Rare Event Simulation using Interacting Particle Systems for Rare Credit Portfolio Losses

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May 15, 2017

Outline

The goal of our project is to highlight IPS models for rare event siulation and estimation specifically in the context of Credit portfolio defaults.
To that effect we talk about the following:

- Intuition and Motivation for IPS.
- Local intensity model of credit risk.
- Merton's model of credit risk.
    - We use an adapted model with stochastic volatility which is explained later.

Outline

Motivation

Merton's model
- Theory
- IPS Adaptation
- Results

Local Intensity model
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Intuition

- Given a Markov chain $S = (S_n)_{0 \leq n \leq T}$. At each n, we have d correlated assets, $S_n = (S_n^1, ..., S_n^d) \in E$.
- We wish to understand the nature of probabilities of rare events which are of the form $L(T) > K$ which can be expressed as below.

Rare Event Probability

$$\{L(T) \geq K\} = \{V_T(S_T) \geq K\} = \{V_T(S_0, ..., S_T) \geq K\}$$

$V(T)$ can be thought of as a real positive function that is a risk measure of these rare events.

- Now using the Cramer theory framework we express the probability as follows:

Risk Probability

$$\mathbb{P}(V_T(S_T) \geq K) = \mathbb{E}\left(\mathbf{1}_{\{V_T(S_T)\geq K\}} e^{\lambda V_T(S_T)} e^{-\lambda V_T(S_T)}\right)$$

which can be rewritten as

$$\mathbb{E}^{(\lambda)}\left(\mathbf{1}_{\{V_T(S_T)\geq K\}} e^{-\lambda V_T(S_T)}\right) \mathbb{E}\left(e^{\lambda V_T(S_T)}\right) = \mathbb{E}^{(\lambda)}(f_T(S_T))\mathbb{E}(e^{\lambda V_T(S_T)})$$

where

$$f_t(S_T) := \mathbf{1}_{\{V_T(S_T)\geq K\}} e^{-\lambda V_T(S_T)}$$

$$d\mathbb{P}^{(\lambda)} \propto e^{\lambda V_T(S_T)} d\mathbb{P}$$

With the convention that $V_0 = 0$, we get the following decomposition

$$e^{\lambda V_T(S_T)} \equiv \prod_{p=1}^{T} e^{\lambda(V_p(S_p) - V_{p-1}(S_{p-1}))}$$

By using the notation

$$\mathcal{X}_k = (S_k, S_{k+1})$$

for $0 \leq k < T$, the above produce can be rewritten as

$$\prod_{p=1}^{T} G_{p-1}(\mathcal{X}_{p-1})$$

where

$$G_{p-1}(\mathcal{X}_{p-1}) := e^{\lambda(V_p(S_p) - V_{p-1}(S_{p-1}))}$$

Using the notation that $F_T(\mathcal{X}_T) = f_T(S_T)$ we get

$$\mathbb{E}^{(\lambda)}(f_T(S_T)) = \frac{\mathbb{E}(F_T(\mathcal{X}_T)\prod_{p=1}^{T} G_p(\mathcal{X}_p))}{\mathbb{E}(\prod_{p=1}^{T} G_p(\mathcal{X}_p))} := \eta_T(F_T) \qquad (1)$$

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Credit Portfolio Model

- Model is based on a modified version Merton's model with the addition of a stochastic volatility term.
- We consider N assets in a portfolio each with a price $S_i(t)$ at time t.

Asset Evolution

$$dS_i(t) = rS_i(t)dt + \sigma_i \sigma(t) S_i(t) dW_i(t)$$

r - risk free interest rate, σ_i - Idiosyncratic volatility, W - Weiner process with the following correlation structure

$$d\langle W_i, W_j \rangle_t = \rho_{ij} dt$$

The stochastic volatility $\sigma(t)$ evolves according to the SDE, correlation structure

$$d\sigma(t) = \kappa(\bar{\sigma} - \sigma(t))dt + \gamma \sqrt{\sigma(t)} dW(t)$$

$$d\langle W_i, W \rangle_t = \rho_\sigma dt$$

Model Implementation

Markov Chain State

$$X_n = \left(\sigma(n\Delta t), (S_i(n\Delta t))_{1\leq i\leq N}, \min_{0\leq m\leq n}((S_i(m\Delta t)))\right)$$

- State X_n is $2N+1$ dimensional.
- Asset evolution done using Euler method with timestep granularity δt.
- For default condition we use a barrier price B_i going below which is treated as a default.
- To increase sampling of these rare events we use IPS methodology with the potential function $G_p(Y_p) = \exp[-\alpha(V(X_p) - V(X_{p-1}))]$ where $V(X_p) = \sum_{i=1}^{N} \log(min_{0\leq m\leq p} S_i(m\Delta t))$.

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Stages of IPS

- Involves 4 stages of operations.
- Initialization of price, volatility.
- n loops of:
  - Selection stage to select paths that could lead to the rare event of higher defaults.
  - Mutation stage to evolve the selected paths as per the SDEs.
- Termination stage to collect result of previous stage and estimate default probabilities.

Initialization

Initial Value

$$\hat{X}_0^{(j)} = (\sigma(0), S_1(0), \cdots, S_N(0), S_1(0), \cdots, S_n(0)), \quad \forall 1 \leq j \leq M$$
$$\hat{W}_0^{(j)} = \hat{X}_0^{(j)}$$

- All M portfolios are started with the same price and volatility.
- Initial minimum is same as initial price.
- Initial history is same as current initialization.

Selection

- Resample with replacement M paths from input M paths according to empirical distribution under the given Gibbs measure.
- $\left(\hat{W}_p^{(j)}, \hat{X}_p^{(j)}\right)$ becomes $\left(\breve{W}_p^{(j)}, \breve{X}_p^{(j)}\right)$.

Empirical distribution

$$\eta_p^M(dW, dX) = \frac{1}{M\hat{\eta}_p^M} \sum_{j=1}^{M} \exp\left[\alpha(V(\hat{X}_p^{(j)})) - V(\hat{W}_p^{(j)})\right] \times \delta_{(\hat{W}_p^j, \hat{X}_p^j)}(dW, dX)$$

Where

$$\hat{\eta}_p^M = \frac{1}{M} \sum_{j=1}^{M} \exp\left[\alpha(V(\hat{X}_p^{(j)})) - V(\hat{W}_p^{(j)})\right]$$

Mutation

- Evolve paths from $\check{X}_p^{(j)}$ to $\hat{X}_{p+1}^{(j)}$.
- Set $\hat{W}_{p+1}^{(j)} = \check{X}_p^{(j)}$.
- Evolve based on SDEs using Euler-Maruyama method with time step δt. Evolution is from t_p to t_{p+1} $(t_p + \Delta t)$.
- $\delta t \ll \Delta t$
- True Dynamics.

Termination

- Teminate at maturity time T by running the selection, mutation step n times. $(n\Delta t = T)$
- Estimate default probability $p_k(T) = \mathbb{P}(L(T) = k)$ using formula below.

Estimation

$$\hat{p}_k^M(T) = \left[\frac{1}{M}\sum_{j=1}^{M} \mathbf{1}_{\{f(\hat{X}_n^{(j)})=k\}} \exp\left[\alpha(V(\hat{W}^{(j)}) - V(\hat{X}_0))\right]\right] \times \left[\prod_{p=0}^{n-1} \hat{\eta}_p^M\right]$$

Where

$$f(X_n^{(j)}) = \sum_{i=1}^{N} \mathbf{1}_{\{X_n^{(j)}(N+1+i) \leq B_i\}}$$

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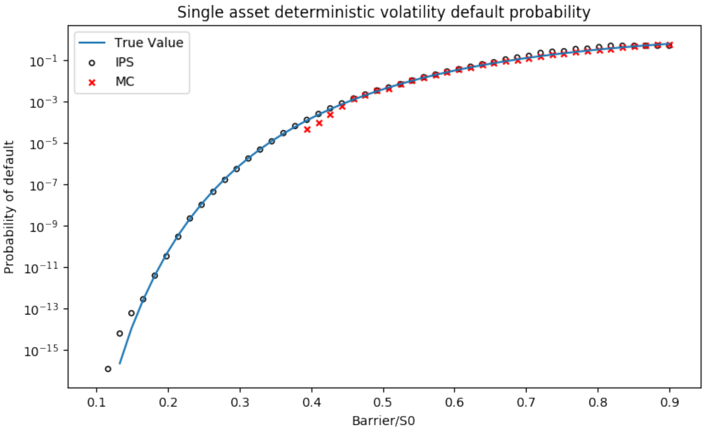
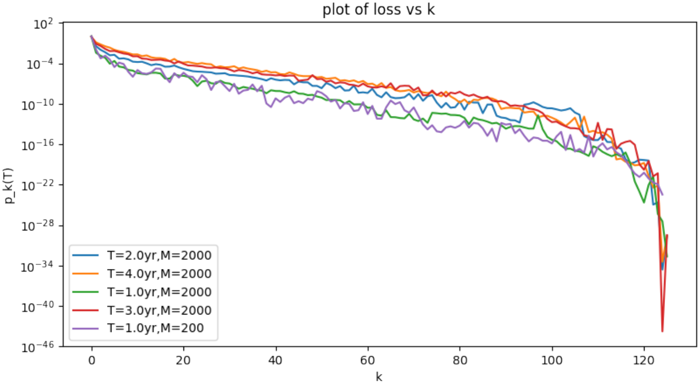
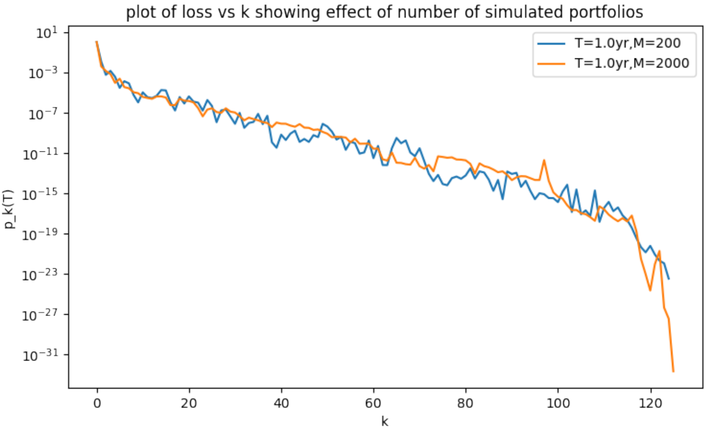
Plots


Single asset deterministic volatility default probability
True Value
IPS
MC
Probability of default
10^{-1}
10^{-3}
10^{-5}
10^{-7}
10^{-9}
10^{-11}
10^{-13}
10^{-15}
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
Barrier/S0

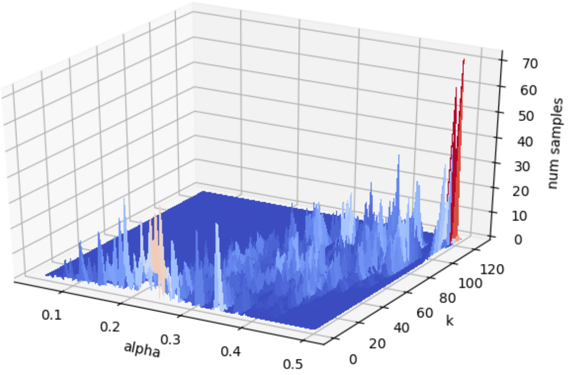
Plots


plot of loss vs k
p_k(T)
k
T=2.0yr,M=2000
T=4.0yr,M=2000
T=1.0yr,M=2000
T=3.0yr,M=2000
T=1.0yr,M=200

Plots


plot of loss vs k showing effect of number of simulated portfolios
T=1.0yr,M=200
T=1.0yr,M=2000
p_k(T)
k
10^{1}
10^{-3}
10^{-7}
10^{-11}
10^{-15}
10^{-19}
10^{-23}
10^{-27}
10^{-31}
0
20
40
60
80
100
120

Plots


70
60
50
40
30
20
10
0
num samples
0.1
0.2
0.3
0.4
0.5
alpha
0
20
40
60
80
100
120
k

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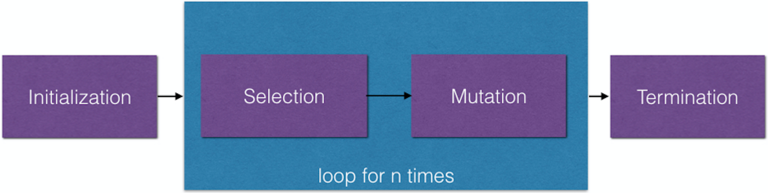

- *Initialization*: Initialize M states (t, L_t) and set up the parameters.
- *Selection*: Choose the states by resampling according weights of a potential function.
- *Mutation*: Evolve the system to the next state from the chosen states.
- *Termination*: Terminate after selecting and mutating for number of firms and estimate the probabilities

Initialization

Initial Value

$$\hat{X}_0^{(j)} = \left(t_0^{(j)}, L_{t_0}^{(j)}\right), \quad 1 \le j \le M$$

- $t_0^{(j)} = 0$ and $L_{t_0}^{(j)} = 0 \ \forall 1 \le j \le M$ considering L as a pure birth process.

Selection

- Sample independent M particles with repetition from input M paths according to empirical distribution under the given Gibbs measure.
- $\left(t_{n-1}^{(j)}, L_{t_{n-1}}^{(j)}\right)$ becomes $\left(\hat{t}_{n-1}^{(j)}, \hat{L}_{t_{n-1}}^{(j)}\right)$.

Empirical distribution

$$\eta_p^M(dX) = \frac{1}{M\hat{\eta}_p^M} \sum_{j=1}^{M} \left(\omega^\alpha(X_p^{(j)})\right) \times \delta_{X_p^{(j)}}(dX)$$

$$\text{Where} \quad \hat{\eta}_p^M = \mathbb{E}_p^m \omega^\alpha(X) = \frac{1}{M} \sum_{j=1}^{M} \left(\omega^\alpha(X_p^{(j)})\right)$$

$$\omega^\alpha(X) = \begin{cases} \exp(\alpha), & \text{if } t < T \\ 1, & \text{otherwise} \end{cases}$$

Mutation

- Evolve paths from $\left(\hat{t}_{n-1}^{(j)}, \hat{L}_{t_{n-1}}^{(j)}\right)$ to $\left(t_n^{(j)}, L_{t_n}^{(j)}\right)$.
- The function λ is used to take each step until t evolves to maturity time T.

Mutation Step

$$t_n^{(j)} = \min\left(\hat{t}_{n-1}^{(j)} + \lambda\left(\hat{t}_{n-1}^{(j)}, \hat{L}_{t_{n-1}}^{(j)}\right), T\right)$$

$$\lambda\left(\hat{X}\right) \sim \frac{1}{1 - \frac{L_t}{N}} \times \exp(-x)$$

$$L_{t_n}^{(j)} = \hat{L}_{t_{n-1}}^{(j)} + 1 \text{ if } t_n^{(j)} \neq T$$

Termination

- Teminate after running n loops of mutation and selection. Here, $n =$ maximum number of defaults that can happen.
- Estimate default probability $p_k(T) = \mathbb{P}(L(T) = k)$ using formula below.

Estimation

$$\tilde{p}_\ell^m(T, \alpha) = \mathbb{E}_n^m \delta_\ell(\mathcal{L}(X)) exp(-\alpha \mathcal{L}(X)) \prod_{i=0}^{n-1} \mathbb{E}_i^m \omega^\alpha(X)$$

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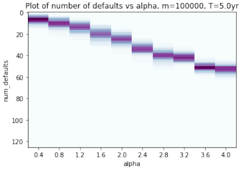
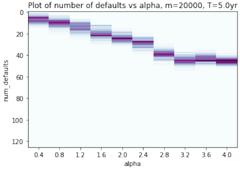
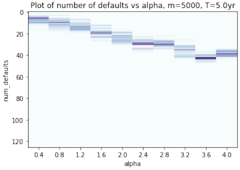
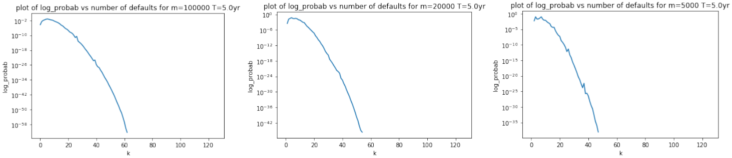
Plots of number of defaults vs α







Plots of log probabilities vs number of defaults



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Thank you!

Questions?