MSO205A PRACTICE PROBLEMS SET 9 SOLUTIONS

Question 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $A, B \in \mathcal{F}$. Define RVs $X, Y : \Omega \to \mathbb{R}$ by

$$X(\omega) = 1_A(\omega) = \begin{cases} 1, \text{ if } \omega \in A, \\ 0, \text{ otherwise} \end{cases}, \quad Y(\omega) = 1_B(\omega), \forall \omega \in \Omega.$$

Show that RVs X, Y are independent if and only if events A, B are independent.

Answer: X is a discrete RV with $\mathbb{P}(X = 1) = \mathbb{P}(A), \mathbb{P}(X = 0) = \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$. Therefore, $X \sim Bernoulli(\mathbb{P}(A))$ with the DF

$$F_X(x) = \begin{cases} 0, \text{ if } x < 0, \\ \mathbb{P}(A^c), \text{ if } 0 \leq x < 1, \\ 1, \text{ if } x \geq 1. \end{cases}$$

Similarly, $Y \sim Bernoulli(\mathbb{P}(B))$ with the DF

$$F_Y(y) = \begin{cases} 0, \text{ if } y < 0, \\ \mathbb{P}(B^c), \text{ if } 0 \leq y < 1, \\ 1, \text{ if } y \geq 1. \end{cases}$$

If X, Y are independent, then $\mathbb{P}(A^c)\mathbb{P}(B^c) = F_X(\frac{1}{2})F_Y(\frac{1}{2}) = F_{X,Y}(\frac{1}{2}, \frac{1}{2}) = \mathbb{P}(X \leq \frac{1}{2}, Y \leq \frac{1}{2}) = \mathbb{P}(A^c \cap B^c)$ and hence A^c and B^c are independent. Therefore, A and B are independent.

Given A, B are independent, we have A^c and B^c are independent and hence the relation $F_X(x)F_Y(y) = F_{X,Y}(x, y), \forall (x, y) \in \mathbb{R}^2$ follows. Therefore, X and Y are independent.

Note: An alternative argument can be worked out using the p.m.f.s rather than DFs.

Question 2. Let $X = (X_1, X_2, \cdots, X_p)$ be a discrete random vector with joint DF F_X, joint p.m.f. f_X and support S_X. Let f_{X_j} denote the marginal p.m.f. of X_j. If $X_1, X_2, \cdots, X_p$ are independent, then show that

$$f_{X_1,X_2,\cdots,X_p}(x_1, x_2, \cdots, x_p) = \prod_{j=1}^{p} f_{X_j}(x_j), \forall x_1, x_2, \cdots, x_p \in \mathbb{R}.$$

(The proof of this statement was left out in the lecture notes.)

Answer: We discuss the proof for the case $p = 2$. Proof for general dimension p is similar.

If X_1, X_2 are independent, by definition,

$$F_{X_1,X_2}(x_1, x_2) = F_{X_1}(x_1)F_{X_2}(x_2), \forall (x_1, x_2) \in \mathbb{R}^2.$$

Now, for $(x_1, x_2) \in \mathbb{R}^2$, we have

$$
\begin{aligned}
& f_{X_1,X_2}(x_1,x_2) \\
&= \mathbb{P}(X_1 = x_1, X_2 = x_2) \\
&= \mathbb{P}\left(\bigcap_{n=1}^{\infty}\left\{x_1 - \frac{1}{n} < X_1 \leq x_1 \text{ and } x_2 - \frac{1}{n} < X_2 \leq x_2\right\}\right) \\
&= \lim_{n\to\infty} \mathbb{P}\left(\left\{x_1 - \frac{1}{n} < X_1 \leq x_1 \text{ and } x_2 - \frac{1}{n} < X_2 \leq x_2\right\}\right) \\
&= \lim_{n\to\infty}\left[F_{X_1,X_2}(x_1,x_2) - F_{X_1,X_2}\left(x_1 - \frac{1}{n}, x_2\right) - F_{X_1,X_2}\left(x_1, x_2 - \frac{1}{n}\right) + F_{X_1,X_2}\left(x_1 - \frac{1}{n}, x_2 - \frac{1}{n}\right)\right] \\
&= \lim_{n\to\infty}\left[\left(F_{X_1}(x_1) - F_{X_1}\left(x_1 - \frac{1}{n}\right)\right) \times \left(F_{X_2}(x_2) - F_{X_2}\left(x_2 - \frac{1}{n}\right)\right)\right] \\
&= f_{X_1}(x_1) f_{X_2}(x_2).
\end{aligned}
$$

This completes the proof.

Question 3. Let $X = (X_1, X_2)$ be a discrete random vector with joint p.m.f.

$$
f_X(x_1,x_2) = \begin{cases} \alpha(2x_1 + x_2), & \text{if } x_1, x_2 \in \{1,2\}, \\ 0, & \text{otherwise} \end{cases}
$$

for some constant $\alpha \in \mathbb{R}$. Find the value of α and identify the marginal p.m.f.s of X_1 and X_2. Are X_1, X_2 independent? If not independent, find the conditional p.m.f. of X_2 given $X_1 = x_1 \in \{1,2\}$.

Answer: For $f_X(x_1,x_2) \geq 0$ for all $(x_1,x_2) \in \mathbb{R}^2$, we must have $\alpha > 0$. Moreover,

$$
\sum_{x_1,x_2 \in \{1,2\}} f_X(x_1,x_2) = 1
$$

yields $18\alpha = 1$. Therefore, $\alpha = \frac{1}{18}$.

The marginal p.m.f. of X_1 is given by

$$
f_{X_1}(x_1) = \begin{cases} \sum_{x_2 \in \{1,2\}} \frac{1}{18}(2x_1 + x_2), & \text{if } x_1 \in \{1,2\}, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{2x_1}{9} + \frac{1}{6}, & \text{if } x_1 \in \{1,2\}, \\ 0, & \text{otherwise} \end{cases}
$$

and the marginal p.m.f. of X_2 is given by

$$
f_{X_2}(x_2) = \begin{cases} \sum_{x_1 \in \{1,2\}} \frac{1}{18}(2x_1 + x_2), & \text{if } x_2 \in \{1,2\}, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{x_2}{9} + \frac{1}{3}, & \text{if } x_2 \in \{1,2\}, \\ 0, & \text{otherwise} \end{cases}
$$

Now, $f_{X_1,X_2}(1,1) = \frac{1}{6} \neq \frac{7}{18} \times \frac{4}{9} = f_{X_1}(1) f_{X_2}(1)$. Hence, X_1 and X_2 are not independent.

Since (X_1, X_2) is supported on $\{1,2\} \times \{1,2\}$, the conditional distribution of X_2 given $X_1 = x_2 \in \{1,2\}$ is supported on $\{1,2\}$ with the conditional p.m.f.

$$
f_{X_2|X_1}(x_2 \mid x_1) = \begin{cases} \frac{f_X(x_1,x_2)}{f_{X_1}(x_1)}, & \text{if } x_2 \in \{1,2\} \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} \frac{2x_1 + x_2}{4x_1 + 3}, & \text{if } x_2 \in \{1,2\} \\ 0, & \text{otherwise.} \end{cases}
$$