MSO205A PRACTICE PROBLEMS SET 1 SOLUTIONS

Question 1. Write down the sample spaces of the following random experiments.

(a) Shuffle a standard deck of cards and draw the first card.

(b) A box contains 3 identical red balls and 2 identical green balls. Draw a ball from the box blindfolded and then check (and note down) the colour of the ball. Put the ball back into the box.

(c) Throw a standard six-sided die two times and add up the two numbers obtained.

Answer:

(a) A standard deck of cards contains cards from four suits, viz. Clubs ($\clubsuit$), Diamonds ($\diamondsuit$), Hearts ($\heartsuit$) and Spades ($\spadesuit$). Each suit has 13 cards distinguished by numbers, viz. 1 (also A or Ace), 2, 3, ..., 9, 10, 11 (also J or Jack), 12 (also Q or Queen), 13 (also K or King). Therefore, each card is identified with a suit and a number and hence the sample space is

$$\Omega = \{(x, y) : x \in \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}, y \in \{1, 2, \cdots, 13\}\}.$$

(b) Since the red balls are identical, we shall not be able to distinguish between them. Similarly, the green balls are indistinguishable from each other. Since we are only noting the colour of the ball drawn, there are only two outcomes in this experiment. The sample space may be written as $\Omega = \{R, G\}$, where R and G stand for the outcomes where a red and a green ball has been observed respectively.

(c) If x and y denote the outcomes of the two throws respectively, then we want to look at the sum $x+y$. Since x and y vary in the set $\{1, 2, 3, 4, 5, 6\}$, the sample space of this experiment is

$$\Omega = \{x + y : x, y \in \{1, 2, 3, 4, 5, 6\}\} = \{2, 3, \cdots, 12\}.$$

Question 2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space associated with a random experiment $\mathcal{E}$.

(a) Let $B \in \mathcal{F}$ be such that $\mathbb{P}(B) = 1$. For any event $A \in \mathcal{F}$ that $\mathbb{P}(A) = \mathbb{P}(A \cap B)$. (Hint: What is $\mathbb{P}(A \cap B^c)$?)

(b) (Boole's inequality) Let $n \geq 2$ be any integer and let $E_1, E_2, \cdots, E_n$ be events in $\mathcal{F}$. Prove that
$$\mathbb{P}(\bigcup_{i=1}^{n} E_i) \leq \sum_{i=1}^{n} \mathbb{P}(E_i).$$
(c) Let $\{E_n\}_n$ be a sequence of events in $\mathcal{F}$. Show that
$$\mathbb{P}(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mathbb{P}(E_n).$$
(Hint: Take $A_1 := E_1$ and for $n \geq 2$, define $A_n := E_n \cap (E_1 \cup E_2 \cup \cdots \cup E_{n-1})^c$. Prove that $\cup_n A_n = \cup_n E_n$. Use the A_n's.)

(d) Let $n \geq 2$ be any integer and let $E_1, E_2, \cdots, E_n$ be events in $\mathcal{F}$. Prove that
$$\mathbb{P}(\bigcap_{i=1}^{n} E_i) \geq \sum_{i=1}^{n} \mathbb{P}(E_i) - (n-1).$$

Answer:

(a) We have $\mathbb{P}(B^c) = 1 - \mathbb{P}(B) = 0$. Now, $A \cap B^c \subseteq B^c$ and hence $\mathbb{P}(A \cap B^c) = 0$ (by the Monotonicity property of $\mathbb{P}$ discussed in the lecture notes).

Since $A \cap B$ and $\mathcal{A} \cap B^c$ are mutually exclusive, we have by finite additivity of $\mathbb{P}$,
$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) = \mathbb{P}(A \cap B).$$

(b) The case $n = 2$ has been discussed in the lecture notes. We apply the principle of Mathematical Induction to prove the result for general $n \geq 3$.

Assume that the result is true for $n-1$ events. Then applying the result for two events, we have
$$\mathbb{P}(\bigcup_{i=1}^{n} E_i) = \mathbb{P}(\left(\bigcup_{i=1}^{n-1} E_i\right) \bigcup E_n) \leq \mathbb{P}\left(\bigcup_{i=1}^{n-1} E_i\right) + \mathbb{P}(E_n) \leq \sum_{i=1}^{n} \mathbb{P}(E_i).$$
The last inequality follows from our assumption. The principle of Mathematical Induction now implies that the result holds for all $n \geq 2$.

(c) Take the A_n's as described in the Hint. By construction, $A_n \subseteq E_n, \forall n$ and $A_n = E_n \cap E_1^c \cap E_2^c \cap \cdots \cap E_{n-1}^c \subseteq E_m^c, \forall n > m$. Therefore for $i \neq j$, $A_i \cap A_j = \emptyset$, i.e. A_n's are mutually exclusive.

Since, $A_n \subseteq E_n, \forall n$, we have $\mathbb{P}(A_n) \leq \mathbb{P}(E_n), \forall n$ and $\cup_{n=1}^{\infty} A_n \subseteq \cup_{n=1}^{\infty} E_n$. If $\omega \in \cup_{n=1}^{\infty} E_n$, then ω is an element of E_n for at least one n. Let k be the natural number such that $\omega \in E_k$ with $\omega \notin E_i, \forall i < k$. Then $\omega \in E_k \cap E_1^c \cap \cdots \cap E_{k-1}^c = A_k$. Hence $\cup_{n=1}^{\infty} E_n \subseteq \cup_{n=1}^{\infty} A_n$. Combining both inclusions, we conclude $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} E_n$.

By the countable additivity of $\mathbb{P}$, we have

$$\mathbb{P}(\bigcup_{i=1}^{\infty} E_i) = \mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i) \leq \sum_{i=1}^{\infty} \mathbb{P}(E_i).$$

(d) This argument is similar to part (b). The case $n = 2$ has been discussed in the lecture notes. We apply the principle of Mathematical Induction to prove the result for general $n \geq 3$.

Assume that the result is true for $n-1$ events. Then applying the result for two events, we have

$$\mathbb{P}(\bigcap_{i=1}^{n} E_i) = \mathbb{P}(\left(\bigcap_{i=1}^{n-1} E_i\right) \bigcap E_n) \geq \mathbb{P}\left(\bigcap_{i=1}^{n-1} E_i\right) + \mathbb{P}(E_n) - 1 \geq \sum_{i=1}^{n} \mathbb{P}(E_i) - (n-1).$$

The last inequality follows from our assumption. The principle of Mathematical Induction now implies that the result holds for all $n \geq 2$.

Question 3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space associate with a random experiment $\mathcal{E}$. Let A, B, C be events such that $\mathbb{P}(A) = 0.3, \mathbb{P}(B) = 0.4, \mathbb{P}(A \cap B) = 0.2, \mathbb{P}(C) = 0.1$. Further assume that A, C are mutually exclusive and B, C are mutually exclusive.

Find the probabilities that

(a) exactly one of the events A or B occurs
(b) at least one of the events A, B or C occurs
(c) none of A and B will occur
(d) A occurs, but C does not occur.

Answer:

(a) The event in question is $(A \cap B^c) \cup (A^c \cap B)$. The events $A \cap B^c$ and $A^c \cap B$ are mutually exclusive and hence by finite additivity of $\mathbb{P}$, we have

$$\mathbb{P}((A \cap B^c) \cup (A^c \cap B)) = \mathbb{P}(A \cap B^c) + \mathbb{P}(A^c \cap B).$$

Now, $\mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A \cap B) = 0.3 - 0.2 = 0.1$. Again, $\mathbb{P}(A^c \cap B) = \mathbb{P}(B) - \mathbb{P}(A \cap B) = 0.4 - 0.2 = 0.2$.

Hence the required probability is $\mathbb{P}((A \cap B^c) \cup (A^c \cap B)) = 0.1 + 0.2 = 0.3$.

(b) The event in question is $A \cup B \cup C$.

Since A, C are mutually exclusive and B, C are mutually exclusive, we have $A \cap C = \emptyset, B \cap C = \emptyset$. Hence, $(A \cup B) \cap C = \emptyset$. Therefore, the required probability is

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A \cup B) + \mathbb{P}(C) = [\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)] + \mathbb{P}(C) = [0.3 + 0.4 - 0.2] + 0.1 = 0.6.$$

(Note: The same probability may also be computed using the Inclusion-Exclusion principle. Since, $A \cap C = \emptyset, B \cap C = \emptyset$, many of the terms in the formula are zero.)

(c) The event in question is $A^c \cap B^c$. Note that $\mathbb{P}(A^c \cap B^c) = \mathbb{P}((A \cup B)^c) = 1 - \mathbb{P}(A \cup B) = 1 - [\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)] = 0.5$.

(d) The event in question is $A \cap C^c$. Since $A \cap C = \emptyset$, we have $A \subseteq C^c$ and hence $A \cap C^c = A$. Therefore, $\mathbb{P}(A \cap C^c) = \mathbb{P}(A) = 0.3$.

Question 4 (Matching problem). A secretary types 3 letters and prepares 3 corresponding envelopes. In a hurry, she places one letter in each envelope at random. What is the probability that at least one letter is in the correct envelope?

Answer: For $i = 1, 2, 3$, consider the events E_i that the i-th letter is in the correct envelope. We need to find $\mathbb{P}(E_1 \cup E_2 \cup E_3)$.

Since one letter has been placed in each envelope, this is an experiment performed at random without replacement. Therefore for any $i = 1, 2, 3$, in the event E_i, the other letters can be placed in any of the remaining $3 - 1 = 2$ envelopes and hence $\mathbb{P}(E_i) = \frac{2!}{3!} = \frac{1}{3}$. Similarly, for $i \neq j$, $\mathbb{P}(E_i \cap E_j) = \frac{1}{6}$ (since 2 letters in the correct envelope means that the last letter must go to the correct envelope) and $\mathbb{P}(E_1 \cap E_2 \cap E_3) = \frac{1}{6}$.

By the Inclusion-Exclusion principle,

$$\mathbb{P}(E_1 \cup E_2 \cup E_3) = \binom{3}{1} \times \frac{1}{3} - \binom{3}{2} \times \frac{1}{6} + \binom{3}{3} \times \frac{1}{6} = 1 - \frac{1}{2} + \frac{1}{6} = \frac{2}{3}.$$

Question 5. Draw 3 cards successively at random without replacement from a standard deck of 52 cards. Find the probability that exactly 2 cards are King and one card is a Queen.

Answer: We are performing the drawing of cards at random. Here, the total number of hands containing 3 cards is $\binom{52}{3}$.

Consider the given event E where exactly 2 cards are King and one card is a Queen. Since a standard deck contains 4 King and 4 Queen cards, number of ways favourable to event E is $\binom{4}{2}\binom{4}{1} = 24$.

The required probability is $24 \times \binom{52}{3}^{-1} = \frac{24 \times 6}{52 \times 51 \times 50} = \frac{6}{13 \times 17 \times 25} = \frac{6}{5525}$.