MSO205A PRACTICE PROBLEMS SET 10 SOLUTIONS

Question 1. Let $X = (X_1, X_2, X_3)$ be a continuous random vector with joint p.d.f.

$$f_X(x_1, x_2, x_3) = \begin{cases} \frac{\alpha}{x_1 x_2}, \text{ if } 0 < x_3 < x_2 < x_1 < 1, \\ 0, \text{ otherwise} \end{cases}$$

for some constant $\alpha \in \mathbb{R}$. Find the value of α and identify the marginal p.d.f.s of X_1, X_2 and X_3. Are X_1, X_2, X_3 independent? If not independent, find the conditional DF and conditional p.d.f. of X_2 given $(X_1, X_3) = (x_1, x_3)$ with $0 < x_3 < x_1 < 1$.

Answer: For $f_X(x_1, x_2, x_3) \geq 0$ for all $(x_1, x_2, x_3) \in \mathbb{R}^3$, we must have $\alpha > 0$. Moreover, we must have

$$\int_{x_1=0}^{1} \int_{x_2=0}^{x_1} \int_{x_3=0}^{x_2} f_X(x_1, x_2, x_3)\, dx_3 dx_2 dx_1 = 1.$$

Now,

$$\begin{aligned} &\int_{x_1=0}^{1} \int_{x_2=0}^{x_1} \int_{x_3=0}^{x_2} f_X(x_1, x_2, x_3)\, dx_3 dx_2 dx_1 \\ &= \int_{x_1=0}^{1} \int_{x_2=0}^{x_1} \frac{\alpha}{x_1}\, dx_2 dx_1 \\ &= \int_{x_1=0}^{1} \alpha\, dx_1 \\ &= \alpha. \end{aligned}$$

This yields $\alpha = 1$.

The marginal p.d.f. of X_1 is given by

$$f_{X_1}(x_1) = \begin{cases} \int_{x_2=0}^{x_1} \int_{x_3=0}^{x_2} \frac{1}{x_1 x_2}\, dx_3 dx_2, \text{ if } x_1 \in (0,1), \\ 0, \text{ otherwise} \end{cases} = \begin{cases} 1, \text{ if } x_1 \in (0,1), \\ 0, \text{ otherwise} \end{cases}$$

The marginal p.d.f. of X_2 is given by

$$f_{X_2}(x_2) = \begin{cases} \int_{x_1=x_2}^{1} \int_{x_3=0}^{x_2} \frac{1}{x_1 x_2}\, dx_3 dx_1, \text{ if } x_2 \in (0,1), \\ 0, \text{ otherwise} \end{cases} = \begin{cases} -\ln x_2, \text{ if } x_2 \in (0,1), \\ 0, \text{ otherwise} \end{cases}$$

The marginal p.d.f. of X_3 is given by

$$f_{X_3}(x_3) = \begin{cases} \int_{x_1=x_3}^{1} \int_{x_2=x_3}^{x_1} \frac{1}{x_1 x_2}\, dx_2 dx_1, \text{ if } x_3 \in (0,1), \\ 0, \text{ otherwise} \end{cases}$$

$$= \begin{cases} \int_{x_1=x_3}^{1} \frac{\ln x_1 - \ln x_3}{x_1}\, dx_1, \text{ if } x_3 \in (0,1), \\ 0, \text{ otherwise} \end{cases}$$

$$= \begin{cases} \int_{x_1=x_3}^{1} \ln x_1\, d(\ln x_1) - \int_{x_1=x_3}^{1} \frac{\ln x_3}{x_1}\, dx_1, \text{ if } x_3 \in (0,1), \\ 0, \text{ otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{2}(\ln x_3)^2, \text{ if } x_3 \in (0,1), \\ 0, \text{ otherwise} \end{cases}$$

Observe that

$$f_X\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right) = 6 \neq 1 \times \left(-\ln\frac{1}{3}\right) \times \frac{1}{2}\left(\ln\frac{1}{4}\right)^2 = f_{X_1}\left(\frac{1}{2}\right) f_{X_2}\left(\frac{1}{3}\right) f_{X_3}\left(\frac{1}{4}\right)$$

and hence X_1, X_2, X_3 are not independent.

The 2-dimensional marginal p.d.f. of (X_1, X_3) is given by

$$f_{X_1,X_3}(x_1,x_3) = \begin{cases} \int_{x_2=x_3}^{x_1} \frac{1}{x_1 x_2}\, dx_2, \text{ if } 0 < x_3 < x_1 < 1, \\ 0, \text{ otherwise} \end{cases} = \begin{cases} \frac{1}{x_1}(\ln x_1 - \ln x_3), \text{ if } 0 < x_3 < x_1 < 1, \\ 0, \text{ otherwise} \end{cases}$$

The conditional p.d.f. of X_2 given $(X_1, X_3) = (x_1, x_3)$ with $0 < x_3 < x_1 < 1$ is

$$f_{X_2|X_1,X_3}(x_2 \mid x_1, x_3) = \frac{f_X(x_1,x_2,x_3)}{f_{X_1,X_3}(x_1,x_3)} = \begin{cases} \frac{1}{x_2(\ln x_1 - \ln x_3)}, \text{ if } x_2 \in (x_3, x_1), \\ 0, \text{ otherwise} \end{cases}$$

and the conditional DF of X_2 given $(X_1, X_3) = (x_1, x_3)$ with $0 < x_3 < x_1 < 1$ is

$$F_{X_2|X_1,X_3}(x_2 \mid x_1, x_3) = \int_{-\infty}^{x_2} f_{X_2|X_1,X_3}(x \mid x_1, x_3)\, dx = \begin{cases} 0, \text{ if } x_2 \leq x_3, \\ \frac{\ln x_2 - \ln x_3}{\ln x_1 - \ln x_3}, \text{ if } x_2 \in (x_3, x_1), \\ 1, \text{ if } x_2 \geq x_1 \end{cases}$$

Question 2. Let $X = (X_1, X_2)$ be a bivariate continuous random vector with joint p.d.f. given by

$$f_X(x_1, x_2) = \begin{cases} 1, & \text{if } 0 < |x_2| \leq x_1 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find the marginal p.d.f.s of X_1 and X_2 and show that X_1, X_2 are not independent.

Answer: The marginal p.d.f. of X_1 is given by

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_X(x_1, x_2)\, dx_2 = \begin{cases} 0, \text{ if } x_1 \leq 0 \text{ or } x_1 \geq 1 \\ \int_{x_2=-x_1}^{x_1} 1\, dx_2, \text{ if } 0 < x_1 < 1 \end{cases} = \begin{cases} 0, \text{ if } x_1 \leq 0 \text{ or } x_1 \geq 1 \\ 2x_1, \text{ if } 0 < x_1 < 1 \end{cases}$$

The marginal p.d.f. of X_2 is given by

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} f_X(x_1, x_2)\, dx_1 = \begin{cases} 0, \text{ if } x_2 \leq -1 \text{ or } x_2 \geq 1 \text{ or } x_2 = 0 \\ \int_{x_1=|x_2|}^{1} 1\, dx_1, \text{ if } 0 < |x_2| < 1 \end{cases} = \begin{cases} 0, \text{ if } x_2 \leq -1 \text{ or } x_2 \geq 1 \text{ or } x_2 = 0 \\ 1 - |x_2|, \text{ if } 0 < |x_2| < 1 \end{cases}$$

Now, $f_X(\frac{1}{2}, \frac{1}{2}) = 1$, but $f_{X_1}(\frac{1}{2}) f_{X_2}(\frac{1}{2}) = 2\frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$. Hence, X_1 and X_2 are not independent.

Question 3. Let $X \sim \mathit{Exponential}(\lambda)$ for some $\lambda > 0$. For $r, s > 0$, show that

$$\mathbb{P}(X > r + s \mid X > r) = \mathbb{P}(X > s).$$

Note: This property is usually referred to as the 'no memory' property of the Exponential distribution.

Answer: The p.d.f. of X is

$$f_X(x) = \begin{cases} \frac{1}{\lambda} \exp(-\frac{x}{\lambda}), \text{ if } x > 0, \\ 0, \text{ otherwise.} \end{cases}$$

Then $\mathbb{P}(X > x) = \int_x^{\infty} f_X(t)\, dt = \exp(-\frac{x}{\lambda}), \forall x > 0$. Then

$$\mathbb{P}(X > r + s \mid X > r) = \frac{\mathbb{P}(X > r + s \text{ and } X > r)}{\mathbb{P}(X > r)} = \frac{\mathbb{P}(X > r + s)}{\mathbb{P}(X > r)} = \exp(-\frac{s}{\lambda}) = \mathbb{P}(X > s).$$

Question 4. Let $X_i \sim \mathit{Gamma}(\alpha_i, \beta), i = 1, 2, \cdots, n$ be independent RVs, with $\alpha_i > 0, \forall i$ and $\beta > 0$. Show that $X_1 + X_2 + \cdots + X_n \sim \mathit{Gamma}(\sum_{i=1}^n \alpha_i, \beta)$.

Answer: Note that the MGF $M_{X_i}(t) = (1 - \beta t)^{-\alpha_i}, \forall t < \frac{1}{\beta}$. Using independence of X_i's, we have

$$M_{X_1 + X_2 + \cdots + X_n}(t) = \prod_{i=1}^{n} M_{X_i}(t) = (1 - \beta t)^{-\sum_{i=1}^n \alpha_i}, \forall t < \frac{1}{\beta}.$$

Since the MGF, if it exists, determines the distribution, we conclude $X_1 + X_2 + \cdots + X_n \sim Gamma(\sum_{i=1}^n \alpha_i, \beta)$.

Question 5. Let $X \sim Gamma(\alpha_1, \beta), Y \sim Gamma(\alpha_2, \beta)$ be independent RVs, for some $\alpha_1, \alpha_2, \beta > 0$. Identify the distribution of $\frac{X}{X+Y}$.

Answer: We generalize the computation discussed in Example 8.7 of the lecture notes.
Using the independence of X and Y, the joint distribution of (X, Y) is given by the joint p.d.f.

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \begin{cases} \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x^{\alpha_1 - 1} y^{\alpha_2 - 1} \beta^{-(\alpha_1+\alpha_2)} \exp(-\frac{x+y}{\beta}), \text{ if } x > 0, y > 0 \\ 0, \text{ otherwise.} \end{cases}$$

Consider the function

$$h(x,y) = \begin{cases} (x+y, \frac{x}{x+y}), \forall x > 0, y > 0, \\ 0, \text{otherwise.} \end{cases}$$

Here, $\{(x,y) \in \mathbb{R}^2 : f_{X,Y}(x,y) > 0\} = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$ and $h : \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\} \to \mathbb{R}^2$ is one-to-one with range $(0,\infty) \times (0,1)$. The inverse function is $h^{-1}(u,v) = (uv, u(1-v))$ for $(u,v) \in (0,\infty) \times (0,1)$ with Jacobian determinant given by

$$J(u,v) = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u.$$

Now, $(U,V) = h(X,Y) = (X+Y, \frac{X}{X+Y})$ has the joint p.d.f. given by

$$\begin{aligned} f_{U,V}(u,v) &= \begin{cases} f_{X,Y}(uv, u(1-v))|J(u,v)|, \text{ if } u > 0, v \in (0,1) \\ 0, \text{ otherwise.} \end{cases} \\ &= \begin{cases} \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} u(uv)^{\alpha_1-1}(u(1-v))^{\alpha_2-1}\beta^{-(\alpha_1+\alpha_2)} \exp(-\frac{u}{\beta}), \text{ if } u > 0, v \in (0,1) \\ 0, \text{ otherwise.} \end{cases} \end{aligned}$$

The marginal p.d.f. f_V is given by

$$\begin{aligned} f_V(v) &= \int_{-\infty}^{\infty} f_{U,V}(u,v)\, du \\ &= \begin{cases} \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} v^{\alpha_1-1}(1-v)^{\alpha_2-1} \int_0^\infty u^{\alpha_1+\alpha_2-1}\beta^{-(\alpha_1+\alpha_2)} \exp(-\frac{u}{\beta})\, du, \text{ if } v \in (0,1) \\ 0, \text{ otherwise} \end{cases} \\ &= \begin{cases} \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} v^{\alpha_1-1}(1-v)^{\alpha_2-1}, \text{ if } v \in (0,1) \\ 0, \text{ otherwise.} \end{cases} \end{aligned}$$

Therefore $V = \frac{X}{X+Y} \sim Beta(\alpha_1, \alpha_2)$.

Question 6. If $X_1, X_2, \cdots, X_n$ are independent RVs with $X_i \sim N(\mu_i, \sigma_i^2)$, then find the distribution of $X_1 + X_2 + \cdots + X_n$.

Answer: First we consider the case $n = 2$. Since $(X_1, X_2) \sim N_2\left(\binom{\mu_1}{\mu_2}, \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}\right)$, we have the linear combination $X_1 + X_2$ follows $N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ distribution. (Note: This can also be checked using the change of variables approach)

Consider the case $n = 3$. Here, $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ and $X_3 \sim N(\mu_3, \sigma_3^2)$ are independent. Hence, using the above case $X_1 + X_2 + X_3 \sim N(\sum_{i=1}^3 \mu_i, \sum_{i=1}^3 \sigma_i^2)$.

By the principle of Mathematical Induction, $X_1 + X_2 + \cdots + X_n \sim N(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2)$.

Question 7. Let X and Y be i.i.d. $N(0,1)$ RVs. Fix $a \neq 0, b \neq 0$ and set $U := aX + bY, V := bX - aY$. Find the joint p.d.f. of U, V. Are U and V independent?

Answer: Since $(X_1, X_2) \sim N_2\left(\binom{0}{0}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$, using change of variables, we have the joint p.d.f of (U, V) as

$$f_{U,V}(u, v) = \frac{1}{2\pi(a^2 + b^2)} \exp\left[-\frac{1}{2}\left(\frac{u^2}{a^2 + b^2} + \frac{v^2}{a^2 + b^2}\right)\right], \forall (u, v) \in \mathbb{R}^2.$$

Since the joint p.d.f. can be factored into functions involving separate variables, U and V are independent.

Alternative method: Note that $(X, Y) \sim N_2\left(\binom{0}{0}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$. For $c_1, c_2 \in \mathbb{R}$, we have $c_1 U + c_2 V = (ac_1 + bc_2)X + (bc_1 - ac_2)Y$, which is a linear combination of X and Y and must have an one-dimensional Normal distribution. Using the characterization of bivariate Normal distribution, (U, V) follows a bivariate Normal distribution. We now determine the parameters. Since, $\mathbb{E}X = \mathbb{E}Y = 0$, we have

$$\mathbb{E}U = a\,\mathbb{E}X + b\,\mathbb{E}Y = 0, \quad \mathbb{E}V = b\,\mathbb{E}X - a\,\mathbb{E}Y = 0.$$

Again $\mathbb{E}X^2 = Var(X) = 1, \mathbb{E}Y^2 = Var(Y) = 1$ and $Cov(X, Y) = 0$. Since $\mathbb{E}U = \mathbb{E}V = 0$,

$$\mathbb{E}U^2 = Var(U) = a^2\mathbb{E}X^2 + b^2\mathbb{E}Y^2 + 2ab\,\mathbb{E}XY = a^2 + b^2$$

and

$$\mathbb{E}V^2 = Var(V) = b^2\mathbb{E}X^2 + a^2\mathbb{E}Y^2 - 2ab\,\mathbb{E}XY = a^2 + b^2.$$

Finally, $Cov(U, V) = \mathbb{E}(aX + bY)(bX - aY) = ab(\mathbb{E}X^2 - \mathbb{E}Y^2) = 0$.

Hence $(U, V) \sim N_2\left(\binom{0}{0}, \begin{pmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{pmatrix}\right)$ with the joint p.d.f.

$$f_{U,V}(u, v) = \frac{1}{2\pi(a^2 + b^2)} \exp\left[-\frac{1}{2}\left(\frac{u^2}{a^2 + b^2} + \frac{v^2}{a^2 + b^2}\right)\right], \forall (u, v) \in \mathbb{R}^2.$$

In particular, U and V are independent.