Exercise 3

In this exercise, you will analyse a dataset obtained from the London transport system (TfL). The data is in a filled called `tfl_readership.csv` (comma-separated-values format). As in Exercise 2, we will load and view the data using `pandas`.

```
# If you are running this on Google Colab, uncomment and run the following lines; otherwise ignore this cell
# from google.colab import drive
# drive.mount('/content/drive')

import math
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
```

```
/var/folders/ds/qvg7h1s93x16vvczwp8vr83m0000gn/T/
ipykernel_56364/188978642.py:4: DeprecationWarning:
Pyarrow will become a required dependency of pandas in the next major
release of pandas (pandas 3.0),
(to allow more performant data types, such as the Arrow string type,
and better interoperability with other libraries)
but was not found to be installed on your system.
If this would cause problems for you,
please provide us feedback at
https://github.com/pandas-dev/pandas/issues/54466

  import pandas as pd
```

```
# Load data
df_tfl = pd.read_csv('tfl_ridership.csv')
# If running on Google Colab change path to
'/content/drive/MyDrive/IB-Data-Science/Exercises/tfl_ridership.csv'

df_tfl.head(13)
```

```
       Year Period       Start          End Days  Bus cash (000s)  \
0   2000/01   P 01  01 Apr '00  29 Apr '00  29d              884
1   2000/01   P 02  30 Apr '00  27 May '00  28d              949
2   2000/01   P 03  28 May '00  24 Jun '00  28d              945
3   2000/01   P 04  25 Jun '00  22 Jul '00  28d              981
4   2000/01   P 05  23 Jul '00  19 Aug '00  28d              958
5   2000/01   P 06  20 Aug '00  16 Sep '00  28d              984
6   2000/01   P 07  17 Sep '00  14 Oct '00  28d             1001
7   2000/01   P 08  15 Oct '00  11 Nov '00  28d              979
8   2000/01   P 09  12 Nov '00  09 Dec '00  28d              971
9   2000/01   P 10  10 Dec '00  06 Jan '01  28d              912
10  2000/01   P 11  07 Jan '01  03 Feb '01  28d              943
```

```
11  2000/01   P 12  04 Feb '01  03 Mar '01  28d              975
12  2000/01   P 13  04 Mar '01  31 Mar '01  28d              974

    Bus Oyster PAYG (000s)  Bus Contactless (000s)  \
0                        0                       0
1                        0                       0
2                        0                       0
3                        0                       0
4                        0                       0
5                        0                       0
6                        0                       0
7                        0                       0
8                        0                       0
9                        0                       0
10                       0                       0
11                       0                       0
12                       0                       0

    Bus One Day Bus Pass (000s)  Bus Day Travelcard (000s)  ...  \
0                           210                        231  ...
1                           214                        205  ...
2                           209                        221  ...
3                           216                        241  ...
4                           225                        248  ...
5                           243                        236  ...
6                           205                        216  ...
7                           199                        221  ...
8                           184                        212  ...
9                           192                        211  ...
10                          193                        186  ...
11                          194                        210  ...
12                          186                        204  ...

    Tube Contactless (000s)  Tube Day Travelcard (000s)  \
0                         0                         655
1                         0                         605
2                         0                         650
3                         0                         708
4                         0                         730
5                         0                         702
6                         0                         639
7                         0                         668
8                         0                         640
9                         0                         631
10                        0                         556
11                        0                         617
12                        0                         584

    Tube Season Travelcard (000s)  Tube Other incl free (000s)  \
0                            1066                          200
```

1	1168	217
2	1154	212
3	1196	214
4	1165	165
5	1164	151
6	1286	196
7	1298	220
8	1302	242
9	993	195
10	1259	234
11	1237	246
12	1262	266

	Tube Total (000s)	TfL Rail (000s)	Overground (000s)	DLR (000s)
\				
0	2509	0	0	96
1	2598	0	0	93
2	2623	0	0	98
3	2761	0	0	105
4	2643	0	0	103
5	2608	0	0	100
6	2763	0	0	107
7	2819	0	0	113
8	2839	0	0	114
9	2359	0	0	90
10	2634	0	0	110
11	2688	0	0	120
12	2699	0	0	119

	Tram (000s)	Air Line (000s)
0	45.8	0.0
1	46.5	0.0
2	47.1	0.0
3	50.8	0.0
4	50.3	0.0
5	49.2	0.0
6	48.8	0.0
7	51.5	0.0

```
8          54.0              0.0
9          55.3              0.0
10         50.1              0.0
11         50.5              0.0
12         47.7              0.0

[13 rows x 26 columns]
```

Each row of our data frame represents the average daily ridership over a 28/29 day period for various types of transport and tickets (bus, tube etc.). We have used the `.head()` command to display the top 13 rows of the data frame (corresponding to one year). Focusing on the "Tube Total" column, notice the dip in ridership in row 9 (presumably due to Christmas/New Year's), and also the slight dip during the summer (rows 4,5).

```
#df_tfl.sample(3)  #random sample of 3 rows
df_tfl.tail(3)  #last 3 rows

        Year Period       Start         End Days  Bus cash (000s)  \
242  2018/19   P 09  11 Nov '18  08 Dec '18  28d                0
243  2018/19   P 10  09 Dec '18  05 Jan '19  28d                0
244  2018/19   P 11  06 Jan '19  02 Feb '19  28d                0

     Bus Oyster PAYG (000s)  Bus Contactless (000s)  \
242                    1110                    1089
243                    1001                     949
244                    1036                    1075

     Bus One Day Bus Pass (000s)  Bus Day Travelcard (000s)  ...  \
242                            0                         41  ...
243                            0                         38  ...
244                            0                         30  ...

     Tube Contactless (000s)  Tube Day Travelcard (000s)  \
242                     1399                         249
243                     1110                         242
244                     1310                         204

     Tube Season Travelcard (000s)  Tube Other incl free (000s)  \
242                           1017                          334
243                            632                          259
244                            924                          305

     Tube Total (000s)  TfL Rail (000s)  Overground (000s)  DLR (000s)
\
242               4221              996                557         355

243               3279              750                414         270

244               3809              929                517         333
```

```
     Tram (000s)  Air Line (000s)
242         84.1              2.6
243         66.3              3.2
244         79.3              2.3

[3 rows x 26 columns]
```

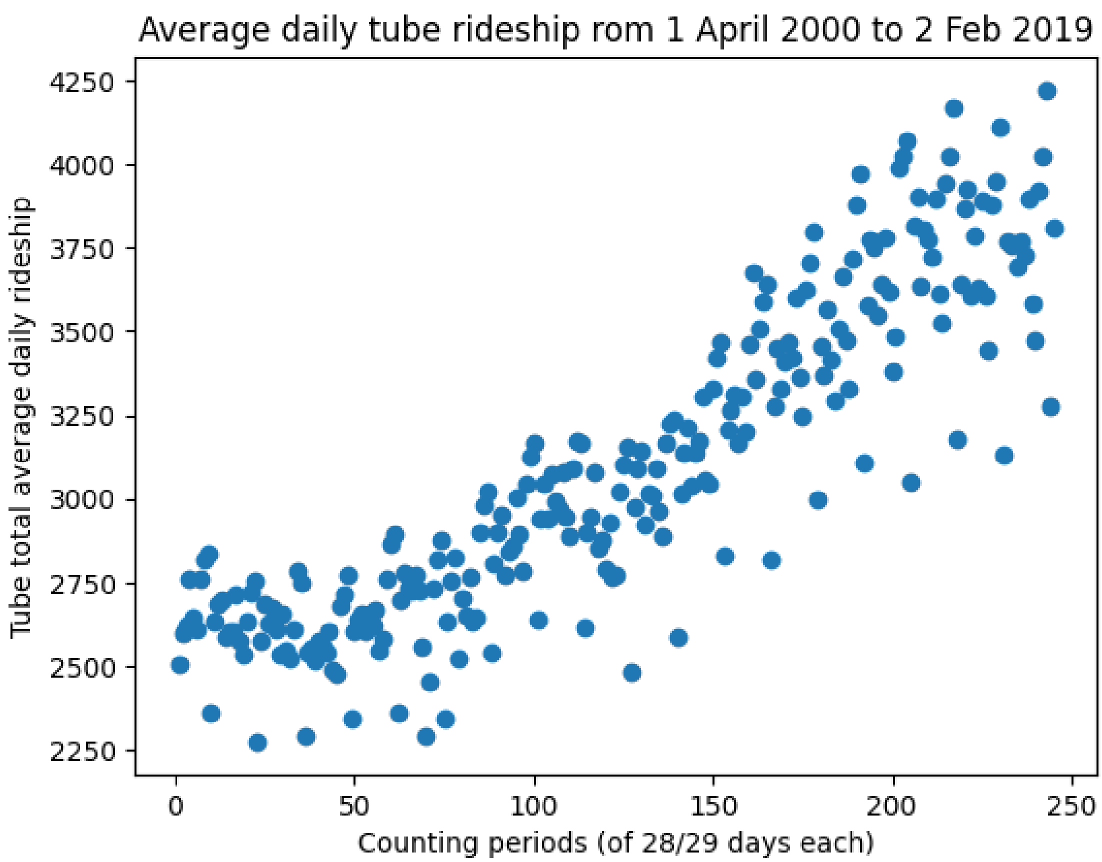
The dataframe contains $N=245$ counting periods (of 28/29 days each) from 1 April 2000 to 2 Feb 2019. We now define a numpy array consisting of the values in the ' Tube Total (000s)' column:

```
yvals = np.array(df_tfl['Tube Total (000s)'])
N = np.size(yvals)
xvals = np.linspace(1,N,N) #an array containing the values 1,2....,N
```

We now have a time series consisting of points (x_i, y_i), for $i=1,\ldots,N$, where y_i is the average daily tube rideship in counting period $x_i=i$.

3a) Plot the data in a scatterplot

```
#Your code for scatterplot here
plt.scatter(xvals, yvals)
plt.title("Average daily tube rideship rom 1 April 2000 to 2 Feb 
2019")
plt.xlabel("Counting periods (of 28/29 days each)")
plt.ylabel("Tube total average daily rideship")
plt.show()
```



3b) Fit a linear model $f(x)=\beta_0+\beta_1 x$ to the data

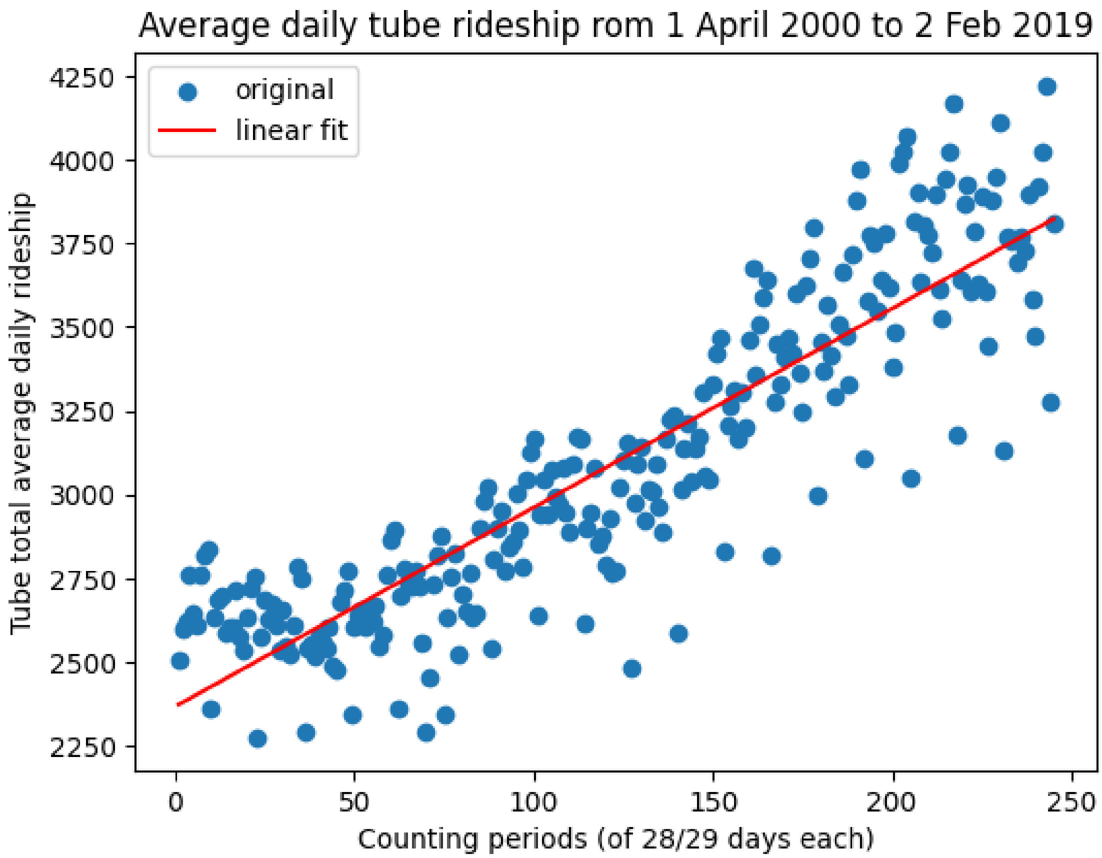
- Print the values of the regression coefficients β_0, β_1 determined using least-squares.
- Plot the fitted model and the scatterplot on the same plot.
- Compute and print the **MSE** and the R^2 coefficient for the fitted model.

All numerical outputs should be displayed to three decimal places.

```
#Your code here
def polyreg(x, y, k):
    # k: degree

    N = x.shape[-1]

    degree = min(k, N-1)
    lsq_matrix = np.column_stack(tuple((x**i for i in 
range(degree+1))))
    lsq_matrix_T = lsq_matrix.T
    optimal_params = 
np.linalg.inv(lsq_matrix_T.dot(lsq_matrix)).dot(lsq_matrix_T).dot(y)

    y_pred = lsq_matrix.dot(optimal_params.T)
    residuals = y - y_pred
```

```
    SSE = np.linalg.norm(residuals)**2
    MSE = SSE/N

    var = np.var(y)
    R2 = 1 - MSE/var

    return optimal_params, y_pred, residuals, MSE, R2

optimal_params, y_pred, residuals, MSE, R2 = polyreg(xvals, yvals, 1)
print(f"b0: {optimal_params[0]} | b1: {optimal_params[1]}")
print(f"MSE: {MSE} | R-squared value: {R2}")


# Your code for scatterplot here
plt.scatter(xvals, yvals, label="original")
plt.plot(xvals, y_pred, label="linear fit", color="red")
plt.title("Average daily tube rideship rom 1 April 2000 to 2 Feb
2019")
plt.xlabel("Counting periods (of 28/29 days each)")
plt.ylabel("Tube total average daily rideship")
plt.legend()
plt.show()
```

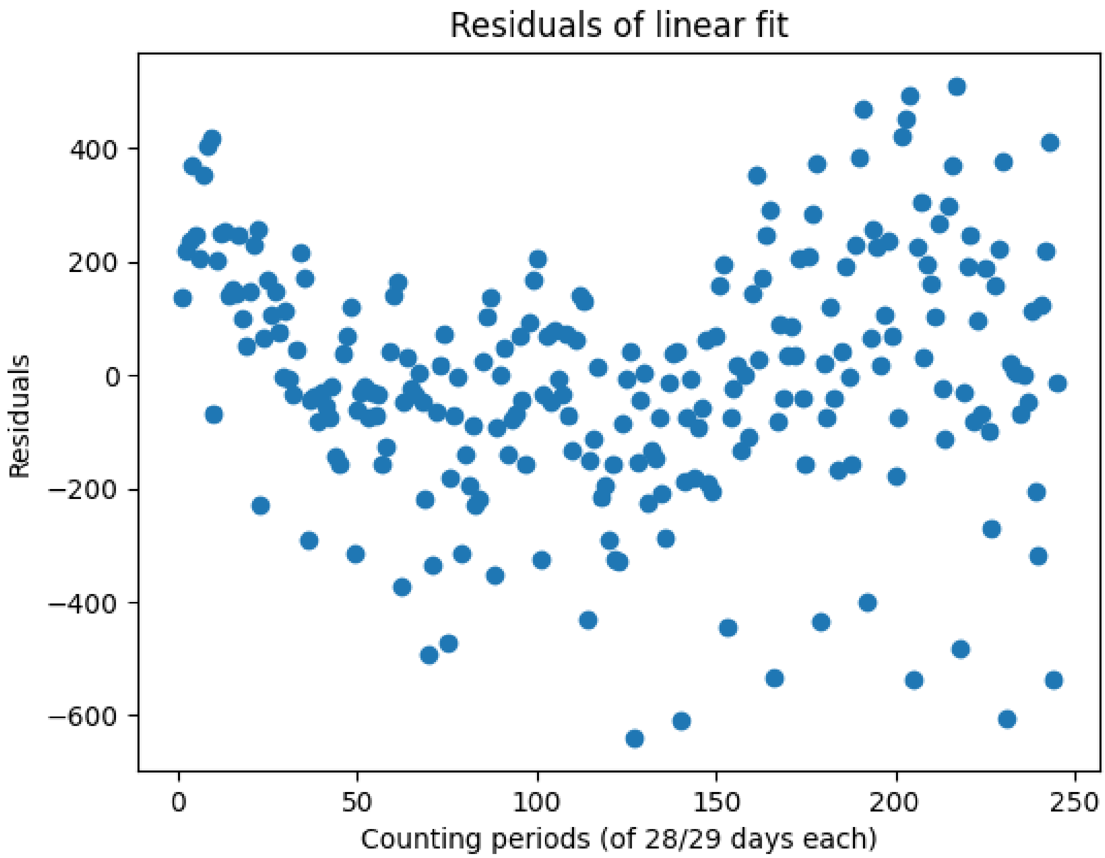
```
b0: 2367.3817664770822 | b1: 5.938990118238413
MSE: 45323.63592122835 | R-squared value: 0.7956113333574003
```



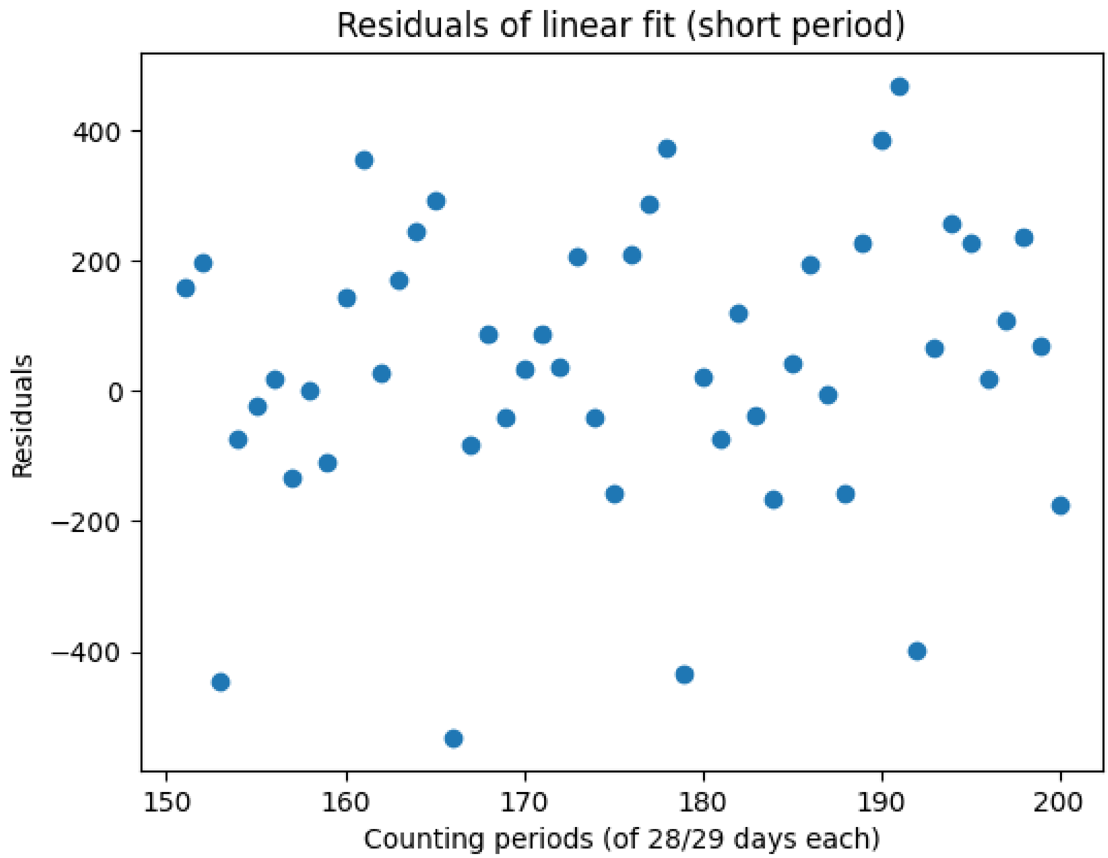
3c) Plotting the residuals

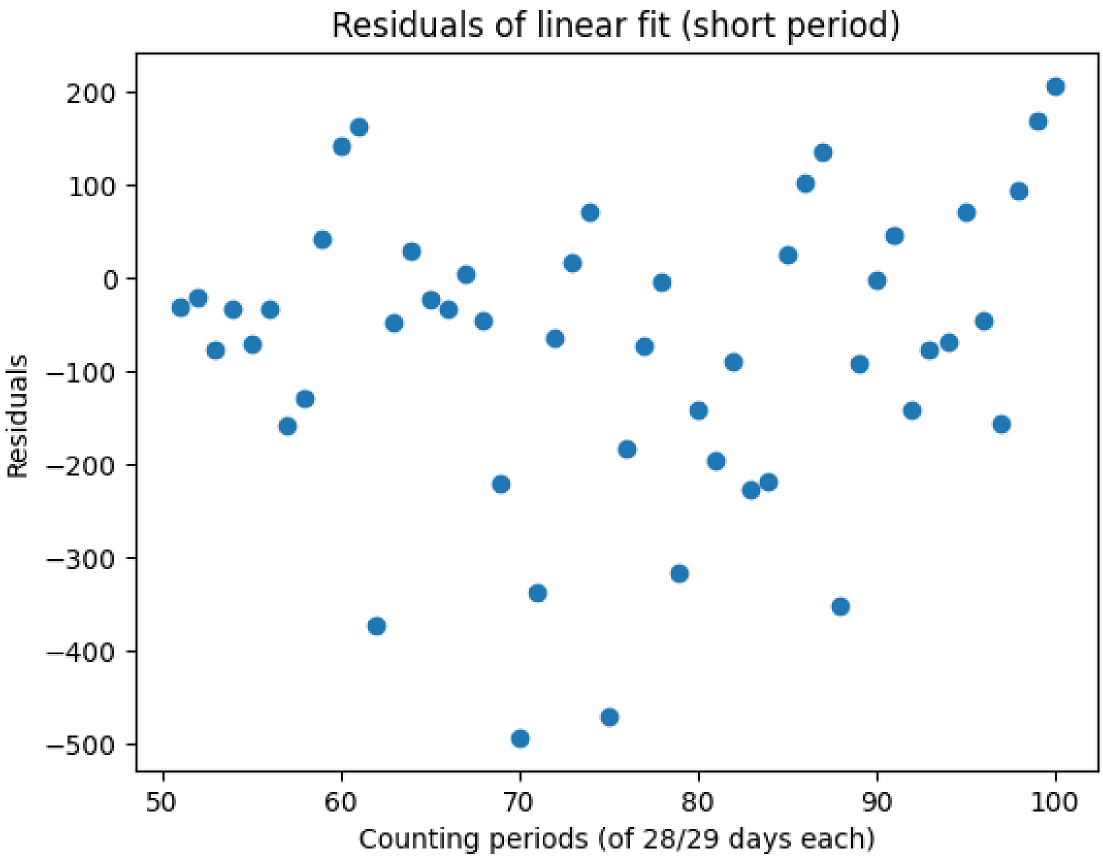
- Plot the residuals on a scatterplot
- Also plot the residuals over a short duration and comment on whether you can discern any periodic components.

```
# Your code here
plt.scatter(xvals, residuals)
plt.title("Residuals of linear fit")
plt.xlabel("Counting periods (of 28/29 days each)")
plt.ylabel("Residuals")
plt.show()
```

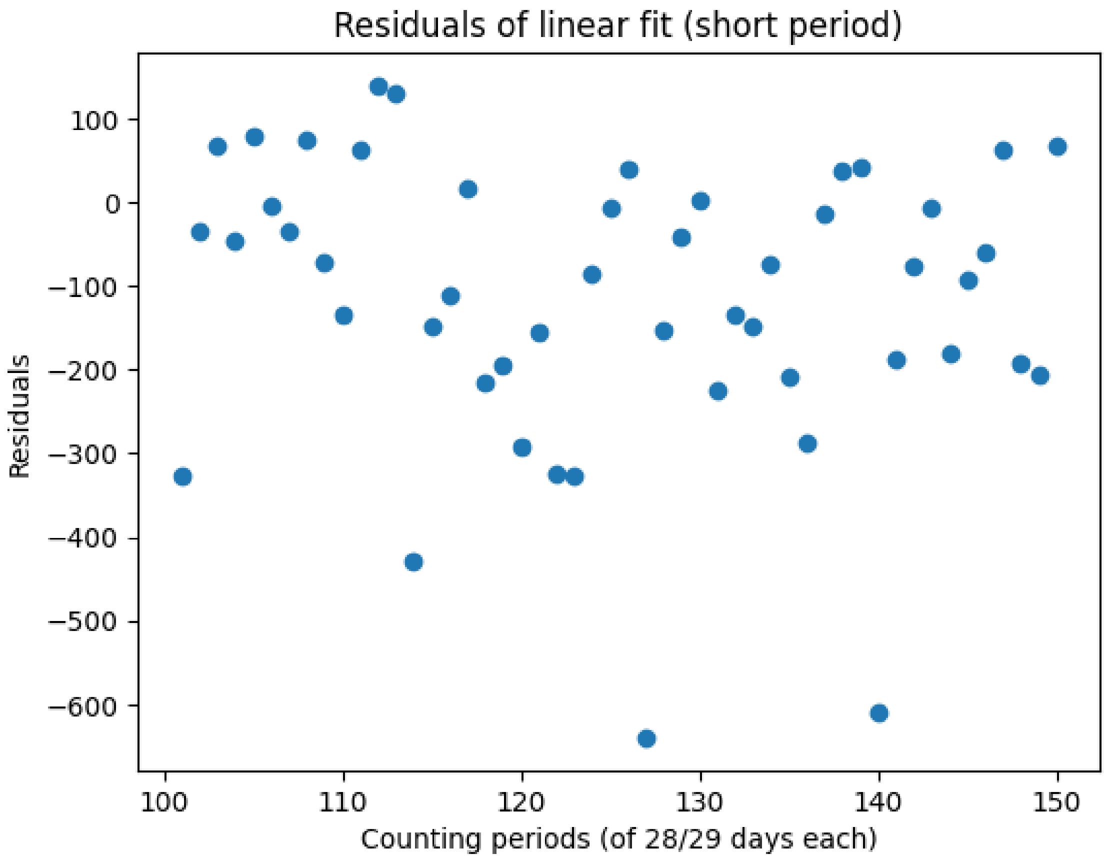



```
# Your code here
plt.scatter(xvals[150:200], residuals[150:200])
plt.title("Residuals of linear fit (short period)")
plt.xlabel("Counting periods (of 28/29 days each)")
plt.ylabel("Residuals")
plt.show()

plt.scatter(xvals[50:100], residuals[50:100])
plt.title("Residuals of linear fit (short period)")
plt.xlabel("Counting periods (of 28/29 days each)")
plt.ylabel("Residuals")
plt.show()

plt.scatter(xvals[100:150], residuals[100:150])
plt.title("Residuals of linear fit (short period)")
plt.xlabel("Counting periods (of 28/29 days each)")
plt.ylabel("Residuals")
plt.show()
```


Residuals of linear fit (short period)
Residuals
400
200
0
−200
−400
150
160
170
180
190
200
Counting periods (of 28/29 days each)



Residuals of linear fit (short period)
200
100
0
−100
−200
−300
−400
−500
Residuals
50
60
70
80
90
100
Counting periods (of 28/29 days each)



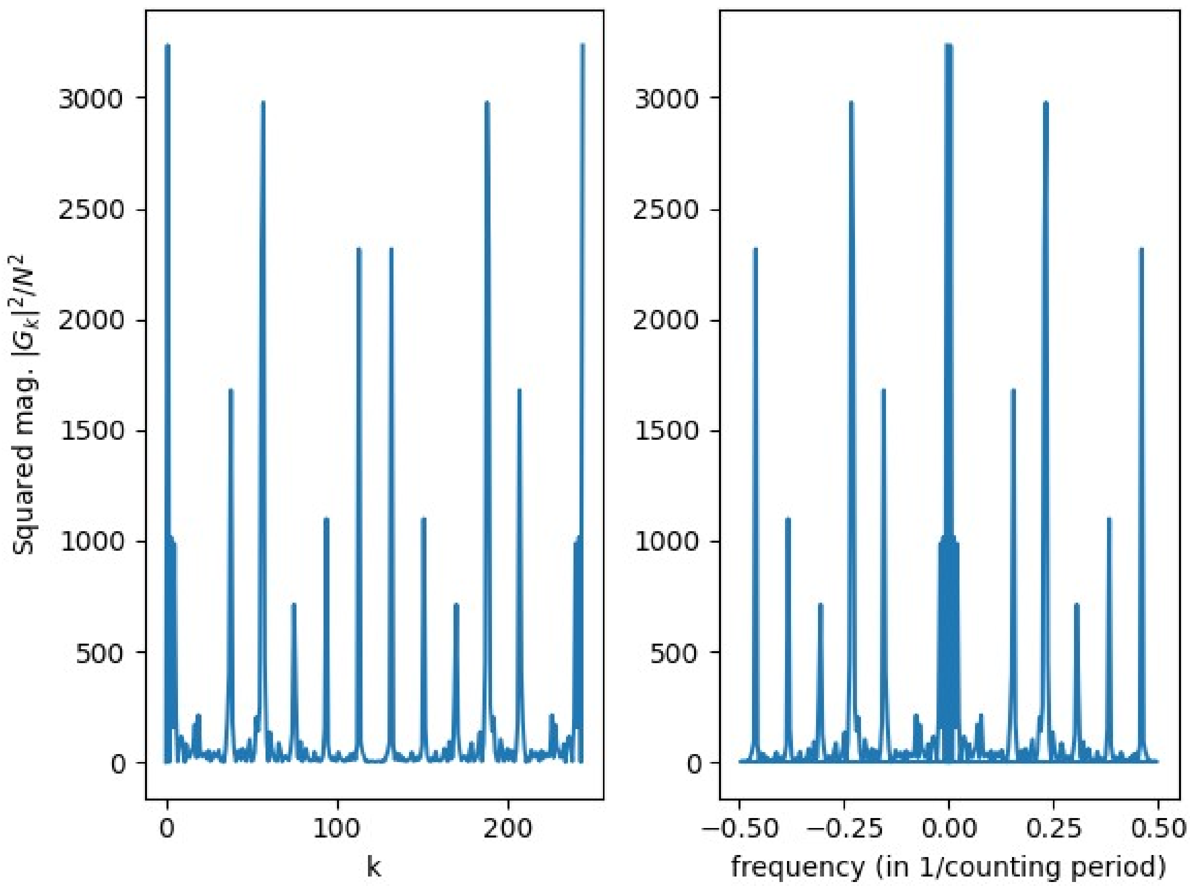
< Comment on periodic components here > The long term residuals appear to have a period of 100 counting peiods (or is quadratic) The short term residuals appear to have some level of periodicity every 10 days

3d) Periodogram

- Compute and plot the peridogram of the residuals. (Recall that the periodogram is the squared-magnitude of the DFT coefficients.)
- Identify the indices/frequencies for which the periogram value exceeds **50%** of the maximum.

```
# Your code to compute and plot the periodogram
T = xvals[100] - xvals[99]
pgram = np.abs(np.fft.fft(residuals, N)/N)**2 #We normalize by N, but this is optional
indices = np.linspace(0, (N-1), num = N)
freqs_in_hz = np.fft.fftfreq(N)/T
freqs_in_rads = freqs_in_hz*2*math.pi

plt.subplot(121)
plt.plot(indices, pgram)
plt.xlabel('k')
plt.ylabel('Squared mag. $|G_k|^2/N^2$')
plt.subplot(122)
```

```
plt.plot(freqs_in_hz, pgram)
plt.xlabel('frequency (in 1/counting period)')  # Since units of T is counting period
plt.tight_layout()
```




```
# Your code to identify the indices for which the periodogram value exceeds 50% of the maximum
top_inds = indices[(pgram > 0.5*np.max(pgram))]
top_freqs_hz = freqs_in_hz[(pgram > 0.5*np.max(pgram))]
print('Top indices:', top_inds, ' Top frequencies in Hz:', top_freqs_hz)
```

```
Top indices: [  1.  38.  56.  57. 113. 132. 188. 189. 207. 244.]  Top frequencies in Hz: [ 0.00408163  0.15510204  0.22857143  0.23265306  0.46122449 -0.46122449
 -0.23265306 -0.22857143 -0.15510204 -0.00408163]
```

3e) To the residuals, fit a model of the form

$$\beta_{1s}\sin(\omega_1 x)+\beta_{1c}\cos(\omega_1 x)+\beta_{2s}\sin(\omega_2 x)+\beta_{2c}\cos(\omega_2 x)+\ldots+\beta_{Ks}\sin(\omega_K x)+\beta_{Kc}\cos(\omega_K x).$$

The frequencies $\omega_1, \ldots, \omega_K$ in the model are those corresponding to the indices identified in Part 2c. (Hint: Each of the sines and cosines will correspond to one column in your X-matrix.)

- Print the values of the regression coefficients obtained using least-squares.

All numerical outputs should be displayed to three decimal places.

```
# Your code here
#Your code here
def sinusoidalreg(x, y, freq):
    # k: degree
    w = 2*math.pi*freq

    N = x.shape[-1]
    sines = tuple((np.sin(i*x) for i in w))
    cosines = tuple((np.cos(i*x) for i in w))

    lsq_matrix = np.column_stack(sines + cosines)
    lsq_matrix_T = lsq_matrix.T
    optimal_params =
np.linalg.inv(lsq_matrix_T.dot(lsq_matrix)).dot(lsq_matrix_T).dot(y)

    y_pred = lsq_matrix.dot(optimal_params.T)
    residuals = y - y_pred

    SSE = np.linalg.norm(residuals)**2
    MSE = SSE/N

    var = np.var(y)
    R2 = 1 - MSE/var

    return optimal_params, y_pred, residuals, MSE, R2

freq = np.array([i for i in top_freqs_hz if i > 0])
optimal_params_sin, y_pred_sin, _, _ , _ = sinusoidalreg(xvals,
residuals, freq)

print("Optimal Params\n")
for i in range(len(optimal_params_sin)):
    if i < len(optimal_params_sin)/2:
        print(f"Bs_{i + 1}:", optimal_params_sin[i])
    else:
        if i == 5:
            print();
        print(f"Bc_{int(i + 1 - len(optimal_params_sin)/2)}:",
optimal_params_sin[i])
```

```
Optimal Params
```

```
Bs_1: -51.25288797104298
Bs_2: 61.6276582480666
Bs_3: -15.580675739589608
Bs_4: 81.65869500973758
Bs_5: 32.47226957213971

Bc_1: 101.5558059835357
Bc_2: -54.005618852677294
Bc_3: -94.79732601829892
Bc_4: 72.38106303401744
Bc_5: 90.5889317701515
```

3f) The combined fit

- Plot the combined fit together with a scatterplot of the data
- Compute and print the final **MSE** and R^2 coefficient. Comment on the improvement over the linear fit.

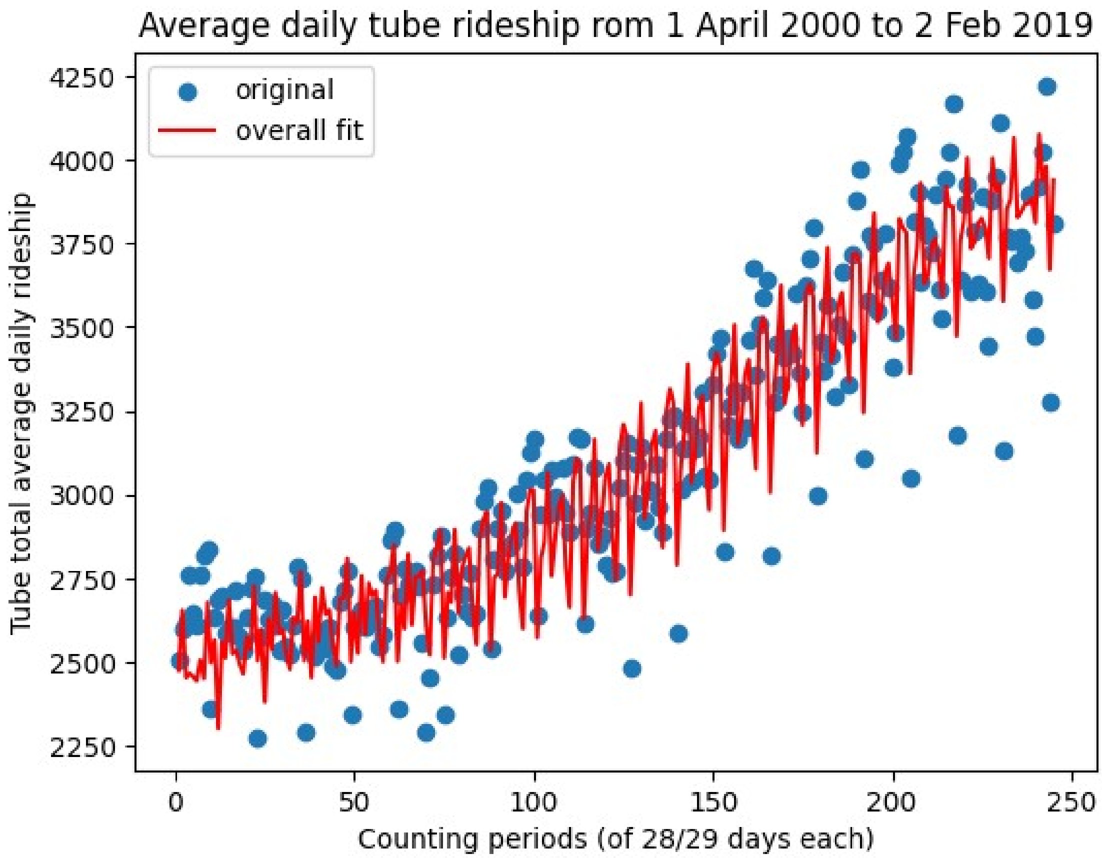
The combined fit, which corresponds to the full model

$$f(x)=\beta_0+\beta_1 x+\beta_{s1}\sin(\omega_1 x)+\beta_{c1}\cos(\omega_1 x)+\ldots+\beta_{sk}\sin(\omega_k x)+\beta_{ck}\cos(\omega_k x),$$

can be obtained by adding the fits in parts 2b) and 2e).

```python
# Your code here
y_pred_total = y_pred + y_pred_sin
residuals_total = y_pred_total - yvals
SSE_total = np.linalg.norm(residuals_total)**2
MSE_total = SSE_total/N
var = np.var(yvals)
R2_total = 1 - MSE_total/var

print(f"MSE: {MSE_total} | R-squared value: {R2_total}")

# Your code for scatterplot here
plt.scatter(xvals, yvals, label="original")
plt.plot(xvals, y_pred_total, label="overall fit", color="red")
plt.title("Average daily tube rideship rom 1 April 2000 to 2 Feb 2019")
plt.xlabel("Counting periods (of 28/29 days each)")
plt.ylabel("Tube total average daily rideship")
plt.legend()
plt.show()
```

```
MSE: 20297.501187772512 | R-squared value: 0.9084676434354128
```



< Add comment on the improvement over the linear fit. >

MSE is almost halved and R2 value is much closer to 1. However, this feels like overfitting of a very noisy data. We will need to evaluate on a test set to double check the validity of this fit.