MSO205A PRACTICE PROBLEMS SET 3 SOLUTIONS

Question 1. Fix $p \in (0,1)$. Suppose we have a coin such that tossing it results in a head with probability p and a tail with probability $1-p$. The coin is tossed twice independently and the number X of heads is observed. Find the law of X, i.e. $\mathbb{P}(X \in A)$ for all subsets A of $\mathbb{R}$.

Answer: Since the coin is tossed twice independently, the events of obtaining a head or a tail in the first toss is independent of the events of obtaining a head or a tail in the second toss. Here, the sample space is $\Omega = \{HH, HT, TH, TT\}$ with

$$\mathbb{P}(HH) = p \times p = p^2, \quad \mathbb{P}(HT) = \mathbb{P}(TH) = p(1-p), \quad \mathbb{P}(TT) = (1-p)^2.$$

If X denotes the number of heads obtained, then possible values of X can only be $0, 1$ or 2 with the following identification of the events

$$X^{-1}(\{0\}) = (X = 0) = \{\omega \in \Omega : X(\omega) = 0\} = \{TT\},$$
$$X^{-1}(\{1\}) = (X = 1) = \{\omega \in \Omega : X(\omega) = 1\} = \{HT, TH\},$$
$$X^{-1}(\{2\}) = (X = 2) = \{\omega \in \Omega : X(\omega) = 2\} = \{HH\}.$$

Then

$$\mathbb{P}(X = 0) = (1-p)^2, \quad \mathbb{P}(X = 1) = 2p(1-p), \quad \mathbb{P}(X = 2) = p^2.$$

and for any subset A of $\mathbb{R}$, we have

$$\mathbb{P} \circ X^{-1}(A) = \mathbb{P}(\{\omega : X(\omega) \in A\}) = \sum_{i \in \{0,1,2\} \cap A} \mathbb{P}(X = i).$$

The above relation describes the law of X.

Question 2. Suppose that an event A is independent of itself. What can you say about $\mathbb{P}(A)$?

Answer: Note that $A \cap A = A$ and using independence, we have $\mathbb{P}(A) = \mathbb{P}(A \cap A) = (\mathbb{P}(A))^2$ and hence $\mathbb{P}(A)$ is either 0 or 1.