MSO205A PRACTICE PROBLEMS SET 5 SOLUTIONS

Question 1. Is it possible that the following function $f : \mathbb{R} \to \mathbb{R}$ is a p.m.f. of an RV? If yes, also compute the corresponding DF. Take f as follows: for some constant $c \in \mathbb{R}$,

$$f(x) = \begin{cases} \frac{c}{(2x-1)(2x+1)}, & \text{if } x \in \{1, 2, 3, \cdots\}, \\ 0, & \text{otherwise.} \end{cases}$$

Answer: For all $x \in \{1, 2, 3, \cdots\}$, we have $2x - 1 > 0$ and $2x + 1 > 0$ and hence by definition, the function f takes non-negative values, provided $c \geq 0$.

For f to be a p.m.f., we must have

$$\sum_{x \in \{1,2,3,\cdots\}} \frac{c}{(2x-1)(2x+1)} = 1.$$

Since, $\frac{1}{(2x-1)(2x+1)} = \frac{1}{2}[\frac{1}{2x-1} - \frac{1}{2x+1}]$, the above condition yields $\frac{c}{2} = 1$, i.e. $c = 2$.

Therefore, f is a p.m.f., provided $c = 2$. In this case the corresponding DF is computed by the formula

$$\begin{aligned} F(x) &:= \sum_{t \in \{1,2,3,\cdots\} \cap (-\infty, x]} f(t), \forall x \in \mathbb{R} \\ &= \begin{cases} 0, & \text{if } x < 1, \\ \sum_{t=1}^{y} \frac{1}{(2t-1)(2t+1)}, & \text{if } x \in [y, y+1), y \in \{1, 2, 3, \cdots\} \end{cases} \\ &= \begin{cases} 0, & \text{if } x < 1, \\ 1 - \frac{1}{2y+1}, & \text{if } x \in [y, y+1), y \in \{1, 2, 3, \cdots\} \end{cases} \end{aligned}$$

Question 2. Consider the following p.d.f. discussed in the lecture notes. Let X be a continuous RV with p.d.f. $f_X : \mathbb{R} \to \mathbb{R}$ given by

$$f_X(x) = \begin{cases} -\frac{4}{3}x, & \text{if } x \in [-1, 0), \\ \frac{x^2}{8}, & \text{if } x \in [0, 2], \\ 0, & \text{otherwise.} \end{cases}$$

Write down the DF F_X. Compute $\mathbb{P}(X > 0), \mathbb{P}(X \leq 1)$ and $\mathbb{P}(X > 0 \mid X \leq 1)$.

Answer: Since X is a continuous RV with p.d.f. f_X, we have

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

$$= \begin{cases} 0, \text{ if } x < -1, \\ \int_{-1}^{x} f_X(t)\,dt, \text{ if } x \in [-1,0), \\ \int_{-1}^{0} f_X(t)\,dt + \int_{0}^{x} f_X(t)\,dt, \text{ if } x \in [0,2], \\ 1, \text{ if } x > 2, \end{cases}$$

$$= \begin{cases} 0, \text{ if } x < -1, \\ \int_{-1}^{x} -\frac{4}{3}t\,dt, \text{ if } x \in [-1,0), \\ \int_{-1}^{0} -\frac{4}{3}t\,dt + \int_{0}^{x} \frac{t^2}{8}\,dt, \text{ if } x \in [0,2], \\ 1, \text{ if } x > 2, \end{cases}$$

$$= \begin{cases} 0, \text{ if } x < -1, \\ \frac{2}{3}(1-x^2), \text{ if } x \in [-1,0), \\ \frac{2}{3} + \frac{x^3}{24} \text{ if } x \in [0,2], \\ 1, \text{ if } x > 2, \end{cases}$$

We have

$$\mathbb{P}(X > 0) = 1 - F_X(0) = 1 - \frac{2}{3} = \frac{1}{3},$$

$$\mathbb{P}(X \le 1) = F_X(1) = \frac{2}{3} + \frac{1}{24} = \frac{17}{24},$$

$$\mathbb{P}(X \ge 0 \mid X \le 1) = \frac{\mathbb{P}((X \ge 0) \cap (X \le 1))}{\mathbb{P}(X \le 1)} = \frac{24}{17}\mathbb{P}(0 \le X \le 1) = \frac{24}{17}[F_X(1) - F_X(0)] = \frac{24}{17}\frac{1}{24} = \frac{1}{17}.$$

Question 3. Suppose that an RV X has the DF F_X given by any of the following functions. In each case, check if X is discrete or continuous. If so, also find the corresponding p.m.f./p.d.f. (as appropriate) of X.

(a)

$$F_X(x) := \begin{cases} 0, \text{ if } x < 0, \\ \frac{x}{4}, \text{ if } 0 \le x < 1, \\ \frac{x}{3}, \text{ if } 1 \le x < 2, \\ \frac{3x}{8}, \text{ if } 2 \le x < \frac{5}{2}, \\ 1, \text{ if } x \ge \frac{5}{2}. \end{cases}$$

(b)

$$F_X(x) = \begin{cases} 0, \text{ if } x < 0, \\ \frac{1}{2}, \text{ if } 0 \le x < 2, \\ \frac{3}{4}, \text{ if } 2 \le x < 3, \\ 1, \text{ if } x \ge 3. \end{cases}$$

(c)

$$F_X(x) := \begin{cases} 0, \text{ if } x < 0, \\ \frac{x^2}{2}, \text{ if } 0 \le x < 1, \\ \frac{x}{2}, \text{ if } 1 \le x < 2, \\ 1, \text{ if } x \ge 2. \end{cases}$$

Answer: (a) The DF F_X is continuously differentiable on $(-\infty, 0), (0, 1), (1, 2), (2, \frac{5}{2})$ and on $(\frac{5}{2}, \infty)$. We check for discontinuities at the points $0, 1, 2$ and $\frac{5}{2}$.

We have

$$\begin{aligned} F_X(0) - F_X(0-) &= 0 - \lim_{x \uparrow 0} \frac{x}{4} = 0, \\ F_X(1) - F_X(1-) &= \frac{1}{3} - \lim_{x \uparrow 1} \frac{x}{4} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}, \\ F_X(2) - F_X(2-) &= \frac{3}{4} - \lim_{x \uparrow 2} \frac{x}{3} = \frac{3}{4} - \frac{2}{3} = \frac{1}{12}, \\ F_X\left(\frac{5}{2}\right) - F_X\left(\frac{5}{2}-\right) &= 1 - \lim_{x \uparrow \frac{5}{2}} \frac{3x}{8} = 1 - \frac{15}{16} = \frac{1}{16}. \end{aligned}$$

Since F_X has jumps at $1, 2$ and at $\frac{5}{2}$, X can not be continuous. Moreover, the sum of the jumps is $\frac{1}{12} + \frac{1}{12} + \frac{1}{16} = \frac{11}{48} < 1$ and hence X is not discrete.

(b) The DF F_X is a constant on $(-\infty, 0), [0, 2), [2, 3)$ and on $[3, \infty)$ and hence is continuous on $(-\infty, 0), (0, 2), (2, 3)$ and on $(3, \infty)$. We check for discontinuities at $0, 2$ and 3. We have

$$F_X(0) - F_X(0-) = \frac{1}{2} - 0 = \frac{1}{2}, \quad F_X(2) - F_X(2-) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}, \quad F_X(3) - F_X(3-) = 1 - \frac{3}{4} = \frac{1}{4}.$$

Therefore, F_X is discontinuous at $0, 2$ and 3 with the sum of the jumps $[F_X(0)-F_X(0-)]+[F_X(2)-F_X(2-)]+[F_X(3)-F_X(3-)]=1$. Hence X is discrete with p.m.f.

$$f_X(x) = \begin{cases} F_X(0)-F_X(0-), \text{ if } x=0, \\ F_X(2)-F_X(2-), \text{ if } x=2, \\ F_X(3)-F_X(3-), \text{ if } x=3, \\ 0, \text{ otherwise.} \end{cases} = \begin{cases} \frac{1}{2}, \text{ if } x=0, \\ \frac{1}{4}, \text{ if } x=2 \text{ or } 3, \\ 0, \text{ otherwise.} \end{cases}$$

(c) The DF F_X is continuously differentiable on $(-\infty,0),(0,1),(1,2)$ and on $(2,\infty)$. We check for discontinuities at the points $0, 1$ and 2.
We have

$$F_X(0)-F_X(0-) = 0 - \lim_{x\uparrow 0} 0 = 0,$$
$$F_X(1)-F_X(1-) = \frac{1}{2} - \lim_{x\uparrow 1}\frac{x^2}{2} = \frac{1}{2}-\frac{1}{2} = 0,$$
$$F_X(2)-F_X(2-) = 1 - \lim_{x\uparrow 2}\frac{x}{2} = 1-1 = 0.$$

Therefore, F_X is continuous. Further F_X is differentiable everywhere except possibly at a finitely many points $0, 1$ and 2. Here, X is continuous with the p.d.f. f_X given by

$$f_X(x) = \begin{cases} F_X'(x), \forall x \in (-\infty,0)\cup(0,1)\cup(1,2)\cup(2,\infty) \\ 0, \forall x \in \{0,1,2\} \end{cases}$$
$$= \begin{cases} 0, \forall x \in (-\infty,0]\cup\{1\}\cup[2,\infty) \\ x, \forall x \in (0,1) \\ \frac{1}{2}, \forall x \in (1,2) \end{cases}$$