end point energies of the beta-decay spectra by using a Kurie plot of the data. I.e., plotting the quantity

$$\sqrt{\frac{N(T_e)}{\sqrt{T_e}F(Z,T_e)}}, \tag{1}$$

as a function of the beta electron kinetic energy T_e. $N(T_e)$ is the number of measured electrons with energy T_e, p_e is the corresponding momentum and $F(Z,T_e)$ is the Fermi function. See Appendix A.1.4 for detailed instructions.

3.3 $\log_{10}(ft)$

Determine the $\log_{10}(ft)$ values for the two beta-decay branches. The ft value is also known as the comparative half life and is obtained by the product of the partial half life $t_{1/2}$ (in seconds) and the Fermi integral $f(Z,Q_\beta)$ (see below).

The partial half life – is the half life of a single beta-decay branch. The total half life is given by

$$t_{1/2} = \frac{\ln 2}{\lambda}, \quad \text{where } \lambda = \sum_i \lambda_i, \tag{2}$$

and λ_i is the decay constant for decay branch i. In our case we have two decay branches, refered to as the low- and high-energy branches. As we can estimate from the beta-decay energy spectrum of ^{137}Cs, the largest fraction of all decays proceeds through the low-energy branch. We will use

$$\lambda_1 = 0.9\lambda, \quad \lambda_2 = 0.1\lambda, \quad t_{1/2}(\text{tot}) = 30 \text{ years}.$$

Compute λ, λ_1, λ_2, $t_{1/2}(1)$, $t_{1/2}(2)$.

$$\lambda = \qquad\qquad [\text{years}]^{-1}$$

The Fermi integral – The Q-value for the decay and the charge of the daughter nucleus completely determine the integrated statistical factor for the decay probability (see Fermi theory of beta decay). The integral is known as the Fermi integral

$$f\left(Z, Q_\beta\right) = \frac{\sqrt{2}}{{m_e}^{7/2} c^7} \int_0^{Q_\beta} \sqrt{T_e}(Q_\beta - T_e)^2 F(Z, T_e) dT_e$$

The f-value can be found in tables or evaluated using approximate analytical expressions (see Appendix A).

	Low-energy branch	High-energy branch
λ_i [years^{-1}]		
$t_{1/2}(i)$ [years]		
$t_{1/2}(i)$ [seconds]		
$\log_{10} t_{1/2}(i)$		
$\log_{10} f(Z, Q_i)$		
$\log_{10}(f t_{1/2}(i))$		

3.4 Selection rules

Classify the beta-decay branches according to the $\log_{10}(ft)$ values and specify the corresponding selection rules. Refer to Tables 1 and 2.

Type of transition	$\log_{10}(f t_{1/2})$
Super-allowed	2.9 – 3.7
Allowed	3.8 – 6.0
First-forbidden nonunique[a]	6 – 9
First-forbidden unique[b]	8 – 10
Second-forbidden	11 – 13
Third-forbidden	17 – 19
Fourth-forbidden	> 22

[a]Mixed Fermi and Gamow-Teller
[b]Pure Gamow-Teller

Table 1: Classification of beta-decay transitions according to their $\log_{10}(ft)$ values.

3.5 Full decay scheme

Employ the selection rules and the initial spin-parity assignments of the nuclear shell model (see the homework problem) and complete the decay scheme with level information.

Table 2: Correlation between ft values and the different classes of beta decays. The table summarizes the $\log_{10}(ft)$ values for almost 1000 measured decay branches.

$\log_{10}(ft)$	Super-allowed	Allowed	First-forbidden		Second-forbidden	
			Nonunique[a]	Unique[b]	Nonunique[a]	Unique[b]
2.8-3.2	6					
3.3-3.7	25	1				
3.8-4.2	2	8				
4.3-4.7		57				
4.8-5.2		120	8			
5.3-5.7		99	5			
5.8-6.2		74	24			
6.3-6.7		47	57			
6.8-7.2		25	40	1		
7.3-7.7		20	57	1		
7.8-8.2		9	38	28		
8.3-8.7		5	25	35		
8.8-9.2		9	21	10		
9.3-9.7			3	7	1	
9.8-10.2			5	2		
10.3-10.7		1	2	1	1	1
10.8-11.2			1	2	3	1
11.3-11.7			1		2	1
11.8-12.2			5		4	1
12.3-12.7			2		3	2
12.8-13.2					3	
> 13.3			1		3	
Total	33	475	295	87	20	6

[a]Mixed Fermi and Gamow-Teller
[b]Pure Gamow-Teller

A Data acquisition and data analysis

A.1 Data analysis – MATLAB operations

The experimental data will be analysed using MATLAB.

A.1.1 Retrieving data (online/in lab)

```
>> data=getk5();
```

To have a binning appropriate to the amount of statistics it is recommended to use the following command during the laboration:

```
>> data=getk5_n1024();
```

A.1.2 Reading data from file (offline)

```
>> data=load('filename');
```

A.1.3 Calibration

- Fitting peaks (using parabola-fit of log-values). Select range approximately at half-maximum;

```
>> rangemin = 1850;
>> rangemax = 1950;
>> [peak1,fit1,fitrange]=fitpeak(rangemin,rangemax,data(:,1),data(:,2))
>> peak1
peak1 =
   1.8919e+03
```

  `peak` is the x-value of the peak (fitted to the data), `fit` is a vector with the y-values of the fit and `fitrange` is a vector with the x-values of the fit.

- Find and fit three peaks and make the linear calibration by fitting the parameters of the energy scale (keV):

```
>> polyfit([peak1 peak2 peak3],[e1 e2 e3],1)
ans =
    0.2586   -5.4891
```

  where `e1`, `e2`, `e3` are the known peak energies.

- Create a vector with raw channels (`ch`), and then an energy calibrated vector (`e`):

```
>> ch=data(:,1);
>> e=ans(1)*ch+ans(2);
```

- Finally, you can plot the energy-calibrated data:

```
>> figure;semilogy(e,data(:,2))
```

- To retrieve values at certain locations you can use the fit function, `fitpeak`.

A.1.4 Make a Kurie plot

- Use the provided (built-in) function '`kurieplot`' to transform the data into a Kurie plot.

```
>> [kuriedata]=kurieplot(e,data,Z)

function [kuriedata]=kurieplot(Te,data,Z)
  % INPUT:
  % Te = Electron kinetic energy vector (in keV) from calibration.
  % data = Raw data in 2-column matrix form [ch,counts]
  % Z = daughter-nucleus proton-number (minus daughter-nucleus proton-number
  %                                     for beta-plus decay)
  % OUTPUT:
  % kuriedata = data for Kurie plot (vector, same length as Te)
```

- Visually inspect and select appropriate ranges, where the Kurie plot looks approximately linear.

```
>> range=500:700;
```

- Then make linear fit(s) using the '`kuriefit`' function:

```
>> [slope,offset] = kuriefit(e,kuriedata,range)
slope =
   -0.0036
offset =
    5.3080

function [slope,offset]=kuriefit(Te,kuriedata,range)
  % INPUT:
  % Te = electron kinetic energy vector (in keV) from calibration.
  % kuriedata = from kurieplot function
  % range = energy range for fit
  % OUTPUT:
  % slope = slope of linear fit to Kurie plot
  % offset = offset of linear fit to Kurie plot
```

- Find the Q-value from the parameters of the linear fit to the Kurie plot.

A.1.5 The f-value

An approximate value for the Fermi integral can be obatined using the 'log10f' function:

```
function log10f=log10f(Z,Q)
  % INPUT:
  % Z = daughter-nucleus proton-number (minus daughter-nucleus proton-number
  %                                     for beta-plus decay)
  % Q = beta-decay Q-value (in keV)

  % OUTPUT:
  % log10f = Approximate value for log_10 (f) (non-relativistic expression)
```

B Fermi theory of beta decay

The Fermi theory of beta decay will be discussed during the data-collection time of the ^{137}Cs beta-decay measurement. The following topics will be covered:

- Perturbation theory and Fermi's golden rule.
- Beta decay phase space: The statistical factor.
- Beta decay matrix elements: The allowed approximation.
- The plane-wave approximation and the Coulomb Fermi function.
- Kurie plots.
- Comparative half lives, or $\log_{10}(ft)$ values.

B.1 The continuous beta spectrum

The beta-decay process transformed one element into another while emitting charged particles, denoted β-particles

$$Z \longrightarrow [Z+1] + \beta^-$$

The process occurred under the release of energy denoted the Q-value. This energy was expected to appear as kinetic energy of the much lighter β-particle. However, a continuous spectrum was observed, see Fig. 3. Energy was apparently not conserved. In addition, angular momentum conservation was also violated when assuming that the emitted particle was a spin-1/2 fermion.

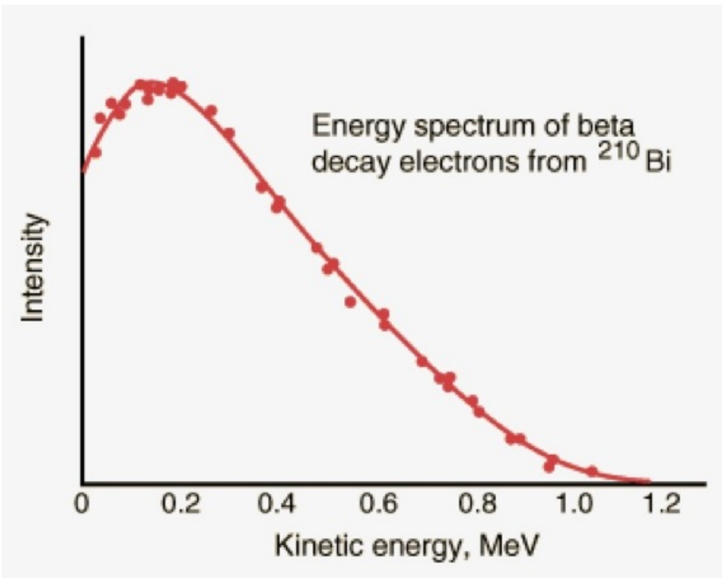

Figure 3: Continuous beta-decay spectrum from ^{210}Bi.

B.2 Fermi theory of beta decay (1934)

The neutrino hypothesis (Pauli, 1930)

In 1930, Wolfgang Pauli postulated the existence of the neutrino to explain the continuous distribution of energy of the electrons emitted in beta decay. Only with the emission of a third particle could momentum and energy be conserved. Pauli's neutrino hypothesis implied that the real, microscopic process was

$$n \to p + e^- + \bar{\nu}_e. \tag{3}$$

Dear Radioactive Ladies and Gentlemen,

As the bearer of these lines, to whom I graciously ask you to listen, will explain to you in more detail, how because of the "wrong" statistics of the N and Li6 nuclei and the continuous beta spectrum, I have hit upon a desperate remedy to save the "exchange theorem" of statistics and the law of conservation of energy. Namely, the possibility that there could exist in the nuclei electrically neutral particles, that I wish to call neutrons, which have spin 1/2 and obey the exclusion principle and which further differ from light quanta in that they do not travel with the velocity of light. The mass of the neutrons should be of the same order of magnitude as the electron mass and in any event not larger than 0.01 proton masses. The continuous beta spectrum would then become understandable by the assumption that in beta decay a neutron is emitted in addition to the electron such that the sum of the energies of the neutron and the electron is constant ...

I agree that my remedy could seem incredible because one should have seen these neutrons much earlier if they really exist. But only the one who dare can win and the difficult situation, due to the continuous structure of the beta spectrum, is lighted by a remark of my honoured predecessor, Mr Debye, who told me recently in Bruxelles: "Oh, It's well better not to think about this at all, like new taxes". From now on, every solution to the issue must be discussed. Thus, dear radioactive people, look and judge.

Unfortunately, I cannot appear in Tubingen personally since I am indispensable here in Zurich because of a ball on the night of 6/7 December.

With my best regards to you, and also to Mr Back.

Your humble servant,

W . Pauli Dec.4, 1930 [abridged translation]

The Fermi theory of beta decay (Fermi, 1933–34)

By 1934, Enrico Fermi had developed a theory of beta decay to include the neutrino, presumed to be massless as well as chargeless.

The main assumptions that lead up to Fermi's theory for beta decay can be sumarized as follows

- Spin $s_\nu = 1/2$

- m_ν=0, $q_\nu = 0$
- The released energy is shared between the electron and the neutrino (while we can neglect the very small recoil energy of the daughter nucleus) so that the electron energy spectrum becomes continuous.
- Fermi postulated that the electron and the neutrino are created in the decay.
- The process in Eq. 3 was treated as a point interaction between the four fermions.[a]
- Treating the beta decay as a weak process that was well described by first-order perturbation theory, Fermi developed an expression for the transition probability that is now referred to as Fermi's Golden Rule:

$$\lambda_{\text{fi}} = \frac{2\pi}{\hbar} \left| \langle f \left| H_\beta \right| i \rangle \right|^2 \rho\left(E_f\right)$$

Straightforward in concept, Fermi's Golden Rule says that the transition rate is proportional to the strength of the coupling between the initial and final states, as described by the transition matrix element $\langle f \left| H_\beta \right| i \rangle$, and factored by the density of final states available to the system, $\rho\left(E_f\right)$. We will discuss these two terms in some detail below.

B.3 Transition matrix element

Fermi assumed that the weak interaction is of such short range that it can be modeled as a point interaction. He specified the interaction in terms of an interaction strength to be determined from experiment

$$H_\beta = G_\beta \tau^\pm \delta\left(\boldsymbol{x} - \boldsymbol{x}'\right)$$

where $\tau^\pm$ is an operator that transforms a neutron to a proton (or vice versa)[b].

The initial state describes the mother nucleus ψ_i. The final state describes the daughter nucleus (with one more proton and one less neutron) together with an electron (or positron) and an anti-neutrino (or neutrino), i.e., $\psi_f \phi_e \phi_{\bar{\nu}}$

$$\langle f \left| H_\beta \right| i \rangle = G_\beta \int {\psi_f}^* {\phi_e}^* {\phi_\nu}^* \tau^\pm \psi_i d\boldsymbol{x}$$

The plane-wave approximation

In the *plane-wave approximation* we will make the assumption that all particles can be treated non-relativistically (which is in general not really correct for the

[a] We know today that Fermi's point interaction picture is an *effective theory* that is appropriate to describe low-energy weak processes. At higher energies, the four-fermion contact interaction must be replaced by a more complete theory, including an exchange of a W or Z boson. This is explained in the electroweak theory of the standard model.

[b] The isospin raising/lowering operator.

leptons, i.e. the electron and neutrino) and that the leptons are emitted as plane waves (i.e. as free particles).

The wave function for a free particle, in a normalization volume V, is given by a plane-wave solution

$$\phi_e = \frac{1}{\sqrt{V}} \exp\left(i\boldsymbol{k}_e \cdot \boldsymbol{x}\right),$$
$$\phi_\nu = \frac{1}{\sqrt{V}} \exp\left(i\boldsymbol{k}_\nu \cdot \boldsymbol{x}\right),$$

where $\boldsymbol{k}$ are wave vectors.

The lepton part of the matrix element is then

$$\phi_e{}^* \phi_\nu{}^* = \frac{\exp\left[-i\left(\boldsymbol{k}_e + \boldsymbol{k}_\nu\right) \cdot \boldsymbol{x}\right]}{V}$$

The allowed approximation

The proper way to analyse the lepton part of the wave function is to make a multipole expansion. However, for our purposes we will simply look at the Taylor series expansion

$$\phi_e{}^* \phi_\nu{}^* = \frac{\exp\left[-i\left(\boldsymbol{k}_e + \boldsymbol{k}_\nu\right) \cdot \boldsymbol{x}\right]}{V} = \frac{1}{V}\left[1 - i\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{2} + \ldots\right] \approx \frac{1}{V}.$$

The last step is known as the *allowed approximation.* It is valid for $\boldsymbol{k}\cdot\boldsymbol{x}$ small and simplifies the matrix element significantly as it removes all dependence on the electron and neutrino energy.

Let us perform an order-of-magnitude estimate: For x given by the nuclear length scale, R $\approx$ a few fm, and k given by the typical energy/momentum scale of nuclear beta decay, $k \approx 1$ MeV/$\hbar c \approx 1/200$ fm^{-1}. We find that $\boldsymbol{k}\cdot\boldsymbol{x} < 0.1$. However, $\boldsymbol{k}\cdot\boldsymbol{x}$ can also be related to the orbital momentum of the leptons (in classical mechanics we have $\boldsymbol{L} = \boldsymbol{x}\times\boldsymbol{k}$. In quantum mechanics, the orbital angular momentum is quantized so that $\boldsymbol{L}^2 = \hbar^2 l(l+1)$, where $l = 0, 1, \ldots$, and $L_z = \hbar m$, where $m = -l, -l+1, \ldots, l-1, l$. Therefore, small $\boldsymbol{k}\cdot\boldsymbol{x}$ corresponds to allowed (l=0) transitions.

The remaining integral over nuclear wave functions is denoted M_{fi} so that

$$\langle f\left|H_\beta\right| i\rangle \approx \frac{G_\beta}{V} \int \psi_f{}^* \tau^{\pm} \psi_i d\boldsymbol{x} = \frac{G_\beta}{V} M_{fi}$$

B.4 Density of states

The energy released in the nuclear beta decay is distributed as kinetic energy and shared between the three particles in the final state. This provides a statistical factor to the decay probability. The more available states (possible distributions of energy) the higher probability. This statistical factor is known as the *density of states*-factor, or the "phase-space"-factor. Assuming that the process under investigation releases an energy E and that there is a number of available states

$N(E)$ for the final state. The density of states is then defined as the number of states per energy interval

$$\rho(E) = \frac{dN}{dE}$$

We will give first an illustrative example of the density of states factor for one free particle in one dimension, and then discuss the relevant case of nuclear beta decay with three particles in the final state.

One particle in one dimension

Let us consider a free particle in one dimension. The energy is $E = p^2/2m$. Let us consider a fraction of phase space, describing a particle's position and momentum. See figure 4. For a classical particle, a state is characterized by a specific position and velocity, i.e. a point in phase space. We can fit infinitely many points into a volume. However, for a particle governed by quantum-mechanics each state requires a cell with a finite volume given by the Heisenberg uncertainty principle. In one dimension, this minimum volume is $2\pi\hbar$. The number of possible particle states that can occupy a phase space volume Lp is therefore

$$N_1 = \frac{pL}{2\pi\hbar}.$$

The density of states factor becomes

$$\rho_1(E) = \frac{dN_1}{dE} = \frac{dN_1}{dp}\frac{dp}{dE} = \frac{L}{2\pi\hbar}\sqrt{\frac{2m}{E}}.$$

Note that there is an extra factor of 2 coming from the two possible states ($\pm p$) per energy. The important result is the $1/\sqrt{E}$ dependence of the number of available states.

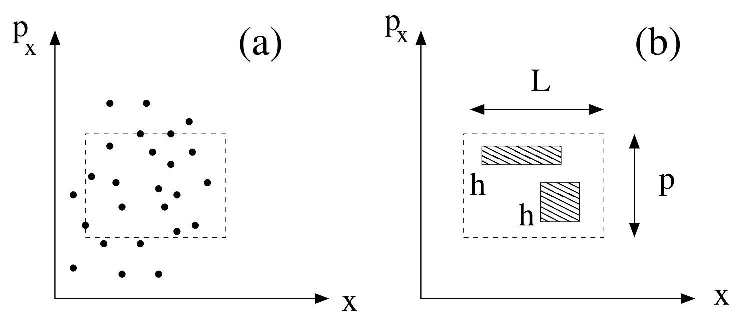

Figure 4: Classical (a) and quantum-mechanical (b) one-dimensional phase space. In the classical case, the state of a particle is represented by a point (x, p_x). In the quantum case, a state must be described by a cell in phase space with the volume $h = 2\pi\hbar$.

Three particles in three dimensions

Let us consider the three particles in the CM frame for which

$$\boldsymbol{p}_e + \boldsymbol{p}_\nu + \boldsymbol{p}_3 = 0.$$

This implies that the momentum of $\boldsymbol{p}_e + \boldsymbol{p}_\nu$ determines the momentum of the third particle, the daughter nucleus, $\boldsymbol{p}_3$. The degrees of freedom in distributing the momentum is therefore restricted to the first two particles. A particle state in three dimensions occupies the minimum volume $(2\pi\hbar)^3$ and the number of available states for a phase-space volume becomes

$$N_3 = \frac{\int d^3\boldsymbol{r}_e d^3\boldsymbol{p}_e d^3\boldsymbol{r}_\nu d^3\boldsymbol{p}_\nu}{(2\pi\hbar)^6} = \frac{V^2}{(2\pi\hbar)^6}\int_0^{\tilde{\boldsymbol{p}}_e} d^3\boldsymbol{p}_e \int_0^{\tilde{\boldsymbol{p}}_\nu} d^3\boldsymbol{p}_\nu,$$

where V is our normalization volume and $\tilde{\boldsymbol{p}}$ is the maximum momentum vector. The phase space factor becomes

$$\rho(E) = \frac{dN_3}{dE} = \frac{V^2}{(2\pi\hbar)^6}\frac{d}{dE}\int p_e^2 dp_e d\Omega_e p_\nu^2 dp_\nu d\Omega_\nu.$$

We note that our transition matrix element V_{fi} has no dependence on the direction of the electron and neutrino momentum. We can therefore integrate out the angular dependence

$$\int d\Omega_e d\Omega_\nu = (4\pi)^2.$$

Let us consider a particular decay event in which the electron receives an energy corresponding to a momentum within the range $[p_e, p_e + dp_e]$. Note, that this will give us the density of states factor for a fixed electron energy $\rho(E, p_e)dp_e$ and plugging this into Fermi's golden rule will give us the partial decay probability $d\lambda = \lambda(p_e)dp_e$. In calculating dN_3/dE we will therefore consider the electron energy fixed

$$\rho(E)dp_e = \frac{V^2(4\pi)^2}{(2\pi\hbar)^6} p_e^2 dp_e p_\nu^2 \frac{dp_\nu}{dE}.$$

To perform the $dp_{\nu,\max}/dE$ differentiation we need an expression for the energy E that is released in the process. Assume that very little energy is transferred to the daughter nucleus due to its large mass. The energy being released (Q-value) is then shared between the electron and the neutrino.

$$E = Q \approx T_e + T_\nu \approx T_e + p_\nu c,$$

assuming that the neutrino is massless. Note that the electron momentum and kinetic energy are related via $p_e^2c^2 = T_e^2 + 2T_e m_e c^2$. This gives

$$\begin{aligned} {p_{\nu,\max}}^2 &= \frac{(Q - T_e)^2}{c^2} \\ \frac{dp_\nu}{dE} &= \frac{1}{c}. \end{aligned}$$

This gives the density-of-states factor. Note that the expression that we have derived is for the specific event in which the electron gets a momentum within the range $[p_e, p_e + \mathrm{dp}_e]$.

$$\rho(Q, p_e)\, dp_e = \frac{V^2 16\pi^2 {p_e}^2 (Q - T_e)^2\, dp_e}{c^3 (2\pi\hbar)^6}.$$

Rewriting this only in energy we get that

$$\rho(Q, T_e)\, dT_e = \frac{V^2 16\pi^2 \sqrt{2m_e T_e}\, (Q - T_e)^2\, m_e dT_e}{c^3 (2\pi\hbar)^6}.$$

B.5 Shape of the beta spectrum

Putting it all together we have

$$d\lambda_{fi} = \frac{m_e^{3/2}}{\sqrt{2}\pi^3 \hbar^7 c^3} G_\beta^2 |M_{fi}|^2 (Q - T_e)^2 \sqrt{T_e} dT_e \tag{4}$$

We denote by $N(T_e) = dN/dT_e$ the number density of beta electrons that are emitted with kinetic energy between $[T_e, T_e + dT_e]$, averaged over all angles. This number will be proportional to the probability of decay into that region of phase space and we will have $N(T_e)\, dT_e = d\lambda$. We have just calculated $d\lambda$ using Fermi's golden rule. Therefore, we will have

$$N(T_e) = \frac{m_e^{3/2}}{\sqrt{2}\pi^3 \hbar^7 c^3} G_\beta^2 |M_{fi}|^2 (Q - T_e)^2 \sqrt{T_e}.$$

The Fermi function

We have disregarded the electric charge of the beta electron and treated it as a free, non-relativistic particle. In principle, both these approximations should be improved upon.

The Coulomb effect can be understood qualitatively as follows: The energy distribution for positrons and electrons in $\beta^{\pm}$-decay is the same at the moment of production. The Coulomb field accelerates the positrons (or deaccelerates the electrons) thus giving fewer (more) slow particles in the spectrum.

This effect is represented by the so called Fermi function, $F(Z, T_e)$, that depends on the charge of the daughter nucleus and the momentum (or energy) of the electron.

$$N(T_e) = \frac{m_e^{3/2}}{\sqrt{2}\pi^3 \hbar^7 c^3} G_\beta^2 |M_{fi}|^2 F(Z, T_e) (Q - T_e)^2 \sqrt{T_e}.$$

One way to estimate the Fermi function is to approximate it as the ratio of a plane wave and a Coulomb wave function at the nuclear radius R. The electron spectrum is now modified as illustrated in Fig. 5.

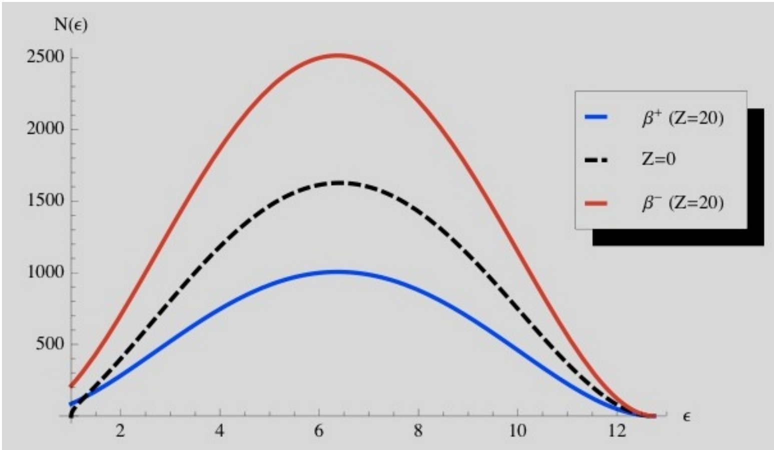

Figure 5: Effects of the Fermi function on the beta energy spectrum.

The Kurie plot

The Q-value for a beta-decay branch corresponds to the end-point of the continuous energy distribution. However, to find this energy from a measurement is almost impossible since the end point is very sensitive to experimental background. Furthermore, if more than one beta-decay branch is present, the end points will disappear in the spectrum of the high-energy branch.

Instead we can employ the fact that we know the shape of the beta spectrum. To linearize the data we can plot the quantity

$$\sqrt{\frac{N(T_e)}{F(Z,T_e)\sqrt{T_e}}} = C\,(Q - T_e)$$

against the electron energy T_e. The factor C is a constant (that includes the still unknown nuclear transition matrix element).

This plot is known as a *Kurie plot* and should produce a straight line intersecting the abcissa at $T_e = Q$. The endpoint energy Q can be found by straight-line extrapolation.

The log-ft value

The total decay probability is given after integration over all electron momenta

$$\lambda = \frac{\ln 2}{t_{1/2}} = \int \frac{d\lambda_{fi}}{dT_e} dT_e = \frac{m_e^{3/2}}{\sqrt{2}\pi^3\hbar^7 c^3} {G_\beta}^2 \left|M_{fi}\right|^2 \int_0^Q F(Z,T_e)\,(Q-T_e)^2 \sqrt{T_e} dT_e$$

where the integral is known as the Fermi integral (except for some dimensional factors). It involves only known quantities

$$f(Z,Q) = \frac{\sqrt{2}}{{m_e}^{7/2} c^7} \int_0^Q F(Z,T_e)\,(Q-T_e)^2 \sqrt{T_e} dT_e.$$

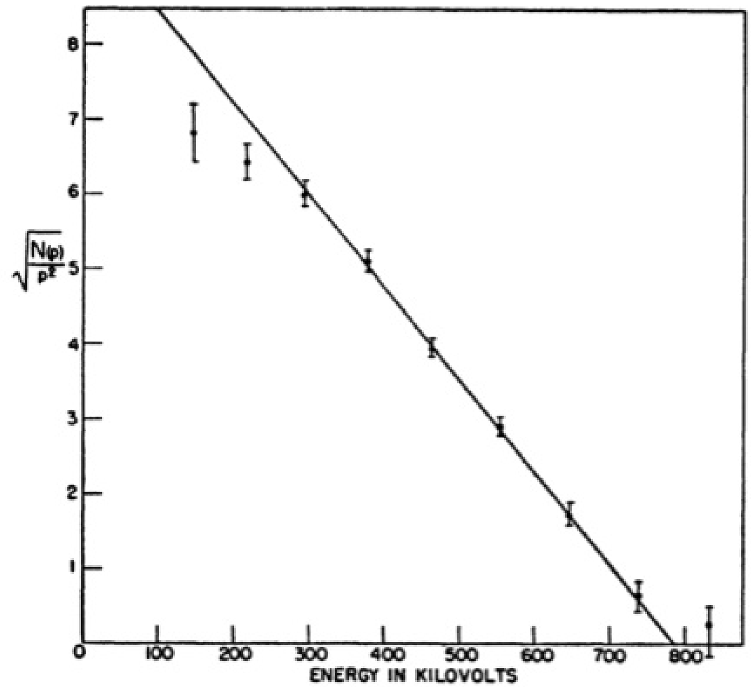

Figure 6: Kurie plot for the neutron decay. From J. M. Robson, Phys. Rev. **83** (1951) 349.

From this we can define the ft-value, which is a constant that characterizes a particular decay. This constant contains the nuclear structure information in terms of the nuclear matrix element

$$ft \equiv f(Z,Q)t_{1/2} = \frac{2\pi^3\hbar^7 \ln 2}{m_e{}^5 c^4 G_\beta{}^2} \frac{1}{|M_{fi}|^2} \equiv \frac{\kappa}{|M_{fi}|^2}.$$

The constant $\kappa = 6147$ sec. As the matrix element can vary over many orders of magnitude, one often use the $\log(ft)$ value instead. Allowed transitions ($l = 0$) have small $\log(ft)$ since the overlap integral is large. Forbidden transitions ($l > 0$) have large $\log(ft)$ values since the overlap integral is small. Refer to Table 1 for approxmiate ranges.