MSO205A PRACTICE PROBLEMS SET 6 SOLUTIONS

Question 1. Verify that the following distribution function $F_X : \mathbb{R} \to \mathbb{R}$ corresponds to a discrete RV X. Take

$$F(x) := \begin{cases} 0, & \text{if } x < 0, \\ \frac{1}{2}, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } x \geq 1. \end{cases}$$

Compute the p.m.f. f_X.

Answer: The DF F_X is a constant on $(-\infty, 0), [0, 1)$ and on $[1, \infty)$ and hence is continuous on $(-\infty, 0), (0, 1)$ and on $(1, \infty)$. We check for discontinuities at 0 and 1. We have

$$F_X(0) - F_X(0-) = \frac{1}{2} - 0 = \frac{1}{2}, \quad F_X(1) - F_X(1-) = 1 - \frac{1}{2} = \frac{1}{2}.$$

Therefore, F_X is discontinuous at 0 and 1 with the sum of the jumps $[F_X(0) - F_X(0-)] + [F_X(1) - F_X(1-)] = 1$. Hence X is discrete with p.m.f.

$$f_X(x) = \begin{cases} F_X(0) - F_X(0-), & \text{if } x = 0, \\ F_X(1) - F_X(1-), & \text{if } x = 1, \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} \frac{1}{2}, & \text{if } x = 0 \text{ or } 1, \\ 0, & \text{otherwise.} \end{cases}$$

Question 2. Verify that the following distribution function $F_X : \mathbb{R} \to \mathbb{R}$ corresponds to a continuous RV X. Take

$$F(x) := \begin{cases} 0, & \text{if } x < 0, \\ \frac{x}{2}, & \text{if } 0 \leq x < 1, \\ \frac{1}{2}, & \text{if } 1 \leq x < 2, \\ \frac{x-1}{2}, & \text{if } 2 \leq x < 3, \\ 1, & \text{if } x \geq 3. \end{cases}$$

Compute the p.d.f. f_X.

Answer: The DF F_X is continuously differentiable on $(-\infty, 0), (0, 1), (1, 2), (2, 3)$ and on $(3, \infty)$. The p.d.f. f_X is given by

$$f_X(x) = \begin{cases} F_X'(x), \forall x \in (-\infty, 0) \cup (0, 1) \cup (1, 2) \cup (2, 3) \cup (3, \infty) \\ 0, \forall x \in \{0, 1, 2, 3\} \end{cases}$$
$$= \begin{cases} 0, \forall x \in (-\infty, 0] \cup [1, 2] \cup [3, \infty) \\ \frac{1}{2}, \forall x \in (0, 1) \cup (2, 3) \end{cases}$$

Question 3. Let X be a continuous RV with the p.d.f.

$$f_X(x) := \begin{cases} \exp(-x), & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find the support of X.

Answer: We consider the following two cases.

(a) Let $x < 0$. Then for $h > 0$ such that $x < x + h < 0$, we have

$$\mathbb{P}(x - h < X \leq x + h) = \int_{x-h}^{x+h} f_X(t)\, dt = 0$$

and hence x is not in the support of X.

(b) Let $x \geq 0$. Then for any $h > 0$,

$$\mathbb{P}(x - h < X \leq x + h) = \int_{x-h}^{x+h} f_X(t)\, dt = \int_{\max\{x-h,0\}}^{x+h} \exp(-t)\, dt > 0$$

and hence x is in the support of X.

Therefore, $S_X = [0, \infty)$.

Question 4. Consider a discrete RV X with the p.m.f.

$$f_X(x) := \begin{cases} \frac{1}{4}\left(\frac{3}{4}\right)^x, & \text{if } x \in \{0, 1, 2, \cdots\}, \\ 0, & \text{otherwise.} \end{cases}$$

Consider the RV $Y = \frac{X}{X+1}$.

(a) First find the DF F_Y and then compute the p.m.f. f_Y.

(b) First find the p.m.f. f_Y and then compute the DF F_Y.

Answer: Given that X is discrete with support $S_X = \{0, 1, 2, \cdots\}$. Then $Y = h(X) = \frac{X}{X+1}$ is also discrete with support $S_Y = \{\frac{n}{n+1} : n = 1, 2, \cdots\} = \{0, \frac{1}{2}, \frac{2}{3}, \cdots\}$, where $h : (-1, \infty) \to \mathbb{R}$ given by $h(x) = \frac{x}{x+1}$ is a differentiable function with $h'(x) = \frac{x}{(x+1)^2} > 0$ on $(0, \infty)$. In particular, h is strictly increasing and one-to-one on S_X with the inverse function given by $h^{-1} : S_Y \to S_X, h(y) = \frac{y}{1-y}, \forall y \in S_Y$.

(a) We have

$$F_Y(y) = \sum_{\substack{x \in S_X \\ h(x) \leq y}} f_X(x) = \sum_{\substack{x \in S_X \\ x \leq \frac{y}{1-y}}} f_X(x) = F_X\left(\frac{y}{1-y}\right).$$

Now,

$$F_X(x) = \begin{cases} 0, \text{ if } x < 0, \\ \sum_{j=0}^{k} \frac{1}{4}\left(\frac{3}{4}\right)^j, \text{ if } x \in [k, k+1), k \in \{0, 1, 2, \cdots\} \end{cases} = \begin{cases} 0, \text{ if } x < 0, \\ 1 - \left(\frac{3}{4}\right)^{k+1}, \text{ if } x \in [k, k+1), k \in \{0, 1, 2, \cdots\} \end{cases}$$

Therefore,

$$F_Y(y) = \begin{cases} 0, \text{ if } y < 0, \\ 1 - \left(\frac{3}{4}\right)^{k+1}, \text{ if } y \in \left[\frac{k}{k+1}, \frac{k+1}{k+2}\right), k \in \{0, 1, 2, \cdots\} \\ 1, \text{ if } y \geq 1. \end{cases}$$

Then

$$f_Y(y) = \begin{cases} 0, \text{ if } y \notin S_Y, \\ F_Y(y) - F_Y(y-), \text{ if } y \in S_Y. \end{cases} = \begin{cases} 0, \text{ if } y \notin S_Y, \\ \frac{1}{4}\left(\frac{3}{4}\right)^{\frac{y}{1-y}}, \text{ if } y \in S_Y. \end{cases}$$

(b) Continue with S_Y and h as described above. For $y \in S_Y$,

$$\mathbb{P}(Y = y) = \mathbb{P}\left(\frac{X}{1+X} = y\right) = \mathbb{P}\left(X = \frac{y}{1-y}\right) = \frac{1}{4}\left(\frac{3}{4}\right)^{\frac{y}{1-y}}.$$

Therefore,

$$f_Y(y) = \begin{cases} 0, \text{ if } y \notin S_Y, \\ \frac{1}{4}\left(\frac{3}{4}\right)^{\frac{y}{1-y}}, \text{ if } y \in S_Y. \end{cases}$$

Hence,

$$F_Y(y) = \sum_{\substack{t \in S_Y \\ t \leq y}} f_Y(t) = \begin{cases} 0, \text{ if } y < 0, \\ \sum\limits_{\substack{t \in S_Y \\ t \leq \frac{k}{k+1}}} f_Y(t), \text{ if } y \in \left[\dfrac{k}{k+1}, \dfrac{k+1}{k+2}\right), k \in \{0, 1, 2, \cdots\} \\ 1, \text{ if } y \geq 1. \end{cases}$$

$$= \begin{cases} 0, \text{ if } y < 0, \\ 1 - \left(\frac{3}{4}\right)^{k+1}, \text{ if } y \in \left[\frac{k}{k+1}, \frac{k+1}{k+2}\right), k \in \{0, 1, 2, \cdots\} \\ 1, \text{ if } y \geq 1. \end{cases}.$$

Question 5. Consider a continuous RV X with the p.d.f.

$$f_X(x) := \begin{cases} \frac{1}{2}, \text{ if } x \in (-1, 0), \\ \frac{1}{3}, \text{ if } x \in (0, \frac{3}{2}), \\ 0, \text{ otherwise.} \end{cases}$$

Consider the RV $Y = X^4$.

(a) First find the DF F_Y and then compute the p.d.f. f_Y.
(b) First find the p.d.f. f_Y and then compute the DF F_Y.

Answer: (a) Since $Y = X^4$ with $\mathbb{P}(X \in (-1, 0) \cap (0, \frac{3}{2}) = 1$, we have $\mathbb{P}(Y \in (0, 1) \cup (0, \frac{81}{16})) = \mathbb{P}(Y \in (0, \frac{81}{16})) = 1$. In particular, $F_Y(y) = 0$ if $y \leq 0$ and $F_Y(y) = 1$ if $y \geq \frac{81}{16}$.

For $y \in (0, \frac{81}{16})$,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(X^4 \leq y) = \mathbb{P}(-y^{\frac{1}{4}} \leq X \leq y^{\frac{1}{4}}) = F_X(y^{\frac{1}{4}}) - F_X(-y^{\frac{1}{4}}).$$

We first compute F_X from f_X. We have

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0, \text{ if } x < -1, \\ \int_{-1}^{x} f_X(t)\,dt, \text{ if } -1 \leq x \leq 0, \\ \int_{-1}^{0} f_X(t)\,dt + \int_{0}^{x} f_X(t)\,dt, \text{ if } 0 < x \leq \frac{3}{2}, \\ 1, \text{ if } x > \frac{3}{2} \end{cases} = \begin{cases} 0, \text{ if } x < -1, \\ \frac{x+1}{2}, \text{ if } -1 \leq x \leq 0, \\ \frac{1}{2} + \frac{x}{3}, \text{ if } 0 < x \leq \frac{3}{2}, \\ 1, \text{ if } x > \frac{3}{2} \end{cases}$$

Then,

$$F_Y(y) = \begin{cases} 0, \text{ if } y < 0, \\ F_X(y^{\frac{1}{4}}) - F_X(-y^{\frac{1}{4}}), \text{ if } 0 \leq y \leq 1, \\ F_X(y^{\frac{1}{4}}) - F_X(-y^{\frac{1}{4}}), \text{ if } 1 < y \leq \frac{81}{16}, \\ 1, \text{ if } y > \frac{81}{16} \end{cases} = \begin{cases} 0, \text{ if } y < 0, \\ \frac{1}{2} + \frac{1}{3}y^{\frac{1}{4}} - \frac{-y^{\frac{1}{4}}+1}{2}, \text{ if } 0 \leq y \leq 1, \\ \frac{1}{2} + \frac{1}{3}y^{\frac{1}{4}}, \text{ if } 1 < y \leq \frac{81}{16}, \\ 1, \text{ if } y > \frac{81}{16} \end{cases} = \begin{cases} 0, \text{ if } y < 0, \\ \frac{5}{6}y^{\frac{1}{4}}, \text{ if } 0 \leq y \leq 1, \\ \frac{1}{2} + \frac{1}{3}y^{\frac{1}{4}}, \text{ if } 1 < y \leq \frac{81}{16}, \\ 1, \text{ if } y > \frac{81}{16} \end{cases}$$

The above expression can now be simplified and then the corresponding p.d.f. can be obtained by differentiation of F_Y. We have

$$f_Y(y) = \begin{cases} 0, \text{ if } y < 0, \\ \frac{5}{24}y^{-\frac{3}{4}}, \text{ if } 0 \leq y \leq 1, \\ \frac{1}{12}y^{-\frac{3}{4}}, \text{ if } 1 < y \leq \frac{81}{16}, \\ 0, \text{ if } y > \frac{81}{16} \end{cases}$$

(b) Given that $\{x \in \mathbb{R} : f_X(x) > 0\} = (-1, 0) \cup (0, \frac{3}{2})$. Observe that the function $h(x) = x^4$ is strictly decreasing on $(-1, 0)$ and strictly increasing on $(0, \frac{3}{2})$ with inverses given by $h_1^{-1}(y) := -y^{\frac{1}{4}}$ and $h_2^{-1}(y) := y^{\frac{1}{4}}$.

Hence Y is a continuous RV with the p.d.f.

$$\begin{aligned} f_Y(y) &= f_X(h_1^{-1}(y))|\frac{d}{dy}h_1^{-1}(y)|1_{(0,1)}(y) + f_X(h_2^{-1}(y))|\frac{d}{dy}h_2^{-1}(y)|1_{(0,\frac{81}{16})}(y) \\ &= f_X(-y^{\frac{1}{4}})\frac{1}{4}y^{-\frac{3}{4}}1_{(0,1)}(y) + f_X(y^{\frac{1}{4}})\frac{1}{4}y^{-\frac{3}{4}}1_{(0,\frac{81}{16})}(y) \\ &= \frac{1}{8}y^{-\frac{3}{4}}1_{(0,1)}(y) + \frac{1}{12}y^{-\frac{3}{4}}1_{(0,\frac{81}{16})}(y) \\ &= \frac{5}{24}y^{-\frac{3}{4}}1_{(0,1)}(y) + \frac{1}{12}y^{-\frac{3}{4}}1_{[1,\frac{81}{16})}(y) \end{aligned}$$

Therefore, the DF of Y can now be computed as

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)\, dt = \begin{cases} 0, \text{ if } y < 0, \\ \frac{5}{6}y^{\frac{1}{4}}, \text{ if } 0 \leq y \leq 1, \\ \frac{5}{6} + \frac{1}{3}\left(y^{\frac{1}{4}} - 1\right), \text{ if } 1 < y \leq \frac{81}{16}, \\ 1, \text{ if } y > \frac{81}{16} \end{cases} = \begin{cases} 0, \text{ if } y < 0, \\ \frac{5}{6}y^{\frac{1}{4}}, \text{ if } 0 \leq y \leq 1, \\ \frac{1}{2} + \frac{1}{3}y^{\frac{1}{4}}, \text{ if } 1 < y \leq \frac{81}{16}, \\ 1, \text{ if } y > \frac{81}{16} \end{cases}$$