MSO205A PRACTICE PROBLEMS SET 11 SOLUTIONS

Question 1. Let $X_i \sim Poisson(\lambda_i), i = 1, 2, \cdots, n$ be independent RVs, with $\lambda_i > 0, \forall i$. Show that $X_1 + X_2 + \cdots + X_n \sim Poisson(\sum_{i=1}^n \lambda_i)$.

(Note: A special case of this result is the following: If $X_1, X_2, \cdots, X_n$ be a random sample from $Poisson(\lambda)$ distribution, then $X_1 + X_2 + \cdots + X_n \sim Poisson(n\lambda)$.)

Answer: Note that the MGF $M_{X_i}(t) = \exp(\lambda_i(e^t - 1)), \forall t \in \mathbb{R}$. Using independence of X_i's, we have

$$M_{X_1+X_2+\cdots+X_n}(t) = \prod_{i=1}^n M_{X_i}(t) = \exp\left(\sum_{i=1}^n \lambda_i(e^t - 1)\right), \forall t \in \mathbb{R}.$$

Since the MGF, if it exists, determines the distribution, we conclude $X_1 + X_2 + \cdots + X_n \sim Poisson(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$.

Question 2. Let $X \sim Poisson(\lambda), Y \sim Poisson(\mu)$ be independent RVs. Find the conditional distribution of X given $X + Y = k$ for $k = 0, 1, \cdots$.

Answer: We have $X + Y \sim Poisson(\lambda + \mu)$ (by problem 2 above). Then, for $k = 0, 1, \cdots$

$$\begin{aligned}
\mathbb{P}(X = x | X + Y = k) &= \frac{\mathbb{P}(X = x \text{ and } X + Y = k)}{\mathbb{P}(X + Y = k)} \\
&= \frac{\mathbb{P}(X = x \text{ and } Y = k - x)}{\mathbb{P}(X + Y = k)} \\
&= \frac{\mathbb{P}(X = x)\mathbb{P}(Y = k - x)}{\mathbb{P}(X + Y = k)}, \text{(using independence of } X \text{ and } Y) \\
&= \begin{cases} \frac{e^{-\lambda}\frac{\lambda^x}{x!}e^{-\mu}\frac{\mu^{k-x}}{(k-x)!}}{e^{-(\lambda+\mu)}\frac{(\lambda+\mu)^k}{k!}}, & \text{if } x \in \{0, 1, \cdots, k\}, \\ 0, \text{otherwise.} \end{cases} \\
&= \begin{cases} \binom{k}{x}\left(\frac{\lambda}{\lambda+\mu}\right)^x\left(1 - \frac{\lambda}{\lambda+\mu}\right)^{k-x}, & \text{if } x \in \{0, 1, \cdots, k\}, \\ 0, \text{otherwise.} \end{cases}
\end{aligned}$$

Therefore, $X \mid X + Y = k \sim Binomial(k, \frac{\lambda}{\lambda+\mu})$.

Question 3. Let X, Y be RVs defined on the same probability space. Fix $a, b, c, d \in \mathbb{R}$ and set $U = a + bX, V = c + dY$. Express $\rho(U, V)$ in terms of $\rho(X, Y)$.

Answer: We have,

$$Cov(U, V) = Cov(a + bX, c + dY) = \mathbb{E}[(a + bX)(c + dY)] - \mathbb{E}(a + bX)\mathbb{E}(c + dY) = bdCov(X, Y)$$

and

$$Var(U) = \mathbb{E}[(a + bX) - (a + b\mathbb{E}X)]^2 = b^2Var(X), Var(V) = d^2Var(Y).$$

If $b = 0$ or $d = 0$, $\rho(U, V)$ is not defined. If $bd \neq 0$, then

$$\rho(U, V) = \frac{Cov(U, V)}{\sqrt{Var(U)Var(V)}} = \frac{bd}{|bd|}\rho(X, Y).$$

Question 4. Compute the factorial moments for Negative Binomial and Hypergeometric distribution.

Answer: Suppose X follows Negative Binomial(r, p) distribution. Then for any non-negative integer k,

$$\begin{aligned}
&\mathbb{E}X(X-1)\cdots(X-k+1) \\
&= \sum_{j=0}^{\infty} j(j-1)\cdots(j-k+1)\binom{j+r-1}{j}p^r(1-p)^j \\
&= \sum_{j=k}^{\infty} j(j-1)\cdots(j-k+1)\binom{j+r-1}{j}p^r(1-p)^j \\
&= p^r(1-p)^k r(r+1)\cdots(r+k-1)\sum_{j=k}^{\infty}\binom{j+r-1}{j-k}(1-p)^{j-k} \\
&= p^r(1-p)^k r(r+1)\cdots(r+k-1)\sum_{j=0}^{\infty}\binom{j+k+r-1}{j}(1-p)^j \\
&= r(r+1)\cdots(r+k-1)p^{-k}(1-p)^k.
\end{aligned}$$

Suppose X follows the Hypergeometric distribution with parameters N, M and n. Then for any non-negative integer k,

$$\begin{aligned}
&\mathbb{E}X(X-1)\cdots(X-k+1) \\
&= \sum_{j=\max\{0,n-(N-M)\}}^{\min\{n,M\}} j(j-1)\cdots(j-k+1)\frac{\binom{M}{j}\binom{N-M}{n-j}}{\binom{N}{n}}
\end{aligned}$$

The above factorial moment is zero for all $k > \min\{n, M\}$. For $1 \leq k \leq \min\{n, M\}$,

$$\begin{aligned}
&\mathbb{E}X(X-1)\cdots(X-k+1) \\
&= \frac{M(M-1)\cdots(M-k+1)}{\binom{N}{n}}\sum_{j=\max\{k,n-(N-M)\}}^{\min\{n,M\}}\binom{M-k}{j-k}\binom{N-M}{n-j} \\
&= \frac{M(M-1)\cdots(M-k+1)}{\binom{N}{n}}\binom{N-k}{n-k}
\end{aligned}$$

Question 5. Suppose a pair of fair die are rolled seven times independently. Find the probability that the sum of the dots obtained is 12 once and 8 twice.

Answer: Let X_1 and X_2 denote the number of times 12 and 8 appear in the seven rolls, respectively.

Since the die are fair, the probability that 12 appears in a roll is $\frac{1}{36}$ (the only favourable event being $(6,6)$) and the corresponding probability for 8 is $\frac{5}{36}$ (favourable events being $(2,6),(3,5),(4,4),(5,3),(6,2)$). Hence, (X_1, X_2) follows the Multinomial distribution with parameters 7 and $\frac{1}{36}, \frac{5}{36}, \frac{36-1-5}{36}$. Therefore, the required probability is

$$\mathbb{P}(X_1 = 1, X_2 = 2) = \frac{7!}{1!2!4!}\left(\frac{1}{36}\right)^1\left(\frac{5}{36}\right)^2\left(\frac{5}{6}\right)^4 = \frac{5 \times 6 \times 7}{2}\frac{5^6}{6^8} = \frac{7}{2}\frac{5^7}{6^7}.$$

Question 6. If $X_1, X_2, \cdots, X_n$ are i.i.d. *Geometric*(p) RVs, for some $p > 0$, then find the distribution of $X_1 + X_2 + \cdots + X_n$.

Answer: First consider the case $n = 2$. Recall that *Geometric*(p) distribution is the same as the negative Binomial$(1, p)$ distribution.

Consider any two independent RVs X and Y with distributions negative Binomial(m, p) and negative Binomial(n, p) respectively. Since the support of X and Y are exactly the set of non-negative integers, $X + Y$ is also discrete with the support contained in the set of non-negative integers. Now for any non-negative integer k, using the independence of X and Y, we have

$$\begin{aligned}
\mathbb{P}(X + Y = k) &= \sum_{j=0}^{k} \mathbb{P}(X = j, Y = k - j) \\
&= \sum_{j=0}^{k} \mathbb{P}(X = j)\mathbb{P}(Y = k - j) \\
&= \sum_{j=0}^{k} \binom{j+m-1}{j}\binom{k-j+n-1}{k-j} p^{m+n}(1-p)^k \\
&= \binom{k+m+n-1}{k} p^{m+n}(1-p)^k,
\end{aligned}$$

i.e. $X + Y$ follows the negative Binomial$(m + n, p)$ distribution. In the setting of the hypothesis, $X_1 + X_2$ follows negative Binomial$(2, p)$ distribution.

Consider the case $n = 3$. Here, $X_1 + X_2$ and X_3 are independent. Hence, using the above case $X_1 + X_2 + X_3$ follows the negative Binomial$(3, p)$ distribution.

By the principle of Mathematical Induction, $X_1 + X_2 + \cdots + X_n$ follows the negative Binomial(n, p) distribution.

Question 7. Let X be a continuous RV with p.d.f. f_X. If X is symmetric about $\mu \in \mathbb{R}$ and if $\mathbb{E}X$ exists, show that

$$\mathbb{E}X = \mu = m = \frac{\mathfrak{z}_{0.25} + \mathfrak{z}_{0.75}}{2},$$

where $m, \mathfrak{z}_{0.25}, \mathfrak{z}_{0.75}$ denotes the median, the lower and upper quartiles respectively. Assume that these are unique.

Answer: Let f_X and F_X denote the p.d.f. and the DF of X, respectively. Since, $X-\mu \stackrel{d}{=} \mu - X$, we have $\mathbb{E}(X-\mu)=\mathbb{E}(\mu-X)$, which implies $\mathbb{E}X=\mu$.
Now, $F_{X-\mu}(x) = F_{\mu-X}(x), \forall x \in \mathbb{R}$. But, $F_{X-\mu}(x) = \mathbb{P}(X-\mu \le x) = \mathbb{P}(X \le \mu + x)$ and $F_{\mu-X}(x) = \mathbb{P}(\mu - X \le x) = \mathbb{P}(X \ge \mu - x)$. In particular, putting $x=0$, we have $\mathbb{P}(X \le \mu) = \mathbb{P}(X \ge \mu) = 1-\mathbb{P}(X<\mu) = 1-\mathbb{P}(X\le\mu)$, since, $\mathbb{P}(X=\mu)=0$ by continuity of F_X and therefore, $F_X(\mu)=\frac{1}{2}$. Hence, $m=\mu$.
Now, $\mathbb{P}(X \le \mathfrak{z}_{0.75}) = 0.75$ gives $\mathbb{P}(X-\mu \le \mathfrak{z}_{0.75}-\mu) = 0.75$ or $\mathbb{P}(\mu - X \le \mathfrak{z}_{0.75}-\mu) = 0.75$, since $X-\mu \stackrel{d}{=} \mu - X$. Therefore, $\mathbb{P}(X \ge 2\mu - \mathfrak{z}_{0.75}) = 0.75$. Using the continuity of F_X, $\mathbb{P}(X \le 2\mu - \mathfrak{z}_{0.75}) = 1-0.75 = 0.25$. Therefore, $\mathfrak{z}_{0.25} = 2\mu - \mathfrak{z}_{0.75}$, which completes the proof.

Question 8. Let X be an RV with $\mathbb{E}|X| < \infty$. Consider the function $g:\mathbb{R}\to\mathbb{R}$ defined by $g(x) := \mathbb{E}|X-x|, x\in\mathbb{R}$. Show that $g(m) \le g(x), \forall x \in \mathbb{R}$, where m is the median of X. (Note: This shows that the mean deviation is minimized at the median).

Answer: Let X be a continuous RV with p.d.f. f_X. Note that $\int_{-\infty}^{m} f_X(t)\,dt = \int_m^\infty f_X(t)\,dt = \frac{1}{2}$. If $x<m$, then

$$\begin{aligned}
&g(x)-g(m)\\
&= \int_{-\infty}^{\infty} |t-x| f_X(t)\,dt - \int_{-\infty}^{\infty} |t-m| f_X(t)\,dt\\
&= \int_{-\infty}^{x} (x-t) f_X(t)\,dt + \int_{x}^{m} (t-x) f_X(t)\,dt + \int_{m}^{\infty} (t-x) f_X(t)\,dt - \int_{-\infty}^{m} (m-t) f_X(t)\,dt - \int_{m}^{\infty} (t-m) f_X(t)\,dt\\
&= \int_{-\infty}^{x} (x-m) f_X(t)\,dt + \int_{x}^{m} (2t-x-m) f_X(t)\,dt + \int_{m}^{\infty} (m-x) f_X(t)\,dt\\
&\ge (m-x)[\mathbb{P}(X\ge m) - \mathbb{P}(X \le x)] + (2x-x-m)\mathbb{P}(x \le X \le m)\\
&= (m-x)[\mathbb{P}(X \ge m) - \mathbb{P}(X \le m)]\\
&= 0.
\end{aligned}$$

A similar argument shows $g(x) \ge g(m)$ if $x > m$. This proves the case when X is continuous with a p.d.f. f_X.
The proof for the discrete case goes in an analogus manner.

Question 9. Let X and Y be i.i.d. $N(0,1)$ RVs. Identify the distribution of $\frac{X}{Y}$ and $\frac{X}{|Y|}$.

Answer: Since, $Y^2 \sim \chi_1^2$, using the independence of X and Y^2, we have

$$\frac{X}{|Y|} = \frac{X}{\sqrt{\frac{Y^2}{1}}} \sim t_1.$$

Note that $\mathbb{P}(Y=0)=0$. For $z\in\mathbb{R}$,

$$\mathbb{P}\left(\frac{X}{Y} \le z\right) = \mathbb{P}\left(\frac{X}{Y} \le z, Y>0\right) + \mathbb{P}\left(\frac{X}{Y} \le z, Y<0\right) = \mathbb{P}\left(\frac{X}{|Y|} \le z, Y>0\right) + \mathbb{P}\left(-\frac{X}{|Y|} \le z, Y<0\right)$$

Using the symmetry $(X,Y) \stackrel{d}{=} (-X,Y)$,

$$\mathbb{P}\left(-\frac{X}{|Y|} \leq z, Y < 0\right) = \mathbb{P}\left(\frac{X}{|Y|} \leq z, Y < 0\right)$$

and hence

$$F_{\frac{X}{Y}}(z) = \mathbb{P}\left(\frac{X}{Y} \leq z\right) = \mathbb{P}\left(\frac{X}{|Y|} \leq z, Y > 0\right) + \mathbb{P}\left(\frac{X}{|Y|} \leq z, Y < 0\right) = \mathbb{P}\left(\frac{X}{|Y|} \leq z\right).$$

So, $\frac{X}{Y} \sim t_1$.

<u>*Question*</u> 10. Let $X \sim F_{m,n}$. Identify the distribution of $\frac{n}{n+mX}$.

Answer: Let $Y_1 \sim \chi^2_m$ and $Y_2 \sim \chi^2_n$ be independent RVs. Then $X \stackrel{d}{=} \frac{n}{m}\frac{Y_1}{Y_2} \sim F_{m,n}$. Therefore,

$$\frac{n}{n+mX} \stackrel{d}{=} \frac{Y_2}{Y_1+Y_2}.$$

Since $Y_1 \sim Gamma(\frac{m}{2}, 2)$ and $Y_2 \sim Gamma(\frac{n}{2}, 2)$ are independent, using Question 5 of Problem set 10 we have

$$\frac{n}{n+mX} \stackrel{d}{=} \frac{Y_2}{Y_1+Y_2} \sim Beta\left(\frac{n}{2}, \frac{m}{2}\right).$$

<u>*Question*</u> 11. Let X and Y be i.i.d. *Exponential*(λ) RVs, for some $\lambda > 0$. Identify the distribution of $\frac{X}{Y}$.

Answer: Since $X \sim Exponential(\lambda) = Gamma(1,\lambda)$, we have $\frac{2X}{\lambda} \sim Gamma(1,2) = \chi^2_2$. Similarly, $\frac{2Y}{\lambda} \sim \chi^2_2$. Then,

$$\frac{X}{Y} = \left(\frac{\frac{2X}{\lambda}}{2}\right)\left(\frac{\frac{2Y}{\lambda}}{2}\right)^{-1} \sim F_{2,2}.$$

<u>*Question*</u> 12. Verify that for a discrete RV X with the DF

$$F_X(x) := \begin{cases} 0, \text{ if } x < 0, \\ \frac{1}{2}, \text{ if } 0 \leq x < 1, \\ 1, \text{ if } x \geq 1, \end{cases}$$

the median is not unique. Given $p \in (0,1)$, construct an example of discrete RV X (by specifying the DF F_X or the p.m.f. f_X) such that the quantile of order p is not unique.

Answer: For any $x \in (0,1)$, we have $F_X(x) = \frac{1}{2}$ and $\mathbb{P}(X = x) = 0$. Hence the inequality $\frac{1}{2} \leq F_X(x) \leq \frac{1}{2} + \mathbb{P}(X = x)$ is satisfied. Therefore, any $x \in (0,1)$ is a median for X.

Moreover, $\frac{1}{2} = F_X(0) \leq \frac{1}{2} + \mathbb{P}(X = 0)$ and hence 0 is also a median for X. Therefore, the median is not unique in this case.

Given $p \in (0,1)$, consider the RV Y given by the DF F_Y (or equivalently, the p.m.f. f_Y)

$$F_Y(x) := \begin{cases} 0, \text{ if } x < 0, \\ p, \text{ if } 0 \leq x < 1, \\ 1, \text{ if } x \geq 1. \end{cases} \quad , \quad f_Y(x) = \begin{cases} p, \text{ if } x = 0, \\ 1-p, \text{ if } x = 1, \\ 0, \text{ otherwise.} \end{cases}$$

Similar to above argument for X, the quantile of order p for Y is not unique.

Question 13. Verify that for a continuous RV X with the DF

$$F_X(x) := \begin{cases} 0, \text{ if } x < 0, \\ \frac{x}{2}, \text{ if } 0 \leq x < 1, \\ \frac{1}{2}, \text{ if } 1 \leq x < 2, \\ \frac{x-1}{2}, \text{ if } 2 \leq x < 3, \\ 1, \text{ if } x \geq 3, \end{cases}$$

the median is not unique. Given $p \in (0,1)$, construct an example of continuous RV X (by specifying the DF F_X or the p.d.f. f_X) such that the quantile of order p is not unique.

Answer: For a continuous RV X, a median x is a solution to the equation $F_X(x) = \frac{1}{2}$. In this case, this equation is solved by all $x \in [1,2]$. Hence, the median is not unique in this case.

Given $p \in (0,1)$, consider the RV Y given by the DF F_Y (or equivalently, the p.d.f. f_Y)

$$F_Y(x) = \begin{cases} 0, \text{ if } x < 0, \\ xp, \text{ if } 0 \leq x < 1, \\ p, \text{ if } 1 \leq x < 2, \\ p(3-x) + x - 2, \text{ if } 2 \leq x < 3, \\ 1, \text{ if } x \geq 3. \end{cases} \quad , \quad f_Y(x) = \begin{cases} p, \forall x \in [0,1) \\ 1-p, \forall x \in [2,3) \\ 0, \text{ otherwise.} \end{cases}$$

Similar to above argument for X, the quantile of order p for Y is not unique.

Question 14. Consider the set

$$A := \left\{ t = (t_1, t_2, \ldots, t_p) \in \mathbb{R}^p : \mathbb{E}\left(e^{\sum_{i=1}^p t_i X_i}\right) < \infty \right\}$$

for a given random vector $X = (X_1, X_2, \cdots, X_p)$ and look at $\Psi_X(t) := \ln M_X(t), t \in A$. Verify that

$$\left[\frac{\partial^2}{\partial t_i \partial t_j} \Psi_X(t)\right]_{(t_1, t_2 \ldots t_p) = (0, \ldots, 0)} = Cov(X_i, X_j).$$

Answer: For $i \neq j$ with $i, j \in \{1, \ldots, p\}$, we have

$$\begin{aligned}
&Cov\left(X_i, X_j\right) \\
&= \mathbb{E}\left(X_i X_j\right) - \mathbb{E}\left(X_i\right) \mathbb{E}\left(X_j\right) \\
&= \left[\frac{\partial^2}{\partial t_i \partial t_j} M_X(t)\right]_{(t_1, t_2 \ldots t_p)=(0,\ldots,0)} - \left[\frac{\partial}{\partial t_i} M_X(t)\right]_{(t_1, t_2 \ldots t_p)=(0,\ldots,0)} \left[\frac{\partial}{\partial t_j} M_X(t)\right]_{(t_1, t_2 \ldots t_p)=(0,\ldots,0)} \\
&= \left[\frac{\partial^2}{\partial t_i \partial t_j} \Psi_X(t)\right]_{(t_1, t_2 \ldots t_p)=(0,\ldots,0)},
\end{aligned}$$

where the last equality follows from the one-dimensional case.
When $i = j$, we have $Cov\left(X_i, X_j\right) = Var(X_i) = \mathbb{E}\left(X_i^2\right) - \left(\mathbb{E}X_i\right)^2 = \left[\frac{\partial^2}{\partial t_i^2} \Psi_X(t)\right]_{(t_1, t_2 \ldots t_p)=(0,\ldots,0)}$, also similar to the one-dimensional case.