The Dual Space

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Hopefully the reader has seen the idea of the *dual space* before, otherwise this section will probably be a little confusing.

Let V be a finite dimensional vector space over the real numbers $\mathbb{R}$ The **dual space** of V, denoted V^*, is the set of all **linear functionals** on V. That is,

$$V^* = \{f : V \to \mathbb{R} \mid f \text{ is linear}\}.$$

A **linear functional** is a function $f : V \to \mathbb{R}$ such that for all vectors $\mathbf{u}, \mathbf{v} \in V$ and scalars $\alpha, \beta \in \mathbb{R}$,

$$f(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha f(\mathbf{u}) + \beta f(\mathbf{v}).$$

The dual space V^* is itself a vector space over $\mathbb{R}$, with vector addition and scalar multiplication defined *pointwise*:

$$(f + g)(\mathbf{v}) = f(\mathbf{v}) + g(\mathbf{v}),$$
$$(\alpha f)(\mathbf{v}) = \alpha\, f(\mathbf{v}),$$

for all $f, g \in V^*$, $\alpha \in \mathbb{R}$, and $\mathbf{v} \in V$.

Suppose that $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis for V, then there is a uniquely associated **dual basis** $\{\mathbf{e}^1, \ldots, \mathbf{e}^n\} \subseteq V^*$ defined by

$$\mathbf{e}^i(\mathbf{e}_j) = \delta^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Every linear functional $f \in V^*$ can be uniquely expressed as

$$f = \sum_{i=1}^{n} f(\mathbf{e}_i)\, \mathbf{e}^i.$$

In $\mathbb{R}^m$, the standard dot product allows us to associate every vector $\mathbf{v} \in \mathbb{R}^m$ with a linear functional $f_{\mathbf{v}} \in (\mathbb{R}^m)^*$ via the rule

$$f_{\mathbf{v}}(\mathbf{y}) = \mathbf{v} \cdot \mathbf{y} = \mathbf{v}^\top \mathbf{y}, \quad \text{for all } \mathbf{y} \in \mathbb{R}^m.$$

This map $\mathbf{v} \mapsto f_{\mathbf{v}}$ is:

- **Linear**: $f_{\alpha\mathbf{v}+\beta\mathbf{w}} = \alpha f_{\mathbf{v}} + \beta f_{\mathbf{w}}$,
- **Injective**: if $f_{\mathbf{v}} = 0$, then $\mathbf{v}^\top \mathbf{y} = 0$ for all $\mathbf{y}$, so in particular $\mathbf{v}^\top \mathbf{v} = \|\mathbf{v}\|^2 = 0$, hence $\mathbf{v} = \mathbf{0}$,
- **Surjective**: given any linear functional $f \in (\mathbb{R}^m)^*$, define $v_i = f(\mathbf{e}_i)$ where $\{\mathbf{e}_i\}$ is the standard basis; then $f(\mathbf{y}) = \sum_i v_i y_i = \mathbf{v}^\top \mathbf{y}$.

Thus, this correspondence is a vector space isomorphism:

$$\mathbb{R}^m \xrightarrow{\sim} (\mathbb{R}^m)^*, \quad \mathbf{v} \longmapsto f_{\mathbf{v}}.$$

Some intuition

It can sometimes help to think of vectors in V as column vectors and vectors in V^* as row vectors. The dot product is now the application of a linear functional to a vector. The transpose is the isomoprhism between V and V^*.

Lemma 1. $\dim V^* = \dim V$ *for finite-dimensional* V

Proof. Let V be a finite-dimensional vector space over a field $\mathbb{R}$, and let $\dim V = n$. Choose a basis $\{e_1, e_2, \ldots, e_n\}$ for V.

Define linear functionals $e^1, e^2, \ldots, e^n \in V^* = \mathrm{Hom}(V, \mathbb{R})$ by

$$e^i(e_j) = \delta^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \quad \text{for all } 1 \leq i, j \leq n.$$

We claim that $\{e^1, \ldots, e^n\}$ is a basis for V^*.

Linear independence: Suppose $\sum_{i=1}^n a_i e^i = 0$ in V^*, where $a_i \in \mathbb{R}$. Applying both sides to e_j gives

$$0 = \left(\sum_{i=1}^n a_i e^i \right) (e_j) = \sum_{i=1}^n a_i e^i(e_j) = \sum_{i=1}^n a_i \delta^i_j = a_j.$$

Thus $a_j = 0$ for all j, so the set $\{e^1, \ldots, e^n\}$ is linearly independent.

Spanning: Let $f \in V^*$ be arbitrary. Define scalars $c_i = f(e_i) \in \mathbb{R}$ for $i = 1, \ldots, n$, and consider the functional

$$g = \sum_{i=1}^n c_i e^i \in V^*.$$

For any basis vector e_j, we have

$$g(e_j) = \sum_{i=1}^n c_i e^i(e_j) = \sum_{i=1}^n c_i \delta^i_j = c_j = f(e_j).$$

Since f and g agree on a basis of V, they agree on all of V; hence $f = g$. Therefore, every $f \in V^*$ is a linear combination of $\{e^1, \ldots, e^n\}$, so this set spans V^*.

Since $\{e^1, \ldots, e^n\}$ is a basis for V^*, we conclude that

$$\dim V^* = n = \dim V.$$

□

The Adjoint

The reason why we have gone to all the trouble of introducing the dual space is that the transpose of a matrix is the natural matrix representation of the **adjoint** of a linear map.

The Adjoint of a Linear Map

Let $T : V \to W$ be a linear map between finite-dimensional real vector spaces. The **dual map** $T^* : W^* \to V^*$ is defined by

$$(T^* f)(\mathbf{v}) = f(T\mathbf{v}) \quad \text{for all } f \in W^*,\ \mathbf{v} \in V.$$

In words: to evaluate $T^* f$ at a vector $\mathbf{v}$, first apply T to $\mathbf{v}$, then apply the functional f to the result.

Theorem 1. *The matrix of the dual map T^* is the transpose of the matrix of T.*

Proof. Choose bases:

- $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for V,
- $\mathcal{C} = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ for W,
- $\mathcal{B}^* = \{\mathbf{v}^1, \ldots, \mathbf{v}^n\}$ and $\mathcal{C}^* = \{\mathbf{w}^1, \ldots, \mathbf{w}^m\}$ for the dual bases.

Suppose the matrix of T with respect to $\mathcal{B}, \mathcal{C}$ is $A = (a_{ij})$, so Then:

$$T(\mathbf{v}_1) = \sum_{i=1}^{m} A_{i1}\, \mathbf{w}_i$$

More generally:

$$T(\mathbf{v}_j) = \sum_{i=1}^{m} a_{ij} \mathbf{w}_i.$$

We compute the matrix of T^* with respect to $\mathcal{C}^*, \mathcal{B}^*$. For any $k \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$,

$$(T^* \mathbf{w}^k)(\mathbf{v}_j) = \mathbf{w}^k(T\mathbf{v}_j) = \mathbf{w}^k \left(\sum_{i=1}^{m} a_{ij} \mathbf{w}_i \right) = a_{kj}.$$

On the other hand, if the matrix of T^* is $B = (b_{\ell k})$, then

$$T^*(\mathbf{w}^1) = \sum_{\ell=1}^{n} b_{\ell 1}\,\mathbf{v}^\ell$$

More generally:

$$T^*(\mathbf{w}^k) = \sum_{\ell=1}^{n} b_{\ell k}\mathbf{v}^\ell,$$

so

$$(T^*\mathbf{w}^k)(\mathbf{v}_j) = b_{jk}.$$

Comparing both expressions gives $b_{jk} = a_{kj}$, so $B = A^\top$.

That is the matrix of the dual map T^* is the transpose of the matrix of T. $\square$

Recall from Chapter 1 that tahe key orthogonal decompositions are:

$$\mathbb{R}^n = \mathrm{Row}(A) \oplus \mathrm{Nul}(A), \quad \mathbb{R}^m = \mathrm{Col}(A) \oplus \mathrm{Nul}(A^\top),$$

or equivalently:

$$\mathrm{Nul}(A) = \mathrm{Row}(A)^\perp, \quad \mathrm{Nul}(A^\top) = \mathrm{Col}(A)^\perp.$$

2. What Is the Annihilator?

Let V be a finite-dimensional vector space, and let $W \subseteq V$ be a subspace. The **annihilator** of W is:

$$W^\circ = \{f \in V^* : f(\mathbf{w}) = 0 \text{ for all } \mathbf{w} \in W\},$$

where V^* is the **dual space** (the space of linear functionals on V).

Lemma 2.

$$\dim(W^\circ) = \dim(V) - \dim(W).$$

Proof. **Step 1: W° is a subspace of V^*.** Clearly $0 \in W^\circ$. If $f, g \in W^\circ$ and $\alpha, \beta \in \mathbb{R}$, then for any $\mathbf{w} \in W$,

$$(\alpha f + \beta g)(\mathbf{w}) = \alpha f(\mathbf{w}) + \beta g(\mathbf{w}) = 0,$$

so $\alpha f + \beta g \in W^\circ$. Hence $W^\circ \leq V^*$.

Step 2: Choose a basis adapted to W. Let $\dim V = n$ and $\dim W = k$. Choose a basis

$$\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$$

for W, and extend it to a basis of V:

$$\mathcal{B} = \{\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}.$$

Let $\mathcal{B}^* = \{\mathbf{w}^1, \ldots, \mathbf{w}^k, \mathbf{v}^{k+1}, \ldots, \mathbf{v}^n\}$ be the dual basis of V^*, so that

$$\mathbf{w}^i(\mathbf{w}_j) = \delta^i_j, \quad \mathbf{v}^i(\mathbf{v}_j) = \delta^i_j, \quad \mathbf{w}^i(\mathbf{v}_j) = 0, \quad \mathbf{v}^i(\mathbf{w}_j) = 0$$

for all valid indices.

Step 3: Characterize W° using the dual basis. Let $f \in V^*$. Write f in the dual basis:

$$f = \sum_{i=1}^{k} a_i \mathbf{w}^i + \sum_{j=k+1}^{n} b_j \mathbf{v}^j.$$

For any $\mathbf{w} \in W$, we have $\mathbf{w} = \sum_{i=1}^k c_i \mathbf{w}_i$, so

$$f(\mathbf{w}) = \sum_{i=1}^{k} a_i c_i.$$

Thus $f(\mathbf{w}) = 0$ for all $\mathbf{w} \in W$ if and only if $a_1 = \cdots = a_k = 0$.

Therefore,

$$W^\circ = \operatorname{span}\{\mathbf{v}^{k+1}, \ldots, \mathbf{v}^n\}.$$

Step 4: Compute the dimension. The set $\{\mathbf{v}^{k+1}, \ldots, \mathbf{v}^n\}$ is linearly independent and spans W°, so it is a basis. Hence

$$\dim(W^\circ) = n - k = \dim(V) - \dim(W).$$

$\square$

3. Connecting Annihilators to the Four Subspaces

Theorem 2.

$$\mathrm{Nul}(A^\top) \cong (\mathrm{Col}(A))^\circ, \qquad \mathrm{Nul}(A) \cong (\mathrm{Row}(A))^\circ,$$

where the annihilator is taken inside the appropriate dual space, and the isomorphism is the one induced by the dot product.

Proof. **Connection Between Null Space and Annihilator via the Dot Product**

Let A be an $m \times n$ real matrix. We will prove that, under the natural identification of vectors in $\mathbb{R}^m$ with linear functionals on $\mathbb{R}^m$ provided by the standard dot product, the null space of $A^\top$ corresponds exactly to the annihilator of the column space of A. That is,

$$\mathrm{Nul}(A^\top) \cong (\mathrm{Col}(A))^\circ.$$

We proceed step by step, explaining the meaning of each concept and how they relate.

Step 1: The dot product gives a canonical isomorphism $\mathbb{R}^m \cong (\mathbb{R}^m)^*$

Step 2: What is the annihilator $(\mathrm{Col}(A))^\circ$?

The **column space** of A is the subspace of $\mathbb{R}^m$ defined by

$$\mathrm{Col}(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m.$$

Its **annihilator** is the set of all linear functionals on $\mathbb{R}^m$ that vanish on every vector in $\mathrm{Col}(A)$:

$$(\mathrm{Col}(A))^\circ = \{f \in (\mathbb{R}^m)^* \mid f(\mathbf{y}) = 0 \text{ for all } \mathbf{y} \in \mathrm{Col}(A)\}.$$

Because every $\mathbf{y} \in \mathrm{Col}(A)$ can be written as $\mathbf{y} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$, we can rephrase this condition as:

$$f \in (\mathrm{Col}(A))^\circ \iff f(A\mathbf{x}) = 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

Step 3: Translate the annihilator condition using the dot product identification

Now consider a vector $\mathbf{v} \in \mathbb{R}^m$, and let $f_{\mathbf{v}}$ be the corresponding functional: $f_{\mathbf{v}}(\mathbf{y}) = \mathbf{v}^\top \mathbf{y}$. We ask: *When does $f_{\mathbf{v}}$ belong to* $(\mathrm{Col}(A))^\circ$*?*

By the characterization above, this happens precisely when

$$f_{\mathbf{v}}(A\mathbf{x}) = 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Substituting the definition of $f_{\mathbf{v}}$, this becomes:

$$\mathbf{v}^\top (A\mathbf{x}) = 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}^n. \tag{1}$$

Step 4: Rewrite the condition using properties of the transpose

The expression $\mathbf{v}^\top A\mathbf{x}$ is a scalar (a 1×1 matrix). Using the associative law for matrix multiplication and the identity $(XY)^\top = Y^\top X^\top$, we observe:

$$\mathbf{v}^\top A\mathbf{x} = (\mathbf{v}^\top A)\mathbf{x} = (A^\top \mathbf{v})^\top \mathbf{x}.$$

Thus, condition (1) is equivalent to:

$$(A^\top \mathbf{v})^\top \mathbf{x} = 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Step 5: A linear functional is zero everywhere iff its representing vector is zero

The map $\mathbf{x} \mapsto (A^\top \mathbf{v})^\top \mathbf{x}$ is a linear functional on $\mathbb{R}^n$. The only linear functional that vanishes on *every* vector in $\mathbb{R}^n$ is the zero functional. But under the same dot product identification in $\mathbb{R}^n$, the zero functional corresponds to the zero vector. Therefore, if this is true for all x then we must have:

$$(A^\top \mathbf{v})^\top \mathbf{x} = 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n \quad \Longleftrightarrow \quad A^\top \mathbf{v} = \mathbf{0}.$$

In other words, $\mathbf{v}$ must lie in the null space of $A^\top$.

Step 6: Conclude the correspondence

We have shown the following chain of equivalences:

$$\begin{aligned}
\mathbf{v} \in \mathrm{Nul}(A^\top) &\iff A^\top \mathbf{v} = \mathbf{0} \\
&\iff (A^\top \mathbf{v})^\top \mathbf{x} = 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n \\
&\iff \mathbf{v}^\top A\mathbf{x} = 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n \\
&\iff f_{\mathbf{v}}(A\mathbf{x}) = 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n \\
&\iff f_{\mathbf{v}} \in (\mathrm{Col}(A))^\circ.
\end{aligned}$$

Therefore, under the isomorphism $\mathbf{v} \leftrightarrow f_{\mathbf{v}}$, the subspace $\mathrm{Nul}(A^\top) \subseteq \mathbb{R}^m$ corresponds exactly to the subspace $(\mathrm{Col}(A))^\circ \subseteq (\mathbb{R}^m)^*$.

□

Geometric Interpretation

This result has a clean geometric meaning: a vector $\mathbf{v} \in \mathbb{R}^m$ is orthogonal (with respect to the dot product) to every vector in the column space of A if and only if $A^\top \mathbf{v} = \mathbf{0}$. But "orthogonal to the column space" is precisely what it means for the functional $f_{\mathbf{v}}(\mathbf{y}) = \mathbf{v} \cdot \mathbf{y}$ to vanish on $\mathrm{Col}(A)$ — i.e., to be in the annihilator. Thus, in $\mathbb{R}^m$, the annihilator $(\mathrm{Col}(A))^\circ$ is naturally identified with the orthogonal complement $\mathrm{Col}(A)^\perp$, and we recover the familiar fundamental theorem of linear algebra:

$$\mathrm{Nul}(A^\top) = \mathrm{Col}(A)^\perp.$$

Theorem 3. $\mathrm{Nul}(A) \cong (\mathrm{Row}(A))^\circ$

Proof. This follows immediately by symmetry. □

Conclusion. Via the standard dot product identification $\mathbb{R}^k \cong (\mathbb{R}^k)^*$, we have natural isomorphisms:

$$\boxed{\mathrm{Nul}(A^\top) \cong (\mathrm{Col}(A))^\circ}, \qquad \boxed{\mathrm{Nul}(A) \cong (\mathrm{Row}(A))^\circ}.$$

The Annihilator reveals that the four subspaces reflect a fundamental duality. The matrix $A : \mathbb{R}^n \to \mathbb{R}^m$ induces a **dual map** $A^* : (\mathbb{R}^m)^* \to (\mathbb{R}^n)^*$, and we have:

$$\ker(A^*) = (\operatorname{Im} A)^\circ, \qquad \operatorname{Im}(A^*) = (\ker A)^\circ.$$

When we identify $(\mathbb{R}^k)^* \cong \mathbb{R}^k$ via the dot product, these become:

$$\operatorname{Nul}(A^\top) = \operatorname{Col}(A)^\perp, \qquad \operatorname{Row}(A) = \operatorname{Nul}(A)^\perp.$$