**MSO205A PRACTICE PROBLEMS SET 8 SOLUTIONS**

*Question* 1. Let $X \sim Binomial(n,p)$ for some integer $n \geq 3$ and $p \in (0,1)$. Compute $\mathbb{E}X(X-1)(X-2)$, if it exists.

Answer: If $\mathbb{E}|X(X-1)(X-2)| < \infty$, then $\mathbb{E}X(X-1)(X-2)$ exists. Now,

$$\begin{aligned}
\mathbb{E}|X(X-1)(X-2)| &= \sum_{k=0}^{n} |k(k-1)(k-2)| \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\
&= n(n-1)(n-2)p^3 \sum_{k=3}^{n} \frac{(n-3)!}{(k-3)!(n-k)!} p^{k-3} (1-p)^{n-k} \\
&= n(n-1)(n-2)p^3 \left(p + (1-p)\right)^{n-3} \\
&= n(n-1)(n-2)p^3 < \infty.
\end{aligned}$$

Hence, $\mathbb{E}X(X-1)(X-2)$ exists and

$$\mathbb{E}X(X-1)(X-2) = \sum_{k=0}^{n} k(k-1)(k-2) \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} = n(n-1)(n-2)p^3.$$

*Question* 2. Verify that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Answer: We have $\Gamma(\frac{1}{2}) = \int_0^\infty \frac{1}{\sqrt{x}} e^{-x}\, dx$. First, we change variables $x = \frac{y^2}{2}$ and hence,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{2} \int_0^\infty e^{-\frac{y^2}{2}}\, dy.$$

Squaring the above relation and going to polar co-ordinates, we have

$$\begin{aligned}
\left[\Gamma\left(\frac{1}{2}\right)\right]^2 &= 2 \left(\int_0^\infty e^{-\frac{x^2}{2}}\, dx\right) \left(\int_0^\infty e^{-\frac{y^2}{2}}\, dy\right) \\
&= 2 \int_0^\infty \int_0^\infty e^{-\frac{x^2+y^2}{2}}\, dx\, dy \\
&= 2 \int_{r=0}^\infty \int_{\theta=0}^{\frac{\pi}{2}} e^{-\frac{r^2}{2}}\, r dr\, d\theta \\
&= \pi
\end{aligned}$$

and hence $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

*Question* 3. Let $X \sim N(\mu, \sigma^2)$ for $\mu \in \mathbb{R}, \sigma > 0$. Compute $\mathbb{E}X^k$ for $k = 2, 3, 4$. [Hint: When $X \sim N(0,1)$, these moments has been computed in the lecture notes.]

Answer: Consider the RV $Y = \frac{X-\mu}{\sigma} \sim N(0,1)$. We have already seen that

$$\mathbb{E}Y = 0, \quad \mathbb{E}Y^2 = 1, \quad \mathbb{E}Y^3 = 0, \quad \mathbb{E}Y^4 = 3.$$

Since, $X = \sigma Y + \mu$, we have

$$\mathbb{E}X^2 = \mathbb{E}(\sigma Y + \mu)^2 = \sigma^2 \mathbb{E}Y^2 + 2\sigma\mu\, \mathbb{E}Y + \mu^2 = \mu^2 + \sigma^2,$$

$$\mathbb{E}X^3 = \mathbb{E}(\sigma Y + \mu)^3 = \sigma^3 \mathbb{E}Y^3 + 3\sigma^2\mu\, \mathbb{E}Y^2 + 3\sigma\mu^2\, \mathbb{E}Y + \mu^3 = \mu^3 + 3\mu\sigma^2,$$

and

$$\mathbb{E}X^4 = \mathbb{E}(\sigma Y + \mu)^4 = \sigma^4 \mathbb{E}Y^4 + 4\sigma^3\mu\, \mathbb{E}Y^3 + 6\sigma^2\mu^2\, \mathbb{E}Y^2 + 4\sigma\mu^3\, \mathbb{E}Y + \mu^4 = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4.$$

Note that we have established the existence of MGF of $X$ and hence existence of all momemts $\mathbb{E}X^k$ follow.

*Question* 4. Fix $\alpha > 0, \beta > 0$ and let $X \sim Beta(\alpha, \beta)$. Compute the MGF of $X$, if it exists.

Answer: Recall that the p.d.f. of $X$ is given by

$$f_X(x) = \begin{cases} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & \text{if } x \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

Since $e^{tX}$ is a non-negative RV for all $t \in \mathbb{R}$, to check the existence of $\mathbb{E}e^{tX}$, we need to check $\mathbb{E}e^{tX} < \infty$. Now,

$$\mathbb{E}e^{tX} = \frac{1}{B(\alpha,\beta)} \int_0^1 e^{tx} x^{\alpha-1}(1-x)^{\beta-1}\, dx \leq \frac{e^t}{B(\alpha,\beta)} \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, dx = e^t < \infty, \forall t \in \mathbb{R}.$$

Therefore, $M_X(t) = \mathbb{E}e^{tX}$ exists for all $t \in \mathbb{R}$. The MGF now can be computed by the Maclaurin's series expansion around the origin as

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbb{E}X^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{B(\alpha+k,\beta)}{B(\alpha,\beta)}.$$

*Question* 5. Let $X \sim Beta(1,1)$. Does the distribution of $X$ match with any other distribution discussed in the lecture notes?

Answer: The p.d.f. of $X$ is given by

$$f_X(x) = \begin{cases} \frac{1}{B(1,1)}, & \text{if } x \in (0,1) \\ 0, & \text{otherwise} \end{cases}$$

with $B(1,1) = \int_0^1 dx = 1$. Hence, $X \sim Uniform(0,1)$.

*Question* 6. An RV $X$ has the MGF given by the following expressions. Identify the distribution of $X$.

(a) $M_X(t) = (1 - \frac{t}{2})^{-3}, \forall t < 2$.
(b) $M_X(t) = \frac{1}{3}e^{-t} + \frac{2}{3}, \forall t \in \mathbb{R}$.

Answer: (a) Recall that an RV $Y \sim Gamma(\alpha, \beta)$ with $\alpha > 0, \beta > 0$ has the p.d.f.

$$f_Y(y) = \begin{cases} \frac{1}{\Gamma(\alpha)} y^{\alpha-1}\beta^{-\alpha}\exp(-\frac{y}{\beta}), & \text{if } y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

and the MGF

$$M_Y(t) = (1 - \beta t)^{-\alpha}, \forall t < \frac{1}{\beta}.$$

Since an MGF, if it exists, determines the distribution, we have $X \sim \Gamma(3, \frac{1}{2})$.

(b) Recall that for a discrete RV $Y$ with support $S_Y$ and p.m.f. $f_Y$, we have

$$M_Y(t) = \sum_{y \in S_y} e^{ty} f_Y(y).$$

Comparing with the given expression for the MGF, we have $S_X = \{-1, 0\}$ and

$$f_X(x) = \begin{cases} \frac{1}{3}, & \text{if } x = -1, \\ \frac{2}{3}, & \text{if } x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since $f_X$ above is a p.m.f. and an MGF, if it exists, determines the distribution, we have $X$ is discrete with support $S_X$ and p.m.f. $f_X$.

*Question* 7. Let $X$ be a continuous RV with $\mathbb{P}(X > 0) = 1$ and such that $\mu_1' = \mathbb{E}X$ exists. Prove that $\mathbb{P}(X > 2\mu_1') \leq \frac{1}{2}$.

Answer: We have $\mu_1' > 0$ (see Question 3, Problem set 5). Then, $\mathbb{P}(X > 2\mu_1') \leq \frac{1}{2\mu_1'}\mu_1' = \frac{1}{2}$.

*Question* 8. Let $x_1, x_2, \cdots, x_k > 0$ be distinct real numbers and let $n$ be a positive integer. Using Jensen's inequality discussed in the lecture notes, show that

$$\left(\frac{x_1 + x_2 + \cdots + x_k}{k}\right)^n \leq \frac{x_1^n + x_2^n + \cdots + x_k^n}{k}$$

Answer: Consider the convex function $h(x) = x^n$ on $[0, \infty)$. Look at the discrete RV $X$ with support $S_X = \{x_1, x_2, \cdots, x_k\}$ and p.m.f.

$$f_X(x) = \begin{cases} \frac{1}{k}, & \text{if } x \in S_X, \\ 0, & \text{otherwise.} \end{cases}$$

By Jensen's inequality, we have $h(\mathbb{E}X) \leq \mathbb{E}h(X)$ and hence,

$$\left(\frac{x_1 + x_2 + \cdots + x_k}{k}\right)^n \leq \frac{x_1^n + x_2^n + \cdots + x_k^n}{k}$$

*Question* 9. Let $x_1, x_2, \cdots, x_k, p_1, p_2, \cdots, p_k > 0$ be such that $\sum_{i=1}^k p_i = 1$. Prove the classical AM-GM-HM inequality using the AM-GM-HM inequality for RVs discussed in the lecture notes,

$$\sum_{i=1}^k x_i p_i \geq \prod_{i=1}^k x_i^{p_i} \geq \frac{1}{\sum_{i=1}^k \frac{p_i}{x_i}}$$

Answer: Consider a discrete RV $X$ with support $S_X = \{x_1, x_2, \cdots, x_k\}$ and p.m.f.

$$f_X(x) = \begin{cases} p_i, \text{ if } x = x_i, i \in \{1, 2, \cdots, k\}, \\ 0, \text{ otherwise.} \end{cases}$$

Then AM of $X$ is $\mathbb{E}X = \sum_{i=1}^k x_i p_i$, GM of $X$ is $e^{\mathbb{E}\ln X} = \prod_{i=1}^k x_i^{p_i}$ and HM of $X$ is $\frac{1}{\mathbb{E}[\frac{1}{X}]} = \frac{1}{\sum_{i=1}^k \frac{p_i}{x_i}}$. From the AM-GM-HM inequality proved for an RV $X$ in the lecture notes, we have the requied inequality.

*Question* 10. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $X = (X_1, X_2, X_3) : \Omega \to \mathbb{R}^3$ be a 3-dimensional random vector. State and prove the non-decreasing property of the joint DF of $X$.

Answer: For all real numbers $a_1 < b_1, a_2 < b_2, a_3 < b_3$, the required non-decreasing property is as follows.

$$\sum_{k=0}^3 (-1)^k \sum_{x \in \Delta_k^3} F_X(x) = \mathbb{P}(a_1 < X_1 \leq b_1, a_2 < X_2 \leq b_2, a_3 < X_3 \leq b_3) \geq 0,$$

where $\Delta_k^3, k = 0, 1, 2, 3$ denote the set of vertices of $\prod_{j=1}^3 (a_j, b_j]$ where exactly $k$ many $a_j$'s appear.

To prove this, consider the following three sets

$$A_1 := (-\infty, a_1] \times (-\infty, b_2] \times (-\infty, b_3],$$

$$A_2 := (-\infty, b_1] \times (-\infty, a_2] \times (-\infty, b_3],$$

and

$$A_3 := (-\infty, b_1] \times (-\infty, b_2] \times (-\infty, a_3].$$

Then,

$$\mathbb{P}(a_1 < X_1 \leq b_1, a_2 < X_2 \leq \mathbb{P}(A_1 \cup A_2 \cup A_3) b_2, a_3 < X_3 \leq b_3) = \mathbb{P}(X_1 \leq b_1, X_2 \leq b_2, X_3 \leq b_3) - \mathbb{P}(A_1 \cup A_2 \cup A_3).$$

The result follows by applying the inclusion-exclusion principle on $\mathbb{P}(A_1 \cup A_2 \cup A_3)$.