MSO205A PRACTICE PROBLEMS SET 7 SOLUTIONS

Question 1. Consider a discrete RV X with the p.m.f.

$$f_X(x) := \begin{cases} \frac{1}{4}\left(\frac{3}{4}\right)^x, \text{ if } x \in \{0,1,2,\cdots\}, \\ 0, \text{ otherwise.} \end{cases}$$

Compute the expectation $\mathbb{E}X$ and the variance $Var(X)$, if these exist.

Answer: We compute the mean and variance using the MGF. We have

$$M_X(t) = \mathbb{E}e^{tX} = \sum_{k=0}^{\infty} e^{tk}\frac{1}{4}\left(\frac{3}{4}\right)^k = \frac{\frac{1}{4}}{1-\left(\frac{3}{4}\right)e^t},$$

exists if $1-\left(\frac{3}{4}\right)e^t > 0$ or equivalently, $t < \ln\left(\frac{4}{3}\right)$. Note that

$$M_X'(t) = \frac{1}{4}\frac{1}{(1-\left(\frac{3}{4}\right)e^t)^2}\left(\frac{3}{4}\right)e^t, \quad M_X''(t) = \frac{2}{4}\frac{1}{(1-\left(\frac{3}{4}\right)e^t)^3}\left(\frac{3}{4}\right)^2 e^{2t} + \frac{1}{4}\frac{1}{(1-\left(\frac{3}{4}\right)e^t)^2}\left(\frac{3}{4}\right)e^t.$$

Evaluating at $t=0$, we have $\mathbb{E}X = 3, \mathbb{E}X^2 = 18+3 = 21$. Then $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = 12$.

Question 2. Consider a continuous RV X with the p.d.f.

$$f_X(x) := \begin{cases} \frac{1}{2}, \text{ if } x \in (-1,0), \\ \frac{1}{3}, \text{ if } x \in (0,\frac{3}{2}), \\ 0, \text{ otherwise.} \end{cases}$$

Compute the expectation $\mathbb{E}X$ and the variance $Var(X)$, if these exist.

Answer: Since $\mathbb{P}(X \in (-1,\frac{3}{2})) = 1$, we have the existence of $\mathbb{E}X$ and $\mathbb{E}X^2$. Now,

$$\mathbb{E}X = \int_{-1}^{0} x f_X(x)\,dx + \int_0^{\frac{3}{2}} x f_X(x)\,dx = \int_{-1}^{0}\frac{x}{2}\,dx + \int_0^{\frac{3}{2}}\frac{x}{3}\,dx = -\frac{1}{4}+\frac{3}{8} = \frac{1}{8}$$

and

$$\mathbb{E}X^2 = \int_{-1}^{0} x^2 f_X(x)\,dx + \int_0^{\frac{3}{2}} x^2 f_X(x)\,dx = \int_{-1}^{0}\frac{x^2}{2}\,dx + \int_0^{\frac{3}{2}}\frac{x^2}{3}\,dx = -\frac{1}{6}+\frac{3}{8} = \frac{5}{24}$$

and hence $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{40-3}{192} = \frac{37}{192}$.

Question 3. Let X be a continuous RV with p.d.f. f_X. Is $|X|$ also a continuous RV?

Answer: Given that $F_X(x) = \int_{-\infty}^x f_X(t)\,dt, \forall x \in \mathbb{R}$. Now,

$$F_{|X|}(y) = \begin{cases} \mathbb{P}(|X| \leq y) = 0, \text{ if } y < 0 \\ \mathbb{P}(|X| \leq 0) = \mathbb{P}(X = 0) = 0, \text{ if } y = 0. \end{cases}$$

Now, consider the case $y > 0$. Here,

$$F_{|X|}(y) = \mathbb{P}(|X| \leq y) = \mathbb{P}(-y \leq X \leq y) = \int_{-y}^{y} f_X(t)\,dt.$$

If f_X is continuous on $\mathbb{R}$, then $F_{|X|}$ is differentiable with

$$F'_{|X|}(y) = \begin{cases} 0, \text{ if } y \leq 0, \\ f_X(y) + f_X(-y), \text{ if } y > 0. \end{cases}$$

In this case, note that $\int_{-\infty}^{\infty} F'_{|X|}(y),\, dy = \int_{-\infty}^{\infty} f_X(y)\,dy = 1$ and hence, $|X|$ is continuous with p.d.f. given by $F'_{|X|}$.

Question 4. Let X an RV with $\mathbb{E}|X| < \infty$.

(i) If $\mathbb{P}(X \geq 0) = 1$ and $\mathbb{E}X = 0$, show that $\mathbb{P}(X = 0) = 1$.
(ii) If $\mathbb{P}(X \geq 1) = 1$, then show that $\mathbb{E}X \geq 1$.
(iii) If X is a discrete RV such that $\mathbb{P}(X \in \{0, 1, 2, \cdots\}) = 1$ and $\mathbb{E}X < 1$, then show that $\mathbb{P}(X = 0) > 0$.

Answer: Let X be discrete with support S_X and p.m.f. f_X. Since $\mathbb{P}(X \geq 0) = 1$, we have $S_X \subseteq [0, \infty)$. If possible, let $0 < \alpha \in S_X$. In particular, $f_X(\alpha) > 0$. Then

$$\mathbb{E}X = \sum_{x \in S_X} x f_X(x) \geq \alpha f_X(\alpha) > 0.$$

This contradicts the fact $\mathbb{E}X = 0$. Hence, $S_X = \{0\}$, i.e. $\mathbb{P}(X = 0) = 1$.

Note that for a continuous RV X with p.d.f. f_X and $\mathbb{P}(X \geq 0) = 1$, we must have $\mathbb{E}X > 0$. To see this, note that $F_X(0) = 1 - \mathbb{P}(X \geq 0) = 0$. Then $F_X(x) = 0, \forall x < 0$ and hence $f_X(x) = 0, \forall x < 0$. By continuity of the DF F_X, given $\epsilon > 0$, there exists $\delta > 0$ such that $F_X(h) = |F_X(h) - F_X(0)| < \epsilon$, for all $0 < h < \delta$.
Then $\mathbb{P}(X > \frac{\delta}{2}) = 1 - F_X(\frac{\delta}{2}) > 1 - \epsilon$ and hence

$$\mathbb{E}X = \int_0^{\infty} x f_X(x)\,dx \geq \int_{\frac{\delta}{2}}^{\infty} x f_X(x)\,dx \geq \frac{\delta}{2}\mathbb{P}(X > \frac{\delta}{2}) > 0.$$

Question 5. Let X be an RV with $\mathbb{E}X^2 < \infty$. Show that $Var(X) = 0$ if and only if $\mathbb{P}(X = \mu_1') = 1$.

Answer: If $\mathbb{P}(X = \mu_1') = 1$, then X is degenerate at μ_1' with $\mathbb{E}X^2 = (\mu_1')^2$. Then $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = 0$.

On the other hand, suppose $Var(X) = \mathbb{E}(X - \mu_1')^2 = 0$. Consider the RV $Y := (X - \mu_1')^2$. Then $\mathbb{P}(Y \geq 0) = 1$ and $\mathbb{E}Y = 0$. By problem 3 above, we conclude $\mathbb{P}(Y = 0) = 1$, i.e. $\mathbb{P}((X - \mu_1')^2 = 0) = 1$ or $\mathbb{P}(X = \mu_1') = 1$.

Question 6. Fix a positive integer n. Find examples of discrete/continuous RVs such that $\mathbb{E}X^n$ exists but $\mathbb{E}X^{n+1}$ does not exist.

Answer: For positive integers $k \geq 2$, observe that $c_k := \sum_{m=1}^{\infty} m^{-k} < \infty$. Then the function $f : \mathbb{R} \to [0, 1]$ given by

$$f(x) = \begin{cases} \frac{1}{c_k} x^{-k}, & \text{if } x \in \{1, 2, \cdots\} \\ 0, & \text{otherwise} \end{cases}$$

is a p.m.f.. Let X_k be a discrete RV with this p.m.f..

Then, for any positive integer n

$$\frac{1}{c_{n+2}} \sum_{m=1}^{\infty} \frac{m^n}{m^{n+2}} = \frac{1}{c_{n+2}} \sum_{m=1}^{\infty} \frac{1}{m^2} < \infty,$$

but

$$\frac{1}{c_{n+2}} \sum_{m=1}^{\infty} \frac{m^{n+1}}{m^{n+2}} = \frac{1}{c_{n+2}} \sum_{m=1}^{\infty} \frac{1}{m} = \infty.$$

Therefore, $\mathbb{E}X_{n+2}^n$ exists, but $\mathbb{E}X_{n+2}^{n+1}$ does not exist.

For real numbers $\alpha > 1$, we have $c_\alpha := \int_1^\infty \frac{1}{x^\alpha}\, dx < \infty$. Then the function $f : \mathbb{R} \to [0, \infty]$ given by

$$f(x) = \begin{cases} \frac{1}{c_\alpha} x^{-\alpha}, & \text{if } x > 1 \\ 0, & \text{otherwise} \end{cases}$$

is a p.d.f.. Let X_α be a continuous RV with this p.d.f..

As verified in the discrete case above, we can check that $\mathbb{E}X_{n+2}^n$ exists, but $\mathbb{E}X_{n+2}^{n+1}$ does not exist.

Question 7. Let X be an RV with $\mathbb{E}|X - a|^n < \infty$, where $n > 1$ is some positive integer and a is some real number. Choose a positive integer m with $m \leq n$ and let b be any real number. Show that $\mathbb{E}(X - b)^m$ exists. Is it true that

$$\mathbb{E}(X - b)^m = \sum_{k=0}^{m} \binom{m}{k} (-b)^{m-k}\, \mathbb{E}X^k \ ?$$

Answer: First, note that $\mathbb{E}|X-a|^k < \infty$ for all $k = 1, 2, \cdots, n$. Now, Observe that for positive integers m with $m \leq n$,

$$\mathbb{E}|X-b|^m = \mathbb{E}|(X-a)+(a-b)|^m \leq 2^{m-1}\left(\mathbb{E}|X-a|^m + |a-b|^m\right) < \infty$$

and hence $\mathbb{E}(X-b)^m$ exists. In particular, for $b=0$, we get the existence of $\mathbb{E}X^m$ for all $m \leq n$.

Applying Binomial Theorem, we have

$$(X-b)^m = \sum_{k=0}^{m} \binom{m}{k} (-b)^{m-k}\, X^k$$

and applying the linearity of expectation, we get the result.

Question 8. Compute the MGF in each of the following cases and hence compute the mean and Variance.

(a) Fix $p \in (0,1)$ and let n be a positive integer. Consider a discrete RV X with the p.m.f.

$$f_X(x) := \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, \text{ if } x \in \{0, 1, 2, \cdots, n\}, \\ 0, \text{ otherwise.} \end{cases}$$

(b) Fix $\lambda > 0$. Consider a continuous RV X with the p.d.f.

$$f_X(x) := \begin{cases} \lambda e^{-\lambda x}, \text{ if } x > 0, \\ 0, \text{ otherwise.} \end{cases}$$

Answer: (a) Note that $\mathbb{P}(0 \leq X \leq n) = 1$ and hence $\mathbb{P}(1 \leq e^{tX} \leq e^{tn}) = 1$ for all $t \in \mathbb{R}$. In particular, $M_X(t) = \mathbb{E}e^{tX}$ exists for all $t \in \mathbb{R}$. Now,

$$M_X(t) = \mathbb{E}e^{tX} = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k} = (1-p+pe^t)^n.$$

Then, we have $\frac{d}{dt}M_X(t) = n(1-p+pe^t)^{n-1}pe^t$ and $\frac{d^2}{dt^2}M_X(t) = n(n-1)(1-p+pe^t)^{n-2}p^2e^{2t} + n(1-p+pe^t)^{n-1}pe^t$. Evaluating at $t=0$ gives $\mathbb{E}X = np$ and $\mathbb{E}X^2 = n(n-1)p^2 + np$. Then $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = n(n-1)p^2 + np - n^2p^2 = np(1-p)$.

(b) Note that

$$\mathbb{E}e^{tX} = \int_0^\infty e^{tx}\lambda e^{-\lambda x}\, dx = \left(1 - \frac{t}{\lambda}\right)^{-1}, \forall t < \lambda,$$

and in particular, $\mathbb{E}e^{tX}$ exists for $t \in (-\lambda, \lambda)$. Therefore, the MGF M_X exists and $M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}, \forall t < \lambda$. Then, we have $\frac{d}{dt}M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-2}\frac{1}{\lambda}$ and $\frac{d^2}{dt^2}M_X(t) = 2\left(1 - \frac{t}{\lambda}\right)^{-3}\frac{1}{\lambda^2}$. Evaluating at $t=0$ gives $\mathbb{E}X = \frac{1}{\lambda}$ and $\mathbb{E}X^2 = \frac{2}{\lambda^2}$. Then $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{1}{\lambda^2}$.