The Rank–Nullity Theorem and the Four Fundamental Subspaces

Finding Bases for the Four Fundamental Subspaces via Row Echelon Reduction

We assume that you know how to perform Gaussian elimination to reduce a matric to Row Echelon form. Let A be an $m \times n$ matrix. Perform Gaussian elimination to reduce A to its (reduced) row echelon form R. The pivot columns and rows of R provide the necessary information to construct bases for all four fundamental subspaces.

The first step is to use elementary row operations to obtain a matrix R in (reduced) row echelon form that is row-equivalent to A. Let the pivot columns be indexed by $j_1, j_2, \ldots, j_r$, where $r = \operatorname{rank}(A)$.

Basis for the Row Space $\operatorname{Row}(A)$

The nonzero rows of R form a basis for $\operatorname{Row}(A)$. This is because elementary row operations do not change the row space.

Basis for the Column space $\operatorname{Col}(A)$

Let A be an $m \times n$ real matrix. We aim to explain why the columns of the **original matrix** A that correspond to the **pivot columns** in a row echelon form of A constitute a basis for the column space $\operatorname{Col}(A)$.

Begin by writing the matrix A in terms of its columns:

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix},$$

where each $\mathbf{a}_j \in \mathbb{R}^m$. The **column space** of A is the subspace of $\mathbb{R}^m$ defined by

$$\operatorname{Col}(A) = \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}.$$

A **basis** for $\operatorname{Col}(A)$ is a subset of these columns that is

(i) linearly independent, and

(ii) spans $\mathrm{Col}(A)$.

The next step is to row reduce A to row echelon form. Perform Gaussian elimination on A to obtain a row echelon form R. Since elementary row operations are invertible, there exists an invertible $m \times m$ matrix E such that

$$R = EA.$$

Let the pivot columns of R be in positions $j_1, j_2, \ldots, j_r$, where $r = \mathrm{rank}(A)$. These are the columns that contain the leading (first nonzero) entries of each nonzero row in R.

3. Properties of pivot columns in row echelon form

In the row echelon matrix R, the following hold:

- The pivot columns $\mathbf{r}_{j_1}, \ldots, \mathbf{r}_{j_r}$ are **linearly independent**.
- Every non-pivot column of R is a **linear combination** of the pivot columns to its left.

These facts follow from the staircase structure of R: each pivot appears in a new row below the previous one, and all entries below a pivot are zero. This allows back-substitution to verify independence and express non-pivot columns as combinations of earlier pivot columns.

Thus, $\{\mathbf{r}_{j_1}, \ldots, \mathbf{r}_{j_r}\}$ is a basis for $\mathrm{Col}(R)$.

However, note that in general

$$\mathrm{Col}(A) \neq \mathrm{Col}(R),$$

so we **cannot** use the columns of R as a basis for $\mathrm{Col}(A)$.

Although the column spaces differ, row operations **preserve all linear dependence relations among the columns**. To see this, observe:

$$A\mathbf{x} = \mathbf{0} \iff EA\mathbf{x} = E\mathbf{0} \iff R\mathbf{x} = \mathbf{0}.$$

Hence,

$$\mathrm{Nul}(A) = \mathrm{Nul}(R).$$

This equality means that a set of columns of A is linearly dependent **if and only if** the corresponding set of columns of R is linearly dependent. In other words, the *pattern* of linear dependence among the columns is identical in A and R.

Because the dependence relations are the same:

- The columns $\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_r}$ of A (in the same positions as the pivot columns of R) are **linearly independent**.
  Reason: If they were linearly dependent, the same dependence would appear among $\mathbf{r}_{j_1}, \dots, \mathbf{r}_{j_r}$, contradicting their independence in R.
- Every non-pivot column $\mathbf{a}_k$ of A (where $k \notin \{j_1, \dots, j_r\}$) is a **linear combination** of $\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_r}$.
  Reason: In R, we have $\mathbf{r}_k = c_1\mathbf{r}_{j_1} + \cdots + c_r\mathbf{r}_{j_r}$ for some scalars c_i. Then the vector $\mathbf{x}$ with $x_k = 1$, $x_{j_i} = -c_i$, and other entries zero satisfies $R\mathbf{x} = \mathbf{0}$. Since $\mathrm{Nul}(A) = \mathrm{Nul}(R)$, we also have $A\mathbf{x} = \mathbf{0}$, which implies $\mathbf{a}_k = c_1\mathbf{a}_{j_1} + \cdots + c_r\mathbf{a}_{j_r}$.

Therefore, the set

$$\{\mathbf{a}_{j_1}, \mathbf{a}_{j_2}, \dots, \mathbf{a}_{j_r}\}$$

is linearly independent and spans $\mathrm{Col}(A)$.

Conclusion

This set is a **basis for the column space of** A. Hence:

Theorem 1. *Let A be an $m \times n$ matrix, and let R be any row echelon form of A. If the pivot columns of R occur in positions $j_1, \dots, j_r$, then the corresponding columns of the original matrix A,*

$$\{\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_r}\},$$

form a basis for $\mathrm{Col}(A)$.

Remark 1. *It is crucial to use the columns of the **original matrix** A, not those of R. While R reveals which columns to select, the actual basis vectors must come from A because* $\mathrm{Col}(A) \neq \mathrm{Col}(R)$ *in general.*

Example

Given a matrix A, find a basis for its column space $\mathrm{Col}(A)$ — that is, a linearly independent set of columns of A that spans $\mathrm{Col}(A)$.

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & 0 & 1 & 2 \\ 0 & 2 & 4 & 1 \end{bmatrix}.$$

This is a 4×4 matrix. Its column space is a subspace of $\mathbb{R}^4$.

Step-by-Step Procedure

Step 1: Row reduce A to row echelon form (REF)

Perform Gaussian elimination:

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & 0 & 1 & 2 \\ 0 & 2 & 4 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1,\, R_3 \leftarrow R_3 - R_1} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & -2 & 1 \\ 0 & 2 & 4 & 1 \end{bmatrix}$$
$$\xrightarrow{\text{swap } R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -2 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 1 \end{bmatrix} \xrightarrow{R_4 \leftarrow R_4 + R_2} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -2 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$
$$\xrightarrow{\text{swap } R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -2 & -2 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is a row echelon form (REF) of A.

Step 2: Identify the pivot columns

The leading (first nonzero) entry in each nonzero row occurs in:

- Row 1: column 1,
- Row 2: column 2,
- Row 3: column 3. Thus, the **pivot columns** are columns 1, 2, and 3. Hence, $\text{rank}(A) = 3$.

Step 3: Select the corresponding columns from the original matrix A

Take columns 1, 2, and 3 of the **original** matrix A:

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 3 \\ 6 \\ 1 \\ 4 \end{bmatrix}.$$

Step 4: Justification

- These columns are **linearly independent** because the corresponding columns in the REF are linearly independent, and row operations preserve linear dependence relations among columns.
- They **span** $\mathrm{Col}(A)$ because every non-pivot column (here, column 4) is a linear combination of the pivot columns, and the same linear relation holds in A as in its REF.

Demonstration: Identical Linear Dependence Relations in A and R

Let us verify explicitly that linear dependence relations among columns are the same in A and R.

Denote the columns of R by $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4$. From the REF:

$$R = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -2 & -2 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so

$$\mathbf{r}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{r}_2 = \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{r}_3 = \begin{bmatrix} 3 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \ \mathbf{r}_4 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}.$$

Observe that column 4 of R can be expressed as a linear combination of columns 1–3. Solving

$$c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + c_3\mathbf{r}_3 = \mathbf{r}_4,$$

we work from the bottom up (using the echelon structure):

- Row 3: $0c_1 + 0c_2 + 2c_3 = 2 \Rightarrow c_3 = 1$, - Row 2: $0c_1 - 2c_2 - 2c_3 = 1 \Rightarrow -2c_2 - 2(1) = 1 \Rightarrow c_2 = -\frac{3}{2}$, - Row 1: $c_1 + 2c_2 + 3c_3 = 1 \Rightarrow c_1 + 2(-\frac{3}{2}) + 3(1) = 1 \Rightarrow c_1 - 3 + 3 = 1 \Rightarrow c_1 = 1$.

Thus,

$$\mathbf{r}_4 = 1 \cdot \mathbf{r}_1 - \tfrac{3}{2} \cdot \mathbf{r}_2 + 1 \cdot \mathbf{r}_3.$$

Now consider the same linear combination of the **original columns** of A:

$$1 \cdot \mathbf{a}_1 - \tfrac{3}{2} \cdot \mathbf{a}_2 + 1 \cdot \mathbf{a}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \tfrac{3}{2} \begin{bmatrix} 2 \\ 4 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 - 3 + 3 \\ 2 - 6 + 6 \\ 1 - 0 + 1 \\ 0 - 3 + 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$

But this is exactly the fourth column of A:

$$\mathbf{a}_4 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$

Hence,

$$\mathbf{a}_4 = 1 \cdot \mathbf{a}_1 - \tfrac{3}{2} \cdot \mathbf{a}_2 + 1 \cdot \mathbf{a}_3.$$

The **same coefficients** that express $\mathbf{r}_4$ as a combination of $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ also express $\mathbf{a}_4$ as a combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$.

This confirms that **linear dependence relations among columns are identical in A and R**, which is why the pivot columns of A are linearly independent and span $\mathrm{Col}(A)$.

Final Answer

A basis for $\mathrm{Col}(A)$ is

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \\ 4 \end{bmatrix} \right\}}.$$

This set contains 3 vectors (matching $\mathrm{rank}(A) = 3$), is linearly independent, and spans $\mathrm{Col}(A)$.

Remark 2. ***Important:*** *Always use the columns of the* ***original matrix*** *A, not the columns of the row echelon form. The column space of the REF is generally different from that of A.*

Right Null Space

Goal

Given an $m \times n$ matrix A, find a **basis** for its null space

$$\text{Nul}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\},$$

i.e., a linearly independent set of vectors in $\mathbb{R}^n$ whose span is exactly the set of all solutions to $A\mathbf{x} = \mathbf{0}$.

Core Idea

The null space consists of all vectors $\mathbf{x}$ such that the linear combination of the columns of A with coefficients $x_1, \dots, x_n$ equals the zero vector:

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}.$$

Row reduction does not change the solution set of this homogeneous system. Thus, we may solve the simpler system $R\mathbf{x} = \mathbf{0}$, where R is the (reduced) row echelon form of A.

Step-by-Step Procedure

Step 1: Reduce A to Reduced Row Echelon Form (RREF)

Use Gaussian elimination with back-substitution to obtain the **reduced row echelon form** R of A.

Since elementary row operations are invertible and preserve the solution set,

$$A\mathbf{x} = \mathbf{0} \quad \Longleftrightarrow \quad R\mathbf{x} = \mathbf{0}.$$

Step 2: Identify Pivot and Free Columns

In R:

- A **pivot column** contains a leading 1 (the first nonzero entry in its row).
- A **free column** has no pivot.

Let:

- $r = \text{rank}(A) =$ number of pivot columns,
- $n - r =$ number of free columns.

The variables corresponding to:

- **Pivot columns** are called **basic variables**,
- **Free columns** are called **free variables**.

The free variables can be assigned arbitrary values; the basic variables are then uniquely determined.

Step 3: Solve $R\mathbf{x} = \mathbf{0}$ in Terms of Free Variables

Write the system $R\mathbf{x} = \mathbf{0}$ as equations. Because R is in RREF, each nonzero row gives an equation of the form

$$x_{\text{pivot}} + \sum_{\text{free } j > \text{pivot}} r_{ij} x_j = 0,$$

so

$$x_{\text{pivot}} = - \sum_{\text{free } j > \text{pivot}} r_{ij} x_j.$$

Thus, every basic variable is expressed as a linear combination of the free variables.

Step 4: Construct Special Solutions (Basis Vectors)

For each free variable x_{f_k} (where $k = 1, 2, \ldots, n - r$), construct a vector $\mathbf{v}_k \in \mathbb{R}^n$ as follows:

(a) Set $x_{f_k} = 1$.

(b) Set all other free variables to 0.

(c) Use the equations from Step 3 to compute the values of all basic variables.

(d) Assemble the full vector $\mathbf{v}_k = (x_1, x_2, \ldots, x_n)^\top$.

Each $\mathbf{v}_k$ is called a **special solution** corresponding to the free variable x_{f_k}.

Step 5: The Set $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-r}\}$ is a Basis

- **Linear independence**: In the subvector corresponding to free variables, $\mathbf{v}_k$ has a 1 in position f_k and 0 elsewhere. Hence, no nontrivial linear combination can be zero.
- **Spanning**: Any solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{0}$ is determined by its free variable values $c_1, \ldots, c_{n-r}$, and then

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_{n-r}\mathbf{v}_{n-r}.$$

Therefore, this set is a basis for $\mathrm{Nul}(A)$.

Worked Example

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 1 & 2 & 3 \end{bmatrix}.$$

Step 1: Reduce to RREF

$$A \xrightarrow{\text{row reduce}} R = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Step 2: Identify pivot and free columns

- Pivot columns: 1 and 2 $\Rightarrow$ basic variables: x_1, x_2,
- Free columns: 3 and 4 $\Rightarrow$ free variables: x_3, x_4,
- Rank $r = 2$, so $\dim(\mathrm{Nul}(A)) = 4 - 2 = 2$.

Step 3: Solve $R\mathbf{x} = \mathbf{0}$

From R:

$$\begin{cases} x_1 + x_3 + 2x_4 = 0 \\ x_2 + x_3 + x_4 = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} x_1 = -x_3 - 2x_4 \\ x_2 = -x_3 - x_4 \end{cases}$$

Step 4: Construct special solutions

- For $x_3 = 1$, $x_4 = 0$:

$$x_1 = -1, \quad x_2 = -1 \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

- For $x_3 = 0$, $x_4 = 1$:

$$x_1 = -2, \quad x_2 = -1 \quad \Rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Left Null Space

Let A be an $m \times n$ real matrix. The **left null space** of A is defined as

$$\mathrm{Nul}(A^\top) = \{\mathbf{y} \in \mathbb{R}^m : A^\top \mathbf{y} = \mathbf{0}\} = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y}^\top A = \mathbf{0}^\top\}.$$

Its dimension is $m - r$, where $r = \mathrm{rank}(A)$. We now describe a systematic method to compute a basis for this subspace using Gaussian elimination.

Method: Using the Elimination Matrix

The key idea is to record the row operations used to reduce A to reduced row echelon form (RREF). This is done by augmenting A with the identity matrix and performing simultaneous row reduction.

Step 1: Form the augmented matrix

Construct the $m \times (n + m)$ matrix

$$[\, A \mid I_m \,],$$

where I_m is the $m \times m$ identity matrix.

Step 2: Row-reduce to RREF

Apply Gaussian elimination (with back-substitution to achieve reduced form) to the left block, performing the same elementary row operations on the entire augmented matrix. The result is

$$[\, R \mid E \,],$$

where:

- R is the reduced row echelon form of A,
- E is an $m \times m$ invertible matrix satisfying $EA = R$.

Step 3: Determine the rank

Let $r = \operatorname{rank}(A)$ be the number of nonzero rows in R. By the definition of RREF, R has the block structure

$$R = \begin{bmatrix} R_1 \\ \mathbf{0} \end{bmatrix},$$

where R_1 is $r \times n$ and $\mathbf{0}$ is $(m - r) \times n$.

Step 4: Partition E

Write E in conformal block form:

$$E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix},$$

where E_1 is $r \times m$ and E_2 is $(m - r) \times m$ (the last $m - r$ rows of E).

Since $EA = R$, we have

$$\begin{bmatrix} E_1 A \\ E_2 A \end{bmatrix} = \begin{bmatrix} R_1 \\ \mathbf{0} \end{bmatrix} \quad \Longrightarrow \quad E_2 A = \mathbf{0}.$$

Step 5: Interpret $E_2 A = \mathbf{0}$

The equation $E_2 A = \mathbf{0}$ means that each row $\mathbf{y}^\top$ of E_2 satisfies

$$\mathbf{y}^\top A = \mathbf{0}^\top \quad \Longleftrightarrow \quad A^\top \mathbf{y} = \mathbf{0}.$$

Thus, every row of E_2 (as a column vector) lies in $\operatorname{Nul}(A^\top)$.

Step 6: Conclude a basis

- E_2 has $m - r$ rows.
- Since E is invertible, its rows are linearly independent; hence the rows of E_2 are linearly independent.
- $\dim(\mathrm{Nul}(A^\top)) = m - r$ (by the Rank–Nullity Theorem applied to $A^\top$).

Therefore, the rows of E_2, when written as column vectors, form a basis for $\mathrm{Nul}(A^\top)$.

Worked Example

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 2 \end{bmatrix}.$$

Step 1: Form $[A \mid I_3]$:

$$[\, A \mid I_3 \,] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 6 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right].$$

Step 2: Row-reduce to RREF:

$$[\, R \mid E \,] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -2 & 1 & 0 \end{array}\right].$$

Step 3: Rank $r = 2$, so $m - r = 1$.
Step 4: The last row of E is $[-2\ 1\ 0]$.
Step 5: Verify:

$$[-2\ 1\ 0]A = [0\ 0\ 0].$$

Step 6: A basis for $\mathrm{Nul}(A^\top)$ is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Summary of the Procedure

To find a basis for $\mathrm{Nul}(A^\top)$:

1. Form the augmented matrix $[\,A \mid I_m\,]$.
2. Row-reduce the left block to RREF, applying the same operations to the right block to obtain $[\,R \mid E\,]$.
3. Let $r = \mathrm{rank}(A)$ (number of nonzero rows in R).
4. Extract the last $m - r$ rows of E.
5. Transpose each of these rows to column vectors; the resulting set is a basis for $\mathrm{Nul}(A^\top)$.

Remark 3. *The left null space consists of all linear combinations of the rows of A that yield the zero vector. The last $m - r$ rows of E encode precisely those combinations, which is why they form a basis.*

Remark 4. *Do **not** use the zero rows of R—they are trivial. The useful information is stored in the corresponding rows of E.*

Step 5: Basis for the Left Null Space $\mathrm{Nul}(A^\top)$

Let A be an $m \times n$ real matrix. The **left null space** of A is defined as

$$\mathrm{Nul}(A^\top) = \{\mathbf{y} \in \mathbb{R}^m : A^\top \mathbf{y} = \mathbf{0}\} = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y}^\top A = \mathbf{0}^\top\}.$$

Its dimension is $m - r$, where $r = \mathrm{rank}(A)$. We now describe a systematic method to compute a basis for this subspace using Gaussian elimination.

Remark 5. *The left null space consists of all linear combinations of the rows of A that yield the zero vector. The last $m - r$ rows of E encode precisely those combinations, which is why they form a basis.*

Remark 6. *Do **not** use the zero rows of R—they are trivial. The useful information is stored in the corresponding rows of E.*