MSO205A PRACTICE PROBLEMS SET 12 SOLUTIONS

Question 1. Let X_1, X_2, X_3 be a random sample from $Bernoulli(p)$ distribution, for some $p \in (0,1)$. Find the p.m.f. of $X_{(2)}$.

Answer: $X_{(2)}$ is a discrete RV supported on $\{0,1\}$. Now,

$$\begin{aligned}
&\mathbb{P}(X_{(2)} = 0) \\
&= \mathbb{P}(\text{the second smallest of } X_1, X_2, X_3 \text{ is } 0) \\
&= \mathbb{P}(\text{at least two of } X_1, X_2, X_3 \text{ are } 0) \\
&= \mathbb{P}(\text{exactly two of } X_1, X_2, X_3 \text{ are } 0) \\
&+ \mathbb{P}(\text{all three of } X_1, X_2, X_3 \text{ are } 0) \\
&= \binom{3}{2}(1-p)^2 p + (1-p)^3, \text{(using independence of } X_1, X_2, X_3) \\
&= (1-p)^2(2p+1).
\end{aligned}$$

Similarly, $\mathbb{P}(X_{(2)} = 1) = \binom{3}{2}p^2(1-p) + p^3 = p^2(3-2p)$. Therefore, $X_{(2)} \sim Bernoulli(p^2(3-2p))$.

Question 2. Let $X_1, \cdots, X_n$ be a random sample from $Uniform(0,1)$ distribution. Identify the distribution of $X_{(r)}$ for $r = 1, \cdots, n$.

Answer: We have $F_{X_{(r)}}(x) = \mathbb{P}(X_{(r)} \le x) = 0, \forall x \le 0$ and $F_{X_{(r)}}(x) = \mathbb{P}(X_{(r)} \le x) = 1, \forall x \ge 1$. For $x \in (0,1)$, $\mathbb{P}(X_i \le x) = x$ and $\mathbb{P}(X_i > x) = 1 - x, \forall i = 1, \cdots, n$. If Y is the number of $X_i, i = 1, \cdots, n$ which fall in $(0, x]$, then $Y \sim Binomial(n, x)$. Then,

$$F_{X_{(r)}}(x) = \mathbb{P}(X_{(r)} \le x) = \mathbb{P}(Y \ge r) = \sum_{k=r}^{n} \binom{n}{k} x^k (1-x)^{n-x}.$$

In this case,

$$\frac{d}{dx} F_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} x^{r-1}(1-x)^{n-r}.$$

Since $F_{X_{(r)}}$ is differentiable everywhere except possibly at the points $0, 1$, we conclude that $X_{(r)}$ is a continuous RV with the p.d.f.

$$f_{X_{(r)}} = \begin{cases} \frac{n!}{(r-1)!(n-r)!} x^{r-1}(1-x)^{n-r}, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

and hence $X_{(r)} \sim Beta(r, n-r+1)$.

Question 3. Let $X_1, \cdots, X_n$ be a random sample from a distribution given by a p.d.f. f. Find the joint p.d.f. of $(X_{(r)}, X_{(s)})$ of $1 \le r < s \le n$.

Answer: In the lecture notes, we have already discussed that the joint p.d.f. of $(X_{(1)}, \cdots, X_{(n)})$ is given by

$$g(y_1, \cdots, y_n) = \begin{cases} n! \prod_{i=1}^n f(y_i), & \text{if } y_1 < \cdots < y_n, \\ 0, & \text{otherwise.} \end{cases}$$

Now, integrating out co-ordinate variables $y_i, i \in 1, 2, \cdots, n, i \neq r, i \neq s$, we have

$$g_{X_{(r)}, X_{(s)}}(y_r, y_s) = \begin{cases} \frac{n!}{(r-1)!(s-r-1)!(n-s)!}(F(y_r))^{r-1}[F(y_s) - F(y_r)]^{s-r-1}(1 - F(y_s))^{n-s} f(y_r) f(y_s), & \text{if } y_r < y_s, \\ 0, & \text{otherwise} \end{cases}$$

where F denotes the common DF for $X_1, \cdots, X_n$.

Question 4. Let $Y \sim N_p(b, K)$. Then for any $c \in \mathbb{R}^n$ and a $n \times p$ real matrix B, consider the n dimensional random vector $Z = c + BY$. Show that $Z \sim N_n(c + Bb, BKB^t)$.

Answer: The Joint MGF of $Y = (Y_1, Y_2, \cdots, Y_p)$ is given by

$$M_Y(u) = \exp\left(u^t b + \frac{1}{2} u^t K u\right), \forall u \in \mathbb{R}^p.$$

We compute the MGF of $Z = (Z_1, \cdots, Z_n)$. For any $v \in \mathbb{R}^n$, we have

$$\begin{aligned} M_Z(v) &= \mathbb{E} \exp(v^t Z) = \mathbb{E} \exp[v^t(c + BY)] \\ &= \exp(v^t c) \mathbb{E} \exp[(v^t B) Y] \\ &= \exp(v^t c) M_Y(B^t v) \\ &= \exp(v^t c) \exp\left(v^t B b + \frac{1}{2} v^t B K B^t v\right) \\ &= \exp\left(v^t(c + Bb) + \frac{1}{2} v^t B K B^t v\right). \end{aligned}$$

Identifying the distribution of Z through the joint MGF, we conclude $Z \sim N_n(c + Bb, BKB^t)$.

Question 5. Let $Y \sim N_p(b, K)$ with K being invertible. Then show that $\sum_{j=1}^p \lambda_j (Y_j - b_j) = 0$ for some scalars $\lambda_1, \lambda_2, \cdots, \lambda_p$ if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_p = 0$.

Answer: Any Variance-Coviance matrix is positive semi-definite. In our case, since K is invertible, we conclude that K is positive definite. Then there exists an orthogonal matrix A such that $D = A^t K A$ is a diagonal matrix, with eigen-values of K as the diagonal entries.

If possible, let $\lambda^t (Y - b) = 0$ for some $\lambda \in \mathbb{R}^p$. Then $\lambda^t (Y - b)(Y - b)^t \lambda = 0$ and in particular,

$$\lambda^t K \lambda = \mathbb{E}\left(\lambda^t (Y - b)(Y - b)^t \lambda\right) = 0.$$

Since K is positive definite, we conclude that $\lambda = 0 \in \mathbb{R}^p$.

If $\lambda = 0 \in \mathbb{R}^p$, we have $\lambda^t (Y - b) = 0$. This concludes the proof.

Question 6. Let $c := \sum_{m=1}^{\infty} m^{-3} < \infty$. Then the function $f : \mathbb{R} \to [0,1]$ given by

$$f(x) = \begin{cases} \frac{1}{c} x^{-3}, \text{ if } x \in \{1, 2, \cdots\} \\ 0, \text{ otherwise} \end{cases}$$

is a p.m.f.. Let X be a discrete RV with this p.m.f. and consider the following sequence of RVs $\{X_n\}_n$ defined by

$$X_n = \begin{cases} X, \text{ if } X \leq n, \\ 0, \text{ otherwise} \end{cases}, \forall n.$$

Show that the sequence of RVs $\{X_n\}_n$ converges in first mean to X, but not in the second mean.

Answer: We have, for any positive integer n

$$\frac{1}{c} \sum_{m=1}^{n} \frac{m}{m^3} = \frac{1}{c} \sum_{m=1}^{n} \frac{1}{m^2} < \infty,$$

and

$$\frac{1}{c} \sum_{m=1}^{n} \frac{m^2}{m^3} = \frac{1}{c} \sum_{m=1}^{n} \frac{1}{m} < \infty.$$

Therefore, $\mathbb{E}X_n$ and $\mathbb{E}X_n^2$ both exist, for all n. Moreover,

$$\mathbb{E}|X_n - X| = \frac{1}{c} \sum_{m=n+1}^{\infty} \frac{m}{m^3} = \frac{1}{c} \sum_{m=n+1}^{\infty} \frac{1}{m^2} \xrightarrow{n \to \infty} 0,$$

but

$$\frac{1}{c} \sum_{m=n+1}^{\infty} \frac{m^2}{m^3} = \frac{1}{c} \sum_{m=n+1}^{\infty} \frac{1}{m} = \infty, \forall n.$$

Hence, $\{X_n\}_n$ converges to X in first mean, but not in the second mean.