# Why $\mathbf{u}_1\mathbf{u}_1^\top$ is the Orthogonal Projection Matrix

## Goal

Let $\mathbf{u}_1 \in \mathbb{R}^n$ be a unit vector (i.e., $\|\mathbf{u}_1\| = 1$). We want to find the matrix $P$ that orthogonally projects any vector $\mathbf{x} \in \mathbb{R}^n$ onto the line spanned by $\mathbf{u}_1$.

## Geometric Definition of Orthogonal Projection

The orthogonal projection $\hat{\mathbf{x}} = P\mathbf{x}$ must satisfy:

1. $\hat{\mathbf{x}}$ lies on the line: $\hat{\mathbf{x}} = c\mathbf{u}_1$ for some scalar $c$,
2. The error $\mathbf{x} - \hat{\mathbf{x}}$ is perpendicular to the line: $(\mathbf{x} - \hat{\mathbf{x}})^\top \mathbf{u}_1 = 0$.

## Deriving the Projection Formula

From condition (2):

$$(\mathbf{x} - c\mathbf{u}_1)^\top \mathbf{u}_1 = 0 \quad \Rightarrow \quad \mathbf{x}^\top \mathbf{u}_1 - c\,\mathbf{u}_1^\top \mathbf{u}_1 = 0.$$

Since $\mathbf{u}_1$ is a unit vector, $\mathbf{u}_1^\top \mathbf{u}_1 = 1$, so

$$c = \mathbf{u}_1^\top \mathbf{x}.$$

Thus, the projection is

$$\hat{\mathbf{x}} = (\mathbf{u}_1^\top \mathbf{x})\,\mathbf{u}_1.$$

## Expressing as a Matrix Multiplication

We now seek a matrix $P$ such that $\hat{\mathbf{x}} = P\mathbf{x}$ for all $\mathbf{x}$. Rewrite the expression using associativity of matrix multiplication:

$$\hat{\mathbf{x}} = (\mathbf{u}_1^\top \mathbf{x})\,\mathbf{u}_1 = \mathbf{u}_1(\mathbf{u}_1^\top \mathbf{x}) = (\mathbf{u}_1\mathbf{u}_1^\top)\mathbf{x}.$$

Since this holds for every $\mathbf{x}$, we identify

$$P = \mathbf{u}_1\mathbf{u}_1^\top.$$

### Verifying Projection Properties

- **Idempotent**:
$$P^2 = (\mathbf{u}_1\mathbf{u}_1^\top)(\mathbf{u}_1\mathbf{u}_1^\top) = \mathbf{u}_1(\mathbf{u}_1^\top\mathbf{u}_1)\mathbf{u}_1^\top = \mathbf{u}_1(1)\mathbf{u}_1^\top = P.$$
- **Symmetric**:
$$P^\top = (\mathbf{u}_1\mathbf{u}_1^\top)^\top = \mathbf{u}_1\mathbf{u}_1^\top = P.$$

  Thus, $P$ is an orthogonal projection matrix.

### Geometric Interpretation

The matrix $\mathbf{u}_1\mathbf{u}_1^\top$ is the **outer product** of $\mathbf{u}_1$ with itself. - The inner product $\mathbf{u}_1^\top\mathbf{x}$ gives the scalar coordinate along $\mathbf{u}_1$, - The outer product $\mathbf{u}_1\mathbf{u}_1^\top$ converts this into a linear transformation that projects any vector onto the line.

### Example: 2D Projection

Let $\mathbf{u}_1 = \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}$. Then

$$P = \mathbf{u}_1\mathbf{u}_1^\top = \begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta\end{bmatrix},$$

the standard orthogonal projection matrix onto a line at angle $\theta$.

### Conclusion

The matrix $\mathbf{u}_1\mathbf{u}_1^\top$ is the unique orthogonal projection matrix onto the line spanned by the unit vector $\mathbf{u}_1$. It arises naturally from the geometric definition of orthogonal projection and satisfies all the required algebraic properties.

## 1 Higher Dimensions

### Theorem

Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a $k$-dimensional subspace, and let $U \in \mathbb{R}^{n\times k}$ be a matrix whose columns $\mathbf{u}_1, \ldots, \mathbf{u}_k$ form an orthonormal basis for $\mathcal{S}$ (i.e., $U^\top U = I_k$).

Then the matrix

$$P = UU^\top$$

is the unique orthogonal projection matrix onto $\mathcal{S}$.

## Proof

We verify that $P$ satisfies the defining properties of an orthogonal projection onto $\mathcal{S}$.

### 1. $P\mathbf{x} \in \mathcal{S}$ for all $\mathbf{x} \in \mathbb{R}^n$

For any $\mathbf{x} \in \mathbb{R}^n$,

$$P\mathbf{x} = UU^\top \mathbf{x} = U(U^\top \mathbf{x}).$$

Let $\mathbf{z} = U^\top \mathbf{x} \in \mathbb{R}^k$. Then

$$P\mathbf{x} = U\mathbf{z} = z_1\mathbf{u}_1 + z_2\mathbf{u}_2 + \cdots + z_k\mathbf{u}_k,$$

which is a linear combination of the basis vectors of $\mathcal{S}$. Hence, $P\mathbf{x} \in \mathcal{S}$.

### 2. The error $\mathbf{x} - P\mathbf{x}$ is orthogonal to $\mathcal{S}$

We show $\mathbf{x} - P\mathbf{x} \perp \mathbf{v}$ for every $\mathbf{v} \in \mathcal{S}$. Since $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ spans $\mathcal{S}$, it suffices to show orthogonality to each basis vector $\mathbf{u}_i$.

Compute the inner product with $\mathbf{u}_i$:

$$\mathbf{u}_i^\top (\mathbf{x} - P\mathbf{x}) = \mathbf{u}_i^\top \mathbf{x} - \mathbf{u}_i^\top UU^\top \mathbf{x}.$$

Note that $\mathbf{u}_i^\top U$ is the $i$-th row of $U^\top U = I_k$, so $\mathbf{u}_i^\top U = \mathbf{e}_i^\top$, where $\mathbf{e}_i$ is the $i$-th standard basis vector in $\mathbb{R}^k$. Thus,

$$\mathbf{u}_i^\top UU^\top \mathbf{x} = \mathbf{e}_i^\top U^\top \mathbf{x} = (U^\top \mathbf{x})_i = \mathbf{u}_i^\top \mathbf{x}.$$

Therefore,

$$\mathbf{u}_i^\top (\mathbf{x} - P\mathbf{x}) = \mathbf{u}_i^\top \mathbf{x} - \mathbf{u}_i^\top \mathbf{x} = 0.$$

Since this holds for all $i = 1, \ldots, k$, the error $\mathbf{x} - P\mathbf{x}$ is orthogonal to every vector in $\mathcal{S}$.

**3.** $P$ **is symmetric and idempotent**

- **Symmetric**: $P^\top = (UU^\top)^\top = UU^\top = P$.
- **Idempotent**: $P^2 = (UU^\top)(UU^\top) = U(U^\top U)U^\top = UI_kU^\top = UU^\top = P$, where we used the orthonormality condition $U^\top U = I_k$.

**4. Uniqueness**

Suppose $Q$ is another matrix such that $Q\mathbf{x} \in \mathcal{S}$ and $\mathbf{x} - Q\mathbf{x} \perp \mathcal{S}$ for all $\mathbf{x}$. Since $Q\mathbf{x} \in \mathcal{S}$, we can write $Q\mathbf{x} = U\mathbf{w}$ for some $\mathbf{w} \in \mathbb{R}^k$. Orthogonality implies $U^\top(\mathbf{x} - Q\mathbf{x}) = \mathbf{0}$, so

$$U^\top\mathbf{x} - U^\top U\mathbf{w} = \mathbf{0} \quad \Rightarrow \quad U^\top\mathbf{x} - I_k\mathbf{w} = \mathbf{0} \quad \Rightarrow \quad \mathbf{w} = U^\top\mathbf{x}.$$

Thus, $Q\mathbf{x} = UU^\top\mathbf{x} = P\mathbf{x}$ for all $\mathbf{x}$, so $Q = P$.

## Conclusion

The matrix $P = UU^\top$ is the unique orthogonal projection matrix onto the subspace $\mathcal{S} \subseteq \mathbb{R}^n$, for any dimension $k = \dim(\mathcal{S}) \geq 1$.