MSO205A PRACTICE PROBLEMS SET 2 SOLUTIONS

Question 1 (Monty Hall Problem). There are 3 doors with one door having an expensive car behind it and each of the other 2 doors having a goat behind them. Monty Hall, being a host of the game, knows what it behind each door. A contestant is asked to select one of the doors and he wins the item (car or goat) behind the selected door. The contestant selects one of the doors at random, and then Monty Hall opens one of the other two doors to reveal a goat behind it (Monty Hall knows the doors behind which there are goats). Monty Hall offers to trade the door the contestant has chosen for the other door that is closed. Should the contestant switch doors if the goal is to win the car? (The problem is based on the American Television game show 'Let's make a deal' hosted by Monty Hall).

Answer: Write the event of choosing doors 1, 2 and 3 as D_1, D_2 and D_3 respectively. Also write C for the event in which the contestant wins the car (i.e. selects the corresponding door).

Without loss of generality assume that the car is behind door 1.

Consider the following two cases.

Case when the contestant decides to switch doors. If the contestant's original choice was door 1, then after switching the new choice is one of door 2 or door 3. In either case, we have $\mathbb{P}(C \mid D_1) = 0$. If the original choice was door 2 or door 3, then Monty Hall must have opened door 3 or door 2 respectively. If the contestant now switches his choice, he chooses door 1 now. In this case, $\mathbb{P}(C \mid D_i) = 1, i = 2, 3$. The doors were originally chosen at random by the contestant. Hence $\mathbb{P}(D_i) = \frac{1}{3}, i = 1, 2, 3$. Further, these events are mutually exclusive and exhaustive. By the Theorem of Total Probability, $\mathbb{P}(C) = \sum_{i=1}^{3} \mathbb{P}(D_i)\, \mathbb{P}(C \mid D_i) = \frac{2}{3}$.

Case when the contestant does not switch: In this case, $\mathbb{P}(C \mid D_1) = 1$ and $\mathbb{P}(C \mid D_i) = 0, i = 2, 3$. Again by the Theorem of Total Probability, we have $\mathbb{P}(C) = \sum_{i=1}^{3} \mathbb{P}(D_i)\, \mathbb{P}(C \mid D_i) = \frac{1}{3}$.

At the time of the original choice, doors were chosen at random. However, after this choice, new information became available and we observe that by switching to the other option, the probability of winning the car increases. The contestant should switch.

Note that once new information becomes available, we should update the prior probabilities.

Question 2. Suppose that a population comprises of 45% females and 55% males. Suppose that 60% of the females and 80% of the males in the population have jobs. Choose a person at random from the population.

(a) Find the probability that the person chosen has a job.
(b) Given that the selected person has a job, find the probability that the person is female.

Answer: Consider the following two mutually exclusive and exhaustive events. Let F and M denote the event that the selected person is a female or a male respectively. Let J denote the event that the selected person has a job.

Given that $\mathbb{P}(F) = \frac{45}{100}, \mathbb{P}(M) = \frac{55}{100}, \mathbb{P}(J \mid F) = \frac{60}{100}, \mathbb{P}(J \mid M) = \frac{80}{100}$.

(a) By the Theorem of Total Probability, we have $\mathbb{P}(J) = \mathbb{P}(F)\mathbb{P}(J \mid F) + \mathbb{P}(M)\mathbb{P}(J \mid M) = \frac{45\times 60 + 55\times 80}{100\times 100} = \frac{71}{100}$.

(b) We need to find $\mathbb{P}(F \mid J)$. We have

$$\mathbb{P}(F \mid J) = \frac{\mathbb{P}(F \cap J)}{\mathbb{P}(J)} = \frac{\mathbb{P}(F)\mathbb{P}(J \mid F)}{\mathbb{P}(J)} = \frac{\frac{45\times 60}{100\times 100}}{\frac{71}{100}} = \frac{27}{71}.$$

Question 3. Let A, B be two independent events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show that

(1) A and B^c are independent.
(2) A^c and B are independent.
(3) A^c and B^c are independent.

Answer:

(1) By definition, we have $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. Then, $\mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A \cap B) = \mathbb{P}(A)[1 - \mathbb{P}(B)] = \mathbb{P}(A)\mathbb{P}(B^c)$. Hence, A and B^c are independent.
(2) By definition, we have $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. Then, $\mathbb{P}(A^c \cap B) = \mathbb{P}(B) - \mathbb{P}(A \cap B) = \mathbb{P}(B)[1 - \mathbb{P}(A)] = \mathbb{P}(A^c)\mathbb{P}(B)$. Hence, A^c and B are independent.
(3) By part (b), for any independent events C and D, the events C^c and D are independent. By part (a), we have A and B^c are independent. Choosing $C = A$ and $D = B^c$ yields the proof.

Question 4. (1) Let $\{E_1, E_2, \cdots, E_n\}$ be a finite collection of mutually independent events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Construct a collection $\{A_1, A_2, \cdots, A_n\}$ of events in the following way. For each $1 \le i \le n$, make a choice for A_i between E_i or E_i^c. For every

$2 \leq k \leq n$ and indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, show that $\{A_{i_1}, A_{i_2}, \cdots, A_{i_k}\}$ is a collection of mutually independent events.

(2) Let $\mathcal{I}$ be an infinite indexing set and let $\{E_i : i \in \mathcal{I}\}$ be a collection of mutually independent events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Construct a collection $\{A_i : i \in \mathcal{I}\}$ of events in the following way. For each $i \in \mathcal{I}$, make a choice for A_i between E_i or E_i^c. For positive integers $k \geq 2$ and distinct indices $i_1, i_2, \cdots, i_k$, show that $\{A_{i_1}, A_{i_2}, \cdots, A_{i_k}\}$ is a collection of mutually independent events.

Answer:

(1) For all $l \in \{2, 3, \cdots, n\}$ and indices $1 \leq i_1 < i_2 < \cdots < i_l \leq n$, we need to show that

$$(*) \qquad \mathbb{P}\left(\bigcap_{j=1}^{l} A_{i_j}\right) = \prod_{j=1}^{l} \mathbb{P}\left(A_{i_j}\right).$$

Since $\{E_1, E_2, \cdots, E_n\}$ is a finite collection of mutually independent events, then for all $l \in \{2, 3, \cdots, n\}$ and indices $1 \leq i_1 < i_2 < \cdots < i_l \leq n$, we have

$$(**) \qquad \mathbb{P}\left(\bigcap_{j=1}^{l} E_{i_j}\right) = \prod_{j=1}^{l} \mathbb{P}\left(E_{i_j}\right).$$

First consider the case $l = 2$. We have $\mathbb{P}(E_{i_1} \cap E_{i_2}) = \mathbb{P}(E_{i_1})\mathbb{P}(E_{i_2})$, i.e. E_{i_1} and E_{i_2} are independent for all $1 \leq i_1 < i_2 \leq n$. By problem 1, each of the collections $\{E_{i_1}^c, E_{i_2}\}$, $\{E_{i_1}, E_{i_2}^c\}$ and $\{E_{i_1}^c, E_{i_2}^c\}$ are collections of mutually independent events. This proves the conditions for $l = 2$, under all choices of A_{i_j}'s.

Now, consider the case $3 \leq l \leq n$. Let $m := \#\{j = 1, 2, \cdots, k : A_{i_j} = E_{i_j}^c\}$. Then $0 \leq m \leq l$.

If $m = 0$, then all A_{i_j}'s are E_{i_j}'s and $(*)$ follows from $(**)$.

If $m = 1$, then suppose $A_{i_\alpha} = E_{i_\alpha}^c$ and $A_{i_j} = E_{i_j}, \forall j \neq \alpha$. Using the case for $m = 0$ and $(**)$, we have

$$\mathbb{P}(\bigcap_{j=1}^{l} E_{i_j}) = \prod_{j=1}^{l} \mathbb{P}(E_{i_j}), \quad \mathbb{P}(\bigcap_{j \in \{1,2,\cdots,\alpha-1,\alpha+1,\cdots,l\}} E_{i_j}) = \prod_{j \in \{1,2,\cdots,\alpha-1,\alpha+1,\cdots,l\}} \mathbb{P}(E_{i_j}).$$

Subtracting the second equality from the first, we have $(*)$.

If $m = 2$, then suppose $A_{i_\alpha} = E_{i_\alpha}^c, A_{i_\beta} = E_{i_\beta}^c$ with $i_\alpha \neq i_\beta$ and $A_{i_j} = E_{i_j}, \forall j \notin \{\alpha, \beta\}$. Using the case for $m = 1$ and $(**)$, we have

$$\mathbb{P}\left(\left(\bigcap_{j \in \{1,2,\cdots,\alpha-1\}} A_{i_j}\right) \cap E_{i_\alpha} \cap \left(\bigcap_{j \in \{\alpha+1,\cdots,l\}} A_{i_j}\right)\right) = \prod_{j \in \{1,2,\cdots,\alpha-1\}} \mathbb{P}(A_{i_j}) \times \mathbb{P}(E_{i_\alpha}) \times \prod_{j \in \{\alpha+1,\cdots,l\}} \mathbb{P}(A_{i_j})$$

and

$$\mathbb{P}(\bigcap_{j\in\{1,2,\cdots,\alpha-1,\alpha+1,\cdots,l\}} A_{i_j}) = \prod_{j\in\{1,2,\cdots,\alpha-1,\alpha+1,\cdots,l\}} \mathbb{P}(A_{i_j}).$$

Subtracting the second relation from the first, we have $(*)$ for $m = 2$.

Proceeding by finite induction, the result is established for all $m \leq l$.

(2) Since, $\{E_i : i \in \mathcal{I}\}$ is a collection of mutually independent events, by definition $\{E_{i_1}, E_{i_2}, \cdots, E_{i_k}\}$ is also a collection of mutually independent events. The required result follows from part (a).

Question 5. A student appears in the examinations of four subjects Biology, Chemistry, Physics and Mathematics. Suppose that the performances of the student in four subjects are independent and that the probabilities of the student obtaining a passing grade in these subjects are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ and $\frac{1}{5}$ respectively. Find the probability that the student will obtain a passing grade in (a) all the subjects, (b) no subject, (c) at least one subject.

Answer: Let B, C, P and M denote the events the student obtains a passing grade in Biology, Chemistry, Physics and Mathematics respectively. Given that these events are independent and that

$$\mathbb{P}(B) = \frac{1}{2}, \quad \mathbb{P}(C) = \frac{1}{3}, \quad \mathbb{P}(P) = \frac{1}{4}, \quad \mathbb{P}(M) = \frac{1}{5}.$$

Then,

$$\mathbb{P}(B^c) = 1 - \mathbb{P}(B) = \frac{1}{2}, \quad \mathbb{P}(C^c) = \frac{2}{3}, \quad \mathbb{P}(P^c) = \frac{3}{4}, \quad \mathbb{P}(M^c) = \frac{4}{5}.$$

(1) The event is $B \cap C \cap P \cap M$. Using independence of these events, the required probability is

$$\mathbb{P}(B \cap C \cap P \cap M) = \mathbb{P}(B)\mathbb{P}(C)\mathbb{P}(P)\mathbb{P}(M) = \frac{1}{120}.$$

(2) The event is $B^c \cap C^c \cap P^c \cap M^c$. Using independence of these events and problem 2(a), the required probability is

$$\mathbb{P}(B^c \cap C^c \cap P^c \cap M^c) = \mathbb{P}(B^c)\mathbb{P}(C^c)\mathbb{P}(P^c)\mathbb{P}(M^c) = \frac{1}{5}.$$

(3) The event is $(B^c \cap C^c \cap P^c \cap M^c)^c$. Using part (b), the required probability is

$$\mathbb{P}\left((B^c \cap C^c \cap P^c \cap M^c)^c\right) = 1 - \mathbb{P}(B^c \cap C^c \cap P^c \cap M^c) = \frac{4}{5}.$$

Question 6. Let Ω be a non-empty set and let $X : \Omega \to \mathbb{R}$ be a function. Given any subset A of $\mathbb{R}$, we consider the subset $X^{-1}(A)$ of Ω defined by

$$X^{-1}(A) := \{\omega \in \Omega : X(\omega) \in A\}.$$

The set $X^{-1}(A)$ shall be referred to as the pre-image of A under the function X. Verify the following properties of the pre-images under X, which follow from the fact that X is a function.

(i) $X^{-1}(\mathbb{R}) = \Omega$.

(ii) $X^{-1}(\emptyset_{\mathbb{R}}) = \emptyset_{\Omega}$, where $\emptyset_{\mathbb{R}}$ and $\emptyset_{\Omega}$ denote the empty sets under $\mathbb{R}$ and Ω, respectively. (Note: When there is no chance of confusion, we simply write $X^{-1}(\emptyset) = \emptyset$.)

(iii) For any two subsets A, B of $\mathbb{R}$ with $A \cap B = \emptyset$, we have $X^{-1}(A) \cap X^{-1}(B) = \emptyset$.

(iv) For any subset A of $\mathbb{R}$, we have $X^{-1}(A^c) = \left(X^{-1}(A)\right)^c$.

(v) Let $\mathcal{I}$ be an indexing set. For any collection $\{A_i : i \in \mathcal{I}\}$ of subsets of $\mathbb{R}$, we have

$$X^{-1}\left(\bigcup_{i\in\mathcal{I}} A_i\right) = \bigcup_{i\in\mathcal{I}} X^{-1}(A_i), \quad X^{-1}\left(\bigcap_{i\in\mathcal{I}} A_i\right) = \bigcap_{i\in\mathcal{I}} X^{-1}(A_i).$$

Answer: (i) Since $X : \Omega \to \mathbb{R}$ is a function, $X(\omega) \in \mathbb{R}, \forall \omega \in \Omega$ and hence $X^{-1}(\mathbb{R}) = \Omega$.

(ii) If possible, let $\omega \in X^{-1}(\emptyset_{\mathbb{R}})$. Then, $X(\omega) \in \emptyset_{\mathbb{R}}$, which is a contradiction and hence we must have $X^{-1}(\emptyset_{\mathbb{R}}) = \emptyset_{\Omega}$.

(iii) If possible, let $\omega \in X^{-1}(A) \cap X^{-1}(B)$. Then, $X(\omega) \in A$ and $X(\omega) \in B$ and hence $X(\omega) \in A \cap B$. This contradicts $A \cap B = \emptyset$. Hence, we must have $X^{-1}(A) \cap X^{-1}(B) = \emptyset$.

(iv) Observe that

$$\begin{aligned} \omega \in X^{-1}(A^c) &\iff X(\omega) \in A^c \\ &\iff X(\omega) \notin A \\ &\iff \omega \notin X^{-1}(A) \\ &\iff \omega \in (X^{-1}(A))^c. \end{aligned}$$

This completes the proof.

(v) Observe that

$$\begin{aligned} \omega \in X^{-1}\left(\bigcup_{i\in\mathcal{I}} A_i\right) &\iff X(\omega) \in \bigcup_{i\in\mathcal{I}} A_i \\ &\iff X(\omega) \in A_i \text{ for some } i \in \mathcal{I} \\ &\iff \omega \in X^{-1}(A_i) \text{ for some } i \in \mathcal{I} \\ &\iff \omega \in \bigcup_{i\in\mathcal{I}} X^{-1}(A_i) \end{aligned}$$

This proves the first statement. Similarly,

$$\begin{aligned}
\omega \in X^{-1}\left(\bigcap_{i \in \mathcal{I}} A_i\right) &\iff X(\omega) \in \bigcap_{i \in \mathcal{I}} A_i \\
&\iff X(\omega) \in A_i \text{ for all } i \in \mathcal{I} \\
&\iff \omega \in X^{-1}(A_i) \text{ for all } i \in \mathcal{I} \\
&\iff \omega \in \bigcap_{i \in \mathcal{I}} X^{-1}(A_i)
\end{aligned}$$

This proves the second statement.