2. Let $X_1, \dots, X_n, \dots$ be a sequence of i.i.d. r.v. Derive the limiting dist^{n}. of :

$$\frac{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - EX_1\right)}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2}} \quad \text{——} \quad \odot$$

, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

& $EX_1^2 < \infty$, assume $Var(X_1) > 0$.

Sol^{n}: By CLT,

$$\frac{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - EX_1\right)}{\sqrt{Var(X_1)}} \xrightarrow{d} Z \sim N(0,1).$$

Now, try to show that

$$\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2 \xrightarrow{P} Var(X_1)$$

$$\Rightarrow \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2} \xrightarrow{P} \sqrt{Var(X_1)}$$

$$\Rightarrow \frac{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2}}{\sqrt{Var(X_1)}} \xrightarrow{P} 1$$

So,

$$\frac{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - EX_1\right)}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2}} = (*)\left\{ \underbrace{\frac{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - EX_1\right)}{\sqrt{Var(X_1)}}}_{\xrightarrow{d} Z\sim N(0,1)} \times \underbrace{\frac{\sqrt{Var(X_1)}}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2}}}_{\xrightarrow{P} 1} \right.$$

$\odot \rightsquigarrow Z \sim N(0,1)$.

Hence, using Slutsky's result, (*) leads $\cancel{\sqrt{n}\left(\frac{1}{n}\sum X_i - EX_1\right)}$

A few examples :-

1. $X_n \sim$ t-distⁿ with n degrees of freedom.

Then $X_n \xrightarrow{d} X$, where $X \sim N(0,1)$.

Proof:- $X_n(\omega) \overset{d}{=} \dfrac{Z_1(\omega)}{\sqrt{Z_2(\omega)/n}}$, where — (1)

$Z_1 \sim N(0,1)$ & $Z_2 \sim \chi^2_n$.

By defⁿ of χ^2_n.

$Z_2 = Y_1^2 + \cdots + Y_n^2$, where

Y_i's are i.i.d $\sim N(0,1)$.

So, By WLLN,

$$\frac{Z_2}{n} = \frac{Y_1^2 + \cdots + Y_n^2}{n} \xrightarrow{P} EY_1^2 = 1. \quad \text{— (2)}$$

So, by Slutsky's theorem,

$$\frac{Z_1}{\sqrt{\frac{Z_2}{n}}} \xrightarrow{d} Z_3(\omega) \sim N(0,1), \text{ as}$$

$Z_1 \sim N(0,1)$ & $\frac{Z_2}{n} \xrightarrow{P} 1$ as $n \to \infty$

$\Rightarrow \sqrt{\frac{Z_2}{n}} \xrightarrow{P} 1$ as $n \to \infty$

as $g(x) = \sqrt{x}$ is a contⁿ fⁿ.

Hence, $X_n \xrightarrow{d} X$, where $X \sim N(0,1)$.