# Orthogonal Projection Matrices and Symmetry

Let $P \in \mathbb{R}^{n \times n}$ be a projection matrix, meaning $P^2 = P$. We say $P$ is an **orthogonal projection** if it projects vectors orthogonally onto its column space. We will use the theory of the **four fundamental subspaces** of a matrix to prove the following key result:

**A projection matrix $P$ is orthogonal if and only if $P^T = P$.**

Recall the four fundamental subspaces of a matrix $P$:

- Column space: $\mathrm{Col}(P)$

- Null space: $\mathrm{Nul}(P)$

- Row space: $\mathrm{Row}(P)$

- Left null space: $\mathrm{Nul}(P^T)$

For any matrix, we have the orthogonal decompositions:

$$\mathbb{R}^n = \mathrm{Col}(P) \oplus \mathrm{Nul}(P^T), \quad \mathbb{R}^n = \mathrm{Row}(P) \oplus \mathrm{Nul}(P)$$

and since $\mathrm{Row}(P) = \mathrm{Col}(P^T)$, the symmetry condition $P^T = P$ implies $\mathrm{Row}(P) = \mathrm{Col}(P)$.

Now, suppose $P$ is a projection, so $P^2 = P$. This means that for any $\mathbf{x} \in \mathbb{R}^n$, we can write:

$$\mathbf{x} = P\mathbf{x} + (\mathbf{x} - P\mathbf{x})$$

where $P\mathbf{x} \in \mathrm{Col}(P)$ and $\mathbf{x} - P\mathbf{x} \in \mathrm{Nul}(P)$, because:

$$P(\mathbf{x} - P\mathbf{x}) = P\mathbf{x} - P^2\mathbf{x} = P\mathbf{x} - P\mathbf{x} = \mathbf{0}.$$

Thus, $\mathbb{R}^n = \mathrm{Col}(P) \oplus \mathrm{Nul}(P)$, and $P$ projects $\mathbf{x}$ onto $\mathrm{Col}(P)$ along $\mathrm{Nul}(P)$.

**Definition:** The projection $P$ is **orthogonal** if $\mathrm{Nul}(P) \perp \mathrm{Col}(P)$. That is, the projection direction is orthogonal to the range.

We now prove the main result in two parts.

$(\Rightarrow)$ Suppose $P$ is an orthogonal projection. Then $\mathrm{Nul}(P) \perp \mathrm{Col}(P)$.

Since $\mathbb{R}^n = \mathrm{Col}(P) \oplus \mathrm{Nul}(P)$ and the sum is orthogonal, we have:

$$\mathrm{Nul}(P) = \mathrm{Col}(P)^{\perp}.$$

But from the fundamental theorem of linear algebra,

$$\mathrm{Col}(P)^{\perp} = \mathrm{Nul}(P^T).$$

Therefore,

$$\mathrm{Nul}(P) = \mathrm{Nul}(P^T).$$

Also, since $P$ is a projection, $\mathrm{Col}(P)$ consists of all vectors $\mathbf{y}$ such that $P\mathbf{y} = \mathbf{y}$. Similarly, $\mathrm{Col}(P^T)$ contains vectors fixed by $P^T$ if $P^T$ is also idempotent, but more importantly, we use subspace equality.

We already have $\mathrm{Nul}(P) = \mathrm{Nul}(P^T)$. Taking orthogonal complements:

$$\mathrm{Nul}(P)^{\perp} = \mathrm{Nul}(P^T)^{\perp} \implies \mathrm{Col}(P^T) = \mathrm{Col}(P),$$

since $\mathrm{Col}(P) = \mathrm{Nul}(P)^{\perp}$ and $\mathrm{Col}(P^T) = \mathrm{Nul}(P^T)^{\perp}$.

So $\mathrm{Col}(P) = \mathrm{Col}(P^T)$ and $\mathrm{Nul}(P) = \mathrm{Nul}(P^T)$.

Now, consider that both $P$ and $P^T$ are projections (note: $(P^T)^2 = (P^2)^T = P^T$, so $P^T$ is also idempotent). Both project onto the same column space $\text{Col}(P)$, and along the same null space $\text{Nul}(P)$. Since a projection is uniquely determined by its range and null space (as complementary subspaces), and here both $P$ and $P^T$ have:

- Range: $\text{Col}(P)$

- Null space: $\text{Nul}(P)$ it follows that $P = P^T$. Hence, $P^T = P$.

($\Leftarrow$) Conversely, suppose $P^T = P$ and $P^2 = P$. We show $P$ is an orthogonal projection.

Since $P$ is symmetric, $\text{Col}(P) = \text{Row}(P)$, and from the fundamental theorem:

$$\text{Nul}(P) = \text{Row}(P)^\perp = \text{Col}(P)^\perp.$$

Thus, $\text{Nul}(P) \perp \text{Col}(P)$, which means the projection is along the orthogonal complement of the column space. Therefore, $P$ is an orthogonal projection.

**Conclusion:** A projection matrix $P$ (i.e., $P^2 = P$) is an orthogonal projection if and only if $P^T = P$.