When is a Projection Orthogonal?
The Role of Symmetry

## Introduction

In linear algebra, a **projection** is a linear transformation that "sends" vectors onto a subspace and leaves vectors in that subspace unchanged. Formally, a square matrix $P$ is a **projection** if

$$P^2 = P.$$

This means that applying the projection twice is the same as applying it once—once you're on the subspace, you stay there.

But not all projections are created equal. Some are **orthogonal** (they drop a perpendicular from a vector to the subspace), while others are **oblique** (they slide along a slanted direction).

The central question is:

*How can we tell, just by looking at the matrix $P$, whether the projection is orthogonal?*

The answer is beautiful and simple:

**Theorem 1** *Let $P$ be an $n \times n$ real matrix such that $P^2 = P$ (i.e., $P$ is a projection). Then $P$ represents an* ***orthogonal projection*** *if and only if $P$ is* ***symmetric****, meaning*

$$P^\top = P.$$

We now prove this in two parts, with clear geometric reasoning.

## Part 1: If $P$ is symmetric and a projection, then it is orthogonal

Assume that $P^2 = P$ and $P^\top = P$. We want to show that the projection is orthogonal.

What does "orthogonal projection" mean geometrically? . It means that for any vector $\mathbf{x}$, the error $\mathbf{x} - P\mathbf{x}$ is perpendicular to the projected vector $P\mathbf{x}$. Equivalently, the entire **column space** of $P$ (the subspace we're projecting onto) is perpendicular to the **null space** of $P$ (the directions along which we project).

### Statement

Let $P$ be an $n \times n$ real matrix that is a **projection**, meaning $P^2 = P$. For any vector $\mathbf{x} \in \mathbb{R}^n$, define the **error** (or **residual**) as

$$\mathbf{e} = \mathbf{x} - P\mathbf{x}.$$

Then $\mathbf{e} \in \mathrm{Nul}(P)$; that is, $P\mathbf{e} = \mathbf{0}$.

### Proof

We compute $P\mathbf{e}$ directly:

$$P\mathbf{e} = P(\mathbf{x} - P\mathbf{x}) = P\mathbf{x} - P(P\mathbf{x}) = P\mathbf{x} - P^2\mathbf{x}.$$

Since $P$ is a projection, $P^2 = P$. Substituting this in gives

$$P\mathbf{e} = P\mathbf{x} - P\mathbf{x} = \mathbf{0}.$$

Therefore, $\mathbf{e} \in \mathrm{Nul}(P)$.

### Intuition

A projection $P$ leaves vectors in its column space unchanged: if $\mathbf{y} \in \mathrm{Col}(P)$, then $P\mathbf{y} = \mathbf{y}$. The error $\mathbf{e} = \mathbf{x} - P\mathbf{x}$ is the part of $\mathbf{x}$ that is "left over" after removing the component that lies in $\mathrm{Col}(P)$. Since this leftover part is annihilated by $P$ (i.e., sent to zero), it must belong to the null space of $P$.

### Example

Let

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Then

$$P\mathbf{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \mathbf{e} = \mathbf{x} - P\mathbf{x} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

Applying $P$ to the error:

$$P\mathbf{e} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so indeed $\mathbf{e} \in \mathrm{Nul}(P)$.

## Conclusion

This simple identity—$P(\mathbf{x} - P\mathbf{x}) = \mathbf{0}$—is a direct consequence of the defining property $P^2 = P$. It underpins the fundamental decomposition

$$\mathbb{R}^n = \mathrm{Col}(P) \oplus \mathrm{Nul}(P)$$

that holds for every projection matrix $P$.

Let's verify this.

Take any vector $\mathbf{u}$ in the column space of $P$. Then $\mathbf{u} = P\mathbf{a}$ for some vector $\mathbf{a}$. Take any vector $\mathbf{v}$ in the null space of $P$. Then $P\mathbf{v} = \mathbf{0}$.

Now compute their dot product:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^\top \mathbf{v} = (P\mathbf{a})^\top \mathbf{v}.$$

Using the property of the transpose, $(P\mathbf{a})^\top = \mathbf{a}^\top P^\top$, so

$$(P\mathbf{a})^\top \mathbf{v} = \mathbf{a}^\top P^\top \mathbf{v}.$$

But we assumed $P^\top = P$, so this becomes

$$\mathbf{a}^\top P\mathbf{v}.$$

And since $\mathbf{v}$ is in the null space, $P\mathbf{v} = \mathbf{0}$, so

$$\mathbf{a}^\top \mathbf{0} = 0.$$

Thus, every vector in the column space is perpendicular to every vector in the null space. This is exactly what it means for the projection to be orthogonal.

**Part 2: If $P$ is an orthogonal projection, then it is symmetric**

Now assume that $P$ is a projection ($P^2 = P$) and that it is **orthogonal**. This means, by definition, that

$$\mathrm{Col}(P) \perp \mathrm{Nul}(P).$$

We want to show that $P^\top = P$. A standard way to prove two matrices are equal is to show that they give the same result when used in a dot product. Specifically, we will show that for *all* vectors $\mathbf{x}$ and $\mathbf{y}$,

$$\mathbf{x}^\top P\mathbf{y} = \mathbf{x}^\top P^\top \mathbf{y}.$$

Since $\mathbf{x}^\top P^\top \mathbf{y} = (P\mathbf{x})^\top \mathbf{y} = \mathbf{y}^\top P\mathbf{x}$, this is equivalent to showing

$$\mathbf{x}^\top P\mathbf{y} = \mathbf{y}^\top P\mathbf{x} \quad \text{for all } \mathbf{x}, \mathbf{y}.$$

To see why this is true, decompose both $\mathbf{x}$ and $\mathbf{y}$ using the projection:

$$\mathbf{x} = P\mathbf{x} + (\mathbf{x} - P\mathbf{x}), \quad \mathbf{y} = P\mathbf{y} + (\mathbf{y} - P\mathbf{y}).$$

Because $P$ is a projection, $P\mathbf{x}$ lies in the column space, and $\mathbf{x} - P\mathbf{x}$ lies in the null space (since $P(\mathbf{x} - P\mathbf{x}) = P\mathbf{x} - P^2\mathbf{x} = \mathbf{0}$). The same holds for $\mathbf{y}$.

Now, because the projection is *orthogonal*, the column space and null space are perpendicular. So: - $P\mathbf{x} \perp (\mathbf{y} - P\mathbf{y})$, - $(\mathbf{x} - P\mathbf{x}) \perp P\mathbf{y}$, - and $(\mathbf{x} - P\mathbf{x}) \perp (\mathbf{y} - P\mathbf{y})$ (though we won't need this).

Now compute $\mathbf{x}^\top P\mathbf{y}$:

$$\mathbf{x}^\top P\mathbf{y} = \big(P\mathbf{x} + (\mathbf{x} - P\mathbf{x})\big)^\top P\mathbf{y} = (P\mathbf{x})^\top P\mathbf{y} + (\mathbf{x} - P\mathbf{x})^\top P\mathbf{y}.$$

But $(\mathbf{x} - P\mathbf{x}) \perp P\mathbf{y}$, so the second term is zero. Thus,

$$\mathbf{x}^\top P\mathbf{y} = (P\mathbf{x})^\top P\mathbf{y}.$$

Similarly, compute $\mathbf{y}^\top P\mathbf{x}$:

$$\mathbf{y}^\top P\mathbf{x} = (P\mathbf{y})^\top P\mathbf{x} = (P\mathbf{x})^\top P\mathbf{y},$$

since the dot product is symmetric: $\mathbf{a}^\top \mathbf{b} = \mathbf{b}^\top \mathbf{a}$.

Therefore,

$$\mathbf{x}^\top P\mathbf{y} = \mathbf{y}^\top P\mathbf{x} \quad \text{for all } \mathbf{x}, \mathbf{y}.$$

As noted earlier, this implies $P = P^\top$.

## Conclusion

We have shown both directions:

- If $P$ is a symmetric projection ($P^2 = P$ and $P^\top = P$), then it is an orthogonal projection.
- If $P$ is an orthogonal projection, then it must be symmetric.

Hence, symmetry ($P^\top = P$) is the precise algebraic condition that captures the geometric idea of orthogonal projection.

This result beautifully ties together:

- **Algebra**: the equation $P^\top = P$,
- **Geometry**: perpendicularity of range and null space,
- **Linear algebra**: the structure of projections and the four fundamental subspaces.

## Linear Regression

Suppose you are a scientist running experiments.

- You perform $m$ experiments (e.g., measuring plant growth under different conditions).
- In each experiment, you record $n$ **features** (inputs), such as sunlight, water, etc.
- You also measure one **outcome** (response), like final height. You believe the outcome is approximately a linear combination of the features:

$$\text{Outcome} \approx \beta_1 \cdot (\text{feature}_1) + \cdots + \beta_n \cdot (\text{feature}_n).$$

Your goal is to find the best coefficients $\beta_1, \ldots, \beta_n$.

## The Design Matrix

Organize your data into:

- The **design matrix** $X$ (size $m \times n$): each row is an experiment, each column is a feature.

- The **outcome vector** $\mathbf{y} \in \mathbb{R}^m$: the measured responses.
- The **coefficient vector** $\boldsymbol{\beta} \in \mathbb{R}^n$: the unknown weights to be estimated. We wish to solve $X\boldsymbol{\beta} = \mathbf{y}$. But if $m > n$ (more experiments than features), this system is **overdetermined**—there is usually no exact solution because $\mathbf{y}$ does not lie in the column space of $X$.

### Orthogonal Projection to the Rescue

Instead, we find the vector $\hat{\mathbf{y}} \in \mathrm{Col}(X)$ that is **closest** to $\mathbf{y}$ in Euclidean distance. Geometrically, this is the **orthogonal projection** of $\mathbf{y}$ onto $\mathrm{Col}(X)$.

The error $\mathbf{e} = \mathbf{y} - \hat{\mathbf{y}}$ must be perpendicular to $\mathrm{Col}(X)$, which means:

$$X^\top(\mathbf{y} - \hat{\mathbf{y}}) = \mathbf{0}.$$

Since $\hat{\mathbf{y}} = X\hat{\boldsymbol{\beta}}$, this gives the **normal equation**:

$$X^\top X\hat{\boldsymbol{\beta}} = X^\top \mathbf{y}.$$

Assuming $X$ has full column rank, $X^\top X$ is invertible, so:

$$\hat{\boldsymbol{\beta}} = (X^\top X)^{-1}X^\top \mathbf{y}.$$

Thus, the projected vector is:

$$\hat{\mathbf{y}} = X(X^\top X)^{-1}X^\top \mathbf{y} = P\mathbf{y},$$

where

$$P = X(X^\top X)^{-1}X^\top$$

is the **orthogonal projection matrix** onto $\mathrm{Col}(X)$. It satisfies $P^2 = P$ and $P^\top = P$.

### Why This Matters

- $\hat{\boldsymbol{\beta}}$ tells you how much each feature contributes to the outcome.
- The residuals $\mathbf{e} = \mathbf{y} - \hat{\mathbf{y}}$ represent unexplained variation.
- Because the projection is orthogonal, $\hat{\mathbf{y}} \perp \mathbf{e}$, so the explained and unexplained parts are uncorrelated—a key property in statistics.

### Numerical Example

We wish to predict house prices based on size (in square feet) using a linear model:

$$\text{Price} = \beta_0 + \beta_1 \cdot \text{Size}.$$

We have data from 4 houses:

| House | Size (sq ft) | Price ($) |
|---|---|---|
| 1 | 1000 | 200,000 |
| 2 | 1500 | 300,000 |
| 3 | 2000 | 400,000 |
| 4 | 2500 | 550,000 |

### Step 1: Set up the design matrix and outcome vector

To include an intercept $\beta_0$, we add a column of 1s to the design matrix:

$$X = \begin{bmatrix} 1 & 1000 \\ 1 & 1500 \\ 1 & 2000 \\ 1 & 2500 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 200{,}000 \\ 300{,}000 \\ 400{,}000 \\ 550{,}000 \end{bmatrix}.$$

### Step 2: Compute $X^\top X$

$$X^\top X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1000 & 1500 & 2000 & 2500 \end{bmatrix} \begin{bmatrix} 1 & 1000 \\ 1 & 1500 \\ 1 & 2000 \\ 1 & 2500 \end{bmatrix} = \begin{bmatrix} 4 & 7000 \\ 7000 & 13{,}500{,}000 \end{bmatrix}.$$

### Step 3: Compute $X^\top \mathbf{y}$

$$X^\top \mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1000 & 1500 & 2000 & 2500 \end{bmatrix} \begin{bmatrix} 200{,}000 \\ 300{,}000 \\ 400{,}000 \\ 550{,}000 \end{bmatrix} = \begin{bmatrix} 1{,}450{,}000 \\ 2{,}825{,}000{,}000 \end{bmatrix}.$$

### Step 4: Solve the normal equation

We solve $(X^\top X)\boldsymbol{\beta} = X^\top \mathbf{y}$ for $\boldsymbol{\beta} = \begin{bmatrix}\beta_0 \\ \beta_1\end{bmatrix}$.

The inverse of $X^\top X$ is:

$$(X^\top X)^{-1} = \frac{1}{(4)(13{,}500{,}000) - (7000)^2} \begin{bmatrix} 13{,}500{,}000 & -7000 \\ -7000 & 4 \end{bmatrix} = \frac{1}{5{,}000{,}000} \begin{bmatrix} 13{,}500{,}000 & - \\ -7000 & \end{bmatrix}$$

Thus,

$$\boldsymbol{\beta} = (X^\top X)^{-1} X^\top \mathbf{y} = \frac{1}{5{,}000{,}000} \begin{bmatrix} 13{,}500{,}000 & -7000 \\ -7000 & 4 \end{bmatrix} \begin{bmatrix} 1{,}450{,}000 \\ 2{,}825{,}000{,}000 \end{bmatrix} = \begin{bmatrix} -40{,}000 \\ 230 \end{bmatrix}.$$

So the best-fit line is:

$$\widehat{\text{Price}} = -40{,}000 + 230 \cdot \text{Size}.$$

### Step 5: Compute predictions and residuals

| Size | Actual Price | Predicted Price | Residual |
|---|---|---|---|
| 1000 | 200,000 | $-40{,}000 + 230(1000) = 190{,}000$ | $+10{,}000$ |
| 1500 | 300,000 | $-40{,}000 + 230(1500) = 305{,}000$ | $-5{,}000$ |
| 2000 | 400,000 | $-40{,}000 + 230(2000) = 420{,}000$ | $-20{,}000$ |
| 2500 | 550,000 | $-40{,}000 + 230(2500) = 535{,}000$ | $+15{,}000$ |

The residual vector is

$$\mathbf{e} = \begin{bmatrix} 10{,}000 \\ -5{,}000 \\ -20{,}000 \\ 15{,}000 \end{bmatrix}.$$

### Step 6: Verify orthogonality

For the projection to be orthogonal, the residuals must be perpendicular to both columns of $X$.

- Dot product with first column of $X$ (intercept):

$$\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} \mathbf{e} = 10{,}000 - 5{,}000 - 20{,}000 + 15{,}000 = 0.$$

- Dot product with second column of $X$ (size):

  $\begin{bmatrix}1000 & 1500 & 2000 & 2500\end{bmatrix} \mathbf{e} = 1000(10{,}000) + 1500(-5{,}000) + 2000(-20{,}000) + 2500(15{,}$

  Both dot products are zero, confirming that the residuals are orthogonal to $\mathrm{Col}(X)$. This verifies that the least-squares solution is indeed the orthogonal projection of $\mathbf{y}$ onto the column space of $X$.