Independent Events in Two-Dice Experiments

Two events are said to be *independent* if:

$$P(A \cap B) = P(A)P(B)$$

Mutually exclusive events are never independent. In this section we look at dice rolling experiments which are both dependent and independent.

The key point is that when events or *not* independent the conditional probability doesn't equal the "raw" probability:

$$P(B|A) \neq P(B)$$

In other words, knowing that B occured gives you some predictive knowledge over whether A will occur. The more general formula is:

$$P(A \cap B) = P(A)P(B|A)$$

Example 1: Sum Equals 7 and First Die is 6 [Independent]

Consider rolling two fair six-sided dice. Define the events:

$$A = \{\text{sum of the dice is } 7\} = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

$$B = \{\text{first die shows } 6\} = \{(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

Step 1: Probabilities

$$P(A) = \frac{6}{36} = \frac{1}{6}, \quad P(B) = \frac{6}{36} = \frac{1}{6}$$

$$A \cap B = \{(6, 1)\} \implies P(A \cap B) = \frac{1}{36}$$

Step 2: Check Independence

Two events are independent if:

$$P(A \cap B) = P(A)P(B)$$

Compute:

$$P(A)P(B) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = P(A \cap B)$$

Thus, A and B are **independent**.

Step 3: Intuition

Knowing that the first die is 6 does not change the probability that the sum of the dice is 7. - Original probability: $P(A) = 1/6$ - Conditional probability: $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{6/36} = 1/6$

Since $P(A \mid B) = P(A)$, the events are independent.

Example 2: Sum is Even and First Die is Even [Independent]

Consider rolling two fair six-sided dice. Define the events:

$$A = \{\text{sum of the dice is even}\}, \quad B = \{\text{first die shows even}\}$$

Step 1: Probabilities

There are 36 equally likely outcomes. Half of them have an even sum:

$$P(A) = \frac{18}{36} = 0.5$$

Half of the outcomes have an even first die:

$$P(B) = \frac{18}{36} = 0.5$$

The intersection is outcomes where first die is even and sum is even:

$$A \cap B = \{(2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6)\}$$

$$P(A \cap B) = \frac{9}{36} = 0.25$$

Step 2: Check Independence

$$P(A)P(B) = 0.5 \cdot 0.5 = 0.25 = P(A \cap B)$$

So A and B are **independent**.

Step 3: Intuition

- Event A depends on the parity of the sum. - Event B fixes the parity of the first die. - Whether the sum is even depends only on whether the two dice have the same parity. - If the first die is even, the sum is even if the second die is even (50% chance). - If the first die is odd, the sum is even if the second die is odd (50% chance).

In both cases, $P(A \mid B) = P(A) = 0.5$. Knowing that the first die is even does not change the probability that the sum is even.

Example 3: Sum Even and Sum Greater than 7 [Dependent]

Consider rolling two fair six-sided dice. Define the events:

$$A = \{\text{sum of the dice is even}\}, \quad B = \{\text{sum of the dice is greater than 7}\}$$

Step 1: Probabilities

- There are 36 equally likely outcomes in total. - Number of outcomes with an even sum: 18

$$P(A) = \frac{18}{36} = 0.5$$

- Number of outcomes with sum greater than 7: 15

$$P(B) = \frac{15}{36} \approx 0.4167$$

- Intersection (sum even **and** sum > 7): sums 8, 10, 12 $\rightarrow$ 9 outcomes

$$P(A \cap B) = \frac{9}{36} = 0.25$$

Step 2: Check Independence

Two events are independent if:

$$P(A \cap B) = P(A)P(B)$$

Compute:

$$P(A)P(B) = 0.5 \cdot 0.4167 = 0.2083 \neq 0.25$$

Conclusion: A and B are **not independent**.

Step 3: Conditional Probability and Intuition

The conditional probability of A given B is:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.25}{0.4167} \approx 0.6$$

- Original probability of sum even: $P(A) = 0.5$ - Probability of sum even given sum > 7: $P(A \mid B) \approx 0.6$

Intuition:

- Event A depends on the parity of the sum. - Event B restricts outcomes to sums greater than 7. - Among sums > 7 (8, 9, 10, 11, 12), there are proportionally more even sums (8, 10, 12) than odd sums (9, 11). - Therefore, knowing that B occurred increases the probability of A, which is exactly why A and B are **dependent**.

Example 4: At Least One 6 and Sum Greater Than 10 [Dependent]

Consider rolling two fair six-sided dice. Define the events:

$$C = \{\text{at least one die shows } 6\}, \quad D = \{\text{sum of the dice is greater than } 10\}$$

Probabilities

- Total outcomes: 36 - Outcomes with at least one 6: 11

$$P(C) = \frac{11}{36} \approx 0.3056$$

- Outcomes with sum > 10: 3

$$P(D) = \frac{3}{36} \approx 0.0833$$

- Intersection (at least one 6 and sum > 10): 3 outcomes

$$P(C \cap D) = \frac{3}{36} = 0.0833$$

Check Independence

Two events are independent if $P(C \cap D) = P(C)P(D)$.

$$P(C)P(D) = 0.3056 \cdot 0.0833 \approx 0.0254 \neq 0.0833$$

Conclusion: C and D are **not independent**.

Conditional Probability and Intuition

$$P(D \mid C) = \frac{P(C \cap D)}{P(C)} = \frac{0.0833}{0.3056} \approx 0.2727$$

- Original probability of sum > 10: $P(D) = 0.0833$ - Probability of sum > 10 given at least one die is 6: $P(D \mid C) \approx 0.2727$

Intuition:

- Event D requires a high sum (> 10). - Event C ensures at least one die is 6. - High sums are more likely if at least one die is 6, so knowing C occurred significantly increases the probability of D, showing dependence.