Example: Grades Distribution with Median < Mean and More Than Half Below Average

Consider the grades of 11 students (out of 100):

$$40,\ 45,\ 50,\ 55,\ 60,\ 62,\ 65,\ 70,\ 75,\ 80,\ 100$$

Median

Since there are 11 observations (odd number), the median is the 6th value:

$$\text{Median} = 62$$

Mean

The sum of the grades is:

$$40+45+50+55+60+62+65+70+75+80+100 = 702$$

Thus, the mean is:

$$\text{Mean} = \frac{702}{11} \approx 63.82$$

Comparison

$$\text{Median} = 62 \quad < \quad \text{Mean} \approx 63.82$$

Students Below Average

Grades below the mean (< 63.82) are:

$$40,\ 45,\ 50,\ 55,\ 60,\ 62$$

This is 6 out of 11 students, or

$$\frac{6}{11} \approx 54.5\,\%$$

Hence, **more than half** of the students scored below the average.

1 The Mean as a Balance Point: An Analogy with Torque in Physics

In descriptive statistics, the **mean** of a data set is often described as its "center of mass" or "balance point." This is not merely a figure of speech—it reflects a deep mathematical analogy with classical mechanics, particularly the concept of **torque** and rotational equilibrium. In this note, we elaborate on this connection, with special attention to the physical meaning of torque and how outliers in data behave like distant masses on a lever.

2 Torque in Classical Mechanics

In physics, **torque** (denoted $\boldsymbol{\tau}$) quantifies the tendency of a force to cause rotation about an axis or pivot point.

The torque vector $\boldsymbol{\tau}$ is defined as the cross product of the position vector $\mathbf{r}$ and the force vector $\mathbf{F}$:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

where:

- $\mathbf{r}$ is the displacement vector from the pivot (axis of rotation) to the point of application of the force,

- $\mathbf{F}$ is the applied force vector,
- $\times$ denotes the vector cross product.

For a point mass or a force applied in a plane, the magnitude of torque is given by:

$$\tau = rF \sin\theta$$

where:

- r is the distance from the pivot (fulcrum) to the point where the force is applied (the *lever arm*),
- F is the magnitude of the applied force,
- θ is the angle between the force vector $\mathbf{F}$ and the position vector $\mathbf{r}$.

In the common case where the force is applied *perpendicular* to the lever arm (e.g., gravity acting downward on a horizontal seesaw), $\theta = 90^\circ$ and $\sin\theta = 1$, so the magnitude of torque simplifies to:

$$\tau = rF$$

However, torque is a vector quantity. Its direction is given by the right-hand rule applied to the cross product:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

Consider a horizontal seesaw along the x-axis with pivot at the origin:

- If a downward force $\mathbf{F} = -F\,\hat{\mathbf{y}}$ is applied at a point on the **right** ($\mathbf{r} = +r\,\hat{\mathbf{x}}$, $r > 0$), then

$$\boldsymbol{\tau} = (r\,\hat{\mathbf{x}}) \times (-F\,\hat{\mathbf{y}}) = -rF\,(\hat{\mathbf{x}} \times \hat{\mathbf{y}}) = -rF\,\hat{\mathbf{z}}$$

The torque points in the $-\hat{\mathbf{z}}$ direction (into the page), corresponding to **clockwise** rotation.

- If the same downward force is applied on the **left** ($\mathbf{r} = -r\,\hat{\mathbf{x}}$, $r > 0$), then

  $$\boldsymbol{\tau} = (-r\,\hat{\mathbf{x}}) \times (-F\,\hat{\mathbf{y}}) = +rF\,\hat{\mathbf{z}}$$

  The torque points in the $+\hat{\mathbf{z}}$ direction (out of the page), corresponding to **counterclockwise** rotation.

In two-dimensional problems, we often treat torque as a signed scalar:

$$\tau = \begin{cases} > 0 & \text{(counterclockwise)} \\ < 0 & \text{(clockwise)} \end{cases}$$

If the force arises from gravity acting on a mass m, then $F = mg$, where $g \approx 9.8\,\mathrm{m\,s^{-2}}$ is the acceleration due to gravity. Thus:

$$\tau = r \cdot mg$$

Since g is constant, the torque is proportional to the product rm, often called the **first moment of mass** about the pivot.

2.1 Rotational Equilibrium

A rigid body is in **rotational equilibrium** when the net torque about the pivot is zero:

$$\sum \tau_i = 0$$

For a seesaw with multiple masses m_i placed at positions x_i along a horizontal beam, and assuming the pivot is at position x_0, the torque due to mass i is:

$$\tau_i = (x_i - x_0) \cdot m_i g$$

(We adopt the sign convention that torques causing counterclockwise rotation are positive, and clockwise are negative; this is captured by the sign of $x_i - x_0$.)

Setting the total torque to zero:

$$\sum_{i=1}^{n} (x_i - x_0) m_i g = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} (x_i - x_0) m_i = 0$$

Solving for the equilibrium pivot position x_0 gives:

$$x_0 = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

This is precisely the formula for the **center of mass**. If all masses are equal (e.g., $m_i = 1$ for all i), then:

$$x_0 = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x}$$

Thus, the **arithmetic mean** $\bar{x}$ is the center of mass of n equal point masses located at the data values x_i.

3 Statistical Interpretation: Mean vs. Median

In statistics, for a data set $\{x_1, x_2, \ldots, x_n\}$, the mean $\bar{x}$ satisfies:

$$\sum_{i=1}^{n} (x_i - \bar{x}) = 0$$

This is mathematically identical to the torque equilibrium condition with unit masses. Each deviation $(x_i - \bar{x})$ acts like a signed lever arm, and the sum of these "torques" vanishes.

By contrast, the **median** is defined solely by order: it is the middle value when data are sorted. It does *not* depend on the magnitude of deviations—only on counts. Hence, the median is unaffected by how far an outlier lies, just as the "middle person" on a line doesn't move if someone at the end steps farther away.

4 The Role of Outliers: A Torque Analogy

Consider a class of 11 students with grades:

$$40,\ 45,\ 50,\ 55,\ 60,\ 62,\ 65,\ 70,\ 75,\ 80,\ 100$$

Most grades cluster between 40 and 80, but the score of 100 is an **outlier** on the high end.

In the mechanical analogy:

- Each grade is a unit mass at position x_i on a number line.
- The outlier at $x = 100$ is far from the cluster, giving it a large lever arm relative to any pivot near the center.
- Even though its mass is the same as the others, its **torque** $(x_i - x_0) \cdot 1$ is large in magnitude.
- To achieve torque balance $(\sum \tau_i = 0)$, the fulcrum (mean) must shift *toward* the outlier.

Computationally:

$$\bar{x} = \frac{702}{11} \approx 63.82, \quad \text{Median} = 62$$

Thus, Median < Mean, and 6 of 11 students (54.5%) score below the mean.

This rightward shift of the mean mirrors how a heavy child sitting far out on a seesaw forces the pivot to move toward them to maintain balance.

5 Connection to Statistical Moments

The analogy extends beyond the first moment. In physics, the k-th moment of mass about a point is $\sum m_i r_i^k$. In statistics:

- The **first moment about zero** is $\frac{1}{n}\sum x_i = \bar{x}$ (mean).
- The **second central moment** is $\frac{1}{n}\sum(x_i - \bar{x})^2$ (variance), analogous to rotational inertia.

Thus, the term "moment" in statistics is borrowed directly from mechanics—underscoring that this is more than metaphor; it is a shared mathematical framework.

6 Conclusion

The statement:

> "Outliers pull the mean like a heavy weight on a lever creates torque, shifting the balance point."

is a physically grounded analogy. The mean is the point of zero net torque for unit masses at data locations. Outliers exert disproportionate influence because torque depends on *distance*, not just mass. This explains why the mean is sensitive to extreme values, while the median—being purely ordinal—is not.

Understanding this connection enriches both statistical intuition and physical reasoning.

7 Why Does Torque "Point Out of the Page"?

Torque is a vector, but unlike force or velocity, its direction does not indicate motion *in* that direction. Instead, the direction of the torque vector encodes the **axis** and **sense** (clockwise or counterclockwise) of rotation. This is done using the **right-hand rule**, and in two-dimensional problems, we often say torque "points out of" or "into" the page. This is a *convention*—not a physical displacement.

8 The Right-Hand Rule

To find the direction of the torque vector $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$:

1. Point the fingers of your **right hand** in the direction of the position vector $\mathbf{r}$ (from pivot to point of force).
2. Curl your fingers toward the direction of the force $\mathbf{F}$ (through the smaller angle).
3. Your extended **thumb** now points in the direction of $\boldsymbol{\tau}$.

In a planar (2D) situation—like a seesaw lying flat on the page—the rotation occurs in the xy-plane. The torque vector is therefore perpendicular to this plane, i.e., along the z-axis.

9 Representing 3D Directions on a 2D Page

Since we draw on flat paper, we use special symbols to represent vectors perpendicular to the page:

- $\odot$: Arrow **coming out of the page** (toward you). Think of the dot as the tip of an arrow pointing at your eye.
- $\otimes$: Arrow **going into the page** (away from you). Think of the cross as the tail feathers of an arrow flying away.

These correspond to:

$$\odot \leftrightarrow +\hat{\mathbf{z}} \quad \text{(out of page)}, \qquad \otimes \leftrightarrow -\hat{\mathbf{z}} \quad \text{(into page)}.$$

10 Example: A Horizontal Seesaw

Consider a seesaw along the x-axis, with pivot at the origin. Gravity acts downward ($-\hat{\mathbf{y}}$).

- **Mass on the left** ($x < 0$): The seesaw rotates **counterclockwise**. Right-hand rule $\rightarrow$ thumb points **out of the page**.

$$\boldsymbol{\tau} = +|\tau|\,\hat{\mathbf{z}} \quad \text{or} \quad \tau = \odot$$

- **Mass on the right** ($x > 0$): The seesaw rotates **clockwise**. Right-hand rule $\rightarrow$ thumb points **into the page**.

$$\boldsymbol{\tau} = -|\tau|\,\hat{\mathbf{z}} \quad \text{or} \quad \tau = \otimes$$

11 Why Use This Convention?

We could simply say "clockwise" or "counterclockwise" in 2D—but in three dimensions, rotation can occur about *any axis*. Representing torque as a vector:

- Allows us to use vector addition (e.g., multiple torques on a rigid body),
- Makes rotational dynamics consistent with linear dynamics (e.g., $\boldsymbol{\tau} = I\boldsymbol{\alpha}$),
- Unambiguously specifies both axis and direction of rotation.

Key Takeaway

> **Torque does not physically push out of the page.** The vector direction is a mathematical convention that encodes the axis and sense of rotation via the right-hand rule.