MSO205A PRACTICE PROBLEMS SET 13 SOLUTIONS

Question 1. Refer to Question 6 of problem set 12. Show by an example that the continuous mapping theorem does not hold for converge in r-th mean/moment.

Answer: In the problem 6 of set 12, we have an example of a sequence of RVs $\{X_n\}_n$ converging in the first mean to an RV X, but not in the second mean. Hence, considering the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) := x^2, x \in \mathbb{R}$, we have $\{X_n\}_n$ converging to X in the first mean and $\{h(X_n)\}_n$ does not converge to $h(X)$ in the first mean.

Question 2. Construct an example of a sequence of RVs $\{X_n\}_n$ converging in law/distribution, but not in probability.

Answer: We consider the example discussed in class. Let $X, X_1, X_2, \cdots$ be independent RVs with $X \sim N(0,1)$ and $X_n \sim N(0, 1+\frac{1}{n})$. Using the pointwise convergence of the MGFs, we have already proved that $X_n \xrightarrow[n\to\infty]{d} X$ and $X_n - X \xrightarrow[n\to\infty]{d} Z \sim N(0,2)$.
To show that $\{X_n\}_n$ does not converge in probability to X. If true, then $X_n - X \xrightarrow[n\to\infty]{P} 0$. However, this would imply $X_n - X \xrightarrow[n\to\infty]{d} 0$, which is a contradiction. Hence, $\{X_n\}_n$ does not converge in probability to X.

Question 3. Let $\{X_n\}_n$ be a sequence of i.i.d. RVs with finite second moment. Show that:

(a) $\frac{2}{n(n+1)} \sum_{j=1}^n jX_j \xrightarrow[n\to\infty]{P} \mathbb{E}X_1$.
(b) $\frac{6}{n(n+1)(2n+1)} \sum_{j=1}^n j^2 X_j \xrightarrow[n\to\infty]{P} \mathbb{E}X_1$.

Answer: (a) Take $Y_n = \frac{2}{n(n+1)} \sum_{j=1}^n jX_j, \forall n = 1, 2, \cdots$. Now,

$$\mathbb{E}Y_n = \frac{2}{n(n+1)} \sum_{j=1}^n j\mathbb{E}X_j = \frac{2\mathbb{E}X_1}{n(n+1)} \sum_{j=1}^n j = \mathbb{E}X_1.$$

Again, using the independence of the X_i's, we have

$$\begin{aligned}
\mathbb{E}\left(Y_n - \mathbb{E}X_1\right)^2 &= \mathbb{E}\left(\frac{2}{n(n+1)} \sum_{j=1}^n j(X_j - \mathbb{E}X_j)\right)^2 \\
&= \frac{4}{n^2(n+1)^2} \sum_{j=1}^n j^2 Var(X_j) \\
&= \frac{4}{n^2(n+1)^2} \frac{n(n+1)(2n+1)}{6} Var(X_1) \\
&\xrightarrow{n\to\infty} 0.
\end{aligned}$$

Hence, $\{Y_n\}_n$ converges to $\mathbb{E}X_1$ in 2nd mean and hence in probability.

(b) Take $Z_n = \frac{6}{n(n+1)(2n+1)} \sum_{j=1}^n j^2 X_j, \forall n = 1, 2, \cdots$. Now,

$$\mathbb{E}Z_n = \frac{6}{n(n+1)(2n+1)} \sum_{j=1}^n j^2 \mathbb{E}X_j = \frac{6\mathbb{E}X_1}{n(n+1)(2n=1)} \sum_{j=1}^n j = \mathbb{E}X_1.$$

Again, using the independence of the X_i's, we have

$$\begin{aligned}
\mathbb{E}\left(Z_n - \mathbb{E}X_1\right)^2 &= \mathbb{E}\left(\frac{6}{n(n+1)(2n+1)} \sum_{j=1}^n j(X_j - \mathbb{E}X_j)\right)^2 \\
&= \frac{36}{n^2(n+1)^2(2n+1)^2} \sum_{j=1}^n j^4 Var(X_j) \\
&= \frac{36 Var(X_1)}{n^2(n+1)^2(2n+1)^2} \sum_{j=1}^n j^4 \\
&\xrightarrow{n\to\infty} 0.
\end{aligned}$$

Hence, $\{Z_n\}_n$ converges to $\mathbb{E}X_1$ in 2nd mean and hence in probability.

Question 4. Let $a, b \in \mathbb{R}$ and let $\{X_n\}_n$ be a sequence of RVs such that $X_n \xrightarrow[n\to\infty]{P} a$ as well as $X_n \xrightarrow[n\to\infty]{P} b$. Show that $a = b$.

Answer: By definition, for any $\epsilon > 0$, $\lim_n \mathbb{P}(|X_n - a| < \epsilon) = 1$. Hence, $\lim_n \mathbb{P}(a - \epsilon < X_n < a + \epsilon) = 1$. Similarly, $\lim_n \mathbb{P}(b - \epsilon < X_n < b + \epsilon) = 1$.

If $a \neq b$, then without loss of generality, take $a < b$ and choose $0 < \epsilon < \frac{b-a}{2}$. Then the intervals $(a-\epsilon, a+\epsilon)$ and $(b-\epsilon, b+\epsilon)$ are disjoint. Hence, $\mathbb{P}(a-\epsilon < X_n < a+\epsilon \text{ and } b-\epsilon < X_n < b+\epsilon) = 0$ for all n.

Now, using the convergence above for any fixed ϵ with $0 < \epsilon < \frac{b-a}{2}$, we have for large n, $\mathbb{P}(a - \epsilon < X_n < a + \epsilon) > \frac{1}{2}$ and $\mathbb{P}(b - \epsilon < X_n < b + \epsilon) > \frac{1}{2}$. Then, by Bonferroni's inequality, $\mathbb{P}(a - \epsilon < X_n < a + \epsilon \text{ and } b - \epsilon < X_n < b + \epsilon) \geq \mathbb{P}(a - \epsilon < X_n < a + \epsilon) + \mathbb{P}(b - \epsilon < X_n < b + \epsilon) - 1 > 0$, which is a contradiction.

Hence, we must have $a = b$.

Question 5. Consider a sequence $\{X_n\}_n$ of RVs with $X_n \sim N(\frac{1}{n}, 1 - \frac{1}{n}), \forall n$. Does this sequence converge in law/distribution?

Answer: We consider the pointwise convergence of the MGFs of X_n's. For all $t \in \mathbb{R}$,

$$M_{X_n}(t) = \exp\left(\frac{t}{n} + \frac{1}{2}\left(1 - \frac{1}{n}\right)t^2\right) \xrightarrow{n\to\infty} \exp\left(\frac{1}{2}t^2\right) = M_{N(0,1)}(t).$$

Hence, we conclude that $\{X_n\}_n$ converges in law to $X \sim N(0, 1)$.

Question 6. Suppose that a continuous RV X has a quantile of order $\frac{1}{3}$ at 5. Consider a random sample of size 100 from the distribution of X. What is the probability (approximately) that more than 40 sample values are more than 5? Express the approximate value in terms of Φ, the DF of $N(0, 1)$ distribution.

Answer: Note that $\mathbb{P}(X \le 5) = \frac{1}{3}$ or $\mathbb{P}(X > 5) = \frac{2}{3}$.

Let $X_1, \cdots, X_{100}$ denote the given random sample. Define

$$Y_i = \begin{cases} 1, \text{ if } X_i > 5, \\ 0, \text{ otherwise} \end{cases}, \forall i = 1, \cdots, 100.$$

Then, Y_i's are i.i.d. with $Y_1 \sim Bernoulli(\mathbb{P}(X > 5)) = Bernoulli(\frac{2}{3})$. Here, $\mathbb{E}Y_1 = \frac{2}{3}$ and $Var(Y_1) = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$.

By the CLT, for large n the distribution of $\sqrt{n}\frac{\frac{1}{n}\sum_{i=1}^n Y_i - \frac{2}{3}}{\sqrt{\frac{2}{9}}}$ is close to $N(0,1)$ in the sense of convergence in distribution. Putting $n = 100$, the distribution of $\frac{30}{\sqrt{2}}\left(\frac{1}{100}\sum_{i=1}^{100} Y_i - \frac{2}{3}\right)$ is close to $N(0,1)$ in the sense of convergence in distribution.

The required probability is

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{100} Y_i > 40\right) &= \mathbb{P}\left(\frac{1}{100}\sum_{i=1}^{100} Y_i - \frac{2}{3} > \frac{40}{100} - \frac{2}{3}\right) \\
&= \mathbb{P}\left(\frac{1}{100}\sum_{i=1}^{100} Y_i - \frac{2}{3} > -\frac{4}{15}\right) \\
&= \mathbb{P}\left(\frac{30}{\sqrt{2}}\left\{\frac{1}{100}\sum_{i=1}^{100} Y_i - \frac{2}{3}\right\} > -4\sqrt{2}\right) \\
&= 1 - \mathbb{P}\left(\frac{30}{\sqrt{2}}\left\{\frac{1}{100}\sum_{i=1}^{100} Y_i - \frac{2}{3}\right\} \le -4\sqrt{2}\right)
\end{aligned}$$

Using the above convergence, an approximate value of the required probability is $1 - \Phi(-4\sqrt{2})$, where Φ is the DF of $N(0,1)$ distribution.

Question 7. Fix $\lambda > 0$. Let $X_1, X_2, \cdots$ be a sequence of i.i.d. RVs with *Exponential*(λ) distribution. Consider the sample mean $\bar{X}_n := \frac{1}{n}\sum_{j=1}^n X_j, \forall n$. Show that

$$\sqrt{n}\left(\frac{1}{\bar{X}_n} - \frac{1}{\lambda}\right) \xrightarrow[n\to\infty]{d} N\left(0, \frac{1}{\lambda^2}\right).$$

Answer: We have $Var(X_1) = \lambda^2 > 0$. Now, $\mathbb{E}X_1 = \lambda$. By the CLT, we have

$$\sqrt{n}\frac{\bar{X}_n - \lambda}{\lambda} \xrightarrow[n\to\infty]{d} X,$$

with $X \sim N(0,1)$. We rewrite the above convergence in preparation for applying the Delta method. We have $\sqrt{n}(\bar{X}_n - \lambda) \xrightarrow[n\to\infty]{d} Y \sim N(0, \lambda^2)$.

Consider the function $g : (0, \infty) \to \mathbb{R}$ defined by $g(x) = \frac{1}{x}, \forall x$. Now, $g'(x) = -\frac{1}{x^2} \neq 0$. By the Delta method,

$$\sqrt{n}(g(\bar{X}_n) - g(\lambda)) \xrightarrow[n\to\infty]{d} -\frac{1}{\lambda^2}Y \sim N\left(0, \frac{1}{\lambda^2}\right)$$

Question 8. Compute the mode of *Binomial*(n, p) distribution.

Answer: We need to find the point(s) $x \in \mathbb{R}$ such that the p.m.f. f_X is maximized at x. Since $f_X(x) = 0$ for $x \notin S_X\{0, 1, \cdots, n\}$, we look at $x \in S_X$. Now,

$$\begin{aligned}
& f_X(x+1) \gtreqless f_X(x) \\
\iff & \binom{n}{x+1} p^{x+1}(1-p)^{n-x-1} \gtreqless \binom{n}{x} p^x (1-p)^{n-x} \\
\iff & \frac{1}{(x+1)} p \gtreqless \frac{1}{n-x}(1-p) \\
\iff & np - xp \gtreqless (1-p)x + (1-p) \\
\iff & (n+1)p - 1 \gtreqless x
\end{aligned}$$

If $(n+1)p - 1$ is an integer, then $f_X((n+1)p) = f_X((n+1)p - 1)$ and this is the maximum value attained for f_X. This is a bi-modal case with the modes given by $(n+1)p - 1$ and $(n+1)p$.
If $(n+1)p - 1$ is not an integer, take the largest integer less than $(n+1)p$, usually written as $[(n+1)p]$. Note that $(n+1)p - 1 < [(n+1)p] < (n+1)p$. Here, the maximum is attained at $x = [(n+1)p]$ (unimodal case).

Question 9. Let $X \sim Poisson(\lambda)$ for some $\lambda > 0$. Compute the coefficient of skewness and excess kurtosis.

Answer: Coefficient of skewness is given by $\frac{\mu_3(X)}{(\mu_2(X))^{3/2}}$ and excess kurtosis by $\frac{\mu_4(X)}{(\mu_2(X))^2} - 3$.
We have seen that $\mathbb{E}X = \lambda, \mu_2(X) = Var(X) = \lambda, \mathbb{E}X^2 = \lambda + \lambda^2$. Recall that $\mathbb{E}X(X-1)(X-2) = \lambda^3$ and $\mathbb{E}X(X-1)(X-2)(X-3) = \lambda^4$. Therefore, $\mathbb{E}X^3 = \lambda^3 + 3\lambda^2 + \lambda$ and $\mathbb{E}X^4 = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$. Then,

$$\mu_3(X) = \mathbb{E}(X-\lambda)^3 = \lambda, \quad \mu_4(X) = \mathbb{E}(X-\lambda)^4 = 3\lambda^2 + \lambda.$$

Coefficient of skewness is $\lambda^{-\frac{1}{2}}$ and excess kurtosis is λ^{-1}.

Question 10. Let X be a p-dimensional random vector, $a \in \mathbb{R}^m$ and A be an $m \times p$ real matrix. Then the Characteristic function of the m-dimensional random vector $Y = a + AX$ given by

$$\Phi_Y(u) = \exp(iu^t a)\, \Phi_X(A^t u), u \in \mathbb{R}^m.$$

Answer: We have, for $u \in \mathbb{R}^m$

$$\Phi_Y(u) = \mathbb{E}\exp(iu^tY) = \mathbb{E}\exp(iu^t(a+AX)) = \exp(iu^ta)\mathbb{E}\exp(i(A^tu)^tX) = \exp(iu^ta)\, \Phi_X(A^tu).$$

Question 11. Show that $\mathbb{E}|X|^\alpha < \infty, \forall \alpha \in (0,1)$ when $X \sim Cauchy(0,1)$.

Answer: We need to show that

$$\int_{-\infty}^{\infty} |x|^\alpha \frac{1}{\pi} \frac{1}{1+x^2}\, dx = \frac{2}{\pi} \int_0^\infty \frac{x^\alpha}{1+x^2}\, dx < \infty.$$

Consider the change of variables $t = \frac{1}{1+x^2}$ or $x = \sqrt{\frac{1-t}{t}}$. We have,

$$\frac{2}{\pi} \int_0^\infty \frac{x^\alpha}{1+x^2}\, dx = \frac{1}{\pi} \int_0^1 t^{-\frac{\alpha+1}{2}} (1-t)^{\frac{\alpha-1}{2}}\, dt.$$

The above integral converges if $0 < \alpha < 1$ and this proves $\mathbb{E}|X|^\alpha < \infty$.