Orthogonal Projection Proof

Definition

A matrix $P \in \mathbb{R}^{n \times n}$ is an **orthogonal projection** if for any $\mathbf{x} \in \mathbb{R}^n$:

1. $P\mathbf{x}$ is in the subspace W
2. $\mathbf{x} - P\mathbf{x}$ is orthogonal to W
3. $P^2 = P$ (idempotent)

Proof that $P^T = P$

Let P be an orthogonal projection matrix. For any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$:

Since $P\mathbf{u} \in W$ and $\mathbf{v} - P\mathbf{v} \in W^\perp$, their dot product is zero:

$$(P\mathbf{u}) \cdot (\mathbf{v} - P\mathbf{v}) = 0$$

In matrix form (using $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T\mathbf{b}$):

$$(P\mathbf{u})^T(\mathbf{v} - P\mathbf{v}) = 0$$

Expand the expression:

$$\mathbf{u}^T P^T \mathbf{v} - \mathbf{u}^T P^T P \mathbf{v} = 0$$

Factor out $\mathbf{u}^T$ and $\mathbf{v}$:

$$\mathbf{u}^T (P^T - P^T P)\mathbf{v} = 0$$

Since this holds for **all** $\mathbf{u}, \mathbf{v}$, we must have:

$$P^T - P^T P = 0 \quad \Rightarrow \quad P^T = P^T P \quad (1)$$

Now consider the reverse case: $\mathbf{u} - P\mathbf{u} \in W^\perp$ and $P\mathbf{v} \in W$, so:

$$(\mathbf{u} - P\mathbf{u}) \cdot (P\mathbf{v}) = 0$$

In matrix form:

$$(\mathbf{u} - P\mathbf{u})^T (P\mathbf{v}) = 0$$

Expand:

$$\mathbf{u}^T P\mathbf{v} - (P\mathbf{u})^T P\mathbf{v} = 0$$

$$\mathbf{u}^T P\mathbf{v} - \mathbf{u}^T P^T P\mathbf{v} = 0$$

$$\mathbf{u}^T (P - P^T P)\mathbf{v} = 0$$

Since this holds for all $\mathbf{u}, \mathbf{v}$:

$$P - P^T P = 0 \quad \Rightarrow \quad P = P^T P \quad (2)$$

From (1) and (2):

$$P^T = P^T P = P$$

Therefore:

$$\boxed{P^T = P}$$

Q.E.D.