الكلية الجامعية بالجموم

قسم الرياضيات

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جامعة أم القرى
الكلية الجامعية بالجموم
قسم الرياضيات
الفصل الاول 1442هـ

رمز المقرر: **4-2304101**
المتطلب : -

الخطة الدراسية لمقرر: تفاضل وتكامل (1)

Week	Lessons	Examples	Problems
1	0.1 Real numbers		1,2
	0.2 Inequalities	1,2,3,4,5,6	3,9,11,12,15,17
2	Absolute values	8,9,13,14	35,37,44,45,59,60
3	0.5 Functions and their graphs	1,2,4(a),5	1,9,13,15,17,18,19,20,21,24,25
	0.6 Operations on functions	1,2,3,4	1,2,5,16,17
4	0.7 Trigonometric Functions	5,6,7	9(a,b,c),11(a),12(a),13(a)
	1.1 Introduction to limits	1,2,3,5	5,710,12,19
	1.2 Limit Theorems	1,2,3,4,5,6,7,8	1,3,5,711,13,14,19,27,41
5	1.5 Limis at infinity	2,3,5,6,7	1,2,3,4,9,43,44,45,46
	1.6 Continuity of functions	1,2,3,4,5,6,7	2,3,5,7,9,13,33
6	2.2 The derivative	1,2,3,4,5,6	1,3,5,8,27,29
	2.3Rules for finding derivatives	1,2,3,4,5	1,3,5,9,13,15,17,23,25,37
7	2.4 Derivatives of trigonometric functions	1,2,3,4	1,3,5,7,9,11,15
	2.5 The chain rule	1,2,3,4,7,9,10	1,5,6,7,9,11,13,21,22,34
8	**Review-Med term exam**		
9	2.6 Higher order derivatives	1,2	8,16,19,29
	2.7 Implicite differentiations	1,2,3,4	2,3,7,8,11,13,15,20
10	3.1 Maxima and minima	1,2,3,4,5	1,3,5,7
	3.2 Monotonicity and concavity	1,2,3,4,7	1,2,4,5,8,13
11	3.3 Local Extrema	1,2,4,5,6	1,7,11,31
12	3.5 Graphing Function	1,2	1,9
13	3.6The mean value theorem for derivatives	1,2,3	1,16,19,21
14	5.1 Sums and Sigma Notation	1,2,3,4	1,4,13,17
	5.2 Areas as Limits of Sums	1,2,3,4	2,6,14
15	5.3Definite Integrals	1,2,3,4	1,4,7,11

مواعيد الاختبارات الدورية

الاختبار الدوري	اليوم	التاريخ
الأول		
الثاني		

التقويم:.

تعليمات اضافية

حضور المحاضرات	نهيب بالطلبة حضور المحاضرات على النحو المطلوب للاستفادة و لعدم الحرمان من المادة طبقا لقوانين الجامعة
الغش في الاختبارات	الغش في الامتحان محرم و منكر كالغش في المعاملات أو أشد و يعرض الطالب للعقوبات طبقا لقوانين الجامعة

مع الأمنيات بالتوفيق والسداد

(b) For every x, $x > 0 \Leftrightarrow x^2 > 0$.

(c) For every x, $x^2 > x$.

(d) For every x, there exists a y such that $y > x^2$.

(e) For every positive number y, there exists another positive number x such that $0 < x < y$.

72. Which of the following are true? Unless it is stated otherwise, assume that x, y, and ε are real numbers.

(a) For every x, $x < x + 1$.

(b) There exists a natural number N such that all prime numbers are less than N. (A **prime number** is a natural number whose only factors are 1 and itself.)

(c) For every $x > 0$, there exists a y such that $y > \frac{1}{x}$.

(d) For every positive x, there exists a natural number n such that $\frac{1}{n} < x$.

(e) For every positive ε, there exists a natural number n such that $\frac{1}{2^n} < \varepsilon$.

73. Prove the following statements.

(a) If n is odd, then n^2 is odd. (*Hint:* If n is odd, then there exists an integer k such that $n = 2k + 1$.)

(b) If n^2 is odd, then n is odd. (*Hint:* Prove the contrapositive.)

74. Prove that n is odd if and only if n^2 is odd. (See Problem 73.)

75. According to the **Fundamental Theorem of Arithmetic,** every natural number greater than 1 can be written as the product of primes in a unique way, except for the order of the factors. For example, $45 = 3 \cdot 3 \cdot 5$. Write each of the following as a product of primes.

(a) 243 (b) 124 (c) 5100

76. Use the Fundamental Theorem of Arithmetic (Problem 75) to show that the square of any natural number greater than 1 can be written as the product of primes in a unique way, except for the order of the factors, with each prime occurring an *even* number of times. For example, $(45)^2 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \cdot 5$.

77. Show that $\sqrt{2}$ is irrational. *Hint:* Try a proof by contradiction. Suppose that $\sqrt{2} = p/q$, where p and q are natural numbers (necessarily different from 1). Then $2 = p^2/q^2$, and so $2q^2 = p^2$. Now use Problem 76 to get a contradiction.

78. Show that $\sqrt{3}$ is irrational (see Problem 77).

79. Show that the sum of two rational numbers is rational.

80. Show that the product of a rational number (other than 0) and an irrational number is irrational. *Hint:* Try proof by contradiction.

81. Which of the following are rational and which are irrational?

(a) $-\sqrt{9}$ (b) 0.375

(c) $(3\sqrt{2})(5\sqrt{2})$ (d) $(1 + \sqrt{3})^2$

82. A number b is called an **upper bound** for a set S of numbers if $x \le b$ for all x in S. For example 5, 6.5, and 13 are upper bounds for the set $S = \{1, 2, 3, 4, 5\}$. The number 5 is the **least upper bound** for S (the smallest of all upper bounds). Similarly, 1.6, 2, and 2.5 are upper bounds for the infinite set $T = \{1.4, 1.49, 1.499, 1.4999, \ldots\}$, whereas 1.5 is its least upper bound. Find the least upper bound of each of the following sets.

(a) $S = \{-10, -8, -6, -4, -2\}$

(b) $S = \{-2, -2.1, -2.11, -2.111, -2.1111, \ldots\}$

(c) $S = \{2.4, 2.44, 2.444, 2.4444, \ldots\}$

(d) $S = \left\{1 - \frac{1}{2}, 1 - \frac{1}{3}, 1 - \frac{1}{4}, 1 - \frac{1}{5}, \ldots\right\}$

(e) $S = \{x : x = (-1)^n + 1/n, n \text{ a positive integer}\}$; that is, S is the set of all numbers x that have the form $x = (-1)^n + 1/n$, where n is a positive integer.

(f) $S = \{x : x^2 < 2, x \text{ a rational number}\}$

EXPL **83. The Axiom of Completeness** for the real numbers says: Every set of real numbers that has an upper bound has a *least* upper bound that is a real number.

(a) Show that the italicized statement is false if the word *real* is replaced by *rational*.

(b) Would the italicized statement be true or false if the word *real* were replaced by *natural*?

Answers to Concepts Review: **1.** rational numbers **2.** dense **3.** "If not Q then not P." **4.** theorems

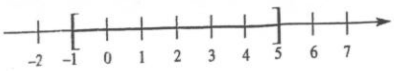
0.2 Inequalities and Absolute Values

Solving equations (for instance, $3x - 17 = 6$ or $x^2 - x - 6 = 0$) is one of the traditional tasks of mathematics; it will be important in this course and we assume that you remember how to do it. But of almost equal significance in calculus is the notion of solving an inequality (e.g., $3x - 17 < 6$ or $x^2 - x - 6 \ge 0$). To **solve** an inequality is to find the set of all real numbers that make the inequality true. In contrast to an equation, whose solution set normally consists of one number or perhaps a finite set of numbers, the solution set of an inequality is usually an entire interval of numbers or, in some cases, the union of such intervals.

Intervals Several kinds of intervals will arise in our work and we introduce special terminology and notation for them. The inequality $a < x < b$, which is actually two inequalities, $a < x$ and $x < b$, describes the **open interval** consisting of all numbers between a and b, not including the end points a and b. We denote this interval by the symbol (a, b) (Figure 1). In contrast, the inequality $a \le x \le b$ describes the corresponding **closed interval**, which does include the end points a and

$(-1, 6) = \{x : -1 < x < 6\}$

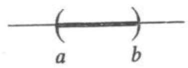
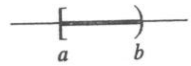
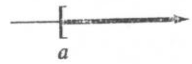
Figure 1



$[-1, 5] = \{x: -1 \le x \le 5\}$

Figure 2

b. This interval is denoted by $[a, b]$ (Figure 2). The table indicates the wide variety of possibilities and introduces our notation.

Set Notation	Interval Notation	Graph
$\{x: a < x < b\}$	(a, b)	
$\{x: a \le x \le b\}$	$[a, b]$	
$\{x: a \le x < b\}$	$[a, b)$	
$\{x: a < x \le b\}$	$(a, b]$	
$\{x: x \le b\}$	$(-\infty, b]$	
$\{x: x < b\}$	$(-\infty, b)$	
$\{x: x \ge a\}$	$[a, \infty)$	
$\{x: x > a\}$	(a, ∞)	
$\mathbb{R}$	$(-\infty, \infty)$	

Solving Inequalities As with equations, the procedure for solving an inequality consists of transforming the inequality one step at a time until the solution set is obvious. We may perform certain operations on both sides of an inequality without changing its solution set. In particular,

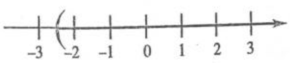
1. We may add the same number to both sides of an inequality.
2. We may multiply both sides of an inequality by the same positive number.
3. We may multiply both sides by the same negative number, but then we must reverse the direction of the inequality sign.

EXAMPLE 1 Solve the inequality $2x - 7 < 4x - 2$ and show the graph of its solution set.

SOLUTION

$$\begin{aligned} 2x - 7 &< 4x - 2 & \\ 2x &< 4x + 5 & &\text{(adding 7)} \\ -2x &< 5 & &\text{(adding } -4x\text{)} \\ x &> -\tfrac{5}{2} & &\text{(multiplying by } -\tfrac{1}{2}\text{)} \end{aligned}$$

The graph appears in Figure 3. ■



$\left(-\frac{5}{2}, \infty\right) = \left\{x: x > -\frac{5}{2}\right\}$

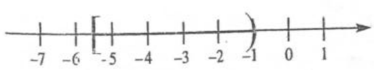
Figure 3

EXAMPLE 2 Solve $-5 \le 2x + 6 < 4$.

SOLUTION

$$\begin{aligned} -5 \le 2x + 6 &< 4 & \\ -11 \le 2x \quad &< -2 & &\text{(adding } -6\text{)} \\ -\tfrac{11}{2} \le x \quad &< -1 & &\text{(multiplying by } \tfrac{1}{2}\text{)} \end{aligned}$$

Figure 4 shows the corresponding graph. ■



$\left[-\frac{11}{2}, -1\right) = \left\{x: -\frac{11}{2} \le x < -1\right\}$

Figure 4

Before tackling a quadratic inequality, we point out that a linear factor of the form $x - a$ is positive for $x > a$ and negative for $x < a$. If follows that a product $(x - a)(x - b)$ can change from being positive to negative, or vice versa, only at a or b. These points, where a factor is zero, are called **split points.** They are the keys to determining the solution sets of quadratic and other more complicated inequalities.

EXAMPLE 3 Solve the quadratic inequality $x^2 - x < 6$.

SOLUTION As with quadratic equations, we move all nonzero terms to one side and factor.

$$
\begin{aligned}
x^2 - x &< 6 \\
x^2 - x - 6 &< 0 \qquad \text{(adding } -6\text{)} \\
(x - 3)(x + 2) &< 0 \qquad \text{(factoring)}
\end{aligned}
$$

Test Point	Sign of $(x - 3)$	Sign of $(x + 2)$	Sign of $(x - 3)(x + 2)$
-3	$-$	$-$	$+$
0	$-$	$+$	$-$
5	$+$	$+$	$+$

We see that -2 and 3 are the split points; they divide the real line into the three intervals $(-\infty, -2), (-2, 3)$, and $(3, \infty)$. On each of these intervals, $(x - 3)(x + 2)$ is of one sign; that is, it is either always positive or always negative. To find this sign in each interval, we use the **test points** -3, 0, and 5 (any points in the three intervals would do). Our results are shown in the margin.

The information we have obtained is summarized in the top half of Figure 5. We conclude that the solution set for $(x - 3)(x + 2) < 0$ is the interval $(-2, 3)$. Its graph is shown in the bottom half of Figure 5. ■

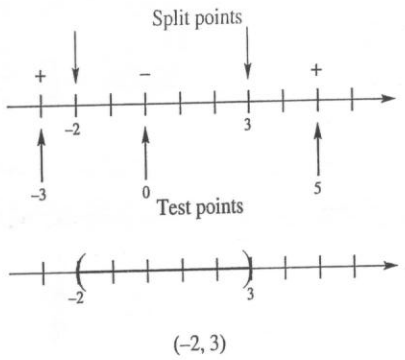


Figure 5

EXAMPLE 4 Solve $3x^2 - x - 2 > 0$.

SOLUTION Since

$$3x^2 - x - 2 = (3x + 2)(x - 1) = 3(x - 1)\left(x + \tfrac{2}{3}\right)$$

the split points are $-\frac{2}{3}$ and 1. These points, together with the test points -2, 0, and 2, establish the information shown in the top part of Figure 6. We conclude that the solution set of the inequality consists of the points in either $\left(-\infty, -\frac{2}{3}\right)$ or $(1, \infty)$. In set language, the solution set is the **union** (symbolized by $\cup$) of these two intervals; that is, it is $\left(-\infty, -\frac{2}{3}\right) \cup (1, \infty)$. ■

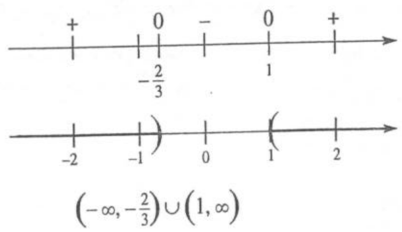


Figure 6

EXAMPLE 5 Solve $\dfrac{x - 1}{x + 2} \geq 0$.

SOLUTION Our inclination to multiply both sides by $x + 2$ leads to an immediate dilemma, since $x + 2$ may be either positive or negative. Should we reverse the inequality sign or leave it alone? Rather than try to untangle this problem (which would require breaking it into two cases), we observe that the quotient $(x - 1)/(x + 2)$ can change sign only at the split points of the numerator and denominator, that is, at 1 and -2. The test points -3, 0, and 2 yield the information displayed in the top part of Figure 7. The symbol u indicates that the quotient is undefined at -2. We conclude that the solution set is $(-\infty, -2) \cup [1, \infty)$. Note that -2 is not in the solution set because the quotient is undefined there. On the other hand, 1 is included because the inequality is true when $x = 1$. ■

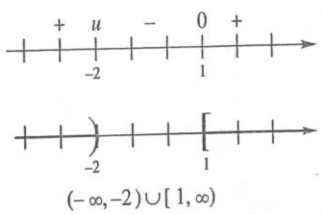


Figure 7

EXAMPLE 6 Solve $(x + 1)(x - 1)^2(x - 3) \leq 0$.

SOLUTION The split points are -1, 1 and 3, which divide the real line into four intervals, as shown in Figure 8. After testing these intervals, we conclude that the solution set is $[-1, 1] \cup [1, 3]$, which is the interval $[-1, 3]$. ■

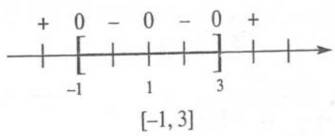


Figure 8

EXAMPLE 7 Solve $2.9 < \dfrac{1}{x} < 3.1$.

SOLUTION It is tempting to multiply through by x, but this again brings up the dilemma that x may be positive or negative. In this case, however, $\frac{1}{x}$ must be between 2.9 and 3.1, which guarantees that x is positive. It is therefore permissible to multiply by x and not reverse the inequalities. Thus,

$$2.9x < 1 < 3.1x$$

At this point, we must break this compound inequality into two inequalities, which we solve separately.

$$2.9x < 1 \quad \text{and} \quad 1 < 3.1x$$

$$x < \frac{1}{2.9} \quad \text{and} \quad \frac{1}{3.1} < x$$

Any value of x that satisfies the original inequality must satisfy both of these inequalities. The solution set thus consists of those values of x satisfying

$$\frac{1}{3.1} < x < \frac{1}{2.9}$$

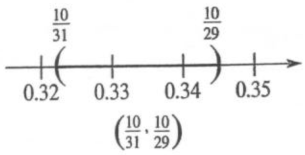


Figure 9

This inequality can be written as

$$\frac{10}{31} < x < \frac{10}{29}$$

The interval $\left(\frac{10}{31}, \frac{10}{29}\right)$ is shown in Figure 9. ■

Absolute Values

The concept of absolute value is extremely useful in calculus, and the reader should acquire skill in working with it. The **absolute value** of a real number x, denoted by $|x|$, is defined by

$$\boxed{\begin{aligned} |x| &= x \quad \text{if } x \geq 0 \\ |x| &= -x \quad \text{if } x < 0 \end{aligned}}$$

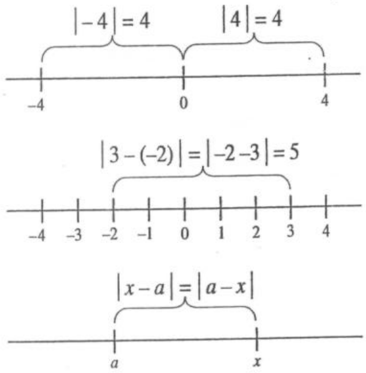


Figure 10

For example, $|6| = 6$, $|0| = 0$, and $|-5| = -(-5) = 5$. This two-pronged definition merits careful study. Note that it does not say that $|-x| = x$ (try $x = -5$ to see why). It is true that $|x|$ is always nonnegative; it is also true that $|-x| = |x|$.

One of the best ways to think of the absolute value of a number is as an undirected distance. In particular, $|x|$ is the distance between x and the origin. Similarly, $|x - a|$ is the distance between x and a (Figure 10).

Properties

Absolute values behave nicely under multiplication and division, but not so well under addition and subtraction.

Properties of Absolute Values

1. $|ab| = |a||b|$
2. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
3. $|a + b| \leq |a| + |b|$ (Triangle Inequality)
4. $|a - b| \geq ||a| - |b||$

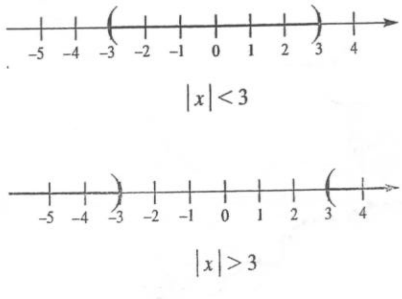


Figure 11

Inequalities Involving Absolute Values

If $|x| < 3$, then the distance between x and the origin must be less than 3. In other words, x must be simultaneously less than 3 *and* greater than -3; that is, $-3 < x < 3$. On the other hand, if $|x| > 3$, then the distance between x and the origin must be at least 3. This can happen when $x > 3$ *or* $x < -3$ (Figure 11). These are special cases of the following general statements that hold when $a > 0$.

(1) $$|x| < a \Leftrightarrow -a < x < a$$
$$|x| > a \Leftrightarrow x < -a \quad \text{or} \quad x > a$$

We can use these facts to solve inequalities involving absolute values, since they provide a way of removing absolute value signs.

EXAMPLE 8 Solve the inequality $|x - 4| < 2$ and show the solution set on the real line. Interpret the absolute value as a distance.

SOLUTION From the equations in (1), with x replaced by $|x - 4|$, we see that

$$|x - 4| < 2 \Leftrightarrow -2 < x - 4 < 2$$

When we add 4 to all three members of this latter inequality, we obtain $2 < x < 6$. The graph is shown in Figure 12.

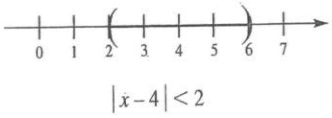


Figure 12

In terms of distance, the symbol $|x - 4|$ represents the distance between x and 4. The inequality says that the distance between x and 4 is less than 2. The numbers x with this property are the numbers between 2 and 6; that is, $2 < x < 6$. ■

The statements in the equations just before Example 8 are valid with $<$ and $>$ replaced by $\leq$ and $\geq$, respectively. We need the second statement in this form in our next example.

EXAMPLE 9 Solve the inequality $|3x - 5| \geq 1$ and show its solution set on the real line.

SOLUTION The given inequality may be written successively as

$$\begin{aligned} 3x - 5 &\leq -1 & \text{or} \quad 3x - 5 &\geq 1 \\ 3x &\leq 4 & \text{or} \quad 3x &\geq 6 \\ x &\leq \tfrac{4}{3} & \text{or} \quad x &\geq 2 \end{aligned}$$

The solution set is the union of two intervals, $\left(-\infty, \frac{4}{3}\right] \cup [2, \infty)$, and is shown in Figure 13. ■

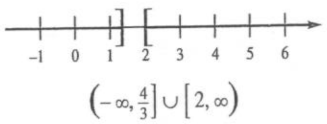


Figure 13

In Chapter 1, we will need to make the kind of manipulations illustrated by the next two examples. Delta (δ) and epsilon (ε) are the fourth and fifth letters, respectively, of the Greek alphabet and are traditionally used to stand for small positive numbers.

EXAMPLE 10 Let ε (epsilon) be a positive number. Show that

$$|x - 2| < \frac{\varepsilon}{5} \Leftrightarrow |5x - 10| < \varepsilon$$

In terms of distance, this says that the distance between x and 2 is less than $\varepsilon/5$ if and only if the distance between $5x$ and 10 is less than ε.

SOLUTION

$$\begin{aligned} |x - 2| < \frac{\varepsilon}{5} \quad &\Leftrightarrow \quad 5|x - 2| < \varepsilon && \text{(multiplying by 5)} \\ &\Leftrightarrow \quad |5||(x - 2)| < \varepsilon && (|5| = 5) \\ &\Leftrightarrow \quad |5(x - 2)| < \varepsilon && (|a||b| = |ab|) \\ &\Leftrightarrow \quad |5x - 10| < \varepsilon \end{aligned}$$ ■

EXAMPLE 11 Let ε be a positive number. Find a positive number δ (delta) such that

$$|x - 3| < \delta \Rightarrow |6x - 18| < \varepsilon$$

SOLUTION

$$\begin{aligned} |6x - 18| < \varepsilon &\Leftrightarrow |6(x - 3)| < \varepsilon \\ &\Leftrightarrow 6|x - 3| < \varepsilon \qquad (|ab| = |a||b|) \\ &\Leftrightarrow |x - 3| < \frac{\varepsilon}{6} \qquad \left(\text{multiplying by } \frac{1}{6}\right) \end{aligned}$$

Finding Delta

Note two facts about our solution to Example 11.

1. The value we find for δ must depend on ε. Our choice is $\delta = \varepsilon/6$.
2. Any positive δ smaller than $\varepsilon/6$ is acceptable. For example $\delta = \varepsilon/7$ or $\delta = \varepsilon/(2\pi)$ are other correct choices.

Therefore, we choose $\delta = \varepsilon/6$. Following the implications backward, we see that

$$|x - 3| < \delta \Rightarrow |x - 3| < \frac{\varepsilon}{6} \Rightarrow |6x - 18| < \varepsilon$$

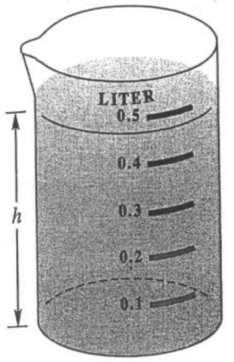


Figure 14

Here is a practical problem that uses the same type of reasoning.

EXAMPLE 12 A $\frac{1}{2}$-liter (500 cubic centimeter) glass beaker has an inner radius of 4 centimeters. How closely must we measure the height h of water in the beaker to be sure that we have $\frac{1}{2}$ liter of water with an error of less than 1%, that is, an error of less than 5 cubic centimeters? See Figure 14.

SOLUTION The volume V of water in the glass is given by the formula $V = 16\pi h$. We want $|V - 500| < 5$ or, equivalently, $|16\pi h - 500| < 5$. Now

$$\begin{aligned} |16\pi h - 500| < 5 &\Leftrightarrow \left|16\pi\left(h - \frac{500}{16\pi}\right)\right| < 5 \\ &\Leftrightarrow \quad 16\pi\left|h - \frac{500}{16\pi}\right| < 5 \\ &\Leftrightarrow \quad \left|h - \frac{500}{16\pi}\right| < \frac{5}{16\pi} \\ &\Leftrightarrow \quad |h - 9.947| < 0.09947 \approx 0.1 \end{aligned}$$

Thus, we must measure the height to an accuracy of about 0.1 centimeter, or 1 millimeter.

Notation for Square Roots

Every positive number has two square roots. For example, the two square roots of 9 are 3 and -3. We sometimes represent these two numbers as ± 3. For $a \geq 0$, the symbol $\sqrt{a}$, called the **principal square root** of a, denotes the nonnegative square root of a. Thus, $\sqrt{9} = 3$ and $\sqrt{121} = 11$. It is incorrect to write $\sqrt{16} = \pm 4$ because $\sqrt{16}$ means the nonnegative square root of 16, that is, 4. The number 7 has two square roots, which are written as $\pm\sqrt{7}$, but $\sqrt{7}$ represents a single real number. Just remember this:

$$a^2 = 16$$

has two solutions, $a = -4$ and $a = 4$, but

$$\sqrt{16} = 4$$

Quadratic Formula Most students will recall the **Quadratic Formula.** The solutions to the quadratic equation $ax^2 + bx + c = 0$ are given by

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

The number $d = b^2 - 4ac$ is called the **discriminant** of the quadratic equation. The equation $ax^2 + bx + c = 0$ has two real solutions if $d > 0$, one real solution if $d = 0$, and no real solutions if $d < 0$. With the Quadratic Formula, we can easily solve quadratic inequalities even if they do not factor by inspection.

EXAMPLE 13 Solve $x^2 - 2x - 4 \leq 0$.

SOLUTION The two solutions of $x^2 - 2x - 4 = 0$ are

$$x_1 = \frac{-(-2) - \sqrt{4 + 16}}{2} = 1 - \sqrt{5} \approx -1.24$$

and

$$x_2 = \frac{-(-2) + \sqrt{4 + 16}}{2} = 1 + \sqrt{5} \approx 3.24$$

Thus,

$$x^2 - 2x - 4 = (x - x_1)(x - x_2) = \left(x - 1 + \sqrt{5}\right)\left(x - 1 - \sqrt{5}\right)$$

The split points $1 - \sqrt{5}$ and $1 + \sqrt{5}$ divide the real line into three intervals (Figure 15). When we test them with the test points -2, 0, and 4, we conclude that the solution set for $x^2 - 2x - 4 \leq 0$ is $\left[1 - \sqrt{5}, 1 + \sqrt{5}\right]$.

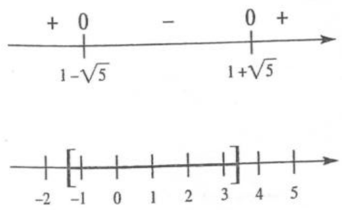


Figure 15

Squares Turning to squares, we notice that

$$\boxed{|x|^2 = x^2 \quad \text{and} \quad |x| = \sqrt{x^2}}$$

> **Notation for Roots**
>
> If n is even and $a \geq 0$ the symbol $\sqrt[n]{a}$ denotes the nonnegative nth root of a. When n is odd, there is only one real nth root of a, denoted by the symbol $\sqrt[n]{a}$. Thus, $\sqrt[4]{16} = 2$, $\sqrt[3]{27} = 3$, and $\sqrt[3]{-8} = -2$.

These follow from the property $|a||b| = |ab|$.

Does the squaring operation preserve inequalities? In general, the answer is no. For instance, $-3 < 2$, but $(-3)^2 > 2^2$. On the other hand, $2 < 3$ and $2^2 < 3^2$. If we are dealing with nonnegative numbers, then $a < b \Leftrightarrow a^2 < b^2$. A useful variant of this (see Problem 63) is

$$\boxed{|x| < |y| \Leftrightarrow x^2 < y^2}$$

EXAMPLE 14 Solve the inequality $|3x + 1| < 2|x - 6|$.

SOLUTION This inequality is more difficult to solve than our earlier examples, because there are two sets of absolute value signs. We can remove both of them by using the last boxed result.

$$\begin{aligned}
|3x + 1| < 2|x - 6| &\Leftrightarrow |3x + 1| < |2x - 12| \\
&\Leftrightarrow (3x + 1)^2 < (2x - 12)^2 \\
&\Leftrightarrow 9x^2 + 6x + 1 < 4x^2 - 48x + 144 \\
&\Leftrightarrow 5x^2 + 54x - 143 < 0 \\
&\Leftrightarrow (x + 13)(5x - 11) < 0
\end{aligned}$$

The split points for this quadratic inequality are -13 and $\frac{11}{5}$; they divide the real line into the three intervals: $(-\infty, -13)$, $\left(-13, \frac{11}{5}\right)$, and $\left(\frac{11}{5}, \infty\right)$. When we use the test points -14, 0, and 3, we discover that only the points in $\left(-13, \frac{11}{5}\right)$ satisfy the inequality. ■

Concepts Review

1. The set $\{x: -1 \leq x < 5\}$ is written in interval notation as ____ and the set $\{x: x \leq -2\}$ is written as ____.

2. If $a/b < 0$, then either $a < 0$ and ____ or $a > 0$ and ____.

3. Which of the following are always true?
(a) $|-x| = x$ (b) $|x|^2 = x^2$
(c) $|xy| = |x||y|$ (d) $\sqrt{x^2} = x$

4. The inequality $|x - 2| \leq 3$ is equivalent to ____ $\leq x \leq$ ____.

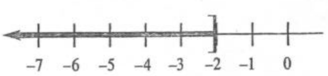
Problem Set 0.2

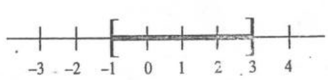
1. Show each of the following intervals on the real line.
(a) $[-1, 1]$ (b) $(-4, 1]$
(c) $(-4, 1)$ (d) $[1, 4]$
(e) $[-1, \infty)$ (f) $(-\infty, 0]$

2. Use the notation of Problem 1 to describe the following intervals.

(a)


(b)


(c)


(d)


In each of Problems 3–26, express the solution set of the given inequality in interval notation and sketch its graph.

3. $x - 7 < 2x - 5$
4. $3x - 5 < 4x - 6$
5. $7x - 2 \leq 9x + 3$
6. $5x - 3 > 6x - 4$
7. $-4 < 3x + 2 < 5$
8. $-3 < 4x - 9 < 11$
9. $-3 < 1 - 6x \leq 4$
10. $4 < 5 - 3x < 7$
11. $x^2 + 2x - 12 < 0$
12. $x^2 - 5x - 6 > 0$
13. $2x^2 + 5x - 3 > 0$
14. $4x^2 - 5x - 6 < 0$
15. $\dfrac{x + 4}{x - 3} \leq 0$
16. $\dfrac{3x - 2}{x - 1} \geq 0$
17. $\dfrac{2}{x} < 5$
18. $\dfrac{7}{4x} \leq 7$
19. $\dfrac{1}{3x - 2} \leq 4$
20. $\dfrac{3}{x + 5} > 2$

21. $(x+2)(x-1)(x-3) > 0$

22. $(2x+3)(3x-1)(x-2) < 0$

23. $(2x-3)(x-1)^2(x-3) \geq 0$

24. $(2x-3)(x-1)^2(x-3) > 0$

25. $x^3 - 5x^2 - 6x < 0$ **26.** $x^3 - x^2 - x + 1 > 0$

27. Tell whether each of the following is true or false.

(a) $-3 < -7$ (b) $-1 > -17$ (c) $-3 < -\frac{22}{7}$

28. Tell whether each of the following is true or false.

(a) $-5 > -\sqrt{26}$ (b) $\frac{6}{7} < \frac{34}{39}$ (c) $-\frac{5}{7} < -\frac{44}{59}$

29. Assume that $a > 0, b > 0$. Prove each statement. *Hint:* Each part requires two proofs: one for $\Rightarrow$ and one for $\Leftarrow$.

(a) $a < b \Leftrightarrow a^2 < b^2$ (b) $a < b \Leftrightarrow \frac{1}{a} > \frac{1}{b}$

30. Which of the following are true if $a \leq b$?

(a) $a^2 \leq ab$ (b) $a - 3 \leq b - 3$

(c) $a^3 \leq a^2 b$ (d) $-a \leq -b$

31. Find all values of x that satisfy both inequalities simultaneously.

(a) $3x + 7 > 1$ and $2x + 1 < 3$

(b) $3x + 7 > 1$ and $2x + 1 > -4$

(c) $3x + 7 > 1$ and $2x + 1 < -4$

32. Find all the values of x that satisfy at least one of the two inequalities.

(a) $2x - 7 > 1$ or $2x + 1 < 3$

(b) $2x - 7 \leq 1$ or $2x + 1 < 3$

(c) $2x - 7 \leq 1$ or $2x + 1 > 3$

33. Solve for x, expressing your answer in interval notation.

(a) $(x+1)(x^2+2x-7) \geq x^2 - 1$

(b) $x^4 - 2x^2 \geq 8$

(c) $(x^2+1)^2 - 7(x^2+1) + 10 < 0$

34. Solve each inequality. Express your solution in interval notation.

(a) $1.99 < \frac{1}{x} < 2.01$ (b) $2.99 < \frac{1}{x+2} < 3.01$

In Problems 35–44, find the solution sets of the given inequalities.

35. $|x-2| \geq 5$ **36.** $|x+2| < 1$

37. $|4x+5| \leq 10$ **38.** $|2x-1| > 2$

39. $\left|\frac{2x}{7} - 5\right| \geq 7$ **40.** $\left|\frac{x}{4} + 1\right| < 1$

41. $|5x-6| > 1$ **42.** $|2x-7| > 3$

43. $\left|\frac{1}{x} - 3\right| > 6$ **44.** $\left|2 + \frac{5}{x}\right| > 1$

In Problems 45–48, solve the given quadratic inequality using the Quadratic Formula.

45. $x^2 - 3x - 4 \geq 0$ **46.** $x^2 - 4x + 4 \leq 0$

47. $3x^2 + 17x - 6 > 0$ **48.** $14x^2 + 11x - 15 \leq 0$

In Problems 49–52, show that the indicated implication is true.

49. $|x-3| < 0.5 \Rightarrow |5x-15| < 2.5$

50. $|x+2| < 0.3 \Rightarrow |4x+8| < 1.2$

51. $|x-2| < \frac{\varepsilon}{6} \Rightarrow |6x-12| < \varepsilon$

52. $|x+4| < \frac{\varepsilon}{2} \Rightarrow |2x+8| < \varepsilon$

In Problems 53–56, find δ (depending on ε) so that the given implication is true.

53. $|x-5| < \delta \Rightarrow |3x-15| < \varepsilon$

54. $|x-2| < \delta \Rightarrow |4x-8| < \varepsilon$

55. $|x+6| < \delta \Rightarrow |6x+36| < \varepsilon$

56. $|x+5| < \delta \Rightarrow |5x+25| < \varepsilon$

57. On a lathe, you are to turn out a disk (thin right circular cylinder) of circumference 10 inches. This is done by continually measuring the diameter as you make the disk smaller. How closely must you measure the diameter if you can tolerate an error of at most 0.02 inch in the circumference?

58. Fahrenheit temperatures and Celsius temperatures are related by the formula $C = \frac{5}{9}(F - 32)$. An experiment requires that a solution be kept at 50°C with an error of at most 3% (or 1.5°). You have only a Fahrenheit thermometer. What error are you allowed on it?

In Problems 59–62, solve the inequalities.

59. $|x-1| < 2|x-3|$ **60.** $|2x-1| \geq |x+1|$

61. $2|2x-3| < |x+10|$ **62.** $|3x-1| < 2|x+6|$

63. Prove that $|x| < |y| \Leftrightarrow x^2 < y^2$ by giving a reason for each of these steps:

$$|x| < |y| \Rightarrow |x||x| \leq |x||y| \quad \text{and} \quad |x||y| < |y||y|$$
$$\Rightarrow |x|^2 < |y|^2$$
$$\Rightarrow x^2 < y^2$$

Conversely,

$$x^2 < y^2 \Rightarrow |x|^2 < |y|^2$$
$$\Rightarrow |x|^2 - |y|^2 < 0$$
$$\Rightarrow (|x| - |y|)(|x| + |y|) < 0$$
$$\Rightarrow |x| - |y| < 0$$
$$\Rightarrow |x| < |y|$$

64. Use the result of Problem 63 to show that

$$0 < a < b \Rightarrow \sqrt{a} < \sqrt{b}$$

65. Use the properties of the absolute value to show that each of the following is true.

(a) $|a-b| \leq |a| + |b|$ (b) $|a-b| \geq |a| - |b|$

(c) $|a+b+c| \leq |a| + |b| + |c|$

66. Use the Triangle Inequality and the fact that $0 < |a| < |b| \Rightarrow 1/|b| < 1/|a|$ to establish the following chain of inequalities.

$$\left|\frac{1}{x^2+3} - \frac{1}{|x|+2}\right| \leq \frac{1}{x^2+3} + \frac{1}{|x|+2} \leq \frac{1}{3} + \frac{1}{2}$$

67. Show that (see Problem 66)

$$\left|\frac{x-2}{x^2+9}\right| \leq \frac{|x|+2}{9}$$

68. Show that

$$|x| \leq 2 \Rightarrow \left|\frac{x^2+2x+7}{x^2+1}\right| \leq 15$$

0.5 Functions and Their Graphs

The concept of function is one of the most basic in all mathematics, and it plays an indispensable role in calculus.

> **Definition**
>
> A **function** f is a rule of correspondence that associates with each object x in one set, called the **domain,** a single value $f(x)$ from a second set. The set of all values so obtained is called the **range** of the function. (See Figure 1.)

Think of a function as a machine that takes as its input a value x and produces an output $f(x)$. (See Figure 2.) Each input value is matched with a *single* output value. It can, however, happen that several different input values give the same output value.

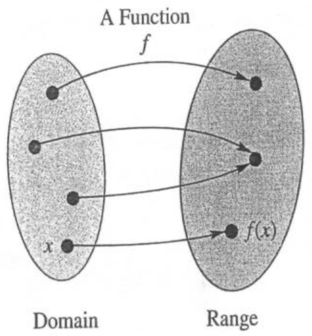


Figure 1

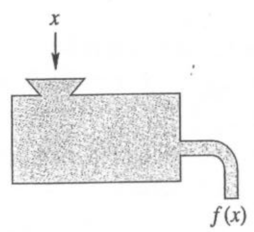


Figure 2

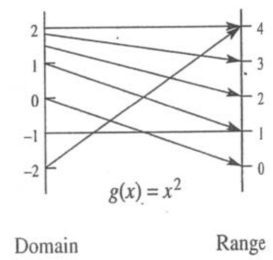


Figure 3

The definition puts no restriction on the domain and range sets. The domain might consist of the set of people in your calculus class, the range the set of grades $\{A, B, C, D, F\}$ that will be given, and the rule of correspondence the assignment of grades. Nearly all functions you encounter in this book will be functions of one or more real numbers. For example, the function g might take a real number x and square it, producing the real number x^2. In this case we have a formula that gives the rule of correspondence, that is, $g(x) = x^2$. A schematic diagram for this function is shown in Figure 3.

Function Notation A single letter like f (or g or F) is used to name a function. Then $f(x)$, read "f of x" or "f at x," denotes the value that f assigns to x. Thus, if $f(x) = x^3 - 4$, then

$$f(2) = 2^3 - 4 = 4$$
$$f(a) = a^3 - 4$$
$$f(a+h) = (a+h)^3 - 4 = a^3 + 3a^2h + 3ah^2 + h^3 - 4$$

Study the following examples carefully. Although some of these examples may look odd now, they will play an important role in Chapter 2.

EXAMPLE 1 For $f(x) = x^2 - 2x$, find and simplify

(a) $f(4)$ (b) $f(4+h)$
(c) $f(4+h) - f(4)$ (d) $[f(4+h) - f(4)]/h$

SOLUTION

(a) $f(4) = 4^2 - 2 \cdot 4 = 8$

(b) $f(4+h) = (4+h)^2 - 2(4+h) = 16 + 8h + h^2 - 8 - 2h$
$= 8 + 6h + h^2$

(c) $f(4+h) - f(4) = 8 + 6h + h^2 - 8 = 6h + h^2$

(d) $\dfrac{f(4+h) - f(4)}{h} = \dfrac{6h + h^2}{h} = \dfrac{h(6+h)}{h} = 6 + h$

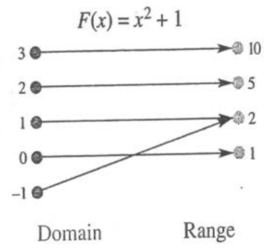


Figure 4

Domain and Range To specify a function completely, we must state, in addition to the rule of correspondence, the domain of the function. For example, if F is the function defined by $F(x) = x^2 + 1$ with domain $\{-1, 0, 1, 2, 3\}$ (Figure 4), then the range is $\{1, 2, 5, 10\}$. The rule of correspondence, together with the domain, determines the range.

When no domain is specified for a function, we assume that it is the largest set of real numbers for which the rule for the function makes sense. This is called the **natural domain.** Numbers that you should remember to exclude from the natural domain are those values that would cause division by zero or the square root of a negative number.

EXAMPLE 2 Find the natural domains for

(a) $f(x) = 1/(x - 3)$ (b) $g(t) = \sqrt{9 - t^2}$

(c) $h(w) = 1/\sqrt{9 - w^2}$

SOLUTION

(a) We must exclude 3 from the domain because it would require division by zero. Thus, the natural domain is $\{x: x \neq 3\}$. This may be read "the set of x's such that x is not equal to 3."
(b) To avoid the square root of a negative number, we must choose t so that $9 - t^2 \geq 0$. Thus, t must satisfy $|t| \leq 3$. The natural domain is therefore $\{t: |t| \leq 3\}$, which we can write using interval notation as $[-3, 3]$.
(c) Now we must avoid division by zero *and* square roots of negative numbers, so we must exclude -3 and 3 from the natural domain. The natural domain is therefore the interval $(-3, 3)$. ■

When the rule for a function is given by an equation of the form $y = f(x)$, we call x the **independent variable** and y the **dependent variable.** *Any* value in the domain may be substituted for the independent variable. Once selected, this value of x completely determines the corresponding value of the dependent variable y.

The input for a function need not be a single real number. In many important applications, a function depends on more than one independent variable. For example, the amount A of a monthly car payment depends on the loan's principal P, the rate of interest r, and the required number n of monthly payments. We could write such a function as $A(P, r, n)$. The value of $A(16000, 0.07, 48)$, that is, the required monthly payment to retire a \$16,000 loan in 48 months at an annual interest rate of 7%, is \$383.14. In this situation, there is no simple mathematical formula that gives the output A in terms of the input variables P, r, and n.

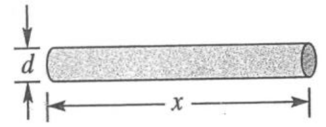


Figure 5

EXAMPLE 3 Let $V(x, d)$ denote the volume of a cylindrical rod of length x and diameter d. (See Figure 5.) Find

(a) a formula for $V(x, d)$
(b) the domain and range of V
(c) $V(4, 0.1)$

SOLUTION

(a) $V(x, d) = x \cdot \pi \left(\dfrac{d}{2}\right)^2 = \dfrac{\pi x d^2}{4}$

(b) Because the length and diameter of the rod must be positive, the domain is the set of all ordered pairs (x, d) where $x > 0$ and $d > 0$. Any positive volume is possible so the range is $(0, \infty)$.

(c) $V(4, 0.1) = \dfrac{\pi \cdot 4 \cdot 0.1^2}{4} = 0.01\pi$ ■

Chapters 1 through 11 will deal mostly with functions of a single independent variable. Beginning in Chapter 12, we will study properties of functions of two or more independent variables.

Graphing Calculator
Remember, use your graphing calculator to reproduce the figures in this book. Experiment with various graphing windows until you are convinced that you understand all important aspects of the graph.

Graphs of Functions When both the domain and range of a function are sets of real numbers, we can picture the function by drawing its graph on a coordinate plane. The **graph of a function** f is simply the graph of the equation $y = f(x)$.

EXAMPLE 4 Sketch the graphs of

(a) $f(x) = x^2 - 2$ (b) $g(x) = 2/(x - 1)$

SOLUTION The natural domains of f and g are, respectively, all real numbers and all real numbers except 1. Following the procedure described in Section 0.4 (make a table of values, plot the corresponding points, connect these points with a smooth curve), we obtain the two graphs shown in Figures 6 and 7a. ■

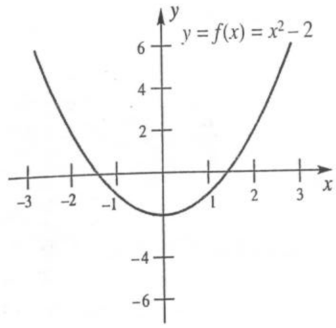


Figure 6

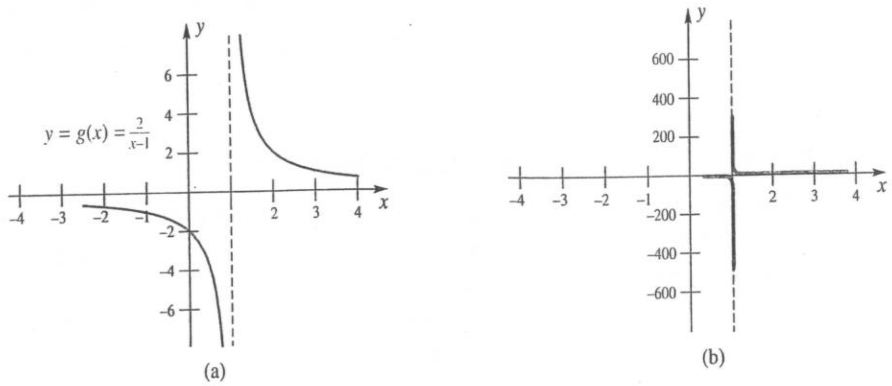


Figure 7

Pay special attention to the graph of g; it points to an oversimplification that we have made and now need to correct. When connecting the plotted points by a smooth curve, do not do so in a mechanical way that ignores special features that may be apparent from the formula for the function. In the case of $g(x) = 2/(x - 1)$, something dramatic happens as x nears 1. In fact, the values of $|g(x)|$ increase without bound; for example, $g(0.99) = 2/(0.99 - 1) = -200$ and $g(1.001) = 2000$. We have indicated this by drawing a dashed vertical line, called an **asymptote**, at $x = 1$. As x approaches 1, the graph gets closer and closer to this line, though this line itself is not part of the graph. Rather, it is a guideline. Notice that the graph of g also has a horizontal asymptote, the x-axis.

Functions like $g(x) = 2/(x - 1)$ can even cause problems when you graph them on a CAS. For example, *Maple*, when asked to plot $g(x) = 2/(x - 1)$ over the domain $[-4, 4]$ responded with the graph shown in Figure 7b. Computer Algebra Systems use an algorithm much like that described in Section 0.4; they choose a number of x-values over the stated domain, find the corresponding y-values, and plot these points with connecting lines. When *Maple* chose a number near 1, the resulting output was large, leading to the y-axis scaling in the figure. *Maple* also connected the points right across the break at $x = 1$. Always be cautious and careful when you use a graphing calculator or a CAS to plot functions.

The domains and ranges for the functions f and g are shown in the table below.

Function	Domain	Range
$f(x) = x^2 - 2$	all real numbers	$\{y: y \geq -2\}$
$g(x) = \dfrac{2}{x - 1}$	$\{x: x \neq 1\}$	$\{y: y \neq 0\}$

Even and Odd Functions We can often predict the symmetries of the graph of a function by inspecting the formula for the function. If $f(-x) = f(x)$ for all x, then the graph is symmetric with respect to the y-axis. Such a function is called an

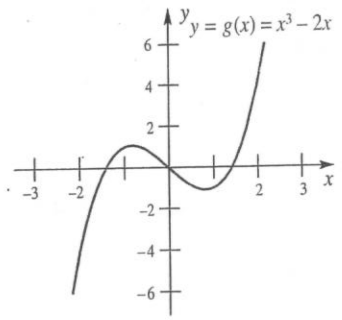


Figure 8

even function, probably because a function that specifies $f(x)$ as a sum of only even powers of x is even. The function $f(x) = x^2 - 2$ (graphed in Figure 6) is even; so are $f(x) = 3x^6 - 2x^4 + 11x^2 - 5$, $f(x) = x^2/(1 + x^4)$, and $f(x) = (x^3 - 2x)/3x$.

If $f(-x) = -f(x)$ for all x, the graph is symmetric with respect to the origin. We call such a function an **odd function.** A function that gives $f(x)$ as a sum of only odd powers of x is odd. Thus, $g(x) = x^3 - 2x$ (graphed in Figure 8) is odd. Note that

$$g(-x) = (-x)^3 - 2(-x) = -x^3 + 2x = -(x^3 - 2x) = -g(x)$$

Consider the function $g(x) = 2/(x - 1)$ from Example 4, which we graphed in Figure 7. It is neither even nor odd. To see this, observe that $g(-x) = 2/(-x - 1)$, which is not equal to either $g(x)$ or $-g(x)$. Note that the graph of $y = g(x)$ is neither symmetric with respect to the y-axis nor the origin.

EXAMPLE 5 Is $f(x) = \dfrac{x^3 + 3x}{x^4 - 3x^2 + 4}$ even, odd, or neither?

SOLUTION Since

$$f(-x) = \frac{(-x)^3 + 3(-x)}{(-x)^4 - 3(-x)^2 + 4} = \frac{-(x^3 + 3x)}{x^4 - 3x^2 + 4} = -f(x)$$

f is an odd function. The graph of $y = f(x)$ (Figure 9) is symmetric with respect to the origin. ■

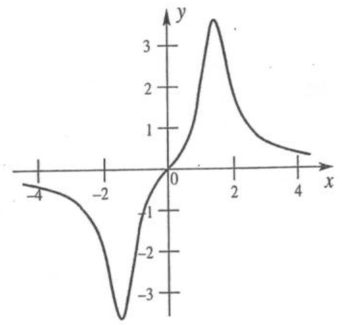


Figure 9

Two Special Functions

Among the functions that will often be used as examples are two very special ones: the **absolute value function,** | |, and the **greatest integer function,** [[ ]]. They are defined by

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

and

$$[\![x]\!] = \text{the greatest integer less than or equal to } x$$

Thus, $|-3.1| = |3.1| = 3.1$, while $[\![-3.1]\!] = -4$ and $[\![3.1]\!] = 3$. We show the graphs of these two functions in Figures 10 and 11. The absolute value function is even, since $|-x| = |x|$. The greatest integer function is neither even nor odd, as you can see from its graph.

We will often appeal to the following special features of these graphs. The graph of $|x|$ has a sharp corner at the origin, while the graph of $[\![x]\!]$ takes a jump at each integer.

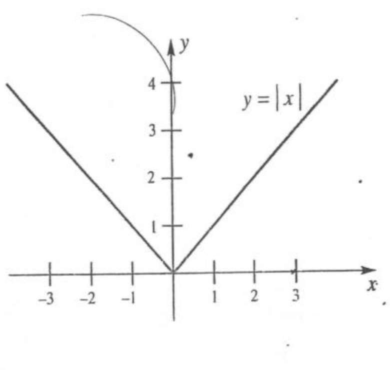


Figure 10

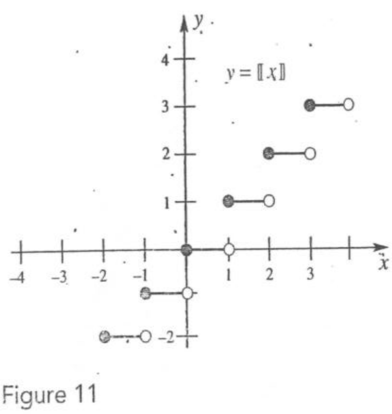


Figure 11

Concepts Review

1. The set of allowable inputs for a function is called the _____ of the function; the set of outputs that are obtained is called the _____ of the function.

2. If $f(x) = 3x^2$, then $f(2u) =$ _____ and $f(x + h) =$ _____.

3. If $f(x)$ gets closer and closer to L as $|x|$ increases indefinitely, then the line $y = L$ is a(an) _____ for the graph of f.

4. If $f(-x) = f(x)$ for all x in the domain of f, then f is called a(an) _____ function; if $f(-x) = -f(x)$ for all x in the domain of f, then f is called a(an) _____ function. In the first case, the graph of f is symmetric with respect to the _____; in the second case, it is symmetric with respect to the _____.

Problem Set 0.5

1. For $f(x) = 1 - x^2$, find each value.

(a) $f(1)$ (b) $f(-2)$ (c) $f(0)$
(d) $f(k)$ (e) $f(-5)$ (f) $f(\frac{1}{4})$
(g) $f(1 + h)$ (h) $f(1 + h) - f(1)$
(i) $f(2 + h) - f(2)$

2. For $F(x) = x^3 + 3x$, find each value.

(a) $F(1)$ (b) $F(\sqrt{2})$ (c) $F(\frac{1}{4})$
(d) $F(1 + h)$ (e) $F(1 + h) - F(1)$
(f) $F(2 + h) - F(2)$

3. For $G(y) = 1/(y - 1)$, find each value.

(a) $G(0)$ (b) $G(0.999)$ (c) $G(1.01)$
(d) $G(y^2)$ (e) $G(-x)$ (f) $G\left(\frac{1}{x^2}\right)$

4. For $\Phi(u) = \dfrac{u + u^2}{\sqrt{u}}$, find each value. ($\Phi$ is the uppercase Greek letter phi.)

(a) $\Phi(1)$ (b) $\Phi(-t)$ (c) $\Phi(\frac{1}{2})$
(d) $\Phi(u + 1)$ (e) $\Phi(x^2)$ (f) $\Phi(x^2 + x)$

5. For

$$f(x) = \frac{1}{\sqrt{x - 3}}$$

find each value.

(a) $f(0.25)$ (b) $f(\pi)$ (c) $f(3 + \sqrt{2})$

[C] 6. For $f(x) = \sqrt{x^2 + 9}/(x - \sqrt{3})$, find each value.

(a) $f(0.79)$ (b) $f(12.26)$ (c) $f(\sqrt{3})$

7. Which of the following determine a function f with formula $y = f(x)$? For those that do, find $f(x)$. *Hint:* Solve for y in terms of x and note that the definition of a function requires a single y for each x.

(a) $x^2 + y^2 = 1$ (b) $xy + y + x = 1, x \neq -1$
(c) $x = \sqrt{2y + 1}$ (d) $x = \dfrac{y}{y + 1}$

8. Which of the graphs in Figure 12 are graphs of functions?

This problem suggests a rule: *For a graph to be the graph of a function, each vertical line must meet the graph in at most one point.*

9. For $f(x) = 2x^2 - 1$, find and simplify $[f(a + h) - f(a)]/h$.

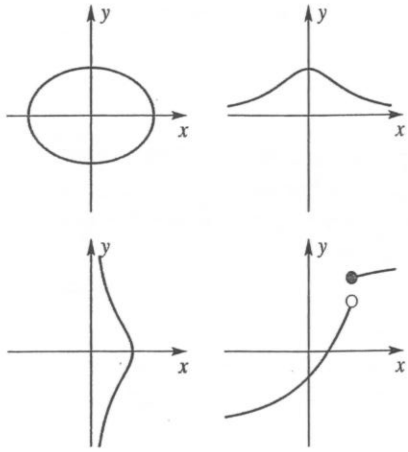


Figure 12

10. For $F(t) = 4t^3$, find and simplify $[F(a + h) - F(a)]/h$.

11. For $g(u) = 3/(u - 2)$, find and simplify $[g(x + h) - g(x)]/h$.

12. For $G(t) = t/(t + 4)$, find and simplify $[G(a + h) - G(a)]/h$.

13. Find the natural domain for each of the following.

(a) $F(z) = \sqrt{2z + 3}$ (b) $g(v) = 1/(4v - 1)$
(c) $\psi(x) = \sqrt{x^2 - 9}$ (d) $H(y) = -\sqrt{625 - y^4}$

14. Find the natural domain in each case.

(a) $f(x) = \dfrac{4 - x^2}{x^2 - x - 6}$ (b) $G(y) = \sqrt{(y + 1)^{-1}}$
(c) $\phi(u) = |2u + 3|$ (d) $F(t) = t^{2/3} - 4$

In Problems 15–30, specify whether the given function is even, odd, or neither, and then sketch its graph.

15. $f(x) = -4$ 16. $f(x) = 3x$
17. $F(x) = 2x + 1$ 18. $F(x) = 3x - \sqrt{2}$
19. $g(x) = 3x^2 + 2x - 1$ 20. $g(u) = \dfrac{u^3}{8}$
21. $g(x) = \dfrac{x}{x^2 - 1}$ 22. $\phi(z) = \dfrac{2z + 1}{z - 1}$
23. $f(w) = \sqrt{w - 1}$ 24. $h(x) = \sqrt{x^2 + 4}$
25. $f(x) = |2x|$ 26. $F(t) = -|t + 3|$
27. $g(x) = \left[\!\left[\dfrac{x}{2}\right]\!\right]$ 28. $G(x) = [\![2x - 1]\!]$

29. $g(t) = \begin{cases} 1 & \text{if } t \le 0 \\ t+1 & \text{if } 0 < t < 2 \\ t^2 - 1 & \text{if } t \ge 2 \end{cases}$

30. $h(x) = \begin{cases} -x^2 + 4 & \text{if } x \le 1 \\ 3x & \text{if } x > 1 \end{cases}$

31. A plant has the capacity to produce from 0 to 100 computers per day. The daily overhead for the plant is \$5000, and the direct cost (labor and materials) of producing one computer is \$805. Write a formula for $T(x)$, the total cost of producing x computers in one day, and also for the unit cost $u(x)$ (average cost per computer). What are the domains of these functions?

[C] **32.** It costs the ABC Company $400 + 5\sqrt{x(x-4)}$ dollars to make x toy stoves that sell for \$6 each.

(a) Find a formula for $P(x)$, the total profit in making x stoves.

(b) Evaluate $P(200)$ and $P(1000)$.

(c) How many stoves does ABC have to make to just break even?

[C] **33.** Find the formula for the amount $E(x)$ by which a number x exceeds its square. Plot a graph of $E(x)$ for $0 \le x \le 1$. Use the graph to estimate the positive number less than or equal to 1 that exceeds its square by the maximum amount.

34. Let p denote the perimeter of an equilateral triangle. Find a formula for $A(p)$, the area of such a triangle.

35. A right triangle has a fixed hypotenuse of length h and one leg that has length x. Find a formula for the length $L(x)$ of the other leg.

36. A right triangle has a fixed hypotenuse of length h and one leg that has length x. Find a formula for the area $A(x)$ of the triangle.

37. The Acme Car Rental Agency charges \$24 a day for the rental of a car plus \$0.40 per mile.

(a) Write a formula for the total rental expense $E(x)$ for one day, where x is the number of miles driven.

(b) If you rent a car for one day, how many miles can you drive for \$120?

38. A right circular cylinder of radius r is inscribed in a sphere of radius $2r$. Find a formula for $V(r)$, the volume of the cylinder, in terms of r.

39. A 1-mile track has parallel sides and equal semicircular ends. Find a formula for the area enclosed by the track, $A(d)$, in terms of the diameter d of the semicircles. What is the natural domain for this function?

40. Let $A(c)$ denote the area of the region bounded from above by the line $y = x + 1$, from the left by the y-axis, from below by the x-axis, and from the right by the line $x = c$. Such a function is called an **accumulation function.** (See Figure 13.) Find

(a) $A(1)$ (b) $A(2)$

(c) $A(0)$ (d) $A(c)$

(e) Sketch the graph of $A(c)$.

(f) What are the domain and range of A?

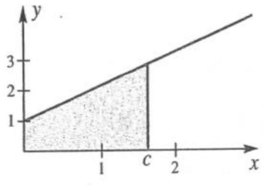


Figure 13

41. Let $B(c)$ denote the area of the region bounded from above by the graph of the curve $y = x(1-x)$, from below by the x-axis, and from the right by the line $x = c$. The domain of B is the interval $[0, 1]$. (See Figure 14.) Given that $B(1) = \frac{1}{6}$,

(a) Find $B(0)$ (b) Find $B\left(\frac{1}{2}\right)$

(c) As best you can, sketch a graph of $B(c)$.

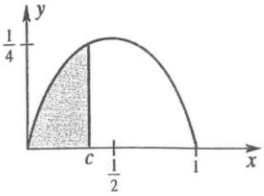


Figure 14

42. Which of the following functions satisfies $f(x+y) = f(x) + f(y)$ for all real numbers x and y?

(a) $f(t) = 2t$ (b) $f(t) = t^2$

(c) $f(t) = 2t + 1$ (d) $f(t) = -3t$

43. Let $f(x+y) = f(x) + f(y)$ for all x and y. Prove that there is a number m such that $f(t) = mt$ for all rational numbers t. *Hint:* First decide what m has to be. Then proceed in steps, starting with $f(0) = 0, f(p) = mp$ for a natural number p, $f(1/p) = m/p$, and so on.

44. A baseball diamond is a square with sides of 90 feet. A player, after hitting a home run, loped around the diamond at 10 feet per second. Let s represent the player's distance from home plate after t seconds.

(a) Express s as a function of t by means of a four-part formula.

(b) Express s as a function of t by means of a three-part formula.

[GC] *To use technology effectively, you need to discover its capabilities, its strengths, and its weaknesses. We urge you to practice graphing functions of various types using your own computer package or calculator. Problems 45–50 are designed for this purpose.*

45. Let $f(x) = (x^3 + 3x - 5)/(x^2 + 4)$.

(a) Evaluate $f(1.38)$ and $f(4.12)$.

(b) Construct a table of values for this function corresponding to $x = -4, -3, \ldots, 3, 4$.

46. Follow the instructions in Problem 45 for $f(x) = (\sin^2 x - 3\tan x)/\cos x$.

47. Draw the graph of $f(x) = x^3 - 5x^2 + x + 8$ on the domain $[-2, 5]$.

(a) Determine the range of f.

(b) Where on this domain is $f(x) \ge 0$?

48. Superimpose the graph of $g(x) = 2x^2 - 8x - 1$ with domain $[-2, 5]$ on the graph of $f(x)$ of Problem 47.

(a) Estimate the x-values where $f(x) = g(x)$.

(b) Where on $[-2, 5]$ is $f(x) \ge g(x)$?

(c) Estimate the largest value of $|f(x) - g(x)|$ on $[-2, 5]$.

49. Graph $f(x) = (3x - 4)/(x^2 + x - 6)$ on the domain $[-6, 6]$.

(a) Determine the x- and y-intercepts.

(b) Determine the range of f for the given domain.

(c) Determine the vertical asymptotes of the graph.

(d) Determine the horizontal asymptote for the graph when the domain is enlarged to the natural domain.

50. Follow the directions in Problem 49 for the function $g(x) = (3x^2 - 4)/(x^2 + x - 6)$

Answers to Concepts Review: **1.** domain; range **2.** $12u^2$; $3(x + h)^2 = 3x^2 + 6xh + 3h^2$ **3.** asymptote **4.** even; odd; y-axis; origin

0.6 Operations on Functions

Just as two numbers a and b can be added to produce a new number $a + b$, so two functions f and g can be added to produce a new function $f + g$. This is just one of several operations on functions that we will describe in this section.

Sums, Differences, Products, Quotients, and Powers Consider functions f and g with formulas

$$f(x) = \frac{x - 3}{2}, \qquad g(x) = \sqrt{x}$$

We can make a new function $f + g$ by having it assign to x the value $f(x) + g(x) = (x - 3)/2 + \sqrt{x}$; that is,

$$(f + g)(x) = f(x) + g(x) = \frac{x - 3}{2} + \sqrt{x}$$

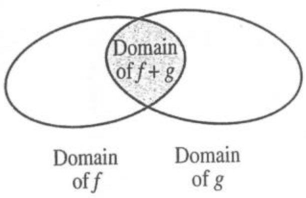


Figure 1

Of course, we must be a little careful about domains. Clearly, x must be a number on which both f and g can work. In other words, the domain of $f + g$ is the intersection (common part) of the domains of f and g (Figure 1).

The functions $f - g$, $f \cdot g$, and f/g are introduced in a completely analogous way. Assuming that f and g have their natural domains, we have the following:

Formula	Domain
$(f + g)(x) = f(x) + g(x) = \dfrac{x - 3}{2} + \sqrt{x}$	$[0, \infty)$
$(f - g)(x) = f(x) - g(x) = \dfrac{x - 3}{2} - \sqrt{x}$	$[0, \infty)$
$(f \cdot g)(x) = f(x) \cdot g(x) = \dfrac{x - 3}{2}\sqrt{x}$	$[0, \infty)$
$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{x - 3}{2\sqrt{x}}$	$(0, \infty)$

We had to exclude 0 from the domain of f/g to avoid division by 0.

We may also raise a function to a power. By f^n, we mean the function that assigns to x the value $[f(x)]^n$. Thus,

$$g^3(x) = [g(x)]^3 = \left(\sqrt{x}\right)^3 = x^{3/2}$$

There is one exception to the above agreement on exponents, namely, when $n = -1$. We reserve the symbol f^{-1} for the inverse function, which will be discussed in Section 6.2. Thus, f^{-1} does not mean $1/f$.

EXAMPLE 1 Let $F(x) = \sqrt[4]{x + 1}$ and $G(x) = \sqrt{9 - x^2}$, with respective natural domains $[-1, \infty)$ and $[-3, 3]$. Find formulas for $F + G$, $F - G$, $F \cdot G$, F/G, and F^5 and give their natural domains.

SOLUTION

Formula	Domain
$(F + G)(x) = F(x) + G(x) = \sqrt[4]{x+1} + \sqrt{9 - x^2}$	$[-1, 3]$
$(F - G)(x) = F(x) - G(x) = \sqrt[4]{x+1} - \sqrt{9 - x^2}$	$[-1, 3]$
$(F \cdot G)(x) = F(x) \cdot G(x) = \sqrt[4]{x+1}\sqrt{9 - x^2}$	$[-1, 3]$
$\left(\frac{F}{G}\right)(x) = \frac{F(x)}{G(x)} = \frac{\sqrt[4]{x+1}}{\sqrt{9 - x^2}}$	$[-1, 3)$
$F^5(x) = [F(x)]^5 = \left(\sqrt[4]{x+1}\right)^5 = (x+1)^{5/4}$	$[-1, \infty)$ ■

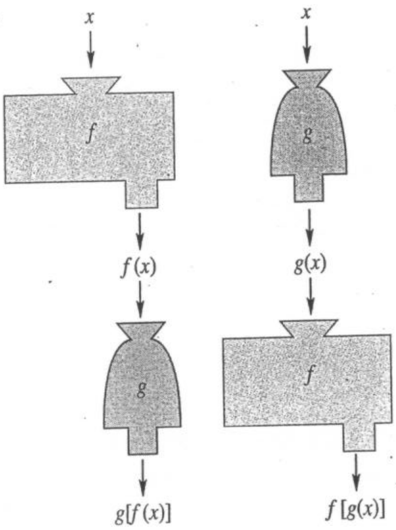


Figure 2

Composition of Functions Earlier, we asked you to think of a function as a machine. It accepts x as input, works on x, and produces $f(x)$ as output. Two machines may often be put together in tandem to make a more complicated machine; so may two functions f and g (Figure 2). If f works on x to produce $f(x)$ and g then works on $f(x)$ to produce $g(f(x))$, we say that we have *composed* g with f. The resulting function, called the **composition** of g with f, is denoted by $g \circ f$. Thus,

$$\boxed{(g \circ f)(x) = g(f(x))}$$

In our previous examples we had $f(x) = (x - 3)/2$ and $g(x) = \sqrt{x}$. We may compose these functions in two ways:

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{x-3}{2}\right) = \sqrt{\frac{x-3}{2}}$$

$$(f \circ g)(x) = f(g(x)) = f\left(\sqrt{x}\right) = \frac{\sqrt{x} - 3}{2}$$

Right away we notice that $g \circ f$ does not equal $f \circ g$. Thus, we say that the composition of functions is not commutative.

We must be careful in describing the domain of a composite function. The domain of $g \circ f$ is equal to the set of those values x that satisfy the following properties:

1. x is in the domain of f.
2. $f(x)$ is in the domain of g.

In other words, x must be a valid input for f, and $f(x)$ must be a valid input for g. In our example, the value $x = 2$ is in the domain of f, but it is not in the domain of $g \circ f$ because this would lead to the square root of a negative number:

$$g(f(2)) = g((2-3)/2) = g\left(-\frac{1}{2}\right) = \sqrt{-\frac{1}{2}}$$

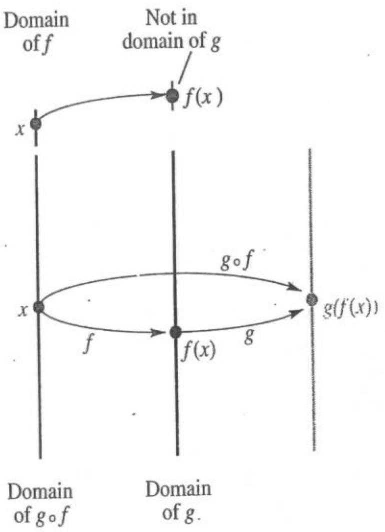


Figure 3

The domain for $g \circ f$ is the interval $[3, \infty)$ because $f(x)$ is nonnegative on this interval, and the input to g must be nonnegative. The domain for $f \circ g$ is the interval $[0, \infty)$ (why?), so we see that the domains of $g \circ f$ and $f \circ g$ can be different. Figure 3 shows how the domain of $g \circ f$ excludes those values of x for which $f(x)$ is not in the domain of g.

EXAMPLE 2 Let $f(x) = 6x/(x^2 - 9)$ and $g(x) = \sqrt{3x}$, with their natural domains. First, find $(f \circ g)(12)$; then find $(f \circ g)(x)$ and give its domain.

SOLUTION

$$(f \circ g)(12) = f(g(12)) = f\left(\sqrt{36}\right) = f(6) = \frac{6 \cdot 6}{6^2 - 9} = \frac{4}{3}$$

$$(f \circ g)(x) = f(g(x)) = f\left(\sqrt{3x}\right) = \frac{6\sqrt{3x}}{\left(\sqrt{3x}\right)^2 - 9}$$

The expression $\sqrt{3x}$ appears in both the numerator and denominator. Any negative number for x will lead to the square root of a negative number. Thus, all negative numbers must be excluded from the domain of $f \circ g$. For $x \geq 0$, we have $\left(\sqrt{3x}\right)^2 = 3x$, allowing us to write

$$(f \circ g)(x) = \frac{6\sqrt{3x}}{3x - 9} = \frac{2\sqrt{3x}}{x - 3}$$

We must also exclude $x = 3$ from the domain of $f \circ g$ because $g(3)$ is not in the domain of f. (It would cause division by 0.) Thus, the domain of $f \circ g$ is $[0, 3) \cup (3, \infty)$. ■

In calculus, we will often need to take a given function and write it as the composition of two simpler functions. Usually, this can be done in a number of ways. For example, $p(x) = \sqrt{x^2 + 4}$ can be written as

$$p(x) = g(f(x)), \qquad \text{where } g(x) = \sqrt{x} \quad \text{and} \quad f(x) = x^2 + 4$$

or as

$$p(x) = g(f(x)), \qquad \text{where } g(x) = \sqrt{x + 4} \quad \text{and} \quad f(x) = x^2$$

(You should check that both of these compositions give $p(x) = \sqrt{x^2 + 4}$ with domain $(-\infty, \infty)$.) The decomposition $p(x) = g(f(x))$ with $f(x) = x^2 + 4$ and $g(x) = \sqrt{x}$ is regarded as simpler and is usually preferred. We can therefore view $p(x) = \sqrt{x^2 + 4}$ as the square root of a function of x. This way of looking at functions will be important in Chapter 2.

EXAMPLE 3 Write the function $p(x) = (x + 2)^5$ as a composite function $g \circ f$.

SOLUTION The most obvious way to decompose p is to write

$$p(x) = g(f(x)), \qquad \text{where } g(x) = x^5 \quad \text{and} \quad f(x) = x + 2$$

We thus view $p(x) = (x + 2)^5$ as the fifth power of a function of x. ■

Translations Observing how a function is built up from simpler ones can be a big aid in graphing. We may ask this question: How are the graphs of

$$y = f(x) \qquad y = f(x - 3) \qquad y = f(x) + 2 \qquad y = f(x - 3) + 2$$

related to each other? Consider $f(x) = |x|$ as an example. The corresponding four graphs are displayed in Figure 4.

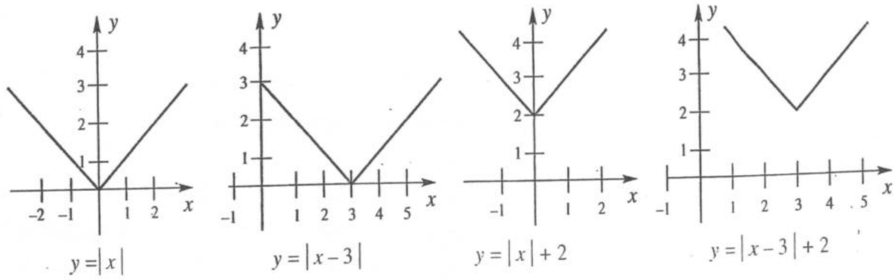


Figure 4

Notice that all four graphs have the same shape; the last three are just translations of the first. Replacing x by $x - 3$ translates the graph 3 units to the right; adding 2 translates it upward by 2 units.

What happened with $f(x) = |x|$ is typical. Figure 5 offers an illustration for the function $f(x) = x^3 + x^2$.

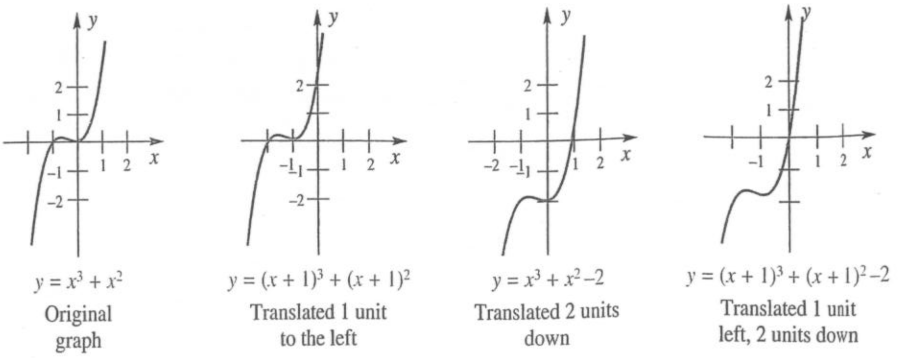


Figure 5

Exactly the same principles apply in the general situation. They are illustrated in Figure 6 with both h and k positive. If $h < 0$, the translation is to the left; if $k < 0$, the translation is downward.

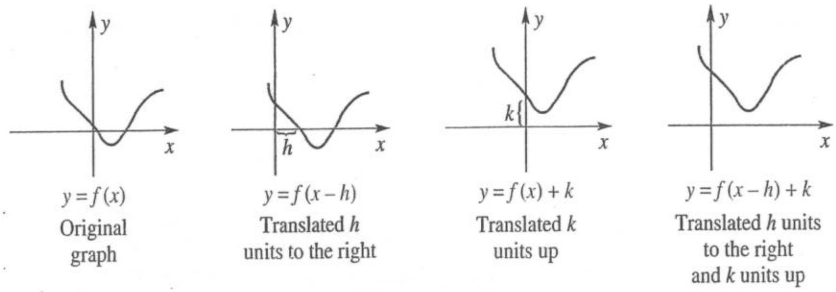


Figure 6

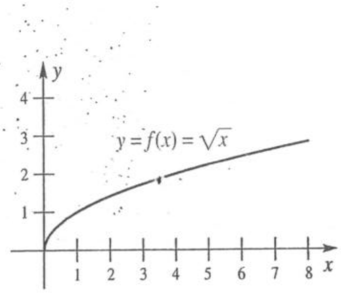


Figure 7

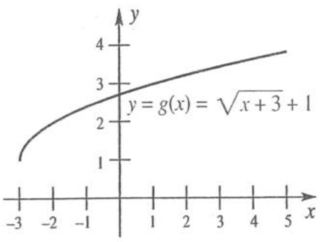


Figure 8

EXAMPLE 4 Sketch the graph of $g(x) = \sqrt{x + 3} + 1$ by first graphing $f(x) = \sqrt{x}$ and then making appropriate translations.

SOLUTION By translating the graph of f (Figure 7) 3 units left and 1 unit up, we obtain the graph of g (Figure 8). ■

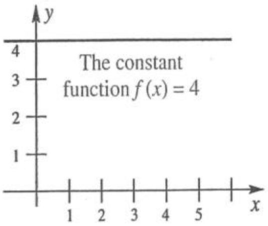


Figure 9

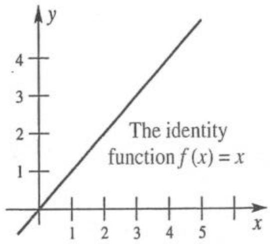


Figure 10

Partial Catalog of Functions

A function of the form $f(x) = k$, where k is a constant (real number), is called a **constant function.** Its graph is a horizontal line (Figure 9). The function $f(x) = x$ is called the **identity function.** Its graph is a line through the origin having slope 1 (Figure 10). From these simple functions, we can build many important functions.

Any function that can be obtained from the constant functions and the identity function by use of the operations of addition, subtraction, and multiplication is called a **polynomial function.** This amounts to saying that f is a polynomial function if it is of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the a's are real numbers and n is a nonnegative integer. If $a_n \neq 0$, n is the **degree** of the polynomial function. In particular, $f(x) = ax + b$ is a first-degree polynomial function, or **linear function,** and $f(x) = ax^2 + bx + c$ is a second-degree polynomial function, or **quadratic function.**

Quotients of polynomial functions are called **rational functions.** Thus, f is a rational function if it is of the form

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

The domain of a rational function consists of those real numbers for which the denominator is nonzero.

An **explicit algebraic function** is one that can be obtained from the constant functions and the identity function via the five operations of addition, subtraction, multiplication, division, and root extraction. Examples are

$$f(x) = 3x^{2/5} = 3\sqrt[5]{x^2} \qquad g(x) = \frac{(x+2)\sqrt{x}}{x^3 + \sqrt[3]{x^2 - 1}}$$

The functions listed so far, together with the trigonometric, inverse trigonometric, exponential, and logarithmic functions (to be introduced later), are the basic raw materials for calculus.

Concepts Review

1. If $f(x) = x^2 + 1$, then $f^3(x) =$ ____.

2. The value of the composite function $f \circ g$ at x is given by $(f \circ g)(x) =$ ____.

3. Compared to the graph of $y = f(x)$, the graph of $y = f(x+2)$ is translated ____ units to the ____.

4. A rational function is defined as ____.

Problem Set 0.6

1. For $f(x) = x + 3$ and $g(x) = x^2$, find each value (if possible).
(a) $(f + g)(2)$ (b) $(f \cdot g)(0)$ (c) $(g/f)(3)$
(d) $(f \circ g)(1)$ (e) $(g \circ f)(1)$ (f) $(g \circ f)(-8)$

2. For $f(x) = x^2 + x$ and $g(x) = 2/(x+3)$, find each value.
(a) $(f - g)(2)$ (b) $(f/g)(1)$ (c) $g^2(3)$
(d) $(f \circ g)(1)$ (e) $(g \circ f)(1)$ (f) $(g \circ g)(3)$

3. For $\Phi(u) = u^3 + 1$ and $\Psi(v) = 1/v$, find each value.
(a) $(\Phi + \Psi)(t)$ (b) $(\Phi \circ \Psi)(r)$
(c) $(\Psi \circ \Phi)(r)$ (d) $\Phi^3(z)$
(e) $(\Phi - \Psi)(5t)$ (f) $((\Phi - \Psi) \circ \Psi)(t)$

4. If $f(x) = \sqrt{x^2 - 1}$ and $g(x) = 2/x$, find formulas for the following and state their domains.
(a) $(f \cdot g)(x)$ (b) $f^4(x) + g^4(x)$
(c) $(f \circ g)(x)$ (d) $(g \circ f)(x)$

5. If $f(s) = \sqrt{s^2 - 4}$ and $g(w) = |1 + w|$, find formulas for $(f \circ g)(x)$ and $(g \circ f)(x)$.

6. If $g(x) = x^2 + 1$, find formulas for $g^3(x)$ and $(g \circ g \circ g)(x)$.

$\boxed{C}$ 7. Calculate $g(3.141)$ if $g(u) = \dfrac{\sqrt{u^3 + 2u}}{2 + u}$.

$\boxed{C}$ 8. Calculate $g(2.03)$ if $g(x) = \dfrac{(\sqrt{x} - \sqrt[3]{x})^4}{1 - x + x^2}$.

$\boxed{C}$ 9. Calculate $[g^2(\pi) - g(\pi)]^{1/3}$ if $g(v) = |11 - 7v|$.

$\boxed{C}$ 10. Calculate $[g^3(\pi) - g(\pi)]^{1/3}$ if $g(x) = 6x - 11$.

11. Find f and g so that $F = g \circ f$. (See Example 3.)
(a) $F(x) = \sqrt{x + 7}$ (b) $F(x) = (x^2 + x)^{15}$

12. Find f and g so that $p = f \circ g$.
(a) $p(x) = \dfrac{2}{(x^2 + x + 1)^3}$ (b) $p(x) = \dfrac{1}{x^3 + 3x}$

13. Write $p(x) = 1/\sqrt{x^2 + 1}$ as a composite of three functions in two different ways.

14. Write $p(x) = 1/\sqrt{x^2 + 1}$ as a composite of four functions.

15. Sketch the graph of $f(x) = \sqrt{x - 2} - 3$ by first sketching $g(x) = \sqrt{x}$ and then translating. (See Example 4.)

16. Sketch the graph of $g(x) = |x + 3| - 4$ by first sketching $h(x) = |x|$ and then translating.

17. Sketch the graph of $f(x) = (x - 2)^2 - 4$ using translations.

18. Sketch the graph of $g(x) = (x + 1)^3 - 3$ using translations.

19. Sketch the graphs of $f(x) = (x - 3)/2$ and $g(x) = \sqrt{x}$ using the same coordinate axes. Then sketch $f + g$ by adding y-coordinates.

20. Follow the directions of Problem 19 for $f(x) = x$ and $g(x) = |x|$.

21. Sketch the graph of $F(t) = \dfrac{|t| - t}{t}$.

22. Sketch the graph of $G(t) = t - [\![t]\!]$.

23. State whether each of the following is an odd function, an even function, or neither. Prove your statements.

(a) The sum of two even functions

(b) The sum of two odd functions

(c) The product of two even functions

(d) The product of two odd functions

(e) The product of an even function and an odd function

24. Let F be any function whose domain contains $-x$ whenever it contains x. Prove each of the following.

(a) $F(x) - F(-x)$ is an odd function.

(b) $F(x) + F(-x)$ is an even function.

(c) F can always be expressed as the sum of an odd and an even function.

25. Is every polynomial of even degree an even function? Is every polynomial of odd degree an odd function? Explain.

26. Classify each of the following as a PF (polynomial function), RF (rational function but not a polynomial function), or neither.

(a) $f(x) = 3x^{1/2} + 1$ (b) $f(x) = 3$

(c) $f(x) = 3x^2 + 2x^{-1}$ (d) $f(x) = \pi x^3 - 3\pi$

(e) $f(x) = \dfrac{1}{x+1}$ (f) $f(x) = \dfrac{x+1}{\sqrt{x+3}}$

27. The relationship between the unit price P (in cents) for a certain product and the demand D (in thousands of units) appears to satisfy

$$P = \sqrt{29 - 3D + D^2}$$

On the other hand, the demand has risen over the t years since 1970 according to $D = 2 + \sqrt{t}$.

(a) Express P as a function of t.

(b) Evaluate P when $t = 15$.

28. After being in business for t years, a manufacturer of cars is making $120 + 2t + 3t^2$ units per year. The sales price in dollars per unit has risen according to the formula $6000 + 700t$. Write a formula for the manufacturer's yearly revenue $R(t)$ after t years.

29. Starting at noon, airplane A flies due north at 400 miles per hour. Starting 1 hour later, airplane B flies due east at 300 miles per hour. Neglecting the curvature of the Earth and assuming that they fly at the same altitude, find a formula for $D(t)$, the distance between the two airplanes t hours after noon. *Hint:* There will be two formulas for $D(t)$, one if $0 < t < 1$ and the other if $t \geq 1$.

≈ C **30.** Find the distance between the airplanes of Problem 29 at 2:30 p.m.

31. Let $f(x) = \dfrac{ax+b}{cx-a}$. Show that $f(f(x)) = x$, provided $a^2 + bc \neq 0$ and $x \neq a/c$.

32. Let $f(x) = \dfrac{x-3}{x+1}$. Show that $f(f(f(x))) = x$, provided $x \neq \pm 1$.

33. Let $f(x) = \dfrac{x}{x-1}$. Find and simplify each value.

(a) $f(1/x)$ (b) $f(f(x))$ (c) $f(1/f(x))$

34. Let $f(x) = \dfrac{x}{\sqrt{x} - 1}$. Find and simplify.

(a) $f\left(\dfrac{1}{x}\right)$ (b) $f(f(x))$

35. Prove that the operation of composition of functions is associative; that is, $f_1 \circ (f_2 \circ f_3) = (f_1 \circ f_2) \circ f_3$.

36. Let $f_1(x) = x$, $f_2(x) = 1/x$, $f_3(x) = 1 - x$, $f_4(x) = 1/(1-x)$, $f_5(x) = (x-1)/x$, and $f_6(x) = x/(x-1)$. Note that $f_3(f_4(x)) = f_3(1/(1-x)) = 1 - 1/(1-x) = x/(x-1) = f_6(x)$; that is, $f_3 \circ f_4 = f_6$. In fact, the composition of any two of these functions is another one in the list. Fill in the composition table in Figure 11.

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6
f_1						
f_2						
f_3				f_6		
f_4						
f_5						
f_6						

Figure 11

Then use this table to find each of the following. From Problem 35, you know that the associative law holds.

(a) $f_3 \circ f_3 \circ f_3 \circ f_3 \circ f_3$ (b) $f_1 \circ f_2 \circ f_3 \circ f_4 \circ f_5 \circ f_6$

(c) F if $F \circ f_6 = f_1$ (d) G if $G \circ f_3 \circ f_6 = f_1$

(e) H if $f_2 \circ f_5 \circ H = f_5$

GC *Use a computer or a graphing calculator in Problems 37–40.*

37. Let $f(x) = x^2 - 3x$. Using the same axes, draw the graphs of $y = f(x)$, $y = f(x - 0.5) - 0.6$, and $y = f(1.5x)$, all on the domain $[-2, 5]$.

38. Let $f(x) = |x^3|$. Using the same axes, draw the graphs of $y = f(x)$, $y = f(3x)$, and $y = f(3(x - 0.8))$, all on the domain $[-3, 3]$.

39. Let $f(x) = 2\sqrt{x} - 2x + 0.25x^2$. Using the same axes, draw the graphs of $y = f(x)$, $y = f(1.5x)$, and $y = f(x-1) + 0.5$, all on the domain $[0, 5]$.

40. Let $f(x) = 1/(x^2 + 1)$. Using the same axes, draw the graphs of $y = f(x)$, $y = f(2x)$, and $y = f(x-2) + 0.6$, all on the domain $[-4, 4]$.

CAS **41.** Your computer algebra system (CAS) may allow the use of parameters in defining functions. In each case, draw the graph of $y = f(x)$ for the specified values of the parameter k, using the same axes and $-5 \leq x \leq 5$.

(a) $f(x) = |kx|^{0.7}$ for $k = 1, 2, 0.5$, and 0.2.

(b) $f(x) = |x - k|^{0.7}$ for $k = 0, 2, -0.5$, and -3.

(c) $f(x) = |x|^k$ for $k = 0.4, 0.7, 1$, and 1.7.

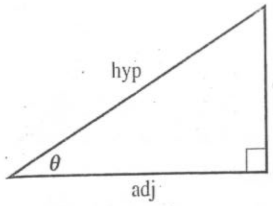
CAS **42.** Using the same axes, draw the graph of $f(x) = |k(x - c)|^n$ for the following choices of parameters.

(a) $c = -1, k = 1.4, n = 0.7$ (b) $c = 2, k = 1.4, n = 1$

(c) $c = 0, k = 0.9, n = 0.6$

Answers to Concepts Review: **1.** $(x^2 + 1)^3$ **2.** $f(g(x))$ **3.** 2; left **4.** a quotient of two polynomial functions

0.7 Trigonometric Functions



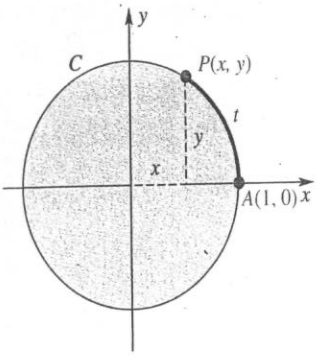
$\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$ $\cos\theta = \dfrac{\text{adj}}{\text{hyp}}$ $\tan\theta = \dfrac{\text{opp}}{\text{adj}}$

Figure 1

You have probably seen the definitions of the trigonometric functions based on right triangles. Figure 1 summarizes the definitions of the sine, cosine, and tangent functions. You should review Figure 1 carefully, because these concepts are needed for many applications later in this book.

More generally, we define the trigonometric functions based on the unit circle. The unit circle, which we denote by C, is the circle with radius 1 and center at the origin; it has equation $x^2 + y^2 = 1$. Let A be the point $(1, 0)$ and let t be a positive number. There is a single point P on the circle C such that the distance, measured in the *counterclockwise* direction around the arc AP, is equal to t. (See Figure 2.) Recall that the circumference of a circle with radius r is $2\pi r$, so the circumference of C is 2π. Thus, if $t = \pi$, then the point P is exactly halfway around the circle from the point A; in this case, P is the point $(-1, 0)$. If $t = 3\pi/2$, then P is the point $(0, -1)$, and if $t = 2\pi$, then P is the point A. If $t > 2\pi$, then it will take more than one complete circuit of the circle C to trace the arc AP.

When $t < 0$, we trace the circle in a *clockwise* direction. There will be a single point P on the circle C such that the arc length measured in the clockwise direction from A is t. Thus, for every real number t, we can associate a unique point $P(x, y)$ on the unit circle. This allows us to make the key definitions of the sine and cosine functions. The functions sine and cosine are written as sin and cos, rather than as a single letter such as f or g. Parentheses around the independent variable are usually omitted unless there is some ambiguity.



The unit circle

Figure 2

Definition **Sine and Cosine Functions**

Let t be a real number that determines the point $P(x, y)$ as indicated above. Then

$$\sin t = y \quad \text{and} \quad \cos t = x$$

Basic Properties of Sine and Cosine A number of facts follow almost immediately from the definitions given above. First, since t can be any real number, the domain for both the sine and cosine functions is $(-\infty, \infty)$. Second, x and y are always between -1 and 1. Thus, the range for both the sine and cosine functions is the interval $[-1, 1]$.

Because the unit circle has circumference 2π, the values t and $t + 2\pi$ determine the *same* point $P(x, y)$. Thus,

$$\sin(t + 2\pi) = \sin t \quad \text{and} \quad \cos(t + 2\pi) = \cos t$$

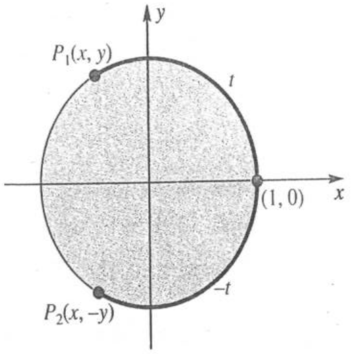


Figure 3

(Notice that parentheses are needed to make it clear that we mean $\sin(t + 2\pi)$, rather than $(\sin t) + 2\pi$. The expression $\sin t + 2\pi$ would be ambiguous.)

The points P_1 and P_2 that correspond to t and $-t$, respectively, are symmetric about the x-axis (Figure 3). Thus, the x-coordinates for P_1 and P_2 are the same, and the y-coordinates differ only in sign. Consequently,

$$\sin(-t) = -\sin t \quad \text{and} \quad \cos(-t) = \cos t$$

In other words, sine is an odd function and cosine is an even function.

The points P_3 and P_4 corresponding to t and $\pi/2 - t$, respectively, are symmetric with respect to the line $y = x$ and thus they have their coordinates interchanged (Figure 4). This means that

$$\sin\left(\frac{\pi}{2} - t\right) = \cos t \quad \text{and} \quad \cos\left(\frac{\pi}{2} - t\right) = \sin t$$

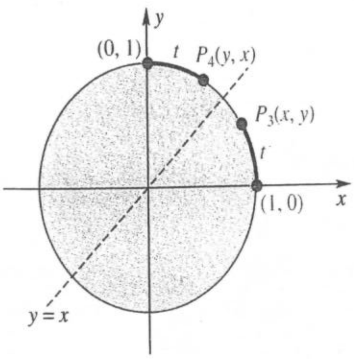


Figure 4

Finally, we mention an important identity connecting the sine and cosine functions:

$$\sin^2 t + \cos^2 t = 1$$

for every real number t. This identity follows from the fact that since the point (x, y) is on the unit circle, x and y satisfy $x^2 + y^2 = 1$.

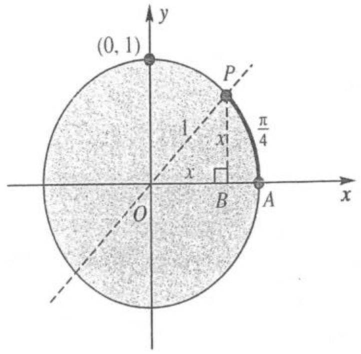


Figure 5

Graphs of Sine and Cosine To graph $y = \sin t$ and $y = \cos t$, we follow our usual procedure of making a table of values, plotting the corresponding points, and connecting these points with a smooth curve. So far, however, we know the values of sine and cosine for only a few values of t. A number of other values can be determined from geometric arguments. For example, if $t = \pi/4$, then t determines the point half of the way counterclockwise around the unit circle between the points $(1, 0)$ and $(0, 1)$. By symmetry, x and y will be on the line $y = x$, so $y = \sin t$ and $x = \cos t$ will be equal. Thus, the two legs of the right triangle OBP are equal, and the hypotenuse is 1 (Figure 5). The Pythagorean Theorem can be applied to give

$$1 = x^2 + x^2 = \cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{4}$$

From this we conclude that $\cos(\pi/4) = 1/\sqrt{2} = \sqrt{2}/2$. Similarly, $\sin(\pi/4) = \sqrt{2}/2$. We can determine $\sin t$ and $\cos t$ for a number of other values of t. Some of these are shown in the table in the margin. Using these results, along with a number of results from a calculator (in radian mode), we obtain the graphs shown in Figure 6.

t	$\sin t$	$\cos t$
0	0	1
$\pi/6$	1/2	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	1/2
$\pi/2$	1	0
$2\pi/3$	$\sqrt{3}/2$	$-1/2$
$3\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$
$5\pi/6$	1/2	$-\sqrt{3}/2$
π	0	-1

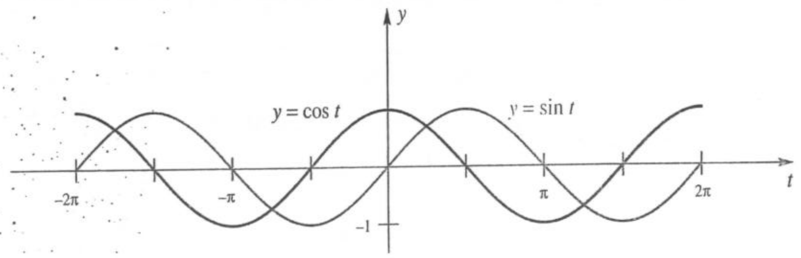


Figure 6

Four things are noticeable from these graphs:

1. Both $\sin t$ and $\cos t$ range from -1 to 1.
2. Both graphs repeat themselves on adjacent intervals of length 2π.
3. The graph of $y = \sin t$ is symmetric about the origin, and $y = \cos t$ is symmetric about the y-axis. (Thus, the sine function is odd and the cosine function is even.)
4. The graph of $y = \sin t$ is the same as that of $y = \cos t$, but translated $\pi/2$ units to the right.

The next example deals with functions of the form $\sin(at)$ or $\cos(at)$, which occur frequently in applications.

EXAMPLE 1 Sketch the graphs of

(a) $y = \sin(2\pi t)$ (b) $y = \cos(2t)$

SOLUTION

(a) As t goes from 0 to 1, the argument $2\pi t$ goes from 0 to 2π. Thus, the graph of this function will repeat itself on adjacent intervals of length 1. From the entries in the following table, we can sketch a graph of $y = \sin(2\pi t)$.

t	$\sin(2\pi t)$	t	$\sin(2\pi t)$
0	$\sin(2\pi\cdot 0) = 0$	$\frac{5}{8}$	$\sin\left(2\pi\cdot\frac{5}{8}\right) = -\frac{\sqrt{2}}{2}$
$\frac{1}{8}$	$\sin\left(2\pi\cdot\frac{1}{8}\right) = \frac{\sqrt{2}}{2}$	$\frac{3}{4}$	$\sin\left(2\pi\cdot\frac{3}{4}\right) = -1$
$\frac{1}{4}$	$\sin\left(2\pi\cdot\frac{1}{4}\right) = 1$	$\frac{7}{8}$	$\sin\left(2\pi\cdot\frac{7}{8}\right) = -\frac{\sqrt{2}}{2}$
$\frac{3}{8}$	$\sin\left(2\pi\cdot\frac{3}{8}\right) = \frac{\sqrt{2}}{2}$	1	$\sin(2\pi\cdot 1) = 0$
$\frac{1}{2}$	$\sin\left(2\pi\cdot\frac{1}{2}\right) = 0$	$\frac{9}{8}$	$\sin\left(2\pi\cdot\frac{9}{8}\right) = \frac{\sqrt{2}}{2}$

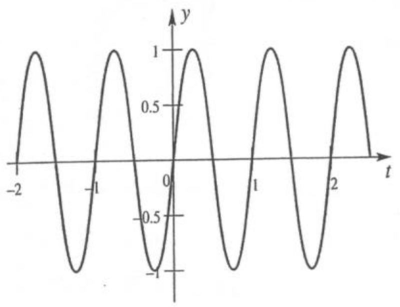


Figure 7

Figure 7 shows a sketch of the graph of $y = \sin(2\pi t)$.

(b) As t goes from 0 to π, the argument $2t$ goes from 0 to 2π. Thus, the graph of $y = \cos(2t)$ will repeat itself on adjacent intervals of length π. Once we construct a table we can sketch a plot of $y = \cos(2t)$. Figure 8 shows the graph.

t	$\cos(2t)$	t	$\cos(2t)$
0	$\cos(2\cdot 0) = 1$	$\frac{5\pi}{8}$	$\cos\left(2\cdot\frac{5\pi}{8}\right) = -\frac{\sqrt{2}}{2}$
$\frac{\pi}{8}$	$\cos\left(2\cdot\frac{\pi}{8}\right) = \frac{\sqrt{2}}{2}$	$\frac{3\pi}{4}$	$\cos\left(2\cdot\frac{3\pi}{4}\right) = 0$
$\frac{\pi}{4}$	$\cos\left(2\cdot\frac{\pi}{4}\right) = 0$	$\frac{7\pi}{8}$	$\cos\left(2\cdot\frac{7\pi}{8}\right) = \frac{\sqrt{2}}{2}$
$\frac{3\pi}{8}$	$\cos\left(2\cdot\frac{3\pi}{8}\right) = -\frac{\sqrt{2}}{2}$	π	$\cos(2\cdot\pi) = 1$
$\frac{\pi}{2}$	$\cos\left(2\cdot\frac{\pi}{2}\right) = -1$	$\frac{9\pi}{8}$	$\cos\left(2\cdot\frac{9\pi}{8}\right) = \frac{\sqrt{2}}{2}$

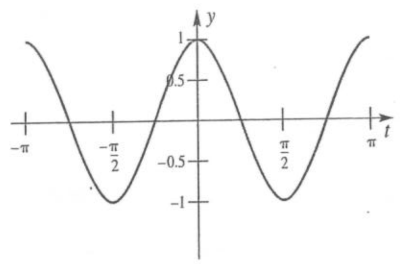


Figure 8

Period and Amplitude of the Trigonometric Functions A function f is **periodic** if there is a positive number p such that

$$f(x + p) = f(x)$$

for all real numbers x in the domain of f. The smallest such positive number p is called the **period** of f. The sine function is periodic because $\sin(x + 2\pi) = \sin x$ for all x. It is also true that

$$\sin(x + 4\pi) = \sin(x - 2\pi) = \sin(x + 12\pi) = \sin x$$

for all x. Thus, $4\pi, -2\pi$, and 12π are all numbers p with the property $\sin(x + p) = \sin x$. The period is defined to be the *smallest* such positive number p. For the sine function, the smallest positive p with the property that $\sin(x + p) = \sin x$ is $p = 2\pi$. We therefore say that the sine function is periodic with period 2π. The cosine function is also periodic with period 2π.

The function sin (at) has period $2\pi/a$ since

$$\sin\left[a\left(t + \frac{2\pi}{a}\right)\right] = \sin[at + 2\pi] = \sin(at)$$

The period of the function $\cos(at)$ is also $2\pi/a$.

EXAMPLE 2 What are the periods of the following functions?

(a) $\sin(2\pi t)$ (b) $\cos(2t)$ (c) $\sin(2\pi t/12)$

SOLUTION

(a) Because the function $\sin(2\pi t)$ is of the form $\sin(at)$ with $a = 2\pi$, its period is $p = \dfrac{2\pi}{2\pi} = 1$.

(b) The function $\cos(2t)$ is of the form $\cos(at)$ with $a = 2$. Thus, the period of $\cos(2t)$ is $p = \dfrac{2\pi}{2} = \pi$.

(c) The function $\sin(2\pi t/12)$ has period $p = \dfrac{2\pi}{2\pi/12} = 12$. ■

If the periodic function f attains a minimum and a maximum, we define the **amplitude** A as half the vertical distance between the highest point and the lowest point on the graph.

EXAMPLE 3 Find the amplitude of the following periodic functions.

(a) $\sin(2\pi t/12)$ (b) $3\cos(2t)$

(c) $50 + 21\sin(2\pi t/12 + 3)$

SOLUTION

(a) Since the range of the function $\sin(2\pi t/12)$ is $[-1, 1]$, its amplitude is $A = 1$.

(b) The function $3\cos(2t)$ will take on values from -3 (which occurs when $t = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \ldots$) to 3 (which occurs when $t = 0, \pm\pi, \pm 2\pi, \ldots$). The amplitude is therefore $A = 3$.

(c) The function $21\sin(2\pi t/12 + 3)$ takes on values from -21 to 21. Thus, $50 + 21\sin(2\pi t/12 + 3)$ takes on values from $50 - 21 = 29$ to $50 + 21 = 71$. The amplitude is therefore 21. ■

In general, for $a > 0$ and $A > 0$,

> $C + A\sin(a(t + b))$ and $C + A\cos(a(t + b))$ have period $\dfrac{2\pi}{a}$ and amplitude A.

Trigonometric functions can be used to model a number of physical phenomena, including daily tide levels and yearly temperatures.

EXAMPLE 4 The normal high temperature for St. Louis, Missouri, ranges from 37°F for January 15 to 89°F for July 15. The normal high temperature follows roughly a sinusoidal curve.

(a) Find values of C, A, a, and b such that

$$T(t) = C + A\sin(a(t + b))$$

where t, expressed in months since January 1, is a reasonable model for the normal high temperature.

(b) Use this model to approximate the normal high temperature for May 15.

SOLUTION

(a) The required function must have period $t = 12$ since the seasons repeat every 12 months. Thus, $\dfrac{2\pi}{a} = 12$, so we have $a = \dfrac{2\pi}{12}$. The amplitude is half the difference between the lowest and highest points; in this case,

$A = \frac{1}{2}(89 - 37) = 26$. The value of C is equal to the midpoint of the low and high temperatures, so $C = \frac{1}{2}(89 + 37) = 63$. The function $T(t)$ must therefore be of the form

$$T(t) = 63 + 26\sin\left(\frac{2\pi}{12}(t + b)\right)$$

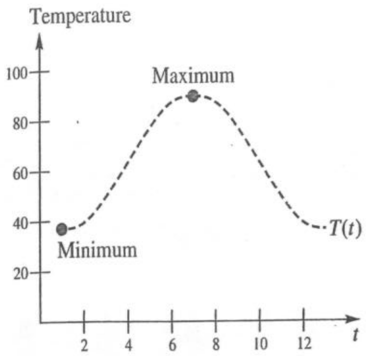


Figure 9

The only constant left to find is b. The lowest normal high temperature is 37, which occurs on January 15, roughly in the middle of January. Thus, our function must satisfy $T(1/2) = 37$, and the function must reach its minimum of 37 when $t = 1/2$. Figure 9 summarizes the information that we have so far. The function $63 + 26\sin(2\pi t/12)$ reaches its minimum when $2\pi t/12 = -\pi/2$, that is, when $t = -3$. We must therefore translate the curve defined by $y = 63 + 26\sin(2\pi t/12)$ to the right by the amount $1/2 - (-3) = 7/2$. In Section 0.6, we showed that replacing x with $x - c$ translates the graph of $y = f(x)$ to the right by c units. Thus, in order to translate the graph of $y = 63 + 26\sin(2\pi t/12)$ to the right by $7/2$ units, we must replace t with $t - 7/2$. Thus,

$$T(t) = 63 + 26\sin\left(\frac{2\pi}{12}\left(t - \frac{7}{2}\right)\right)$$

Figure 10 shows a plot of the normal high temperature T as a function of time t, where t is given in months.

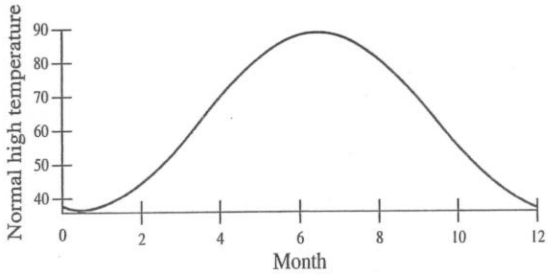


Figure 10

(b) To estimate the normal high temperature for May 15, we substitute $t = 4.5$ (because the middle of May is four and one-half months into the year) and obtain

$$T(4.5) = 63 + 26\sin(2\pi(4.5 - 3.5)/12) = 76$$

The normal high temperature for St. Louis on May 15 is actually 75°F. Thus, our model overpredicts by 1°, which is remarkably accurate considering how little information was given. ▪

Models and Modeling

It is important to keep in mind that all models such as this are simplifications of reality. (That is why they are called *models*.) Although such models are inherently simplifications of reality, many of them are still useful for prediction.

Four Other Trigonometric Functions

We could get by with just the sine and cosine functions, but it is convenient to introduce four additional trigonometric functions: tangent, cotangent, secant, and cosecant.

$$\tan t = \frac{\sin t}{\cos t} \qquad \cot t = \frac{\cos t}{\sin t}$$

$$\sec t = \frac{1}{\cos t} \qquad \csc t = \frac{1}{\sin t}$$

What we know about sine and cosine will automatically give us knowledge about these four new functions.

EXAMPLE 5 Show that tangent is an odd function.

SOLUTION

$$\tan(-t) = \frac{\sin(-t)}{\cos(-t)} = \frac{-\sin t}{\cos t} = -\tan t \qquad \blacksquare$$

EXAMPLE 6 Verify that the following are identities.

$$\boxed{1 + \tan^2 t = \sec^2 t \qquad 1 + \cot^2 t = \csc^2 t}$$

SOLUTION

$$1 + \tan^2 t = 1 + \frac{\sin^2 t}{\cos^2 t} = \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} = \sec^2 t$$

$$1 + \cot^2 t = 1 + \frac{\cos^2 t}{\sin^2 t} = \frac{\sin^2 t + \cos^2 t}{\sin^2 t} = \frac{1}{\sin^2 t} = \csc^2 t \qquad \blacksquare$$

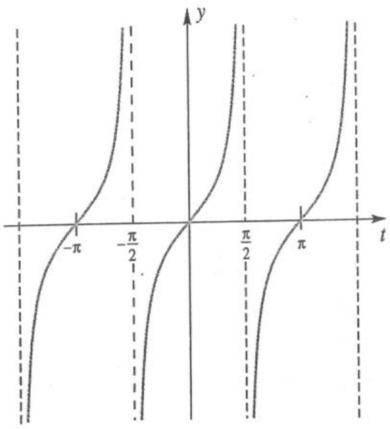


Figure 11

When we study the tangent function (Figure 11), we are in for two minor surprises. First, we notice that there are vertical asymptotes at $\pm\pi/2, \pm 3\pi/2, \ldots$. We should have anticipated this since $\cos t = 0$ at these values of t, which means that $\sin t/\cos t$ would involve a division by zero. Second, it appears that the tangent is periodic (which we expected), but with period π (which we might not have expected). You will see the analytic reason for this in Problem 33.

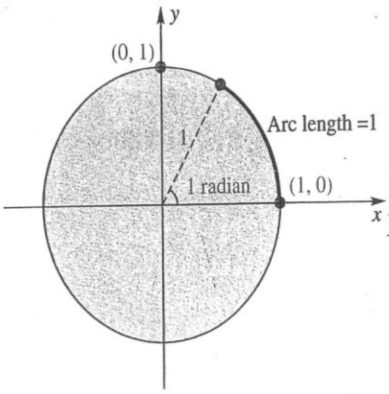


Figure 12

Relation to Angle Trigonometry Angles are commonly measured either in degrees or in radians. One radian is by definition the angle corresponding to an arc of length 1 on the unit circle. See Figure 12. The angle corresponding to a complete revolution measures 360°, but only 2π radians. Equivalently, a straight angle measures 180° or π radians, a fact worth remembering.

$$\boxed{180^\circ = \pi \text{ radians} \approx 3.1415927 \text{ radians}}$$

This leads to the results

$$1 \text{ radian} \approx 57.29578^\circ \qquad 1^\circ \approx 0.0174533 \text{ radian}$$

Degrees	Radians
0	0
30	$\pi/6$
45	$\pi/4$
60	$\pi/3$
90	$\pi/2$
120	$2\pi/3$
135	$3\pi/4$
150	$5\pi/6$
180	π
360	2π

Figure 13

Figure 13 shows some other common conversions between degrees and radians.

The division of a revolution into 360 parts is quite arbitrary (due to the ancient Babylonians, who liked multiples of 60). The division into 2π parts is more fundamental and lies behind the almost universal use of radian measure in calculus. Notice, in particular, that the length s of the arc cut off on a circle of radius r by a central angle of t radians satisfies (see Figure 14)

$$\frac{s}{2\pi r} = \frac{t}{2\pi}$$

That is, the fraction of the total circumference $2\pi r$ corresponding to an angle t is the same as the fraction of the unit circle corresponding to the same angle t. This implies that $s = rt$.

When $r = 1$, this gives $s = t$. This means that *the length of the arc on the unit circle cut off by a central angle of t radians is t*. This is correct even if t is negative, provided that we interpret length to be negative when measured in the clockwise direction.

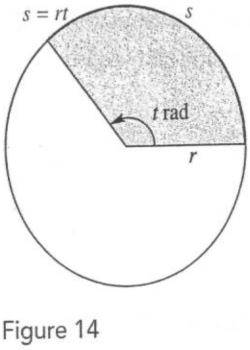
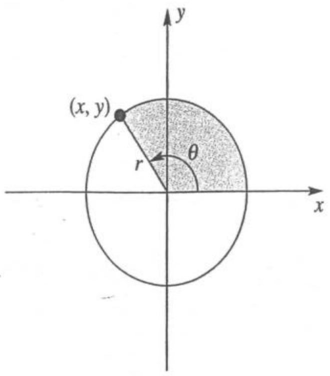


Figure 14

EXAMPLE 7 Find the distance traveled by a bicycle with wheels of radius 30 centimeters when the wheels turn through 100 revolutions.

Another View

We have based our discussion of trigonometry on the unit circle. We could as well have used a circle of radius r.



Then

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

SOLUTION We use the fact that $s = rt$, recognizing that 100 revolutions correspond to $100 \cdot (2\pi)$ radians.

$$s = (30)(100)(2\pi) = 6000\pi \approx 18{,}849.6 \text{ centimeters} \approx 188.5 \text{ meters}$$ ■

Now we can make the connection between angle trigonometry and unit circle trigonometry. If θ is an angle measuring t radians, that is, if θ is an angle that cuts off an arc of length t from the unit circle, then

$$\boxed{\sin\theta = \sin t \qquad \cos\theta = \cos t}$$

In calculus, when we meet an angle measured in degrees, we almost always change it to radians before doing any calculations. For example,

$$\boxed{\sin 31.6^\circ = \sin\left(31.6 \cdot \frac{\pi}{180} \text{radian}\right) \approx \sin 0.552}$$

List of Important Identities We will not take space to verify all the following identities. We simply assert their truth and suggest that most of them will be needed somewhere in this book.

Trigonometric Identities The following are true for all x and y, provided that both sides are defined at the chosen x and y.

Odd–even identities

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\tan(-x) = -\tan x$$

Cofunction identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x$$

Pythagorean identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Addition identities

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Double-angle identities

$$\sin 2x = 2 \sin x \cos x$$

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x\end{aligned}$$

Half-angle identities

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 - \cos x}{2}}$$

$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 + \cos x}{2}}$$

Sum identities

$$\sin x + \sin y = 2\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

Product identities

$$\sin x \sin y = -\tfrac{1}{2}[\cos(x+y) - \cos(x-y)]$$

$$\cos x \cos y = \tfrac{1}{2}[\cos(x+y) + \cos(x-y)]$$

$$\sin x \cos y = \tfrac{1}{2}[\sin(x+y) + \sin(x-y)]$$

Concepts Review

1. The natural domain of the sine function is ____; its range is ____.

2. The period of the cosine function is ____; the period of the sine function is ____; the period of the tangent function is ____.

3. Since $\sin(-x) = -\sin x$, the sine function is ____, and since $\cos(-x) = \cos x$, the cosine function is ____.

4. If $(-4, 3)$ lies on the terminal side of an angle θ whose vertex is at the origin and initial side is along the positive x-axis, then $\cos\theta =$ ____.

Problem Set 0.7

1. Convert the following degree measures to radians (leave π in your answer).
(a) $30°$ (b) $45°$ (c) $-60°$
(d) $240°$ (e) $-370°$ (f) $10°$

2. Convert the following radian measures to degrees.
(a) $\frac{7}{6}\pi$ (b) $\frac{3}{4}\pi$ (c) $-\frac{1}{3}\pi$.
(d) $\frac{4}{3}\pi$ (e) $-\frac{35}{18}\pi$ (f) $\frac{3}{18}\pi$

C 3. Convert the following degree measures to radians ($1° = \pi/180 \approx 1.7453 \times 10^{-2}$ radian).
(a) $33.3°$ (b) $46°$ (c) $-66.6°$
(d) $240.11°$ (e) $-369°$ (f) $11°$

C 4. Convert the following radian measures to degrees (1 radian $= 180/\pi \approx 57.296$ degrees).
(a) 3.141 (b) 6.28 (c) 5.00
(d) 0.001 (e) -0.1 (f) 36.0

C 5. Calculate (be sure that your calculator is in radian or degree mode as needed).
(a) $\dfrac{56.4 \tan 34.2°}{\sin 34.1°}$ (b) $\dfrac{5.34 \tan 21.3°}{\sin 3.1° + \cot 23.5°}$
(c) $\tan 0.452$ (d) $\sin(-0.361)$

C 6. Calculate.
(a) $\dfrac{234.1 \sin 1.56}{\cos 0.34}$ (b) $\sin^2 2.51 + \sqrt{\cos 0.51}$

C 7. Calculate.
(a) $\dfrac{56.3 \tan 34.2°}{\sin 56.1°}$ (b) $\left(\dfrac{\sin 35°}{\sin 26° + \cos 26°}\right)^3$

8. Verify the values of $\sin t$ and $\cos t$ in the table used to construct Figure 6.

9. Evaluate without using a calculator.
(a) $\tan\dfrac{\pi}{6}$ (b) $\sec \pi$ (c) $\sec\dfrac{3\pi}{4}$
(d) $\csc\dfrac{\pi}{2}$ (e) $\cot\dfrac{\pi}{4}$ (f) $\tan\left(-\dfrac{\pi}{4}\right)$

10. Evaluate without using a calculator.
(a) $\tan\dfrac{\pi}{3}$ (b) $\sec\dfrac{\pi}{3}$ (c) $\cot\dfrac{\pi}{3}$
(d) $\csc\dfrac{\pi}{4}$ (e) $\tan\left(-\dfrac{\pi}{6}\right)$ (f) $\cos\left(-\dfrac{\pi}{3}\right)$

11. Verify that the following are identities (see Example 6).
(a) $(1 + \sin z)(1 - \sin z) = \dfrac{1}{\sec^2 z}$
(b) $(\sec t - 1)(\sec t + 1) = \tan^2 t$
(c) $\sec t - \sin t \tan t = \cos t$
(d) $\dfrac{\sec^2 t - 1}{\sec^2 t} = \sin^2 t$

12. Verify that the following are identities (see Example 6).
(a) $\sin^2 v + \dfrac{1}{\sec^2 v} = 1$
(b) $\cos 3t = 4\cos^3 t - 3\cos t$ *Hint:* Use a double-angle identity.
(c) $\sin 4x = 8 \sin x \cos^3 x - 4 \sin x \cos x$ *Hint:* Use a double-angle identity twice.
(d) $(1 + \cos\theta)(1 - \cos\theta) = \sin^2\theta$

13. Verify the following are identities.
(a) $\dfrac{\sin u}{\csc u} + \dfrac{\cos u}{\sec u} = 1$
(b) $(1 - \cos^2 x)(1 + \cot^2 x) = 1$
(c) $\sin t(\csc t - \sin t) = \cos^2 t$
(d) $\dfrac{1 - \csc^2 t}{\csc^2 t} = \dfrac{-1}{\sec^2 t}$

14. Sketch the graphs of the following on $[-\pi, 2\pi]$.
(a) $y = \sin 2x$ (b) $y = 2\sin t$
(c) $y = \cos\left(x - \dfrac{\pi}{4}\right)$ (d) $y = \sec t$

15. Sketch the graphs of the following on $[-\pi, 2\pi]$.
(a) $y = \csc t$ (b) $y = 2\cos t$

CHAPTER 1 Limits

1.1 Introduction to Limits

The topics discussed in the previous chapter are part of what is called *precalculus*. They provide the foundation for calculus, but they are not calculus. Now we are ready for an important new idea, the notion of *limit*. It is this idea that distinguishes calculus from other branches of mathematics. In fact, we define calculus this way:

Calculus is the study of limits.

Problems Leading to the Limit Concept The concept of **limit** is central to many problems in the physical, engineering, and social sciences. Basically the question is this: what happens to the function $f(x)$ as x gets close to some constant c? There are variations on this theme, but the basic idea is the same in many circumstances.

Suppose that as an object steadily moves forward we know its position at any given time. We denote the position at time t by $s(t)$. How fast is the object moving at time $t = 1$? We can use the formula "distance equals rate times time" to find the speed (rate of change of position) over any interval of time; in other words

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

We call this the "average" speed over the interval since, no matter how small the interval is, we never know whether the speed is constant over this interval. For example, over the interval $[1, 2]$, the average speed is $\frac{s(2) - s(1)}{2 - 1}$; over the interval $[1, 1.2]$, the average speed is $\frac{s(1.2) - s(1)}{1.2 - 1}$; over the interval $[1, 1.02]$, the average speed is $\frac{s(1.02) - s(1)}{1.02 - 1}$, etc. How fast is the object traveling at time $t = 1$? To give meaning to this "instantaneous" velocity we must talk about the *limit* of the average speed over smaller and smaller intervals.

Figure 1

We can find areas of rectangles and triangles using formulas from geometry, but what about regions with curved boundaries, such as a circle? Archimedes had this idea over two thousand years ago. Imagine regular polygons inscribed in a circle as shown in Figure 1. Archimedes was able to find the area of a regular polygon with n sides, and by taking the regular polygon with more and more sides, he was able to approximate the area of a circle to any desired level of accuracy. In other words, the area of the circle is the *limit* of the areas of the inscribed polygons as n (the number of sides in the polygon) increases without bound.

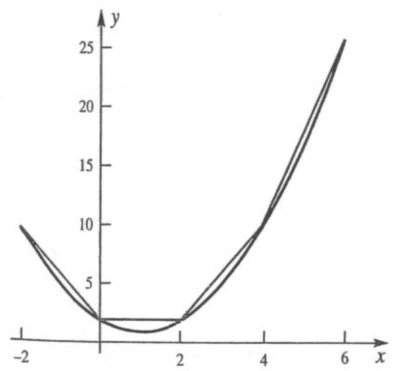


Figure 2

Consider the graph of the function $y = f(x)$ for $a \leq x \leq b$. If the graph is a straight line, the length of the curve is easy to find using the distance formula. But what if the graph is curved? We can find numerous points along the curve and connect them with line segments as shown in Figure 2. If we add up the lengths of these line segments we should get a sum that is approximately the length of the curve. In fact, by "length of the curve" we mean the *limit* of the sum of the lengths of these line segments as the number of line segments increases without bound.

The last three paragraphs describe situations that lead to the concept of *limit*. There are many others, and we will study them throughout this book. We begin with an intuitive explanation of limits. The precise definition is given in the next section.

An Intuitive Understanding Consider the function defined by

$$f(x) = \frac{x^3 - 1}{x - 1}$$

Note that it is not defined at $x = 1$ since at this point $f(x)$ has the form $\frac{0}{0}$, which is meaningless. We can, however, still ask what is happening to $f(x)$ as x approaches 1. More precisely, is $f(x)$ approaching some specific number as x approaches 1? To get at the answer, we can do three things. We can calculate some values of $f(x)$ for x near 1, we can show these values in a schematic diagram, and we can sketch the graph of $y = f(x)$. All this has been done, and the results are shown in Figure 3.

x	$y = \frac{x^3-1}{x-1}$
1.25	3.813
1.1	3.310
1.01	3.030
1.001	3.003
↓	↓
1.000	?
↑	↑
0.999	2.997
0.99	2.970
0.9	2.710
0.75	2.313

Table of values

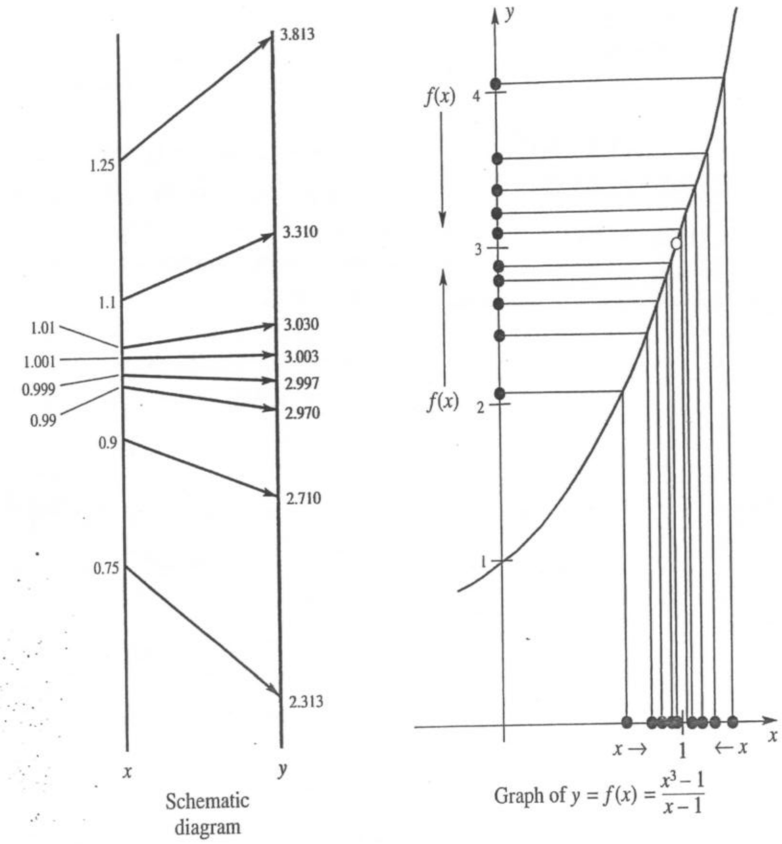


Figure 3

All the information we have assembled seems to point to the same conclusion: $f(x)$ approaches 3 as x approaches 1. In mathematical symbols, we write

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$$

This is read "the limit as x approaches 1 of $(x^3 - 1)/(x - 1)$ is 3."

Being good algebraists (thus knowing how to factor the difference of cubes), we can provide more and better evidence.

$$\begin{aligned}\lim_{x \to 1} \frac{x^3 - 1}{x - 1} &= \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{x - 1} \\ &= \lim_{x \to 1} (x^2 + x + 1) = 1^2 + 1 + 1 = 3\end{aligned}$$

Note that $(x - 1)/(x - 1) = 1$ as long as $x \neq 1$. This justifies the second step. The third step should seem reasonable; a rigorous justification will come later.

To be sure that we are on the right track, we need to have a clearly understood meaning for the word *limit*. Here is our first attempt at a definition.

> **Definition** **Intuitive Meaning of Limit**
>
> To say that $\lim_{x \to c} f(x) = L$ means that when x is near but different from c then $f(x)$ is near L.

Notice that we do not require anything *at c*. The function f need not even be defined at c; it was not in the example $f(x) = (x^3 - 1)/(x - 1)$ just considered. The notion of limit is associated with the behavior of a function *near c*, not *at c*.

A cautious reader is sure to object to our use of the word *near*. What does *near* mean? How near is near? For precise answers, you will have to study the next section; however, some further examples will help to clarify the idea.

More Examples

Our first example is almost trivial, but nonetheless important.

EXAMPLE 1 Find $\lim_{x \to 3}(4x - 5)$.

SOLUTION When x is near 3, $4x - 5$ is near $4 \cdot 3 - 5 = 7$. We write

$$\lim_{x \to 3}(4x - 5) = 7$$

EXAMPLE 2 Find $\lim_{x \to 3} \frac{x^2 - x - 6}{x - 3}$.

SOLUTION Note that $(x^2 - x - 6)/(x - 3)$ is not defined at $x = 3$, but this is all right. To get an idea of what is happening as x approaches 3, we could use a calculator to evaluate the given expression, for example, at 3.1, 3.01, 3.001, and so on. But it is much better to use a little algebra to simplify the problem.

$$\lim_{x \to 3} \frac{x^2 - x - 6}{x - 3} = \lim_{x \to 3} \frac{(x - 3)(x + 2)}{x - 3} = \lim_{x \to 3}(x + 2) = 3 + 2 = 5$$

The cancellation of $x - 3$ in the second step is legitimate because the definition of limit ignores the behavior *at* $x = 3$. Remember, $\frac{x - 3}{x - 3} = 1$ as long as x is not equal to 3.

x	$\frac{\sin x}{x}$
1.0	0.84147
0.1	0.99833
0.01	0.99998
↓	↓
0	?
↑	↑
−0.01	0.99998
−0.1	0.99833
−1.0	0.84147

Figure 4

EXAMPLE 3 Find $\lim_{x \to 0} \frac{\sin x}{x}$.

SOLUTION No algebraic trick will simplify our task; certainly, we cannot cancel the x's. A calculator will help us to get an idea of the limit. Use your own calculator (radian mode) to check the values in the table of Figure 4. Figure 5 shows a plot of $y = (\sin x)/x$. Our conclusion, though we admit it is a bit shaky, is that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

We will give a rigorous demonstration in Section 1.4.

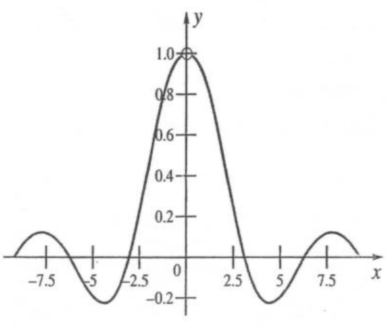


Figure 5

Some Warning Flags

Things are not quite as simple as they may appear. Calculators may mislead us; so may our own intuition. The examples that follow suggest some possible pitfalls.

x	$x^2 - \frac{\cos x}{10{,}000}$
± 1	0.99995
± 0.5	0.24991
± 0.1	0.00990
± 0.01	0.000000005
$\downarrow$	$\downarrow$
0	?

Figure 6

EXAMPLE 4 **(Your calculator may fool you.)** Find $\lim_{x\to 0}\left[x^2 - \frac{\cos x}{10{,}000}\right]$.

SOLUTION Following the procedure used in Example 3, we construct the table of values shown in Figure 6. The conclusion it suggests is that the desired limit is 0. But this is wrong. If we recall the graph of $y = \cos x$, we realize that $\cos x$ approaches 1 as x approaches 0. Thus,

$$\lim_{x\to 0}\left[x^2 - \frac{\cos x}{10{,}000}\right] = 0^2 - \frac{1}{10{,}000} = -\frac{1}{10{,}000}$$

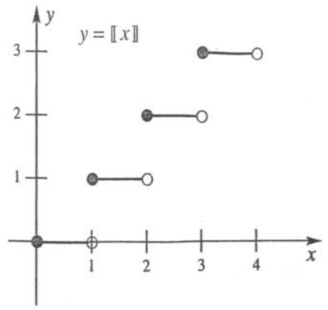


Figure 7

EXAMPLE 5 **(No limit at a jump)** Find $\lim_{x\to 2}[\![x]\!]$.

SOLUTION Recall that $[\![x]\!]$ denotes the greatest integer less than or equal to x (see Section 0.5). The graph of $y = [\![x]\!]$ is shown in Figure 7. For all numbers x less than 2 but near 2, $[\![x]\!] = 1$, but for all numbers x greater than 2 but near 2, $[\![x]\!] = 2$. Is $[\![x]\!]$ near a single number L when x is near 2? No. No matter what number we propose for L, there will be x's arbitrarily close to 2 on one side or the other, where $[\![x]\!]$ differs from L by at least $\frac{1}{2}$. Our conclusion is that $\lim_{x\to 2}[\![x]\!]$ does not exist. If you check back, you will see that we have not claimed that every limit we can write must exist.

EXAMPLE 6 **(Too many wiggles)** Find $\lim_{x\to 0}\sin(1/x)$.

SOLUTION This example poses the most subtle limit question asked yet. Since we do not want to make too big a story out of it, we ask you to do two things. First, pick a sequence of x-values approaching 0. Use your calculator to evaluate $\sin(1/x)$ at these x's. Unless you happen on some very lucky choices, your values will oscillate wildly.

Second, consider trying to graph $y = \sin(1/x)$. No one will ever do this very well, but the table of values in Figure 8 gives a good clue about what is happening. In any neighborhood of the origin, the graph wiggles up and down between -1 and 1 infinitely many times (Figure 9). Clearly, $\sin(1/x)$ is not near a single number L when x is near 0. We conclude that $\lim_{x\to 0}\sin(1/x)$ does not exist.

x	$\sin\frac{1}{x}$
$2/\pi$	1
$2/(2\pi)$	0
$2/(3\pi)$	-1
$2/(4\pi)$	0
$2/(5\pi)$	1
$2/(6\pi)$	0
$2/(7\pi)$	-1
$2/(8\pi)$	0
$2/(9\pi)$	1
$2/(10\pi)$	0
$2/(11\pi)$	-1
$2/(12\pi)$	0
$\downarrow$	$\downarrow$
0	?

Figure 8

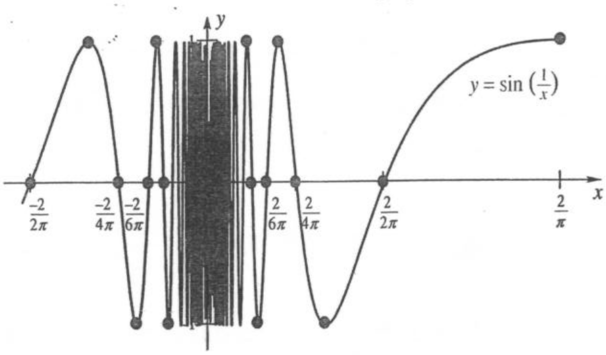


Figure 9

One-Sided Limits

When a function takes a jump (as does $[\![x]\!]$ at each integer in Example 5), then the limit does not exist at the jump points. Such functions suggest the introduction of **one-sided limits.** Let the symbol $x \to c^+$ mean that x approaches c from the right, and let $x \to c^-$ mean that x approaches c from the left.

Definition Right- and Left-Hand Limits

To say that $\lim_{x\to c^+} f(x) = L$ means that when x is near but to the right of c then $f(x)$ is near L. Similarly, to say that $\lim_{x\to c^-} f(x) = L$ means that when x is near but to the left of c then $f(x)$ is near L.

Thus, while $\lim_{x\to 2}[\![x]\!]$ does not exist, it is correct to write (look at the graph in Figure 7)

$$\lim_{x\to 2^-}[\![x]\!] = 1 \quad \text{and} \quad \lim_{x\to 2^+}[\![x]\!] = 2$$

We believe that you will find the following theorem quite reasonable.

Theorem A

$\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to c^-} f(x) = L$ and $\lim_{x\to c^+} f(x) = L$.

Figure 10 should give additional insight. Two of the limits do not exist, although all but one of the one-sided limits exist.

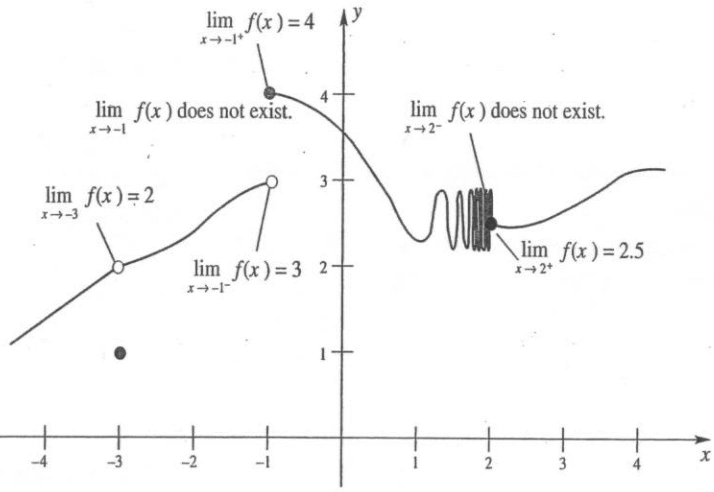


Figure 10

Concepts Review

1. $\lim_{x\to c} f(x) = L$ means that $f(x)$ gets close to ____ when x gets sufficiently close to (but is different from) ____.

2. Let $f(x) = (x^2 - 9)/(x - 3)$ and note that $f(3)$ is undefined. Nevertheless, $\lim_{x\to 3} f(x) =$ ____.

3. $\lim_{x\to c^+} f(x) = L$ means that $f(x)$ gets near to ____ when x approaches c from the ____.

4. If both $\lim_{x\to c^-} f(x) = M$ and $\lim_{x\to c^+} f(x) = M$, then ____.

Problem Set 1.1

In Problems 1–6, find the indicated limit.

1. $\lim_{x\to 3}(x - 5)$

2. $\lim_{t\to -1}(1 - 2t)$

3. $\lim_{x\to -2}(x^2 + 2x - 1)$

4. $\lim_{x\to -2}(x^2 + 2t - 1)$

5. $\lim_{t\to -1}(t^2 - 1)$

6. $\lim_{t\to -1}(t^2 - x^2)$

In Problems 7–18, find the indicated limit. In most cases, it will be wise to do some algebra first (see Example 2).

7. $\lim_{x\to 2}\dfrac{x^2 - 4}{x - 2}$

8. $\lim_{t\to -7}\dfrac{t^2 + 4t - 21}{t + 7}$

9. $\lim_{x\to -1}\dfrac{x^3 - 4x^2 + x + 6}{x + 1}$

10. $\lim_{x\to 0}\dfrac{x^4 + 2x^3 - x^2}{x^2}$

11. $\lim_{x\to -t}\dfrac{x^2 - t^2}{x + t}$

12. $\lim_{x\to 3}\dfrac{x^2 - 9}{x - 3}$

13. $\lim_{t\to 2}\dfrac{\sqrt{(t + 4)(t - 2)^4}}{(3t - 6)^2}$

14. $\lim_{t\to 7^+}\dfrac{\sqrt{(t - 7)^3}}{t - 7}$

15. $\lim_{x\to 3}\dfrac{x^4 - 18x^2 + 81}{(x - 3)^2}$

16. $\lim_{u\to 1}\dfrac{(3u + 4)(2u - 2)^3}{(u - 1)^2}$

17. $\lim_{h\to 0}\dfrac{(2 + h)^2 - 4}{h}$

18. $\lim_{h\to 0}\dfrac{(x + h)^2 - x^2}{h}$

GC *In Problems 19–28, use a calculator to find the indicated limit. Use a graphing calculator to plot the function near the limit point.*

19. $\lim_{x\to 0}\dfrac{\sin x}{2x}$

20. $\lim_{t\to 0}\dfrac{1 - \cos t}{2t}$

21. $\lim_{x\to 0}\dfrac{(x - \sin x)^2}{x^2}$

22. $\lim_{x\to 0}\dfrac{(1 - \cos x)^2}{x^2}$

23. $\lim_{t\to 1}\dfrac{t^2 - 1}{\sin(t - 1)}$

24. $\lim_{x\to 3}\dfrac{x - \sin(x - 3) - 3}{x - 3}$

25. $\lim_{x\to \pi}\dfrac{1 + \sin(x - 3\pi/2)}{x - \pi}$

26. $\lim_{t\to 0}\dfrac{1 - \cot t}{1/t}$

27. $\lim_{x\to \pi/4}\dfrac{(x - \pi/4)^2}{(\tan x - 1)^2}$

28. $\lim_{u\to \pi/2}\dfrac{2 - 2\sin u}{3u}$

29. For the function f graphed in Figure 11, find the indicated limit or function value, or state that it does not exist.

(a) $\lim_{x\to -3} f(x)$ (b) $f(-3)$ (c) $f(-1)$
(d) $\lim_{x\to -1} f(x)$ (e) $f(1)$ (f) $\lim_{x\to 1} f(x)$
(g) $\lim_{x\to 1^-} f(x)$ (h) $\lim_{x\to 1^+} f(x)$ (i) $\lim_{x\to -1^+} f(x)$

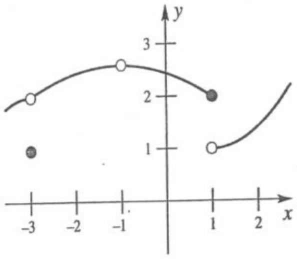


Figure 11

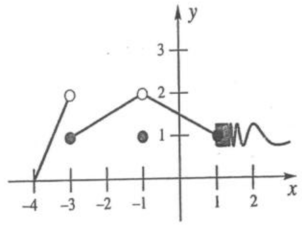


Figure 12

30. Follow the directions of Problem 29 for the function f graphed in Figure 12.

31. For the function f graphed in Figure 13, find the indicated limit or function value, or state that it does not exist.

(a) $f(-3)$ (b) $f(3)$ (c) $\lim_{x\to -3^-} f(x)$
(d) $\lim_{x\to -3^+} f(x)$ (e) $\lim_{x\to -3} f(x)$ (f) $\lim_{x\to 3^+} f(x)$

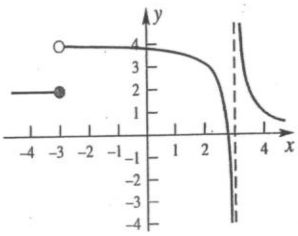


Figure 13

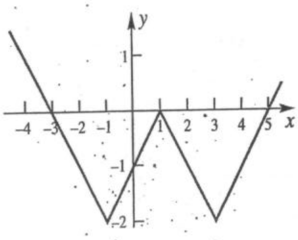


Figure 14

32. For the function f graphed in Figure 14, find the indicated limit or function value, or state that it does not exist.

(a) $\lim_{x\to -1^-} f(x)$ (b) $\lim_{x\to -1^+} f(x)$ (c) $\lim_{x\to -1} f(x)$
(d) $f(-1)$ (e) $\lim_{x\to 1} f(x)$ (f) $f(1)$

33. Sketch the graph of

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } 0 \le x < 1 \\ 1 + x & \text{if } x \ge 1 \end{cases}$$

Then find each of the following or state that it does not exist.

(a) $\lim_{x\to 0} f(x)$ (b) $\lim_{x\to 1} f(x)$
(c) $f(1)$ (d) $\lim_{x\to 1^+} f(x)$

34. Sketch the graph of

$$g(x) = \begin{cases} -x + 1 & \text{if } x < 1 \\ x - 1 & \text{if } 1 < x < 2 \\ 5 - x^2 & \text{if } x \ge 2 \end{cases}$$

Then find each of the following or state that it does not exist.

(a) $\lim_{x\to 1} g(x)$ (b) $g(1)$
(c) $\lim_{x\to 2} g(x)$ (d) $\lim_{x\to 2^+} g(x)$

35. Sketch the graph of $f(x) = x - [\![x]\!]$; then find each of the following or state that it does not exist.

(a) $f(0)$ (b) $\lim_{x\to 0} f(x)$
(c) $\lim_{x\to 0^-} f(x)$ (d) $\lim_{x\to 1/2} f(x)$

36. Follow the directions of Problem 35 for $f(x) = x/|x|$.

37. Find $\lim_{x\to 1}(x^2 - 1)/|x - 1|$ or state that it does not exist.

38. Evaluate $\lim_{x\to 0}\left(\sqrt{x+2} - \sqrt{2}\right)/x$. *Hint:* Rationalize the numerator by multiplying the numerator and denominator by $\sqrt{x+2} + \sqrt{2}$.

39. Let

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{cases}$$

Find each value, if possible.

(a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 0} f(x)$

40. Sketch, as best you can, the graph of a function f that satisfies all the following conditions.

(a) Its domain is the interval $[0, 4]$.
(b) $f(0) = f(1) = f(2) = f(3) = f(4) = 1$
(c) $\lim_{x\to 1} f(x) = 2$ (d) $\lim_{x\to 2} f(x) = 1$
(e) $\lim_{x\to 3^-} f(x) = 2$ (f) $\lim_{x\to 3^+} f(x) = 1$

41. Let

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ x^4 & \text{if } x \text{ is irrational} \end{cases}$$

For what values of a does $\lim_{x\to a} f(x)$ exist?

42. The function $f(x) = x^2$ had been carefully graphed, but during the night a mysterious visitor changed the values of f at a million different places. Does this affect the value of $\lim_{x\to a} f(x)$ at any a? Explain.

43. Find each of the following limits or state that it does not exist.

(a) $\lim_{x\to 1} \dfrac{|x-1|}{x-1}$ (b) $\lim_{x\to 1^-} \dfrac{|x-1|}{x-1}$
(c) $\lim_{x\to 1^-} \dfrac{x^2 - |x-1| - 1}{|x-1|}$ (d) $\lim_{x\to 1^-}\left[\dfrac{1}{x-1} - \dfrac{1}{|x-1|}\right]$

44. Find each of the following limits or state that it does not exist.

(a) $\lim_{x\to 1^+} \sqrt{x - [\![x]\!]}$ (b) $\lim_{x\to 0^+} [\![1/x]\!]$
(c) $\lim_{x\to 0^+} x(-1)^{[\![1/x]\!]}$ (d) $\lim_{x\to 0^+} [\![x]\!](-1)^{[\![1/x]\!]}$

45. Find each of the following limits or state that it does not exist.

(a) $\lim_{x\to 0^+} x[\![1/x]\!]$ (b) $\lim_{x\to 0^+} x^2[\![1/x]\!]$
(c) $\lim_{x\to 3^-}([\![x]\!] + [\![-x]\!])$ (d) $\lim_{x\to 3^+}([\![x]\!] + [\![-x]\!])$

46. Find each of the following limits or state that it does not exist.

(a) $\lim_{x\to 3} [\![x]\!]/x$ (b) $\lim_{x\to 0^+} [\![x]\!]/x$
(c) $\lim_{x\to 1.8} [\![x]\!]$ (d) $\lim_{x\to 1.8} [\![x]\!]/x$

CAS *Many software packages have programs for calculating limits, although you should be warned that they are not infallible. To develop confidence in your program, use it to recalculate some of the limits in Problems 1–28. Then for each of the following, find the limit or state that it does not exist.*

47. $\lim_{x\to 0} \sqrt{x}$ **48.** $\lim_{x\to 0^+} x^x$

49. $\lim_{x\to 0} \sqrt{|x|}$ **50.** $\lim_{x\to 0} |x|^x$

1.3 Limit Theorems

Most readers will agree that proving the existence and values of limits using the ε–δ definition of the preceding section is both time consuming and difficult. That is why the theorems of this section are so welcome. Our first theorem is the big one. With it, we can handle most limit problems that we will face for quite some time.

One-Sided Limits

Although stated in terms of two-sided limits, Theorem A remains true for both left- and right-hand limits.

Theorem A Main Limit Theorem

Let n be a positive integer, k be a constant, and f and g be functions that have limits at c. Then

1. $\lim_{x\to c} k = k$;
2. $\lim_{x\to c} x = c$;
3. $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$;
4. $\lim_{x\to c} [f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$;
5. $\lim_{x\to c} [f(x) - g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$;
6. $\lim_{x\to c} [f(x)\cdot g(x)] = \lim_{x\to c} f(x)\cdot \lim_{x\to c} g(x)$;
7. $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided $\lim_{x\to c} g(x) \neq 0$;
8. $\lim_{x\to c} [f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n$;
9. $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$, provided $\lim_{x\to c} f(x) > 0$ when n is even.

These important results are remembered best if learned in words. For example, Statement 4 translates as *The limit of a sum is the sum of the limits.*

Of course, Theorem A needs to be proved. We postpone that job till the end of the section, choosing first to show how this multipart theorem is used.

Applications of the Main Limit Theorem In the next examples, the circled numbers refer to the numbered statements from Theorem A. Each equality is justified by the indicated statement.

EXAMPLE 1 Find $\lim_{x\to 3} 2x^4$.

$$\lim_{x\to 3} 2x^4 \overset{③}{=} 2 \lim_{x\to 3} x^4 \overset{⑧}{=} 2\left[\lim_{x\to 3} x\right]^4 \overset{②}{=} 2[3]^4 = 162$$

■

EXAMPLE 2 Find $\lim_{x\to 4}(3x^2 - 2x)$.

SOLUTION

$$\lim_{x\to 4}(3x^2 - 2x) \overset{⑤}{=} \lim_{x\to 4} 3x^2 - \lim_{x\to 4} 2x \overset{③}{=} 3 \lim_{x\to 4} x^2 - 2 \lim_{x\to 4} x$$

$$\overset{⑧}{=} 3\left(\lim_{x\to 4} x\right)^2 - 2 \lim_{x\to 4} x \overset{②}{=} 3(4)^2 - 2(4) = 40$$

■

EXAMPLE 3 Find $\lim_{x \to 4} \frac{\sqrt{x^2 + 9}}{x}$.

SOLUTION

$$\lim_{x \to 4} \frac{\sqrt{x^2+9}}{x} \overset{7}{=} \frac{\lim_{x \to 4} \sqrt{x^2+9}}{\lim_{x \to 4} x} \overset{9,2}{=} \frac{\sqrt{\lim_{x \to 4} (x^2+9)}}{4} \overset{4}{=} \frac{1}{4}\sqrt{\lim_{x \to 4} x^2 + \lim_{x \to 4} 9}$$

$$\overset{8,1}{=} \frac{1}{4}\sqrt{\left[\lim_{x \to 4} x\right]^2 + 9} \overset{2}{=} \frac{1}{4}\sqrt{4^2 + 9} = \frac{5}{4}$$

EXAMPLE 4 If $\lim_{x \to 3} f(x) = 4$ and $\lim_{x \to 3} g(x) = 8$, find

$$\lim_{x \to 3}\left[f^2(x) \cdot \sqrt[3]{g(x)}\,\right]$$

SOLUTION

$$\lim_{x \to 3} [f^2(x) \cdot \sqrt[3]{g(x)}] \overset{6}{=} \lim_{x \to 3} f^2(x) \cdot \lim_{x \to 3} \sqrt[3]{g(x)}$$

$$\overset{8,9}{=} \left[\lim_{x \to 3} f(x)\right]^2 \cdot \sqrt[3]{\lim_{x \to 3} g(x)}$$

$$= [4]^2 \cdot \sqrt[3]{8} = 32$$

Recall that a polynomial function f has the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

whereas a rational function f is the quotient of two polynomial functions, that is,

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

Theorem B Substitution Theorem

If f is a polynomial function or a rational function, then

$$\lim_{x \to c} f(x) = f(c)$$

provided $f(c)$ is defined. In the case of a rational function, this means that the value of the denominator at c is not zero.

Evaluating a Limit "by Substitution"

When we apply Theorem B, the Substitution Theorem, we say we evaluate the limit *by substitution*. Not all limits can be evaluated by substitution; consider $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$. The Substitution Theorem does not apply here because the denominator is 0 when $x = 1$, but the limit does exist.

The proof of Theorem B follows from repeated applications of Theorem A. Note that Theorem B allows us to find limits for polynomial and rational functions by simply substituting c for x throughout, provided the denominator of the rational function is not zero at c.

EXAMPLE 5 Find $\lim_{x \to 2} \frac{7x^5 - 10x^4 - 13x + 6}{3x^2 - 6x - 8}$.

SOLUTION

$$\lim_{x\to 2}\frac{7x^5 - 10x^4 - 13x + 6}{3x^2 - 6x - 8} = \frac{7(2)^5 - 10(2)^4 - 13(2) + 6}{3(2)^2 - 6(2) - 8} = -\frac{11}{2}$$ ■

EXAMPLE 6 Find $\lim_{x\to 1}\frac{x^3 + 3x + 7}{x^2 - 2x + 1} = \lim_{x\to 1}\frac{x^3 + 3x + 7}{(x-1)^2}$.

SOLUTION Neither Theorem B nor Statement 7 of Theorem A applies, since the limit of the denominator is 0. However, since the limit of the numerator is 11, we see that as x nears 1 we are dividing a number near 11 by a positive number near 0. The result is a large positive number. In fact, the resulting number can be made as large as you like by letting x get close enough to 1. We say that the limit does not exist. (Later in this chapter (see Section 1.5) we will allow ourselves to say that the limit is $+\infty$.) ■

In many cases, Theorem B cannot be applied because substitution of c causes the denominator to be 0. In cases like this, it sometimes happens that the function can be simplified, for example by factoring. For example, we can write

$$\frac{x^2 + 3x - 10}{x^2 + x - 6} = \frac{(x-2)(x+5)}{(x-2)(x+3)} = \frac{x+5}{x+3}$$

We have to be careful with this last step. The fraction $(x+5)/(x+3)$ is equal to the one on the left side of the equal sign only if x is not equal to 2. If $x = 2$, the left side is undefined (because the denominator is 0), whereas the right side is equal to $(2+5)/(2+3) = 7/5$. This brings up the question about whether the limits

$$\lim_{x\to 2}\frac{x^2 + 3x - 10}{x^2 + x - 6} \quad \text{and} \quad \lim_{x\to 2}\frac{x+5}{x+3}$$

are equal. The answer is contained in the following theorem.

Theorem C

If $f(x) = g(x)$ for all x in an open interval containing the number c, except possibly at the number c itself, and if $\lim_{x\to c} g(x)$ exists, then $\lim_{x\to c} f(x)$ exists and $\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$.

EXAMPLE 7 Find $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}$.

SOLUTION

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1} = \lim_{x\to 1}\frac{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}{\sqrt{x}-1} = \lim_{x\to 1}\left(\sqrt{x}+1\right) = \sqrt{1}+1 = 2$$ ■

EXAMPLE 8 Find $\lim_{x\to 2}\frac{x^2 + 3x - 10}{x^2 + x - 6}$.

SOLUTION Theorem B does not apply because the denominator is 0 when $x = 2$. When we substitute $x = 2$ in the numerator we also get 0, so the quotient takes on the meaningless form 0/0 at $x = 2$. When this happens we should look for some sort of simplification such as factoring.

$$\lim_{x\to 2}\frac{x^2 + 3x - 10}{x^2 + x - 6} = \lim_{x\to 2}\frac{(x-2)(x+5)}{(x-2)(x+3)} = \lim_{x\to 2}\frac{x+5}{x+3} = \frac{7}{5}$$

The second to last equality is justified by Theorem C since

$$\frac{(x-2)(x+5)}{(x-2)(x+3)} = \frac{x+5}{x+3}$$

for all x except $x = 2$. Once we apply Theorem C, we can evaluate the limit by substitution (i.e., by applying Theorem B). ■

> **Optional?**
>
> How much theorem proving should be done in a first calculus course? Mathematics teachers argue long and hard about this and about the right balance between
>
> - logic and intuition
> - proof and explanation
> - theory and application
>
> A great scientist of long ago had some sage advice.
>
> "He who loves practice without theory is like the sailor who boards ship without a rudder and compass and never knows where he may cast."
>
> *Leonardo da Vinci*

Proof of Theorem A (Optional) You should not be too surprised when we say that the proofs of some parts of Theorem A are quite sophisticated. Because of this, we prove only the first five parts here, deferring the others to the Appendix (Section A.2, Theorem A). To get your feet wet, you might try Problems 35 and 36.

Proofs of Statements 1 and 2 These statements result from $\lim_{x\to c}(mx + b) = mc + b$ (Example 4 of Section 1.2) using first $m = 0$ and then $m = 1, b = 0$. ■

Proof of Statement 3 If $k = 0$, the result is trivial, so we suppose that $k \neq 0$. Let $\varepsilon > 0$ be given. By hypothesis, $\lim_{x\to c} f(x)$ exists; call its value L. By definition of limit, there is a number δ such that

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \frac{\varepsilon}{|k|}$$

Someone is sure to complain that we put $\varepsilon/|k|$ rather than ε at the end of the inequality above. Well, isn't $\varepsilon/|k|$ a positive number? Yes. Doesn't the definition of limit require that for *any* positive number there be a corresponding δ? Yes.

Now, for δ so determined (again by a preliminary analysis that we have not shown here), we assert that $0 < |x - c| < \delta$ implies that

$$|kf(x) - kL| = |k||f(x) - L| < |k|\frac{\varepsilon}{|k|} = \varepsilon$$

This shows that

$$\lim_{x\to c} kf(x) = kL = k\lim_{x\to c} f(x)$$ ■

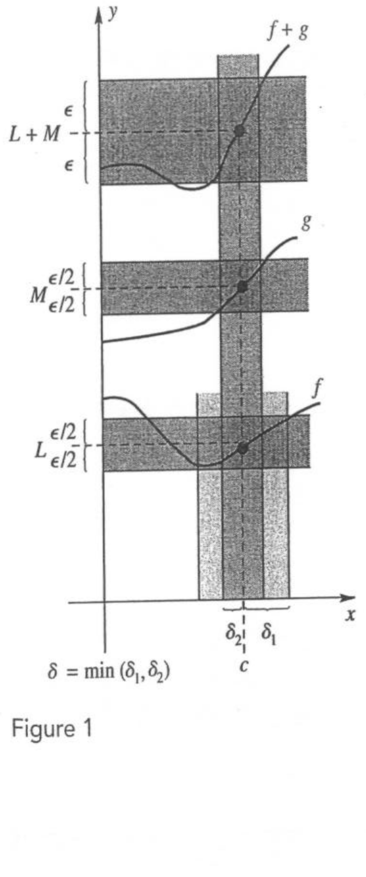


Figure 1

Proof of Statement 4 Refer to Figure 1. Let $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. If ε is any given positive number, then $\varepsilon/2$ is positive. Since $\lim_{x\to c} f(x) = L$, there is a positive number δ_1 such that

$$0 < |x - c| < \delta_1 \Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$$

Since $\lim_{x\to c} g(x) = M$, there is a positive number δ_2 such that

$$0 < |x - c| < \delta_2 \Rightarrow |g(x) - M| < \frac{\varepsilon}{2}$$

Choose $\delta = \min\{\delta_1, \delta_2\}$; that is, choose δ to be the smaller of δ_1 and δ_2. Then $0 < |x - c| < \delta$ implies that

$$\begin{aligned}|f(x) + g(x) - (L + M)| &= |[f(x) - L] + [g(x) - M]| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

In this chain, the first inequality is the Triangle Inequality (Section 0.2); the second results from the choice of δ. We have just shown that

$$0 < |x - c| < \delta \Rightarrow |f(x) + g(x) - (L + M)| < \varepsilon$$

Thus,

$$\lim_{x\to c}[f(x) + g(x)] = L + M = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$ ■

Proof of Statement 5

$$\begin{aligned}\lim_{x\to c}[f(x)-g(x)] &= \lim_{x\to c}[f(x)+(-1)g(x)]\\ &= \lim_{x\to c} f(x) + \lim_{x\to c}(-1)g(x)\\ &= \lim_{x\to c} f(x) + (-1)\lim_{x\to c} g(x)\\ &= \lim_{x\to c} f(x) - \lim_{x\to c} g(x)\end{aligned}$$ ■

The Squeeze Theorem You have likely heard someone say, "I was caught between a rock and a hard place." This is what happens to g in the following theorem (see Figure 2).

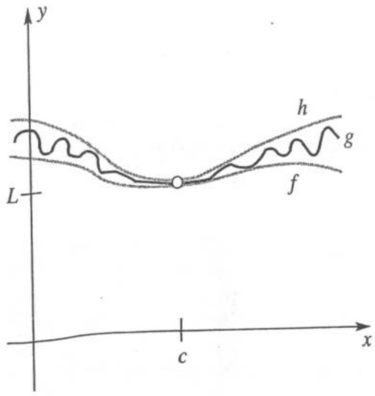


Figure 2

Theorem D Squeeze Theorem

Let f, g, and h be functions satisfying $f(x) \le g(x) \le h(x)$ for all x near c, except possibly at c. If $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$.

Proof (Optional) Let $\varepsilon > 0$ be given. Choose δ_1 such that

$$0 < |x - c| < \delta_1 \Rightarrow L - \varepsilon < f(x) < L + \varepsilon$$

and δ_2 such that

$$0 < |x - c| < \delta_2 \Rightarrow L - \varepsilon < h(x) < L + \varepsilon$$

Choose δ_3 so that

$$0 < |x - c| < \delta_3 \Rightarrow f(x) \le g(x) \le h(x)$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then

$$0 < |x - c| < \delta \Rightarrow L - \varepsilon < f(x) \le g(x) \le h(x) < L + \varepsilon$$

We conclude that $\lim_{x\to c} g(x) = L$. ■

EXAMPLE 9 Assume that we have proved $1 - x^2/6 \le (\sin x)/x \le 1$ for all x near but different from 0. What can we conclude about $\lim_{x\to 0}\frac{\sin x}{x}$?

SOLUTION Let $f(x) = 1 - x^2/6$, $g(x) = (\sin x)/x$, and $h(x) = 1$. It follows that $\lim_{x\to 0} f(x) = 1 = \lim_{x\to 0} h(x)$ and so, by Theorem D,

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$ ■

Concepts Review

1. If $\lim_{x\to 3} f(x) = 4$, then $\lim_{x\to 3}(x^2 + 3)f(x) =$ ____.

2. If $\lim_{x\to 2} g(x) = -2$, then $\lim_{x\to 2}\sqrt{g^2(x) + 12} =$ ____.

3. If $\lim_{x\to c} f(x) = 4$ and $\lim_{x\to c} g(x) = -2$, then $\lim_{x\to c}\frac{f^2(x)}{g(x)} =$ ____ and $\lim_{x\to c}\left[g(x)\sqrt{f(x)} + 5x\right] =$ ____.

4. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = L$, then $\lim_{x\to c}[f(x) - L]g(x) =$ ____.

Problem Set 1.3

In Problems 1–12, use Theorem A to find each of the limits. Justify each step by appealing to a numbered statement, as in Examples 1–4.

1. $\lim_{x\to 1}(2x + 1)$

2. $\lim_{x\to -1}(3x^2 - 1)$

3. $\lim_{x\to 0}[(2x + 1)(x - 3)]$

4. $\lim_{x\to\sqrt{2}}[(2x^2 + 1)(7x^2 + 13)]$

5. $\lim_{x\to 2}\frac{2x + 1}{5 - 3x}$

6. $\lim_{x\to -3}\frac{4x^3 + 1}{7 - 2x^2}$

7. $\lim_{x\to 3} \sqrt{3x-5}$

8. $\lim_{x\to -3} \sqrt{5x^2+2x}$

9. $\lim_{t\to -2} (2t^3+15)^{13}$

10. $\lim_{w\to -2} \sqrt{-3w^3+7w^2}$

11. $\lim_{y\to 2} \left(\frac{4y^3+8y}{y+4}\right)^{1/3}$

12. $\lim_{w\to 5} (2w^4-9w^3+19)^{-1/2}$

In Problems 13–24, find the indicated limit or state that it does not exist. In many cases, you will want to do some algebra before trying to evaluate the limit.

13. $\lim_{x\to 2} \frac{x^2-4}{x^2+4}$

14. $\lim_{x\to 2} \frac{x^2-5x+6}{x-2}$

15. $\lim_{x\to -1} \frac{x^2-2x-3}{x+1}$

16. $\lim_{x\to -1} \frac{x^2+x}{x^2+1}$

17. $\lim_{x\to -1} \frac{x^3-6x^2+11x-6}{x^3+4x^2-19x+14}$

18. $\lim_{x\to 2} \frac{x^2+7x+10}{x+2}$

19. $\lim_{x\to 1} \frac{x^2+x-2}{x^2-1}$

20. $\lim_{x\to -3} \frac{x^2-14x-51}{x^2-4x-21}$

21. $\lim_{u\to -2} \frac{u^2-ux+2u-2x}{u^2-u-6}$

22. $\lim_{x\to 1} \frac{x^2+ux-x-u}{x^2+2x-3}$

23. $\lim_{x\to \pi} \frac{2x^2-6x\pi+4\pi^2}{x^2-\pi^2}$

24. $\lim_{w\to -2} \frac{(w+2)(w^2-w-6)}{w^2+4w+4}$

In Problems 25–30, find the limits if $\lim_{x\to a} f(x)=3$ and $\lim_{x\to a} g(x)=-1$ (see Example 4).

25. $\lim_{x\to a} \sqrt{f^2(x)+g^2(x)}$

26. $\lim_{x\to a} \frac{2f(x)-3g(x)}{f(x)+g(x)}$

27. $\lim_{x\to a} \sqrt[3]{g(x)}\left[f(x)+3\right]$

28. $\lim_{x\to a} \left[f(x)-3\right]^4$

29. $\lim_{t\to a} \left[|f(t)|+|3g(t)|\right]$

30. $\lim_{u\to a} \left[f(u)+3g(u)\right]^3$

In Problems 31–34, find $\lim_{x\to 2} [f(x)-f(2)]/(x-2)$ for each given function f.

31. $f(x)=3x^2$

32. $f(x)=3x^2+2x+1$

33. $f(x)=\frac{1}{x}$

34. $f(x)=\frac{3}{x^2}$

35. Prove Statement 6 of Theorem A. *Hint:*

$$\begin{aligned}|f(x)g(x)-LM| &= |f(x)g(x)-Lg(x)+Lg(x)-LM|\\ &= |g(x)[f(x)-L]+L[g(x)-M]|\\ &\le |g(x)||f(x)-L|+|L||g(x)-M|\end{aligned}$$

Now show that if $\lim_{x\to c} g(x)=M$, then there is a number δ_1 such that

$$0<|x-c|<\delta_1 \Rightarrow |g(x)|<|M|+1$$

36. Prove Statement 7 of Theorem A by first giving an ε–δ proof that $\lim_{x\to c}[1/g(x)] = 1/\left[\lim_{x\to c} g(x)\right]$ and then applying Statement 6.

37. Prove that $\lim_{x\to c} f(x)=L \Leftrightarrow \lim_{x\to c}[f(x)-L]=0$.

38. Prove that $\lim_{x\to c} f(x)=0 \Leftrightarrow \lim_{x\to c}|f(x)|=0$.

39. Prove that $\lim_{x\to c}|x|=|c|$.

40. Find examples to show that if

(a) $\lim_{x\to c}\left[f(x)+g(x)\right]$ exists, this does not imply that either $\lim_{x\to c} f(x)$ or $\lim_{x\to c} g(x)$ exists;

(b) $\lim_{x\to c}\left[f(x)\cdot g(x)\right]$ exists, this does not imply that either $\lim_{x\to c} f(x)$ or $\lim_{x\to c} g(x)$ exists.

In Problems 41–48, find each of the right-hand and left-hand limits or state that they do not exist.

41. $\lim_{x\to -3^+} \frac{\sqrt{3+x}}{x}$

42. $\lim_{x\to -\pi^+} \frac{\sqrt{\pi^3+x^3}}{x}$

43. $\lim_{x\to 3^+} \frac{x-3}{\sqrt{x^2-9}}$

44. $\lim_{x\to 1^-} \frac{\sqrt{1+x}}{4+4x}$

45. $\lim_{x\to 2^+} \frac{(x^2+1)[\![x]\!]}{(3x-1)^2}$

46. $\lim_{x\to 3^-} (x-[\![x]\!])$

47. $\lim_{x\to 0^-} \frac{x}{|x|}$

48. $\lim_{x\to 3^+} [\![x^2+2x]\!]$

49. Suppose that $f(x)g(x)=1$ for all x and $\lim_{x\to a} g(x)=0$. Prove that $\lim_{x\to a} f(x)$ does not exist.

50. Let R be the rectangle joining the midpoints of the sides of the quadrilateral Q having vertices $(\pm x, 0)$ and $(0, \pm 1)$. Calculate

$$\lim_{x\to 0^+} \frac{\text{perimeter of } R}{\text{perimeter of } Q}$$

51. Let $y=\sqrt{x}$ and consider the points $M, N, O,$ and P with coordinates $(1,0), (0,1), (0,0),$ and (x,y) on the graph of $y=\sqrt{x}$, respectively. Calculate

(a) $\lim_{x\to 0^+} \frac{\text{perimeter of } \Delta NOP}{\text{perimeter of } \Delta MOP}$ (b) $\lim_{x\to 0^+} \frac{\text{area of } \Delta NOP}{\text{area of } \Delta MOP}$

Answers to Concepts Review: 1. 48 **2.** 4 **3.** -8; $-4+5c$ **4.** 0

1.4 Limits Involving Trigonometric Functions

Theorem B of the previous section says that limits of polynomial functions can always be found by substitution, and limits of rational functions can be found by substitution as long as the denominator is not zero at the limit point. This substitution rule applies to the trigonometric functions as well. This result is stated next.

Problem Set 1.4

In Problems 1–14, evaluate each limit.

1. $\lim_{x\to 0} \frac{\cos x}{x+1}$

2. $\lim_{\theta\to \pi/2} \theta\cos\theta$

3. $\lim_{t\to 0} \frac{\cos^2 t}{1+\sin t}$

4. $\lim_{x\to 0} \frac{3x\tan x}{\sin x}$

5. $\lim_{x\to 0} \frac{\sin x}{2x}$

6. $\lim_{\theta\to 0} \frac{\sin 3\theta}{2\theta}$

7. $\lim_{\theta\to 0} \frac{\sin 3\theta}{\tan\theta}$

8. $\lim_{\theta\to 0} \frac{\tan 5\theta}{\sin 2\theta}$

9. $\lim_{\theta\to 0} \frac{\cot(\pi\theta)\sin\theta}{2\sec\theta}$

10. $\lim_{t\to 0} \frac{\sin^2 3t}{2t}$

11. $\lim_{t\to 0} \frac{\tan^2 3t}{2t}$

12. $\lim_{t\to 0} \frac{\tan 2t}{\sin 2t - 1}$

13. $\lim_{t\to 0} \frac{\sin 3t + 4t}{t\sec t}$

14. $\lim_{\theta\to 0} \frac{\sin^2\theta}{\theta^2}$

In Problems 15–19, plot the functions $u(x)$, $l(x)$, and $f(x)$. Then use these graphs along with the Squeeze Theorem to determine $\lim_{x\to 0} f(x)$.

15. $u(x) = |x|, l(x) = -|x|, f(x) = x\sin(1/x)$

16. $u(x) = |x|, l(x) = -|x|, f(x) = x\sin(1/x^2)$

17. $u(x) = |x|, l(x) = -|x|, f(x) = (1-\cos^2 x)/x$

18. $u(x) = 1, l(x) = 1 - x^2, f(x) = \cos^2 x$

19. $u(x) = 2, l(x) = 2 - x^2, f(x) = 1 + \frac{\sin x}{x}$

20. Prove that $\lim_{t\to c} \cos t = \cos c$ using an argument similar to the one used in the proof that $\lim_{t\to c} \sin t = \sin c$.

21. Prove statements 3 and 4 of Theorem A using Theorem 1.3A.

22. Prove statements 5 and 6 of Theorem A using Theorem 1.3A.

23. From area $(OBP) \le$ area (sector OAP) $\le$ area (OBP) + area $(ABPQ)$ in Figure 4, show that

$$\cos t \le \frac{t}{\sin t} \le 2 - \cos t$$

and thus obtain another proof that $\lim_{t\to 0^+} (\sin t)/t = 1$.

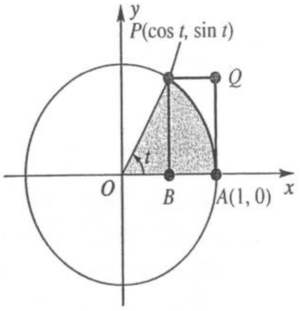


Figure 4

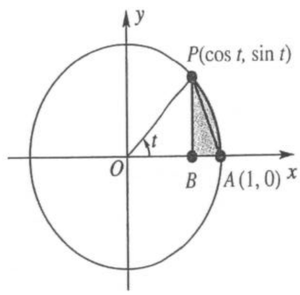


Figure 5

24. In Figure 5, let D be the area of triangle ABP and E the area of the shaded region.

(a) Guess the value of $\lim_{t\to 0^+} \frac{D}{E}$ by looking at the figure.

(b) Find a formula for D/E in terms of t.

$\boxed{C}$ (c) Use a calculator to get an accurate estimate of $\lim_{t\to 0^+} \frac{D}{E}$.

Answers to Concepts Review: **1.** 0 **2.** 1 **3.** the denominator is zero when $t = 0$ **4.** 1

1.5 Limits at Infinity; Infinite Limits

The deepest problems and most profound paradoxes of mathematics are often intertwined with the use of the concept of the infinite. Yet mathematical progress can in part be measured in terms of our understanding the concept of infinity. We have already used the symbols ∞ and $-\infty$ in our notation for certain intervals. Thus, $(3, \infty)$ is our way of denoting the set of all real numbers greater than 3. Note that we have never referred to ∞ as a number. For example, we have never added it to a number or divided it by a number. We will use the symbols ∞ and $-\infty$ in a new way in this section, but they will still not represent numbers.

Limits at Infinity Consider the function $g(x) = x/(1+x^2)$ whose graph is shown in Figure 1. We ask this question: What happens to $g(x)$ as x gets larger and larger? In symbols, we ask for the value of $\lim_{x\to\infty} g(x)$.

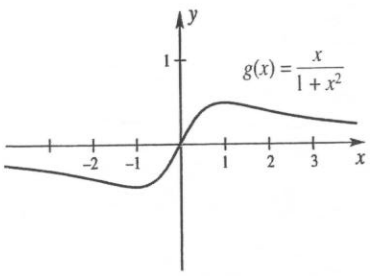


Figure 1

When we write $x \to \infty$, we are *not* implying that somewhere far, far to the right on the x-axis there is a number—bigger than all other numbers—that x is approaching. Rather, we use $x \to \infty$ as a shorthand way of saying that x gets larger and larger without bound.

In the table in Figure 2, we have listed values of $g(x) = x/(1+x^2)$ for several values of x. It appears that $g(x)$ gets smaller and smaller as x gets larger and larger. We write

x	$\frac{x}{1+x^2}$
10	0.099
100	0.010
1000	0.001
10000	0.0001
↓	↓
∞	?

Figure 2

$$\lim_{x\to\infty}\frac{x}{1+x^2}=0$$

Experimenting with negative numbers far to the left of zero on the real number line would lead us to write

$$\lim_{x\to-\infty}\frac{x}{1+x^2}=0$$

Rigorous Definitions of Limits as $x\to\pm\infty$ In analogy with our ε–δ definition for ordinary limits, we make the following definition.

> **Definition** **Limit as $x\to\infty$**
>
> Let f be defined on $[c,\infty)$ for some number c. We say that $\lim_{x\to\infty} f(x)=L$ if for each $\varepsilon>0$ there is a corresponding number M such that
>
> $$x>M \Rightarrow |f(x)-L|<\varepsilon$$

You will note that M can, and usually does, depend on ε. In general, the smaller ε is, the larger M will have to be. The graph in Figure 3 may help you to understand what we are saying.

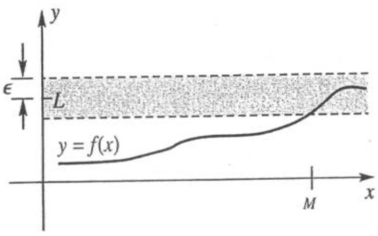


Figure 3

> **Definition** **Limit as $x\to-\infty$**
>
> Let f be defined on $(-\infty,c]$ for some number c. We say that $\lim_{x\to-\infty} f(x)=L$ if for each $\varepsilon>0$ there is a corresponding number M such that
>
> $$x<M \Rightarrow |f(x)-L|<\varepsilon$$

EXAMPLE 1 Show that if k is a positive integer, then

$$\lim_{x\to\infty}\frac{1}{x^k}=0 \quad\text{and}\quad \lim_{x\to-\infty}\frac{1}{x^k}=0$$

SOLUTION Let $\varepsilon>0$ be given. After a preliminary analysis (as in Section 1.2), we chose $M=\sqrt[k]{1/\varepsilon}$. Then $x>M$ implies that

$$\left|\frac{1}{x^k}-0\right|=\frac{1}{x^k}<\frac{1}{M^k}=\varepsilon$$

The proof of the second statement is similar. ■

Having given the definitions of these new kinds of limits, we must face the question of whether the Main Limit Theorem (Theorem 1.3A) holds for them. The answer is yes, and the proof is similar to the original one. Note how we use this theorem in the following examples.

EXAMPLE 2 Prove that $\lim_{x\to\infty}\frac{x}{1+x^2}=0$.

SOLUTION Here we use a standard trick: divide the numerator and denominator by the highest power of x that appears in the denominator, that is, x^2.

$$\lim_{x\to\infty}\frac{x}{1+x^2}=\lim_{x\to\infty}\frac{\frac{x}{x^2}}{\frac{1+x^2}{x^2}}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{x^2}+1}$$

$$=\frac{\lim_{x\to\infty}\frac{1}{x}}{\lim_{x\to\infty}\frac{1}{x^2}+\lim_{x\to\infty}1}=\frac{0}{0+1}=0$$ ■

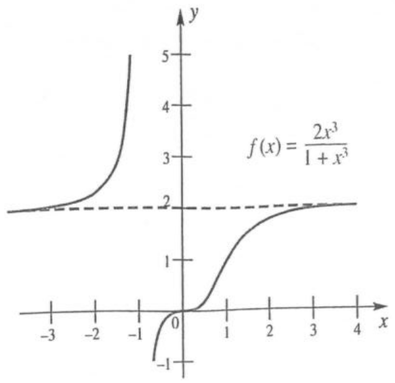


Figure 4

EXAMPLE 3 Find $\lim_{x\to-\infty} \frac{2x^3}{1+x^3}$.

SOLUTION The graph of $f(x) = 2x^3/(1+x^3)$ is shown in Figure 4. To find the limit, divide both the numerator and denominator by x^3.

$$\lim_{x\to-\infty} \frac{2x^3}{1+x^3} = \lim_{x\to-\infty} \frac{2}{1/x^3+1} = \frac{2}{0+1} = 2$$

Limits of Sequences The domain for some functions is the set of natural numbers $\{1, 2, 3, \dots\}$. In this situation, we usually write a_n rather than $a(n)$ to denote the nth term of the sequence, or $\{a_n\}$ to denote the whole sequence. For example, we might define the sequence by $a_n = n/(n+1)$. Let's consider what happens as n gets large. A little calculation shows that

$$a_1 = \frac{1}{2}, \quad a_2 = \frac{2}{3}, \quad a_3 = \frac{3}{4}, \quad a_4 = \frac{4}{5}, \quad \dots, \quad a_{100} = \frac{100}{101}, \quad \dots$$

It looks as if these values are approaching 1, so it seems reasonable to say that for this sequence $\lim_{n\to\infty} a_n = 1$. The next definition gives meaning to this idea of the limit of a sequence.

Definition Limit of a Sequence

Let a_n be defined for all natural numbers greater than or equal to some number c. We say that $\lim_{n\to\infty} a_n = L$ if for each $\varepsilon > 0$ there is a corresponding natural number M such that

$$n > M \Rightarrow |a_n - L| < \varepsilon$$

Notice that this definition is nearly identical to the definition of $\lim_{x\to\infty} f(x)$. The only difference is that now we are requiring that the argument to the function be a natural number. As we might expect, the Main Limit Theorem (Theorem 1.3A) holds for sequences.

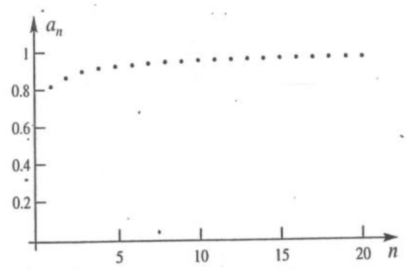


Figure 5

EXAMPLE 4 Find $\lim_{n\to\infty} \sqrt{\frac{n+1}{n+2}}$.

SOLUTION Figure 5 shows a graph of $a_n = \sqrt{\frac{n+1}{n+2}}$. Applying Theorem 1.3A gives

$$\lim_{n\to\infty} \sqrt{\frac{n+1}{n+2}} = \left(\lim_{n\to\infty} \frac{n+1}{n+2}\right)^{1/2} = \left(\lim_{n\to\infty} \frac{1+1/n}{1+2/n}\right)^{1/2} = \left(\frac{1+0}{1+0}\right)^{1/2} = 1$$

We will need the concept of the limit of a sequence in Section 3.7 and in Chapter 4. Sequences are covered more thoroughly in Chapter 9.

Infinite Limits Consider the function $f(x) = 1/(x-2)$, which is graphed in Figure 6. As x gets close to 2 from the left, the function seems to decrease without bound. Similarly, as x approaches 2 from the right, the function seems to increase without bound. It therefore makes no sense to talk about $\lim_{x\to2} 1/(x-2)$, but we think it is reasonable to write

$$\lim_{x\to2^-} \frac{1}{x-2} = -\infty \quad \text{and} \quad \lim_{x\to2^+} \frac{1}{x-2} = \infty$$

Here is the precise definition.

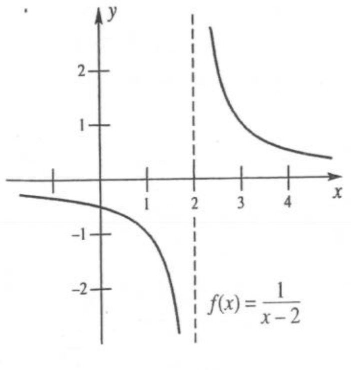


Figure 6

Definition Infinite Limit

We say that $\lim_{x\to c^+} f(x) = \infty$ if for every positive number M, there exists a corresponding $\delta > 0$ such that

$$0 < x - c < \delta \Rightarrow f(x) > M$$

In other words, $f(x)$ can be made as large as we wish (greater than any M that we choose) by taking x to be sufficiently close to but to the right of c. There are corresponding definitions of

$$\lim_{x\to c^+} f(x) = -\infty \quad \lim_{x\to c^-} f(x) = \infty \quad \lim_{x\to c^-} f(x) = -\infty$$
$$\lim_{x\to\infty} f(x) = \infty \quad \lim_{x\to\infty} f(x) = -\infty \quad \lim_{x\to-\infty} f(x) = \infty \quad \lim_{x\to-\infty} f(x) = -\infty$$

(See Problems 51 and 52.)

EXAMPLE 5 Find $\lim_{x\to 1^-} \frac{1}{(x-1)^2}$ and $\lim_{x\to 1^+} \frac{1}{(x-1)^2}$.

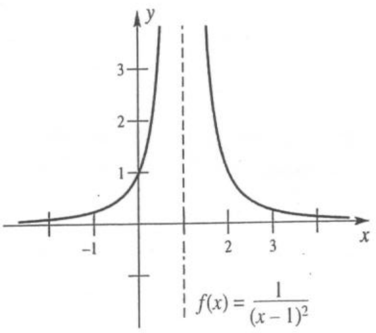


Figure 7

SOLUTION The graph of $f(x) = 1/(x-1)^2$ is shown in Figure 7. As $x \to 1^+$, the denominator remains positive but goes to zero, while the numerator is 1 for all x. Thus, the ratio $1/(x-1)^2$ can be made arbitrarily large by restricting x to be near, but to the right of, 1. Similarly, as $x \to 1^-$, the denominator is positive and can be made arbitrarily close to 0. Thus $1/(x-1)^2$ can be made arbitrarily large by restricting x to be near, but to the left of, 1. We therefore conclude that

$$\lim_{x\to 1^+} \frac{1}{(x-1)^2} = \infty \quad \text{and} \quad \lim_{x\to 1^-} \frac{1}{(x-1)^2} = \infty$$

Since both limits are ∞, we could also write

$$\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$$

EXAMPLE 6 Find $\lim_{x\to 2^+} \frac{x+1}{x^2-5x+6}$.

SOLUTION

$$\lim_{x\to 2^+} \frac{x+1}{x^2-5x+6} = \lim_{x\to 2^+} \frac{x+1}{(x-3)(x-2)}$$

As $x \to 2^+$ we see that $x + 1 \to 3$, $x - 3 \to -1$, and $x - 2 \to 0^+$; thus, the numerator is approaching 3, but the denominator is negative and approaching 0. We conclude that

$$\lim_{x\to 2^+} \frac{x+1}{(x-3)(x-2)} = -\infty$$

Relation to Asymptotes Asymptotes were discussed briefly in Section 0.5, but now we can say more about them. The line $x = c$ is a **vertical asymptote** of the graph of $y = f(x)$ if any of the following four statements is true.

1. $\lim_{x\to c^+} f(x) = \infty$
2. $\lim_{x\to c^+} f(x) = -\infty$
3. $\lim_{x\to c^-} f(x) = \infty$
4. $\lim_{x\to c^-} f(x) = -\infty$

Thus, in Figure 6, the line $x = 2$ is a vertical asymptote. Likewise, the lines $x = 2$ and $x = 3$, although not shown graphically, are vertical asymptotes in Example 6.

In a similar vein, the line $y = b$ is a **horizontal asymptote** of the graph of $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b$$

The line $y = 0$ is a horizontal asymptote in both Figures 6 and 7.

EXAMPLE 7 Find the vertical and horizontal asymptotes of the graph of $y = f(x)$ if

$$f(x) = \frac{2x}{x-1}$$

Do Infinite Limits Exist?

In previous sections we required that a limit be equal to a real number. For example, we said that $\lim_{x\to 2^+} \frac{1}{x-2}$ does not exist because $1/(x-2)$ does not approach a real number as x approaches 2 from the right. Many mathematicians maintain that this limit does not exist even though we write $\lim_{x\to 2^+} \frac{1}{x-2} = \infty$; to say that the limit is ∞ is to describe the particular way in which the limit does not exist. Here we will use the phrase "exists in the infinite sense" to describe such limits.

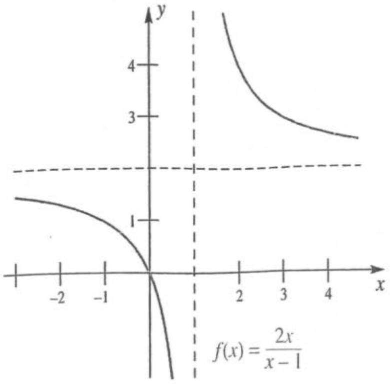


Figure 8

SOLUTION We often have a vertical asymptote at a point where the denominator is zero, and in this case we do because

$$\lim_{x\to 1^+} \frac{2x}{x-1} = \infty \quad \text{and} \quad \lim_{x\to 1^-} \frac{2x}{x-1} = -\infty$$

On the other hand,

$$\lim_{x\to\infty} \frac{2x}{x-1} = \lim_{x\to\infty} \frac{2}{1-1/x} = 2 \quad \text{and} \quad \lim_{x\to-\infty} \frac{2x}{x-1} = 2$$

and so $y = 2$ is a horizontal asymptote. The graph of $y = 2x/(x-1)$ is shown in Figure 8. ■

Concepts Review

1. To say that $x \to \infty$ means that ____; to say that $\lim_{x\to\infty} f(x) = L$ means that ____. Give your answers in informal language.

2. To say that $\lim_{x\to c^+} f(x) = \infty$ means that ____; to say that $\lim_{x\to c^-} f(x) = -\infty$ means that ____. Give your answers in informal language.

3. If $\lim_{x\to\infty} f(x) = 6$, then the line ____ is a ____ asymptote of the graph of $y = f(x)$.

4. If $\lim_{x\to 6^+} f(x) = \infty$, then the line ____ is a ____ asymptote of the graph of $y = f(x)$.

Problem Set 1.5

In Problems 1–42, find the limits.

1. $\lim_{x\to\infty} \frac{x}{x-5}$

2. $\lim_{x\to\infty} \frac{x^2}{5-x^3}$

3. $\lim_{t\to-\infty} \frac{t^2}{7-t^2}$

4. $\lim_{t\to-\infty} \frac{t}{t-5}$

5. $\lim_{x\to\infty} \frac{x^2}{(x-5)(3-x)}$

6. $\lim_{x\to\infty} \frac{x^2}{x^2-8x+15}$

7. $\lim_{x\to\infty} \frac{x^3}{2x^3-100x^2}$

8. $\lim_{\theta\to-\infty} \frac{\pi\theta^5}{\theta^5-5\theta^4}$

9. $\lim_{x\to\infty} \frac{3x^3-x^2}{\pi x^3-5x^2}$

10. $\lim_{\theta\to\infty} \frac{\sin^2\theta}{\theta^2-5}$

11. $\lim_{x\to\infty} \frac{3\sqrt{x^3}+3x}{\sqrt{2x^3}}$

12. $\lim_{x\to\infty} \sqrt[3]{\frac{\pi x^3+3x}{\sqrt{2}x^3+7x}}$

13. $\lim_{x\to\infty} \sqrt[3]{\frac{1+8x^2}{x^2+4}}$

14. $\lim_{x\to\infty} \sqrt{\frac{x^2+x+3}{(x-1)(x+1)}}$

15. $\lim_{n\to\infty} \frac{n}{2n+1}$

16. $\lim_{n\to\infty} \frac{n^2}{n^2+1}$

17. $\lim_{n\to\infty} \frac{n^2}{n+1}$

18. $\lim_{n\to\infty} \frac{n}{n^2+1}$

19. $\lim_{x\to\infty} \frac{2x+1}{\sqrt{x^2+3}}$. *Hint:* Divide numerator and denominator by x. Note that, for $x > 0$, $\sqrt{x^2+3}/x = \sqrt{(x^2+3)/x^2}$.

20. $\lim_{x\to\infty} \frac{\sqrt{2x+1}}{x+4}$

21. $\lim_{x\to\infty} \left(\sqrt{2x^2+3} - \sqrt{2x^2-5}\right)$. *Hint:* Multiply and divide by $\sqrt{2x^2+3} + \sqrt{2x^2-5}$.

22. $\lim_{x\to\infty} \left(\sqrt{x^2+2x} - x\right)$

23. $\lim_{y\to-\infty} \frac{9y^3+1}{y^2-2y+2}$. *Hint:* Divide numerator and denominator by y^2.

24. $\lim_{x\to\infty} \frac{a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n}{b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n}$, where $a_0 \neq 0$, $b_0 \neq 0$, and n is a natural number.

25. $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}}$

26. $\lim_{n\to\infty} \frac{n^2}{\sqrt{n^3+2n+1}}$

27. $\lim_{x\to 4^+} \frac{x}{x-4}$

28. $\lim_{t\to -3^+} \frac{t^2-9}{t+3}$

29. $\lim_{t\to 3^-} \frac{t^2}{9-t^2}$

30. $\lim_{x\to\sqrt[3]{5}^+} \frac{x^2}{5-x^3}$

31. $\lim_{x\to 5^-} \frac{x^2}{(x-5)(3-x)}$

32. $\lim_{\theta\to\pi^+} \frac{\theta^2}{\sin\theta}$

33. $\lim_{x\to 3^-} \frac{x^3}{x-3}$

34. $\lim_{\theta\to(\pi/2)^+} \frac{\pi\theta}{\cos\theta}$

35. $\lim_{x\to 3^-} \frac{x^2-x-6}{x-3}$

36. $\lim_{x\to 2^+} \frac{x^2+2x-8}{x^2-4}$

37. $\lim_{x\to 0^+} \frac{[\![x]\!]}{x}$

38. $\lim_{x\to 0^-} \frac{[\![x]\!]}{x}$

39. $\lim_{x\to 0^-} \frac{|x|}{x}$

40. $\lim_{x\to 0^+} \frac{|x|}{x}$

41. $\lim_{x\to 0^-} \frac{1+\cos x}{\sin x}$

42. $\lim_{x\to\infty} \frac{\sin x}{x}$

GC *In Problems 43–48, find the horizontal and vertical asymptotes for the graphs of the indicated functions. Then sketch their graphs.*

43. $f(x) = \frac{3}{x+1}$

44. $f(x) = \frac{3}{(x+1)^2}$

45. $F(x) = \frac{2x}{x-3}$

46. $F(x) = \frac{3}{9-x^2}$

47. $g(x) = \frac{14}{2x^2+7}$

48. $g(x) = \frac{2x}{\sqrt{x^2+5}}$

49. The line $y = ax + b$ is called an **oblique asymptote** to the graph of $y = f(x)$ if either $\lim_{x\to\infty} [f(x) - (ax+b)] = 0$ or $\lim_{x\to-\infty} [f(x) - (ax+b)] = 0$. Find the oblique asymptote for

$$f(x) = \frac{2x^4 + 3x^3 - 2x - 4}{x^3 - 1}$$

Hint: Begin by dividing the denominator into the numerator.

50. Find the oblique asymptote for

$$f(x) = \frac{3x^3 + 4x^2 - x + 1}{x^2 + 1}$$

51. Using the symbols M and δ, give precise definitions of each expression.

(a) $\lim_{x\to c^+} f(x) = -\infty$ (b) $\lim_{x\to c^-} f(x) = \infty$

52. Using the symbols M and N, give precise definitions of each expression.

(a) $\lim_{x\to\infty} f(x) = \infty$ (b) $\lim_{x\to-\infty} f(x) = \infty$

53. Give a rigorous proof that if $\lim_{x\to\infty} f(x) = A$ and $\lim_{x\to\infty} g(x) = B$, then

$$\lim_{x\to\infty} [f(x) + g(x)] = A + B$$

54. We have given meaning to $\lim_{x\to A} f(x)$ for $A = a, a^-, a^+, -\infty, \infty$. Moreover, in each case, this limit may be L (finite), $-\infty$, ∞, or may fail to exist in any sense. Make a table illustrating each of the 20 possible cases.

55. Find each of the following limits or indicate that it does not exist even in the infinite sense.

(a) $\lim_{x\to\infty} \sin x$ (b) $\lim_{x\to\infty} \sin\frac{1}{x}$

(c) $\lim_{x\to\infty} x\sin\frac{1}{x}$ (d) $\lim_{x\to\infty} x^{3/2}\sin\frac{1}{x}$

(e) $\lim_{x\to\infty} x^{-1/2}\sin x$ (f) $\lim_{x\to\infty} \sin\left(\frac{\pi}{6} + \frac{1}{x}\right)$

(g) $\lim_{x\to\infty} \sin\left(x + \frac{1}{x}\right)$ (h) $\lim_{x\to\infty} \left[\sin\left(x + \frac{1}{x}\right) - \sin x\right]$

56. Einstein's Special Theory of Relativity says that the mass $m(v)$ of an object is related to its velocity v by

$$m(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

Here m_0 is the rest mass and c is the velocity of light. What is $\lim_{v\to c^-} m(v)$?

GC *Use a computer or a graphing calculator to find the limits in Problems 57–64. Begin by plotting the function in an appropriate window.*

57. $\lim_{x\to\infty} \frac{3x^2+x+1}{2x^2-1}$

58. $\lim_{x\to-\infty} \sqrt{\frac{2x^2-3x}{5x^2+1}}$

59. $\lim_{x\to-\infty} \left(\sqrt{2x^2+3x} - \sqrt{2x^2-5}\right)$

60. $\lim_{x\to\infty} \frac{2x+1}{\sqrt{3x^2+1}}$

61. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{10}$

62. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{x}$

63. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{x^2}$

64. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{\sin x}$

CAS *Find the one-sided limits in Problems 65–71. Begin by plotting the function in an appropriate window. Your computer may indicate that some of these limits do not exist, but, if so, you should be able to interpret the answer as either ∞ or $-\infty$.*

65. $\lim_{x\to 3^-} \frac{\sin|x-3|}{x-3}$

66. $\lim_{x\to 3^-} \frac{\sin|x-3|}{\tan(x-3)}$

67. $\lim_{x\to 3^-} \frac{\cos(x-3)}{x-3}$

68. $\lim_{x\to\frac{\pi}{2}^+} \frac{\cos x}{x-\pi/2}$

69. $\lim_{x\to 0^+} (1+\sqrt{x})^{1/\sqrt{x}}$

70. $\lim_{x\to 0^+} (1+\sqrt{x})^{1/x}$

71. $\lim_{x\to 0^+} (1+\sqrt{x})^{x}$

Answers to Concepts Review: **1.** x increases without bound; $f(x)$ gets close to L as x increases without bound **2.** $f(x)$ increases without bound as x approaches c from the right; $f(x)$ decreases without bound as x approaches c from the left **3.** $y = 6$; horizontal **4.** $x = 6$; vertical

1.6 Continuity of Functions

In mathematics and science, we use the word *continuous* to describe a process that goes on without abrupt changes. In fact, our experience leads us to assume that this is an essential feature of many natural processes. It is this notion as it pertains to functions that we now want to make precise. In the three graphs shown in Figure 1, only the third graph exhibits continuity at c. In the first two graphs, either $\lim_{x\to c} f(x)$ does not exist, or it exists but does not equal $f(c)$. Only in the third graph does $\lim_{x\to c} f(x) = f(c)$.

A Discontinuous Machine

A good example of a discontinuous machine is the postage machine, which (in 2005) charged \$0.37 for a 1-ounce letter but \$0.60 for a letter the least little bit over 1 ounce.

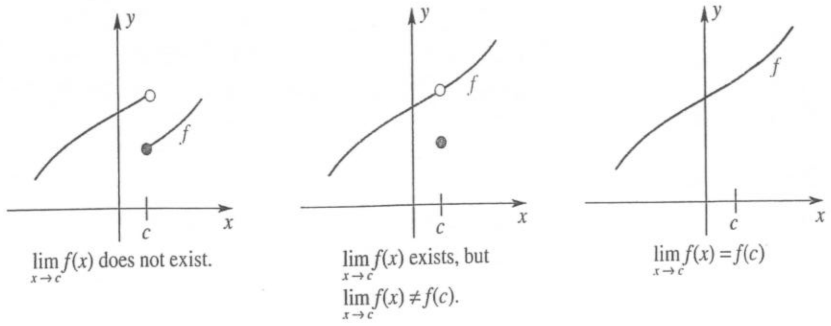


Figure 1

Here is the formal definition.

Definition Continuity at a Point

Let f be defined on an open interval containing c. We say that f is **continuous** at c if

$$\lim_{x\to c} f(x) = f(c)$$

We mean by this definition to require three things:

1. $\lim_{x\to c} f(x)$ exists,
2. $f(c)$ exists (i.e., c is in the domain of f), and
3. $\lim_{x\to c} f(x) = f(c)$.

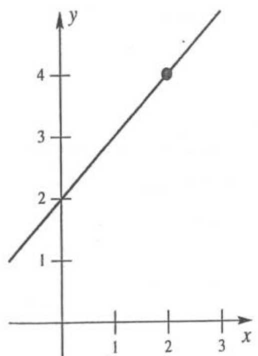
If any one of these three fails, then f is **discontinuous** at c. Thus, the functions represented by the first and second graphs of Figure 1 are discontinuous at c. They do appear, however, to be continuous at other points of their domains.

EXAMPLE 1 Let $f(x) = \dfrac{x^2 - 4}{x - 2}$, $x \neq 2$. How should f be defined at $x = 2$ in order to make it continuous there?

SOLUTION

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2} = \lim_{x\to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x\to 2}(x + 2) = 4$$

Therefore, we define $f(2) = 4$. The graph of the resulting function is shown in Figure 2. In fact, we see that $f(x) = x + 2$ for all x. ■



$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

Figure 2

A point of discontinuity c is called **removable** if the function can be defined or redefined at c so as to make the function continuous. Otherwise, a point of discontinuity is called **nonremovable.** The function f in Example 1 has a removable discontinuity at 2 because we could define $f(2) = 4$ and the function would be continuous there.

Continuity of Familiar Functions

Most functions that we will meet in this book are either (1) continuous everywhere or (2) continuous everywhere except at a few points. In particular, Theorem 1.3B implies the following result.

Theorem A **Continuity of Polynomial and Rational Functions**

A polynomial function is continuous at every real number c. A rational function is continuous at every real number c in its domain, that is, everywhere except where its denominator is zero.

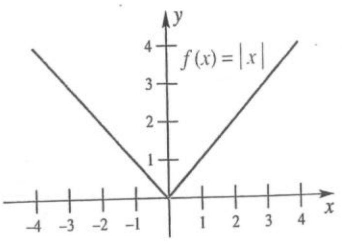


Figure 3

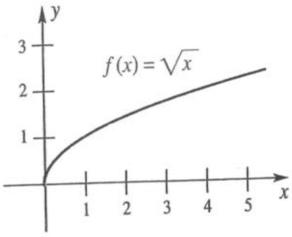


Figure 4

Recall the absolute value function $f(x) = |x|$; its graph is shown in Figure 3. For $x < 0, f(x) = -x$, a polynomial; for $x > 0, f(x) = x$, another polynomial. Thus, $|x|$ is continuous at all numbers different from 0 by Theorem A. But

$$\lim_{x\to 0}|x| = 0 = |0|$$

(see Problem 27 of Section 1.2). Therefore, $|x|$ is also continuous at 0; it is continuous everywhere.

By the Main Limit Theorem (Theorem 1.3A)

$$\lim_{x\to c}\sqrt[n]{x} = \sqrt[n]{\lim_{x\to c}x} = \sqrt[n]{c}$$

provided $c > 0$ when n is even. This means that $f(x) = \sqrt[n]{x}$ is continuous at each point where it makes sense to talk about continuity. In particular, $f(x) = \sqrt{x}$ is continuous at each real number $c > 0$ (Figure 4). We summarize.

Theorem B Continuity of Absolute Value and *n*th Root Functions

The absolute value function is continuous at every real number c. If n is odd, the nth root function is continuous at every real number c; if n is even, the nth-root function is continuous at every positive real number c.

Continuity under Function Operations Do the standard function operations preserve continuity? Yes, according to the next theorem. In it, f and g are functions, k is a constant, and n is a positive integer.

Theorem C Continuity under Function Operations

If f and g are continuous at c, then so are $kf, f + g, f - g, f \cdot g, f/g$ (provided that $g(c) \neq 0$), f^n, and $\sqrt[n]{f}$ (provided that $f(c) > 0$ if n is even).

Proof All these results are easy consequences of the corresponding facts for limits from Theorem 1.3A. For example, that theorem, combined with the fact that f and g are continuous at c, gives

$$\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = f(c)g(c)$$

This is precisely what it means to say that $f \cdot g$ is continuous at c. ■

EXAMPLE 2 At what numbers is $F(x) = (3|x| - x^2)/(\sqrt{x} + \sqrt[3]{x})$ continuous?

SOLUTION We need not even consider nonpositive numbers, since F is not defined at such numbers. For any positive number, the functions $\sqrt{x}, \sqrt[3]{x}, |x|$, and x^2 are all continuous (Theorems A and B). It follows from Theorem C that $3|x|$, $3|x| - x^2$, $\sqrt{x} + \sqrt[3]{x}$, and finally,

$$\frac{(3|x| - x^2)}{(\sqrt{x} + \sqrt[3]{x})}$$

are continuous at each positive number. ■

The continuity of the trigonometric functions follows from Theorem 1.4A.

Theorem D Continuity of Trigonometric Functions

The sine and cosine functions are continuous at every real number c. The functions $\tan x$, $\cot x$, $\sec x$, and $\csc x$ are continuous at every real number c in their domains.

Proof Theorem 1.4A says that for every real number c in the function's domain, $\lim_{x \to c} \sin x = \sin c$, $\lim_{x \to c} \cos x = \cos c$, and so forth, for all six of the trigonometric functions. These are exactly the conditions required for these functions to be continuous at every real number in their respective domains. ■

EXAMPLE 3 Determine all points of discontinuity of $f(x) = \dfrac{\sin x}{x(1-x)}$, $x \neq 0, 1$. Classify each point of discontinuity as removable or nonremovable.

SOLUTION By Theorem D, the numerator is continuous at every real number. The denominator is also continuous at every real number, but when $x = 0$ or $x = 1$, the denominator is 0. Thus, by Theorem C, f is continuous at every real number except $x = 0$ and $x = 1$. Since

$$\lim_{x \to 0} \frac{\sin x}{x(1-x)} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{(1-x)} = (1)(1) = 1$$

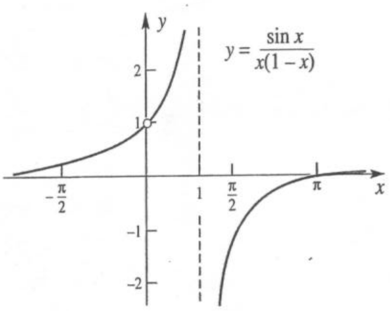


Figure 5

we could define $f(0) = 1$ and the function would continuous there. Thus, $x = 0$ is a removable discontinuity. Also, since

$$\lim_{x \to 1^+} \frac{\sin x}{x(1-x)} = -\infty \quad \text{and} \quad \lim_{x \to 1^-} \frac{\sin x}{x(1-x)} = \infty$$

there is no way to define $f(1)$ to make f continuous at $x = 1$. Thus $x = 1$ is a nonremovable discontinuity. A graph of $y = f(x)$ is shown in Figure 5. ■

There is another functional operation, composition, that will be very important in later work. It, too, preserves continuity.

Theorem E Composite Limit Theorem

If $\lim_{x \to c} g(x) = L$ and if f is continuous at L, then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L)$$

In particular, if g is continuous at c and f is continuous at $g(c)$, then the composite $f \circ g$ is continuous at c.

Proof of Theorem E (Optional)

Proof Let $\varepsilon > 0$ be given. Since f is continuous at L, there is a corresponding $\delta_1 > 0$ such that

$$|t - L| < \delta_1 \Rightarrow |f(t) - f(L)| < \varepsilon$$

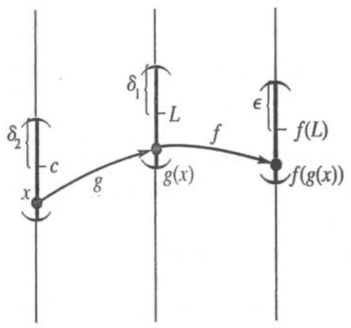


Figure 6

and so (see Figure 6)

$$|g(x) - L| < \delta_1 \Rightarrow |f(g(x)) - f(L)| < \varepsilon$$

But because $\lim_{x \to c} g(x) = L$, for a given $\delta_1 > 0$ there is a corresponding $\delta_2 > 0$ such that

$$0 < |x - c| < \delta_2 \Rightarrow |g(x) - L| < \delta_1$$

When we put these two facts together, we have

$$0 < |x - c| < \delta_2 \Rightarrow |f(g(x)) - f(L)| < \varepsilon$$

This shows that

$$\lim_{x \to c} f(g(x)) = f(L)$$

The second statement in Theorem E follows from the observation that if g is continuous at c then $L = g(c)$. ■

EXAMPLE 4 Show that $h(x) = |x^2 - 3x + 6|$ is continuous at each real number.

SOLUTION Let $f(x) = |x|$ and $g(x) = x^2 - 3x + 6$. Both are continuous at each real number, and so their composite

$$h(x) = f(g(x)) = |x^2 - 3x + 6|$$

is also. ■

EXAMPLE 5 Show that

$$h(x) = \sin \frac{x^4 - 3x + 1}{x^2 - x - 6}$$

is continuous except at 3 and -2.

SOLUTION $x^2 - x - 6 = (x - 3)(x + 2)$. Thus, the rational function

$$g(x) = \frac{x^4 - 3x + 1}{x^2 - x - 6}$$

is continuous except at 3 and -2 (Theorem A). We know from Theorem D that the sine function is continuous at every real number. Thus, from Theorem E, we conclude that, since $h(x) = \sin(g(x))$, h is also continuous except at 3 and -2. ■

Continuity on an Interval So far, we have been discussing continuity at a point. We now wish to discuss continuity on an interval. Continuity on an interval ought to mean continuity at each point of that interval. This is exactly what it does mean for an *open* interval.

When we consider a closed interval $[a, b]$, we face a problem. It might be that f is not even defined to the left of a (e.g., this occurs for $f(x) = \sqrt{x}$ at $a = 0$), so, strictly speaking, $\lim_{x \to a} f(x)$ does not exist. We choose to get around this problem by calling f continuous on $[a, b]$ if it is continuous at each point of (a, b) and if $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$. We summarize in a formal definition.

Definition Continuity on an Interval

The function f is **right continuous** at a if $\lim_{x \to a^+} f(x) = f(a)$ and **left continuous** at b if $\lim_{x \to b^-} f(x) = f(b)$.

We say f is **continuous on an open interval** if it is continuous at each point of that interval. It is **continuous on the closed interval** $[a, b]$ if it is continuous on (a, b), right continuous at a, and left continuous at b.

For example, it is correct to say that $f(x) = 1/x$ is continuous on $(0, 1)$ and that $g(x) = \sqrt{x}$ is continuous on $[0, 1]$.

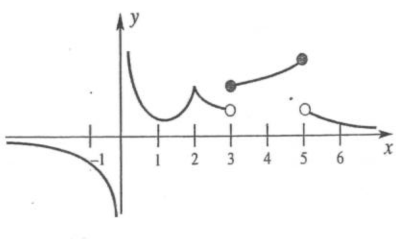


Figure 7

EXAMPLE 6 Using the definition above, describe the continuity properties of the function whose graph is sketched in Figure 7.

SOLUTION The function appears to be continuous on the open intervals $(-\infty, 0)$ $(0, 3)$, and $(5, \infty)$, and also on the closed interval $[3, 5]$. ■

EXAMPLE 7 What is the largest interval over which the function defined by $g(x) = \sqrt{4 - x^2}$ is continuous?

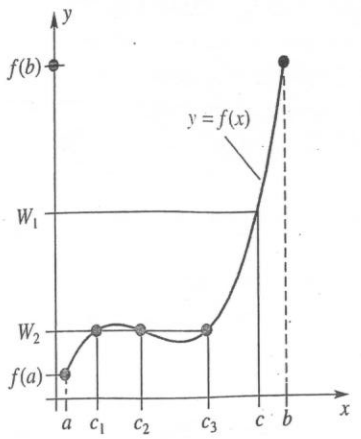


Figure 8

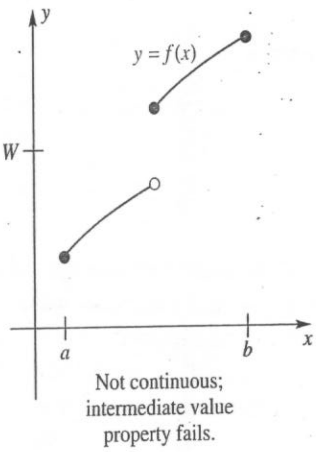


Figure 9

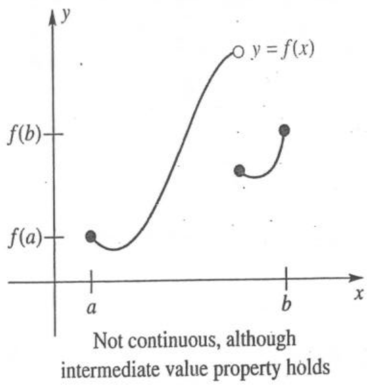


Figure 10

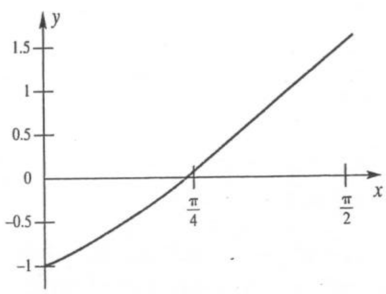


Figure 11

SOLUTION The domain of g is the interval $[-2, 2]$. If c is in the open interval $(-2, 2)$, then g is continuous at c by Theorem E; hence, g is continuous on $(-2, 2)$. The one-sided limits are

$$\lim_{x\to-2^+} \sqrt{4 - x^2} = \sqrt{4 - \left(\lim_{x\to-2^+} x\right)^2}\sqrt{4 - 4} = 0 = g(-2)$$

and

$$\lim_{x\to2^-} \sqrt{4 - x^2} = \sqrt{4 - \left(\lim_{x\to2^-} x\right)^2} = \sqrt{4 - 4} = 0 = g(2)$$

This implies that g is right continuous at -2 and left continuous at 2. Thus, g is continuous on its domain, the closed interval $[-2, 2]$. ■

Intuitively, for f to be continuous on $[a, b]$ means that the graph of f on $[a, b]$ should have no jumps, so we should be able to "draw" the graph of f from the point $(a, f(a))$ to the point $(b, f(b))$ without lifting our pencil from the paper. Thus, the function f should take on every value between $f(a)$ and $f(b)$. This property is stated more precisely in Theorem F.

Theorem F Intermediate Value Theorem

Let f be a function defined on $[a, b]$ and let W be a number between $f(a)$ and $f(b)$. If f is continuous on $[a, b]$, then there is at least one number c between a and b such that $f(c) = W$.

Figure 8 shows the graph of a function $f(x)$ that is continuous on $[a, b]$. The Intermediate Value Theorem says that for every W in $(f(a), f(b))$ there must be a c in $[a, b]$ such that $f(c) = W$. In other words, f takes on every value between $f(a)$ and $f(b)$. Continuity is needed for this theorem, for otherwise it is possible to find a function f and a number W between $f(a)$ and $f(b)$ such that there is no c in $[a, b]$ that satisfies $f(c) = W$. Figure 9 shows an example of such a function.

It seems clear that continuity is sufficient, although a formal proof of this result turns out to be difficult. We leave the proof to more advanced works.

The converse of this theorem, which is not true in general, says that if f takes on every value between $f(a)$ and $f(b)$ then f is continuous. Figures 8 and 10 show functions that take on all values between $f(a)$ and $f(b)$, but the function in Figure 10 is not continuous on $[a, b]$. Just because a function has the intermediate value property does not mean that it must be continuous.

The Intermediate Value Theorem can be used to tell us something about the solutions of equations, as the next example shows.

EXAMPLE 8 Use the Intermediate Value Theorem to show that the equation $x - \cos x = 0$ has a solution between $x = 0$ and $x = \pi/2$.

SOLUTION Let $f(x) = x - \cos x$, and let $W = 0$. Then $f(0) = 0 - \cos 0 = -1$ and $f(\pi/2) = \pi/2 - \cos \pi/2 = \pi/2$. Since f is continuous on $[0, \pi/2]$ and since $W = 0$ is between $f(0)$ and $f(\pi/2)$, the Intermediate Value Theorem implies the existence of a c in the interval $(0, \pi/2)$ with the property that $f(c) = 0$. Such a c is a solution to the equation $x - \cos x = 0$. Figure 11 suggests that there is exactly one such c.

We can go one step further. The midpoint of the interval $[0, \pi/2]$ is the point $x = \pi/4$. When we evaluate $f(\pi/4)$, we get

$$f(\pi/4) = \frac{\pi}{4} - \cos\frac{\pi}{4} = \frac{\pi}{4} - \frac{\sqrt{2}}{2} \approx 0.0782914$$

which is greater than 0. Thus, $f(0) < 0$ and $f(\pi/4) > 0$, so another application of the Intermediate Value Theorem tells us that there exists a c between 0 and $\pi/4$ such that $f(c) = 0$. We have thus narrowed down the interval containing the desired c from $[0, \pi/2]$ to $[0, \pi/4]$. There is nothing stopping us from selecting the midpoint of $[0, \pi/4]$ and evaluating f at that point, thereby narrowing even further the interval containing c. This process could be continued indefinitely until we find that c is in a sufficiently small interval. This method of zeroing in on a solution is called the *bisection method*, and we will study it further in Section 3.7. ■

The Intermediate Value Theorem can also lead to some surprising results.

EXAMPLE 9 Use the Intermediate Value Theorem to show that on a circular wire ring there are always two points opposite from each other with the same temperature.

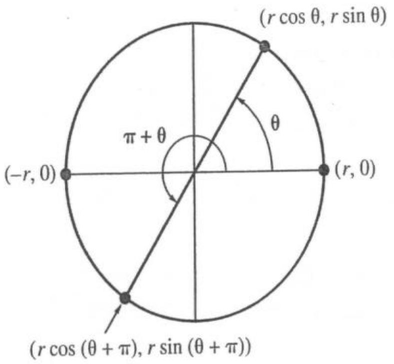


Figure 12

SOLUTION Choose coordinates for this problem so that the center of the ring is the origin, and let r be the radius of the ring. (See Figure 12.) Define $T(x, y)$ to be the temperature at the point (x, y). Consider a diameter of the circle that makes an angle θ with the x-axis, and define $f(\theta)$ to be the temperature difference between the points that make angles of θ and $\theta + \pi$; that is,

$$f(\theta) = T(r\cos\theta, r\sin\theta) - T(r\cos(\theta+\pi), r\sin(\theta+\pi))$$

With this definition

$$f(0) = T(r, 0) - T(-r, 0)$$
$$f(\pi) = T(-r, 0) - T(r, 0) = -\left[T(r, 0) - T(-r, 0)\right] = -f(0)$$

Thus, either $f(0)$ and $f(\pi)$ are both zero, or one is positive and the other is negative. If both are zero, then we have found the required two points. Otherwise, we can apply the Intermediate Value Theorem. Assuming that temperature varies continuously, we conclude that there exists a c between 0 and π such that $f(c) = 0$. Thus, for the two points at the angles c and $c + \pi$, the temperatures are the same. ■

Concepts Review

1. A function f is continuous at c if ____ $= f(c)$.

2. The function $f(x) = [x]$ is discontinuous at ____.

3. A function f is said to be continuous on a closed interval $[a, b]$ if it is continuous at every point of (a, b) and if ____ and ____.

4. The Intermediate Value Theorem says that if a function f is continuous on $[a, b]$ and W is a number between $f(a)$ and $f(b)$, then there is a number c between ____ and ____ such that ____.

Problem Set 1.6

In Problems 1–15, state whether the indicated function is continuous at 3. If it is not continuous, tell why.

1. $f(x) = (x-3)(x-4)$

2. $g(x) = x^2 - 9$

3. $h(x) = \dfrac{3}{x-3}$

4. $g(t) = \sqrt{t-4}$

5. $h(t) = \dfrac{|t-3|}{t-3}$

6. $h(t) = \dfrac{|\sqrt{(t-3)^4}|}{t-3}$

7. $f(t) = |t|$

8. $g(t) = |t-2|$

9. $h(x) = \dfrac{x^2-9}{x-3}$

10. $f(x) = \dfrac{21-7x}{x-3}$

11. $r(t) = \begin{cases} \dfrac{t^3-27}{t-3} & \text{if } t \neq 3 \\ 27 & \text{if } t = 3 \end{cases}$

12. $r(t) = \begin{cases} \dfrac{t^3-27}{t-3} & \text{if } t \neq 3 \\ 23 & \text{if } t = 3 \end{cases}$

13. $f(t) = \begin{cases} t-3 & \text{if } t \le 3 \\ 3-t & \text{if } t > 3 \end{cases}$

14. $f(t) = \begin{cases} t^2-9 & \text{if } t \le 3 \\ (3-t)^2 & \text{if } t > 3 \end{cases}$

15. $f(x) = \begin{cases} -3x+7 & \text{if } x \le 3 \\ -2 & \text{if } x > 3 \end{cases}$

16. From the graph of g (see Figure 13), indicate the values where g is discontinuous. For each of these values state whether g is continuous from the right, left, or neither.

2.2 The Derivative

We have seen that *slope of the tangent line* and *instantaneous velocity* are manifestations of the same basic idea. Rate of growth of an organism (biology), marginal profit (economics), density of a wire (physics), and dissolution rates (chemistry) are other versions of the same basic concept. Good mathematical sense suggests that we study this concept independently of these specialized vocabularies and diverse applications. We choose the neutral name *derivative*. Add it to *function* and *limit* as one of the key words in calculus.

Definition Derivative

The **derivative** of a function f is another function f' (read "f prime") whose value at any number x is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

If this limit does exist, we say that f is **differentiable** at x. Finding a derivative is called **differentiation**; the part of calculus associated with the derivative is called **differential calculus.**

Finding Derivatives

We illustrate with several examples.

EXAMPLE 1 Let $f(x) = 13x - 6$. Find $f'(4)$.

SOLUTION

$$f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{[13(4+h) - 6] - [13(4) - 6]}{h}$$

$$= \lim_{h\to 0} \frac{13h}{h} = \lim_{h\to 0} 13 = 13$$

EXAMPLE 2 If $f(x) = x^3 + 7x$, find $f'(x)$.

SOLUTION

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\left[(x+h)^3 + 7(x+h)\right] - \left[x^3 + 7x\right]}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 + 7h}{h} \\ &= \lim_{h\to 0} (3x^2 + 3xh + h^2 + 7) \\ &= 3x^2 + 7 \end{aligned}$$

EXAMPLE 3 If $f(x) = 1/x$, find $f'(x)$.

SOLUTION

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h}$$

$$= \lim_{h\to 0} \left[\frac{x - (x+h)}{(x+h)x} \cdot \frac{1}{h}\right] = \lim_{h\to 0} \left[\frac{-h}{(x+h)x} \cdot \frac{1}{h}\right]$$

$$= \lim_{h\to 0} \frac{-1}{(x+h)x} = -\frac{1}{x^2}$$

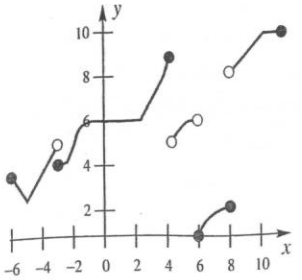


Figure 13

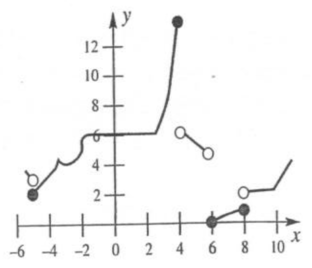


Figure 14

17. From the graph of h given in Figure 14, indicate the intervals on which h is continuous.

In Problems 18–23, the given function is not defined at a certain point. How should it be defined in order to make it continuous at that point? (See Example 1.)

18. $f(x) = \dfrac{x^2 - 49}{x - 7}$ **19.** $f(x) = \dfrac{2x^2 - 18}{3 - x}$

20. $g(\theta) = \dfrac{\sin\theta}{\theta}$ **21.** $H(t) = \dfrac{\sqrt{t} - 1}{t - 1}$

22. $\phi(x) = \dfrac{x^4 + 2x^2 - 3}{x + 1}$ **23.** $F(x) = \sin\dfrac{x^2 - 1}{x + 1}$

In Problems 24–35, at what points, if any, are the functions discontinuous?

24. $f(x) = \dfrac{3x + 7}{(x - 30)(x - \pi)}$

25. $f(x) = \dfrac{33 - x^2}{x\pi + 3x - 3\pi - x^2}$

26. $h(\theta) = |\sin\theta + \cos\theta|$ **27.** $r(\theta) = \tan\theta$

28. $f(u) = \dfrac{2u + 7}{\sqrt{u + 5}}$ **29.** $g(u) = \dfrac{u^2 + |u - 1|}{\sqrt[3]{u + 1}}$

30. $F(x) = \dfrac{1}{\sqrt{4 + x^2}}$ **31.** $G(x) = \dfrac{1}{\sqrt{4 - x^2}}$

32. $f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } x > 1 \end{cases}$

33. $g(x) = \begin{cases} x^2 & \text{if } x < 0 \\ -x & \text{if } 0 \le x \le 1 \\ x & \text{if } x > 1 \end{cases}$

34. $f(t) = [\![t]\!]$ **35.** $g(t) = [\![t + \frac{1}{2}]\!]$

36. Sketch the graph of a function f that satisfies all the following conditions.

(a) Its domain is $[-2, 2]$.

(b) $f(-2) = f(-1) = f(1) = f(2) = 1$.

(c) It is discontinuous at -1 and 1.

(d) It is right continuous at -1 and left continuous at 1.

37. Sketch the graph of a function that has domain $[0, 2]$ and is continuous on $[0, 2)$ but not on $[0, 2]$.

38. Sketch the graph of a function that has domain $[0, 6]$ and is continuous on $[0, 2]$ and $(2, 6]$ but is not continuous on $[0, 6]$.

39. Sketch the graph of a function that has domain $[0, 6]$ and is continuous on $(0, 6)$ but not on $[0, 6]$.

40. Let

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{cases}$$

Sketch the graph of this function as best you can and decide where it is continuous.

In Problems 41–48, determine whether the function is continuous at the given point c. If the function is not continuous, determine whether the discontinuity is removable or nonremovable.

41. $f(x) = \sin x;\ c = 0$ **42.** $f(x) = \dfrac{x^2 - 100}{x - 10};\ c = 10$

43. $f(x) = \dfrac{\sin x}{x};\ c = 0$ **44.** $f(x) = \dfrac{\cos x}{x};\ c = 0$

45. $g(x) = \begin{cases} \dfrac{\sin x}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$ **46.** $F(x) = x\sin\dfrac{1}{x};\ c = 0$

47. $f(x) = \sin\dfrac{1}{x};\ c = 0$ **48.** $f(x) = \dfrac{4 - x}{2 - \sqrt{x}};\ c = 4$

49. A cell phone company charges \$0.12 for connecting a call plus \$0.08 per minute or any part thereof (e.g., a phone call lasting 2 minutes and 5 seconds costs \$0.12 + 3 × \$0.08). Sketch a graph of the cost of making a call as a function of the length of time t that the call lasts. Discuss the continuity of this function.

50. A rental car company charges \$20 for one day, allowing up to 200 miles. For each additional 100 miles, or any fraction thereof, the company charges \$18. Sketch a graph of the cost for renting a car for one day as a function of the miles driven. Discuss the continuity of this function.

51. A cab company charges \$2.50 for the first $\frac{1}{4}$ mile and \$0.20 for each additional $\frac{1}{8}$ mile. Sketch a graph of the cost of a cab ride as a function of the number of miles driven. Discuss the continuity of this function.

52. Use the Intermediate Value Theorem to prove that $x^3 + 3x - 2 = 0$ has a real solution between 0 and 1.

53. Use the Intermediate Value Theorem to prove that $(\cos t)t^3 + 6\sin^5 t - 3 = 0$ has a real solution between 0 and 2π.

GC **54.** Use the Intermediate Value Theorem to show that $x^3 - 7x^2 + 14x - 8 = 0$ has at least one solution in the interval $[0, 5]$. Sketch the graph of $y = x^3 - 7x^2 + 14x - 8$ over $[0, 5]$. How many solutions does this equation really have?

GC **55.** Use the Intermediate Value Theorem to show that $\sqrt{x} - \cos x = 0$ has a solution between 0 and $\pi/2$. Zoom in on the graph of $y = \sqrt{x} - \cos x$ to find an interval having length 0.1 that contains this solution.

56. Show that the equation $x^5 + 4x^3 - 7x + 14 = 0$ has at least one real solution.

57. Prove that f is continuous at c if and only if $\lim_{t\to 0} f(c + t) = f(c)$.

58. Prove that if f is continuous at c and $f(c) > 0$ there is an interval $(c - \delta, c + \delta)$ such that $f(x) > 0$ on this interval.

59. Prove that if f is continuous on $[0, 1]$ and satisfies $0 \le f(x) \le 1$ there, then f has a *fixed point*; that is, there is a number c in $[0, 1]$ such that $f(c) = c$. *Hint:* Apply the Intermediate Value Theorem to $g(x) = x - f(x)$.

Thus, f' is the function given by $f'(x) = -1/x^2$. Its domain is all real numbers except $x = 0$. ■

EXAMPLE 4 Find $F'(x)$ if $F(x) = \sqrt{x}, x > 0$.

SOLUTION

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

By this time you will have noticed that finding a derivative always involves taking the limit of a quotient where both numerator and denominator are approaching zero. Our task is to simplify this quotient so that we can cancel a factor h from the numerator and denominator, thereby allowing us to evaluate the limit by substitution. In the present example, this can be accomplished by rationalizing the numerator.

$$F'(x) = \lim_{h\to 0}\left[\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right]$$
$$= \lim_{h\to 0} \frac{x+h-x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$
$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$
$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Thus, F', the derivative of F, is given by $F'(x) = 1/\left(2\sqrt{x}\right)$. Its domain is $(0, \infty)$. ■

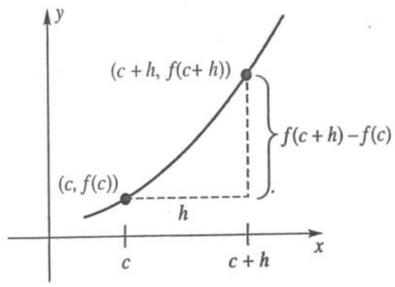


Figure 1

Equivalent Forms for the Derivative There is nothing sacred about use of the letter h in defining $f'(c)$. Notice, for example, that

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$
$$= \lim_{p\to 0} \frac{f(c+p) - f(c)}{p}$$
$$= \lim_{s\to 0} \frac{f(c+s) - f(c)}{s}$$

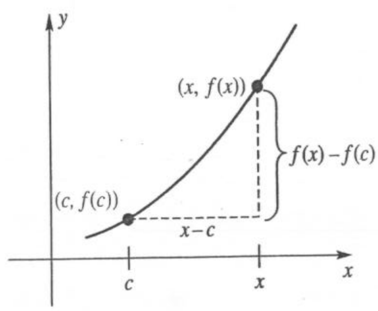


Figure 2

A more radical change, but still just a change of notation, may be understood by comparing Figures 1 and 2. Note how x takes the place of $c + h$, and so $x - c$ replaces h. Thus,

$$\boxed{f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}}$$

Note that in all cases the number at which f' is evaluated is held fixed during the limit operation.

EXAMPLE 5 Use the last boxed result to find $g'(c)$ if $g(x) = 2/(x + 3)$.

SOLUTION

$$g'(c) = \lim_{x\to c} \frac{g(x) - g(c)}{x - c} = \lim_{x\to c} \frac{\dfrac{2}{x+3} - \dfrac{2}{c+3}}{x - c}$$

$$= \lim_{x\to c}\left[\frac{2(c+3) - 2(x+3)}{(x+3)(c+3)} \cdot \frac{1}{x - c}\right]$$

$$= \lim_{x\to c}\left[\frac{-2(x-c)}{(x+3)(c+3)} \cdot \frac{1}{x - c}\right]$$

$$= \lim_{x\to c} \frac{-2}{(x+3)(c+3)} = \frac{-2}{(c+3)^2}$$

Here we manipulated the quotient until we could cancel a factor of $x - c$ from the numerator and denominator. Then we could evaluate the limit. ■

EXAMPLE 6 Each of the following is a derivative, but of what function and at what point?

(a) $\displaystyle\lim_{h\to 0} \frac{(4+h)^2 - 16}{h}$ (b) $\displaystyle\lim_{x\to 3} \frac{\dfrac{2}{x} - \dfrac{2}{3}}{x - 3}$

SOLUTION

(a) This is the derivative of $f(x) = x^2$ at $x = 4$.

(b) This is the derivative of $f(x) = 2/x$ at $x = 3$. ■

Differentiability Implies Continuity If a curve has a tangent line at a point, then that curve cannot take a jump or wiggle too badly at the point. The precise formulation of this fact is an important theorem.

> **Theorem A Differentiability Implies Continuity**
>
> If $f'(c)$ exists, then f is continuous at c.

Proof We need to show that $\lim_{x\to c} f(x) = f(c)$. We begin by writing $f(x)$ in a fancy way.

$$f(x) = f(c) + \frac{f(x) - f(c)}{x - c}\cdot(x - c), \qquad x \neq c$$

Therefore,

$$\lim_{x\to c} f(x) = \lim_{x\to c}\left[f(c) + \frac{f(x) - f(c)}{x - c}\cdot(x - c)\right]$$

$$= \lim_{x\to c} f(c) + \lim_{x\to c} \frac{f(x) - f(c)}{x - c}\cdot\lim_{x\to c}(x - c)$$

$$= f(c) + f'(c)\cdot 0$$

$$= f(c)$$ ■

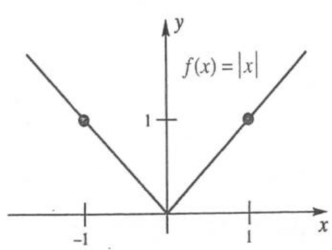


Figure 3

The converse of this theorem is false. If a function f is continuous at c, it does not follow that f has a derivative at c. This is easily seen by considering $f(x) = |x|$ at the origin (Figure 3). This function is certainly continuous at zero. However, it does not have a derivative there, as we now show. Note that

$$\frac{f(0+h)-f(0)}{h}=\frac{|0+h|-|0|}{h}=\frac{|h|}{h}$$

Thus,

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{|h|}{h}=\lim_{h\to 0^+}\frac{h}{h}=1$$

whereas

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{|h|}{h}=\lim_{h\to 0^-}\frac{-h}{h}=-1$$

Since the right- and left-hand limits are different,

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

does not exist. Therefore, $f'(0)$ does not exist.

A similar argument shows that at any point where the graph of a continuous function has a sharp corner the function is not differentiable. The graph in Figure 4 indicates a number of ways for a function to be nondifferentiable at a point.

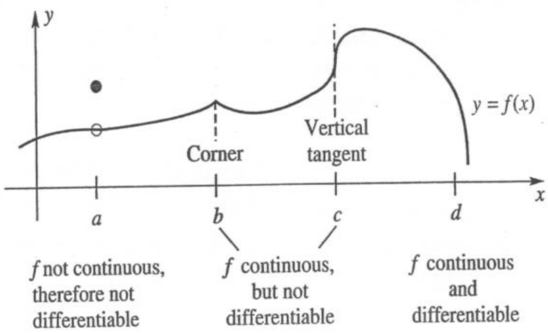


Figure 4

For the function shown in Figure 4 the derivative does not exist at the point c where the tangent line is vertical. This is because

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=\infty$$

This corresponds to the fact that the slope of a vertical line is undefined.

Increments If the value of a variable x changes from x_1 to x_2, then $x_2 - x_1$, the change in x, is called an **increment** of x and is commonly denoted by Δx (read "delta x"). Note that Δx does *not* mean Δ times x. If $x_1 = 4.1$ and $x_2 = 5.7$, then

$$\Delta x = x_2 - x_1 = 5.7 - 4.1 = 1.6$$

If $x_1 = c$ and $x_2 = c + h$, then

$$\Delta x = x_2 - x_1 = c + h - c = h$$

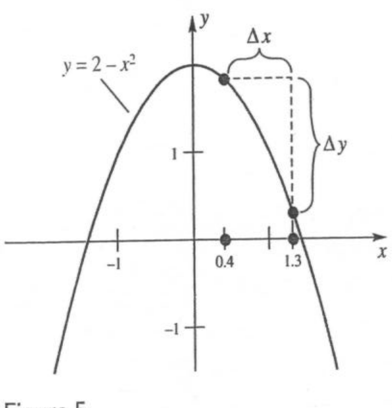


Figure 5

Suppose next that $y = f(x)$ determines a function. If x changes from x_1 to x_2, then y changes from $y_1 = f(x_1)$ to $y_2 = f(x_2)$. Thus, corresponding to the increment $\Delta x = x_2 - x_1$ in x, there is an increment in y given by

$$\Delta y = y_2 - y_1 = f(x_2) - f(x_1)$$

EXAMPLE 7 Let $y = f(x) = 2 - x^2$. Find Δy when x changes from 0.4 to 1.3 (see Figure 5).

SOLUTION

$$\Delta y = f(1.3) - f(0.4) = \left[2 - (1.3)^2\right] - \left[2 - (0.4)^2\right] = -1.53$$

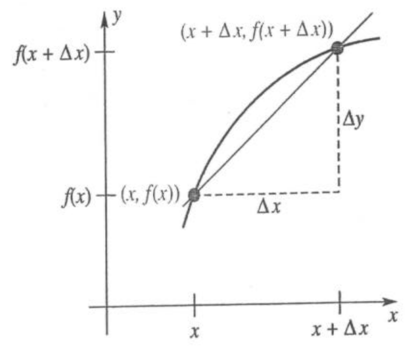


Figure 6

Leibniz Notation for the Derivative Suppose now that the independent variable changes from x to $x + \Delta x$. The corresponding change in the dependent variable, y, will be

$$\Delta y = f(x + \Delta x) - f(x)$$

and the ratio

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

represents the slope of a secant line through $(x, f(x))$, as shown in Figure 6. As $\Delta x \to 0$, the slope of this secant line approaches that of the tangent line, and for this latter slope we use the symbol dy/dx. Thus,

$$\boxed{\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)}$$

Gottfried Wilhelm Leibniz, a contemporary of Isaac Newton, called dy/dx a quotient of two infinitesimals. The meaning of the word *infinitesimal* is vague, and we will not use it. However, dy/dx is a standard symbol for the derivative and we will use it frequently from now on.

The Graph of the Derivative The derivative $f'(x)$ gives the slope of the tangent line to the graph of $y = f(x)$ at the value of x. Thus, when the tangent line is sloping up to the right, the derivative is positive, and when the tangent line is sloping down to the right, the derivative is negative. We can therefore get a rough picture of the derivative given just the graph of the function.

EXAMPLE 8 Given the graph of $y = f(x)$ shown in the first part of Figure 7, sketch a graph of the derivative $f'(x)$.

SOLUTION For $x < 0$, the tangent line to the graph of $y = f(x)$ has positive slope. A rough calculation from the plot suggests that when $x = -2$, the slope is about 3. As we move from left to right along the curve in Figure 7, we see that the slope is still positive (for a while) but that the tangent lines become flatter and flatter. When $x = 0$, the tangent line is horizontal, telling us that $f'(0) = 0$. For x between 0 and 2, the tangent lines have negative slope, indicating that the derivative will be negative over this interval. When $x = 2$, we are again at a point where the tangent line is horizontal, so the derivative is equal to zero when $x = 2$. For $x > 2$, the tangent line again has positive slope. The graph of the derivative $f'(x)$ is shown in the last part of Figure 7.

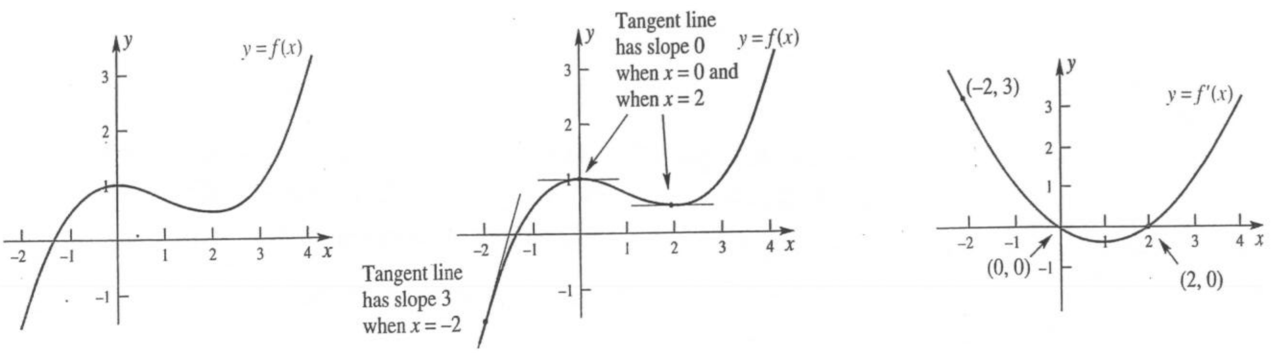


Figure 7

Concepts Review

1. The derivative of f at x is given by $f'(x) = \lim_{h \to 0}$ _____. Equivalently, $f'(x) = \lim_{t \to x}$ _____.

2. The slope of the tangent line to the graph of $y = f(x)$ at the point $(c, f(c))$ is _____.

3. If f is differentiable at c, then f is _____ at c. The converse is false, as is shown by the example $f(x) =$ _____.

4. If $y = f(x)$, we now have two different symbols for the derivative of y with respect to x. They are _____ and _____.

Problem Set 2.2

In Problems 1–4, use the definition

$$f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

to find the indicated derivative.

1. $f'(1)$ if $f(x) = x^2$
2. $f'(2)$ if $f(t) = (2t)^2$
3. $f'(3)$ if $f(t) = t^2 - t$
4. $f'(4)$ if $f(s) = \dfrac{1}{s - 1}$

In Problems 5–22, use $f'(x) = \lim_{h \to 0}[f(x + h) - f(x)]/h$ to find the derivative at x.

5. $s(x) = 2x + 1$
6. $f(x) = \alpha x + \beta$
7. $r(x) = 3x^2 + 4$
8. $f(x) = x^2 + x + 1$
9. $f(x) = ax^2 + bx + c$
10. $f(x) = x^4$
11. $f(x) = x^3 + 2x^2 + 1$
12. $g(x) = x^4 + x^2$
13. $h(x) = \dfrac{2}{x}$
14. $S(x) = \dfrac{1}{x + 1}$
15. $F(x) = \dfrac{6}{x^2 + 1}$
16. $F(x) = \dfrac{x - 1}{x + 1}$
17. $G(x) = \dfrac{2x - 1}{x - 4}$
18. $G(x) = \dfrac{2x}{x^2 - x}$
19. $g(x) = \sqrt{3x}$
20. $g(x) = \dfrac{1}{\sqrt{3x}}$
21. $H(x) = \dfrac{3}{\sqrt{x - 2}}$
22. $H(x) = \sqrt{x^2 + 4}$

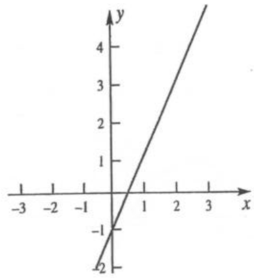
In Problems 23–26, use $f'(x) = \lim_{t \to x}[f(t) - f(x)]/[t - x]$ to find $f'(x)$ (see Example 5).

23. $f(x) = x^2 - 3x$
24. $f(x) = x^3 + 5x$
25. $f(x) = \dfrac{x}{x - 5}$
26. $f(x) = \dfrac{x + 3}{x}$

In Problems 27–36, the given limit is a derivative, but of what function and at what point? (See Example 6.)

27. $\displaystyle\lim_{h \to 0} \frac{2(5 + h)^3 - 2(5)^3}{h}$
28. $\displaystyle\lim_{h \to 0} \frac{(3 + h)^2 + 2(3 + h) - 15}{h}$
29. $\displaystyle\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$
30. $\displaystyle\lim_{x \to 3} \frac{x^3 + x - 30}{x - 3}$
31. $\displaystyle\lim_{t \to x} \frac{t^2 - x^2}{t - x}$
32. $\displaystyle\lim_{p \to x} \frac{p^3 - x^3}{p - x}$
33. $\displaystyle\lim_{x \to t} \frac{\frac{2}{x} - \frac{2}{t}}{x - t}$
34. $\displaystyle\lim_{x \to y} \frac{\sin x - \sin y}{x - y}$
35. $\displaystyle\lim_{h \to 0} \frac{\cos(x + h) - \cos x}{h}$
36. $\displaystyle\lim_{h \to 0} \frac{\tan(t + h) - \tan t}{h}$

In Problems 37–44, the graph of a function $y = f(x)$ is given. Use this graph to sketch the graph of $y = f'(x)$.

37.


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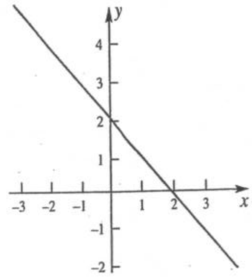

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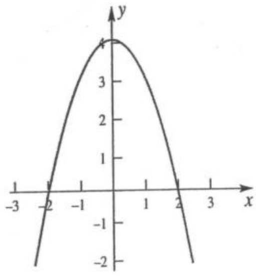

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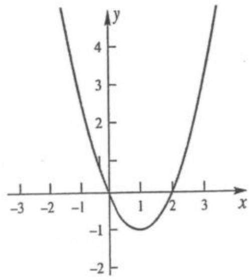

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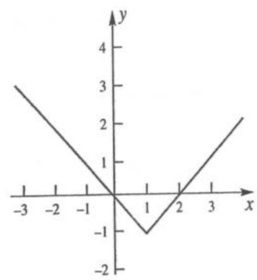

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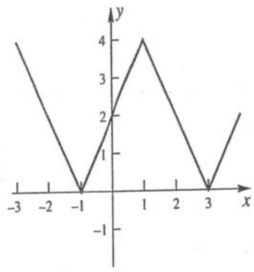
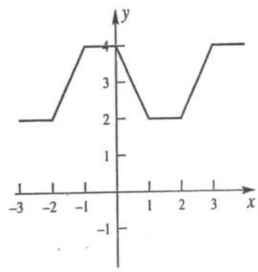
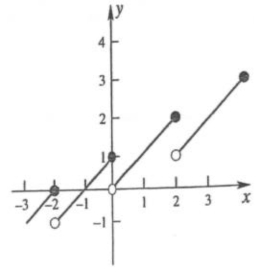


43.


44.


In Problems 45–50, find Δy for the given values of x_1 and x_2 (see Example 7).

45. $y = 3x + 2, x_1 = 1, x_2 = 1.5$
46. $y = 3x^2 + 2x + 1, x_1 = 0.0, x_2 = 0.1$
47. $y = \dfrac{1}{x}, x_1 = 1.0, x_2 = 1.2$
48. $y = \dfrac{2}{x + 1}, x_1 = 0, x_2 = 0.1$

[C] **49.** $y = \dfrac{3}{x+1}, x_1 = 2.34, x_2 = 2.31$

[C] **50.** $y = \cos 2x, x_1 = 0.571, x_2 = 0.573$

In Problems 51–56, first find and simplify

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Then find dy/dx by taking the limit of your answer as $\Delta x \to 0$.

51. $y = x^2$

52. $y = x^3 - 3x^2$

53. $y = \dfrac{1}{x+1}$

54. $y = 1 + \dfrac{1}{x}$

55. $y = \dfrac{x-1}{x+1}$

56. $y = \dfrac{x^2 - 1}{x}$

57. From Figure 8, estimate $f'(0), f'(2), f'(5)$, and $f'(7)$.

58. From Figure 9, estimate $g'(-1), g'(1), g'(4)$, and $g'(6)$.

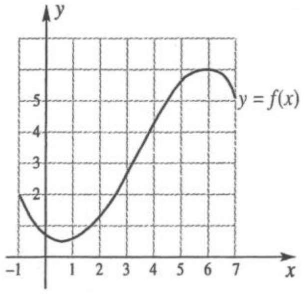


Figure 8

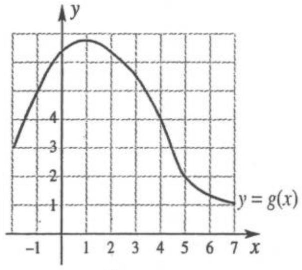


Figure 9

59. Sketch the graph of $y = f'(x)$ on $-1 < x < 7$ for the function f in Figure 8.

60. Sketch the graph of $y = g'(x)$ on $-1 < x < 7$ for the function g in Figure 9.

61. Consider the function $y = f(x)$, whose graph is sketched in Figure 10.

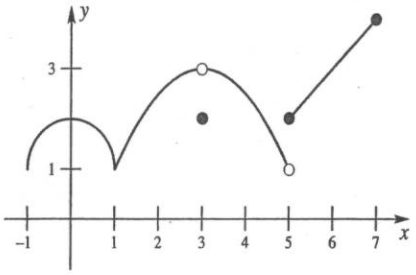


Figure 10

(a) Estimate $f(2), f'(2), f(0.5)$, and $f'(0.5)$.

(b) Estimate the average rate of change in f on the interval $0.5 \le x \le 2.5$.

(c) Where on the interval $-1 < x < 7$ does $\lim_{u \to x} f(u)$ fail to exist?

(d) Where on the interval $-1 < x < 7$ does f fail to be continuous?

(e) Where on the interval $-1 < x < 7$ does f fail to have a derivative?

(f) Where on the interval $-1 < x < 7$ is $f'(x) = 0$?

(g) Where on the interval $-1 < x < 7$ is $f'(x) = 1$?

62. Figure 14 in Section 2.1 shows the position s of an elevator as a function of time t. At what points does the derivative exist? Sketch the derivative of s.

63. Figure 15 in Section 2.1 shows the normal high temperature for St. Louis, Missouri. Sketch the derivative.

64. Figure 11 shows two functions. One is the function f, and the other is its derivative f'. Which one is which?

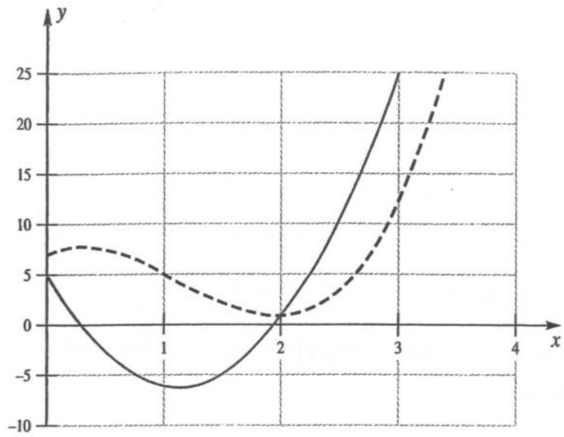


Figure 11

65. Figure 12 shows three functions. One is the function f; another is its derivative f', which we will call g; and the third is the derivative of g. Which one is which?

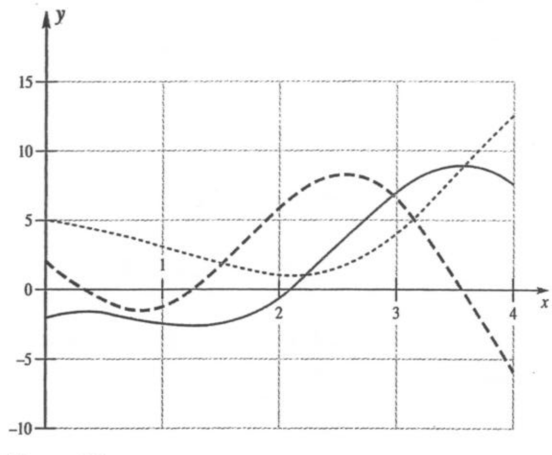


Figure 12

[EXPL] **66.** Suppose that $f(x + y) = f(x)f(y)$ for all x and y. Show that if $f'(0)$ exists then $f'(a)$ exists and $f'(a) = f(a)f'(0)$.

67. Let $f(x) = \begin{cases} mx + b & \text{if } x < 2 \\ x^2 & \text{if } x \ge 2 \end{cases}$

Determine m and b so that f is differentiable everywhere.

[EXPL] **68.** The **symmetric derivative** $f_s(x)$ is defined by

$$f_s(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

Show that if $f'(x)$ exists then $f_s(x)$ exists, but that the converse is false.

69. Let f be differentiable and let $f'(x_0) = m$. Find $f'(-x_0)$ if

(a) f is an odd function.

(b) f is an even function.

70. Prove that the derivative of an odd function is an even function and that the derivative of an even function is an odd function.

CAS *Use a CAS to do Problems 71 and 72.*

EXPL **71.** Draw the graphs of $f(x) = x^3 - 4x^2 + 3$ and its derivative $f'(x)$ on the interval $[-2, 5]$ using the same axes.

(a) Where on this interval is $f'(x) < 0$?

(b) Where on this interval is $f(x)$ decreasing as x increases?

(c) Make a conjecture. Experiment with other intervals and other functions to support this conjecture.

EXPL **72.** Draw the graphs of $f(x) = \cos x - \sin(x/2)$ and its derivative $f'(x)$ on the interval $[0, 9]$ using the same axes.

(a) Where on this interval is $f'(x) > 0$?

(b) Where on this interval is $f(x)$ increasing as x increases?

(c) Make a conjecture. Experiment with other intervals and other functions to support this conjecture.

Answers to Concepts Review: **1.** $[f(x+h) - f(x)]/h$; $[f(t) - f(x)]/(t - x)$ **2.** $f'(c)$ **3.** continuous; $|x|$ **4.** $f'(x)$; $\dfrac{dy}{dx}$

2.3 Rules for Finding Derivatives

The process of finding the derivative of a function directly from the definition of the derivative, that is, by setting up the difference quotient

$$\frac{f(x+h) - f(x)}{h}$$

and evaluating its limit, can be time consuming and tedious. We are going to develop tools that will allow us to shortcut this lengthy process—that will, in fact, allow us to find derivatives of the most complicated looking functions.

Recall that the derivative of a function f is another function f'. We saw in the previous section that, if $f(x) = x^3 + 7x$ is the formula for f, then $f'(x) = 3x^2 + 7$ is the formula for f'. When we take the derivative of f, we say that we are differentiating f. The derivative *operates* on f to produce f'. We often use the symbol D_x to indicate the operation of differentiating (Figure 1). The D_x symbol says that we are to take the derivative (with respect to the variable x) of what follows. Thus, we write $D_x f(x) = f'(x)$ or (in the case just mentioned) $D_x(x^3 + 7x) = 3x^2 + 7$. This D_x is an example of an **operator.** As Figure 1 suggests, an operator is a function whose input is a function and whose output is another function.

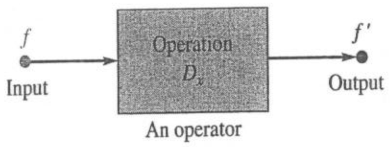


Figure 1

With Leibniz notation, introduced in the last section, we now have three notations for the derivative. If $y = f(x)$, we can denote the derivative of f by

$$f'(x) \quad \text{or} \quad D_x f(x) \quad \text{or} \quad \frac{dy}{dx}$$

We will use the notation $\dfrac{d}{dx}$ to mean the same as the operator D_x.

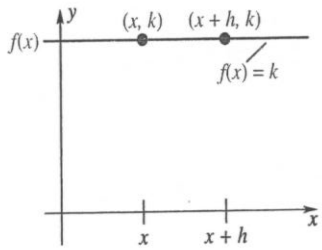


Figure 2

The Constant and Power Rules The graph of the constant function $f(x) = k$ is a horizontal line (Figure 2), which therefore has slope zero everywhere. This is one way to understand our first theorem.

Theorem A Constant Function Rule

If $f(x) = k$, where k is a constant, then for any x, $f'(x) = 0$; that is,

$$D_x(k) = 0$$

Proof

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{k - k}{h} = \lim_{h \to 0} 0 = 0 \qquad \blacksquare$$

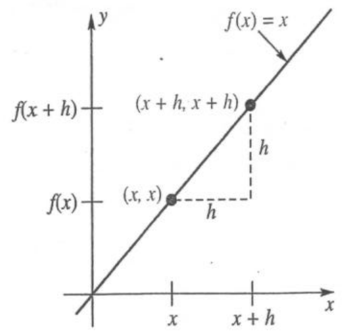


Figure 3

The graph of $f(x) = x$ is a line through the origin with slope 1 (Figure 3); so we should expect the derivative of this function to be 1 for all x.

Theorem B Identity Function Rule

If $f(x) = x$, then $f'(x) = 1$; that is,

$$D_x(x) = 1$$

Proof

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x + h - x}{h} = \lim_{h \to 0} \frac{h}{h} = 1 \quad \blacksquare$$

Before stating our next theorem, we recall something from algebra: how to raise a binomial to a power.

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$\vdots$$
$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots + nab^{n-1} + b^n$$

Theorem C Power Rule

If $f(x) = x^n$, where n is a positive integer, then $f'(x) = nx^{n-1}$; that is,

$$D_x(x^n) = nx^{n-1}$$

Proof

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}\left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right]}{\cancel{h}}$$

Within the brackets, all terms except the first have h as a factor, and so for every value of x, each of these terms has limit zero as h approaches zero. Thus,

$$f'(x) = nx^{n-1} \quad \blacksquare$$

As illustrations of Theorem C, note that

$$D_x(x^3) = 3x^2 \qquad D_x(x^9) = 9x^8 \qquad D_x(x^{100}) = 100x^{99}$$

D_x Is a Linear Operator The operator D_x behaves very well when applied to constant multiples of functions or to sums of functions.

Theorem D Constant Multiple Rule

If k is a constant and f is a differentiable function, then $(kf)'(x) = k \cdot f'(x)$; that is,

$$D_x[k \cdot f(x)] = k \cdot D_x f(x)$$

In words, *a constant multiplier k can be passed across the operator D_x.*

Proof Let $F(x) = k \cdot f(x)$. Then

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{k \cdot f(x+h) - k \cdot f(x)}{h}$$

$$= \lim_{h \to 0} k \cdot \frac{f(x+h) - f(x)}{h} = k \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= k \cdot f'(x)$$

The next-to-last step was the critical one. We could shift k past the limit sign because of the Main Limit Theorem Part 3. ■

Examples that illustrate this result are

$$D_x(-7x^3) = -7D_x(x^3) = -7 \cdot 3x^2 = -21x^2$$

and

$$D_x\left(\tfrac{4}{3}x^9\right) = \tfrac{4}{3}D_x(x^9) = \tfrac{4}{3} \cdot 9x^8 = 12x^8$$

Theorem E Sum Rule

If f and g are differentiable functions, then $(f + g)'(x) = f'(x) + g'(x)$; that is,

$$D_x\left[f(x) + g(x)\right] = D_x f(x) + D_x g(x)$$

In words, *the derivative of a sum is the sum of the derivatives.*

Proof Let $F(x) = f(x) + g(x)$. Then

$$F'(x) = \lim_{h \to 0} \frac{\left[f(x+h) + g(x+h)\right] - \left[f(x) + g(x)\right]}{h}$$

$$= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right]$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x)$$

Again, the next-to-last step was the critical one. It is justified by the Main Limit Theorem Part 4. ■

Any operator L with the properties stated in Theorems D and E is called *linear*; that is, L is a **linear operator** if for all functions f and g:

1. $L(kf) = kL(f)$, for every constant k;
2. $L(f + g) = L(f) + L(g)$.

Linear operators will appear again and again in this book; D_x is a particularly important example. A linear operator always satisfies the difference rule $L(f - g) = L(f) - L(g)$ stated next for D_x.

Theorem F Difference Rule

If f and g are differentiable functions, then $(f - g)'(x) = f'(x) - g'(x)$; that is,

$$D_x\left[f(x) - g(x)\right] = D_x f(x) - D_x g(x)$$

The proof of Theorem F is left as an exercise (Problem 54).

Linear Operator

The fundamental meaning of the word *linear*, as used in mathematics, is that given in this section. An operator L is linear if it satisfies the two key conditions:

- $L(ku) = kL(u)$
- $L(u + v) = L(u) + L(v)$

Linear operators play a central role in the *linear algebra* course, which many readers of this book will take.

Functions of the form $f(x) = mx + b$ are called *linear functions* because of their connections with lines. This terminology can be confusing because linear functions are not linear in the operator sense. To see this, note that

$$f(kx) = m(kx) + b$$

whereas

$$kf(x) = k(mx + b)$$

Thus, $f(kx) \neq kf(x)$ unless b happens to be zero.

EXAMPLE 1 Find the derivatives of $5x^2 + 7x - 6$ and $4x^6 - 3x^5 - 10x^2 + 5x + 16$.

SOLUTION

$$
\begin{aligned}
D_x(5x^2 + 7x - 6) &= D_x(5x^2 + 7x) - D_x(6) && \text{(Theorem F)} \\
&= D_x(5x^2) + D_x(7x) - D_x(6) && \text{(Theorem E)} \\
&= 5D_x(x^2) + 7D_x(x) - D_x(6) && \text{(Theorem D)} \\
&= 5 \cdot 2x + 7 \cdot 1 - 0 && \text{(Theorems C, B, A)} \\
&= 10x + 7
\end{aligned}
$$

To find the next derivative, we note that the theorems on sums and differences extend to any finite number of terms. Thus,

$$
\begin{aligned}
&D_x(4x^6 - 3x^5 - 10x^2 + 5x + 16) \\
&\quad = D_x(4x^6) - D_x(3x^5) - D_x(10x^2) + D_x(5x) + D_x(16) \\
&\quad = 4D_x(x^6) - 3D_x(x^5) - 10D_x(x^2) + 5D_x(x) + D_x(16) \\
&\quad = 4(6x^5) - 3(5x^4) - 10(2x) + 5(1) + 0 \\
&\quad = 24x^5 - 15x^4 - 20x + 5
\end{aligned}
$$

The method of Example 1 allows us to find the derivative of any polynomial. If you know the Power Rule and do what comes naturally, you are almost sure to get the right result. Also, with practice, you will find that you can write the derivative immediately, without having to write any intermediate steps.

Product and Quotient Rules

Now we are in for a surprise. So far, we have seen that the limit of a sum or difference is equal to the sum or difference of the limits (Theorem 1.3A, Parts 4 and 5), the limit of a product or quotient is the product or quotient of the limits (Theorem 1.3A, Parts 6 and 7), and the derivative of a sum or difference is the sum or difference of the derivatives (Theorems E and F). So what could be more natural than to have the derivative of a product be the product of the derivatives?

This may seem natural, but it is wrong. To see why, let's look at the following example.

EXAMPLE 2 Let $g(x) = x, h(x) = 1 + 2x,$ and $f(x) = g(x) \cdot h(x) = x(1 + 2x)$. Find $D_x f(x), D_x g(x)$, and $D_x h(x)$, and show that $D_x f(x) \neq [D_x g(x)][D_x h(x)]$.

SOLUTION

$$
\begin{aligned}
D_x f(x) &= D_x[x(1 + 2x)] \\
&= D_x(x + 2x^2) \\
&= 1 + 4x \\
D_x g(x) &= D_x x = 1 \\
D_x h(x) &= D_x(1 + 2x) = 2
\end{aligned}
$$

Notice that

$$D_x(g(x))D_x(h(x)) = 1 \cdot 2 = 2$$

whereas

$$D_x f(x) = D_x[g(x)h(x)] = 1 + 4x$$

Thus, $D_x f(x) \neq [D_x g(x)][D_x h(x)]$.

That the derivative of a product should be the product of the derivatives seemed so natural that it even fooled Gottfried Wilhelm von Leibniz, one of the discoverers of calculus. In a manuscript of November 11, 1675, he computed the derivative of the product of two functions and said (without checking) that it was equal to the product of the derivatives. Ten days later, he caught the error and gave the correct product rule, which we present as Theorem G.

> **Memorization**
>
> Some people say that memorization is passé, that only logical reasoning is important in mathematics. They are wrong. Some things (including the rules of this section) must become so much a part of our mental apparatus that we can use them without stopping to reflect.
>
> "Civilization advances by extending the number of important operations which we can perform without thinking about them."
>
> *Alfred N. Whitehead*

Theorem G Product Rule

If f and g are differentiable functions, then

$$(f \cdot g)'(x) = f(x)g'(x) + g(x)f'(x)$$

That is,

$$D_x\left[f(x)g(x)\right] = f(x)D_xg(x) + g(x)D_xf(x)$$

This rule should be memorized in words as follows: *The derivative of a product of two functions is the first times the derivative of the second plus the second times the derivative of the first.*

Proof Let $F(x) = f(x)g(x)$. Then

$$\begin{aligned}
F'(x) &= \lim_{h\to0} \frac{F(x+h) - F(x)}{h} \\
&= \lim_{h\to0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to0} \left[f(x+h)\cdot\frac{g(x+h) - g(x)}{h} + g(x)\cdot\frac{f(x+h) - f(x)}{h}\right] \\
&= \lim_{h\to0} f(x+h)\cdot\lim_{h\to0}\frac{g(x+h) - g(x)}{h} + g(x)\cdot\lim_{h\to0}\frac{f(x+h) - f(x)}{h} \\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

The derivation just given relies first on the trick of adding and subtracting the same thing, that is, $f(x+h)g(x)$. Second, at the very end, we use the fact that

$$\lim_{h\to0} f(x+h) = f(x)$$

This is just an application of Theorem 2.2A (which says that differentiability at a point implies continuity there) and the definition of continuity at a point. ■

EXAMPLE 3 Find the derivative of $(3x^2 - 5)(2x^4 - x)$ by use of the Product Rule. Check the answer by doing the problem a different way.

SOLUTION

$$\begin{aligned}
D_x\left[(3x^2 - 5)(2x^4 - x)\right] &= (3x^2 - 5)D_x(2x^4 - x) + (2x^4 - x)D_x(3x^2 - 5) \\
&= (3x^2 - 5)(8x^3 - 1) + (2x^4 - x)(6x) \\
&= 24x^5 - 3x^2 - 40x^3 + 5 + 12x^5 - 6x^2 \\
&= 36x^5 - 40x^3 - 9x^2 + 5
\end{aligned}$$

To check, we first multiply and then take the derivative.

$$(3x^2 - 5)(2x^4 - x) = 6x^6 - 10x^4 - 3x^3 + 5x$$

Thus,

$$D_x\left[(3x^2-5)(2x^4-x)\right] = D_x(6x^6) - D_x(10x^4) - D_x(3x^3) + D_x(5x)$$
$$= 36x^5 - 40x^3 - 9x^2 + 5$$

Theorem H Quotient Rule

Let f and g be differentiable functions with $g(x) \neq 0$. Then

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

That is,

$$D_x\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_xf(x) - f(x)D_xg(x)}{g^2(x)}$$

We strongly urge you to memorize this in words, as follows: *The derivative of a quotient is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

Proof Let $F(x) = f(x)/g(x)$. Then

$$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$$

$$= \lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0}\frac{g(x)f(x+h) - f(x)g(x+h)}{h}\cdot\frac{1}{g(x)g(x+h)}$$

$$= \lim_{h\to 0}\left[\frac{g(x)f(x+h) - g(x)f(x) + f(x)g(x) - f(x)g(x+h)}{h}\cdot\frac{1}{g(x)g(x+h)}\right]$$

$$= \lim_{h\to 0}\left\{\left[g(x)\frac{f(x+h)-f(x)}{h} - f(x)\frac{g(x+h)-g(x)}{h}\right]\frac{1}{g(x)g(x+h)}\right\}$$

$$= \left[g(x)f'(x) - f(x)g'(x)\right]\frac{1}{g(x)g(x)}$$ ■

EXAMPLE 4 Find $\dfrac{d}{dx}\dfrac{(3x-5)}{(x^2+7)}$.

SOLUTION

$$\frac{d}{dx}\left[\frac{3x-5}{x^2+7}\right] = \frac{(x^2+7)\frac{d}{dx}(3x-5) - (3x-5)\frac{d}{dx}(x^2+7)}{(x^2+7)^2}$$

$$= \frac{(x^2+7)(3) - (3x-5)(2x)}{(x^2+7)^2}$$

$$= \frac{-3x^2+10x+21}{(x^2+7)^2}$$ ■

EXAMPLE 5 Find D_xy if $y = \dfrac{2}{x^4+1} + \dfrac{3}{x}$.

SOLUTION

$$\begin{aligned} D_xy &= D_x\left(\frac{2}{x^4+1}\right) + D_x\left(\frac{3}{x}\right) \\ &= \frac{(x^4+1)D_x(2) - 2D_x(x^4+1)}{(x^4+1)^2} + \frac{xD_x(3) - 3D_x(x)}{x^2} \\ &= \frac{(x^4+1)(0) - (2)(4x^3)}{(x^4+1)^2} + \frac{(x)(0) - (3)(1)}{x^2} \\ &= \frac{-8x^3}{(x^4+1)^2} - \frac{3}{x^2} \end{aligned}$$

■

EXAMPLE 6 Show that the Power Rule holds for negative integral exponents; that is,

$$\boxed{D_x\left(x^{-n}\right) = -nx^{-n-1}}$$

$$D_x(x^{-n}) = D_x\left(\frac{1}{x^n}\right) = \frac{x^n \cdot 0 - 1 \cdot nx^{n-1}}{x^{2n}} = \frac{-nx^{n-1}}{x^{2n}} = -nx^{-n-1}$$

■

We saw as part of Example 5 that $D_x(3/x) = -3/x^2$. Now we have another way to see the same thing.

Concepts Review

1. The derivative of a product of two functions is the first times _____ plus the _____ times the derivative of the first. In symbols, $D_x\left[f(x)g(x)\right] =$ _____.

2. The derivative of a quotient is the _____ times the derivative of the numerator minus the numerator times the derivative of the _____, all divided by the _____. In symbols, $D_x\left[f(x)/g(x)\right] =$ _____.

3. The second term (the term involving h) in the expansion of $(x+h)^n$ is _____. It is this fact that leads to the formula $D_x\left[x^n\right] =$ _____.

4. L is called a linear operator if $L(kf) =$ _____ and $L(f+g) =$ _____. The derivative operator denoted by _____ is such an operator.

Problem Set 2.3

In Problems 1–44, find D_xy using the rules of this section.

1. $y = 2x^2$
2. $y = 3x^3$
3. $y = \pi x$
4. $y = \pi x^3$
5. $y = 2x^{-2}$
6. $y = -3x^{-4}$
7. $y = \dfrac{\pi}{x}$
8. $y = \dfrac{\alpha}{x^3}$
9. $y = \dfrac{100}{x^5}$
10. $y = \dfrac{3\alpha}{4x^5}$
11. $y = x^2 + 2x$
12. $y = 3x^4 + x^3$
13. $y = x^4 + x^3 + x^2 + x + 1$
14. $y = 3x^4 - 2x^3 - 5x^2 + \pi x + \pi^2$
15. $y = \pi x^7 - 2x^5 - 5x^{-2}$
16. $y = x^{12} + 5x^{-2} - \pi x^{-10}$
17. $y = \dfrac{3}{x^3} + x^{-4}$
18. $y = 2x^{-6} + x^{-1}$
19. $y = \dfrac{2}{x} - \dfrac{1}{x^2}$
20. $y = \dfrac{3}{x^3} - \dfrac{1}{x^4}$
21. $y = \dfrac{1}{2x} + 2x$
22. $y = \dfrac{2}{3x} - \dfrac{2}{3}$
23. $y = x(x^2 + 1)$
24. $y = 3x(x^3 - 1)$
25. $y = (2x + 1)^2$
26. $y = (-3x + 2)^2$
27. $y = (x^2 + 2)(x^3 + 1)$
28. $y = (x^4 - 1)(x^2 + 1)$
29. $y = (x^2 + 17)(x^3 - 3x + 1)$
30. $y = (x^4 + 2x)(x^3 + 2x^2 + 1)$
31. $y = (5x^2 - 7)(3x^2 - 2x + 1)$
32. $y = (3x^2 + 2x)(x^4 - 3x + 1)$

33. $y = \dfrac{1}{3x^2 + 1}$

34. $y = \dfrac{2}{5x^2 - 1}$

35. $y = \dfrac{1}{4x^2 - 3x + 9}$

36. $y = \dfrac{4}{2x^3 - 3x}$

37. $y = \dfrac{x - 1}{x + 1}$

38. $y = \dfrac{2x - 1}{x - 1}$

39. $y = \dfrac{2x^2 - 1}{3x + 5}$

40. $y = \dfrac{5x - 4}{3x^2 + 1}$

41. $y = \dfrac{2x^2 - 3x + 1}{2x + 1}$

42. $y = \dfrac{5x^2 + 2x - 6}{3x - 1}$

43. $y = \dfrac{x^2 - x + 1}{x^2 + 1}$

44. $y = \dfrac{x^2 - 2x + 5}{x^2 + 2x - 3}$

45. If $f(0) = 4, f'(0) = -1, g(0) = -3$, and $g'(0) = 5$, find

(a) $(f \cdot g)'(0)$ (b) $(f + g)'(0)$ (c) $(f/g)'(0)$

46. If $f(3) = 7, f'(3) = 2, g(3) = 6$, and $g'(3) = -10$, find

(a) $(f - g)'(3)$ (b) $(f \cdot g)'(3)$ (c) $(g/f)'(3)$

47. Use the Product Rule to show that $D_x[f(x)]^2 = 2 \cdot f(x) \cdot D_x f(x)$.

EXPL 48. Develop a rule for $D_x[f(x)g(x)h(x)]$.

49. Find the equation of the tangent line to $y = x^2 - 2x + 2$ at the point $(1, 1)$.

50. Find the equation of the tangent line to $y = 1/(x^2 + 4)$ at the point $(1, 1/5)$.

51. Find all points on the graph of $y = x^3 - x^2$ where the tangent line is horizontal.

52. Find all points on the graph of $y = \frac{1}{3}x^3 + x^2 - x$ where the tangent line has slope 1.

53. Find all points on the graph of $y = 100/x^5$ where the tangent line is perpendicular to the line $y = x$.

54. Prove Theorem F in two ways.

55. The height s in feet of a ball above the ground at t seconds is given by $s = -16t^2 + 40t + 100$.

(a) What is its instantaneous velocity at $t = 2$?

(b) When is its instantaneous velocity 0?

56. A ball rolls down a long inclined plane so that its distance s from its starting point after t seconds is $s = 4.5t^2 + 2t$ feet. When will its instantaneous velocity be 30 feet per second?

≈ 57. There are two tangent lines to the curve $y = 4x - x^2$ that go through $(2, 5)$. Find the equations of both of them. *Hint*: Let (x_0, y_0) be a point of tangency. Find two conditions that (x_0, y_0) must satisfy. See Figure 4.

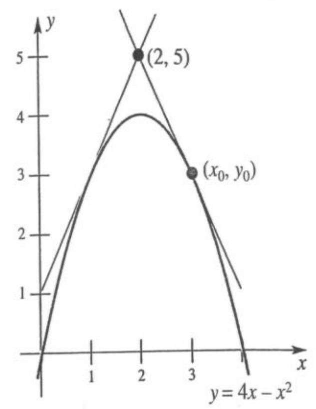


Figure 4

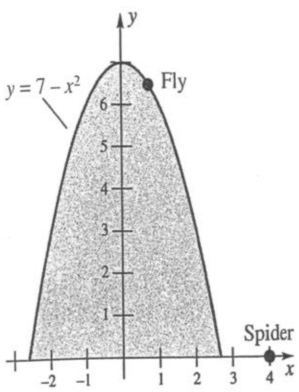


Figure 5

≈ 58. A space traveler is moving from left to right along the curve $y = x^2$. When she shuts off the engines, she will continue traveling along the tangent line at the point where she is at that time. At what point should she shut off the engines in order to reach the point $(4, 15)$?

≈ 59. A fly is crawling from left to right along the top of the curve $y = 7 - x^2$ (Figure 5). A spider waits at the point $(4, 0)$. Find the distance between the two insects when they first see each other.

60. Let $P(a, b)$ be a point on the first quadrant portion of the curve $y = 1/x$ and let the tangent line at P intersect the x-axis at A. Show that triangle AOP is isosceles and determine its area.

61. The radius of a spherical watermelon is growing at a constant rate of 2 centimeters per week. The thickness of the rind is always one-tenth of the radius. How fast is the volume of the rind growing at the end of the fifth week? Assume that the radius is initially 0.

CAS 62. Redo Problems 29–44 on a computer and compare your answers with those you get by hand.

Answers to Concepts Review: **1.** the derivative of the second; second; $f(x)D_x g(x) + g(x)D_x f(x)$ **2.** denominator; denominator; square of the denominator; $[g(x)D_x f(x) - f(x)D_x g(x)]/g^2(x)$ **3.** $nx^{n-1}h$; nx^{n-1} **4.** $kL(f)$; $L(f) + L(g)$; D_x

---

2.4 Derivatives of Trigonometric Functions

Figure 1 reminds us of the definition of the sine and cosine functions. In what follows, t should be thought of as a number measuring the length of an arc on the unit circle or, equivalently, as the number of radians in the corresponding angle. Thus, $f(t) = \sin t$ and $g(t) = \cos t$ are functions for which both domain and range are sets of real numbers. We may consider the problem of finding their derivatives.

The Derivative Formulas We choose to use x rather than t as our basic variable. To find $D_x(\sin x)$, we appeal to the definition of derivative and use the Addition Identity for $\sin(x + h)$.

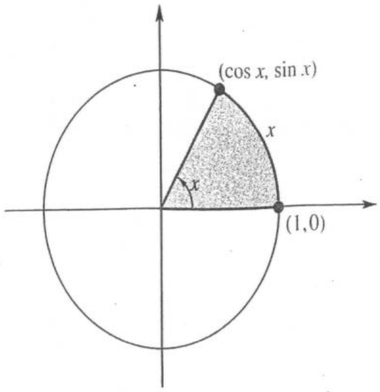


Figure 1

$$D_x(\sin x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0}\left(-\sin x\,\frac{1-\cos h}{h} + \cos x\,\frac{\sin h}{h}\right)$$

$$= (-\sin x)\left[\lim_{h\to 0}\frac{1-\cos h}{h}\right] + (\cos x)\left[\lim_{h\to 0}\frac{\sin h}{h}\right]$$

Notice that the two limits in this last expression are exactly the limits we studied in Section 1.4. In Theorem 1.4B we proved that

$$\lim_{h\to 0}\frac{\sin h}{h} = 1 \quad\text{and}\quad \lim_{h\to 0}\frac{1-\cos h}{h} = 0$$

Thus,

$$D_x(\sin x) = (-\sin x)\cdot 0 + (\cos x)\cdot 1 = \cos x$$

Similarly,

$$D_x(\cos x) = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$= \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$

$$= \lim_{h\to 0}\left(-\cos x\,\frac{1-\cos h}{h} - \sin x\,\frac{\sin h}{h}\right)$$

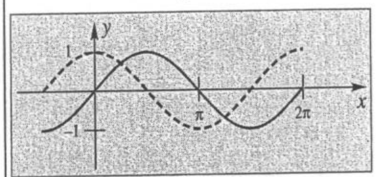
$$= (-\cos x)\cdot 0 - (\sin x)\cdot 1$$

$$= -\sin x$$

We summarize these results in an important theorem.

Could You Have Guessed?

The solid curve below is the graph of $y = \sin x$. Note that the slope is 1 at 0, 0 at $\pi/2$, -1 at π, and so on. When we graph the slope function (the derivative), we obtain the dashed curve. Could you have guessed that $D_x \sin x = \cos x$?



Try plotting these two functions in the same window on your CAS or graphing calculator.

Theorem A

The functions $f(x) = \sin x$ and $g(x) = \cos x$ are both differentiable and,

$$D_x(\sin x) = \cos x \qquad D_x(\cos x) = -\sin x$$

EXAMPLE 1 Find $D_x(3\sin x - 2\cos x)$.

SOLUTION

$$D_x(3\sin x - 2\cos x) = 3D_x(\sin x) - 2D_x(\cos x)$$
$$= 3\cos x + 2\sin x$$

EXAMPLE 2 Find the equation of the tangent line to the graph of $y = 3\sin x$ at the point $(\pi, 0)$. (See Figure 2.)

SOLUTION The derivative is $\frac{dy}{dx} = 3\cos x$, so when $x = \pi$, the slope is $3\cos\pi = -3$. Using the point-slope form for a line we find that the equation of the tangent line is

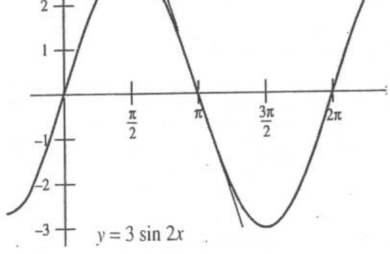


Figure 2

$$y - 0 = -3(x - \pi)$$
$$y = -3x + 3\pi$$

The Product and Quotient Rules are useful when evaluating derivatives of functions involving the trigonometric functions.

EXAMPLE 3 Find $D_x(x^2 \sin x)$.

SOLUTION The Product Rule is needed here.

$$D_x(x^2 \sin x) = x^2 D_x(\sin x) + \sin x(D_x x^2) = x^2 \cos x + 2x \sin x$$

EXAMPLE 4 Find $\dfrac{d}{dx}\left(\dfrac{1 + \sin x}{\cos x}\right)$.

SOLUTION For this problem, the Quotient Rule is needed.

$$\begin{aligned}\frac{d}{dx}\left(\frac{1 + \sin x}{\cos x}\right) &= \frac{\cos x\left(\frac{d}{dx}(1 + \sin x)\right) - (1 + \sin x)\left(\frac{d}{dx}\cos x\right)}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin x + \sin^2 x}{\cos^2 x}\\ &= \frac{1 + \sin x}{\cos^2 x}\end{aligned}$$

EXAMPLE 5 At time t seconds, the center of a bobbing cork is $y = 2 \sin t$ centimeters above (or below) water level. What is the velocity of the cork at $t = 0, \pi/2, \pi$?

SOLUTION The velocity is the derivative of position, and $\dfrac{dy}{dt} = 2 \cos t$. Thus, when $t = 0, \dfrac{dy}{dt} = 2 \cos 0 = 2$, when $t = \pi/2, \dfrac{dy}{dt} = 2 \cos \dfrac{\pi}{2} = 0$, and when $t = \pi, \dfrac{dy}{dt} = 2 \cos \pi = -2$.

Since the tangent, cotangent, secant, and cosecant functions are defined in terms of the sine and cosine functions, the derivatives of these functions can be obtained from Theorem A by applying the Quotient Rule. The results are summarized in Theorem B; for proofs, see Problems 5–8.

Theorem B

For all points x in the function's domain,

$$D_x \tan x = \sec^2 x \qquad D_x \cot x = -\csc^2 x$$
$$D_x \sec x = \sec x \tan x \qquad D_x \csc x = -\csc x \cot x$$

EXAMPLE 6 Find $D_x(x^n \tan x)$ for $n \geq 1$.

SOLUTION We apply the Product Rule along with Theorem B.

$$\begin{aligned}D_x(x^n \tan x) &= x^n D_x(\tan x) + \tan x(D_x x^n)\\ &= x^n \sec^2 x + nx^{n-1} \tan x\end{aligned}$$

EXAMPLE 7 Find the equation of the tangent line to the graph of $y = \tan x$ at the point $(\pi/4, 1)$.

SOLUTION The derivative of $y = \tan x$ is $\frac{dy}{dx} = \sec^2 x$. When $x = \pi/4$, the derivative is equal to $\sec^2 \frac{\pi}{4} = \left(\frac{2}{\sqrt{2}}\right)^2 = 2$. Thus the required line has slope 2 and passes through $(\pi/4, 1)$. Thus

$$y - 1 = 2\left(x - \frac{\pi}{4}\right)$$

$$y = 2x - \frac{\pi}{2} + 1$$

■

EXAMPLE 8 Find all points on the graph of $y = \sin^2 x$ where the tangent line is horizontal.

SOLUTION The tangent line is horizontal when the derivative is equal to zero. To get the derivative of $\sin^2 x$, we use the Product Rule.

$$\frac{d}{dx}\sin^2 x = \frac{d}{dx}(\sin x \sin x) = \sin x \cos x + \sin x \cos x = 2 \sin x \cos x$$

The product of $\sin x$ and $\cos x$ is equal to zero when either $\sin x$ or $\cos x$ is equal to zero; that is, at $x = 0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}, \ldots$. ■

Concepts Review

1. By definition, $D_x(\sin x) = \lim_{h\to 0}$ _____.

2. To evaluate the limit in the preceding statement, we first use the Addition Identity for the sine function and then do a little algebra to obtain

$$D_x(\sin x) = (-\sin x)\left(\lim_{h\to 0}\frac{1 - \cos h}{h}\right) + (\cos x)\left(\lim_{h\to 0}\frac{\sin h}{h}\right)$$

The two displayed limits have the values _____ and _____, respectively.

3. The result of the calculation in the preceding statement is the important derivative formula $D_x(\sin x) =$ _____. The corresponding derivative formula $D_x(\cos x) =$ _____ is obtained in a similar manner.

4. At $x = \pi/3$, $D_x(\sin x)$ has the value _____. Thus, the equation of the tangent line to $y = \sin x$ at $x = \pi/3$ is _____.

Problem Set 2.4

In Problems 1–18, find $D_x y$.

1. $y = 2\sin x + 3\cos x$
2. $y = \sin^2 x$
3. $y = \sin^2 x + \cos^2 x$
4. $y = 1 - \cos^2 x$
5. $y = \sec x = 1/\cos x$
6. $y = \csc x = 1/\sin x$
7. $y = \tan x = \frac{\sin x}{\cos x}$
8. $y = \cot x = \frac{\cos x}{\sin x}$
9. $y = \frac{\sin x + \cos x}{\cos x}$
10. $y = \frac{\sin x + \cos x}{\tan x}$
11. $y = \sin x \cos x$
12. $y = \sin x \tan x$
13. $y = \frac{\sin x}{x}$
14. $y = \frac{1 - \cos x}{x}$
15. $y = x^2 \cos x$
16. $y = \frac{x\cos x + \sin x}{x^2 + 1}$
17. $y = \tan^2 x$
18. $y = \sec^3 x$

C 19. Find the equation of the tangent line to $y = \cos x$ at $x = 1$.

20. Find the equation of the tangent line to $y = \cot x$ at $x = \frac{\pi}{4}$.

21. Use the trigonometric identity $\sin 2x = 2\sin x \cos x$ along with the Product Rule to find $D_x \sin 2x$.

22. Use the trigonometric identity $\cos 2x = 2\cos^2 x - 1$ along with the Product Rule to find $D_x \cos 2x$.

23. A Ferris wheel of radius 30 feet is rotating counterclockwise with an angular velocity of 2 radians per second. How fast is a seat on the rim rising (in the vertical direction) when it is 15 feet above the horizontal line through the center of the wheel? *Hint:* Use the result of Problem 21.

24. A Ferris wheel of radius 20 feet is rotating counterclockwise with an angular velocity of 1 radian per second. One seat on the rim is at $(20, 0)$ at time $t = 0$.

(a) What are its coordinates at $t = \pi/6$?

(b) How fast is it rising (vertically) at $t = \pi/6$?

(c) How fast is it rising when it is rising at the fastest rate?

25. Find the equation of the tangent line to $y = \tan x$ at $x = 0$.

26. Find all points on the graph of $y = \tan^2 x$ where the tangent line is horizontal.

27. Find all points on the graph of $y = 9 \sin x \cos x$ where the tangent line is horizontal.

28. Let $f(x) = x - \sin x$. Find all points on the graph of $y = f(x)$ where the tangent line is horizontal. Find all points on the graph of $y = f(x)$ where the tangent line has slope 2.

29. Show that the curves $y = \sqrt{2} \sin x$ and $y = \sqrt{2} \cos x$ intersect at right angles at a certain point with $0 < x < \pi/2$.

30. At time t seconds, the center of a bobbing cork is $3 \sin 2t$ centimeters above (or below) water level. What is the velocity of the cork at $t = 0, \pi/2, \pi$?

31. Use the definition of the derivative to show that $D_x(\sin x^2) = 2x \cos x^2$.

32. Use the definition of the derivative to show that $D_x(\sin 5x) = 5 \cos 5x$.

$\boxed{\text{GC}}$ *Problems 33 and 34 are computer or graphing calculator exercises.*

33. Let $f(x) = x \sin x$.

(a) Draw the graphs of $f(x)$ and $f'(x)$ on $[\pi, 6\pi]$.

(b) How many solutions does $f(x) = 0$ have on $[\pi, 6\pi]$? How many solutions does $f'(x) = 0$ have on this interval?

(c) What is wrong with the following conjecture? If f and f' are both continuous and differentiable on $[a, b]$, if $f(a) = f(b) = 0$, and if $f(x) = 0$ has exactly n solutions on $[a, b]$, then $f'(x) = 0$ has exactly $n - 1$ solutions on $[a, b]$.

(d) Determine the maximum value of $|f(x) - f'(x)|$ on $[\pi, 6\pi]$.

34. Let $f(x) = \cos^3 x - 1.25 \cos^2 x + 0.225$. Find $f'(x_0)$ at that point x_0 in $[\pi/2, \pi]$ where $f(x_0) = 0$.

Answers to Concepts Review: **1.** $[\sin(x + h) - \sin x]/h$ **2.** 0; 1 **3.** $\cos x$; $-\sin x$ **4.** $\frac{1}{2}$; $y - \sqrt{3}/2 = \frac{1}{2}(x - \pi/3)$

2.5 The Chain Rule

Imagine trying to find the derivative of

$$F(x) = (2x^2 - 4x + 1)^{60}$$

We could find the derivative, but we would first have to multiply together the 60 quadratic factors of $2x^2 - 4x + 1$ and then differentiate the resulting polynomial. Or, how about trying to find the derivative of

$$G(x) = \sin 3x$$

We might be able to use some trigonometric identities to reduce it to something that depends on $\sin x$ and $\cos x$ and then use the rules from the previous section.

Fortunately, there is a better way. After learning the *Chain Rule*, we will be able to write the answers

$$F'(x) = 60(2x^2 - 4x + 1)^{59}(4x - 4)$$

and

$$G'(x) = 3 \cos 3x$$

The Chain Rule is so important that we will seldom again differentiate any function without using it.

Differentiating a Composite Function If David can type twice as fast as Mary and Mary can type three times as fast as Joe, then David can type $2 \times 3 = 6$ times as fast as Joe.

Consider the composite function $y = f(g(x))$. If we let $u = g(x)$, we can then think of f as a function of u. Suppose that $f(u)$ changes twice as fast as u, and $u = g(x)$ changes three times as fast as x. How fast is y changing? The statements

"$y = f(u)$ changes twice as fast as u" and "$u = g(x)$ changes three times as fast as x" can be restated as

$$\frac{dy}{du} = 2 \quad \text{and} \quad \frac{du}{dx} = 3$$

Just as in the previous paragraph, it seems as if the rates should multiply; that is, the rate of change of y with respect to x should equal the rate of change of y with respect to u times the rate of change of u with respect to x. In other words,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

This is in fact true, and we will sketch the proof at the end of this section. The result is called the **Chain Rule.**

Theorem A Chain Rule

Let $y = f(u)$ and $u = g(x)$. If g is differentiable at x and f is differentiable at $u = g(x)$, then the composite function $f \circ g$, defined by $(f \circ g)(x) = f(g(x))$, is differentiable at x and

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

That is,

$$D_x(f(g(x))) = f'(g(x))g'(x)$$

or

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

You can remember the Chain Rule this way: *The derivative of a composite function is the derivative of the outer function evaluated at the inner function, times the derivative of the inner function.*

Applications of the Chain Rule We begin with the example $(2x^2 - 4x + 1)^{60}$ introduced at the beginning of this section.

EXAMPLE 1 If $y = (2x^2 - 4x + 1)^{60}$, find $D_x y$.

SOLUTION We think of y as the 60th power of a function of x; that is

$$y = u^{60} \quad \text{and} \quad u = 2x^2 - 4x + 1$$

The outer function is $f(u) = u^{60}$ and the inner function is $u = g(x) = 2x^2 - 4x + 1$. Thus,

$$\begin{aligned} D_x y &= D_x f(g(x)) \\ &= f'(u)g'(x) \\ &= (60u^{59})(4x - 4) \\ &= 60(2x^2 - 4x + 1)^{59}(4x - 4) \end{aligned}$$

■

EXAMPLE 2 If $y = 1/(2x^5 - 7)^3$, find $\dfrac{dy}{dx}$.

SOLUTION Think of it this way.

$$y = \frac{1}{u^3} = u^{-3} \quad \text{and} \quad u = 2x^5 - 7$$

Thus,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= (-3u^{-4})(10x^4) \\ &= \frac{-3}{u^4}\cdot 10x^4 \\ &= \frac{-30x^4}{(2x^5-7)^4}\end{aligned}$$

The Last First

Here is an informal rule that may help you in using the derivative rules.

The last step in calculation corresponds to the first step in differentiation.

For example, the last step in calculating $(2x+1)^3$ is to cube $2x+1$, so you would first apply the Chain Rule to the cube function. The last step in calculating

$$\frac{x^2-1}{x^2+1}$$

is to take the quotient, so the first rule to use in differentiating is the Quotient Rule.

EXAMPLE 3 Find $D_t\left(\frac{t^3-2t+1}{t^4+3}\right)^{13}$.

SOLUTION The last step in calculating this expression would be to raise the expression on the inside to the power 13. Thus, we begin by applying the Chain Rule to the function $y = u^{13}$, where $u = (t^3 - 2t + 1)/(t^4 + 3)$. The Chain Rule followed by the Quotient Rule gives

$$\begin{aligned}D_t\left(\frac{t^3-2t+1}{t^4+3}\right)^{13} &= 13\left(\frac{t^3-2t+1}{t^4+3}\right)^{13-1} D_t\left(\frac{t^3-2t+1}{t^4+3}\right) \\ &= 13\left(\frac{t^3-2t+1}{t^4+3}\right)^{12}\frac{(t^4+3)(3t^2-2)-(t^3-2t+1)(4t^3)}{(t^4+3)^2} \\ &= 13\left(\frac{t^3-2t+1}{t^4+3}\right)^{12}\frac{-t^6+6t^4-4t^3+9t^2-6}{(t^4+3)^2}\end{aligned}$$

The Chain Rule simplifies computation of many derivatives involving the trigonometric functions. Although it is possible to differentiate $y = \sin 2x$ using trigonometric identities (see Problem 21 of the previous section), it is much easier to use the Chain Rule.

EXAMPLE 4 If $y = \sin 2x$, find $\frac{dy}{dx}$.

SOLUTION The last step in calculating this expression would be to take the sine of the quantity $2x$. Thus we use the Chain Rule on the function $y = \sin u$ where $u = 2x$.

$$\frac{dy}{dx} = (\cos 2x)\left(\frac{d}{dx}2x\right) = 2\cos 2x$$

EXAMPLE 5 Find $F'(y)$ where $F(y) = y \sin y^2$.

SOLUTION The last step in calculating this expression would be to multiply y and $\sin y^2$, so we begin by applying the Product Rule. The Chain Rule is needed when we differentiate $\sin y^2$.

$$\begin{aligned}F'(y) &= yD_y[\sin y^2] + (\sin y^2)D_y(y) \\ &= y(\cos y^2)D_y(y^2) + (\sin y^2)(1) \\ &= 2y^2\cos y^2 + \sin y^2\end{aligned}$$

EXAMPLE 6 Find $D_x\left(\dfrac{x^2(1-x)^3}{1+x}\right)$.

SOLUTION The last step in calculating this expression would be to take the quotient. Thus, the Quotient Rule is the first to be applied. But notice that when we take the derivative of the numerator, we must apply the Product Rule and then the Chain Rule.

$$\begin{aligned}
D_x\left(\frac{x^2(1-x)^3}{1+x}\right) &= \frac{(1+x)D_x(x^2(1-x)^3) - x^2(1-x)^3D_x(1+x)}{(1+x)^2} \\
&= \frac{(1+x)[x^2D_x(1-x)^3 + (1-x)^3D_x(x^2)] - x^2(1-x)^3(1)}{(1+x)^2} \\
&= \frac{(1+x)[x^2(3(1-x)^2(-1)) + (1-x)^3(2x)] - x^2(1-x)^3}{(1+x)^2} \\
&= \frac{(1+x)[-3x^2(1-x)^2 + 2x(1-x)^3] - x^2(1-x)^3}{(1+x)^2} \\
&= \frac{(1+x)(1-x)^2x(2-5x) - x^2(1-x)^3}{(1+x)^2}
\end{aligned}$$

EXAMPLE 7 Find $\dfrac{d}{dx}\dfrac{1}{(2x-1)^3}$.

SOLUTION

$$\frac{d}{dx}\frac{1}{(2x-1)^3} = \frac{d}{dx}(2x-1)^{-3} = -3(2x-1)^{-3-1}\frac{d}{dx}(2x-1) = -\frac{6}{(2x-1)^4}$$

In this last example we were able to avoid use of the Quotient Rule. If you use the Quotient Rule, you would notice that the derivative of the numerator is 0, which simplifies the calculation. (You should check that the Quotient Rule gives the same answer as above.) As a general rule, if the numerator of a fraction is a constant, then do not use the Quotient Rule; instead write the quotient as the product of the constant and the expression in the denominator raised to a negative power, and then use the Chain Rule.

> **Notations for the Derivative**
>
> In this section, we have used all the various notations for the derivative, namely,
>
> $$f'(x)$$
>
> $$\frac{dy}{dx}$$
>
> and
>
> $$D_xf(x)$$
>
> You should by now be familiar with all of these notations. They will all be used in the remainder of the book.

EXAMPLE 8 Express the following derivatives in terms of the function $F(x)$. Assume that F is differentiable.

$$\text{(a) } D_x(F(x^3)) \quad \text{and} \quad \text{(b) } D_x[(F(x))^3]$$

SOLUTION

(a) The last step in calculating this expression would be to apply the function F. (Here the inner function is $u = x^3$ and the outer function is $F(u)$.) Thus

$$D_x(F(x^3)) = F'(x^3)D_x(x^3) = 3x^2\,F'(x^3)$$

(b) For this expression we would first evaluate $F(x)$ and then cube the result. (Here the inner function is $u = F(x)$ and the outer function is u^3.) Thus we apply the Power Rule first, then the Chain Rule.

$$D_x[(F(x))^3] = 3[F(x)]^2D_x(F(x)) = 3[F(x)]^2F'(x)$$

Applying the Chain Rule More than Once Sometimes when we apply the Chain Rule to a composite function we find that differentiation of the inner function also requires the Chain Rule. In cases like this, we simply have to use the Chain Rule a second time.

EXAMPLE 9 Find $D_x \sin^3(4x)$.

SOLUTION Remember, $\sin^3(4x) = [\sin(4x)]^3$, so we view this as the cube of a function of x. Thus, using our rule "derivative of the outer function evaluated at the inner function times the derivative of the inner function," we have

$$D_x \sin^3(4x) = D_x[\sin(4x)]^3 = 3[\sin(4x)]^{3-1}D_x[\sin(4x)]$$

Now we apply the Chain Rule once again for the derivative of the inner function.

$$\begin{aligned} D_x \sin^3(4x) &= 3[\sin(4x)]^{3-1}D_x \sin(4x) \\ &= 3[\sin(4x)]^2 \cos(4x) D_x(4x) \\ &= 3[\sin(4x)]^2 \cos(4x)(4) \\ &= 12\cos(4x)\sin^2(4x) \end{aligned}$$

EXAMPLE 10 Find $D_x \sin[\cos(x^2)]$.

SOLUTION

$$\begin{aligned} D_x \sin[\cos(x^2)] &= \cos[\cos(x^2)] \cdot [-\sin(x^2)] \cdot 2x \\ &= -2x \sin(x^2)\cos[\cos(x^2)] \end{aligned}$$

EXAMPLE 11 Suppose that the graphs of $y = f(x)$ and $y = g(x)$ are as shown in Figure 1. Use these graphs to approximate (a) $(f - g)'(2)$ and (b) $(f \circ g)'(2)$.

SOLUTION

(a) By Theorem 2.3F, $(f - g)'(2) = f'(2) - g'(2)$. From Figure 1, we can determine that $f'(2) \approx 1$ and $g'(2) \approx -\frac{1}{2}$. Thus,

$$(f - g)'(2) \approx 1 - \left(-\frac{1}{2}\right) = \frac{3}{2}.$$

(b) From Figure 1 we can determine that $f'(1) \approx \frac{1}{2}$. Thus, by the Chain Rule,

$$(f \circ g)'(2) = f'(g(2))g'(2) = f'(1)g'(2) \approx \frac{1}{2}\left(-\frac{1}{2}\right) = -\frac{1}{4}$$

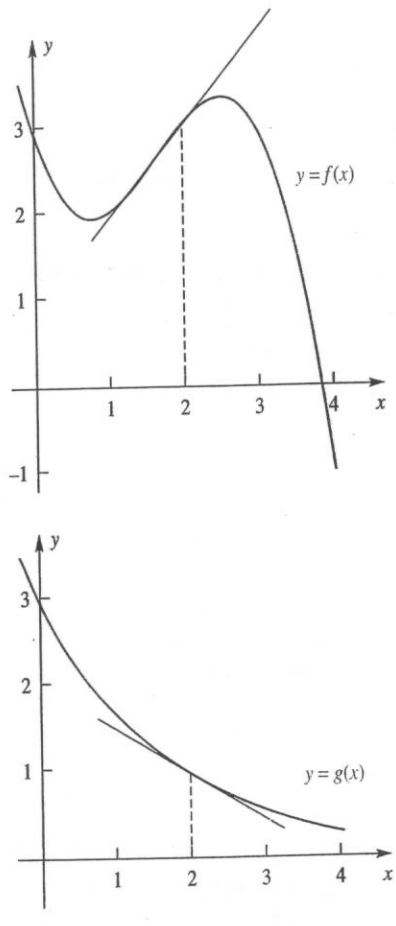


Figure 1

A Partial Proof of the Chain Rule

We can now give a sketch of the proof of the Chain Rule.

Proof We suppose that $y = f(u)$ and $u = g(x)$, that g is differentiable at x, and that f is differentiable at $u = g(x)$. When x is given an increment Δx, there are corresponding increments in u and y given by

$$\begin{aligned} \Delta u &= g(x + \Delta x) - g(x) \\ \Delta y &= f(g(x + \Delta x)) - f(g(x)) \\ &= f(u + \Delta u) - f(u) \end{aligned}$$

Thus,

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \end{aligned}$$

Since g is differentiable at x, it is continuous there (Theorem 2.2A), and so $\Delta x \to 0$ forces $\Delta u \to 0$. Hence,

$$\frac{dy}{dx} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{dy}{du} \cdot \frac{du}{dx}$$

This proof was very slick, but unfortunately it contains a subtle flaw. There are functions $u = g(x)$ that have the property that $\Delta u = 0$ for some points in every neighborhood of x (the constant function $g(x) = k$ is a good example). This means the division by Δu at our first step might not be legal. There is no simple way to get around this difficulty, though the Chain Rule is valid even in this case. We give a complete proof of the Chain Rule in the appendix (Section A.2, Theorem B). ■

Concepts Review

1. If $y = f(u)$, where $u = g(t)$, then $D_t y = D_u y \cdot$ ____. In function notation, $(f \circ g)'(t) =$ ____ ____.

2. If $w = G(v)$, where $v = H(s)$, then $D_s w =$ ____ $D_s v$. In function notation $(G \circ H)'(s) =$ ____ ____.

3. $D_x \cos[(f(x))^2] = -\sin($____$) \cdot D_x($____$)$.

4. If $y = (2x + 1)^3 \sin(x^2)$, then $D_x y = (2x + 1)^3 \cdot$ ____ $+ \sin(x^2) \cdot$ ____.

Problem Set 2.5

In Problems 1–20, find $D_x y$.

1. $y = (1 + x)^{15}$
2. $y = (7 + x)^5$
3. $y = (3 - 2x)^5$
4. $y = (4 + 2x^2)^7$
5. $y = (x^3 - 2x^2 + 3x + 1)^{11}$
6. $y = (x^2 - x + 1)^{-7}$
7. $y = \dfrac{1}{(x + 3)^5}$
8. $y = \dfrac{1}{(3x^2 + x - 3)^9}$
9. $y = \sin(x^2 + x)$
10. $y = \cos(3x^2 - 2x)$
11. $y = \cos^3 x$
12. $y = \sin^4(3x^2)$
13. $y = \left(\dfrac{x + 1}{x - 1}\right)^3$
14. $y = \left(\dfrac{x - 2}{x - \pi}\right)^{-3}$
15. $y = \cos\left(\dfrac{3x^2}{x + 2}\right)$
16. $y = \cos^3\left(\dfrac{x^2}{1 - x}\right)$
17. $y = (3x - 2)^2(3 - x^2)^2$
18. $y = (2 - 3x^2)^4(x^7 + 3)^3$
19. $y = \dfrac{(x + 1)^2}{3x - 4}$
20. $y = \dfrac{2x - 3}{(x^2 + 4)^2}$

In Problems 21–28, find the indicated derivative.

21. y' where $y = (x^2 + 4)^2$
22. y' where $y = (x + \sin x)^2$
23. $D_t\left(\dfrac{3t - 2}{t + 5}\right)^3$
24. $D_s\left(\dfrac{s^2 - 9}{s + 4}\right)$
25. $\dfrac{d}{dt}\left(\dfrac{(3t - 2)^3}{t + 5}\right)$
26. $\dfrac{d}{d\theta}(\sin^3 \theta)$
27. $\dfrac{dy}{dx}$, where $y = \left(\dfrac{\sin x}{\cos 2x}\right)^3$
28. $\dfrac{dy}{dt}$, where $y = [\sin t \tan(t^2 + 1)]$

In Problems 29–32, evaluate the indicated derivative.

29. $f'(3)$ if $f(x) = \left(\dfrac{x^2 + 1}{x + 2}\right)^3$
30. $G'(1)$ if $G(t) = (t^2 + 9)^3(t^2 - 2)^4$

$\boxed{C}$ 31. $F'(1)$ if $F(t) = \sin(t^2 + 3t + 1)$

32. $g'\left(\frac{1}{2}\right)$ if $g(s) = \cos \pi s \sin^2 \pi s$

In Problems 33–40, apply the Chain Rule more than once to find the indicated derivative.

33. $D_x[\sin^4(x^2 + 3x)]$
34. $D_t[\cos^5(4t - 19)]$
35. $D_t[\sin^3(\cos t)]$
36. $D_u\left[\cos^4\left(\dfrac{u + 1}{u - 1}\right)\right]$
37. $D_\theta[\cos^4(\sin \theta^2)]$
38. $D_x[x \sin^2(2x)]$
39. $\dfrac{d}{dx}\{\sin[\cos(\sin 2x)]\}$
40. $\dfrac{d}{dt}\{\cos^2[\cos(\cos t)]\}$

In Problems 41–46, use Figures 2 and 3 to approximate the indicated expressions.

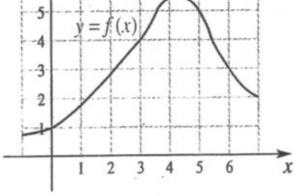


Figure 2

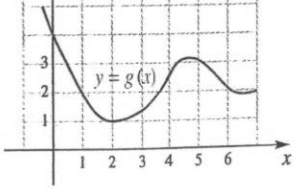


Figure 3

41. $(f + g)'(4)$
42. $(f - 2g)'(2)$
43. $(fg)'(2)$
44. $(f/g)'(2)$
45. $(f \circ g)'(6)$
46. $(g \circ f)'(3)$

In Problems 47–58, express the indicated derivative in terms of the function $F(x)$. Assume that F is differentiable.

47. $D_x(F(2x))$
48. $D_x(F(x^2 + 1))$

49. $D_t((F(t))^{-2})$

50. $\frac{d}{dz}\left(\frac{1}{(F(z))^2}\right)$

51. $\frac{d}{dz}(1 + (F(2z)))^2$

52. $\frac{d}{dy}\left(y^2 + \frac{1}{F(y^2)}\right)$

53. $\frac{d}{dx}F(\cos x)$

54. $\frac{d}{dx}\cos F(x)$

55. $D_x \tan F(2x)$

56. $\frac{d}{dx}g(\tan 2x)$

57. $D_x(F(x)\sin^2 F(x))$

58. $D_x \sec^3 F(x)$

59. Given that $f(0) = 1$ and $f'(0) = 2$, find $g'(0)$ where $g(x) = \cos f(x)$.

60. Given that $F(0) = 2$ and $F'(0) = -1$, find $G'(0)$ where $G(x) = \frac{x}{1 + \sec F(2x)}$.

61. Given that $f(1) = 2, f'(1) = -1, g(1) = 0$ and $g'(1) = 1$, find $F'(1)$ where $F(x) = f(x)\cos g(x)$.

62. Find the equation of the tangent line to the graph of $y = 1 + x\sin 3x$ at $\left(\frac{\pi}{3}, 1\right)$. Where does this line cross the x-axis?

63. Find all points on the graph of $y = \sin^2 x$ where the tangent line has slope 1.

64. Find the equation of the tangent line to $y = (x^2 + 1)^3(x^4 + 1)^2$ at $(1, 32)$.

65. Find the equation of the tangent line to $y = (x^2 + 1)^{-2}$ at $\left(1, \frac{1}{4}\right)$.

66. Where does the tangent line to $y = (2x + 1)^3$ at $(0, 1)$ cross the x-axis?

67. Where does the tangent line to $y = (x^2 + 1)^{-2}$ at $\left(1, \frac{1}{4}\right)$ cross the x-axis?

68. A point P is moving in the plane so that its coordinates after t seconds are $(4\cos 2t, 7\sin 2t)$, measured in feet.

(a) Show that P is following an elliptical path. *Hint:* Show that $(x/4)^2 + (y/7)^2 = 1$, which is an equation of an ellipse.

(b) Obtain an expression for L, the distance of P from the origin at time t.

(c) How fast is the distance between P and the origin changing when $t = \pi/8$? You will need the fact that $D_u(\sqrt{u}) = 1/(2\sqrt{u})$ (see Example 4 of Section 2.2).

69. A wheel centered at the origin and of radius 10 centimeters is rotating counterclockwise at a rate of 4 revolutions per second. A point P on the rim is at $(10, 0)$ at $t = 0$.

(a) What are the coordinates of P at time t seconds?

(b) At what rate is P rising (or falling) at time $t = 1$?

70. Consider the wheel-piston device in Figure 4. The wheel has radius 1 foot and rotates counterclockwise at 2 radians per second. The connecting rod is 5 feet long. The point P is at $(1, 0)$ at time $t = 0$.

(a) Find the coordinates of P at time t.

(b) Find the y-coordinate of Q at time t (the x-coordinate is always zero).

(c) Find the velocity of Q at time t. You will need the fact that $D_u(\sqrt{u}) = 1/(2\sqrt{u})$.

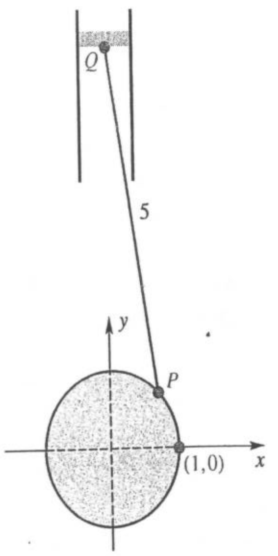


Figure 4

71. Do Problem 70, assuming that the wheel is rotating at 60 revolutions per minute and t is measured in seconds.

72. The dial of a standard clock has a 10-centimeter radius. One end of an elastic string is attached to the rim at 12 and the other to the tip of the 10-centimeter minute hand. At what rate is the string stretching at 12:15 (assuming that the clock is not slowed down by this stretching)?

$\boxed{C}$ 73. The hour and minute hands of a clock are 6 and 8 inches long, respectively. How fast are the tips of the hands separating at 12:20 (see Figure 5). *Hint:* Law of Cosines.

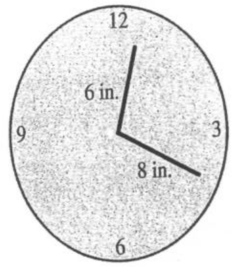


Figure 5

$\boxed{\approx}$ $\boxed{GC}$ 74. Find the approximate time between 12:00 and 1:00 when the distance s between the tips of the hands of the clock of Figure 5 is increasing most rapidly, that is, when the derivative ds/dt is largest.

75. Let x_0 be the smallest positive value of x at which the curves $y = \sin x$ and $y = \sin 2x$ intersect. Find x_0 and also the acute angle at which the two curves intersect at x_0 (see Problem 40 of Section 0.7).

76. An isosceles triangle is topped by a semicircle, as shown in Figure 6. Let D be the area of triangle AOB and E be the area of the shaded region. Find a formula for D/E in terms of t and then calculate

$$\lim_{t\to 0^+}\frac{D}{E} \quad \text{and} \quad \lim_{t\to \pi^-}\frac{D}{E}$$

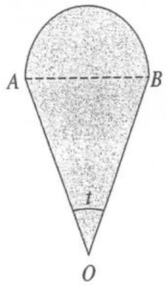


Figure 6

77. Show that $D_x|x| = |x|/x, x \neq 0$. *Hint:* Write $|x| = \sqrt{x^2}$ and use the Chain Rule with $u = x^2$.

78. Apply the result of Problem 77 to find $D_x|x^2 - 1|$.

79. Apply the result of Problem 77 to find $D_x|\sin x|$.

80. In Chapter 6 we will study a function L satisfying $L'(x) = 1/x$. Find each of the following derivatives.

(a) $D_x(L(x^2))$ (b) $D_x(L(\cos^4 x))$

81. Let $f(0) = 0$ and $f'(0) = 2$. Find the derivative of $f(f(f(f(x))))$ at $x = 0$.

82. Suppose that f is a differentiable function.

(a) Find $\dfrac{d}{dx}f(f(x))$. (b) Find $\dfrac{d}{dx}f(f(f(x)))$.

(c) Let $f^{[n]}$ denote the function defined as follows: $f^{[1]} = f$ and $f^{[n]} = f \circ f^{[n-1]}$ for $n \geq 2$. Thus $f^{[2]} = f \circ f, f^{[3]} = f \circ f \circ f$, etc. Based on your results from parts (a) and (b), make a conjecture regarding $\dfrac{d}{dx}f^{[n]}$. Prove your conjecture.

83. Give a second proof of the Quotient Rule. Write

$$D_x\left(\frac{f(x)}{g(x)}\right) = D_x\left(f(x)\frac{1}{g(x)}\right)$$

and use the Product Rule and the Chain Rule.

84. Suppose that f is differentiable and that there are real numbers x_1 and x_2 such that $f(x_1) = x_2$ and $f(x_2) = x_1$. Let $g(x) = f(f(f(f(x))))$. Show that $g'(x_1) = g'(x_2)$.

Answers to Concepts Review: 1. $D_t u; f'(g(t))g'(t)$ **2.** $D_v w; G'(H(s))H'(s)$ **3.** $(f(x))^2; (f(x))^2$ **4.** $2x\cos(x^2); 6(2x + 1)^2$

2.6 Higher-Order Derivatives

The operation of differentiation takes a function f and produces a new function f'. If we now differentiate f', we produce still another function, denoted by f'' (read "f double prime") and called the **second derivative** of f. It in turn, may be differentiated, thereby producing f''', which is called the **third derivative** of f, and so on. The **fourth derivative** is denoted $f^{(4)}$, the **fifth derivative** is denoted $f^{(5)}$, and so on.

If, for example

$$f(x) = 2x^3 - 4x^2 + 7x - 8$$

then

$$f'(x) = 6x^2 - 8x + 7$$
$$f''(x) = 12x - 8$$
$$f'''(x) = 12$$
$$f^{(4)}(x) = 0$$

Since the derivative of the zero function is zero, the fourth derivative and all *higher-order derivatives* of f will be zero.

We have introduced three notations for the derivative (now also called the *first derivative*) of $y = f(x)$. They are

$$f'(x) \qquad D_x y \qquad \frac{dy}{dx}$$

called, respectively, the *prime notation*, the *D notation*, and the *Leibniz notation*. There is a variation of the prime notation, y', that we will also use occasionally. All these notations have extensions for higher-order derivates, as shown in the accompanying table. Note especially the Leibniz notation, which, though complicated, seemed most appropriate to Leibniz. What, thought he, is more natural than to write

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) \quad \text{as} \quad \frac{d^2y}{dx^2}$$

Leibniz's notation for the second derivative is read *the second derivative of y with respect to x.*

Notations for Derivatives of $y = f(x)$

Derivative	f' Notation	y' Notation	D Notation	Leibniz Notation
First	$f'(x)$	y'	$D_x y$	$\frac{dy}{dx}$
Second	$f''(x)$	y''	$D_x^2 y$	$\frac{d^2y}{dx^2}$
Third	$f'''(x)$	y'''	$D_x^3 y$	$\frac{d^3y}{dx^3}$
Fourth	$f^{(4)}(x)$	$y^{(4)}$	$D_x^4 y$	$\frac{d^4y}{dx^4}$
⋮	⋮	⋮	⋮	⋮
nth	$f^{(n)}(x)$	$y^{(n)}$	$D_x^n y$	$\frac{d^ny}{dx^n}$

EXAMPLE 1 If $y = \sin 2x$, find d^3y/dx^3, d^4y/dx^4, and $d^{12}y/dx^{12}$.

SOLUTION

$$\frac{dy}{dx} = 2\cos 2x$$
$$\frac{d^2y}{dx^2} = -2^2 \sin 2x$$
$$\frac{d^3y}{dx^3} = -2^3 \cos 2x$$
$$\frac{d^4y}{dx^4} = 2^4 \sin 2x$$
$$\frac{d^5y}{dx^5} = 2^5 \cos 2x$$
$$\vdots$$
$$\frac{d^{12}y}{dx^{12}} = 2^{12} \sin 2x$$

■

Velocity and Acceleration In Section 2.1, we used the notion of instantaneous velocity to motivate the definition of the derivative. Let's review this notion by means of an example. Also, from now on we will use the single word *velocity* in place of the more cumbersome phrase *instantaneous velocity.*

EXAMPLE 2 An object moves along a coordinate line so that its position s satisfies $s = 2t^2 - 12t + 8$, where s is measured in centimeters and t in seconds with $t \geq 0$. Determine the velocity of the object when $t = 1$ and when $t = 6$. When is the velocity 0? When is it positive?

SOLUTION If we use the symbol $v(t)$ for the velocity at time t, then

$$v(t) = \frac{ds}{dt} = 4t - 12$$

Thus,

$$v(1) = 4(1) - 12 = -8 \text{ centimeters per second}$$
$$v(6) = 4(6) - 12 = 12 \text{ centimeters per second}$$

The velocity is 0 when $4t - 12 = 0$, that is, when $t = 3$. The velocity is positive when $4t - 12 > 0$, or when $t > 3$. All this is shown schematically in Figure 1.

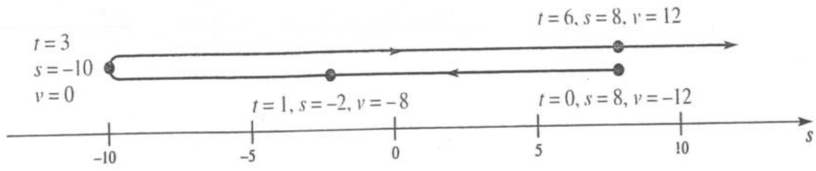


Figure 1

The object is, of course, moving along the s-axis, not on the colored path above it. But the colored path shows what happens to the object. Between $t = 0$ and $t = 3$, the velocity is negative; the object is moving to the left (backing up). By the time $t = 3$, it has "slowed" to a zero velocity. It then starts moving to the right as its velocity becomes positive. Thus, negative velocity corresponds to moving in the direction of decreasing s; positive velocity corresponds to moving in the direction of increasing s. A rigorous discussion of these points will be given in Chapter 3. ■

There is a technical distinction between the words *velocity* and *speed*. Velocity has a sign associated with it; it may be positive or negative. **Speed** is defined to be the absolute value of the velocity. Thus, in the example above, the speed at $t = 1$ is $|-8| = 8$ centimeters per second. The meter in most cars is a *speed*ometer; it always gives nonnegative values.

Now we want to give a physical interpretation of the second derivative d^2s/dt^2. It is, of course, just the first derivative of the velocity. Thus, it measures the rate of change of velocity with respect to time, which has the name **acceleration.** If it is denoted by a, then

$$\boxed{a = \frac{dv}{dt} = \frac{d^2s}{dt^2}}$$

Measuring Time

If $t = 0$ corresponds to the present moment, then $t < 0$ corresponds to the past, and $t > 0$ to the future. In many problems, it will be obvious that we are concerned only with the future. However, since the statement of Example 3 does not specify this, it seems reasonable to allow t to have negative as well as positive values.

In Example 2, $s = 2t^2 - 12t + 8$. Thus,

$$v = \frac{ds}{dt} = 4t - 12$$

$$a = \frac{d^2s}{dt^2} = 4$$

This means that the velocity is increasing at a constant rate of 4 centimeters per second every second, which we write as 4 centimeters per second per second, or as 4 cm/sec^2.

EXAMPLE 3 An object moves along a horizontal coordinate line in such a way that its position at time t is specified by

$$s = t^3 - 12t^2 + 36t - 30$$

Here s is measured in feet and t in seconds.

(a) When is the velocity 0?
(b) When is the velocity positive?
(c) When is the object moving to the left (that is, in the negative direction)?
(d) When is the acceleration positive?

SOLUTION

(a) $v = ds/dt = 3t^2 - 24t + 36 = 3(t - 2)(t - 6)$. Thus, $v = 0$ at $t = 2$ and at $t = 6$.

(b) $v > 0$ when $(t - 2)(t - 6) > 0$. We learned how to solve quadratic inequalities in Section 0.2. The solution is $\{t: t < 2 \text{ or } t > 6\}$ or, in interval notation, $(-\infty, 2) \cup (6, \infty)$; see Figure 2.

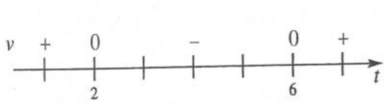


Figure 2

(c) The object is moving to the left when $v < 0$; that is, when $(t - 2)(t - 6) < 0$. This inequality has as its solution the interval $(2, 6)$.
(d) $a = dv/dt = 6t - 24 = 6(t - 4)$. Thus, $a > 0$ when $t > 4$. The motion of the object is shown schematically in Figure 3.

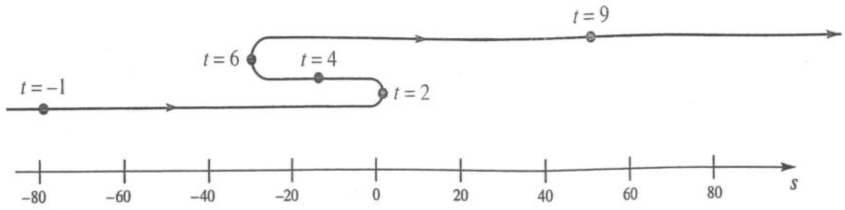


Figure 3

Falling-Body Problems If an object is thrown straight upward (or downward) from an initial height of s_0 feet with an initial velocity of v_0 feet per second and if s is its height above the ground in feet after t seconds, then

$$s = -16t^2 + v_0t + s_0$$

This assumes that the experiment takes place near sea level and that air resistance can be neglected. The diagram in Figure 4 portrays the situation we have in mind. Notice that positive velocity means that the object is moving upward.

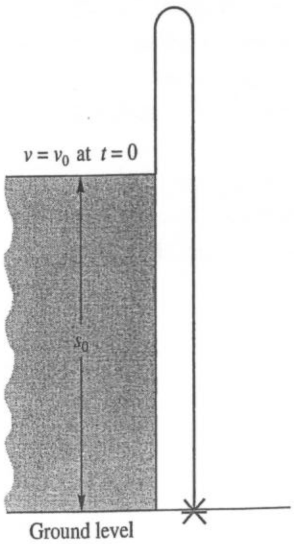


Figure 4

EXAMPLE 4 From the top of a building 160 feet high, a ball is thrown upward with an initial velocity of 64 feet per second.

(a) When does it reach its maximum height?
(b) What is its maximum height?
(c) When does it hit the ground?
(d) With what speed does it hit the ground?
(e) What is its acceleration at $t = 2$?

SOLUTION Let $t = 0$ correspond to the instant when the ball was thrown. Then $s_0 = 160$ and $v_0 = 64$ (v_0 is positive because the ball was thrown *upward*). Thus,

$$s = -16t^2 + 64t + 160$$

$$v = \frac{ds}{dt} = -32t + 64$$

$$a = \frac{dv}{dt} = -32$$

(a) The ball reached its maximum height at the time its velocity was 0, that is, when $-32t + 64 = 0$ or when $t = 2$ seconds.
(b) At $t = 2$, $s = -16(2)^2 + 64(2) + 160 = 224$ feet.
(c) The ball hit the ground when $s = 0$, that is, when

$$-16t^2 + 64t + 160 = 0$$

Dividing by -16 yields

$$t^2 - 4t - 10 = 0$$

The quadratic formula then gives

$$t = \frac{4 \pm \sqrt{16 + 40}}{2} = \frac{4 \pm 2\sqrt{14}}{2} = 2 \pm \sqrt{14}$$

Only the positive answer makes sense. Thus, the ball hit the ground at $t = 2 + \sqrt{14} \approx 5.74$ seconds.
(d) At $t = 2 + \sqrt{14}$, $v = -32(2 + \sqrt{14}) + 64 \approx -119.73$. Thus, the ball hit the ground with a speed of 119.73 feet per second.

(e) The acceleration is always -32 feet per second per second. This is the acceleration of gravity near sea level. ■

Concepts Review

1. If $y = f(x)$, then the third derivative of y with respect to x can be denoted by any one of the following four symbols:_____.

2. If $s = f(t)$ denotes the position of a particle on a coordinate line at time t, then its velocity is given by _____, its speed is given by _____, and its acceleration is given by _____.

3. If $s = f(t)$ denotes the position of an object at time t, then the object is moving to the right if _____.

4. Assume that an object is thrown straight upward so that its height s at time t is given by $s = f(t)$. The object reaches its maximum height when $ds/dt =$ _____, after which, ds/dt _____.

Problem Set 2.6

In Problems 1–8, find d^3y/dx^3.

1. $y = x^3 + 3x^2 + 6x$
2. $y = x^5 + x^4$
3. $y = (3x + 5)^3$
4. $y = (3 - 5x)^5$
5. $y = \sin(7x)$
6. $y = \sin(x^3)$
7. $y = \dfrac{1}{x-1}$
8. $y = \dfrac{3x}{1-x}$

In Problems 9–16, find $f''(2)$.

9. $f(x) = x^2 + 1$
10. $f(x) = 5x^3 + 2x^2 + x$
11. $f(t) = \dfrac{2}{t}$
12. $f(u) = \dfrac{2u^2}{5-u}$
13. $f(\theta) = (\cos\theta\pi)^{-2}$
14. $f(t) = t\sin(\pi/t)$
15. $f(s) = s(1 - s^2)^3$
16. $f(x) = \dfrac{(x+1)^2}{x-1}$

17. Let $n! = n(n-1)(n-2)\cdots3\cdot2\cdot1$. Thus, $4! = 4\cdot3\cdot2\cdot1 = 24$ and $5! = 5\cdot4\cdot3\cdot2\cdot1$. We give $n!$ the name ***n* factorial.** Show that $D_x^n(x^n) = n!$.

18. Find a formula for

$$D_x^n(a_{n-1}x^{n-1} + \cdots + a_1x + a_0)$$

19. Without doing any calculating, find each derivative.

(a) $D_x^4(3x^3 + 2x - 19)$ (b) $D_x^{12}(100x^{11} - 79x^{10})$

(c) $D_x^{11}(x^2 - 3)^5$

20. Find a formula for $D_x^n(1/x)$.

21. If $f(x) = x^3 + 3x^2 - 45x - 6$, find the value of f'' at each zero of f', that is, at each point c where $f'(c) = 0$.

22. Suppose that $g(t) = at^2 + bt + c$ and $g(1) = 5$, $g'(1) = 3$, and $g''(1) = -4$. Find a, b, and c.

In Problems 23–28, an object is moving along a horizontal coordinate line according to the formula $s = f(t)$, where s, the directed distance from the origin, is in feet and t is in seconds. In each case, answer the following questions (see Examples 2 and 3).

(a) What are $v(t)$ and $a(t)$, the velocity and acceleration, at time t?

(b) When is the object moving to the right?

(c) When is it moving to the left?

(d) When is its acceleration negative?

(e) Draw a schematic diagram that shows the motion of the object.

23. $s = 12t - 2t^2$
24. $s = t^3 - 6t^2$
25. $s = t^3 - 9t^2 + 24t$
26. $s = 2t^3 - 6t + 5$
27. $s = t^2 + \dfrac{16}{t}, t > 0$
28. $s = t + \dfrac{4}{t}, t > 0$

29. If $s = \frac{1}{2}t^4 - 5t^3 + 12t^2$, find the velocity of the moving object when its acceleration is zero.

30. If $s = \frac{1}{10}(t^4 - 14t^3 + 60t^2)$, find the velocity of the moving object when its acceleration is zero.

31. Two objects move along a coordinate line. At the end of t seconds their directed distances from the origin, in feet, are given by $s_1 = 4t - 3t^2$ and $s_2 = t^2 - 2t$, respectively.

(a) When do they have the same velocity?

(b) When do they have the same speed?

(c) When do they have the same position?

32. The positions of two objects, P_1 and P_2, on a coordinate line at the end of t seconds are given by $s_1 = 3t^3 - 12t^2 + 18t + 5$ and $s_2 = -t^3 + 9t^2 - 12t$, respectively. When do the two objects have the same velocity?

33. An object thrown directly upward is at a height of $s = -16t^2 + 48t + 256$ feet after t seconds (see Example 4).

(a) What is its initial velocity?

(b) When does it reach its maximum height?

(c) What is its maximum height?

C (d) When does it hit the ground?

C (e) With what speed does it hit the ground?

34. An object thrown directly upward from ground level with an initial velocity of 48 feet per second is $s = 48t - 16t^2$ feet high at the end of t seconds.

(a) What is the maximum height attained?

(b) How fast is the object moving, and in which direction, at the end of 1 second?

(c) How long does it take to return to its original position?

C 35. A projectile is fired directly upward from the ground with an initial velocity of v_0 feet per second. Its height in t seconds is given by $s = v_0t - 16t^2$ feet. What must its initial velocity be for the projectile to reach a maximum height of 1 mile?

36. An object thrown directly downward from the top of a cliff with an initial velocity of v_0 feet per second falls $s = v_0t + 16t^2$ feet in t seconds. If it strikes the ocean below in 3 seconds with a speed of 140 feet per second, how high is the cliff?

37. An object moves along a horizontal coordinate line in such a way that its position at time t is specified by $s = t^3 - 3t^2 - 24t - 6$. Here s is measured in centimeters and t in seconds. When is the object slowing down; that is, when is its *speed* decreasing?

38. Explain why an object moving along a line is slowing down when its velocity and acceleration have opposite signs (see Problem 37).

EXPL **39.** Leibniz obtained a general formula for $D_x^n(uv)$, where u and v are both functions of x. See if you can find it. *Hint:* Begin by considering the cases $n = 1, n = 2$, and $n = 3$.

40. Use the formula of Problem 39 to find $D_x^4(x^4 \sin x)$.

GC **41.** Let $f(x) = x[\sin x - \cos(x/2)]$.

(a) Draw the graphs of $f(x), f'(x), f''(x)$, and $f'''(x)$ on $[0, 6]$ using the same axes.

(b) Evaluate $f'''(2.13)$.

GC **42.** Repeat Problem 41 for $f(x) = (x + 1)/(x^2 + 2)$.

Answers to Concepts Review: **1.** $f'''(x)$; $D_x^3 y$; d^3y/dx^3; y''' **2.** ds/dt; $|ds/dt|$; d^2s/dt^2 **3.** $f'(t) > 0$ **4.** 0; <0

2.7 Implicit Differentiation

In the equation

$$y^3 + 7y = x^3$$

we cannot solve for y in terms of x. It still may be the case, however, that there is exactly one y corresponding to each x. For example, we may ask what y-values (if any) correspond to $x = 2$. To answer this question, we must solve

$$y^3 + 7y = 8$$

Certainly, $y = 1$ is one solution, and it turns out that $y = 1$ is the *only* real solution. Given $x = 2$, the equation $y^3 + 7y = x^3$ determines a corresponding y-value. We say that the equation defines y as an **implicit** function of x. The graph of this equation, shown in Figure 1, certainly looks like the graph of a differentiable function. The new element is that we do not have an equation of the form $y = f(x)$. Based on the graph, we assume that y is some unknown function of x. If we denote this function by $y(x)$, we can write the equation as

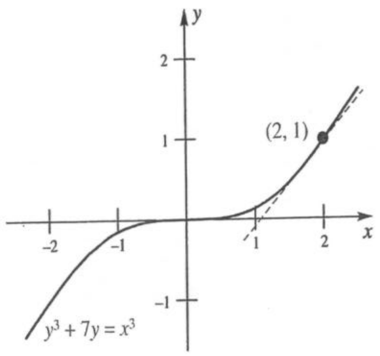


Figure 1

$$[y(x)]^3 + 7y(x) = x^3$$

Even though we do not have a formula for $y(x)$, we can nevertheless get a relation between x, $y(x)$, and $y'(x)$, by differentiating both sides of the equation with respect to x. Remembering to apply the Chain Rule, we get

$$\frac{d}{dx}(y^3) + \frac{d}{dx}(7y) = \frac{d}{dx}x^3$$

$$3y^2\frac{dy}{dx} + 7\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx}(3y^2 + 7) = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{3y^2 + 7}$$

Note that our expression for dy/dx involves both x and y, a fact that is often a nuisance. But if we wish only to find a slope at a point where we know both coordinates, no difficulty exists. At $(2, 1)$,

$$\frac{dy}{dx} = \frac{3(2)^2}{3(1)^2 + 7} = \frac{12}{10} = \frac{6}{5}$$

The slope is $\frac{6}{5}$.

The method just illustrated for finding dy/dx without first solving the given equation for y explicitly in terms of x is called **implicit differentiation.** But is the method legitimate—does it give the right answer?

An Example That Can Be Checked To give some evidence for the correctness of the method, consider the following example, which can be worked two ways.

EXAMPLE 1 Find dy/dx if $4x^2y - 3y = x^3 - 1$.

SOLUTION

Method 1 We can solve the given equation explicitly for y as follows:

$$y(4x^2 - 3) = x^3 - 1$$

$$y = \frac{x^3 - 1}{4x^2 - 3}$$

Thus,

$$\frac{dy}{dx} = \frac{(4x^2 - 3)(3x^2) - (x^3 - 1)(8x)}{(4x^2 - 3)^2} = \frac{4x^4 - 9x^2 + 8x}{(4x^2 - 3)^2}$$

Method 2 Implicit Differentiation We equate the derivatives of the two sides.

$$\frac{d}{dx}(4x^2y - 3y) = \frac{d}{dx}(x^3 - 1)$$

We obtain, after using the Product Rule on the first term,

$$4x^2 \cdot \frac{dy}{dx} + y \cdot 8x - 3\frac{dy}{dx} = 3x^2$$

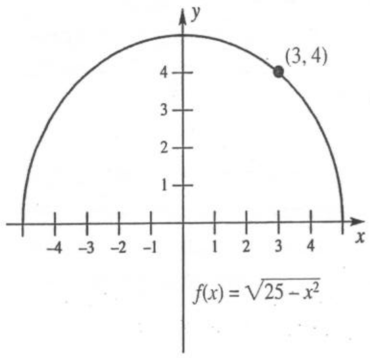
$$\frac{dy}{dx}(4x^2 - 3) = 3x^2 - 8xy$$

$$\frac{dy}{dx} = \frac{3x^2 - 8xy}{4x^2 - 3}$$

These two answers look different. For one thing, the answer obtained from Method 1 involves x only, whereas the answer from Method 2 involves both x and y. Remember, however, that the original equation could be solved for y in terms of x to give $y = (x^3 - 1)/(4x^2 - 3)$. When we substitute $y = (x^3 - 1)/(4x^2 - 3)$ into the expression just obtained for dy/dx, we get the following:

$$\frac{dy}{dx} = \frac{3x^2 - 8xy}{4x^2 - 3} = \frac{3x^2 - 8x\dfrac{x^3 - 1}{4x^2 - 3}}{4x^2 - 3}$$

$$= \frac{12x^4 - 9x^2 - 8x^4 + 8x}{(4x^2 - 3)^2} = \frac{4x^4 - 9x^2 + 8x}{(4x^2 - 3)^2}$$ ■



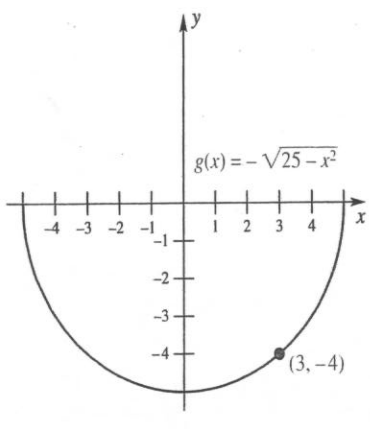


Figure 2

Some Subtle Difficulties If an equation in x and y determines a function $y = f(x)$ and if this function is differentiable, then the method of implicit differentiation will yield a correct expression for dy/dx. But notice there are two big *ifs* in this statement.

Consider the equation

$$x^2 + y^2 = 25$$

which determines both the function $y = f(x) = \sqrt{25 - x^2}$ and the function $y = g(x) = -\sqrt{25 - x^2}$. Their graphs are shown in Figure 2.

Happily, both of these functions are differentiable on $(-5, 5)$. Consider f first. It satisfies

$$x^2 + [f(x)]^2 = 25$$

When we differentiate implicitly and solve for $f'(x)$, we obtain

$$2x + 2f(x)f'(x) = 0$$
$$f'(x) = -\frac{x}{f(x)} = -\frac{x}{\sqrt{25 - x^2}}$$

A similar treatment of $g(x)$ yields

$$g'(x) = -\frac{x}{g(x)} = \frac{x}{\sqrt{25 - x^2}}$$

For practical purposes, we can obtain both of these results simultaneously by implicit differentiation of $x^2 + y^2 = 25$. This gives

$$2x + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{x}{y} = \begin{cases} \dfrac{-x}{\sqrt{25 - x^2}} & \text{if } y = f(x) \\ \dfrac{-x}{-\sqrt{25 - x^2}} & \text{if } y = g(x) \end{cases}$$

Naturally, the results are identical with those obtained above.

Note that it is often enough to know that $dy/dx = -x/y$ in order to apply our results. Suppose we want to know the slopes of the tangent lines to the circle $x^2 + y^2 = 25$ when $x = 3$. For $x = 3$, the corresponding y-values are 4 and -4. The slopes at $(3, 4)$ and $(3, -4)$, obtained by substituting in $-x/y$, are $-\frac{3}{4}$ and $\frac{3}{4}$, respectively (see Figure 2).

To complicate matters, we point out that

$$x^2 + y^2 = 25$$

determines many other functions. For example, consider the function h defined by

$$h(x) = \begin{cases} \sqrt{25 - x^2} & \text{if } -5 \le x \le 3 \\ -\sqrt{25 - x^2} & \text{if } 3 < x \le 5 \end{cases}$$

It too satisfies $x^2 + y^2 = 25$, since $x^2 + [h(x)]^2 = 25$. But it is not even continuous at $x = 3$, so it certainly does not have a derivative there (see Figure 3).

While the subject of implicit functions leads to difficult technical questions (treated in advanced calculus), the problems we study have straightforward solutions.

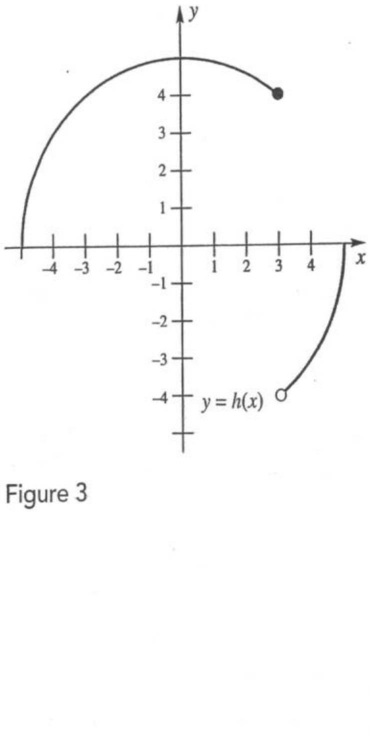


Figure 3

More Examples In the examples that follow, we assume that the given equation determines one or more differentiable functions whose derivatives can be found by implicit differentiation. Note that in each case we begin by taking the derivative of each side of the given equation with respect to the appropriate variable. Then we use the Chain Rule as needed.

EXAMPLE 2 Find dy/dx if $x^2 + 5y^3 = x + 9$.

SOLUTION

$$\frac{d}{dx}(x^2 + 5y^3) = \frac{d}{dx}(x + 9)$$
$$2x + 15y^2\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1 - 2x}{15y^2}$$

■

EXAMPLE 3 Find the equation of the tangent line to the curve

$$y^3 - xy^2 + \cos xy = 2$$

at the point (0, 1).

SOLUTION For simplicity, let us use the notation y' for dy/dx. When we differentiate both sides and equate the results, we obtain

$$3y^2y' - x(2yy') - y^2 - (\sin xy)(xy' + y) = 0$$
$$y'(3y^2 - 2xy - x\sin xy) = y^2 + y\sin xy$$
$$y' = \frac{y^2 + y\sin xy}{3y^2 - 2xy - x\sin xy}$$

At (0, 1), $y' = \frac{1}{3}$. Thus, the equation of the tangent line at (0, 1) is

$$y - 1 = \tfrac{1}{3}(x - 0)$$

or

$$y = \tfrac{1}{3}x + 1$$

■

The Power Rule Again We have learned that $D_x(x^n) = nx^{n-1}$, where n is any nonzero integer. We now extend this to the case where n is any nonzero rational number.

Theorem A Power Rule

Let r be any nonzero rational number. Then, for $x > 0$,

$$D_x(x^r) = rx^{r-1}$$

If r can be written in lowest terms as $r = p/q$, where q is odd, then $D_x(x^r) = rx^{r-1}$ for all x.

Proof Since r is rational, r may be written as p/q, where p and q are integers with $q > 0$. Let

$$y = x^r = x^{p/q}$$

Then

$$y^q = x^p$$

and, by implicit differentiation,

$$qy^{q-1}D_xy = px^{p-1}$$

Thus,

$$D_xy = \frac{px^{p-1}}{qy^{q-1}} = \frac{p}{q}\frac{x^{p-1}}{(x^{p/q})^{q-1}} = \frac{p}{q}\frac{x^{p-1}}{x^{p-p/q}}$$
$$= \frac{p}{q}x^{p-1-p+p/q} = \frac{p}{q}x^{p/q-1} = rx^{r-1}$$

We have obtained the desired result, but, to be honest, we must point out a flaw in our argument. In the implicit differentiation step, we assumed that D_xy exists, that is, that $y = x^{p/q}$ is differentiable. We can fill this gap, but since it is hard work we relegate the complete proof to the appendix (Section A.2, Theorem C). ■

EXAMPLE 4 If $y = 2x^{5/3} + \sqrt{x^2 + 1}$, find D_xy.

SOLUTION Using Theorem A and the Chain Rule, we have

$$D_xy = 2D_xx^{5/3} + D_x(x^2+1)^{1/2}$$
$$= 2\cdot\frac{5}{3}x^{5/3-1} + \frac{1}{2}(x^2+1)^{1/2-1}\cdot(2x)$$
$$= \frac{10}{3}x^{2/3} + \frac{x}{\sqrt{x^2+1}}$$

Concepts Review

1. The implicit relation $yx^3 - 3y = 9$ can be solved explicitly for y giving $y =$ ____.

2. Implicit differentiation of $y^3 + x^3 = 2x$ with respect to x gives ____ $+ 3x^2 = 2$.

3. Implicit differentiation of $xy^2 + y^3 - y = x^3$ with respect to x gives ____ $=$ ____.

4. The Power Rule with rational exponents says that $D_x(x^{p/q}) =$ ____. This rule, together with the Chain Rule, implies that $D_x[(x^2 - 5x)^{5/3}] =$ ____.

Problem Set 2.7

Assuming that each equation in Problems 1–12 defines a differentiable function of x, find D_xy by implicit differentiation.

1. $y^2 - x^2 = 1$
2. $9x^2 + 4y^2 = 36$
3. $xy = 1$
4. $x^2 + \alpha^2y^2 = 4\alpha^2$, where α is a constant.
5. $xy^2 = x - 8$
6. $x^2 + 2x^2y + 3xy = 0$
7. $4x^3 + 7xy^2 = 2y^3$
8. $x^2y = 1 + y^2x$
9. $\sqrt{5xy} + 2y = y^2 + xy^3$
10. $x\sqrt{y+1} = xy + 1$
11. $xy + \sin(xy) = 1$
12. $\cos(xy^2) = y^2 + x$

In Problems 13–18, find the equation of the tangent line at the indicated point (see Example 3).

13. $x^3y + y^3x = 30$; $(1, 3)$
14. $x^2y^2 + 4xy = 12y$; $(2, 1)$
15. $\sin(xy) = y$; $(\pi/2, 1)$
16. $y + \cos(xy^2) + 3x^2 = 4$; $(1, 0)$
17. $x^{2/3} - y^{2/3} - 2y = 2$; $(1, -1)$
18. $\sqrt{y} + xy^2 = 5$; $(4, 1)$

In Problems 19–32, find dy/dx.

19. $y = 3x^{5/3} + \sqrt{x}$
20. $y = \sqrt[3]{x} - 2x^{7/2}$
21. $y = \sqrt[3]{x} + \dfrac{1}{\sqrt[3]{x}}$
22. $y = \sqrt[4]{2x+1}$
23. $y = \sqrt[4]{3x^2 - 4x}$
24. $y = (x^3 - 2x)^{1/3}$
25. $y = \dfrac{1}{(x^3+2x)^{2/3}}$
26. $y = (3x - 9)^{-5/3}$
27. $y = \sqrt{x^2 + \sin x}$
28. $y = \sqrt{x^2 \cos x}$
29. $y = \dfrac{1}{\sqrt[3]{x^2 \sin x}}$
30. $y = \sqrt[4]{1 + \sin 5x}$
31. $y = \sqrt[4]{1 + \cos(x^2 + 2x)}$
32. $y = \sqrt{\tan^2 x + \sin^2 x}$
33. If $s^2t + t^3 = 1$, find ds/dt and dt/ds.
34. If $y = \sin(x^2) + 2x^3$, find dx/dy.
35. Sketch the graph of the circle $x^2 + 4x + y^2 + 3 = 0$ and then find equations of the two tangent lines that pass through the origin.
36. Find the equation of the **normal line** (line perpendicular to the tangent line) to the curve $8(x^2+y^2)^2 = 100(x^2 - y^2)$ at $(3, 1)$.
37. Suppose that $xy + y^3 = 2$. Then implicit differentiation twice with respect to x yields in turn:
    (a) $xy' + y + 3y^2y' = 0$;
    (b) $xy'' + y' + y' + 3y^2y'' + 6y(y')^2 = 0$.
    Solve (a) for y' and substitute in (b), and then solve for y''.
38. Find y'' if $x^3 - 4y^2 + 3 = 0$ (see Problem 37).
39. Find y'' at $(2, 1)$ if $2x^2y - 4y^3 = 4$ (see Problem 37).
40. Use implicit differentiation twice to find y'' at $(3, 4)$ if $x^2 + y^2 = 25$.
41. Show that the normal line to $x^3 + y^3 = 3xy$ at $\left(\frac{3}{2}, \frac{3}{2}\right)$ passes through the origin.
42. Show that the hyperbolas $xy = 1$ and $x^2 - y^2 = 1$ intersect at right angles.
43. Show that the graphs of $2x^2 + y^2 = 6$ and $y^2 = 4x$ intersect at right angles.
44. Suppose that curves C_1 and C_2 intersect at (x_0, y_0) with slopes m_1 and m_2, respectively, as in Figure 4. Then (see Problem 40 of Section 0.7) the positive angle θ from C_1 (i.e., from the tangent line to C_1 at (x_0, y_0)) to C_2 satisfies

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1m_2}$$

CHAPTER 3
Applications of the Derivative

3.1
Maxima and Minima

Often in life, we are faced with the problem of finding the *best* way to do something. For example, a farmer wants to choose the mix of crops that is likely to produce the largest profit. A doctor wishes to select the smallest dosage of a drug that will cure a certain disease. A manufacturer would like to minimize the cost of distributing its products. Often such a problem can be formulated so that it involves maximizing or minimizing a function over a specified set. If so, the methods of calculus provide a powerful tool for solving the problem.

Suppose then that we are given a function $f(x)$ and a domain S as in Figure 1. We now pose three questions:

1. Does $f(x)$ have a maximum or minimum value on S?
2. If it does have a maximum or a minimum, where are they attained?
3. If they exist, what are the maximum and minimum values?

Answering these questions is the principal goal of this section. We begin by introducing a precise vocabulary.

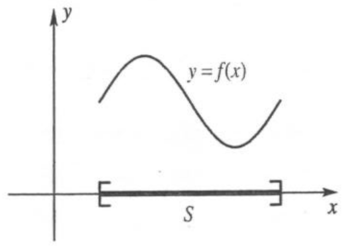


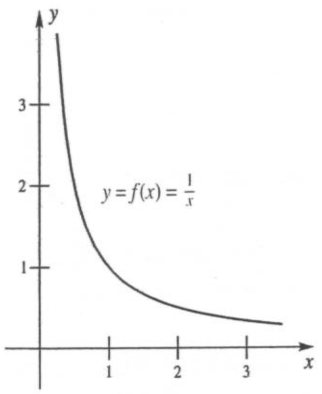
Figure 1

Definition

Let S, the domain of f, contain the point c. We say that

(i) $f(c)$ is the **maximum value** of f on S if $f(c) \geq f(x)$ for all x in S;
(ii) $f(c)$ is the **minimum value** of f on S if $f(c) \leq f(x)$ for all x in S;
(iii) $f(c)$ is an **extreme value** of f on S if it is either the maximum value or the minimum value;
(iv) the function we want to maximize or minimize is the **objective function.**

The Existence Question *Does f have a maximum (or minimum) value on S?* The answer depends first of all on the set S. Consider $f(x) = 1/x$ on $S = (0, \infty)$; it has neither a maximum value nor a minimum value (Figure 2). On the other hand, the same function on $S = [1, 3]$ has the maximum value of $f(1) = 1$ and the minimum value of $f(3) = \frac{1}{3}$. On $S = (1, 3]$, f has no maximum value and the minimum value of $f(3) = \frac{1}{3}$.



On $(0, \infty)$, no max or min
On $[1, 3]$, max = 1, min = $\frac{1}{3}$
On $(1, 3]$, no max, min = $\frac{1}{3}$

Figure 2

The answer also depends on the type of function. Consider the discontinuous function g (Figure 3) defined by

$$g(x) = \begin{cases} x & \text{if } 1 \leq x < 2 \\ x - 2 & \text{if } 2 \leq x \leq 3 \end{cases}$$

On $S = [1, 3]$, g has no maximum value (it gets arbitrarily close to 2 but never attains it). However, g has the minimum value $g(2) = 0$.

There is a nice theorem that answers the existence question for many of the problems that come up in practice. Though it is intuitively obvious, a rigorous proof is quite difficult; we leave that for more advanced textbooks.

Theorem A Max–Min Existence Theorem

If f is continuous on a closed interval $[a, b]$, then f attains both a maximum value and a minimum value there.

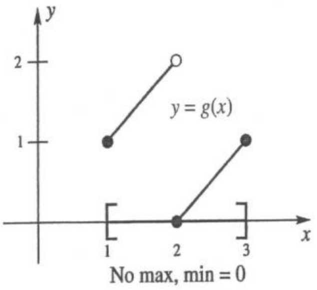


Figure 3

Note the key words in Theorem A; f is required to be *continuous* and the set S is required to be a *closed interval.*

Where Do Extreme Values Occur? Usually, the objective function will have an interval I as its domain. But this interval may be any of the nine types discussed in Section 0.2. Some of them contain their end points; some do not. For instance, $I = [a, b]$ contains both its end points; $[a, b)$ contains only its left end point; (a, b) contains neither end point. Extreme values of functions defined on closed intervals often occur at end points (see Figure 4).

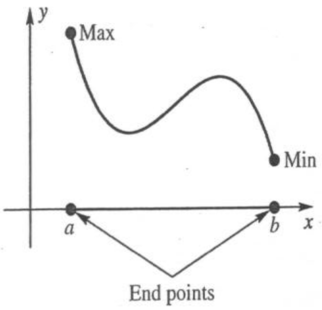


Figure 4

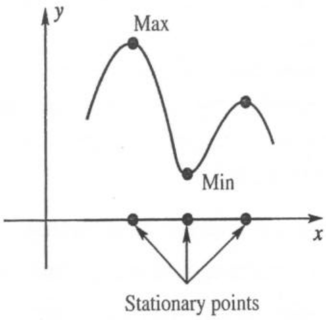


Figure 5

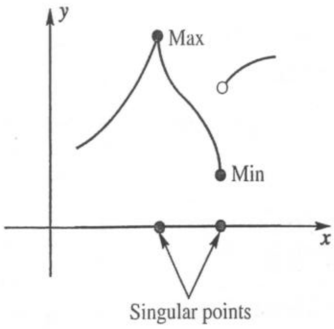


Figure 6

If c is a point at which $f'(c) = 0$, we call c a **stationary point.** The name derives from the fact that at a stationary point the graph of f levels off, since the tangent line is horizontal. Extreme values often occur at stationary points (see Figure 5).

Finally, if c is an interior point of I where f' fails to exist, we call c a **singular point.** It is a point where the graph of f has a sharp corner, a vertical tangent, or perhaps takes a jump, or near where the graph wiggles very badly. Extreme values can occur at singular points (Figure 6), though in practical problems this is quite rare.

These three kinds of points (end points, stationary points, and singular points) are the key points of max–min theory. Any point of one of these three types in the domain of a function f is called a **critical point** of f.

EXAMPLE 1 Find the critical points of $f(x) = -2x^3 + 3x^2$ on $\left[-\frac{1}{2}, 2\right]$.

SOLUTION The end points are $-\frac{1}{2}$ and 2. To find the stationary points, we solve $f'(x) = -6x^2 + 6x = 0$ for x, obtaining 0 and 1. There are no singular points. Thus, the critical points are $-\frac{1}{2}$, 0, 1, and 2. ■

Theorem B Critical Point Theorem

Let f be defined on an interval I containing the point c. If $f(c)$ is an extreme value, then c must be a critical point; that is, either c is

(i) an end point of I;

(ii) a stationary point of f; that is, a point where $f'(c) = 0$; or

(iii) a singular point of f; that is, a point where $f'(c)$ does not exist.

Proof Consider first the case where $f(c)$ is the maximum value of f on I and suppose that c is neither an end point nor a singular point. We must show that c is a stationary point.

Now, since $f(c)$ is the maximum value, $f(x) \le f(c)$ for all x in I; that is,

$$f(x) - f(c) \le 0$$

Thus, if $x < c$, so that $x - c < 0$, then

$$\frac{f(x) - f(c)}{x - c} \ge 0 \tag{1}$$

whereas if $x > c$, then

$$(2) \qquad \frac{f(x) - f(c)}{x - c} \leq 0$$

But $f'(c)$ exists because c is not a singular point. Consequently, when we let $x \to c^-$ in (1) and $x \to c^+$ in (2), we obtain, respectively, $f'(c) \geq 0$ and $f'(c) \leq 0$. We conclude that $f'(c) = 0$, as desired.

The case where $f(c)$ is the minimum value is handled similarly. ■

In the proof just given, we used the fact that the inequality $\leq$ is preserved under the operation of taking limits.

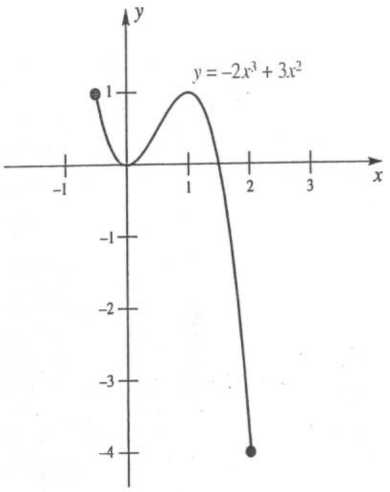


Figure 7

Terminology
Notice the way that terms are used in Example 3. The maximum is 1, which is equal to $f\left(-\frac{1}{2}\right)$ and $f(1)$. We say that the maximum is attained at $-\frac{1}{2}$ and at 1. Similarly, the minimum is -4, which is attained at 2.

What Are the Extreme Values? In view of Theorems A and B, we can now state a very simple procedure for finding the maximum value and minimum value of a continuous function f on a *closed interval I.*

Step 1: Find the critical points of f on I.

Step 2: Evaluate f at each of these critical points. The largest of these values is the maximum value; the smallest is the minimum value.

EXAMPLE 2 Find the maximum and minimum values of $f(x) = x^3$ on $[-2, 2]$.

SOLUTION The derivative is $f'(x) = 3x^2$, which is defined on $(-2, 2)$ and is zero only when $x = 0$. The critical points are therefore $x = 0$ and the end points $x = -2$ and $x = 2$. Evaluating f at the critical points yields $f(-2) = -8$, $f(0) = 0$, and $f(2) = 8$. Thus, the maximum value of f is 8 (attained at $x = 2$) and the minimum is -8 (attained at $x = -2$). ■

Notice that in Example 2, $f'(0) = 0$, but f did not attain a minimum or a maximum at $x = 0$. This does not contradict Theorem B. Theorem B does not say that if c is a critical point then $f(c)$ is a minimum or maximum; it says that if $f(c)$ is a minimum or a maximum, then c is a critical point.

EXAMPLE 3 Find the maximum and minimum values of

$$f(x) = -2x^3 + 3x^2$$

on $\left[-\frac{1}{2}, 2\right]$.

SOLUTION In Example 1, we identified $-\frac{1}{2}$, 0, 1, and 2 as the critical points. Now $f\left(-\frac{1}{2}\right) = 1$, $f(0) = 0$, $f(1) = 1$, and $f(2) = -4$. Thus, the maximum value is 1 (attained at both $x = -\frac{1}{2}$ and $x = 1$), and the minimum value is -4 (attained at $x = 2$). The graph of f is shown in Figure 7. ■

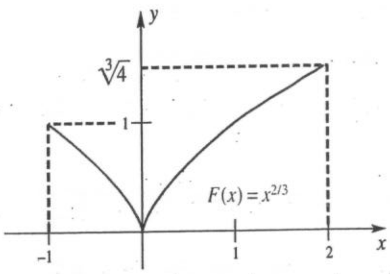


Figure 8

EXAMPLE 4 The function $F(x) = x^{2/3}$ is continuous everywhere. Find its maximum and minimum values on $[-1, 2]$.

SOLUTION $F'(x) = \frac{2}{3}x^{-1/3}$, which is never 0. However, $F'(0)$ does not exist, so 0 is a critical point, as are the end points -1 and 2. Now $F(-1) = 1$, $F(0) = 0$, and $F(2) = \sqrt[3]{4} \approx 1.59$. Thus, the maximum value is $\sqrt[3]{4}$; the minimum value is 0. The graph is shown in Figure 8. ■

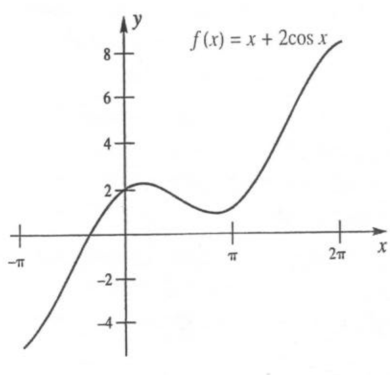


Figure 9

EXAMPLE 5 Find the maximum and minimum values of $f(x) = x + 2\cos x$ on $[-\pi, 2\pi]$.

SOLUTION Figure 9 shows a plot of $y = f(x)$. The derivative is $f'(x) = 1 - 2\sin x$, which is defined on $(-\pi, 2\pi)$ and is zero when $\sin x = 1/2$. The only values of x in the interval $[-\pi, 2\pi]$ that satisfy $\sin x = 1/2$ are $x = \pi/6$ and $x = 5\pi/6$. These two numbers, together with the end points $-\pi$ and 2π, are the critical points. Now, evaluate f at each critical point:

$$f(-\pi) = -2 - \pi \approx -5.14 \qquad f(\pi/6) = \sqrt{3} + \frac{\pi}{6} \approx 2.26$$

$$f(5\pi/6) = -\sqrt{3} + \frac{5\pi}{6} \approx 0.89 \qquad f(2\pi) = 2 + 2\pi \approx 8.28$$

Thus, $-2 - \pi$ is the minimum (attained at $x = -\pi$), and the maximum is $2 + 2\pi$ (attained at $x = 2\pi$). ■

Concepts Review

1. A _____ function on a _____ interval will always have both a maximum value and a minimum value on that interval.

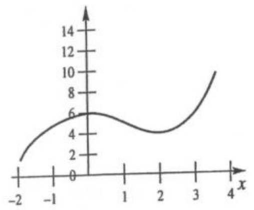
2. The term _____ value denotes either a maximum or a minimum value.

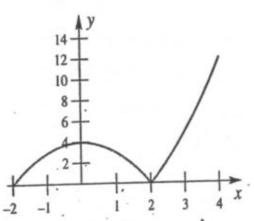
3. A function can attain an extreme value only at a critical point. Critical points are of three types: _____, _____, and _____.

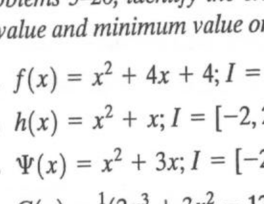
4. A stationary point for f is a number c such that _____; a singular point for f is a number c such that _____.

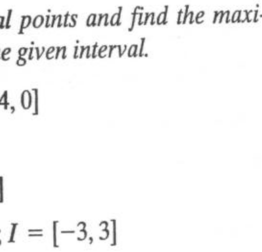
Problem Set 3.1

In Problems 1–4, find all critical points and find the minimum and maximum of the function. Each function has domain $[-2, 4]$.

1. 

2. 

3. 

4. 

In Problems 5–26, identify the critical points and find the maximum value and minimum value on the given interval.

5. $f(x) = x^2 + 4x + 4; I = [-4, 0]$

6. $h(x) = x^2 + x; I = [-2, 2]$

7. $\Psi(x) = x^2 + 3x; I = [-2, 1]$

8. $G(x) = \frac{1}{5}(2x^3 + 3x^2 - 12x); I = [-3, 3]$

9. $f(x) = x^3 - 3x + 1; I = \left(-\frac{3}{2}, 3\right)$ *Hint:* Sketch the graph.

10. $f(x) = x^3 - 3x + 1; I = \left[-\frac{3}{2}, 3\right]$

11. $h(r) = \dfrac{1}{r}; I = [-1, 3]$

12. $g(x) = \dfrac{1}{1 + x^2}; I = [-3, 1]$

13. $f(x) = x^4 - 2x^2 + 2; I = [-2, 2]$

14. $f(x) = x^5 - \dfrac{25}{3}x^3 + 20x - 1; I = [-3, 2]$

15. $g(x) = \dfrac{1}{1 + x^2}; I = (-\infty, \infty)$ *Hint:* Sketch the graph.

16. $f(x) = \dfrac{x}{1 + x^2}; I = [-1, 4]$

17. $r(\theta) = \sin\theta; I = \left[-\dfrac{\pi}{4}, \dfrac{\pi}{6}\right]$

18. $s(t) = \sin t - \cos t; I = [0, \pi]$

19. $a(x) = |x - 1|; I = [0, 3]$

20. $f(s) = |3s - 2|; I = [-1, 4]$

21. $g(x) = \sqrt[3]{x}; I = [-1, 27]$

22. $s(t) = t^{2/5}; I = [-1, 32]$

23. $H(t) = \cos t; I = [0, 8\pi]$

24. $g(x) = x - 2\sin x; I = [-2\pi, 2\pi]$

25. $g(\theta) = \theta^2 \sec\theta; I = \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

26. $h(t) = \dfrac{t^{5/3}}{2 + t}; I = [-1, 8]$

GC 27. Identify the critical points and find the extreme values on the interval $[-1, 5]$ for each function:

(a) $f(x) = x^3 - 6x^2 + x + 2$ (b) $g(x) = |f(x)|$

GC 28. Identify the critical points and find the extreme values on the interval $[-1, 5]$ for each function:

(a) $f(x) = \cos x + x\sin x + 2$ (b) $g(x) = |f(x)|$

In Problems 29–36, sketch the graph of a function with the given properties.

29. f is differentiable, has domain $[0, 6]$, reaches a maximum of 6 (attained when $x = 3$) and a minimum of 0 (attained when $x = 0$). Additionally, $x = 5$ is a stationary point.

30. f is differentiable, has domain $[0, 6]$, reaches a maximum of 4 (attained when $x = 6$) and a minimum of -2 (attained when $x = 1$). Additionally, $x = 2, 3, 4, 5$ are stationary points.

31. f is continuous, but not necessarily differentiable, has domain $[0, 6]$, reaches a maximum of 6 (attained when $x = 5$), and a minimum of 2 (attained when $x = 3$). Additionally, $x = 1$ and $x = 5$ are the only stationary points.

32. f is continuous, but not necessarily differentiable, has domain $[0, 6]$, reaches a maximum of 4 (attained when $x = 4$), and a minimum of 2 (attained when $x = 2$). Additionally, f has no stationary points.

33. f is differentiable, has domain $[0, 6]$, reaches a maximum of 4 (attained at two different values of x, neither of which is an end point), and a minimum of 1 (attained at three different values of x, exactly one of which is an end point.)

34. f is continuous but not necessarily differentiable, has domain $[0, 6]$, reaches a maximum of 6 (attained when $x = 0$) and a minimum of 0 (attained when $x = 6$). Additionally, f has two stationary points and two singular points in $(0, 6)$.

35. f has domain $[0, 6]$, but is not necessarily continuous, and f does not attain a maximum.

36. f has domain $[0, 6]$, but is not necessarily continuous, and f attains neither a maximum nor a minimum.

Answers to Concepts Review: **1.** continuous; closed **2.** extreme **3.** end points; stationary points; singular points **4.** $f'(c) = 0$; $f'(c)$ does not exist

3.2 Monotonicity and Concavity

Consider the graph in Figure 1. No one will be surprised when we say that f is decreasing to the left of c and increasing to the right of c. But to make sure that we agree on terminology, we give precise definitions.

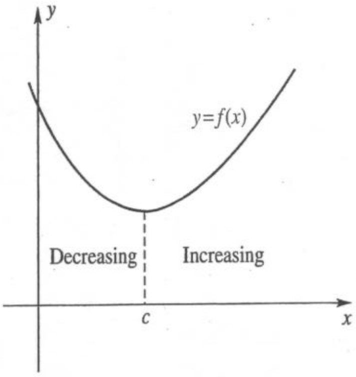


Figure 1

Definition

Let f be defined on an interval I (open, closed, or neither). We say that

(i) f is **increasing** on I if, for every pair of numbers x_1 and x_2 in I,

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

(ii) f is **decreasing** on I if, for every pair of numbers x_1 and x_2 in I,

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

(iii) f is **strictly monotonic** on I if it is either increasing on I or decreasing on I.

How shall we decide where a function is increasing? We could draw its graph and look at it, but a graph is usually drawn by plotting a few points and connecting those points with a smooth curve. Who can be sure that the graph does not wiggle between the plotted points? Even computer algebra systems and graphing calculators plot by simply connecting points. We need a better procedure.

The First Derivative and Monotonicity Recall that the first derivative $f'(x)$ gives us the slope of the tangent line to the graph of f at the point x. Thus, if $f'(x) > 0$, then the tangent line is rising to the right, suggesting that f is increasing. (See Figure 2.) Similarly, if $f'(x) < 0$, then the tangent line is falling to the right, suggesting that f is decreasing. We can also look at this in terms of motion along a line. Suppose an object is at position $s(t)$ at time t and that its velocity is always positive, that is, $s'(t) = ds/dt > 0$. Then it seems reasonable that the object will continue to move to the right as long as the derivative stays positive. In other words, $s(t)$ will be an *increasing* function of t. These observations do not prove Theorem A, but they make the result plausible. We postpone a rigorous proof until Section 3.6.

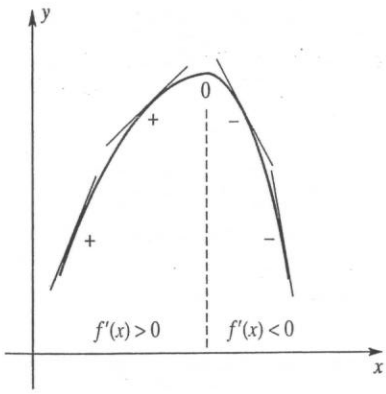


Figure 2

Theorem A Monotonicity Theorem

Let f be continuous on an interval I and differentiable at every interior point of I.

(i) If $f'(x) > 0$ for all x interior to I, then f is increasing on I.

(ii) If $f'(x) < 0$ for all x interior to I, then f is decreasing on I.

This theorem usually allows us to determine precisely where a differentiable function increases and where it decreases. It is a matter of solving two inequalities.

EXAMPLE 1 If $f(x) = 2x^3 - 3x^2 - 12x + 7$, find where f is increasing and where it is decreasing.

SOLUTION We begin by finding the derivative of f.

$$f'(x) = 6x^2 - 6x - 12 = 6(x + 1)(x - 2)$$

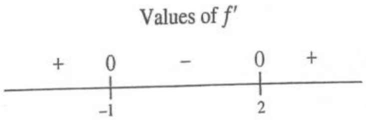


Figure 3

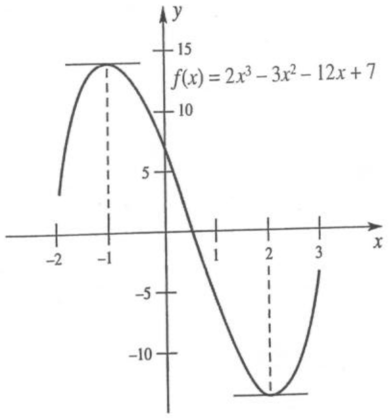


Figure 4

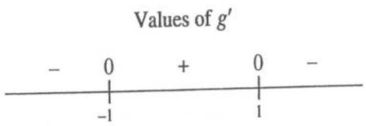


Figure 5

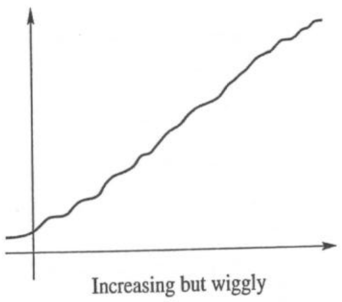


Figure 6

We need to determine where

$$(x + 1)(x - 2) > 0$$

and also where

$$(x + 1)(x - 2) < 0$$

This problem was discussed in great detail in Section 0.2, a section worth reviewing now. The split points are -1 and 2; they split the x-axis into three intervals: $(-\infty, -1)$, $(-1, 2)$, and $(2, \infty)$. Using the test points $-2, 0,$ and 3, we conclude that $f'(x) > 0$ on the first and last of these intervals and that $f'(x) < 0$ on the middle interval (Figure 3). Thus, by Theorem A, f is increasing on $(-\infty, -1]$ and $[2, \infty)$; it is decreasing on $[-1, 2]$. Note that the theorem allows us to include the end points of these intervals, even though $f'(x) = 0$ at those points. The graph of f is shown in Figure 4. ■

EXAMPLE 2 Determine where $g(x) = x/(1 + x^2)$ is increasing and where it is decreasing.

SOLUTION

$$g'(x) = \frac{(1 + x^2) - x(2x)}{(1 + x^2)^2} = \frac{1 - x^2}{(1 + x^2)^2} = \frac{(1 - x)(1 + x)}{(1 + x^2)^2}$$

Since the denominator is always positive, $g'(x)$ has the same sign as the numerator $(1 - x)(1 + x)$. The split points, -1 and 1, determine the three intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. When we test them, we find that $g'(x) < 0$ on the first and last of these intervals and that $g'(x) > 0$ on the middle one (Figure 5). We conclude from Theorem A that g is decreasing on $(-\infty, -1]$ and $[1, \infty)$ and that it is increasing on $[-1, 1]$. We postpone graphing g until later, but if you want to see the graph, turn to Figure 11 and Example 4. ■

The Second Derivative and Concavity A function may be increasing and still have a very wiggly graph (Figure 6). To analyze wiggles, we need to study how the tangent line turns as we move from left to right along the graph. If the tangent line turns steadily in the counterclockwise direction, we say that the graph is *concave up*; if the tangent turns in the clockwise direction, the graph is *concave down*. Both definitions are better stated in terms of functions and their derivatives.

Definition

Let f be differentiable on an open interval I. We say that f (as well as its graph) is **concave up** on I if f' is increasing on I, and we say that f is **concave down** on I if f' is decreasing on I.

The diagrams in Figure 7 will help to clarify these notions. Note that a curve that is concave *up* is shaped like a *cup*.

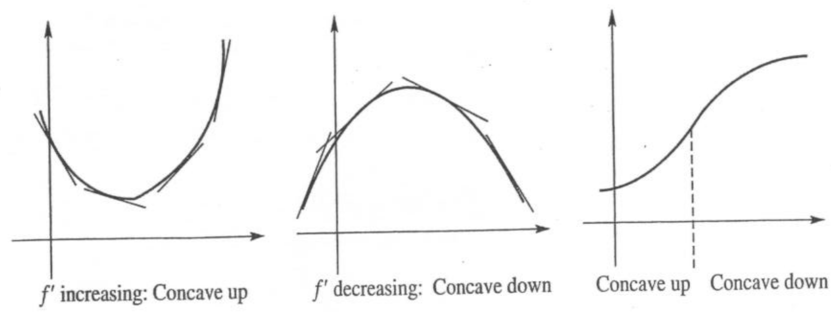


Figure 7

Conditions in Theorems A and B

The conditions regarding the derivatives in Theorems A and B are sufficient to guarantee the conclusions stated. These conditions are not, however, necessary. It is possible that a function is increasing on some interval even though the derivative isn't always positive on that interval. If we consider the function $f(x) = x^3$ over the interval $[-4, 4]$ we note that it is increasing but its derivative is not always positive on that interval $(f'(0) = 0)$. The function $g(x) = x^4$ is concave up on the interval $[-4, 4]$, but the second derivative, $g''(x) = 12x^2$, is not always positive on that interval.

In view of Theorem A, we have a simple criterion for deciding where a curve is concave up and where it is concave down. We simply keep in mind that the second derivative of f is the first derivative of f'. Thus, f' is increasing if f'' is positive; it is decreasing if f'' is negative.

Theorem B Concavity Theorem

Let f be twice differentiable on the open interval I.

(i) If $f''(x) > 0$ for all x in I, then f is concave up on I.

(ii) If $f''(x) < 0$ for all x in I, then f is concave down on I.

For most functions, this theorem reduces the problem of determining concavity to the problem of solving inequalities. By now we are experts at this.

EXAMPLE 3 Where is $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$ increasing, decreasing, concave up, and concave down?

SOLUTION

$$f'(x) = x^2 - 2x - 3 = (x + 1)(x - 3)$$

$$f''(x) = 2x - 2 = 2(x - 1)$$

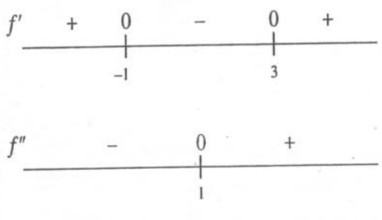


Figure 8

By solving the inequalities $(x + 1)(x - 3) > 0$ and its opposite, $(x + 1)(x - 3) < 0$, we conclude that f is increasing on $(-\infty, -1]$ and $[3, \infty)$ and decreasing on $[-1, 3]$ (Figure 8). Similarly, solving $2(x - 1) > 0$ and $2(x - 1) < 0$ shows that f is concave up on $(1, \infty)$ and concave down on $(-\infty, 1)$. The graph of f is shown in Figure 9. ■

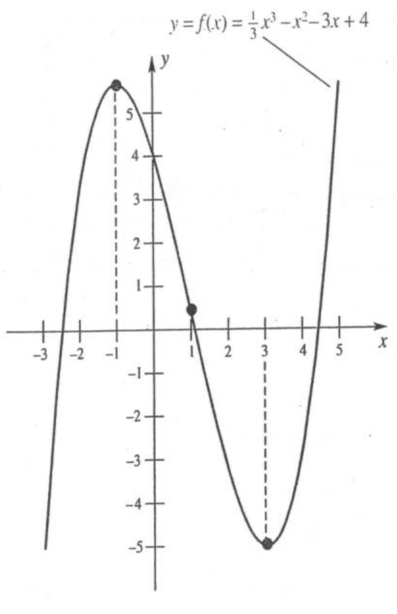


Figure 9

EXAMPLE 4 Where is $g(x) = x/(1 + x^2)$ concave up and where is it concave down? Sketch the graph of g.

SOLUTION We began our study of this function in Example 2. There we learned that g is decreasing on $(-\infty, -1]$ and $[1, \infty)$ and increasing on $[-1, 1]$. To analyze concavity, we calculate g''.

$$g'(x) = \frac{1 - x^2}{(1 + x^2)^2}$$

$$g''(x) = \frac{(1 + x^2)^2(-2x) - (1 - x^2)(2)(1 + x^2)(2x)}{(1 + x^2)^4}$$

$$= \frac{(1 + x^2)[(1 + x^2)(-2x) - (1 - x^2)(4x)]}{(1 + x^2)^4}$$

$$= \frac{2x^3 - 6x}{(1 + x^2)^3}$$

$$= \frac{2x(x^2 - 3)}{(1 + x^2)^3}$$

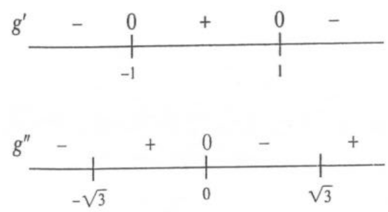


Figure 10

Since the denominator is always positive, we need only solve $x(x^2 - 3) > 0$ and its opposite. The split points are $-\sqrt{3}$, 0, and $\sqrt{3}$. These three split points determine four intervals. After testing them (Figure 10), we conclude that g is concave up on $\left(-\sqrt{3}, 0\right)$ and $\left(\sqrt{3}, \infty\right)$ and that it is concave down on $\left(-\infty, -\sqrt{3}\right)$ and $\left(0, \sqrt{3}\right)$.

To sketch the graph of g, we make use of all the information obtained so far, plus the fact that g is an odd function whose graph is symmetric with respect to the origin (Figure 11).

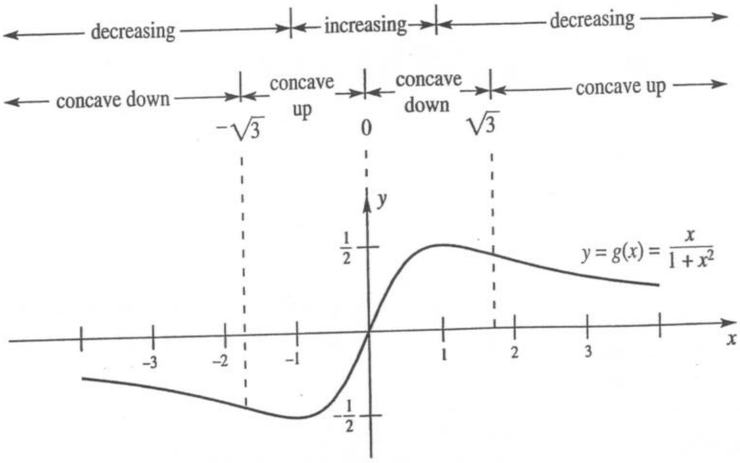


Figure 11

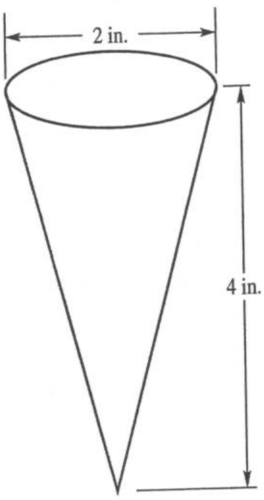


Figure 12

EXAMPLE 5 Suppose that water is poured into the conical container, as shown in Figure 12, at the constant rate of $\frac{1}{2}$ cubic inch per second. Determine the height h of the water as a function of time t and plot $h(t)$ from time $t = 0$ until the time that the container is full.

SOLUTION Before we solve this problem, let's think about what the graph will look like. At first, the height will increase rapidly, since it takes very little water to fill the bottom of the cone. As the container fills up, the height will increase less rapidly. What do these statements say about the function $h(t)$, its derivative $h'(t)$, and its second derivative $h''(t)$? Since the water is steadily pouring in, the height will always increase, so $h'(t)$ will be positive. The height will increase more slowly as the water level rises. Thus, the function $h'(t)$ is decreasing so $h''(t)$ is negative. The graph of $h(t)$ is therefore increasing (because $h'(t)$ is positive) and concave down (because $h''(t)$ is negative).

Now, once we have an intuitive idea about what the graph should look like (increasing and concave down), let's solve the problem analytically. The volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$, where V, r, and h are all functions of time. The functions h and r are related; notice the similar triangles in Figure 13. Using properties of similar triangles, we have

$$\frac{r}{h} = \frac{1}{4}$$

Thus, $r = h/4$. The volume of the water inside the cone is thus

$$V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}\left(\frac{h}{4}\right)^2 h = \frac{\pi}{48}h^3$$

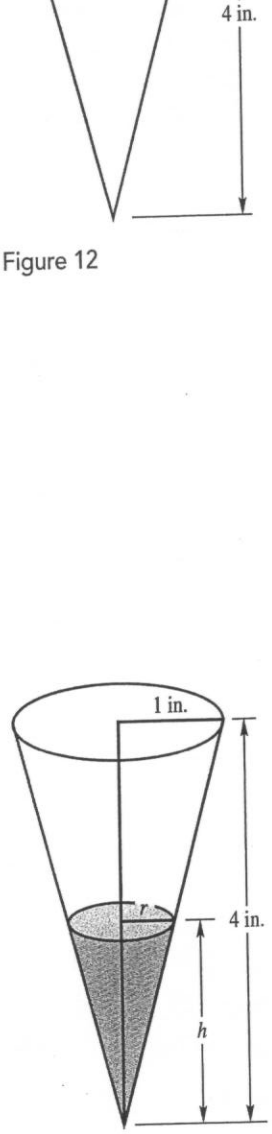


Figure 13

On the other hand, since water is flowing into the container at the rate of $\frac{1}{2}$ cubic inch per second, the volume at time t is $V = \frac{1}{2}t$, where t is measured in seconds. Equating these two expressions for V gives

$$\frac{1}{2}t = \frac{\pi}{48}h^3$$

When $h = 4$, we have $t = \frac{2\pi}{48}4^3 = \frac{8}{3}\pi \approx 8.4$; thus, it takes about 8.4 seconds to fill the container. Now solve for h in the above equation relating h and t to obtain

$$h(t) = \sqrt[3]{\frac{24}{\pi}t}$$

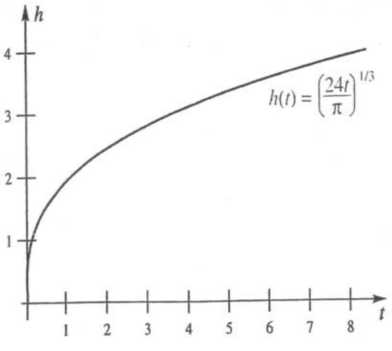


Figure 14

The first and second derivatives of h are

$$h'(t) = D_t\sqrt[3]{\frac{24}{\pi}t} = \frac{8}{\pi}\left(\frac{24}{\pi}t\right)^{-2/3} = \frac{2}{\sqrt[3]{9\pi t^2}}$$

which is positive, and

$$h''(t) = D_t\frac{2}{\sqrt[3]{9\pi t^2}} = -\frac{4}{3\sqrt[3]{9\pi t^5}}$$

which is negative. The graph of $h(t)$ is shown in Figure 14. As expected, the graph of h is increasing and concave down. ■

EXAMPLE 6 A news agency reported in May 2004 that unemployment in eastern Asia was continuing to increase at an increasing rate. On the other hand, the price of food was increasing, but at a slower rate than before. Interpret these statements in terms of increasing/decreasing functions and concavity.

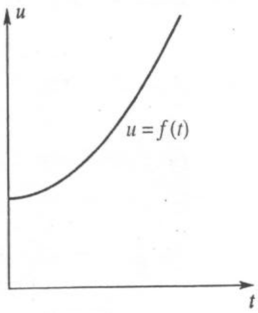


Figure 15

SOLUTION Let $u = f(t)$ denote the number of people unemployed at time t. Although u actually jumps by unit amounts, we will follow standard practice in representing u by a smooth curve as in Figure 15. To say unemployment is increasing is to say that $du/dt > 0$. To say that it is increasing at an increasing rate is to say that the function du/dt is *increasing*; but this means that the derivative of du/dt must be positive. Thus, $d^2u/dt^2 > 0$. In Figure 15, notice that the slope of the tangent line increases as t increases. Unemployment is increasing and concave up.

Similarly, if $p = g(t)$ represents the price of food (e.g., the typical cost of one day's groceries for one person) at time t, then dp/dt is positive but *decreasing*. Thus, the derivative of dp/dt is negative, so $d^2p/dt^2 < 0$. In Figure 16, notice that the slope of the tangent line decreases as t increases. The price of food is increasing but concave down. ■

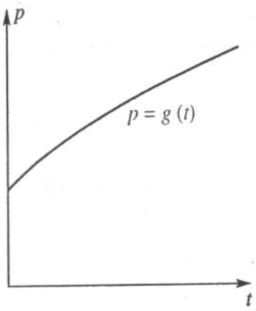


Figure 16

Inflection Points

Let f be continuous at c. We call $(c, f(c))$ an **inflection point** of the graph of f if f is concave up on one side of c and concave down on the other side. The graph in Figure 17 indicates a number of possibilities.

Terminology
While a function's minimum or maximum is a *number*, an inflection point is always an *ordered pair* $(c, f(c))$.

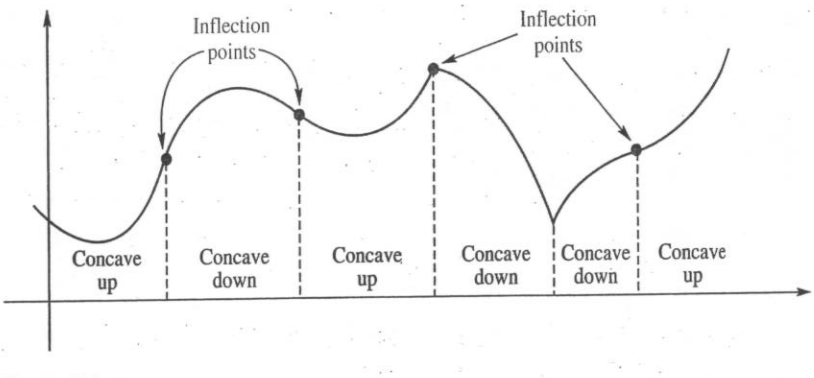


Figure 17

As you might guess, *points where $f''(x) = 0$ or where $f''(x)$ does not exist are the candidates for points of inflection.* We use the word *candidate* deliberately. Just as a candidate for political office may fail to be elected, so, for example, may a point where $f''(x) = 0$ fail to be a point of inflection. Consider $f(x) = x^4$, which has the graph shown in Figure 18. It is true that $f''(0) = 0$; yet the origin is not a point of inflection. Therefore, in searching for inflection points, we begin by identifying those points where $f''(x) = 0$ (and where $f''(x)$ does not exist). Then we check to see if they really are inflection points.

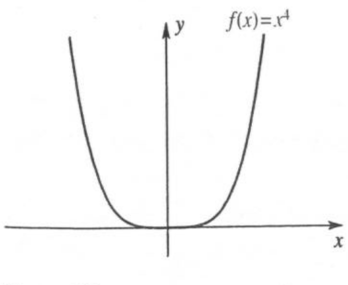


Figure 18

Look back at the graph in Example 4. You will see that it has three inflection points. They are $\left(-\sqrt{3}, -\sqrt{3}/4\right)$, $(0,0)$, and $\left(\sqrt{3}, \sqrt{3}/4\right)$.

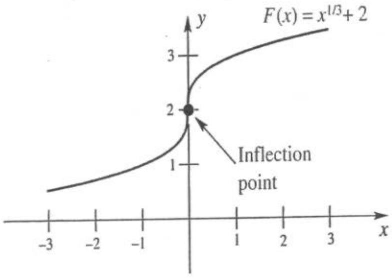


Figure 19

EXAMPLE 7 Find all points of inflection of $F(x) = x^{1/3} + 2$.

SOLUTION

$$F'(x) = \frac{1}{3x^{2/3}}, \qquad F''(x) = \frac{-2}{9x^{5/3}}$$

The second derivative, $F''(x)$, is never 0; however, it fails to exist at $x = 0$. The point $(0, 2)$ is an inflection point since $F''(x) > 0$ for $x < 0$ and $F''(x) < 0$ for $x > 0$. The graph is sketched in Figure 19. ■

Concepts Review

1. If $f'(x) > 0$ everywhere, then f is ____ everywhere; if $f''(x) > 0$ everywhere, then f is ____ everywhere.

2. If ____ and ____ on an open interval I, then f is both increasing and concave down on I.

3. A point on the graph of a continuous function where the concavity changes is called ____.

4. In trying to locate the inflection points for the graph of a function f, we should look at numbers c, where either ____ or ____.

Problem Set 3.2

In Problems 1–10, use the Monotonicity Theorem to find where the given function is increasing and where it is decreasing.

1. $f(x) = 3x + 3$
2. $g(x) = (x + 1)(x - 2)$
3. $h(t) = t^2 + 2t - 3$
4. $f(x) = x^3 - 1$
5. $G(x) = 2x^3 - 9x^2 + 12x$
6. $f(t) = t^3 + 3t^2 - 12$
7. $h(z) = \dfrac{z^4}{4} - \dfrac{4z^3}{6}$
8. $f(x) = \dfrac{x - 1}{x^2}$
9. $H(t) = \sin t, 0 \le t \le 2\pi$
10. $R(\theta) = \cos^2 \theta, 0 \le \theta \le 2\pi$

In Problems 11–18, use the Concavity Theorem to determine where the given function is concave up and where it is concave down. Also find all inflection points.

11. $f(x) = (x - 1)^2$
12. $G(w) = w^2 - 1$
13. $T(t) = 3t^3 - 18t$
14. $f(z) = z^2 - \dfrac{1}{z^2}$
15. $q(x) = x^4 - 6x^3 - 24x^2 + 3x + 1$
16. $f(x) = x^4 + 8x^3 - 2$
17. $F(x) = 2x^2 + \cos^2 x$
18. $G(x) = 24x^2 + 12\sin^2 x$

In Problems 19–28, determine where the graph of the given function is increasing, decreasing, concave up, and concave down. Then sketch the graph (see Example 4).

19. $f(x) = x^3 - 12x + 1$
20. $g(x) = 4x^3 - 3x^2 - 6x + 12$
21. $g(x) = 3x^4 - 4x^3 + 2$
22. $F(x) = x^6 - 3x^4$
23. $G(x) = 3x^5 - 5x^3 + 1$
24. $H(x) = \dfrac{x^2}{x^2 + 1}$
25. $f(x) = \sqrt{\sin x}$ on $[0, \pi]$
26. $g(x) = x\sqrt{x - 2}$
27. $f(x) = x^{2/3}(1 - x)$
28. $g(x) = 8x^{1/3} + x^{4/3}$

In Problems 29–34, sketch the graph of a continuous function f on [0, 6] that satisfies all the stated conditions.

29. $f(0) = 1; f(6) = 3$; increasing and concave down on $(0, 6)$

30. $f(0) = 8; f(6) = -2$; decreasing on $(0, 6)$; inflection point at the ordered pair $(2, 3)$, concave up on $(2, 6)$

31. $f(0) = 3; f(3) = 0; f(6) = 4;$
$f'(x) < 0$ on $(0, 3); f'(x) > 0$ on $(3, 6);$
$f''(x) > 0$ on $(0, 5); f''(x) < 0$ on $(5, 6)$

32. $f(0) = 3; f(2) = 2; f(6) = 0;$
$f'(x) < 0$ on $(0, 2) \cup (2, 6); f'(2) = 0;$
$f''(x) < 0$ on $(0, 1) \cup (2, 6); f''(x) > 0$ on $(1, 2)$

33. $f(0) = f(4) = 1; f(2) = 2; f(6) = 0;$
$f'(x) > 0$ on $(0, 2); f'(x) < 0$ on $(2, 4) \cup (4, 6);$
$f'(2) = f'(4) = 0; f''(x) > 0$ on $(0, 1) \cup (3, 4);$
$f''(x) < 0$ on $(1, 3) \cup (4, 6)$

34. $f(0) = f(3) = 3; f(2) = 4; f(4) = 2; f(6) = 0;$
$f'(x) > 0$ on $(0, 2); f'(x) < 0$ on $(2, 4) \cup (4, 5);$
$f'(2) = f'(4) = 0; f'(x) = -1$ on $(5, 6);$
$f''(x) < 0$ on $(0, 3) \cup (4, 5); f''(x) > 0$ on $(3, 4)$

35. Prove that a quadratic function has no point of inflection.

36. Prove that a cubic function has exactly one point of inflection.

37. Prove that, if $f'(x)$ exists and is continuous on an interval I and if $f'(x) \neq 0$ at all interior points of I, then either f is

increasing throughout I or decreasing throughout I. *Hint:* Use the Intermediate Value Theorem to show that there cannot be two points x_1 and x_2 of I where f' has opposite signs.

38. Suppose that f is a function whose derivative is $f'(x) = (x^2 - x + 1)/(x^2 + 1)$. Use Problem 37 to prove that f is increasing everywhere.

39. Use the Monotonicity Theorem to prove each statement if $0 < x < y$.

(a) $x^2 < y^2$ (b) $\sqrt{x} < \sqrt{y}$ (c) $\dfrac{1}{x} > \dfrac{1}{y}$

40. What conditions on a, b, and c will make $f(x) = ax^3 + bx^2 + cx + d$ always increasing?

41. Determine a and b so that $f(x) = a\sqrt{x} + b/\sqrt{x}$ has the point $(4, 13)$ as an inflection point.

42. Suppose that the cubic function $f(x)$ has three real zeros, r_1, r_2, and r_3. Show that its inflection point has x-coordinate $(r_1 + r_2 + r_3)/3$. *Hint:* $f(x) = a(x - r_1)(x - r_2)(x - r_3)$.

43. Suppose that $f'(x) > 0$ and $g'(x) > 0$ for all x. What simple additional conditions (if any) are needed to guarantee that:

(a) $f(x) + g(x)$ is increasing for all x;

(b) $f(x) \cdot g(x)$ is increasing for all x;

(c) $f(g(x))$ is increasing for all x?

44. Suppose that $f''(x) > 0$ and $g''(x) > 0$ for all x. What simple additional conditions (if any) are needed to guarantee that

(a) $f(x) + g(x)$ is concave up for all x;

(b) $f(x) \cdot g(x)$ is concave up for all x;

(c) $f(g(x))$ is concave up for all x?

GC *Use a graphing calculator or a computer to do Problems 45–48.*

45. Let $f(x) = \sin x + \cos(x/2)$ on the interval $I = (-2, 7)$.

(a) Draw the graph of f on I.

(b) Use this graph to estimate where $f'(x) < 0$ on I.

(c) Use this graph to estimate where $f''(x) < 0$ on I.

(d) Plot the graph of f' to confirm your answer to part (b).

(e) Plot the graph of f'' to confirm your answer to part (c).

46. Repeat Problem 45 for $f(x) = x\cos^2(x/3)$ on $(0, 10)$.

47. Let $f'(x) = x^3 - 5x^2 + 2$ on $I = [-2, 4]$. Where on I is f increasing?

48. Let $f''(x) = x^4 - 5x^3 + 4x^2 + 4$ on $I = [-2, 3]$. Where on I is f concave down?

49. Translate each of the following into the language of derivatives of distance with respect to time. For each part, sketch a plot of the car's position s against time t, and indicate the concavity.

(a) The speed of the car is proportional to the distance it has traveled.

(b) The car is speeding up.

(c) I didn't say the car was slowing down; I said its rate of increase in speed was slowing down.

(d) The car's speed is increasing 10 miles per hour every minute.

(e) The car is slowing very gently to a stop.

(f) The car always travels the same distance in equal time intervals.

50. Translate each of the following into the language of derivatives, sketch a plot of the appropriate function and indicate the concavity.

(a) Water is evaporating from the tank at a constant rate.

(b) Water is being poured into the tank at 3 gallons per minute but is also leaking out at $\frac{1}{2}$ gallon per minute.

(c) Since water is being poured into the conical tank at a constant rate, the water level is rising at a slower and slower rate.

(d) Inflation held steady this year but is expected to rise more and more rapidly in the years ahead.

(e) At present the price of oil is dropping, but this trend is expected to slow and then reverse direction in 2 years.

(f) David's temperature is still rising, but the penicillin seems to be taking effect.

51. Translate each of the following statements into mathematical language, sketch a plot of the appropriate function, and indicate the concavity.

(a) The cost of a car continues to increase and at a faster and faster rate.

(b) During the last 2 years, the United States has continued to cut its consumption of oil, but at a slower and slower rate.

(c) World population continues to grow, but at a slower and slower rate.

(d) The angle that the Leaning Tower of Pisa makes with the vertical is increasing more and more rapidly.

(e) Upper Midwest firm's profit growth slows.

(f) The XYZ Company has been losing money, but will soon turn this situation around.

52. Translate each statement from the following newspaper column into a statement about derivatives.

(a) In the United States, the ratio R of government debt to national income remained unchanged at around 28% up to 1981, but

(b) then it began to increase more and more sharply, reaching 36% during 1983.

53. Coffee is poured into the cup shown in Figure 20 at the rate of 2 cubic inches per second. The top diameter is 3.5 inches, the bottom diameter is 3 inches, and the height of the cup is 5 inches. This cup holds about 23 fluid ounces. Determine the height h of the coffee as a function of time t, and sketch the graph of $h(t)$ from time $t = 0$ until the time that the cup is full.

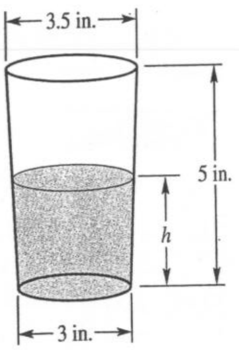


Figure 20

54. Water is being pumped into a cylindrical tank at a constant rate of 5 gallons per minute, as shown in Figure 21. The tank has diameter 3 feet and length 9.5 feet. The volume of the tank is $\pi r^2 l = \pi \times 1.5^2 \times 9.5 \approx 67.152$ cubic feet ≈ 500 gallons. Without doing any calculations, sketch a graph of the height h of the water as a function of time t (see Example 6). Where is h concave up? Concave down?

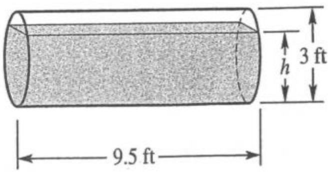


Figure 21

55. A liquid is poured into the container shown in Figure 22 at the rate of 3 cubic inches per second. The container holds about 24 cubic inches. Sketch a graph of the height h of the liquid as a function of time t. In your graph, pay special attention to the concavity of h.

56. A 20-gallon barrel, as shown in Figure 23, leaks at the constant rate of 0.1 gallon per day. Sketch a plot of the height h of the water as a function of time t, assuming that the barrel is full at time $t = 0$. In your graph, pay special attention to the concavity of h.

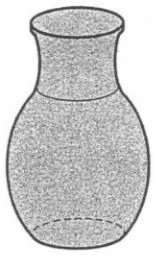


Figure 22

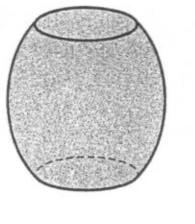


Figure 23

57. What are you able to deduce about the shape of a vase based on each of the following tables, which give measurements of the volume of the water as a function of the depth.

(a)

Depth	1	2	3	4	5	6
Volume	4	8	11	14	20	28

(b)

Depth	1	2	3	4	5	6
Volume	4	9	12	14	20	28

Answers to Concepts Review: **1.** increasing; concave up **2.** $f'(x) > 0; f''(x) < 0$ **3.** an inflection point **4.** $f''(c) = 0; f''(c)$ does not exist

3.3 Local Extrema and Extrema on Open Intervals

We recall from Section 3.1 that the maximum value (if it exists) of a function f on a set S is the largest value that f attains on the whole set S. It is sometimes referred to as the **global maximum value,** or the *absolute maximum value* of f. Thus, for the function f with domain $S = [a, b]$ whose graph is sketched in Figure 1, $f(a)$ is the global maximum value. But what about $f(c)$? It may not be king of the country, but at least it is chief of its own locality. We call it a **local maximum value,** or a *relative maximum value*. Of course, a global maximum value is automatically a local maximum value. Figure 2 illustrates a number of possibilities. Note that the global maximum value (if it exists) is simply the largest of the local maximum values. Similarly, the global minimum value is the smallest of the local minimum values.

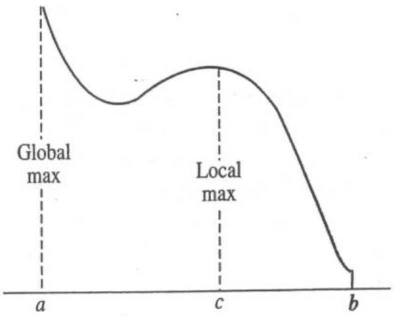


Figure 1

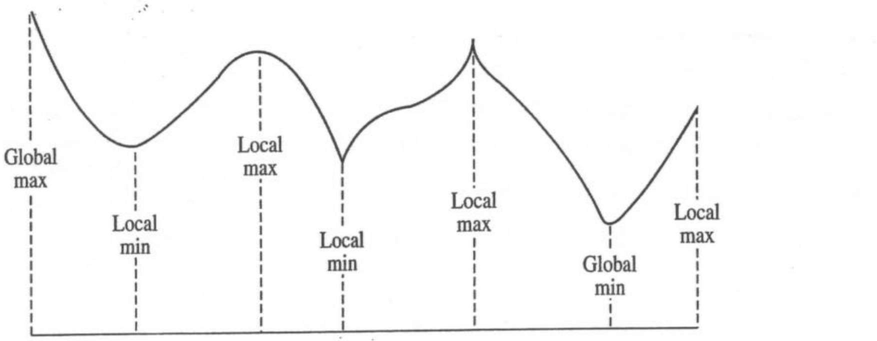


Figure 2

Here is the formal definition of local maxima and local minima. Recall that the symbol $\cap$ denotes the intersection (common part) of two sets.

Definition

Let S, the domain of f, contain the point c. We say that

(i) $f(c)$ is a **local maximum value** of f if there is an interval (a, b) containing c such that $f(c)$ is the maximum value of f on $(a, b) \cap S$;

(ii) $f(c)$ is a **local minimum value** of f if there is an interval (a, b) containing c such that $f(c)$ is the minimum value of f on $(a, b) \cap S$;

(iii) $f(c)$ is a **local extreme value** of f if it is either a local maximum or a local minimum value.

Where Do Local Extreme Values Occur? The Critical Point Theorem (Theorem 3.1B) holds with the phrase *extreme value* replaced by *local extreme value*; the proof is essentially the same. Thus, the critical points (end points, stationary points, and singular points) are the candidates for points where local extrema may occur. We say *candidates* because we are not claiming that there must be a local extremum at every critical point. The left graph in Figure 3 makes this clear. However, if the derivative is positive on one side of the critical point and negative on the other (and if the function is continuous), then we have a local extremum, as shown in the middle and right graphs of Figure 3.

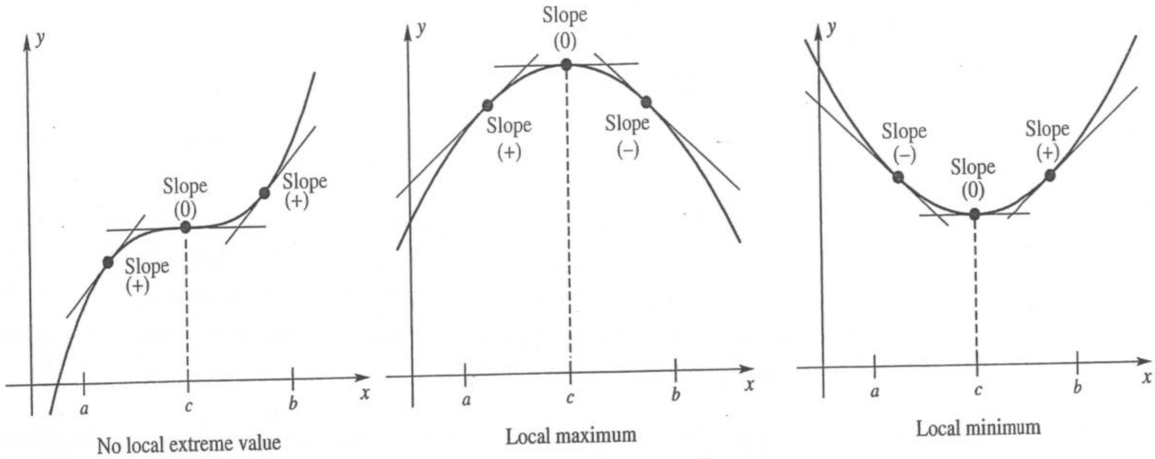


Figure 3

Theorem A	**First Derivative Test**

Let f be continuous on an open interval (a, b) that contains a critical point c.

(i) If $f'(x) > 0$ for all x in (a, c) and $f'(x) < 0$ for all x in (c, b), then $f(c)$ is a local maximum value of f.

(ii) If $f'(x) < 0$ for all x in (a, c) and $f'(x) > 0$ for all x in (c, b), then $f(c)$ is a local minimum value of f.

(iii) If $f'(x)$ has the same sign on both sides of c, then $f(c)$ is not a local extreme value of f.

Proof of (i) Since $f'(x) > 0$ for all x in (a, c), f is increasing on $(a, c]$ by the Monotonicity Theorem. Again, since $f'(x) < 0$ for all x in (c, b), f is decreasing on $[c, b)$. Thus, $f(x) < f(c)$ for all x in (a, b), except of course at $x = c$. We conclude that $f(c)$ is a local maximum.

The proofs of (ii) and (iii) are similar. ■

EXAMPLE 1 Find the local extreme values of the function $f(x) = x^2 - 6x + 5$ on $(-\infty, \infty)$.

SOLUTION The polynomial function f is continuous everywhere, and its derivative, $f'(x) = 2x - 6$, exists for all x. Thus, the only critical point for f is the single solution of $f'(x) = 0$; that is, $x = 3$.

Since $f'(x) = 2(x - 3) < 0$ for $x < 3$, f is decreasing on $(-\infty, 3]$; and because $2(x - 3) > 0$ for $x > 3$, f is increasing on $[3, \infty)$. Therefore, by the First Derivative Test, $f(3) = -4$ is a local minimum value of f. Since 3 is the only critical point, there are no other extreme values. The graph of f is shown in Figure 4. Note that $f(3)$ is actually the (global) minimum value in this case. ■

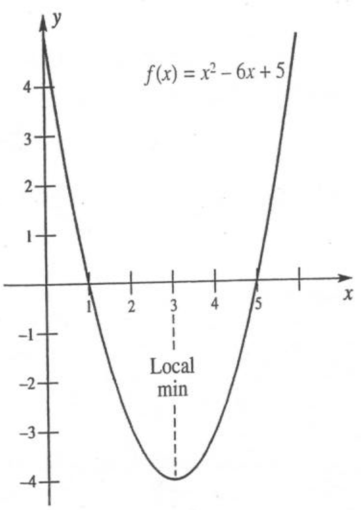


Figure 4

EXAMPLE 2 Find the local extreme values of $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$ on $(-\infty, \infty)$.

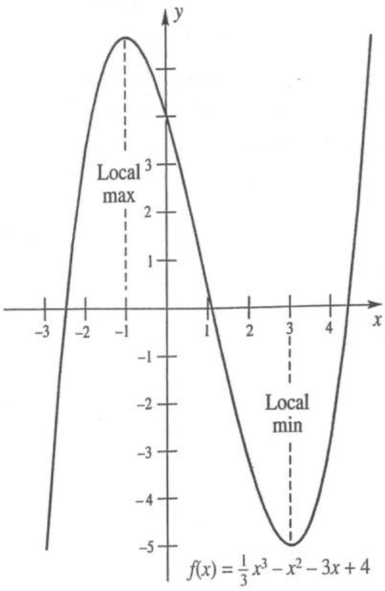


Figure 5

SOLUTION Since $f'(x) = x^2 - 2x - 3 = (x + 1)(x - 3)$, the only critical points of f are -1 and 3. When we use the test points -2, 0, and 4, we learn that $(x + 1)(x - 3) > 0$ on $(-\infty, -1)$ and $(3, \infty)$ and $(x + 1)(x - 3) < 0$ on $(-1, 3)$. By the First Derivative Test, we conclude that $f(-1) = \frac{17}{3}$ is a local maximum value and that $f(3) = -5$ is a local minimum value (Figure 5).

EXAMPLE 3 Find the local extreme values of $f(x) = (\sin x)^{2/3}$ on $(-\pi/6, 2\pi/3)$.

SOLUTION

$$f'(x) = \frac{2 \cos x}{3(\sin x)^{1/3}}, \qquad x \neq 0$$

The points 0 and $\pi/2$ are critical points, since $f'(0)$ does not exist and $f'(\pi/2) = 0$. Now $f'(x) < 0$ on $(-\pi/6, 0)$ and on $(\pi/2, 2\pi/3)$, while $f'(x) > 0$ on $(0, \pi/2)$. By the First Derivative Test, we conclude that $f(0) = 0$ is a local minimum value and that $f(\pi/2) = 1$ is a local maximum value. The graph of f is shown in Figure 6.

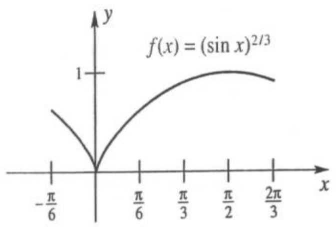


Figure 6

The Second Derivative Test There is another test for local maxima and minima that is sometimes easier to apply than the First Derivative Test. It involves evaluating the second derivative at the stationary points. It does not apply to singular points.

Theorem B Second Derivative Test

Let f' and f'' exist at every point in an open interval (a, b) containing c, and suppose that $f'(c) = 0$.

(i) If $f''(c) < 0$, then $f(c)$ is a local maximum value of f.
(ii) If $f''(c) > 0$, then $f(c)$ is a local minimum value of f.

Proof of (i) It is tempting to say that, since $f''(c) < 0$, f is concave downward near c and to claim that this proves (i). However, to be sure that f is concave downward in a neighborhood of c, we need $f''(x) < 0$ in that neighborhood (not just at c), and nothing in our hypothesis guarantees that. We must be a bit more careful. By definition and hypothesis,

$$f''(c) = \lim_{x \to c} \frac{f'(x) - f'(c)}{x - c} = \lim_{x \to c} \frac{f'(x) - 0}{x - c} < 0$$

so we can conclude that there is a (possibly small) interval (α, β) around c where

$$\frac{f'(x)}{x - c} < 0, \qquad x \neq c$$

But this inequality implies that $f'(x) > 0$ for $\alpha < x < c$ and $f'(x) < 0$ for $c < x < \beta$. Thus, by the First Derivative Test, $f(c)$ is a local maximum value.

The proof of (ii) is similar. ■

EXAMPLE 4 For $f(x) = x^2 - 6x + 5$, use the Second Derivative Test to identify local extrema.

SOLUTION This is the function of Example 1. Note that

$$f'(x) = 2x - 6 = 2(x - 3)$$

$$f''(x) = 2$$

Thus, $f'(3) = 0$ and $f''(3) > 0$. Therefore, by the Second Derivative Test, $f(3)$ is a local minimum value.

EXAMPLE 5 For $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$, use the Second Derivative Test to identify local extrema.

SOLUTION This is the function of Example 2.

$$f'(x) = x^2 - 2x - 3 = (x + 1)(x - 3)$$

$$f''(x) = 2x - 2$$

The critical points are -1 and 3 ($f'(-1) = f'(3) = 0$). Since $f''(-1) = -4$ and $f''(3) = 4$, we conclude by the Second Derivative Test that $f(-1)$ is a local maximum value and that $f(3)$ is a local minimum value. ■

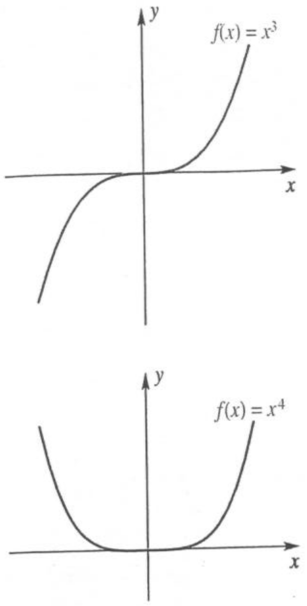


Figure 7

Unfortunately, the Second Derivative Test sometimes fails, since $f''(x)$ may be 0 at a stationary point. For both $f(x) = x^3$ and $f(x) = x^4$, $f'(0) = 0$ and $f''(0) = 0$ (see Figure 7). The first does not have a local maximum or minimum value at 0; the second has a local minimum there. This shows that if $f''(x) = 0$ at a stationary point we are unable to draw a conclusion about maxima or minima without more information.

Extrema on Open Intervals The problems that we studied in this section and in Section 3.1 often assumed that the set on which we wanted to maximize or minimize a function was a *closed* interval. However, the intervals that arise in practice are not always closed; they are sometimes open, or even open on one end and closed on the other. We can still handle these problems if we correctly apply the theory developed in this section. Keep in mind that maximum (minimum) with no qualifying adjective means global maximum (minimum).

EXAMPLE 6 Find (if any exist) the minimum and maximum values of $f(x) = x^4 - 4x$ on $(-\infty, \infty)$.

SOLUTION

$$f'(x) = 4x^3 - 4 = 4(x^3 - 1) = 4(x - 1)(x^2 + x + 1)$$

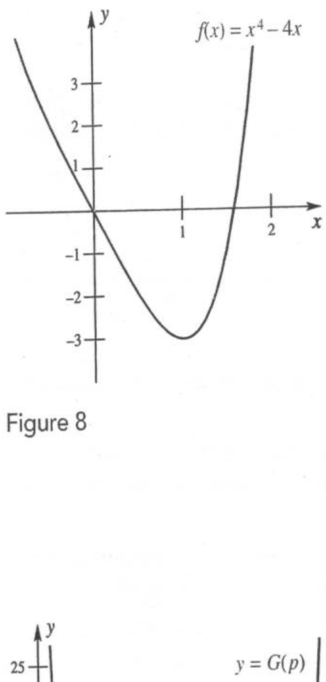


Figure 8

Since $x^2 + x + 1 = 0$ has no real solutions (quadratic formula), there is only one critical point, $x = 1$. For $x < 1, f'(x) < 0$, whereas for $x > 1, f'(x) > 0$. We conclude that $f(1) = -3$ is a local minimum value for f; and since f is decreasing on the left of 1 and increasing on the right of 1, it must actually be the minimum value of f.

The facts stated above imply that f cannot have a maximum value. The graph of f is shown in Figure 8. ■

EXAMPLE 7 Find (if any exist) the maximum and minimum values of

$$G(p) = \frac{1}{p(1 - p)}$$

on $(0, 1)$.

SOLUTION

$$G'(p) = \frac{d}{dp}[p(1 - p)]^{-1} = \frac{2p - 1}{p^2(1 - p)^2}$$

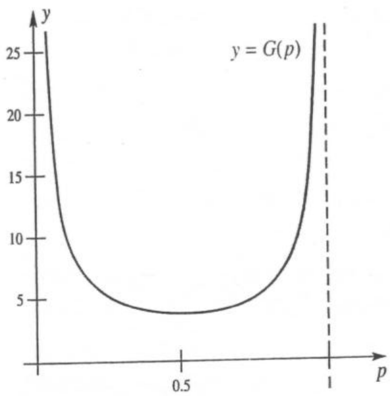


Figure 9

The only critical point is $p = 1/2$. For every value of p in the interval $(0, 1)$ the denominator is positive; thus, the numerator determines the sign. If p is in the interval $(0, 1/2)$, then the numerator is negative; hence, $G'(p) < 0$. Similarly, if p is in the interval $(1/2, 1)$, $G'(p) > 0$. Thus, by the First Derivative Test, $G(1/2) = 4$ is a local minimum. Since there are no end points or singular points to check, $G(1/2)$ is a global minimum. There is no maximum. The graph of $y = G(p)$ is shown in Figure 9. ■

Concepts Review

1. If f is continuous at c, $f'(x) > 0$ near to c on the left, and $f'(x) < 0$ near to c on the right, then $f(c)$ is a local ____ value for f.

2. If $f'(x) = (x + 2)(x - 1)$, then $f(-2)$ is a local ____ value for f and $f(1)$ is a local ____ value for f.

3. If $f'(c) = 0$ and $f''(c) < 0$, we expect to find a local ____ value for f at c.

4. If $f(x) = x^3$, then $f(0)$ is neither a ____ nor a ____, even though $f''(0) =$ ____.

Problem Set 3.3

In Problems 1–10, identify the critical points. Then use (a) the First Derivative Test and (if possible) (b) the Second Derivative Test to decide which of the critical points give a local maximum and which give a local minimum.

1. $f(x) = x^3 - 6x^2 + 4$

2. $f(x) = x^3 - 12x + \pi$

3. $f(\theta) = \sin 2\theta, 0 < \theta < \dfrac{\pi}{4}$

4. $f(x) = \frac{1}{2}x + \sin x, 0 < x < 2\pi$

5. $\Psi(\theta) = \sin^2 \theta, -\pi/2 < \theta < \pi/2$

6. $r(z) = z^4 + 4$

7. $f(x) = \dfrac{x}{x^2 + 4}$

8. $g(z) = \dfrac{z^2}{1 + z^2}$

9. $h(y) = y^2 - \dfrac{1}{y}$

10. $f(x) = \dfrac{3x + 1}{x^2 + 1}$

In Problems 11–20, find the critical points and use the test of your choice to decide which critical points give a local maximum value and which give a local minimum value. What are these local maximum and minimum values?

11. $f(x) = x^3 - 3x$

12. $g(x) = x^4 + x^2 + 3$

13. $H(x) = x^4 - 2x^3$

14. $f(x) = (x - 2)^5$

15. $g(t) = \pi - (t - 2)^{2/3}$

16. $r(s) = 3s + s^{2/5}$

17. $f(t) = t - \dfrac{1}{t}, t \neq 0$

18. $f(x) = \dfrac{x^2}{\sqrt{x^2 + 4}}$

19. $\Lambda(\theta) = \dfrac{\cos \theta}{1 + \sin \theta}, 0 < \theta < 2\pi$

20. $g(\theta) = |\sin \theta|, 0 < \theta < 2\pi$

In Problems 21–30, find, if possible, the (global) maximum and minimum values of the given function on the indicated interval.

21. $f(x) = \sin^2 2x$ on $[0, 2]$

22. $f(x) = \dfrac{2x}{x^2 + 4}$ on $[0, \infty)$

23. $g(x) = \dfrac{x^2}{x^3 + 32}$ on $[0, \infty)$

24. $h(x) = \dfrac{1}{x^2 + 4}$ on $[0, \infty)$

25. $F(x) = 6\sqrt{x} - 4x$ on $[0, 4]$

26. $F(x) = 6\sqrt{x} - 4x$ on $[0, \infty)$

27. $f(x) = \dfrac{64}{\sin x} + \dfrac{27}{\cos x}$ on $(0, \pi/2)$

28. $g(x) = x^2 + \dfrac{16x^2}{(8 - x)^2}$ on $(8, \infty)$

29. $H(x) = |x^2 - 1|$ on $[-2, 2]$

30. $h(t) = \sin t^2$ on $[0, \pi]$

In Problems 31–36, the first derivative f' is given. Find all values of x that make the function f (a) a local minimum and (b) a local maximum.

31. $f'(x) = x^3(1 - x)^2$

32. $f'(x) = -(x - 1)(x - 2)(x - 3)(x - 4)$

33. $f'(x) = (x - 1)^2(x - 2)^2(x - 3)(x - 4)$

34. $f'(x) = (x - 1)^2(x - 2)^2(x - 3)^2(x - 4)^2$

35. $f'(x) = (x - A)^2(x - B)^2, A \neq B$

36. $f'(x) = x(x - A)(x - B), 0 < A < B$

In Problems 37–42, sketch a graph of a function with the given properties. If it is impossible to graph such a function, then indicate this and justify your answer.

37. f is differentiable, has domain $[0, 6]$, and has two local maxima and two local minima on $(0, 6)$.

38. f is differentiable, has domain $[0, 6]$, and has three local maxima and two local minima on $(0, 6)$.

39. f is continuous, but not necessarily differentiable, has domain $[0, 6]$, and has one local minimum and one local maximum on $(0, 6)$.

40. f is continuous, but not necessarily differentiable, has domain $[0, 6]$, and has one local minimum and no local maximum on $(0, 6)$.

41. f has domain $[0, 6]$, but is not necessarily continuous, and has three local maxima and no local minimum on $(0, 6)$.

42. f has domain $[0, 6]$, but is not necessarily continuous, and has two local maxima and no local minimum on $(0, 6)$.

43. Consider $f(x) = Ax^2 + Bx + C$ with $A > 0$. Show that $f(x) \geq 0$ for all x if and only if $B^2 - 4AC \leq 0$.

44. Consider $f(x) = Ax^3 + Bx^2 + Cx + D$ with $A > 0$. Show that f has one local maximum and one local minimum if and only if $B^2 - 3AC > 0$.

45. What conclusions can you draw about f from the information that $f'(c) = f''(c) = 0$ and $f'''(c) > 0$?

Answers to Concepts Review: **1.** maximum **2.** maximum; minimum **3.** maximum **4.** local maximum; local minimum; 0

65. The XYZ Company manufactures wicker chairs. With its present machines, it has a maximum yearly output of 500 units. If it makes x chairs, it can set a price of $p(x) = 200 - 0.15x$ dollars each and will have a total yearly cost of $C(x) = 5000 + 6x - 0.002x^2$ dollars. The company has the opportunity to buy a new machine for \$4000 with which the company can make up to an additional 250 chairs per year. The cost function for values of x between 500 and 750 is thus $C(x) = 9000 + 6x - 0.002x^2$. Basing your analysis on the profit for the next year, answer the following questions.

(a) Should the company purchase the additional machine?

(b) What should be the level of production?

66. Repeat Problem 65, assuming that the additional machine costs \$3000.

$\boxed{C}$ **67.** The ZEE Company makes zingos, which it markets at a price of $p(x) = 10 - 0.001x$ dollars, where x is the number produced each month. Its total monthly cost is $C(x) = 200 + 4x - 0.01x^2$. At peak production, it can make 300 units. What is its maximum monthly profit and what level of production gives this profit?

$\boxed{C}$ **68.** If the company of Problem 67 expands its facilities so that it can produce up to 450 units each month, its monthly cost function takes the form $C(x) = 800 + 3x - 0.01x^2$ for $300 < x \le 450$. Find the production level that maximizes monthly profit and evaluate this profit. Sketch the graph of the monthly profit function $P(x)$ on $0 \le x \le 450$.

$\boxed{\text{EXPL}}$ **69.** The arithmetic mean of the numbers a and b is $(a + b)/2$, and the geometric mean of two positive numbers a and b is $\sqrt{ab}$. Suppose that $a > 0$ and $b > 0$.

(a) Show that $\sqrt{ab} \le (a + b)/2$ holds by squaring both sides and simplifying.

(b) Use calculus to show that $\sqrt{ab} \le (a + b)/2$. *Hint:* Consider a to be fixed. Square both sides of the inequality and divide through by b. Define the function $F(b) = (a + b)^2/4b$. Show that F has its minimum at a.

(c) The geometric mean of three positive numbers a, b, and c is $(abc)^{1/3}$. Show that the analogous inequality holds:

$$(abc)^{1/3} \le \frac{a + b + c}{3}$$

Hint: Consider a and c to be fixed and define $F(b) = (a + b + c)^3/27b$. Show that F has a minimum at $b = (a + c)/2$ and that this minimum is $[(a + c)/2]^2$. Then use the result from (b).

$\boxed{\text{EXPL}}$ **70.** Show that of all three-dimensional boxes with a given surface area, the cube has the greatest volume. *Hint:* The surface area is $S = 2(lw + lh + hw)$ and the volume is $V = lwh$. Let $a = lw$, $b = lh$, and $c = hw$. Use the previous problem to show that $(V^2)^{1/3} \le S/6$. When does equality hold?

Answers to Concepts Review: **1.** $0 < x < \infty$ **2.** $2x + 200/x$ **3.** $y_i - bx_i$ **4.** marginal revenue; marginal cost

3.5 Graphing Functions Using Calculus

Our treatment of graphing in Section 0.4 was elementary. We proposed plotting enough points so that the essential features of the graph were clear. We mentioned that symmetries of the graph could reduce the effort involved. We suggested that one should be alert to possible asymptotes. But if the equation to be graphed is complicated or if we want a very accurate graph, the techniques of that section are inadequate.

Calculus provides a powerful tool for analyzing the fine structure of a graph, especially in identifying those points where the character of the graph changes. We can locate local maximum points, local minimum points, and inflection points; we can determine precisely where the graph is increasing or where it is concave up. Inclusion of all these ideas in our graphing procedure is the program for this section.

Polynomial Functions A polynomial function of degree 1 or 2 is easy to graph by hand; one of degree 50 could be next to impossible. If the degree is of modest size, such as 3 to 6, we can use the tools of calculus to great advantage.

EXAMPLE 1 Sketch the graph of $f(x) = \dfrac{3x^5 - 20x^3}{32}$.

SOLUTION Since $f(-x) = -f(x)$, f is an odd function, and therefore its graph is symmetric with respect to the origin. Setting $f(x) = 0$, we find the x-intercepts to be 0 and $\pm\sqrt{20/3} \approx \pm 2.6$. We can go this far without calculus.

When we differentiate f, we obtain

$$f'(x) = \frac{15x^4 - 60x^2}{32} = \frac{15x^2(x - 2)(x + 2)}{32}$$

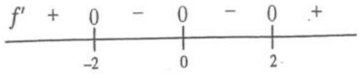


Figure 1

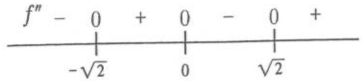


Figure 2

Thus, the critical points are $-2, 0$, and 2; we quickly discover (Figure 1) that $f'(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$ and that $f'(x) < 0$ on $(-2, 0)$ and $(0, 2)$. These facts tell us where f is increasing and where it is decreasing; they also confirm that $f(-2) = 2$ is a local maximum value and that $f(2) = -2$ is a local minimum value.

Differentiating again, we get

$$f''(x) = \frac{60x^3 - 120x}{32} = \frac{15x\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)}{8}$$

By studying the sign of $f''(x)$ (Figure 2), we deduce that f is concave upward on $\left(-\sqrt{2}, 0\right)$ and $\left(\sqrt{2}, \infty\right)$ and concave downward on $\left(-\infty, -\sqrt{2}\right)$ and $\left(0, \sqrt{2}\right)$. Thus, there are three points of inflection: $\left(-\sqrt{2}, 7\sqrt{2}/8\right) \approx (-1.4, 1.2)$, $(0, 0)$, and $\left(\sqrt{2}, -7\sqrt{2}/8\right) \approx (1.4, -1.2)$.

Much of this information is collected at the top of Figure 3, which we use to sketch the graph directly below it.

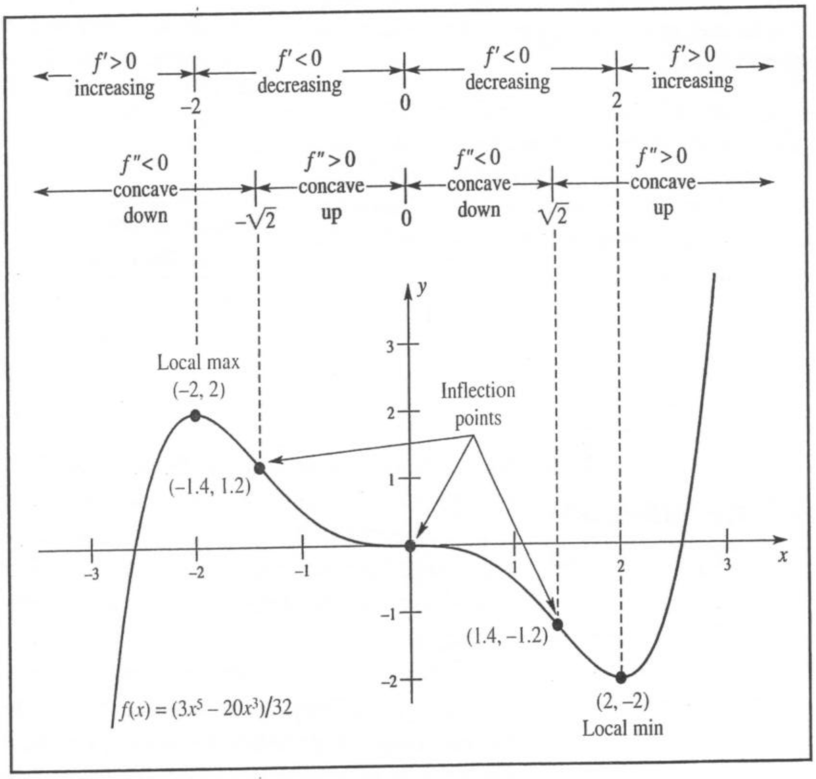


Figure 3

■

Rational Functions A rational function, being the quotient of two polynomial functions, is considerably more complicated to graph than a polynomial. In particular, we can expect dramatic behavior near where the denominator would be zero.

EXAMPLE 2 Sketch the graph of $f(x) = \dfrac{x^2 - 2x + 4}{x - 2}$.

SOLUTION This function is neither even nor odd, so we do not have any of the usual symmetries. There are no x-intercepts, since the solutions to $x^2 - 2x + 4 = 0$ are not real numbers. The y-intercept is -2. We anticipate a vertical asymptote at $x = 2$. In fact,

$$\lim_{x \to 2^-} \frac{x^2 - 2x + 4}{x - 2} = -\infty \qquad \text{and} \qquad \lim_{x \to 2^+} \frac{x^2 - 2x + 4}{x - 2} = \infty$$

Differentiation twice gives

$$f'(x) = \frac{x(x-4)}{(x-2)^2} \quad \text{and} \quad f''(x) = \frac{8}{(x-2)^3}$$

The stationary points are therefore $x = 0$ and $x = 4$.

Thus, $f'(x) > 0$ on $(-\infty, 0) \cup (4, \infty)$ and $f'(x) < 0$ on $(0, 2) \cup (2, 4)$. (Remember, $f'(x)$ does not exist when $x = 2$.) Also, $f''(x) > 0$ on $(2, \infty)$ and $f''(x) < 0$ on $(-\infty, 2)$. Since $f''(x)$ is never 0, there are no inflection points. On the other hand, $f(0) = -2$ and $f(4) = 6$ give local maximum and minimum values, respectively.

It is a good idea to check on the behavior of $f(x)$ for large $|x|$. Since

$$f(x) = \frac{x^2 - 2x + 4}{x - 2} = x + \frac{4}{x - 2}$$

the graph of $y = f(x)$ gets closer and closer to the line $y = x$ as $|x|$ gets larger and larger. We call the line $y = x$ an **oblique asymptote** for the graph of f (see Problem 49 of Section 1.5).

With all this information, we are able to sketch a rather accurate graph (Figure 4).

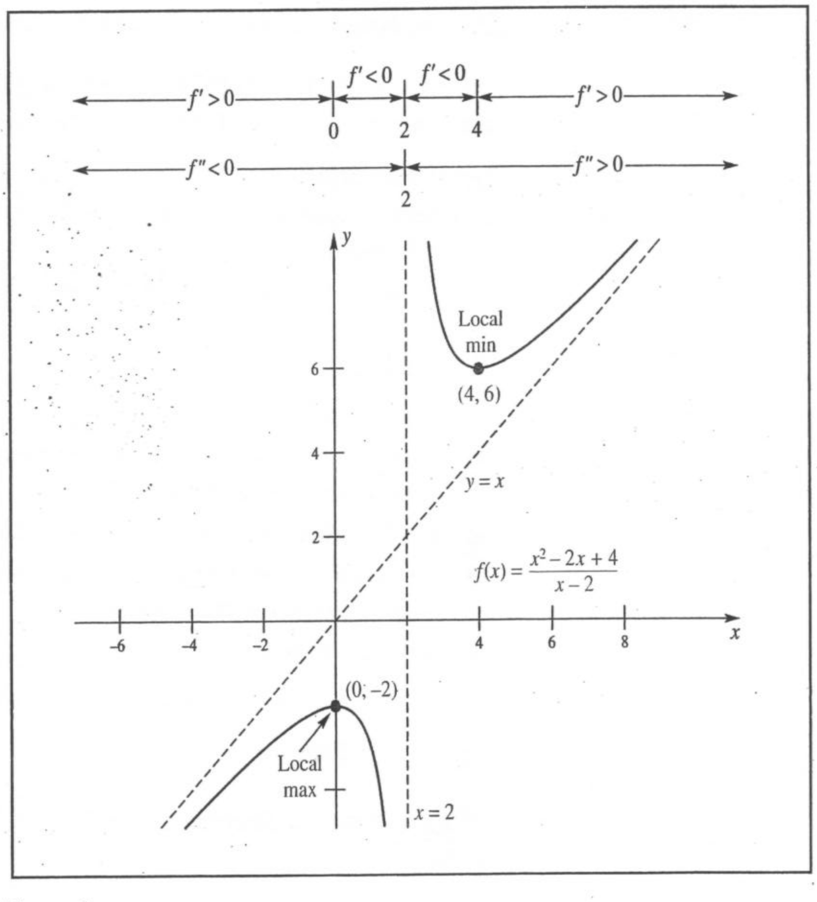


Figure 4

Functions Involving Roots

There is an endless variety of functions involving roots. Here is one example.

EXAMPLE 3 Analyze the function

$$F(x) = \frac{\sqrt{x}(x - 5)^2}{4}$$

and sketch its graph.

SOLUTION The domain of F is $[0, \infty)$ and the range is $[0, \infty)$, so the graph of F is confined to the first quadrant and the positive coordinate axes. The x-intercepts are 0 and 5; the y-intercept is 0. From

$$F'(x) = \frac{5(x-1)(x-5)}{8\sqrt{x}}, \qquad x > 0$$

we find the stationary points 1 and 5. Since $F'(x) > 0$ on $(0, 1)$ and $(5, \infty)$, while $F'(x) < 0$ on $(1, 5)$, we conclude that $F(1) = 4$ is a local maximum value and $F(5) = 0$ is a local minimum value.

So far, it has been clear sailing. But on calculating the second derivative, we obtain

$$F''(x) = \frac{5(3x^2 - 6x - 5)}{16x^{3/2}}, \qquad x > 0$$

which is quite complicated. However, $3x^2 - 6x - 5 = 0$ has one solution in $(0, \infty)$, namely $1 + 2\sqrt{6}/3 \approx 2.6$.

Using the test points 1 and 3, we conclude that $f''(x) < 0$ on $(0, 1 + 2\sqrt{6}/3$ and $f''(x) > 0$ on $(1 + 2\sqrt{6}/3, \infty)$. It then follows that the point $\left(1 + 2\sqrt{6}/3, F(1 + 2\sqrt{6}/3)\right)$, which is approximately $(2.6, 2.3)$, is an inflection point.

As x grows large, $F(x)$ grows without bound and much faster than any linear function; there are no asymptotes. The graph is sketched in Figure 5. ■

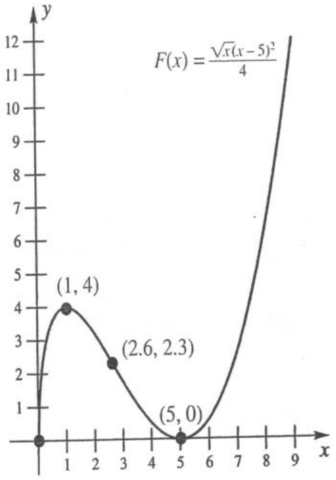


Figure 5

Summary of the Method In graphing functions, there is no substitute for common sense. However, the following procedure will be helpful in most cases.

Step 1: Precalculus analysis.

(a) Check the *domain* and *range* of the function to see if any regions of the plane are excluded.

(b) Test for *symmetry* with respect to the y-axis and the origin. (Is the function even or odd?)

(c) Find the *intercepts*.

Step 2: Calculus analysis.

(a) Use the first derivative to find the critical points and to find out where the graph is *increasing* and *decreasing*.

(b) Test the critical points for *local maxima* and *minima*.

(c) Use the second derivative to find out where the graph is *concave upward* and *concave downward* and to locate *inflection points*.

(d) Find the *asymptotes*.

Step 3: Plot a few points (including all critical points and inflection points).

Step 4: Sketch the graph.

EXAMPLE 4 Sketch the graphs of $f(x) = x^{1/3}$ and $g(x) = x^{2/3}$ and their derivatives.

SOLUTION The domain for both functions is $(-\infty, \infty)$. (Remember, the cube root exists for every real number.) The range for $f(x)$ is $(-\infty, \infty)$ since every real number is the cube root of some other number. Writing $g(x)$ as $g(x) = x^{2/3} = (x^{1/3})^2$, we see that $g(x)$ must be nonnegative; it's range is $[0, \infty)$. Since $f(-x) = (-x)^{1/3} = -x^{1/3} = -f(x)$, we see that f is an odd function. Similarly, since $g(-x) = (-x)^{2/3} = ((-x)^2)^{1/3} = (x^2)^{1/3} = g(x)$, we see that g is an even function. The first derivatives are

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

and

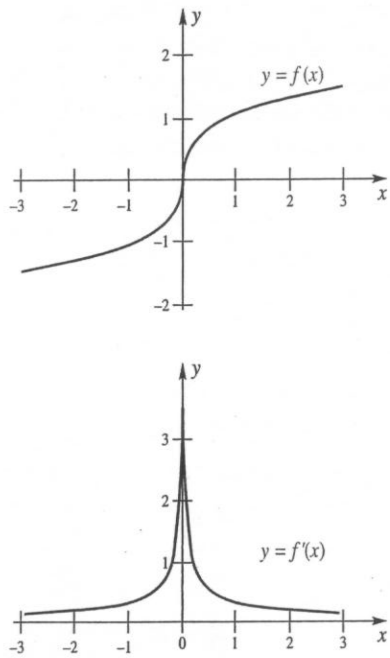


Figure 6

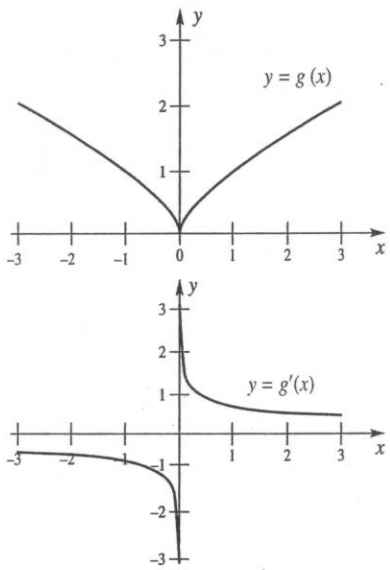


Figure 7

$$g'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$$

and the second derivatives are

$$f''(x) = -\frac{2}{9}x^{-5/3} = -\frac{2}{9x^{5/3}}$$

and

$$g''(x) = -\frac{2}{9}x^{-4/3} = -\frac{2}{9x^{4/3}}$$

For both functions the only critical point, in this case a point where the derivative doesn't exist, is $x = 0$.

Note that $f'(x) > 0$ for all x, except $x = 0$. Thus, f is increasing on $(-\infty, 0]$ and also on $[0, \infty)$, but because f is continuous on $(-\infty, \infty)$, we can conclude that f is always increasing. Consequently, f has no local maxima or minima. Since $f''(x)$ is positive when x is negative and negative when x is positive (and undefined when $x = 0$), we conclude that f is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$. The point $(0, 0)$ is an inflection point because that is where the concavity changes.

Now consider $g(x)$. Note that $g'(x)$ is negative when x is negative and positive when x is positive. Since g is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$, $g(0) = 0$ is a local mimimum. Note also that $g''(x)$ is negative as long as $x \neq 0$. Thus g is concave down on $(-\infty, 0)$ and concave down on $(0, \infty)$, so $(0, 0)$ is not an inflection point. The graphs of $f(x)$, $f'(x)$, $g(x)$ and $g'(x)$ are shown in Figures 6 and 7. ■

Note that in the above example both functions had one critical point, $x = 0$, where the derivative was undefined. Yet the graphs of the functions are fundamentally different. The graph of $y = f(x)$ has a tangent line at all points, but it is vertical when $x = 0$. (If the tangent line is vertical, then the derivative doesn't exist at that point.) The graph of $y = g(x)$ has a sharp point, called a **cusp,** at $x = 0$.

Using the Derivative's Graph to Graph a Function Knowing just a function's derivative can tell us a lot about the function itself and what its graph looks like.

EXAMPLE 5 Figure 8 shows a plot of $y = f'(x)$. Find all local extrema and points of inflection of f on the interval $[-1, 3]$. Given that $f(1) = 0$, sketch the graph of $y = f(x)$.

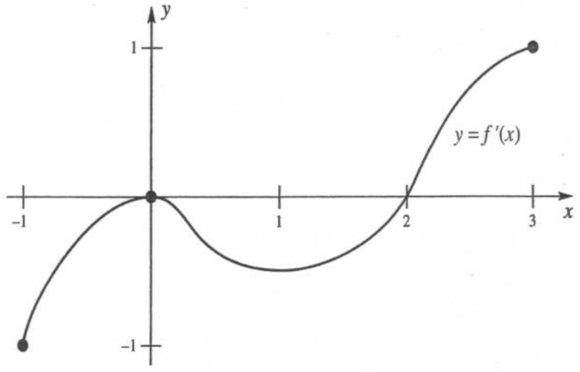


Figure 8

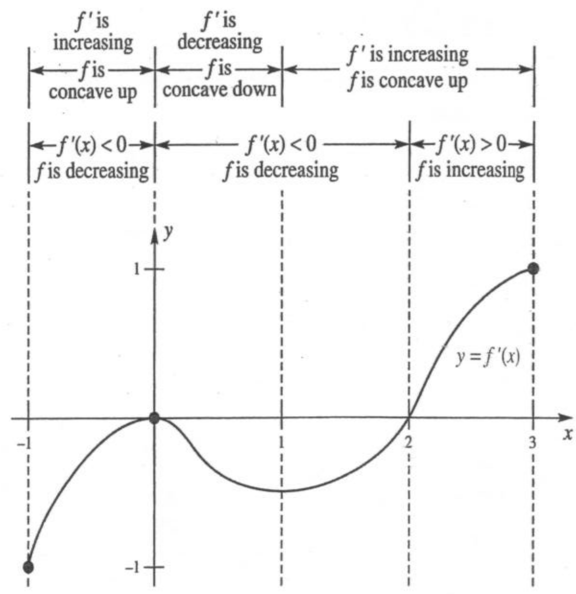


Figure 9

$f(-1)$	Local maximum
$f(2)$	Local minimum
$f(3)$	Local maximum
$(0, f(0))$	Inflection point
$(1, f(1))$	Inflection point

SOLUTION The derivative is negative on the intervals $(-1, 0)$ and $(0, 2)$, and positive on the interval $(2, 3)$. Thus, f is decreasing on $[-1, 0]$ and on $[0, 2]$ so there is a local maximum at the left end point $x = -1$. Since $f'(x)$ is positive on $(2, 3)$, f is increasing on $[2, 3]$ so there is a local maximum at the right end point $x = 3$. Since f is decreasing on $[-1, 2]$ and increasing on $[2, 3]$, there is a local minimum at $x = 2$. Figure 9 summarizes this information.

Inflection points for f occur when the concavity of f changes. Since f' is increasing on $(-1, 0)$ and on $(1, 3)$, f is concave up on $(-1, 0)$ and on $(1, 3)$. Since f' is decreasing on $(0, 1)$, f is concave down on $(0, 1)$. Thus, f changes concavity at $x = 0$ and $x = 1$. The inflection points are therefore $(0, f(0))$ and $(1, f(1))$.

The information given above, together with the fact that $f(1) = 0$, can be used to sketch the graph of $y = f(x)$. (The sketch cannot be too precise because we still have limited information about f.) A sketch is shown in Figure 10.

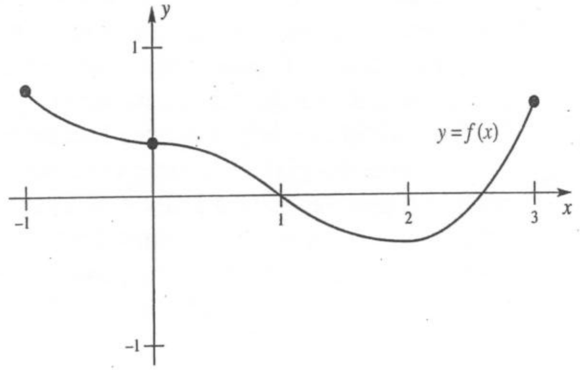


Figure 10

Concepts Review

1. The graph of f is symmetric with respect to the y-axis if $f(-x) =$ ____ for every x; the graph is symmetric with respect to the origin if $f(-x) =$ ____ for every x.

2. If $f'(x) < 0$ and $f''(x) > 0$ for all x in an interval I, then the graph of f is both ____ and ____ on I.

3. The graph of $f(x) = x^3/[(x + 1)(x - 2)(x - 3)]$ has as vertical asymptotes the lines ____ and as a horizontal asymptote the line ____.

4. We call $f(x) = 3x^5 - 2x^2 + 6$ a(n) ____ function, and we call $g(x) = (3x^5 - 2x^2 + 6)/(x^2 - 4)$ a(n) ____ function.

Problem Set 3.5

In Problems 1–27, make an analysis as suggested in the summary above and then sketch the graph.

1. $f(x) = x^3 - 3x + 5$
2. $f(x) = 2x^3 - 3x - 10$
3. $f(x) = 2x^3 - 3x^2 - 12x + 3$
4. $f(x) = (x - 1)^3$
5. $G(x) = (x - 1)^4$
6. $H(t) = t^2(t^2 - 1)$
7. $f(x) = x^3 - 3x^2 + 3x + 10$
8. $F(s) = \dfrac{4s^4 - 8s^2 - 12}{3}$
9. $g(x) = \dfrac{x}{x + 1}$
10. $g(s) = \dfrac{(s - \pi)^2}{s}$
11. $f(x) = \dfrac{x}{x^2 + 4}$
12. $\Lambda(\theta) = \dfrac{\theta^2}{\theta^2 + 1}$
13. $h(x) = \dfrac{x}{x - 1}$
14. $P(x) = \dfrac{1}{x^2 + 1}$
15. $f(x) = \dfrac{(x - 1)(x - 3)}{(x + 1)(x - 2)}$
16. $w(z) = \dfrac{z^2 + 1}{z}$
17. $g(x) = \dfrac{x^2 + x - 6}{x - 1}$
18. $f(x) = |x|^3$ *Hint:* $\dfrac{d}{dx}|x| = \dfrac{x}{|x|}$
19. $R(z) = z|z|$
20. $H(q) = q^2|q|$
21. $g(x) = \dfrac{|x| + x}{2}(3x + 2)$
22. $h(x) = \dfrac{|x| - x}{2}(x^2 - x + 6)$
23. $f(x) = |\sin x|$
24. $f(x) = \sqrt{\sin x}$
25. $h(t) = \cos^2 t$
26. $g(t) = \tan^2 t$
27. [C] $f(x) = \dfrac{5.235x^3 - 1.245x^2}{7.126x - 3.141}$

28. Sketch the graph of a function f that has the following properties:
(a) f is everywhere continuous; (b) $f(0) = 0, f(1) = 2$;
(c) f is an even function; (d) $f'(x) > 0$ for $x > 0$;
(e) $f''(x) > 0$ for $x > 0$.

29. Sketch the graph of a function f that has the following properties:
(a) f is everywhere continuous; (b) $f(2) = -3, f(6) = 1$;
(c) $f'(2) = 0, f'(x) > 0$ for $x \neq 2, f'(6) = 3$;
(d) $f''(6) = 0, f''(x) > 0$ for $2 < x < 6, f''(x) < 0$ for $x > 6$.

30. Sketch the graph of a function g that has the following properties:
(a) g is everywhere *smooth* (continuous with a continuous first derivative);
(b) $g(0) = 0$; (c) $g'(x) < 0$ for all x;
(d) $g''(x) < 0$ for $x < 0$ and $g''(x) > 0$ for $x > 0$.

31. Sketch the graph of a function f that has the following properties:
(a) f is everywhere continuous;
(b) $f(-3) = 1$;
(c) $f'(x) < 0$ for $x < -3, f'(x) > 0$ for $x > -3, f''(x) < 0$ for $x \neq -3$.

32. Sketch the graph of a function f that has the following properties:
(a) f is everywhere continuous;
(b) $f(-4) = -3, f(0) = 0, f(3) = 2$;
(c) $f'(-4) = 0, f'(3) = 0, f'(x) > 0$ for $x < -4, f'(x) > 0$ for $-4 < x < 3, f'(x) < 0$ for $x > 3$;
(d) $f''(-4) = 0, f''(0) = 0, f''(x) < 0$ for $x < -4, f''(x) > 0$ for $-4 < x < 0, f''(x) < 0$ for $x > 0$.

33. Sketch the graph of a function f that
(a) has a continuous first derivative;
(b) is decreasing and concave up for $x < 3$;
(c) has an extremum at $(3, 1)$;
(d) is increasing and concave up for $3 < x < 5$;
(e) has an inflection point at $(5, 4)$;
(f) is increasing and concave down for $5 < x < 6$;
(g) has an extremum at $(6, 7)$;
(h) is decreasing and concave down for $x > 6$.

GC *Linear approximations provide particularly good approximations near points of inflection. Using a graphing calculator, investigate this behavior in Problems 34–36.*

34. Graph $y = \sin x$ and its linear approximation $L(x) = x$ at $x = 0$.

35. Graph $y = \cos x$ and its linear approximation $L(x) = -x + \pi/2$ at $x = \pi/2$.

36. Find the linear approximation to the curve $y = (x - 1)^5 + 3$ at its point of inflection. Graph both the function and its linear approximation in the neighborhood of the inflection point.

37. Suppose $f'(x) = (x - 2)(x - 3)(x - 4)$ and $f(2) = 2$. Sketch a graph of $y = f(x)$.

38. Suppose $f'(x) = (x - 3)(x - 2)^2(x - 1)$ and $f(2) = 0$. Sketch a graph of $y = f(x)$.

39. Suppose $h'(x) = x^2(x - 1)^2(x - 2)$ and $h(0) = 0$. Sketch a graph of $y = h(x)$.

40. Consider a general quadratic curve $y = ax^2 + bx + c$. Show that such a curve has no inflection points.

41. Show that the curve $y = ax^3 + bx^2 + cx + d$ where $a \neq 0$, has exactly one inflection point.

42. Consider a general quartic curve $y = ax^4 + bx^3 + cx^2 + dx + e$, where $a \neq 0$. What is the maximum number of inflection points that such a curve can have?

EXPL CAS *In Problems 43–47, the graph of $y = f(x)$ depends on a parameter c. Using a CAS, investigate how the extremum and inflection points depend on the value of c. Identify the values of c at which the basic shape of the curve changes.*

43. $f(x) = x^2\sqrt{x^2 - c^2}$ **44.** $f(x) = \dfrac{cx}{4 + (cx)^2}$

45. $f(x) = \dfrac{1}{(cx^2 - 4)^2 + cx^2}$ **46.** $f(x) = \dfrac{1}{x^2 + 4x + c}$

47. $f(x) = c + \sin cx$

48. What conclusions can you draw about f from the information that $f'(c) = f''(c) = 0$ and $f'''(c) > 0$?

49. Let $g(x)$ be a function that has two derivatives and satisfies the following properties:
(a) $g(1) = 1$;
(b) $g'(x) > 0$ for all $x \neq 1$;
(c) g is concave down for all $x < 1$ and concave up for all $x > 1$;
(d) $f(x) = g(x^4)$;

Sketch a possible graph of $f(x)$ and justify your answer.

50. Let $H(x)$ have three continuous derivatives, and be such that $H(1) = H'(1) = H''(1) = 0$, but $H'''(1) \neq 0$. Does $H(x)$ have a local maximum, local minimum, or a point of inflection at $x = 1$? Justify your answer.

51. In each case, is it possible for a function F with two continuous derivatives to satisfy the following properties? If so sketch such a function. If not, justify your answer.
(a) $F'(x) > 0, F''(x) > 0$, while $F(x) < 0$ for all x.
(b) $F''(x) < 0$, while $F(x) > 0$.
(c) $F''(x) < 0$, while $F'(x) > 0$.

GC **52.** Use a graphing calculator or a CAS to plot the graphs of each of the following functions on the indicated interval. Determine the coordinates of any of the global extrema and any inflection points. You should be able to give answers that are accurate to at least one decimal place. Restrict the y-axis window to $-5 \leq y \leq 5$.

(a) $f(x) = x^2 \tan x; \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(b) $f(x) = x^3 \tan x; \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(c) $f(x) = 2x + \sin x; [-\pi, \pi]$

(d) $f(x) = x - \dfrac{\sin x}{2}; [-\pi, \pi]$

GC **53.** Each of the following functions is periodic. Use a graphing calculator or a CAS to plot the graph of each of the following functions over one full period with the center of the interval located at the origin. Determine the coordinates of any of the global extrema and any inflection points. You should be able to give answers that are accurate to at least one decimal place.

(a) $f(x) = 2\sin x + \cos^2 x$ (b) $f(x) = 2\sin x + \sin^2 x$

(c) $f(x) = \cos 2x - 2\cos x$ (d) $f(x) = \sin 3x - \sin x$

(e) $f(x) = \sin 2x - \cos 3x$

54. Let f be a continuous function with $f(-3) = f(0) = 2$. If the graph of $y = f'(x)$ is as shown in Figure 11, sketch a possible graph for $y = f(x)$.

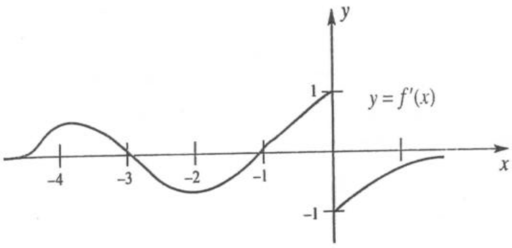


Figure 11

55. Let f be a continuous function and let f' have the graph shown in Figure 12. Sketch a possible graph for f and answer the following questions.

(a) Where is f increasing? Decreasing?

(b) Where is f concave up? Concave down?

(c) Where does f attain a local maximum? A local minimum?

(d) Where are there inflection points for f?

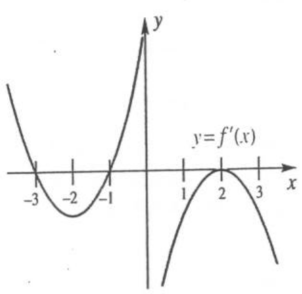


Figure 12

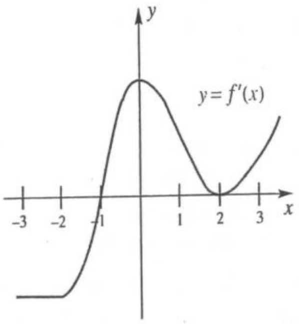


Figure 13

56. Repeat Problem 55 for Figure 13.

57. Let f be a continuous function with $f(0) = f(2) = 0$. If the graph of $y = f'(x)$ is as shown in Figure 14, sketch a possible graph for $y = f(x)$.

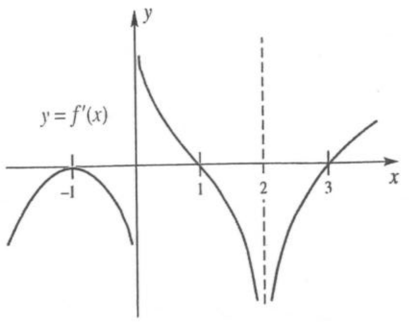


Figure 14

58. Suppose that $f'(x) = (x - 3)(x - 1)^2(x + 2)$ and $f(1) = 2$. Sketch a possible graph of f.

GC **59.** Use a graphing calculator or a CAS to plot the graph of each of the following functions on $[-1, 7]$. Determine the coordinates of any global extrema and any inflection points. You should be able to give answers that are accurate to at least one decimal place.

(a) $f(x) = x\sqrt{x^2 - 6x + 40}$

(b) $f(x) = \sqrt{|x|}(x^2 - 6x + 40)$

(c) $f(x) = \sqrt{x^2 - 6x + 40}/(x - 2)$

(d) $f(x) = \sin[(x^2 - 6x + 40)/6]$

GC **60.** Repeat Problem 59 for the following functions.

(a) $f(x) = x^3 - 8x^2 + 5x + 4$

(b) $f(x) = |x^3 - 8x^2 + 5x + 4|$

(c) $f(x) = (x^3 - 8x^2 + 5x + 4)/(x - 1)$

(d) $f(x) = (x^3 - 8x^2 + 5x + 4)/(x^3 + 1)$

Answers to Concepts Review: **1.** $f(x)$; $-f(x)$ **2.** decreasing; concave up **3.** $x = -1, x = 2, x = 3$; $y = 1$ **4.** polynomial; rational

3.6 The Mean Value Theorem for Derivatives

In geometric language, the Mean Value Theorem is easy to state and understand. It says that, if the graph of a continuous function has a nonvertical tangent line at every point between A and B, then there is at least one point C on the graph between A and B at which the tangent line is parallel to the secant line AB. In Figure 1, there is just one such point C; in Figure 2, there are several. First we state the theorem in the language of functions; then we prove it.

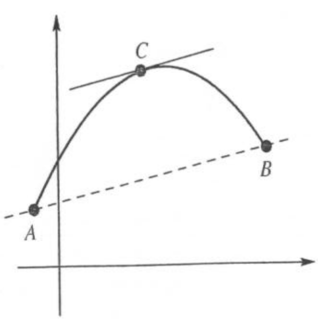


Figure 1

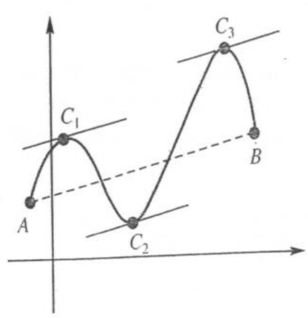


Figure 2

Theorem A Mean Value Theorem for Derivatives

If f is continuous on a closed interval $[a, b]$ and differentiable on its interior (a, b), then there is at least one number c in (a, b) where

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

or, equivalently, where

$$f(b) - f(a) = f'(c)(b - a)$$

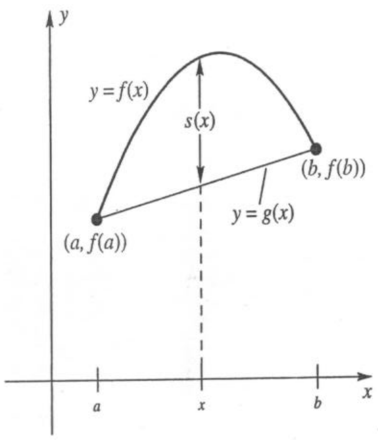


Figure 3

The Key to a Proof

The key to this proof is that c is the value at which $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ *and* $s'(c) = 0$. Many proofs have one or two key ideas; if you understand the key, you will understand the proof.

Proof Our proof rests on a careful analysis of the function $s(x) = f(x) - g(x)$, introduced in Figure 3. Here $y = g(x)$ is the equation of the line through $(a, f(a))$ and $(b, f(b))$. Since this line has slope $[f(b) - f(a)]/(b - a)$ and goes through $(a, f(a))$, the point-slope form for its equation is

$$g(x) - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

This, in turn, yields a formula for $s(x)$:

$$s(x) = f(x) - g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Note immediately that $s(b) = s(a) = 0$ and that, for x in (a, b),

$$s'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

Now we make a crucial observation. If we knew that there was a number c in (a, b) satisfying $s'(c) = 0$, we would be all done. For then the last equation would say that

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a}$$

which is equivalent to the conclusion of the theorem.

To see that $s'(c) = 0$ for some c in (a, b), reason as follows. Clearly, s is continuous on $[a, b]$, being the difference of two continuous functions. Thus, by the Max–Min Existence Theorem (Theorem 3.1A), s must attain both a maximum and a minimum value on $[a, b]$. If both of these values happen to be 0, then $s(x)$ is identically 0 on $[a, b]$, and consequently $s'(x) = 0$ for all x in (a, b), much more than we need.

If either the maximum value or the minimum value is different from 0, then that value is attained at an interior point c, since $s(a) = s(b) = 0$. Now s has a derivative at each point of (a, b), and so, by the Critical Point Theorem (Theorem 3.1B), $s'(c) = 0$. That is all we needed to know. ■

The Theorem Illustrated

EXAMPLE 1 Find the number c guaranteed by the Mean Value Theorem for $f(x) = 2\sqrt{x}$ on $[1, 4]$.

SOLUTION

$$f'(x) = 2 \cdot \frac{1}{2}x^{-1/2} = \frac{1}{\sqrt{x}}$$

and

$$\frac{f(4) - f(1)}{4 - 1} = \frac{4 - 2}{3} = \frac{2}{3}$$

Thus, we must solve

$$\frac{1}{\sqrt{c}} = \frac{2}{3}$$

The single solution is $c = \frac{9}{4}$ (Figure 4).

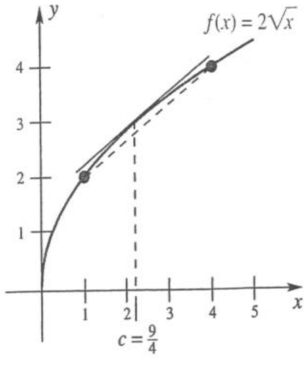


Figure 4

EXAMPLE 2 Let $f(x) = x^3 - x^2 - x + 1$ on $[-1, 2]$. Find all numbers c satisfying the conclusion to the Mean Value Theorem.

SOLUTION Figure 5 shows a graph of the function f. From this graph, it appears that there are two numbers c_1 and c_2 with the required property. We now find

$$f'(x) = 3x^2 - 2x - 1$$

and

$$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{3 - 0}{3} = 1$$

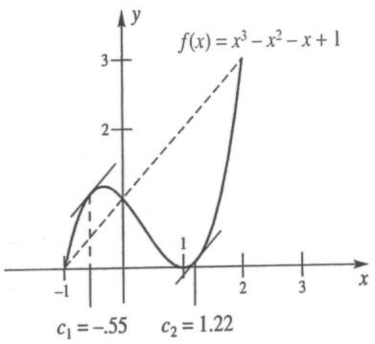


Figure 5

Therefore, we must solve

$$3c^2 - 2c - 1 = 1$$

or, equivalently,

$$3c^2 - 2c - 2 = 0$$

By the Quadratic Formula, there are two solutions, $\left(2 \pm \sqrt{4 + 24}\right)/6$, which correspond to $c_1 \approx -0.55$ and $c_2 \approx 1.22$. Both numbers are in the interval $(-1, 2)$.

EXAMPLE 3 Let $f(x) = x^{2/3}$ on $[-8, 27]$. Show that the conclusion to the Mean Value Theorem fails and figure out why.

SOLUTION

$$f'(x) = \frac{2}{3}x^{-1/3}, \qquad x \neq 0$$

and

$$\frac{f(27) - f(-8)}{27 - (-8)} = \frac{9 - 4}{35} = \frac{1}{7}$$

We must solve

$$\frac{2}{3}c^{-1/3} = \frac{1}{7}$$

which gives

$$c = \left(\frac{14}{3}\right)^3 \approx 102$$

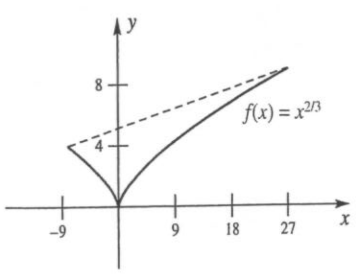


Figure 6

But $c = 102$ is not in the interval $(-8, 27)$ as required. As the graph of $y = f(x)$ suggests (Figure 6), $f'(0)$ fails to exist, so the problem is that $f(x)$ is not everywhere differentiable on $(-8, 27)$.

If the function $s(t)$ represents the position of an object at time t, then the Mean Value Theorem states that over any interval of time, there is some time for which the instantaneous velocity equals the average velocity.

EXAMPLE 4 Suppose that an object has position function $s(t) = t^2 - t - 2$. Find the average velocity over the interval $[3, 6]$ and find the time at which the instantaneous velocity equals the average velocity.

SOLUTION The average velocity over the interval $[3, 6]$ is equal to $(s(6) - s(3))/(6 - 3) = 8$. The instantaneous velocity is $s'(t) = 2t - 1$. To find the point where average velocity equals instantaneous velocity, we equate $8 = 2t - 1$ and solve to get $t = 9/2$. ■

The Theorem Used

In Section 3.2, we promised a rigorous proof of the Monotonicity Theorem (Theorem 3.2A). This is the theorem that relates the sign of the derivative of a function to whether that function is increasing or decreasing.

Proof of the Monotonicity Theorem We suppose that f is continuous on I and that $f'(x) > 0$ at each point x in the interior of I. Consider any two points x_1 and x_2 of I with $x_1 < x_2$. By the Mean Value Theorem applied to the interval $[x_1, x_2]$, there is a number c in (x_1, x_2) satisfying

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Since $f'(c) > 0$, we see that $f(x_2) - f(x_1) > 0$; that is, $f(x_2) > f(x_1)$. This is what we mean when we say that f is increasing on I.

The case where $f'(x) < 0$ on I is handled similarly. ■

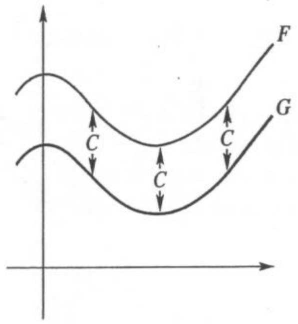


Figure 7

Our next theorem will be used repeatedly in this and the next chapter. In words, it says that *two functions with the same derivative differ by a constant*, possibly the zero constant (see Figure 7).

Theorem B

If $F'(x) = G'(x)$ for all x in (a, b), then there is a constant C such that

$$F(x) = G(x) + C$$

for all x in (a, b).

Geometry and Algebra

As with most topics in this book, you should try to see things from an algebraic and a geometrical point of view. Geometrically, Theorem B says that if F and G have the same derivative then the graph of G is a vertical translation of the graph of F.

Proof Let $H(x) = F(x) - G(x)$. Then

$$H'(x) = F'(x) - G'(x) = 0$$

for all x in (a, b). Choose x_1, as some (fixed) point in (a, b), and let x be any other point there. The function H satisfies the hypotheses of the Mean Value Theorem on the closed interval with end points x_1, and x. Thus, there is a number c between x_1 and x such that

$$H(x) - H(x_1) = H'(c)(x - x_1)$$

But $H'(c) = 0$ by hypothesis. Therefore, $H(x) - H(x_1) = 0$ or, equivalently, $H(x) = H(x_1)$ for all x in (a, b). Since $H(x) = F(x) - G(x)$, we conclude that $F(x) - G(x) = H(x_1)$. Now let $C = H(x_1)$, and we have the conclusion $F(x) = G(x) + C$. ■

Concepts Review

1. The Mean Value Theorem for Derivatives says that if f is ____ on $[a, b]$ and differentiable on ____ then there is a point c in (a, b) such that ____.

2. The function, $f(x) = |\sin x|$ would satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 1]$ but would not satisfy them on the interval $[-1, 1]$ because ____.

3. If two functions F and G have the same derivative on the interval (a, b), then there is a constant C such that ____.

4. Since $D_x(x^4) = 4x^3$, it follows that every function F that satisfies $F'(x) = 4x^3$ has the form $F(x) =$ ____.

Problem Set 3.6

In each of the Problems 1–21, a function is defined and a closed interval is given. Decide whether the Mean Value Theorem applies to the given function on the given interval. If it does, find all possible values of c; if not, state the reason. In each problem, sketch the graph of the given function on the given interval.

1. $f(x) = |x|; [1, 2]$ **2.** $g(x) = |x|; [-2, 2]$

3. $f(x) = x^2 + x; [-2, 2]$ **4.** $g(x) = (x + 1)^3; [-1, 1]$

5. $H(s) = s^2 + 3s - 1; [-3, 1]$

6. $F(x) = \dfrac{x^3}{3}; [-2, 2]$

7. $f(z) = \frac{1}{3}(z^3 + z - 4); [-1, 2]$

8. $F(t) = \dfrac{1}{t - 1}; [0, 2]$ **9.** $h(x) = \dfrac{x}{x - 3}; [0, 2]$

10. $f(x) = \dfrac{x - 4}{x - 3}; [0, 4]$ **11.** $h(t) = t^{2/3}; [0, 2]$

12. $h(t) = t^{2/3}; [-2, 2]$ **13.** $g(x) = x^{5/3}; [0, 1]$

14. $g(x) = x^{5/3}; [-1, 1]$ **15.** $S(\theta) = \sin\theta; [-\pi, \pi]$

16. $C(\theta) = \csc\theta; [-\pi, \pi]$ **17.** $T(\theta) = \tan\theta; [0, \pi]$

18. $f(x) = x + \dfrac{1}{x}; \left[-1, \frac{1}{2}\right]$ **19.** $f(x) = x + \dfrac{1}{x}; [1, 2]$

20. $f(x) = [\![x]\!]; [1, 2]$ **21.** $f(x) = x + |x|; [-2, 1]$

22. (Rolle's Theorem) *If f is continuous on [a, b] and differentiable on (a, b), and if f(a) = f(b), then there is at least one number c in (a, b) such that f′(c) = 0.* Show that Rolle's Theorem is just a special case of the Mean Value Theorem. (Michel Rolle (1652–1719) was a French mathematician.)

23. For the function graphed in Figure 8, find (approximately) all points c that satisfy the conclusion to the Mean Value Theorem for the interval $[0, 8]$.

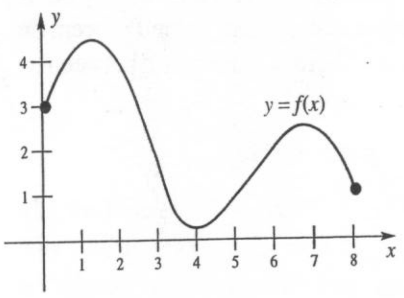


Figure 8

24. Show that if f is the quadratic function defined by $f(x) = \alpha x^2 + \beta x + \gamma, \alpha \neq 0$, then the number c of the Mean Value Theorem is always the midpoint of the given interval $[a, b]$.

25. Prove: If f is continuous on (a, b) and if $f'(x)$ exists and satisfies $f'(x) > 0$ except at one point x_0 in (a, b), then f is increasing on (a, b). *Hint:* Consider f on each of the intervals $(a, x_0]$ and $[x_0, b)$ separately.

26. Use Problem 25 to show that each of the following is increasing on $(-\infty, \infty)$.

(a) $f(x) = x^3$ (b) $f(x) = x^5$

(c) $f(x) = \begin{cases} x^3, & x \le 0 \\ x, & x > 0 \end{cases}$

27. Use the Mean Value Theorem to show that $s = 1/t$ decreases on any interval over which it is defined.

28. Use the Mean Value Theorem to show that $s = 1/t^2$ decreases on any interval to the right of the origin.

29. Prove that if $F'(x) = 0$ for all x in (a, b) then there is a constant C such that $F(x) = C$ for all x in (a, b). *Hint:* Let $G(x) = 0$ and apply Theorem B.

30. Suppose that you know that $\cos(0) = 1$, $\sin(0) = 0$, $D_x \cos x = -\sin x$, and $D_x \sin x = \cos x$, but nothing else about the sine and cosine functions. Show that $\cos^2 x + \sin^2 x = 1$. *Hint:* Let $F(x) = \cos^2 x + \sin^2 x$ and use Problem 29.

31. Prove that if $F'(x) = D$ for all x in (a, b) then there is a constant C such that $F(x) = Dx + C$ for all x in (a, b). *Hint:* Let $G(x) = Dx$ and apply Theorem B.

32. Suppose that $F'(x) = 5$ and $F(0) = 4$. Find a formula for $F(x)$. *Hint:* See Problem 31.

33. Prove: Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a)$ and $f(b)$ have opposite signs and if $f'(x) \neq 0$ for all x in (a, b), then the equation $f(x) = 0$ has one and only one solution between a and b. *Hint:* Use the Intermediate Value Theorem and Rolle's Theorem (Problem 22).

34. Show that $f(x) = 2x^3 - 9x^2 + 1 = 0$ has exactly one solution on each of the intervals $(-1, 0)$, $(0, 1)$, and $(4, 5)$. *Hint:* Apply Problem 33.

35. Let f have a derivative on an interval I. Prove that between successive distinct zeros of f' there can be at most one zero of f. *Hint:* Try a proof by contradiction and use Rolle's Theorem (Problem 22).

36. Let g be continuous on $[a, b]$ and suppose that $g''(x)$ exists for all x in (a, b). Prove that if there are three values of x in $[a, b]$ for which $g(x) = 0$ then there is at least one value of x in (a, b) such that $g''(x) = 0$.

37. Let $f(x) = (x - 1)(x - 2)(x - 3)$. Prove by using Problem 36 that there is at least one value in the interval $[0, 4]$ where $f''(x) = 0$ and two values in the same interval where $f'(x) = 0$.

38. Prove that if $|f'(x)| \le M$ for all x in (a, b) and if x_1 and x_2 are any two points in (a, b) then

$$|f(x_2) - f(x_1)| \le M|x_2 - x_1|$$

Note: A function satisfying the above inequality is said to satisfy a *Lipschitz condition* with constant M. (Rudolph Lipschitz (1832–1903) was a German mathematician.)

39. Show that $f(x) = \sin 2x$ satisfies a Lipschitz condition with constant 2 on the interval $(-\infty, \infty)$. See Problem 38.

40. A function f is said to be **nondecreasing** on an interval I if $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$ for x_1 and x_2 in I. Similarly, f is **nonincreasing** on I if $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$ for x_1 and x_2 in I.

(a) Sketch the graph of a function that is nondecreasing but not increasing.

(b) Sketch the graph of a function that is nonincreasing but not decreasing.

41. Prove that, if f is continuous on I and if $f'(x)$ exists and satisfies $f'(x) \ge 0$ on the interior of I, then f is nondecreasing on I. Similarly, if $f'(x) \le 0$, then f is nonincreasing on I.

CHAPTER 5

Integration

"There are in this world optimists who feel that any symbol that starts off with an integral sign must necessarily denote something that will have every property that they should like an integral to possess. This of course is quite annoying to us rigorous mathematicians; what is even more annoying is that by doing so they often come up with the right answer."

E. J. McShane
Bulletin of the American Mathematical Society, v. 69, p. 611, 1963

Introduction

The second fundamental problem addressed by calculus is the problem of areas, that is, the problem of determining the area of a region of the plane bounded by various curves. Like the problem of tangents considered in Chapter 2, many practical problems in various disciplines require the evaluation of areas for their solution, and the solution of the problem of areas necessarily involves the notion of limits. On the surface the problem of areas appears unrelated to the problem of tangents. However, we will see that the two problems are very closely related; one is the inverse of the other. Finding an area is equivalent to finding an antiderivative or, as we prefer to say, finding an integral. The relationship between areas and antiderivatives is called the Fundamental Theorem of Calculus. When we have proved it, we will be able to find areas at will, provided only that we can integrate (i.e., antidifferentiate) the various functions we encounter.

We would like to have at our disposal a set of integration rules similar to the differentiation rules developed in Chapter 2. We can find the derivative of any differentiable function using those differentiation rules. Unfortunately, integration is generally more difficult; indeed, some fairly simple functions are not themselves derivatives of simple functions. For example, e^{x^2} is not the derivative of any finite combination of elementary functions. Nevertheless, we will expend some effort in Section 5.6 and Sections 6.1–6.4 to develop techniques for integrating as many functions as possible. Later, in Chapter 6, we will examine how to approximate areas bounded by graphs of functions that we cannot antidifferentiate.

5.1 Sums and Sigma Notation

When we begin calculating areas in the next section, we will often encounter sums of values of functions. We need to have a convenient notation for representing sums of arbitrary (possibly large) numbers of terms, and we need to develop techniques for evaluating some such sums.

We use the symbol $\sum$ to represent a sum; it is an enlarged Greek capital letter S called *sigma*.

DEFINITION 1

Sigma notation

If m and n are integers with $m \le n$, and if f is a function defined at the integers $m, m+1, m+2, \ldots, n$, the symbol $\sum_{i=m}^{n} f(i)$ represents the sum of the values of f at those integers:

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n).$$

The explicit sum appearing on the right side of this equation is the **expansion** of the sum represented in sigma notation on the left side.

EXAMPLE 1 $$\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

The i that appears in the symbol $\sum_{i=m}^{n} f(i)$ is called an **index of summation**. To evaluate $\sum_{i=m}^{n} f(i)$, replace the index i with the integers $m, m+1, \ldots, n$, successively, and sum the results. Observe that the value of the sum does not depend on what we call the index; the index does not appear on the right side of the definition. If we use another letter in place of i in the sum in Example 1, we still get the same value for the sum:

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

The index of summation is a *dummy variable* used to represent an arbitrary point where the function is evaluated to produce a term to be included in the sum. On the other hand, the sum $\sum_{i=m}^{n} f(i)$ does depend on the two numbers m and n, called the **limits of summation**; m is the **lower limit**, and n is the **upper limit**.

EXAMPLE 2 **(Examples of sums using sigma notation)**

$$\sum_{j=1}^{20} j = 1 + 2 + 3 + \cdots + 18 + 19 + 20$$

$$\sum_{i=0}^{n} x^i = x^0 + x^1 + x^2 + \cdots + x^{n-1} + x^n$$

$$\sum_{m=1}^{n} 1 = \underbrace{1 + 1 + 1 + \cdots + 1}_{n \text{ terms}}$$

$$\sum_{k=-2}^{3} \frac{1}{k+7} = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}$$

Sometimes we use a subscripted variable a_i to denote the ith term of a general sum instead of using the functional notation $f(i)$:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_n.$$

In particular, an **infinite series** is such a sum with infinitely many terms:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

When no final term follows the $\cdots$, it is understood that the terms go on forever. We will study infinite series in Chapter 9.

When adding finitely many numbers, the order in which they are added is unimportant; any order will give the same sum. If all the numbers have a common factor, then that factor can be removed from each term and multiplied after the sum is evaluated: $ca + cb = c(a + b)$. These laws of arithmetic translate into the following *linearity* rule for finite sums; if A and B are constants, then

$$\sum_{i=m}^{n}\big(Af(i) + Bg(i)\big) = A\sum_{i=m}^{n} f(i) + B\sum_{i=m}^{n} g(i).$$

Both of the sums $\sum_{j=m}^{m+n} f(j)$ and $\sum_{i=0}^{n} f(i+m)$ have the same expansion, namely, $f(m) + f(m+1) + \cdots + f(m+n)$. Therefore, the two sums are equal.

$$\sum_{j=m}^{m+n} f(j) = \sum_{i=0}^{n} f(i+m).$$

This equality can also be derived by substituting $i + m$ for j everywhere j appears on the left side, noting that $i + m = m$ reduces to $i = 0$, and $i + m = m + n$ reduces to $i = n$. It is often convenient to make such a **change of index** in a summation.

EXAMPLE 3 Express $\sum_{j=3}^{17} \sqrt{1+j^2}$ in the form $\sum_{i=1}^{n} f(i)$.

Solution Let $j = i + 2$. Then $j = 3$ corresponds to $i = 1$ and $j = 17$ corresponds to $i = 15$. Thus,

$$\sum_{j=3}^{17} \sqrt{1+j^2} = \sum_{i=1}^{15} \sqrt{1+(i+2)^2}.$$

Evaluating Sums

There is a **closed form** expression for the sum S of the first n positive integers, namely,

$$S = \sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

To see this, write the sum forwards and backwards and add the two to get

$$\begin{array}{rcccccccccc}
S = & 1 & + & 2 & + & 3 & + \cdots + & (n-1) & + & n \\
S = & n & + & (n-1) & + & (n-2) & + \cdots + & 2 & + & 1 \\
\hline
2S = & (n+1) & + & (n+1) & + & (n+1) & + \cdots + & (n+1) & + & (n+1) = n(n+1)
\end{array}$$

The formula for S follows when we divide by 2.

It is not usually this easy to evaluate a general sum in closed form. We can only simplify $\sum_{i=m}^{n} f(i)$ for a small class of functions f. The only such formulas we will need in the next sections are collected in Theorem 1.

THEOREM 1

Summation formulas

(a) $\displaystyle\sum_{i=1}^{n} 1 = \underbrace{1+1+1+\cdots+1}_{n \text{ terms}} = n.$

(b) $\displaystyle\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}.$

(c) $\displaystyle\sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}.$

(d) $\displaystyle\sum_{i=1}^{n} r^{i-1} = 1+r+r^2+r^3+\cdots+r^{n-1} = \frac{r^n-1}{r-1}$ if $r \neq 1$.

PROOF Formula (a) is trivial; the sum of n ones is n. One proof of formula (b) was given above.

To prove (c) we write n copies of the identity

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1,$$

one for each value of k from 1 to n, and add them up:

$$\begin{array}{rcl c ccccc}
2^3 & - & 1^3 & = & 3\times 1^2 & + & 3\times 1 & + & 1\\
3^3 & - & 2^3 & = & 3\times 2^2 & + & 3\times 2 & + & 1\\
4^3 & - & 3^3 & = & 3\times 3^2 & + & 3\times 3 & + & 1\\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots\\
n^3 & - & (n-1)^3 & = & 3(n-1)^2 & + & 3(n-1) & + & 1\\
(n+1)^3 & - & n^3 & = & 3n^2 & + & 3n & + & 1\\
\hline
(n+1)^3 & - & 1^3 & = & 3\left(\sum_{i=1}^{n} i^2\right) & + & 3\left(\sum_{i=1}^{n} i\right) & + & n\\
 & & & = & 3\left(\sum_{i=1}^{n} i^2\right) & + & \dfrac{3n(n+1)}{2} & + & n.
\end{array}$$

We used formula (b) in the last line. The final equation can be solved for the desired sum to give formula (c). Note the cancellations that occurred when we added up the left sides of the n equations. The term 2^3 in the first line cancelled the -2^3 in the second line, and so on, leaving us with only two terms, the $(n+1)^3$ from the nth line and the -1^3 from the first line:

$$\sum_{k=1}^{n}\left((k+1)^3 - k^3\right) = (n+1)^3 - 1^3.$$

This is an example of what we call a **telescoping sum**. In general, a sum of the form $\sum_{i=m}^{n}\big(f(i+1) - f(i)\big)$ telescopes to the closed form $f(n+1) - f(m)$ because all but the first and last terms cancel out.

To prove formula (d), let $s = \sum_{i=1}^{n} r^{i-1}$ and subtract s from rs:

$$\begin{aligned}(r-1)s = rs - s &= (r + r^2 + r^3 + \cdots + r^n) - (1 + r + r^2 + \cdots + r^{n-1})\\ &= r^n - 1.\end{aligned}$$

The result follows on division by $r - 1$.

Other proofs of (b) – (d) are suggested in Exercises 36–38.

EXAMPLE 4 Evaluate $\displaystyle\sum_{k=m+1}^{n} (6k^2 - 4k + 3)$, where $1 \le m < n$.

Solution Using the rules of summation and various summation formulas from Theorem 1, we calculate

$$\begin{aligned}\sum_{k=1}^{n}(6k^2-4k+3) &= 6\sum_{k=1}^{n}k^2-4\sum_{k=1}^{n}k+3\sum_{k=1}^{n}1\\ &= 6\,\frac{n(n+1)(2n+1)}{6}-4\,\frac{n(n+1)}{2}+3n\\ &= 2n^3+n^2+2n\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{k=m+1}^{n}(6k^2-4k+3) &= \sum_{k=1}^{n}(6k^2-4k+3)-\sum_{k=1}^{m}(6k^2-4k+3)\\ &= 2n^3+n^2+2n-2m^3-m^2-2m.\end{aligned}$$

Remark Maple can find closed form expressions for some sums. For example,

```
> sum(i^4, i=1..n); factor(%);
```

$$\frac{1}{5}(n+1)^5-\frac{1}{2}(n+1)^4+\frac{1}{3}(n+1)^3-\frac{1}{30}n-\frac{1}{30}$$

$$\frac{1}{30}n(2n+1)(n+1)(3n^2+3n-1)$$

EXERCISES 5.1

Expand the sums in Exercises 1–6.

1. $\sum_{i=1}^{4} i^3$

2. $\sum_{j=1}^{100} \frac{j}{j+1}$

3. $\sum_{i=1}^{n} 3^i$

4. $\sum_{i=0}^{n-1} \frac{(-1)^i}{i+1}$

5. $\sum_{j=3}^{n} \frac{(-2)^j}{(j-2)^2}$

6. $\sum_{j=1}^{n} \frac{j^2}{n^3}$

Write the sums in Exercises 7–14 using sigma notation. (Note that the answers are not unique.)

7. $5+6+7+8+9$

8. $2+2+2+\cdots+2$ (200 terms)

9. $2^2-3^2+4^2-5^2+\cdots-99^2$

10. $1+2x+3x^2+4x^3+\cdots+100x^{99}$

11. $1+x+x^2+x^3+\cdots+x^n$

12. $1-x+x^2-x^3+\cdots+x^{2n}$

13. $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots+\frac{(-1)^{n-1}}{n^2}$

14. $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\cdots+\frac{n}{2^n}$

Express the sums in Exercises 15–16 in the form $\sum_{i=1}^{n} f(i)$.

15. $\sum_{j=0}^{99} \sin(j)$

16. $\sum_{k=-5}^{m} \frac{1}{k^2+1}$

Find closed form values for the sums in Exercises 17–28.

17. $\sum_{i=1}^{n} (i^2+2i)$

18. $\sum_{j=1}^{1,000} (2j+3)$

19. $\sum_{k=1}^{n} (\pi^k-3)$

20. $\sum_{i=1}^{n} (2^i-i^2)$

21. $\sum_{m=1}^{n} \ln m$

22. $\sum_{i=0}^{n} e^{i/n}$

23. The sum in Exercise 8.

24. The sum in Exercise 11.

25. The sum in Exercise 12.

26. The sum in Exercise 10. *Hint:* Differentiate the sum $\sum_{i=0}^{100} x^i$.

27. The sum in Exercise 9. *Hint:* The sum is

$$\sum_{k=1}^{49}\Big((2k)^2-(2k+1)^2\Big)=\sum_{k=1}^{49}(-4k-1).$$

28. The sum in Exercise 14. *Hint:* apply the method of proof of Theorem 1(d) to this sum.

29. Verify the formula for the value of a telescoping sum:

$$\sum_{i=m}^{n}\Big(f(i+1)-f(i)\Big)=f(n+1)-f(m).$$

Why is the word "telescoping" used to describe this sum?

In Exercises 30–32, evaluate the given telescoping sums.

30. $\displaystyle\sum_{n=1}^{10}\left(n^4-(n-1)^4\right)$ **31.** $\displaystyle\sum_{j=1}^{m}(2^j-2^{j-1})$

32. $\displaystyle\sum_{i=m}^{2m}\left(\frac{1}{i}-\frac{1}{i+1}\right)$

33. Show that $\dfrac{1}{j(j+1)}=\dfrac{1}{j}-\dfrac{1}{j+1}$, and hence evaluate $\displaystyle\sum_{j=1}^{n}\frac{1}{j(j+1)}$.

34. Figure 5.1 shows a square of side n subdivided into n^2 smaller squares of side 1. How many small squares are shaded? Obtain the closed form expression for $\sum_{i=1}^{n} i$ by considering the sum of the areas of the shaded squares.

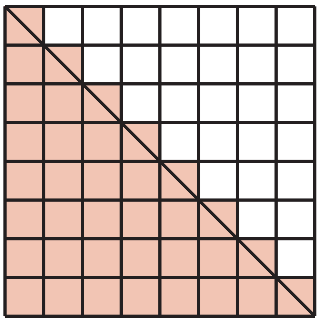


Figure 5.1

35. Write n copies of the identity $(k+1)^2-k^2=2k+1$, one for each integer k from 1 to n, and add them up to obtain the formula

$$\sum_{i=1}^{n} i=\frac{n(n+1)}{2}$$

in a manner similar to the proof of Theorem 1(c).

36. Use mathematical induction to prove Theorem 1(b).

37. Use mathematical induction to prove Theorem 1(c).

38. Use mathematical induction to prove Theorem 1(d).

39. Figure 5.2 shows a square of side $\sum_{i=1}^{n} i=n(n+1)/2$ subdivided into a small square of side 1 and $n-1$ L-shaped regions whose short edges are $2, 3, \dots, n$. Show that the area of the L-shaped region with short side i is i^3, and hence verify that

$$\sum_{i=1}^{n} i^3=\frac{n^2(n+1)^2}{4}.$$

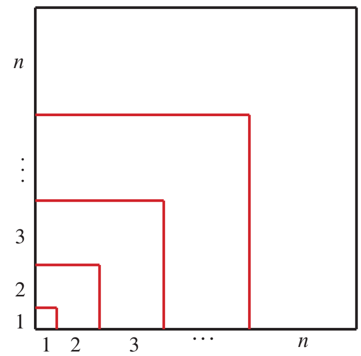


Figure 5.2

40. Write n copies of the identity

$$(k+1)^4-k^4=4k^3+6k^2+4k+1,$$

one for each integer k from 1 to n, and add them up to obtain the formula

$$\sum_{i=1}^{n} i^3=\frac{n^2(n+1)^2}{4}$$

in a manner similar to the proof of Theorem 1(c).

41. Use mathematical induction to verify the formula for the sum of cubes given in Exercise 40.

42. Extend the method of Exercise 40 to find a closed form expression for $\sum_{i=1}^{n} i^4$. You will probably want to use Maple or other computer algebra software to do all the algebra.

43. Use Maple or another computer algebra system to find $\sum_{i=1}^{n} i^k$ for $k=5, 6, 7, 8$. Observe the term involving the highest power of n in each case. Predict the highest-power term in $\sum_{i=1}^{n} i^{10}$ and verify your prediction.

5.2 Areas as Limits of Sums

We began the study of derivatives in Chapter 2 by defining what is meant by a tangent line to a curve at a particular point. We would like to begin the study of integrals by defining what is meant by the **area** of a plane region, but a definition of area is much more difficult to give than a definition of tangency. Let us assume (as we did, for example, in Section 3.3) that we know intuitively what area means and list some of its properties. (See Figure 5.3.)

(i) The area of a plane region is a nonnegative real number of *square units*.

(ii) The area of a rectangle with width w and height h is $A=wh$.

(iii) The areas of congruent plane regions are equal.

(iv) If region S is contained in region R, then the area of S is less than or equal to that of R.

(v) If region R is a union of (finitely many) nonoverlapping regions, then the area of R is the sum of the areas of those regions.

Using these five properties we can calculate the area of any **polygon** (a region bounded by straight line segments). First, we note that properties (iii) and (v) show that the area of a parallelogram is the same as that of a rectangle having the same base width and height. Any triangle can be butted against a congruent copy of itself to form a parallelogram, so a triangle has area half the base width times the height. Finally, any polygon can be subdivided into finitely many nonoverlapping triangles so its area is the sum of the areas of those triangles.

We can't go beyond polygons without taking limits. If a region has a curved boundary, its area can only be approximated by using rectangles or triangles; calculating the exact area requires the evaluation of a limit. We showed how this could be done for a circle in Section 1.1.

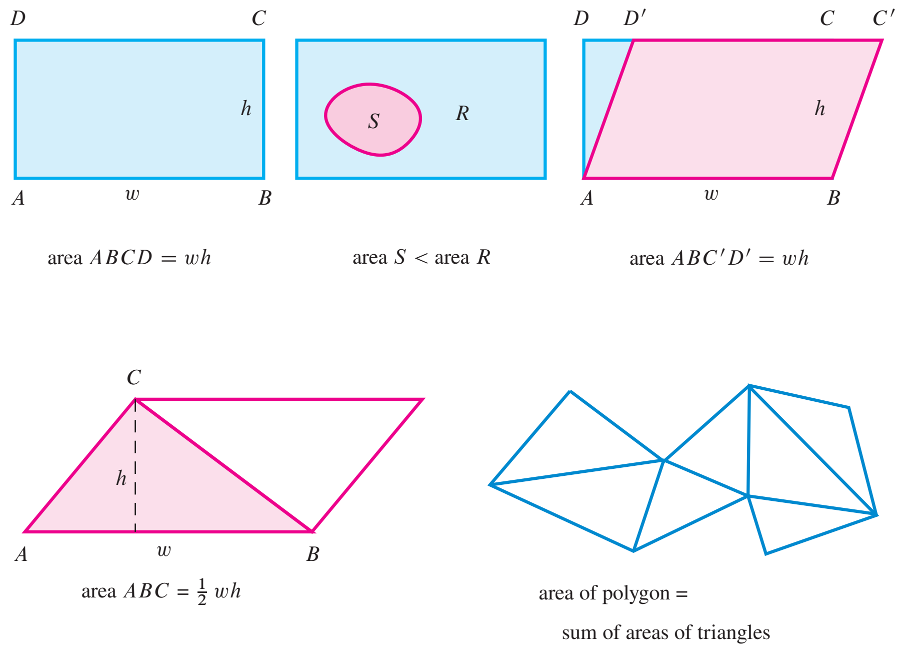


Figure 5.3 Properties of area

The Basic Area Problem

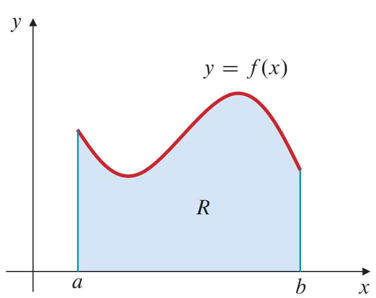


Figure 5.4 The basic area problem: find the area of region R

In this section we are going to consider how to find the area of a region R lying under the graph $y = f(x)$ of a nonnegative-valued, continuous function f, above the x-axis and between the vertical lines $x = a$ and $x = b$, where $a < b$. (See Figure 5.4.) To accomplish this, we proceed as follows. Divide the interval $[a, b]$ into n subintervals by using division points:

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b.$$

Denote by Δx_i the length of the ith subinterval $[x_{i-1}, x_i]$:

$$\Delta x_i = x_i - x_{i-1}, \qquad (i = 1, 2, 3, \ldots, n).$$

Vertically above each subinterval $[x_{i-1}, x_i]$ build a rectangle whose base has length Δx_i and whose height is $f(x_i)$. The area of this rectangle is $f(x_i)\,\Delta x_i$. Form the sum of these areas:

$$S_n = f(x_1)\,\Delta x_1 + f(x_2)\,\Delta x_2 + f(x_3)\,\Delta x_3 + \cdots + f(x_n)\,\Delta x_n = \sum_{i=1}^{n} f(x_i)\,\Delta x_i .$$

The rectangles are shown shaded in Figure 5.5 for a decreasing function f. For an increasing function, the tops of the rectangles would lie above the graph of f rather than below it. Evidently, S_n is an approximation to the area of the region R, and the approximation gets better as n increases, provided we choose the points $a = x_0 < x_1 < \cdots < x_n = b$ in such a way that the width Δx_i of the widest rectangle approaches zero.

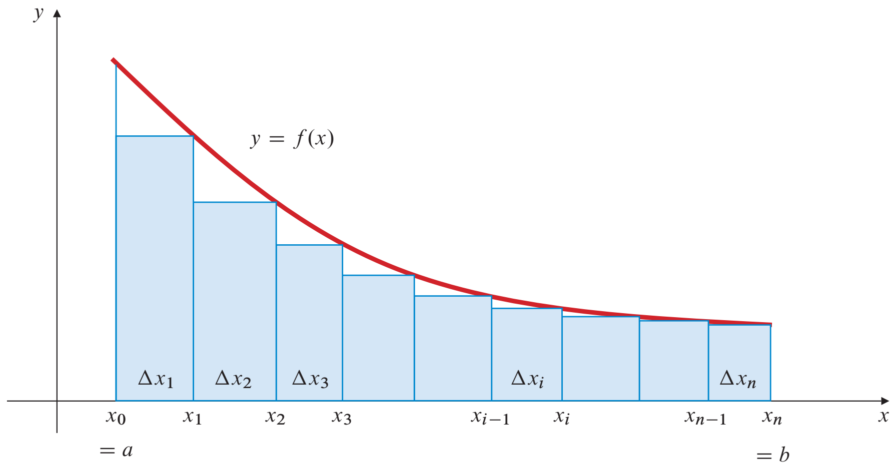


Figure 5.5 Approximating the area under the graph of a decreasing function using rectangles

Observe in Figure 5.6, for example, that subdividing a subinterval into two smaller subintervals reduces the error in the approximation by reducing that part of the area under the curve that is not contained in the rectangles. It is reasonable, therefore, to calculate the area of R by finding the limit of S_n as $n \to \infty$ with the restriction that the largest of the subinterval widths Δx_i must approach zero:

$$\text{Area of } R = \lim_{\substack{n\to\infty \\ \max \Delta x_i \to 0}} S_n.$$

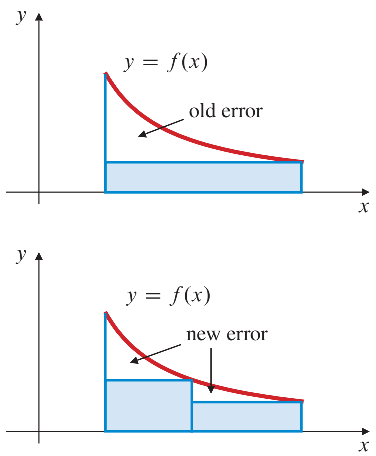


Figure 5.6 Using more rectangles makes the error smaller

Sometimes, but not always, it is useful to choose the points x_i $(0 \le i \le n)$ in $[a, b]$ in such a way that the subinterval lengths Δx_i are all equal. In this case we have

$$\Delta x_i = \Delta x = \frac{b-a}{n}, \qquad x_i = a + i\,\Delta x = a + \frac{i}{n}(b-a).$$

Some Area Calculations

We devote the rest of this section to some examples in which we apply the technique described above for finding areas under graphs of functions by approximating with rectangles. Let us begin with a region for which we already know the area so we can satisfy ourselves that the method does give the correct value.

EXAMPLE 1 Find the area A of the region lying under the straight line $y = x + 1$, above the x-axis, and between the lines $x = 0$ and $x = 2$.

Solution The region is shaded in Figure 5.7(a). It is a *trapezoid* (a four-sided polygon with one pair of parallel sides) and has area 4 square units. (It can be divided into a rectangle and a triangle, each of area 2 square units.) We will calculate the area as a limit of sums of areas of rectangles constructed as described above. Divide the interval $[0, 2]$ into n subintervals *of equal length* by points

$$x_0 = 0,\ x_1 = \frac{2}{n},\ x_2 = \frac{4}{n},\ x_3 = \frac{6}{n},\ \ldots\ x_n = \frac{2n}{n} = 2.$$

The value of $y = x + 1$ at $x = x_i$ is $x_i + 1 = \dfrac{2i}{n} + 1$ and the ith subinterval, $\left[\dfrac{2(i-1)}{n}, \dfrac{2i}{n}\right]$, has length $\Delta x_i = \dfrac{2}{n}$. Observe that $\Delta x_i \to 0$ as $n \to \infty$.

The sum of the areas of the approximating rectangles shown in Figure 5.7(a) is

$$\begin{aligned} S_n &= \sum_{i=1}^{n}\left(\frac{2i}{n}+1\right)\frac{2}{n} \\ &= \left(\frac{2}{n}\right)\left[\frac{2}{n}\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right] \qquad \text{(Use parts (b) and (a) of Theorem 1.)} \\ &= \left(\frac{2}{n}\right)\left[\frac{2}{n}\,\frac{n(n+1)}{2} + n\right] \\ &= 2\,\frac{n+1}{n} + 2. \end{aligned}$$

Therefore, the required area A is given by

$$A = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(2\,\frac{n+1}{n}+2\right) = 2+2 = 4 \text{ square units.}$$

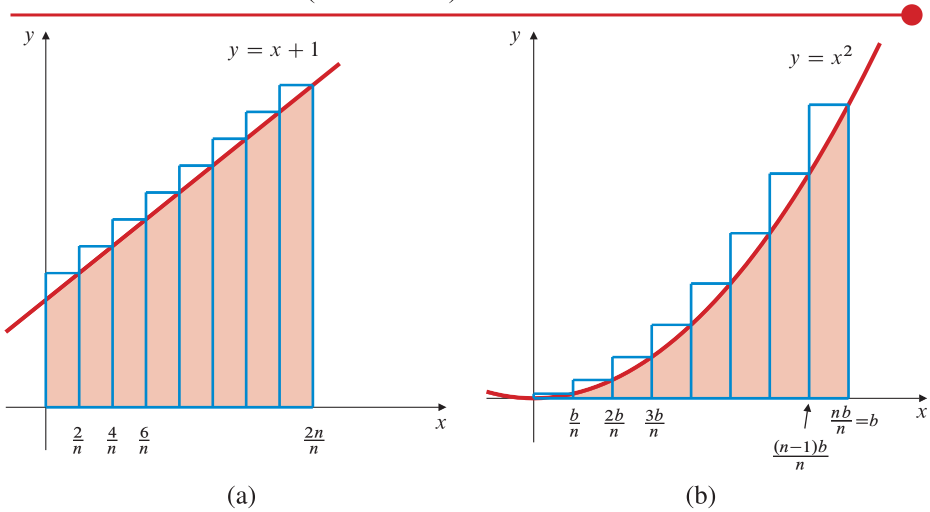


Figure 5.7
(a) The region of Example 1
(b) The region of Example 2

EXAMPLE 2 Find the area of the region bounded by the parabola $y = x^2$ and the straight lines $y = 0$, $x = 0$, and $x = b$, where $b > 0$.

Solution The area A of the region is the limit of the sum S_n of areas of the rectangles shown in Figure 5.7(b). Again we have used equal subintervals, each of length b/n. The height of the ith rectangle is $(ib/n)^2$. Thus,

$$S_n = \sum_{i=1}^{n}\left(\frac{ib}{n}\right)^2\frac{b}{n} = \frac{b^3}{n^3}\sum_{i=1}^{n} i^2 = \frac{b^3}{n^3}\,\frac{n(n+1)(2n+1)}{6},$$

by formula (c) of Theorem 1. Hence, the required area is

$$A = \lim_{n\to\infty} S_n = \lim_{n\to\infty} b^3\,\frac{(n+1)(2n+1)}{6n^2} = \frac{b^3}{3} \text{ square units.}$$

Finding an area under the graph of $y = x^k$ over an interval I becomes more and more difficult as k increases if we continue to try to subdivide I into subintervals of equal length. (See Exercise 14 at the end of this section for the case $k = 3$.) It is, however, possible to find the area for arbitrary k if we subdivide the interval I into subintervals whose lengths increase in geometric progression. Example 3 illustrates this.

EXAMPLE 3 Let $b > a > 0$, and let k be any real number except -1. Show that the area A of the region bounded by $y = x^k$, $y = 0$, $x = a$, and $x = b$ is

$$A = \frac{b^{k+1} - a^{k+1}}{k+1} \text{ square units.}$$

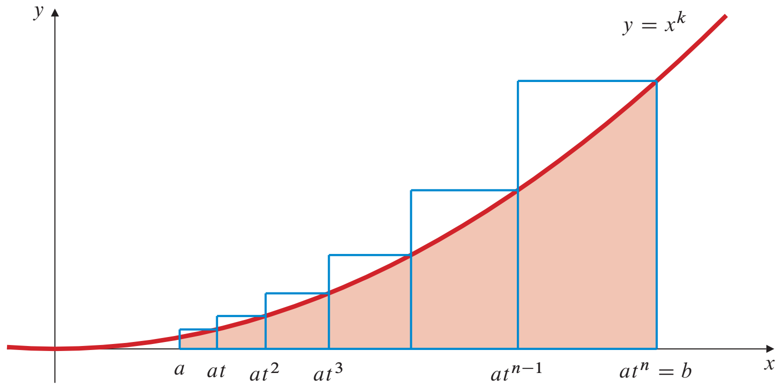


Figure 5.8 For this partition the subinterval lengths increase exponentially

BEWARE! This is a long and rather difficult example. Either skip over it or take your time and check each step carefully.

Solution Let $t = (b/a)^{1/n}$ and let

$$x_0 = a,\ x_1 = at,\ x_2 = at^2,\ x_3 = at^3,\ \ldots\ x_n = at^n = b.$$

These points subdivide the interval $[a, b]$ into n subintervals of which the ith, $[x_{i-1}, x_i]$, has length $\Delta x_i = at^{i-1}(t-1)$. If $f(x) = x^k$, then $f(x_i) = a^k t^{ki}$. The sum of the areas of the rectangles shown in Figure 5.8 is:

$$\begin{aligned}
S_n &= \sum_{i=1}^{n} f(x_i)\,\Delta x_i \\
&= \sum_{i=1}^{n} a^k t^{ki}\, at^{i-1}(t-1) \\
&= a^{k+1}(t-1)\,t^k \sum_{i=1}^{n} t^{(k+1)(i-1)} \\
&= a^{k+1}(t-1)\,t^k \sum_{i=1}^{n} r^{(i-1)} \qquad \text{where } r = t^{k+1} \\
&= a^{k+1}(t-1)\,t^k \frac{r^n - 1}{r-1} \qquad \text{(by Theorem 1(d))} \\
&= a^{k+1}(t-1)\,t^k \frac{t^{(k+1)n} - 1}{t^{k+1} - 1}.
\end{aligned}$$

Now replace t with its value $(b/a)^{1/n}$ and rearrange factors to obtain

$$\begin{aligned}
S_n &= a^{k+1}\left(\left(\frac{b}{a}\right)^{1/n} - 1\right)\left(\frac{b}{a}\right)^{k/n} \frac{\left(\dfrac{b}{a}\right)^{k+1} - 1}{\left(\dfrac{b}{a}\right)^{(k+1)/n} - 1} \\
&= \left(b^{k+1} - a^{k+1}\right) c^{k/n} \frac{c^{1/n} - 1}{c^{(k+1)/n} - 1}, \qquad \text{where } c = \frac{b}{a}.
\end{aligned}$$

Of the three factors in the final line above, the first does not depend on n, and the second, $c^{k/n}$, approaches $c^0 = 1$ as $n \to \infty$. The third factor is an indeterminate form of type $[0/0]$, which we evaluate using l'Hôpital's Rule. First let $u = 1/n$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{c^{1/n} - 1}{c^{(k+1)/n} - 1} &= \lim_{u\to 0+} \frac{c^u - 1}{c^{(k+1)u} - 1} \qquad \left[\frac{0}{0}\right] \\
&= \lim_{u\to 0+} \frac{c^u \ln c}{(k+1)\,c^{(k+1)u} \ln c} = \frac{1}{k+1}.
\end{aligned}$$

Therefore, the required area is

$$A = \lim_{n\to\infty} S_n = \left(b^{k+1} - a^{k+1}\right) \times 1 \times \frac{1}{k+1} = \frac{b^{k+1} - a^{k+1}}{k+1} \text{ square units.}$$

As you can see, it can be rather difficult to calculate areas bounded by curves by the methods developed above. Fortunately, there is an easier way, as we will discover in Section 5.5.

Remark For technical reasons it was necessary to assume $a > 0$ in Example 3. The result is also valid for $a = 0$ provided $k > -1$. In this case we have $\lim_{a\to 0+} a^{k+1} = 0$, so the area under $y = x^k$, above $y = 0$, between $x = 0$ and $x = b > 0$ is $A = b^{k+1}/(k+1)$ square units. For $k = 2$ this agrees with the result of Example 2.

EXAMPLE 4 Identify the limit $L = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{n-i}{n^2}$ as an area, and evaluate it.

Solution We can rewrite the ith term of the sum so that it depends on i/n:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 - \frac{i}{n}\right)\frac{1}{n}.$$

The terms now appear to be the areas of rectangles of base $1/n$ and heights $1 - x_i$, $(1 \le i \le n)$, where

$$x_1 = \frac{1}{n}, \quad x_2 = \frac{2}{n}, \quad x_3 = \frac{3}{n}, \quad \ldots, \quad x_n = \frac{n}{n}.$$

Thus, the limit L is the area under the curve $y = 1 - x$ from $x = 0$ to $x = 1$. (See Figure 5.9.) This region is a triangle having area $1/2$ square unit, so $L = 1/2$.

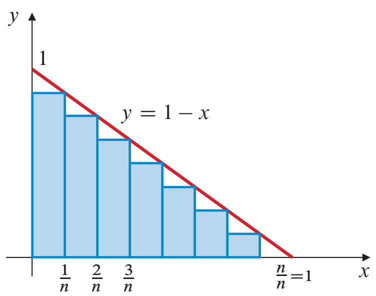


Figure 5.9 Recognizing a sum of areas

EXERCISES 5.2

Use the techniques of Examples 1 and 2 (with subintervals of equal length) to find the areas of the regions specified in Exercises 1–13.

1. Below $y = 3x$, above $y = 0$, from $x = 0$ to $x = 1$.
2. Below $y = 2x + 1$, above $y = 0$, from $x = 0$ to $x = 3$.
3. Below $y = 2x - 1$, above $y = 0$, from $x = 1$ to $x = 3$.
4. Below $y = 3x + 4$, above $y = 0$, from $x = -1$ to $x = 2$.
5. Below $y = x^2$, above $y = 0$, from $x = 1$ to $x = 3$.
6. Below $y = x^2 + 1$, above $y = 0$, from $x = 0$ to $x = a > 0$.
7. Below $y = x^2 + 2x + 3$, above $y = 0$, from $x = -1$ to $x = 2$.
8. Above $y = x^2 - 1$, below $y = 0$.
9. Above $y = 1 - x$, below $y = 0$, from $x = 2$ to $x = 4$.
10. Above $y = x^2 - 2x$, below $y = 0$.
11. Below $y = 4x - x^2 + 1$, above $y = 1$.
12. Below $y = e^x$, above $y = 0$, from $x = 0$ to $x = b > 0$.
13. Below $y = 2^x$, above $y = 0$, from $x = -1$ to $x = 1$.
14. Use the formula $\sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$, from Exercises 39–41 of Section 5.1, to find the area of the region lying under $y = x^3$, above the x-axis, and between the vertical lines at $x = 0$ and $x = b > 0$.
15. Use the subdivision of $[a, b]$ given in Example 3 to find the area under $y = 1/x$, above $y = 0$, from $x = a > 0$ to $x = b > a$. Why should your answer not be surprising?

In Exercises 16–19, interpret the given sum S_n as a sum of areas of rectangles approximating the area of a certain region in the plane and hence evaluate $\lim_{n\to\infty} S_n$.

16. $S_n = \sum_{i=1}^{n} \frac{2}{n}\left(1 - \frac{i}{n}\right)$
17. $S_n = \sum_{i=1}^{n} \frac{2}{n}\left(1 - \frac{2i}{n}\right)$
18. $S_n = \sum_{i=1}^{n} \frac{2n + 3i}{n^2}$
19. $S_n = \sum_{j=1}^{n} \frac{1}{n}\sqrt{1 - (j/n)^2}$

5.3 The Definite Integral

In this section we generalize and make more precise the procedure used for finding areas developed in Section 5.2, and we use it to define the *definite integral* of a function f on an interval I. Let us assume, for the time being, that $f(x)$ is defined and continuous on the closed, finite interval $[a, b]$. We no longer assume that the values of f are nonnegative.

Partitions and Riemann Sums

Let P be a finite set of points arranged in order between a and b on the real line, say

$$P = \{x_0, x_1, x_2, x_3, \ldots, x_{n-1}, x_n\},$$

where $a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$. Such a set P is called a **partition** of $[a, b]$; it divides $[a, b]$ into n subintervals of which the ith is $[x_{i-1}, x_i]$. We call these the subintervals of the partition P. The number n depends on the particular partition, so we write $n = n(P)$. The length of the ith subinterval of P is

$$\Delta x_i = x_i - x_{i-1}, \qquad \text{(for } 1 \le i \le n),$$

and we call the greatest of these numbers Δx_i the **norm** of the partition P and denote it $\|P\|$:

$$\|P\| = \max_{1 \le i \le n} \Delta x_i.$$

Since f is continuous on each subinterval $[x_{i-1}, x_i]$ of P, it takes on maximum and minimum values at points of that interval (by Theorem 8 of Section 1.4). Thus, there are numbers l_i and u_i in $[x_{i-1}, x_i]$ such that

$$f(l_i) \le f(x) \le f(u_i) \qquad \text{whenever } x_{i-1} \le x \le x_i.$$

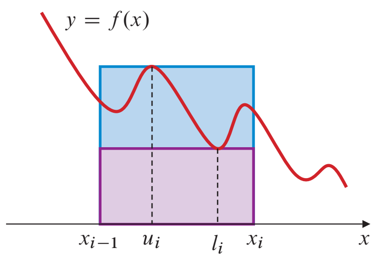


Figure 5.10

If $f(x) \ge 0$ on $[a, b]$, then $f(l_i)\,\Delta x_i$ and $f(u_i)\,\Delta x_i$ represent the areas of rectangles having the interval $[x_{i-1}, x_i]$ on the x-axis as base, and having tops passing through the lowest and highest points, respectively, on the graph of f on that interval. (See Figure 5.10.) If A_i is that part of the area under $y = f(x)$ and above the x-axis that lies in the vertical strip between $x = x_{i-1}$ and $x = x_i$, then

$$f(l_i)\,\Delta x_i \le A_i \le f(u_i)\,\Delta x_i.$$

If f can have negative values, then one or both of $f(l_i)\,\Delta x_i$ and $f(u_i)\,\Delta x_i$ can be negative and will then represent the negative of the area of a rectangle lying below the x-axis. In any event, we always have $f(l_i)\,\Delta x_i \le f(u_i)\,\Delta x_i$.

DEFINITION 2

Upper and lower Riemann sums

The **lower (Riemann) sum**, $L(f, P)$, and the **upper (Riemann) sum**, $U(f, P)$, for the function f and the partition P are defined by:

$$\begin{aligned} L(f, P) &= f(l_1)\,\Delta x_1 + f(l_2)\,\Delta x_2 + \cdots + f(l_n)\,\Delta x_n \\ &= \sum_{i=1}^{n} f(l_i)\,\Delta x_i, \\ U(f, P) &= f(u_1)\,\Delta x_1 + f(u_2)\,\Delta x_2 + \cdots + f(u_n)\,\Delta x_n \\ &= \sum_{i=1}^{n} f(u_i)\Delta x_i. \end{aligned}$$

Figure 5.11 illustrates these Riemann sums as sums of *signed* areas of rectangles; any such areas that lie below the x-axis are counted as negative.

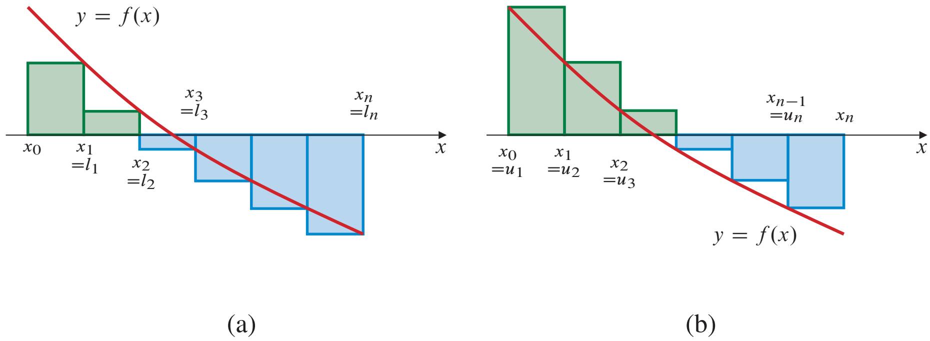


Figure 5.11 (a) A lower Riemann sum and (b) an upper Riemann sum for a decreasing function f. The areas of rectangles shaded in green are counted as positive; those shaded in blue are counted as negative

EXAMPLE 1 Calculate lower and upper Riemann sums for the function $f(x) = 1/x$ on the interval $[1, 2]$, corresponding to the partition P of $[1, 2]$ into four subintervals of equal length.

Solution The partition P consists of the points $x_0 = 1$, $x_1 = 5/4$, $x_2 = 3/2$, $x_3 = 7/4$, and $x_4 = 2$. Since $1/x$ is decreasing on $[1, 2]$, its minimum and maximum values on the ith subinterval $[x_{i-1}, x_i]$ are $1/x_i$ and $1/x_{i-1}$, respectively. Thus, the lower and upper Riemann sums are

$$L(f, P) = \frac{1}{4}\left(\frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{2}\right) = \frac{533}{840} \approx 0.6345,$$
$$U(f, P) = \frac{1}{4}\left(1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right) = \frac{319}{420} \approx 0.7595.$$

EXAMPLE 2 Calculate the lower and upper Riemann sums for the function $f(x) = x^2$ on the interval $[0, a]$ (where $a > 0$), corresponding to the partition P_n of $[0, a]$ into n subintervals of equal length.

Solution Each subinterval of P_n has length $\Delta x = a/n$, and the division points are given by $x_i = ia/n$ for $i = 0, 1, 2, \ldots, n$. Since x^2 is increasing on $[0, a]$, its minimum and maximum values over the ith subinterval $[x_{i-1}, x_i]$ occur at $l_i = x_{i-1}$ and $u_i = x_i$, respectively. Thus, the lower Riemann sum of f for P_n is

$$\begin{aligned} L(f, P_n) &= \sum_{i=1}^{n} (x_{i-1})^2 \Delta x = \frac{a^3}{n^3} \sum_{i=1}^{n} (i-1)^2 \\ &= \frac{a^3}{n^3} \sum_{j=0}^{n-1} j^2 = \frac{a^3}{n^3}\,\frac{(n-1)n(2(n-1)+1)}{6} = \frac{(n-1)(2n-1)a^3}{6n^2}, \end{aligned}$$

where we have used Theorem 1(c) of Section 5.1 to evaluate the sum of squares. Similarly, the upper Riemann sum is

$$\begin{aligned} U(f, P_n) &= \sum_{i=1}^{n} (x_i)^2 \Delta x \\ &= \frac{a^3}{n^3} \sum_{i=1}^{n} i^2 = \frac{a^3}{n^3}\,\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)a^3}{6n^2}. \end{aligned}$$

The Definite Integral

If we calculate $L(f, P)$ and $U(f, P)$ for partitions P having more and more points spaced closer and closer together, we expect that, in the limit, these Riemann sums will converge to a common value that will be the area bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$ if $f(x) \ge 0$ on $[a, b]$. This is indeed the case, but we cannot fully prove it yet.

If P_1 and P_2 are two partitions of $[a, b]$ such that every point of P_1 also belongs to P_2, then we say that P_2 is a **refinement** of P_1. It is not difficult to show that in this case

$$L(f, P_1) \le L(f, P_2) \le U(f, P_2) \le U(f, P_1);$$

adding more points to a partition increases the lower sum and decreases the upper sum. (See Exercise 18 at the end of this section.) Given any two partitions, P_1 and P_2, we can form their **common refinement** P, which consists of all of the points of P_1 and P_2. Thus,

$$L(f, P_1) \le L(f, P) \le U(f, P) \le U(f, P_2).$$

Hence, every lower sum is less than or equal to every upper sum. Since the real numbers are complete, there must exist *at least one* real number I such that

$$L(f, P) \le I \le U(f, P) \qquad \text{for every partition } P.$$

If there is *only one* such number, we will call it the definite integral of f on $[a, b]$.

DEFINITION 3

The definite integral

Suppose there is exactly one number I such that for every partition P of $[a, b]$ we have

$$L(f, P) \le I \le U(f, P).$$

Then we say that the function f is **integrable** on $[a, b]$, and we call I the **definite integral** of f on $[a, b]$. The definite integral is denoted by the symbol

$$I = \int_a^b f(x)\,dx.$$

The definite integral of $f(x)$ over $[a, b]$ is a *number*; it is not a function of x. It depends on the numbers a and b and on the particular function f, but not on the variable x (which is a **dummy variable** like the variable i in the sum $\sum_{i=1}^{n} f(i)$). Replacing x with another variable does not change the value of the integral:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$$

While we normally write the definite integral of $f(x)$ as

$$\int_a^b f(x)\,dx,$$

it is equally correct to write it as

$$\int_a^b dx\, f(x).$$

This latter form will become quite useful when we deal with multiple integrals in Chapter 14.

The various parts of the symbol $\displaystyle\int_a^b f(x)\,dx$ have their own names:

(i) $\int$ is called the **integral sign**; it resembles the letter S since it represents the limit of a sum.

(ii) a and b are called the **limits of integration**; a is the **lower limit**, b is the **upper limit**.

(iii) The function f is the **integrand**; x is the **variable of integration**.

(iv) dx is the **differential** of x. It replaces Δx in the Riemann sums. If an integrand depends on more than one variable, the differential tells you which one is the variable of integration.

EXAMPLE 3 Show that $f(x) = x^2$ is integrable over the interval $[0, a]$, where $a > 0$, and evaluate $\displaystyle\int_0^a x^2\,dx$.

Solution We evaluate the limits as $n \to \infty$ of the lower and upper sums of f over $[0, a]$ obtained in Example 2 above.

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} \frac{(n-1)(2n-1)a^3}{6n^2} = \frac{a^3}{3},$$
$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \frac{(n+1)(2n+1)a^3}{6n^2} = \frac{a^3}{3}.$$

If $L(f, P_n) \le I \le U(f, P_n)$, we must have $I = a^3/3$. Thus, $f(x) = x^2$ is integrable over $[0, a]$, and

$$\int_0^a f(x)\,dx = \int_0^a x^2\,dx = \frac{a^3}{3}.$$

For all partitions P of $[a, b]$, we have

$$L(f, P) \le \int_a^b f(x)\,dx \le U(f, P).$$

If $f(x) \ge 0$ on $[a, b]$, then the area of the region R bounded by the graph of $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$ is A square units, where $A = \int_a^b f(x)\,dx$. If $f(x) \le 0$ on $[a, b]$, the area of R is $-\int_a^b f(x)\,dx$ square units. For general f, $\int_a^b f(x)\,dx$ is the area of that part of R lying above the x-axis minus the area of that part lying below the x-axis. (See Figure 5.12.) You can think of $\int_a^b f(x)\,dx$ as a "sum" of "areas" of infinitely many rectangles with heights $f(x)$ and "infinitesimally small widths" dx; it is a limit of the upper and lower Riemann sums.

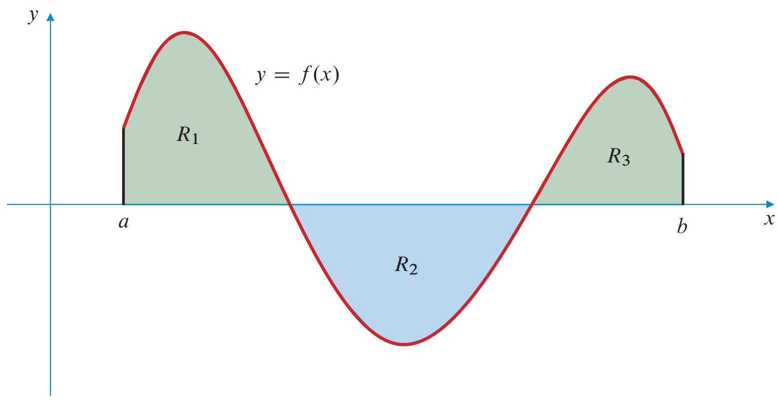


Figure 5.12 $\int_a^b f(x)\,dx$ equals area R_1 − area R_2 + area R_3

General Riemann Sums

Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$ having norm $\|P\| = \max_{1\le i\le n} \Delta x_i$. In each subinterval $[x_{i-1}, x_i]$ of P, pick a point c_i (called a *tag*). Let $c = (c_1, c_2, \ldots, c_n)$ denote the set of these tags. The sum

$$R(f, P, c) = \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$
$$= f(c_1)\,\Delta x_1 + f(c_2)\,\Delta x_2 + f(c_3)\,\Delta x_3 + \cdots + f(c_n)\,\Delta x_n$$

is called the **Riemann sum** of f on $[a, b]$ corresponding to partition P and tags c.

Note in Figure 5.13 that $R(f, P, c)$ is a sum of *signed* areas of rectangles between the x-axis and the curve $y = f(x)$. For any choice of the tags c, the Riemann sum $R(f, P, c)$ satisfies

$$L(f, P) \le R(f, P, c) \le U(f, P).$$

Therefore, if f is integrable on $[a,b]$, then its integral is the limit of such Riemann sums, where the limit is taken as the number $n(P)$ of subintervals of P increases to infinity in such a way that the lengths of all the subintervals approach zero. That is,

$$\lim_{\substack{n(P)\to\infty \\ \|P\|\to 0}} R(f,P,c) = \int_a^b f(x)\,dx.$$

As we will see in Chapter 7, many applications of integration depend on recognizing that a limit of Riemann sums is a definite integral.

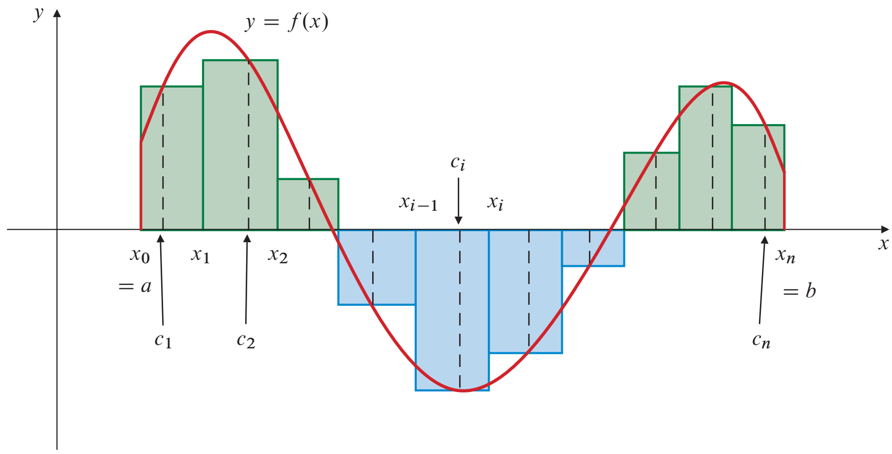


Figure 5.13 The Riemann sum $R(f,P,c)$ is the sum of areas of the rectangles shaded in green minus the sum of the areas of the rectangles shaded in blue

THEOREM 2

If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Remark The assumption that f is continuous in Theorem 2 may seem a bit superfluous since continuity was required throughout the above discussion leading to the definition of the definite integral. We cannot, however, prove this theorem yet. Its proof makes subtle use of the completeness property of the real numbers and is given in Appendix IV in the context of an extended definition of definite integral that is meaningful for a larger class of functions that are not necessarily continuous. (The integral studied in Appendix IV is called the **Riemann integral**.)

We can, however, make the following observation. In order to prove that f is integrable on $[a,b]$, it is sufficient that, for any given positive number ϵ, we should be able to find a partition P of $[a,b]$ for which $U(f,P) - L(f,P) < \epsilon$. This condition prevents there being more than one number I that is both greater than every lower sum and less than every upper sum. It is not difficult to find such a partition if the function f is nondecreasing (or if it is nonincreasing) on $[a,b]$. (See Exercise 17 at the end of this section.) Therefore, nondecreasing and nonincreasing continuous functions are integrable; so, therefore, is any continuous function that is the sum of a nondecreasing and a nonincreasing function. This class of functions includes any continuous functions we are likely to encounter in concrete applications of calculus but, unfortunately, does not include all continuous functions.

Meanwhile, in Sections 5.4 and 6.5 we will extend the definition of the definite integral to certain kinds of functions that are not continuous, or where the interval of integration is not closed or not bounded.

EXAMPLE 4 Express the limit $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(1+\frac{2i-1}{n}\right)^{1/3}$ as a definite integral.

Solution We want to interpret the sum as a Riemann sum for $f(x) = (1+x)^{1/3}$. The factor $2/n$ suggests that the interval of integration has length 2 and is partitioned into n equal subintervals, each of length $2/n$. Let $c_i = (2i-1)/n$ for $i = 1, 2, 3, \ldots, n$. As $n \to \infty$, $c_1 = 1/n \to 0$ and $c_n = (2n-1)/n \to 2$. Thus, the interval is $[0, 2]$, and the points of the partition are $x_i = 2i/n$. Observe that $x_{i-1} = (2i-2)/n < c_i < 2i/n = x_i$ for each i, so that the sum is indeed a Riemann sum for $f(x)$ over $[0, 2]$. Since f is continuous on that interval, it is integrable there, and

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(1 + \frac{2i-1}{n}\right)^{1/3} = \int_0^2 (1+x)^{1/3}\,dx.$$

EXERCISES 5.3

In Exercises 1–6, let P_n denote the partition of the given interval $[a, b]$ into n subintervals of equal length $\Delta x_i = (b-a)/n$. Evaluate $L(f, P_n)$ and $U(f, P_n)$ for the given functions f and the given values of n.

1. $f(x) = x$ on $[0, 2]$, with $n = 8$

2. $f(x) = x^2$ on $[0, 4]$, with $n = 4$

3. $f(x) = e^x$ on $[-2, 2]$, with $n = 4$

4. $f(x) = \ln x$ on $[1, 2]$, with $n = 5$

5. $f(x) = \sin x$ on $[0, \pi]$, with $n = 6$

6. $f(x) = \cos x$ on $[0, 2\pi]$, with $n = 4$

In Exercises 7–10, calculate $L(f, P_n)$ and $U(f, P_n)$ for the given function f over the given interval $[a, b]$, where P_n is the partition of the interval into n subintervals of equal length $\Delta x = (b-a)/n$. Show that

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n).$$

Hence, f is integrable on $[a, b]$. (Why?) What is $\int_a^b f(x)\,dx$?

7. $f(x) = x,\ [a, b] = [0, 1]$

8. $f(x) = 1 - x,\ [a, b] = [0, 2]$

9. $f(x) = x^3,\ [a, b] = [0, 1]$

10. $f(x) = e^x,\ [a, b] = [0, 3]$

In Exercises 11–16, express the given limit as a definite integral.

11. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\sqrt{\frac{i}{n}}$

12. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\sqrt{\frac{i-1}{n}}$

13. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n}\sin\left(\frac{\pi i}{n}\right)$

14. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\ln\left(1 + \frac{2i}{n}\right)$

15. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\tan^{-1}\left(\frac{2i-1}{2n}\right)$

16. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n^2+i^2}$

17. If f is continuous and nondecreasing on $[a, b]$, and P_n is the partition of $[a, b]$ into n subintervals of equal length ($\Delta x_i = (b-a)/n$ for $1 \le i \le n$), show that

$$U(f, P_n) - L(f, P_n) = \frac{(b-a)\big(f(b) - f(a)\big)}{n}.$$

Since we can make the right side as small as we please by choosing n large enough, f must be integrable on $[a, b]$.

18. Let $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ be a partition of $[a, b]$, and let P' be a refinement of P having one more point, x', satisfying, say, $x_{i-1} < x' < x_i$ for some i between 1 and n. Show that

$$L(f, P) \le L(f, P') \le U(f, P') \le U(f, P)$$

for any continuous function f. (*Hint:* Consider the maximum and minimum values of f on the intervals $[x_{i-1}, x_i]$, $[x_{i-1}, x']$, and $[x', x_i]$.) Hence, deduce that

$$L(f, P) \le L(f, P'') \le U(f, P'') \le U(f, P) \text{ if } P''$$

is *any* refinement of P.

5.4 Properties of the Definite Integral

It is convenient to extend the definition of the definite integral $\int_a^b f(x)\,dx$ to allow $a = b$ and $a > b$ as well as $a < b$. The extension still involves partitions P having $x_0 = a$ and $x_n = b$ with intermediate points occurring in order between these end points, so that if $a = b$, then we must have $\Delta x_i = 0$ for every i, and hence the integral is zero. If $a > b$, we have $\Delta x_i < 0$ for each i, so the integral will be negative for positive functions f and vice versa.