Scheme and Solutions : CIE – 2 02|12|2024

Linear Algebra and Probability theory (MA231TC)

Sl. No.	Quiz	M
1	$\sqrt{14}$	1
2	2	1
3	1	1
4	16	1
5	$\langle f,f\rangle = \int_0^{\pi/4}\cos^2 x\,dx = \frac{1}{2}\int_0^{\pi/4}(1+\cos 2x)dx = \frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{2}\right)$	1
	$\lVert f\rVert = \sqrt{\frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{4}\right)} = 0.72$	1
6	$P = \frac{1}{\sqrt{13}}\begin{bmatrix}2 & 3\\ 3 & -2\end{bmatrix}, D = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\ A = PDP^T = \frac{1}{13}\begin{bmatrix}22 & -6\\ -6 & 17\end{bmatrix}$	1+1
7	$(c^3+1)/9 = 1,\ c = 2$	1+1

TEST

1) Let $w_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, w_2 = \begin{bmatrix}-1\\4\\4\\-1\end{bmatrix}, w_3 = \begin{bmatrix}4\\-2\\2\\0\end{bmatrix}$ w_1, w_2 and w_3 are LI, hence form a basis for 3-dimensional subspace of $\mathbb{R}^4$.

We use Gram-Schmidt process to construct orthonormal basis $\{u_1, u_2, u_3\}$

$$u_1 = \frac{w_1}{\lVert w_1\rVert} = \left(\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\right)^T \quad (1)$$

Let $y_2 = w_2 - \langle w_2, u_1\rangle u_1$, $u_2 = \frac{y_2}{\lVert y_2\rVert}$

where $\langle w_2, u_1\rangle = w_2^T u_1 = -\frac{1}{2} + \frac{4}{2} + \frac{4}{2} - \frac{1}{2} = 3$

$$\therefore\ y_2 = w_2 - 3u_1 \left(-1-\tfrac{3}{2}\ \ 4-\tfrac{3}{2}\ \ 4-\tfrac{3}{2}\ \ -1-\tfrac{3}{2}\right)^T = \left(\tfrac{-5}{2}\ \tfrac{5}{2}\ \tfrac{5}{2}\ -\tfrac{5}{2}\right)^T$$

$$\lVert y_2\rVert = \sqrt{\tfrac{25}{4}+\tfrac{25}{4}+\tfrac{25}{4}+\tfrac{25}{4}} = \sqrt{\tfrac{100}{4}} = 5 \ \therefore u_2 = \frac{y_2}{\lVert y_2\rVert} = \left(-\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ -\tfrac{1}{2}\right)^T \quad (2)$$

Let $y_3 = w_3 - \langle w_3, u_1\rangle u_1 - \langle w_3, u_2\rangle u_2$ and $u_3 = \frac{y_3}{\lVert y_3\rVert}$

Where $\langle w_3, u_1\rangle = w_3^T u_1 = \frac{4}{2} - \frac{2}{2} + \frac{2}{2} + 0 = 2$

$\langle w_3, u_2\rangle = w_3^T u = -\frac{4}{2} - \frac{2}{2} + \frac{2}{2} + 0 = -2$

$$\therefore\ y_3 = w_3 - 2u_1 + 2u_2 = (4\ -2\ 2\ 0)^T - 2\left(\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\right)^T + 2\left(-\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\right)^T$$

$$y_3 = (2\ -2\ \ 2\ -2)^T \Rightarrow \|y_3\| = \sqrt{4+4+4+4} = \sqrt{16} = 4$$

$$\therefore\ u_3 = \frac{y_3}{\|y_3\|} = \left(\tfrac{1}{2}\ -\tfrac{1}{2}\ \ \tfrac{1}{2}\ -\tfrac{1}{2}\right)^T$$ ③

Thus $\{u_1, u_2, u_3\}$ forms orthonormal basis for $col(A)$ ②

$$Proj_{C(A)}(x) = QQ^Tx = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/4 & -1/4 & 1/4 & 1/4 \\ -1/4 & 3/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 3/4 & -1/4 \\ 1/4 & 1/4 & -1/4 & 3/4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 7/4 \\ 5/4 \\ 5/4 \end{bmatrix}$$

$$y = x - Proj_{C(A)}(x) = (1\ \ 2\ \ 1\ \ 1)^T - (3/4\ \ 7/4\ \ 5/4\ \ 5/4)^T = (1/4\ \ 1/4\ -1/4\ -1/4)^T$$ ①

shortest distance

$$\|y\| = \sqrt{(1/4)^2 + (1/4)^2 + (-1/4)^2 + (-1/4)^2} = 1/2$$ ①

2 a) ⓘ $u_i^T u_j = 0 \quad \forall\, i \neq j$ and $u_i^T u_j = 1 \quad \forall\, i = j$ ②

ⓘⓘ $$[x]_B = \begin{bmatrix} \langle x, u_1 \rangle \\ \langle x, u_2 \rangle \\ \langle x, u_3 \rangle \end{bmatrix} = \begin{bmatrix} 4/3 + 1 + 2/3 \\ -2/\sqrt{5} + 6/\sqrt{5} + 0 \\ 8/\sqrt{45} + 6/\sqrt{45} - \frac{5}{\sqrt{45}} \end{bmatrix} = \begin{bmatrix} 3 \\ 4/\sqrt{5} \\ 9/\sqrt{45} \end{bmatrix} = \begin{bmatrix} 3 \\ 4/\sqrt{5} \\ 3/\sqrt{5} \end{bmatrix}$$ ②

2 b) i) $|A - \lambda I| = 0 \Rightarrow \lambda^2 - 8\lambda + 12 = 0 \Rightarrow \lambda = 2, 6$ ①

For $\lambda = 2$, $[A - \lambda I]x = 0 \Rightarrow \begin{bmatrix} 4-2 & 2 \\ 2 & 4-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \Rightarrow x_1 = -x_2 \Rightarrow x = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

For $\lambda = 6$, $[A - \lambda I]x = 0 \Rightarrow \begin{bmatrix} 4-6 & 2 \\ 2 & 4-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \Rightarrow x_1 = x_2 \Rightarrow x = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\therefore X = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix},\quad X^{-1} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix},\quad D_\lambda = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$$ ①

(ii) ②

$$B = A^2 - 2I \Rightarrow (2^2 - 2), (6^2 - 2)$$

① $= 2, 34$ are the eigenvalues

eigenvectors are same A. ①

3) $$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 5 \end{bmatrix} \Rightarrow \begin{array}{l} |AA^T - \lambda I| = 0 \\ \lambda^3 - 7\lambda^2 + 6\lambda = 0 \end{array}$$ ②

$$\Rightarrow \lambda(\lambda^2 - 7\lambda + 6) = 0, \lambda = 0, 1, 6$$

for $\lambda = 6$, $[AA^T - \lambda I]x = 0 \Rightarrow \begin{bmatrix} -5 & 0 & 1 \\ 0 & -5 & 2 \\ 1 & 2 & -1 \end{bmatrix} \Rightarrow \frac{x_1}{1} = \frac{-x_2}{-2} = \frac{x_3}{5}, x = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

for $\lambda = 1$, $[AA^T - \lambda I]x = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 4 \end{bmatrix} \Rightarrow \frac{x_1}{-4} = \frac{-x_2}{-2} = \frac{x_3}{0}, x = \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

for $\lambda = 0$, $[AA^T - \lambda I]x = 0 \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 5 \end{bmatrix} \Rightarrow \frac{x_1}{1} = -\frac{x_2}{-2} = \frac{x_3}{-1}, \; x = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ 3

$\therefore U = \begin{bmatrix} 1/\sqrt{30} & -2/\sqrt{5} & 1/\sqrt{6} \\ 2/\sqrt{30} & 1/\sqrt{5} & 2/\sqrt{6} \\ 5/\sqrt{30} & 0 & -1/\sqrt{6} \end{bmatrix}$ 1, $\Sigma = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ 1 $\therefore V^T = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$ 1

Now consider

$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}, \; |A - \lambda I| = 0 \Rightarrow \lambda^2 - 7\lambda - 6 = 0$

$\Rightarrow \lambda = 6, 1$

for $\lambda = 6$

$[A^TA - \lambda I]x = 0 \Rightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \Rightarrow \frac{x_1}{-1} = -\frac{x_2}{2} \Rightarrow x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

for $\lambda = 1$

$[A^TA - \lambda I]x = 0 \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \Rightarrow \frac{x_4}{4} = -\frac{x_2}{2} \Rightarrow x = \begin{bmatrix} 4 \\ -2 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$ 2

4 a)

x :	−200	−100	0	100	200	400	1
P(x) :	0.3077	0.1758	0.0110	0.3516	0.0879	0.0659	3
F(x) :	0.3077	0.4835	0.4945	0.8462	0.9341	1	1

$E[X] = 0$ 1

O	O	W	W	B	O
O	W	W	B	B	B
0	−100	−200	100	400	200
$\frac{2C_2}{14C_2}$	$8C_1\,2C_1$	$8C_2$	$8C_1 4C_1$	$4C_2$	$2C_1 4C_1$

4 b) (i) $\int_{-1}^{1} k(3 - x^2)\,dx = 1$

$\Rightarrow k\left[3x - \frac{x^3}{3}\right]_{-1}^{1} = 1 \Rightarrow k\left[3(1+1) - \frac{1}{3}(1+1)\right] = 1$

$\Rightarrow k = \frac{3}{16}$ 2

(ii) $P(X < \frac{1}{2}) = \int_{-1}^{1/2} \frac{3}{16}(3 - x^2)\,dx = \frac{3}{16}\left[3x - \frac{x^3}{3}\right]\Big|_{-1}^{1/2} = \frac{99}{128}$ 2

5 a) $F(x) = \begin{cases} 0 & ; -\infty \le x \le 0 \\ \int_0^x x\,dx & ; 0 \le x \le 1 \\ \int_0^1 x\,dx + \int_1^x (2-x)\,dx & ; 1 \le x \le 2 \\ 1 & ; x > 2 \end{cases} = \begin{cases} 0 & ; -\infty \le x \le 0 \\ x^2/2 & ; 0 \le x \le 1 \\ \frac{1}{2} + \left(2x - \frac{x^2}{2} - \frac{3}{2}\right) & ; 1 \le x \le 2 \\ 1 & ; x > 2 \end{cases}$ 1 2 1

1

$\therefore P(X \ge 1.5) = 1 - P(X < 1.5) = 1 - F(1.5) = 1 - 2(1.5) - \frac{(1.5)^2}{2} - 1 = 1 - 0.8750$

$= 0.125$

5 b) 2

X|Y		Y 1	2	3	
X	1	C	2C	3C	6C
	2	2C	4C	6C	12C
	3	3C	6C	9C	18C
		6C	12C	18C	1

$\sum_i \sum_j P_{ij} = 36C = 1 \Rightarrow C = 1/36$ 1

X :	1	2	3
P(X) :	6/36	12/36	18/36
:	1/6	1/3	1/2

Y :	1	2	3
P(Y) :	6/36	12/36	18/36

1

$E[XY] = \sum_i \sum_j x_i y_j P_{ij} = 1(C) + 2(2C) + 3(3C) + 2(2C) + 4(4C) + 6(6C) + 3(3C)$
$+ 6(6C) + 9(9C)$ 1

$= (1 + 4 + 9 + 4 + 16 + 36 + 9 + 36 + 81)C = 196 \frac{1}{36} = \frac{49}{9} = 5.44$