MSO205A PRACTICE PROBLEMS SET 4 SOLUTIONS

Question 1. Which of the following functions are distribution functions?

(a) $F_1(x) := \begin{cases} 0, & \text{if } x \leq 1, \\ 1, & \text{if } x > 1. \end{cases}$

(b) $F_2(x) := \begin{cases} 1, & \text{if } x < 1, \\ 0, & \text{if } x \geq 1. \end{cases}$

(c) $F_3(x) := \begin{cases} 0, & \text{if } x < 1, \\ 1, & \text{if } x \geq 1. \end{cases}$

(d) $F_4(x) := \begin{cases} 0, & \text{if } x < 0, \\ \frac{1}{4} + \frac{x}{2}, & \text{if } 0 \leq x \leq 1, \\ \frac{1}{2} + \frac{x}{4}, & \text{if } 1 < x < 2, \\ 1, & \text{if } x \geq 2. \end{cases}$.

Answer:

(a) Note that $\lim_{h \downarrow 0} F_1(1+h) = 1 \neq 0 = F_1(1)$. Therefore, F_1 is not right continuous at 1 and hence cannot be a distribution function.

(b) Note that $F_2(0) = 1 > 0 = F_2(1)$ and therefore F_2 is not non-decreasing. Hence F_2 cannot be a distribution function.

(c) We have F_3 is a constant on $(-\infty, 1)$ and on $[1, \infty)$. Moreover, for $x < 1$ and $y > 1$, we have

$$0 = F_3(x) < 1 = F_3(1) = F_3(y).$$

Therefore, F_3 is non-decreasing.

F_3 is continuous on $(-\infty, 1)$ and on $(1, \infty)$. We check right continuity at 1. We have $\lim_{h \downarrow 0} F_3(1+h) = 1 = F_3(1)$. Therefore F_3 is right continuous on $\mathbb{R}$.

Finally, $\lim_{x \to -\infty} F_3(x) = 0$ and $\lim_{x \to \infty} F_3(x) = 1$. Hence, F_3 is a distribution function.

(d) We have F_4 is non-decreasing on $(-\infty, 0), [0, 1], (1, 2)$ and on $[2, \infty)$. Moreover, for $x < 0, y \in (0, 1), z \in (1, 2), w > 2$, we have

$$F_4(x) = 0 < \frac{1}{4} = F_4(0) < F_4(y) < \frac{1}{4} + \frac{1}{2} = F_4(1) < F_4(z) < 1 = F_4(2) = F_4(w).$$

Therefore F_4 is non-decreasing on $\mathbb{R}$.

By definition, F_4 is continuous on $(-\infty, 0), (0, 1), (1, 2)$ and on $(2, \infty)$. So possible discontinuities may arise at the points $0, 1, 2$. We check the right continuity at these points. We have

$$\begin{aligned} F_4(0+) &= \lim_{h \downarrow 0} F_4(0 + h) = \lim_{h \downarrow 0} \left[\frac{1}{4} + \frac{h}{2}\right] = \frac{1}{4} = F_4(0), \\ F_4(1+) &= \lim_{h \downarrow 0} F_4(1 + h) = \lim_{h \downarrow 0} \left[\frac{1}{2} + \frac{1+h}{4}\right] = \frac{1}{2} + \frac{1}{4} = F_4(1), \\ F_4(2+) &= \lim_{h \downarrow 0} F_4(2 + h) = 1 = F_4(2). \end{aligned}$$

Therefore F_4 is right continuous on $\mathbb{R}$.

Finally $\lim_{x \to -\infty} F_4(x) = \lim_{x \to -\infty} 0 = 0$ and $\lim_{x \to \infty} F_4(x) = \lim_{x \to \infty} 1 = 1$. Hence, F_4 is a distribution function.

Question 2. Let X be an RV defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with law $\mathbb{P}_X$ and DF F_X. Consider the set $D := \{x \in \mathbb{R} : F_X \text{ is discontinuous at } x\}$. Show that it is either finite or countably infinite. (Hint: for each $n = 1, 2, \cdots$, consider the set $D_n := \{x \in \mathbb{R} : F_X(x+) - F_X(x-) > \frac{1}{n}\} = \{x \in \mathbb{R} : F_X(x) - F_X(x-) > \frac{1}{n}\} = \{x \in \mathbb{R} : \mathbb{P}(X = x) > \frac{1}{n}\}$. Then $D = \cup_n D_n$. What can you say about D_n?)

Answer: First note that each D_n is a finite set. If not, then for some positive integer m, D_m is infinite and we can choose a sequence $\{x_k\}_k$ from D_m with distinct points. Write $A = \{x_k : k \geq 1\}$. Then $\mathbb{P}(X = x_k) > \frac{1}{m}, \forall k$ and

$$1 \geq \mathbb{P}(X \in A) = \sum_{k=1}^{\infty} \mathbb{P}(X = x_k) > \sum_{k=1}^{\infty} \frac{1}{m} = \infty,$$

which is a contradiction. Therefore, each D_n is a finite set. In fact, by the above argument, we can show that $\#D_n < n$.

Now, $D = \cup_{n=1}^{\infty} D_n$ and hence D is either finite or countably infinite.