Solutions :- Mid Semester Exam, MSO205.

1. True.

2 points

Let $Y = |X|^q$ & $g(y) = |y|^{\frac{p}{q}}$

Since $q \in (0, p) \Rightarrow \frac{p}{q} > 1$

Hence, $g(\cdot)$ is a convex function (Proof: Not required as proof is straight forward)

Therefore by Jensen's inequality, we have

$$g(E(Y)) \leq E(g(Y))$$

3 points.

$$\Rightarrow |E(Y)|^{\frac{p}{q}} \leq E|Y|^{\frac{p}{q}} \quad \text{since } g(y) = |y|^{\frac{p}{q}}$$

$$\Rightarrow |E(|X|^q)|^{\frac{p}{q}} \leq E|X|^p \quad \text{since } Y = |X|^q.$$

$$\Rightarrow \{E|X|^q\}^{\frac{p}{q}} \leq E|X|^p \text{ as } E|X|^q \geq 0.$$

$$\Rightarrow \{E|X|^q\}^{\cancel{\frac{p}{q}}} \leq \{E|X|^p\}^{\frac{q}{p}}$$

2. True.

Note that in this prob. space,

2 points. $P(A) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$ & $P(B) = \frac{2}{3} - 0 = \frac{2}{3}$.

1 point. Further, $A \cap B = \left(\frac{1}{2}, \frac{3}{4}\right) \cap \left(0, \frac{2}{3}\right)$

$$:= \left(\frac{1}{2}, \frac{2}{3}\right).$$

Now, in this prob. space,

2 points.

$P(A \cap B) = P\left(\left(\frac{1}{2}, \frac{2}{3}\right)\right) = \frac{1}{6}$.

So, $P(A \cap B) = \frac{1}{6} = P(A) \times P(B) = \frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$.

Hence, A & B are independent.

3. X : RV.

$$M_X(t) = \frac{1}{7}e^{2t} + \frac{3}{7}e^{3t} + \frac{2}{7}e^{5t} + \frac{1}{7}e^{8t}.$$

$$\Rightarrow P[X=2] = \frac{1}{7},\ P[X=3] = \frac{3}{7},\ P[X=5] = \frac{2}{7}$$

$$\& \ P[X=8] = \frac{1}{7} \quad \text{———} \quad (*)$$

Sine MGF uniquely determines the ~~CDF~~ PMF of discrete R.V. (if MGF exists), so $(*)$ is the p.m.f. of X.

3 points.

Hence,

$$P[X=2] + P[X=7] + P[X=8]$$

2 points.

$$= \frac{1}{7} + 0 + \frac{1}{7} = \frac{2}{7}$$

4. Note that by the def^{n}. of induced prob. measure

1 point $\{$ $P \circ X^{-1}(A) = P[\omega \in \Omega : X(\omega) \in A]$.

$$= P[\omega \in \Omega : 0 \le \tan(\pi(\omega - \tfrac{1}{2})) \le 1].$$

$$= P[\omega \in \Omega : \tan^{-1} 0 \le \pi(\omega - \tfrac{1}{2}) \le \tan^{-1}(1)].$$

4 points.

$$= P[\omega \in \Omega : 0 \le \pi(\omega - \tfrac{1}{2}) \le \tfrac{\pi}{4}] \quad \text{since } \Omega = [0,1].$$

$$= P[\omega \in \Omega : 0 \le \omega - \tfrac{1}{2} \le \tfrac{1}{4}]$$

$$= P[\omega \in \Omega : \tfrac{1}{2} \le \omega \le \tfrac{3}{4}] = P((\tfrac{1}{2}, \tfrac{3}{4})) = \tfrac{3}{4} - \tfrac{1}{2}$$

$$= \tfrac{1}{4}.$$

Alternative Method for 4:-

Let us take the set $B = (-\infty, x]$.

Now, to derive the CDF of X, consider

$P_0 X^{-1}(B) =$ ~~$P[\omega \in \Omega : X(\omega) \in I$~~

$= P[\omega \in \Omega : X(\omega) \in B]$.

1 points

$= P[\omega \in \Omega : -\infty < \tan(\pi(\omega - \frac{1}{2})) \le x]$.

$= P[\omega \in \Omega : -\frac{\pi}{2} < \pi(\omega - \frac{1}{2}) \le \tan^{-1} x]$.

$= P[\omega \in \Omega : -\frac{1}{2} < \omega - \frac{1}{2} \le \frac{1}{\pi} \tan^{-1} x]$.

$= P[\omega \in \Omega : 0 < \omega \le \frac{1}{2} + \frac{1}{\pi} \tan^{-1} x]$.

2 points

$= P((\frac{1}{2} + \frac{1}{\pi} \tan^{-1} x, 0)) = \underbrace{\frac{1}{2} + \frac{1}{\pi} \tan^{-1} x}$.

It is differentiable f^n of x

So, the p.d.f. of X is

$$f_X(x) = \frac{d}{dx}\left[\frac{1}{2} + \frac{1}{\pi} \tan^{-1} x\right]$$

2 points

$$= \frac{1}{\pi} \frac{1}{1+x^2}, \quad x \in \mathbb{R}.$$

Therefore ~~Now~~, $P_0 X^{-1}(A) = \int_A f_X(x)\,dx = \int_0^1 \frac{1}{\pi(1+x^2)}\,dx$

$$= \frac{1}{\pi}\left[\tan^{-1} x\right]_0^1$$

$$= \frac{1}{\pi} \times \frac{\pi}{4} = \frac{1}{4}.$$

5. Not true.

Observe that

$$\sum_{n\geq 1}^{\infty} P[|X| > \sqrt{n}]$$

1 point

$$= \sum_{n=1}^{\infty} P[X^2 > n].$$

2 points

$$\leq \int_0^{\infty} P[X^2 > x]\, dx$$

2 points

$= EX^2$ since X^2 is a non-negative R.V.

$< \infty$ → Proof is not required.

Providing proof is also fine.

5 points

OR proper counter example

True

6. Note that $X \sim Bin(n, \frac{1}{2})$.

1 point

Now, $M_X(t) = E[e^{tX}]$

$$= \sum_{k=0}^{n} e^{tk} \left(\tfrac{1}{2}\right)^{n} \binom{n}{k}.$$

2 points.

$$= \sum_{k=0}^{n} \binom{n}{k} e^{tk} \left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{2}\right)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{e^t}{2}\right)^k \left(\frac{1}{2}\right)^{n-k}$$

$$= \left(\frac{1}{2} + \frac{e^t}{2}\right)^n. \quad \text{—— } (*)$$

2 points

Denote $Y = n - X$.

Now, we have $M_Y(t) = E[e^{t(n-X)}]$

$= e^{nt} M_{-t}(X) = e^{nt}\left(\frac{1}{2} + \frac{e^{-t}}{2}\right)^n$

$= \left(\frac{1}{2} + \frac{e^t}{2}\right)^n. \quad \text{—— } (**)$

$(*)$ & $(**)$ $\Rightarrow X \overset{d}{=} n - X \Leftrightarrow X - \frac{n}{2} \Leftrightarrow \frac{n}{2} - X$