The Four Fundamental Subspaces and Their Orthogonality Relations

In linear algebra, the **four fundamental subspaces** associated with a real matrix $A \in \mathbb{R}^{m\times n}$ provide deep insight into the structure of linear systems. These are:

1. Column Space ($\mathrm{Col}(A)$)
2. Null Space ($\mathrm{Nul}(A)$)
3. Row Space ($\mathrm{Row}(A)$)
4. Left Null Space ($\mathrm{Nul}(A^\top)$)

We will define each subspace and then rigorously prove the key orthogonality relations:

$$\boxed{\mathrm{Nul}(A) = \mathrm{Row}(A)^\perp} \quad \text{and} \quad \boxed{\mathrm{Nul}(A^\top) = \mathrm{Col}(A)^\perp}$$

1. Definitions of the Four Fundamental Subspaces

Let A be an $m \times n$ real matrix.

1.1 Column Space

The column space of A is the set of all linear combinations of its columns:

$$\mathrm{Col}(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

- Dimension: $\mathrm{rank}(A) = r$ - Basis: Pivot columns of A

1.2 Null Space

The null space of A is the set of vectors mapped to zero:

$$\mathrm{Nul}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^n$$

- Dimension: $n-r$ (Rank-Nullity Theorem) - Basis: Found by solving $A\mathbf{x} = \mathbf{0}$

1.3 Row Space

The row space is the span of the rows of A, or equivalently, the column space of $A^\top$:

$$\mathrm{Row}(A) = \mathrm{Col}(A^\top) \subseteq \mathbb{R}^n$$

- Dimension: r - Basis: Nonzero rows in the reduced row echelon form (RREF) of A

1.4 Left Null Space

The left null space consists of vectors $\mathbf{y}$ such that $A^\top\mathbf{y} = \mathbf{0}$:

$$\mathrm{Nul}(A^\top) = \left\{\mathbf{y} \in \mathbb{R}^m \mid A^\top\mathbf{y} = \mathbf{0}\right\} \subseteq \mathbb{R}^m$$

- Dimension: $m - r$ - Basis: Found via elimination on $A^\top$, or from bottom rows of E in $EA = R$

2. Orthogonality Relations

We now prove the two fundamental orthogonal complement relationships.

2.1 $\mathrm{Nul}(A) = \mathrm{Row}(A)^\perp$

We show that a vector $\mathbf{x} \in \mathbb{R}^n$ satisfies $A\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x}$ is orthogonal to every vector in the row space of A.

Let the rows of A be $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m \in \mathbb{R}^n$. Then:

$$A\mathbf{x} = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{x} \\ \mathbf{r}_2 \cdot \mathbf{x} \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{x} \end{bmatrix}$$

Thus,

$$A\mathbf{x} = \mathbf{0} \iff \mathbf{r}_i \cdot \mathbf{x} = 0 \quad \text{for all } i = 1, \ldots, m$$

Any vector $\mathbf{v} \in \text{Row}(A)$ is a linear combination:

$$\mathbf{v} = c_1\mathbf{r}_1 + \cdots + c_m\mathbf{r}_m$$

Then:

$$\mathbf{v} \cdot \mathbf{x} = \sum_{i=1}^{m} c_i(\mathbf{r}_i \cdot \mathbf{x}) = 0$$

So $\mathbf{x} \perp \mathbf{v}$, hence $\mathbf{x} \in \text{Row}(A)^\perp$.

Conversely, if $\mathbf{x} \in \text{Row}(A)^\perp$, then in particular $\mathbf{x} \perp \mathbf{r}_i$ for each row $\mathbf{r}_i$, so $A\mathbf{x} = \mathbf{0}$, meaning $\mathbf{x} \in \text{Nul}(A)$.

Therefore:

$$\boxed{\text{Nul}(A) = \text{Row}(A)^\perp}$$

2.2 $\text{Nul}(A^\top) = \text{Col}(A)^\perp$

Now let $\mathbf{y} \in \mathbb{R}^m$. Consider $A^\top\mathbf{y}$. The j-th entry of $A^\top\mathbf{y}$ is:

$$(A^\top\mathbf{y})_j = (\text{column } j \text{ of } A) \cdot \mathbf{y} = \mathbf{a}_j \cdot \mathbf{y}$$

So:

$$A^\top\mathbf{y} = \mathbf{0} \iff \mathbf{a}_j \cdot \mathbf{y} = 0 \quad \text{for all } j = 1, \ldots, n$$

That is, $\mathbf{y}$ is orthogonal to every column of A.

Any vector $\mathbf{w} \in \mathrm{Col}(A)$ is of the form:

$$\mathbf{w} = d_1\mathbf{a}_1 + \cdots + d_n\mathbf{a}_n$$

Then:

$$\mathbf{y} \cdot \mathbf{w} = \sum_{j=1}^{n} d_j(\mathbf{y} \cdot \mathbf{a}_j) = 0$$

So $\mathbf{y} \perp \mathbf{w}$, hence $\mathbf{y} \in \mathrm{Col}(A)^\perp$.

Conversely, if $\mathbf{y} \in \mathrm{Col}(A)^\perp$, then $\mathbf{y} \cdot \mathbf{a}_j = 0$ for all j, so $A^\top\mathbf{y} = \mathbf{0}$, so $\mathbf{y} \in \mathrm{Nul}(A^\top)$.

Thus:

$$\boxed{\mathrm{Nul}(A^\top) = \mathrm{Col}(A)^\perp}$$

3. Summary

The four fundamental subspaces satisfy the following orthogonal decompositions:

$$\mathbb{R}^n = \mathrm{Row}(A) \oplus \mathrm{Nul}(A), \qquad \mathbb{R}^m = \mathrm{Col}(A) \oplus \mathrm{Nul}(A^\top)$$

where $\oplus$ denotes an orthogonal direct sum.

These results are central to the **Fundamental Theorem of Linear Algebra**, illustrating how a matrix partitions the domain and codomain into mutually orthogonal invariant subspaces.