Converse Proof: Symmetric Idempotent Matrices are Orthogonal Projections

Theorem

If a matrix $P \in \mathbb{R}^{n\times n}$ satisfies:

1. $P^T = P$ (symmetry)
2. $P^2 = P$ (idempotence)

then P is an orthogonal projection matrix.

Proof

Let P be a matrix satisfying $P^T = P$ and $P^2 = P$. We need to show that P represents an orthogonal projection, which means for any vector $\mathbf{x} \in \mathbb{R}^n$:

1. $P\mathbf{x}$ lies in some subspace $W \subseteq \mathbb{R}^n$
2. $\mathbf{x} - P\mathbf{x}$ is orthogonal to every vector in W

Step 1: Define the subspace W

Let $W = \text{range}(P) = \{P\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$, the column space of P.

For any $\mathbf{x} \in \mathbb{R}^n$, $P\mathbf{x} \in W$ by definition of W.

Step 2: Show $\mathbf{x} - P\mathbf{x}$ is orthogonal to W

We need to prove that for any $\mathbf{x} \in \mathbb{R}^n$ and any $\mathbf{y} \in W$, the following holds:

$$(\mathbf{x} - P\mathbf{x}) \cdot \mathbf{y} = 0$$

Since $\mathbf{y} \in W$, there exists some $\mathbf{z} \in \mathbb{R}^n$ such that $\mathbf{y} = P\mathbf{z}$.

Computing the dot product:

$$\begin{aligned}
(\mathbf{x} - P\mathbf{x}) \cdot \mathbf{y} &= (\mathbf{x} - P\mathbf{x})^T (P\mathbf{z}) \\
&= \mathbf{x}^T P\mathbf{z} - (P\mathbf{x})^T P\mathbf{z} \\
&= \mathbf{x}^T P\mathbf{z} - \mathbf{x}^T P^T P\mathbf{z} \\
&= \mathbf{x}^T P\mathbf{z} - \mathbf{x}^T P P\mathbf{z} \quad (\text{since } P^T = P) \\
&= \mathbf{x}^T P\mathbf{z} - \mathbf{x}^T P^2\mathbf{z} \\
&= \mathbf{x}^T P\mathbf{z} - \mathbf{x}^T P\mathbf{z} \quad (\text{since } P^2 = P) \\
&= 0
\end{aligned}$$

Therefore, $\mathbf{x} - P\mathbf{x}$ is orthogonal to every vector in W.

Step 3: Verify the decomposition

For any $\mathbf{x} \in \mathbb{R}^n$, we can write:

$$\mathbf{x} = P\mathbf{x} + (\mathbf{x} - P\mathbf{x})$$

where:

- $P\mathbf{x} \in W$
- $\mathbf{x} - P\mathbf{x}$ is orthogonal to W (as shown in Step 2)

This is precisely the defining property of an orthogonal projection.

Step 4: Confirm P is a projection

We also need to verify that P acts as a projection onto W. For any $\mathbf{w} \in W$, there exists $\mathbf{z}$ such that $\mathbf{w} = P\mathbf{z}$. Then:

$$P\mathbf{w} = P(P\mathbf{z}) = P^2\mathbf{z} = P\mathbf{z} = \mathbf{w}$$

So P fixes all vectors in W, as expected for a projection.

Conclusion

We have shown that if $P^T = P$ and $P^2 = P$, then for any vector $\mathbf{x}$:

- $P\mathbf{x}$ lies in the subspace $W = \text{range}(P)$
- $\mathbf{x} - P\mathbf{x}$ is orthogonal to W

Therefore, P represents an orthogonal projection onto its range.

$$\boxed{P^T = P \text{ and } P^2 = P \implies P \text{ is an orthogonal projection}}$$

Q.E.D.