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Unit-1

→ Set Theory
→ Relation
→ Functions
→ Mathematical Induction

Set → A set is an unordered, well defined collection of ~~discrete~~ ~~of~~ distinct objects.

There are two methods to represent set →

1 Roaster / set builder method
2 Tabular method

$P = \{x \mid x$ is a positive integer and $x < 10\}$
$P = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Some common sets are :-

$N = \{0, 1, 2, 3, ----\}$ the set of natural numbers
$Z = \{---- -2, -1, 0, 1, 2 ----\}$ the set of integers
$Z^+ = \{1, 2, 3, ----\}$ the set of positive integers
$Q = \{P/q \mid P \in Z; q \in Z$, and $q \neq 0\}$, the set of rational numbers
$R = \{$ the set of real numbers $\}$

Finite set → If a set contains a finite number of distinguishable element.

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Infinite set → If a set contain an infinite number of elements. Ex → The set of +ve integers.

NOTE → $S = \{0, \{1,2\}, p, \{q\}\}$ is also a set which contains set in it

Equal set → Two sets will be equal if both the sets are subsets of each other.

Two sets are equal if and only if they have same elements. That is, if A and B are set of the +ve integers then A & B are equal if and only if $\forall x \; (x \in A \leftrightarrow x \in B)$, $A = B$, A and B are equal.

→ A is subset of $B \equiv \forall x (x \in A \rightarrow x \in B)$ is true.

Empty set (null set) → The set that has no elements, denoted by ϕ or $\{\}$.

Eg:- set of all +ve integers, greater than their square is an empty set.

Finite set → let S be a set. If there are exactly n distinct elements in S where n is a non-negative integer, we say that S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by $|S|$.

Eg:- Null set has no elements, $|\phi| = 0$.

Singleton set → If a set consists of one element, then it is said to be singleton set.

Power set → A set which contains all the subsets of a given set.

Eg → $\{1, 2, 3\}$

Power set $= \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

Power set of a null set : $\{\phi\}$

$$P(\phi) = \{\phi\}$$

$$P(\{\phi\}) = \{\phi, \{\phi\}\}$$

If a set has n elements, its power set has 2^n elements.

Properties of subset :-

① Null set is the subset of every set.
② Every set is the subset of itself.

Cardinality → Total no. of elements in a set.

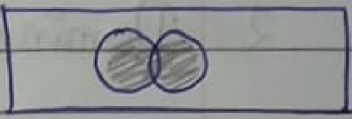
Proper Subset →

$$\boxed{A \subset B \Leftrightarrow A \subseteq B \land A \neq B}$$

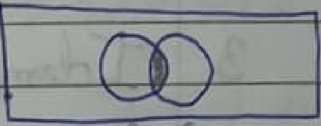
Set operations →

Union $A \cup B = \{x \mid x \in A \lor x \in B\}$

$A \cup B$

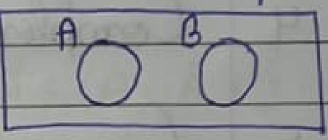


Intersection $A \cap B = \{x \mid x \in A \land x \in B\}$



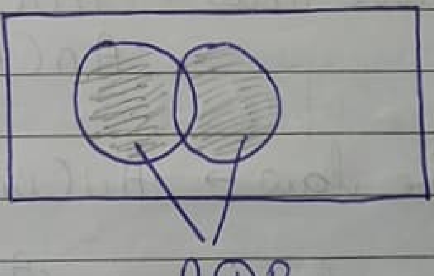
$A \cap B$

disjoint set when no common element is present
$A \cap B = \phi$



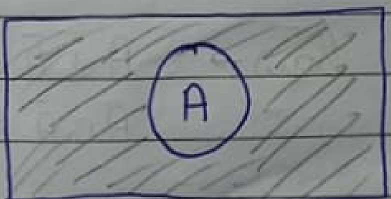
Set difference $\oplus$

$$A \oplus B = (A-B) \cup (B-A)$$
$$= \{x \mid x \in (A-B) \lor x \in (B-A)\}$$



$A \oplus B$

Complement $\bar{A} = \{x \mid x \notin A\}$ $\qquad \bar{A} = U - A$



T : U
F : ϕ or $\{\}$
$\land$: $\cap$
$\lor$: $\cup$

Power set → Collection of all subset of a given set

Set identities →

1 Identity law → $A \cup \phi = A$
$A \cap U = A$

2 Domination law → $A \cup U = U$
$A \cap \phi = \phi$

3 Idempotent law → $A \cup A = A$
$A \cap A = A$

4 Complementation law → $(\overline{\overline{A}}) = A$

5 Commutative law → $A \cup B = B \cup A$
$A \cap B = B \cap A$

6 Associative law → $A \cup (B \cup C) = (A \cup B) \cup C$
$A \cap (B \cap C) = (A \cap B) \cap C$

7 Distributive law → $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

8 Absorption law → $A \cup (A \cap B) = A$
$A \cap (A \cup B) = A$

9 Complement law → $A \cup \bar{A} = U$
$A \cap \bar{A} = \phi$

10 DeMorgan's law → $(\overline{A \cup B}) = \bar{A} \cap \bar{B}$
$(\overline{A \cap B}) = \bar{A} \cup \bar{B}$

Q- $A \cup (B-A) = A \cup B$

LHS

$A \cup (B-A) = \{x \mid x \in (A \cup (B-A))\}$

$= \{x \mid x \in A \text{ or } x \in (B-A)\}$

$= \{x \mid x \in A \text{ or } (x \in B \text{ and } x \notin A)\}$

$= \{x \mid (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \notin A)\}$ distributive

$= \{x \mid x \in (A \cup B) \text{ and } x \in (A \cup \bar{A})\}$

$= \{x \mid x \in (A \cup B) \wedge x \in U\}$ complement

$= \{x \mid x \in (A \cup B) \wedge U\}$

$= \{x \mid x \in (A \cup B) \cap U\}$

$= \{x \mid x \in (A \cup B)\}$ Identity

$(A \cup (B-A)) \subseteq (A \cup B)$ — ①

$(A \cup B) \subseteq (A \cup (B-A))$ — ②

To prove this eq^n take RHS and just go in reverse $direc^n$ of proof of RHS.

Q- $A - \phi = A$

LHS

$$A - \phi = \{x \mid x \in (A - \phi)\}$$
$$= \{x \mid (x \in A) \text{ and } (x \notin \phi)\}$$
$$= \{x \mid (x \in A) \cap (x \notin \phi)\}$$
$$= \{x \mid x \in (A \cap \overset{U}{\cancel{\phi}})\}$$
$$= \{x \mid x \in A\}$$

$A - \phi \subseteq A$ — ①

$A \subseteq A - \phi$ — ② → Same method.

Multiset → Multiset are the set in which an element can occur more than one.

$$A = \{a, a, a, b, b, c\}$$
$$= \{\underline{3} \cdot a, \underline{2} \cdot b, \underline{1} \cdot c\}$$

multiplicity of the element

Operations of multiset → $P = \{4 \cdot a, 2 \cdot b, 1 \cdot c\}$
$Q = \{3 \cdot a, 3 \cdot b, 2 \cdot d\}$

$P \cup Q = \{4 \cdot a, 3 \cdot b, 1 \cdot c, 2 \cdot d\}$

~~Mod~~

$P \cap Q = \{3 \cdot a, 2 \cdot b\}$

$P - Q = \{1 \cdot a, 1 \cdot c\}$

$P + Q = \{7 \cdot a, 5 \cdot b, 1 \cdot c, 2 \cdot d\}$

Generalized union & Intersection →

$$A_1 \cup A_2 \cup A_3 \cup ---- \cup A_n = \sum_{i=1}^{n} \cup A_i$$

$$= \bigcup_{i=1}^{n} A_i$$

$$A_1 \cup A_2 \cup A_3 ---- \cup A_n ---- \cup ---- = \bigcup_{i=1}^{\infty} A_i$$

$$A_1 \cap A_2 \cap A_3 ---- \cap A_n = \bigcap_{i=1}^{n} A_i$$

$$A_1 \cap A_2 \cap A_3 -------- = \bigcap_{i=1}^{\infty} A_i$$

Q- Let $A_i = \{i, i+1, i+2, ----\}$

$\bigcup_{i=1}^{n} A_i = ?$ & $\bigcap_{i=1}^{n} A_i = ?$

Sol^n $i = 1$

$A_1 = \{1, 2, 3 -----\}$

$A_2 = \{2, 3, 4 ------\}$

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$A_3 = \{3, 4, 5 ----- \}$

$\vdots$

$A_n = \{n, n+1, n+2 ---- \}$

$\bigcup_{i=1}^{n} A_i = \{1, 2, 3, 4, 5 ----- n, n+1, ---- \}$

$= Z^+$

$\bigcap_{i=1}^{n} A_i = \{n, n+1, n+2 ---- \infty\}$

$= A_n$

Q- find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ for $i \in Z^+$

a) $A_i = \{-i, -i+1, ---- -1, 0, 1, --- i-1, i\}$

b) $A_i = \{-i, i\}$

c) $A_i = [-i, i]$ that is the set of real numbers x with $-i \leq x \leq i$

d) $A_i = [i, \infty)$ that is the set of real number x with $x \geq i$.

a) $\bigcup_{i=1}^{\infty} A_i = \{-i, -i+1, ---- -1, 0, 1, ---- i-1, i\}$

$= Z$

$$\bigcap_{i=1}^{\infty} A_i = \{-1, 0, 1\}$$

b) $$\bigcup_{i=1}^{\infty} A_i = \{-i, -i+1, \dots -1, 1, \dots\dots i-1, i\}$$

$$= Z - \{0\}$$

$$\bigcap_{i=1}^{\infty} A_i = \phi$$

c $$\bigcup_{i=1}^{\infty} A_i = R$$

$$\bigcap_{i=1}^{\infty} A_i = [-1, 1]$$

d) $$\bigcup_{i=1}^{\infty} A_i = R^+$$

$$\bigcap_{i=1}^{\infty} A_i = [1, \infty)$$

Cartesian Product →

$A = \{1, 2\}$
$B = \{a, b\}$

$A \times B = \{(1,a), (1,b), (2,a), (2,b)\}$

$A \times B = B \times A$ if (i) $A = B$ or (ii) either $A = \phi$ or $B = \phi$

Properties of Cartesian Product →

1. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
2. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
3. $A \times (B - C) = (A \times B) - (A \times C)$
4. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

1 $A \times (B \cup C) = (A \times B) \cup (A \times C)$

let $\{(a,b) \mid (a,b) \in A \times (B \cup C)\}$

$= \{(a,b) \mid (a \in A) \text{ and } b \in (B \cup C)\}$

$= \{(a,b) \mid (a \in A) \text{ and } ((b \in B \text{ or } b \in C)\}$

$= \{(a,b) \mid (a \in A \text{ and } b \in B) \text{ or } (a \in A \text{ and } b \in C)\}$

$= \{(a,b) \mid (a,b) \in A \times B \text{ or } (a,b) \in A \times C\}$

$= \{(a,b) \mid (a,b) \in ((A \times B) \cup (A \times C))\}$

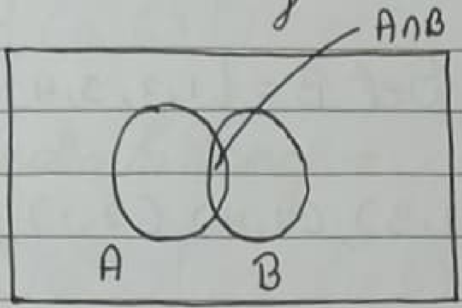
$= A \times (B \cup C) \subseteq (A \times B) \cup (A \times C) \quad - (1)$

$(A \times B) \cup (A \times C) \subseteq A \times (B \cup C) \quad - (2)$

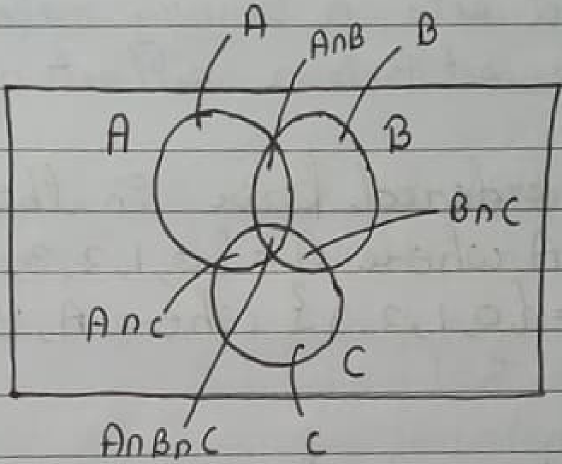
↓

Same method.

Venn diagram →



$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

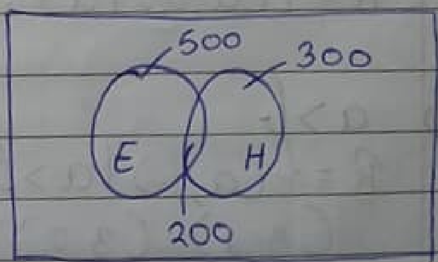


$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C)$$
$$\cancel{-n(C \cap A)} \quad - n(A \cap C) + n(A \cap B \cap C)$$

Q-

$\cancel{n(E) = 700}$
$\cancel{n(H) = 500}$
$\cancel{n(E \cup H) = 1000}$

People = 1000
English = 700
Hindi = 500



$$n(E \cup H) = n(E) + n(H) - n(E \cap H)$$
$$1000 = 700 + 500 - n(E \cap H)$$
$$n(E \cap H) = 200$$

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Relation →

Set $A = \{1, 2\}$ Set $B = \{1, 2, 3, 4\}$

$A \times B = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4)\}$

$R = \{(A, B) \mid \overset{\text{such that}}{A^2 = B}\} = \{(1,1), (2,4)\}$

Let A & B be sets, a binary relation from a set A to set B is a subset of A × B.

Q- List the ordered pair in the relation R from set A where $A = \{0, 1, 2, 3, 4\}$ to set B where $B = \{0, 1, 2, 3\}$ where $A, B \in R$ iff

i) $a = b$

$R = \{(a, b) \mid a = b\} = \{(0,0), (1,1), (2,2), (3,3)\}$

ii) $a + b = 4$

$R = \{(a, b) \mid a + b = 4\} = \{(1,3), (2,2), (3,1), (4,0)\}$

iii) $a > b$

$R = \{(a, b) \mid a > b\} = \{(1,0), (2,0), (2,1), (3,0), (3,1), (3,2), (4,0), (4,1), (4,2), (4,3)\}$

iv) a / b ↗ a mod b = a divides b

$R = \{(a, b) \mid a \mid b\} = \{(1,1), (1,2), (1,3), (2,2), (3,3), (0,1), (0,2)\, (0,3)\}$

v) $\gcd(a, b) = 1$

$R = \{(a, b) \mid \gcd(a, b) = 1\} = \{(2,3), (3,2), (1,2), (2,1), (4,3), (1,1), (4,1), (3,1), (2,1)\}$

vi) $LCM(a,b) = 2$

$R = \{(a,b) \mid LCM(a,b) = 2\} = \{(2,2), (1,2), (2,1)\}$

Domain & Range →

$R = \{(a,b) \mid \gcd(a,b) = 1\} \Rightarrow \{(2,3), (3,2), (4,3), (1,2), (1,1), (1,3), (4,1), (2,1), (3,1)\}$

Domain $= \{1,2,3,4\}$

Range $= \{1,2,3\}$

The set A $\{a \in A \mid (a,b) \in R \ \exists\, b \in B\}$ is called the domain of R and set denoted by DOM(R)

The set B $\{b \in B \mid (a,b) \in R \ \exists\, a \in A\}$ is called the range of R and denoted by Ran(R).

Q- The relation R on set $\{1,2,3,4,5\}$ is defined by the rule $(x,y) \in R$ if 3 divides $x-y$ find the element of R

$R = \{(x,y) \mid 3 \text{ divides } x-y\}$

$= \{(1,4), (2,5), (1,1), (2,2), (3,3), (4,4), (5,5)\}$

$R^{-1} = \{(y,x) \mid 3 \text{ divides } y-x\}$

$= \{(4,1), (5,2), (1,1), (2,2), (3,3), (4,4)$
$(5,5)\}$

$DOM(R) = \{1,2,3,4,5\}$ $\quad$ $Ran(R) = \{1,2,3,4,5\}$

$DOM(R^{-1}) = \{1,2,3,4,5\}$ $\quad$ $Ran(R^{-1}) = \{1,2,3,4,5\}$

Operations on Relation →

$R \cup S = \{(x,y) \mid (x,y) \in R \vee (x,y) \in S\}$
$R \cap S = \{(x,y) \mid (x,y) \in R \wedge (x,y) \in S\}$
$R - S = \{(x,y) \mid (x,y) \in R \wedge (x,y) \notin S\}$
$\bar{R} = \{(x,y) \notin R\}$
$R \Delta S \mid R \oplus S = \{(x,y) \mid (x,y) \in R - S \vee (x,y) \in S - R\}$

Q- Let the set of real numbers are

$R_1 = \{(a,b) \in R_1 \mid a > b\}$
$R_2 = \{(a,b) \in R_2 \mid a \geq b\}$
$R_3 = \{(a,b) \in R_3 \mid a < b\}$
$R_4 = \{(a,b) \in R_4 \mid a \leq b\}$
$R_5 = \{(a,b) \in R_5 \mid a = b\}$
$R_6 = \{(a,b) \in R_6 \mid a \neq b\}$

a) $R_2 \cup R_4$
b) $R_3 \cap R_6$
c) $R_6 - R_3$
d) $R_3 \oplus R_5$

a) $R_2 \cup R_4 = \{(a,b) \mid (a,b) \in (a \geq b) \vee (a,b) \in (a \leq b)\}$
b) $R_3 \cap R_6 = \{(a,b) \mid (a,b) \in (a < b) \wedge (a,b) \in (a \neq b)\}$
c) $R_6 - R_3 = \{(a,b) \mid (a,b) \in a \neq b \wedge (a,b) \notin (a < b)\}$
d) $R_3 \oplus R_5 = \{(a,b) \mid (a,b) \in ((a,b) \in a < b \wedge (a,b) \notin a = b) \vee (a,b) \in ((a,b) \in a = b \wedge (a,b) \notin a < b)\}$

Types of relation →

a) Inverse relation → $R^{-1} = \{(b,a) \mid (a,b) \in R\}$
Eg: R = a is father of b
R^{-1} = b is son/daughter of a

b) Identity relation → $I_A = \{(a,a) \mid a \in A\}$

c) Congruence Module Relation →

$$a \equiv b \bmod m$$
$$a - b = K \cdot m$$

Ex → $7 \equiv 1 \bmod 3$

$$7 - 1 = K \cdot 3$$
$$\boxed{6 = 2 \cdot 3}$$

Q- Let R_1 and R_2 be the "congruent modulo 3" and "the congruent modulo 4" relations, respectively, on the set of integers that is $R_1 = \{(a,b) \mid (a,b) \in a \equiv b \bmod 3)\}$ and $R_2 = \{(a,b) \mid (a,b) \in a \equiv b (\bmod 4)\}$
find →

a) $R_1 \cup R_2 = \{(a,b) \mid (a,b) \in a \equiv b (\bmod 3) \vee (a,b) \in a \equiv b (\bmod 4)\}$

b) $R_1 \cap R_2 = \{(a,b) \mid (a,b) \in a \equiv b (\bmod 3) \wedge (a,b) \in a \equiv b (\bmod 4)\}$

c) $R_1 - R_2 = \{(a,b) \mid (a,b) \in a \equiv b (\bmod 3) \wedge (a,b) \notin a \equiv b (\bmod 4)\}$

d) $R_1 \oplus R_2 = \{(a,b) \mid (a,b) \in ((a,b) \in a \equiv b (\bmod 3) \wedge (a,b) \notin a \equiv b (\bmod 4)) \vee ((a,b) \in a \equiv b (\bmod 4) \wedge (a,b) \notin a \equiv b (\bmod 3))\}$

Q- Let R be the relation $R = \{(a,b) \mid a \text{ divides } b\}$ on the set of positive integers.

find a) $R^{-1} = \{(b,a) \mid (b,a) \in b \text{ divides } a\}$

b) $\bar{R} = \{(a,b) \mid (a,b) \notin a \text{ divides } b\}$

Properties of relation →

1 Reflexive relation → A relation R on a set A is called reflexive, if $(a,a) \in R \; \forall a \in A$

$R = \{(a,a) \mid a \in A\}$

Eg → $A = \{1,2,3\}$

$I_A = \{(1,1),(2,2),(3,3)\}$ ↑ compulsary

Reflexive relation $= \{\underline{(1,1),(2,2),(3,3)}, \underline{(1,2),(2,3)}, \underline{(3,2)}\}$

↓ not compulsary

Every I_A is reflexive but every reflexive is not necessary I_A.

Q- $R = \{(a,b) \mid a/b\} \; \forall a,b \in Z$

Reflexive relation

But not Identity relation

2 Irreflexive relation →

$R = \{(a,b) \notin R \mid \forall a \in A\}$

3 Non-Reflexive relation → A relation R on a set A is called non-reflexive if it is reflexive and irreflexive both.

Eg → $R = \{\underbrace{(1,1),(2,2)}_{\text{reflexive}}, \underbrace{(3,3)}_{\text{irreflexive}}\}$

4 Symmetric relation →

$\forall a \forall b \{(a,b) \in R \rightarrow (b,a) \in R\} \; \forall a,b \in A$

Eg $A = \{1,2,3\}$

$R = \{(1,1),(1,2),(2,1),(3,1)\}$ Not symmetric

$R = \{(1,1),(1,2),(2,1),(3,1),(1,3)\}$ symmetric

5 Asymmetric relation →

$\forall a \forall b \{(a,b) \in R \rightarrow (b,a) \notin R\}$ where $a, b \in A$

6 Transitive relation →

$\forall a \forall b \forall c \{((a,b) \in R \wedge (b,c) \in R) \rightarrow (a,c) \in R\}$

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Antisymmetric →
$\forall a \forall b (((a,b) \in R \wedge (b,a) \in R) \rightarrow a = b)$

Ex → $R = \{(1,2), (2,1), (1,1), (2,2)\}$

Equivalence relation → A relation is said to be equivalent relation if it is reflexive, symmetric, transitive.

Q- Let m be a positive integer with $m > 1$.
Show that the relation
$$R = \{(a,b) \mid a \equiv b \pmod{m}\}$$
is an equivalence relation on the set of integer.

Solⁿ → $Z \times Z$
$$R = \{(a,b) \mid a \equiv b \pmod{m}\}$$

$$a \equiv b \bmod m$$
$$\boxed{a - b = K \cdot m}$$

① Reflexivity: $\forall a \{(a,a) \in R \mid a \in A\}$

$$a - b = K \cdot m$$
$$a - a = K \cdot m$$
$$0 = K \cdot m$$
$$\boxed{0 = 0 \cdot m}$$

② Symmetricity: $\forall a \forall b \{(a,b) \in R \rightarrow (b,a) \in R\}$

let this is true — To prove

let $(a,b) \in R$

$$a - b = K \cdot m$$
$$-(a-b) = -Km$$
$$(b-a) = (-K)m$$
$$(b,a) \in R$$

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③ Transitivity: $\forall a \forall b \forall c \{((a,b) \in R \wedge (b,c) \in R) \rightarrow (a,c) \in R\}$

let $(a,b) \in R$ and $(b,c) \in R$

$a - b = k_1 m$ — ①

$b - c = k_2 m$ — ②

adding ① & ②

$a - c = (k_1 + k_2) m$

$(a,c) \in R$.

Q- Let R and S be a relation from A to B, show that -

(i) if $R \subseteq S$, then $R^{-1} \subseteq S^{-1}$

(ii) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

(iii) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

(i) let $R \subseteq S$

if $(a,b) \in R^{-1}$ then $(b,a) \in R$ and $(b,a) \in S$

then $(a,b) \in S^{-1}$

$\therefore \boxed{R^{-1} \subseteq S^{-1}}$

(ii) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

LHS → $(a,b) \in (R \cap S)^{-1}$

$(b,a) \in (R \cap S)$

$(b,a) \in R$ and $(b,a) \in S$

$(a,b) \in R^{-1}$ and $(a,b) \in S^{-1}$

$(a,b) \in R^{-1} \cap S^{-1}$

$(R \cap S)^{-1} \subseteq R^{-1} \cap S^{-1}$ — ①

$R^{-1} \cap S^{-1} \subseteq (R \cap S)^{-1}$ — ②

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(iii)

Q- Let R be a relation on A, Prove that,

① if R is reflexive, so is R^{-1}
let prove

② R is symmetric if and only if $R = R^{-1}$

③ R is antisymmetric if and only if R

① If R is reflexive
$\forall a. (a,a) \in R$
$(a,a) \in R^{-1}$

② If R is symmetric, $\forall a \forall b (aRb \to bRa)$

let $(a,b) \in R^{-1}$ - ①
$(b,a) \in R$
$(a,b) \in R$ - ②

using ① & ② $\boxed{R^{-1} \subseteq R}$ - (i) let $(a,b) \in R$

$(b,a) \in R^{-1}$

$(b,a) \in R$

$\boxed{R \subseteq R^{-1}}$ $-$ (ii)

Composite Relations :-

Reflexive
symmetric
transitive

$$\overset{D}{A} \times \overset{R}{\underline{B}} \qquad \overset{D}{\underline{B}} \times \overset{R}{C} \qquad A \times C$$

$$R \qquad S \qquad S \circ R$$

S of R

If range of one relation is same as the domain of another relation, then they will get cancelled out.

Eg→ $A = \{1, 2\}$, $B = \{1, 2, 3\}$, $C = \{1, 2, 3, 4\}$

$A \times B$

$R = \{(1, 1), (2, 3), (1, 3)\}$

$B \times C$

$S = \{(1, 2), (2, 3), (2, 4), (3, 2), (3, 4)\}$

$S \circ R = \{(1, 2), (2, 2), (2, 4), (1, 2), (1, 4)\}$

→ Relations can be expressed in matrix form.

$A = \{1, 2, 3\}$ $\qquad$ $B = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2), (2,3), (2,4), (3,3), (3,4)\}$

$\underline{A} \times B$

$$M_R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$M_{R_1} = \begin{matrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{matrix} \qquad M_{R_2} = \begin{matrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

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Composite relations → Let R be a relation from a set A to set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pair (a, c), where $a \in A$, $c \in C$ and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $R \circ S$.

Ex → what is the composite of the relations R and S, where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1,0); (2,0), (3,1), (3,2), (4,1)\}$

Sol^n → $S \circ R = \{(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)\}$

Let R be a relation on the set A. The powers $R^n, n = 1, 2, 3 \dots$ are defined recursively by

$$R^1 = R,\ R^2 = R \circ R,\ R^3 = R^2 \circ R \dots\ R^{n+1} = R^n \circ R.$$

The Matrix representation composition → Suppose that R is a relation from A to B and S is a relation from B to C. Suppose that m, n and P are the elements of A, B, C respectively. The ordered pair (a_i, c_j) belongs to $S \circ R$ if and only if there is an element b_k such that (a_i, b_k) belongs to R and (b_k, c_j) belongs to S from the defination.

from this defination

$$\boxed{M_{S \circ R} = M_R \odot M_S}$$

Q- find the matrix representing the Relation R^2, where the matrix represent R is

$$M_R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \cdot \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array} = $$

(1,2) (1,2)

Matrix representing union & intersection →

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} \quad \text{and} \quad M_{R_2 \cap R_2} = M_{R_1} \wedge M_{R_2}$$

Representing relations using Matrices:-

A relation between finite sets can be represented using a zero-one matrix. Suppose that R is a relation from $A = \{a_1, a_2, ---- a_n\}$ to $B = \{b_1, b_2, b_3 ---- b_n\}$ the relation R can be represented by its matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Example 1→ Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$ and $a > b$.

What is the matrix representing R if $a_1 = 1$, $a_2 = 2$ and $a_3 = 3$ and $b_1 = 1$, $b_2 = 2$?

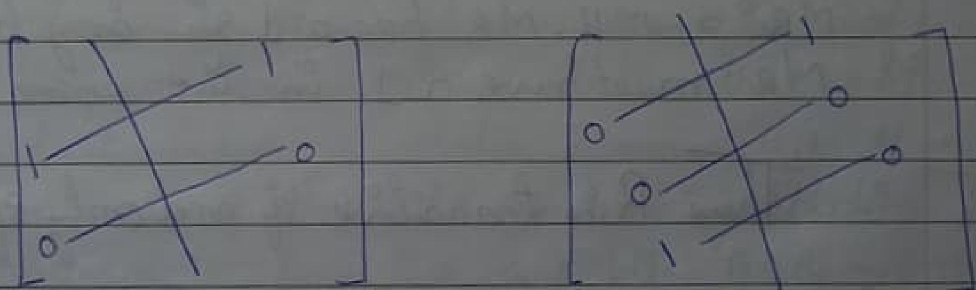
Sol^n because $R = \{(2,1), (3,1), (3,2)\}$, the matrix for R is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Reflexive relation → R is reflexive if and only if $(a_i, a_i) \in R$ for $i = 1, 2, \ldots n$ Hence, R is reflexive iff $m_{ii} = 1$ for $i = 1, 2, \ldots n$.
In ~~other~~ other words, R is ~~if~~ reflexive if all the elements on the main diagonal of M_R are equal to 0.

$$M_R = \begin{bmatrix} 1 & \cdots & \cdots \\ 0 & 1 & \cdots \\ 0 & \cdots & 1 \end{bmatrix}$$

Symmetric Relation Matrix → The relation R is a symmetric if $(a, b) \in R$ implies that $(b, a) \in R$.
Consequently, the relation R on a set $A = \{a_1, a_2 \ldots a_n\}$ is symmetric if and only if $m_{ij} = 1$ whenever $m_{ji} = 1$.
Consequently, R is symmetric iff $m_{ij} = m_{ji}$, for all pair of integers i and j with $i = 1, 2 \ldots n$ and $j = 1, 2, \ldots n$.



a) symmetric   b) antisymmetric.

The relation R is antisymmetric iff $(a, b) \in R$ and $(b, a) \in R$ imply that $a = b$. Consequently, the matrix of an antisymmetric relation has the property that if $m_{ij} = 1$ with $i \neq j$, then $m_{ij} = 0$ or in other words, either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$

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Example → Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$ is R reflexive, symmetric and/or antisymmetric ?

Solⁿ → For reflexivity, because all the diagonal of this matrix are equal to 1, R is reflexive
More over, because M_R is symmetric, it follows that R is symmetric but R is not ~~symm~~ antisymmetric.

Transitive → A relation R is transitive if and only if its matrix $M_R = [m_{ij}]$ has the property if $m_{ij} = 1$ and $m_{jk} = 1$, then $m_{ik} = 1$. Thus statement simply means R is transitive if $M_R \cdot M_R$ has 1 position i.k. Thus, the transitivity of R means that if $MR^2 = MR \cdot MR$ has a 1 in any position then M_R must have a 1 in the same position ~~there~~

Thus R is transitive if and only if $M_R^2 + M_R = M_R$

Eg → Let $A = \{1, 2, 3, 4\}$ and let; R be a relation on A whose matrix is:

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$ Show that R is transitive.

Solⁿ →

Exam

Solⁿ

Sol^n → $M_R^2 = M_R \cdot M_R$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_R^2 + M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= M_R$$

$\therefore$ The relation R is transitive.

Closure of Relations →

Reflexive closure → The reflexive closure $R^{(r)}$ of a relation R is the smallest reflexive relation that contained R as a subset. Given a relation R on a set A, the reflexive closure if R can be formed by adding to R all pairs of the form (a, a) with $A \in R$ not already in R

$$R^{(r)} = R \cup I_A$$

where, $I_A = \{(a,a) : a \in A\}$

Example → A relation $R = \{(1,2), (2,1), (1,1), (2,2)\}$ is defined on a set $S = \{1, 2, 3, 4\}$. find the reflexive closure of R.

Sol^n → $R^{(r)} = R \cup I_A$

$= \{(1,2), (2,1), (1,1), (2,2)\} \cup \{(1,1), (2,2), (3,3), (4,4)\}$

$$R^{(r)} = \{(1,1),(1,2),(1,1),(2,2),(3,3),(4,4)\}$$

Example → what is the reflexive closure of the relation $R = \{(a,b) \mid a < b\}$ on the set of integers?

Sol^n → $R \cup I_A = \{(a,b) \mid a < b\} \cup \{(a,a) \mid a \in Z\} = \{(a,b) \mid a \leq b\}$

Symmetric Closure → The symmetric closure $R^{(s)}$ is the smallest symmetric relation that contains R as a subset. A symmetric relation contains (x,y) if it contains (y,x). Since the inverse relation R^{-1} contains (y,x) if (x,y) is in R, the symmetric closure of relation is

$$R^{(s)} = R \cup R^{-1}$$

where $R^{-1} = \{(y,x) : (x,y) \in R\}$

Example → If $R = \{(1,2),(4,3),(2,2),(2,1),(3,1)\}$ be a relation on $S = \{1,2,3,4\}$. Find the symmetric closure.

Sol^n →

$$R^{(s)} = R \cup R^{-1}$$

$$R^{-1} = \{(3,4),(1,3)\}$$

$$R^{(s)} = \{(1,2),(2,1),(4,3),(3,4),(3,1),(1,3),(2,2)\}$$

07/10/2020 official group

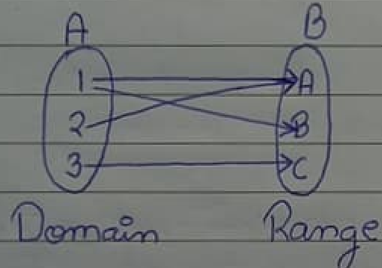
FUNCTIONS →

let there be two sets A & B :-

$A = \{1, 2, 3\}$ $\quad$ $B = \{A, B, C\}$

$A \times B = \{(1,A), (1,B), (1,C), (2,A), (2,B), (2,C), (3,A), (3,B), (3,C)\}$

$f = \{(1,A), (1,B), (2,A), (3,C)\}$

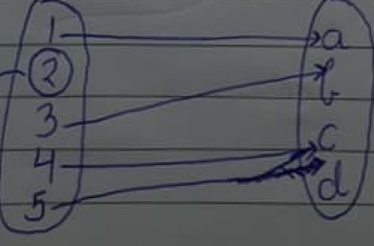


{ Not a function because more than one image of 1 is present }

* No element from set of domain should be left out. It should have an image.

* Elements can have same image.

Ex → $(4, c)$; $(5, c)$



No image is present

∴ not a function

Every function is a relation, but every relation might not be a function.

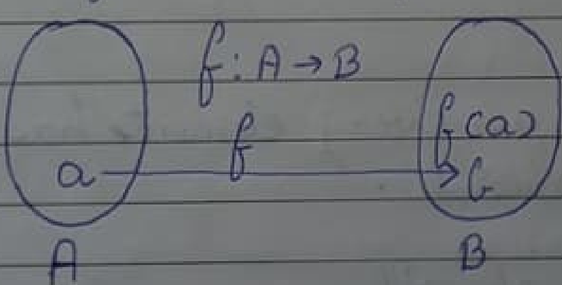
Defination of Functions :-

* Given any set A, B, a function f from (or "mapping") A to B ($f: A \to B$) is an assignment of exactly one element $f(x) \in B$ to each element $x \in A$.

$\boxed{\text{Conditions}}$:-

- $\forall a \in A$ must be related to $\exists b \in B$
- $(a,b) \in f \wedge (a,c) \in f \Rightarrow b = c$

Pictorial representation



$f(a) = b$

f such that A mapped to B and b is the image of a

A ↓ domain

B ↓ codomain

☆ Some functions terminology

☆ If $f: A \to B$, and $f(a) = b$ (where $a \in A$ & $b \in B$), then,

→ A is domain of f.

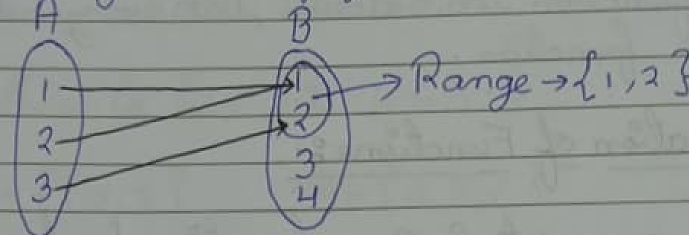
→ B is the codomain of f

→ b is the image of a under f

→ a is the pre-image of b under f.

- In general, b may have more than one pre-image.

→ The range $R \subseteq B$ of f is $\{ b \mid \exists a \, f(a) = b \}$



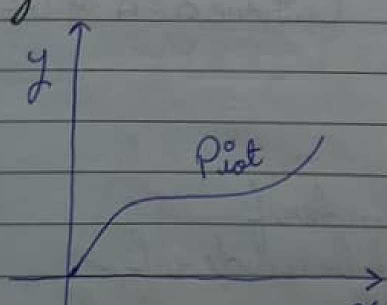
Q- from the set of integers to the set of real numbers.
Determine whether f is a function from Z to R if

a) $f(n) = \pm n$, (b) $f(n) = \sqrt{n^2+1}$

c) $f(n) = \dfrac{1}{n^2-4}$.

a) Not a function, as every element has two images, which is not possible.
b) Not a function.
c) If $n = 2$, then it will give ∞, therefore, there is an element 2 in the domain for which there is no domain, which is not possible.
∴ Not a function.

Graphical Representation :-



$f(a) = b$
$f(a)$ ↓ preimage of b
b ↓ image of a

Q- Deter
if a

a) Not

b) Not
c) Not

Types

1 One
2 Onto
3 One
4 Man

1 One

∀a

∀a ∀

where
fun

A
a
b
c

Q- Determine whether f is a function from R to R if a) $f(x) = \frac{1}{x}$, b) $f(x) = \sqrt{x}$, c) $f(x) = \pm\sqrt{x^2+1}$

a) Not a function

b) Not a function } because two images will be formed
c) Not a function }

Types of functions:-

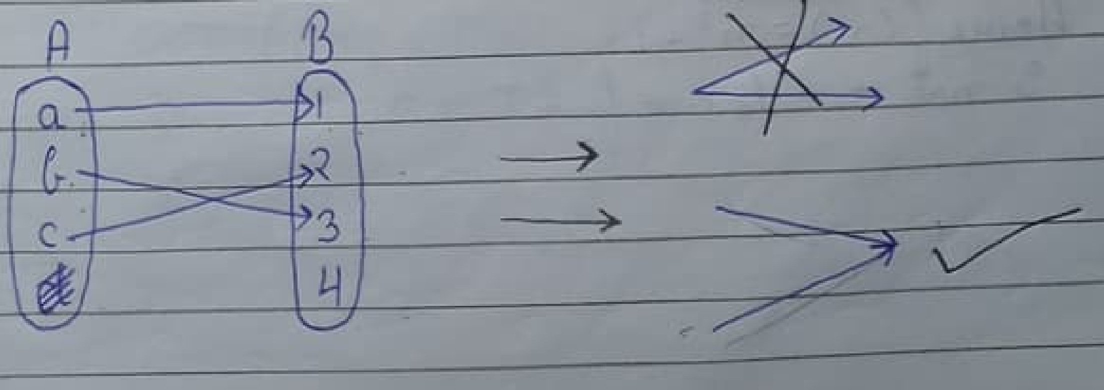
1 One to one
2 Onto
3 One-one-onto
4 Many to one

1 One to one function:- other name is injective function

$\forall a \forall b ((f(a) = f(b)) \to a = b)$ ← to disprove one-one

$\forall a \forall b (a \neq b \to (f(a) \neq f(b)))$ ← to prove one-one

where the universe of discourse is the domain of function.
a, b both are elements of A



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Q- Determine whether each of these functions from Z to Z is one-to one.

a) $f(n) = n-1$
b) $f(n) = n^2+1$
c) $f(n) = n^3$
d) $f(n) = \lceil n/2 \rceil$

a) $f : Z \to Z$
$f(n) = n-1$
$\forall a \forall b (a \neq b \to (f(a) \neq f(b)))$
let $a, b \in Z$
$a \neq b$ is true — ①

subtracting ① from both sides in eq^n ①
$a-1 \neq b-1$
$f(a) \neq f(b)$

Hence $f(n) = n-1$ is one to one f^n.

b) $f(n) = n^2+1$

$\forall a \forall b ((f(a) = f(b)) \to a = b)$
$a, b \in Z$
let $f(a) = f(b)$ is true — ①
since $f(2) = f(-2) = 5$
but $\boxed{2 \neq -2}$
Hence $f(n) = n^2+1$
is not one to one func^n

not a one to one
bcz for 1 image is 2
and for -1 image is 2,
so NO.

c)

d)

to

c) $f(x) = x^3$

$\forall a \forall b (a \neq b \rightarrow (f(a) \neq f(b)))$

$a, b \in \mathbb{Z}$

$a \neq b$ is true

└ ①

cubing both sides in eq^n ① :-

$a^3 \neq b^3$

$f(a) \neq f(b)$

Hence $f(x) = x^3$ is one to one f^n.

$\lceil \; \rceil$ or $\lfloor \; \rfloor$

ceiling floor

to convert real numbers into integers

$\lceil 8.1 \rceil = 9$

$\lfloor 8.999 \rfloor = 8$

d) $f(x) = \lceil x/2 \rceil$

$\forall a \forall b \, ((f(a) = f(b)) \rightarrow a = b)$

let $f(a) = f(b)$

since $\lceil 7.1 \rceil = \lceil 7.9 \rceil = 8$

but $7.1 \neq 7.9$

Hence $f(x) = \lceil x/2 \rceil$

is not one - one f^n

Increasing & Decreasing function :-

<u>Increasing</u> $f(x) \leq f(y)$ whenever $x < y$ where x & y are in one domain

<u>Decreasing</u> $f(x) \geq f(y)$ whenever $x < y$

Strictly increasing & strictly decreasing function

$f(x) < f(y)$ whenever $x < y$
$f(x) > f(y)$ whenever $x < y$

<u>Q-</u> let $f: R \to R$ and let $f(x) > 0 \ \forall x \in R$ Show that $f(x)$ is strictly decreasing if and only if function $g(x) = 1/f(x)$ is strictly increasing.

<u>Sol^n</u> → let $f(x)$ is strictly decreasing
Given:- $f(x) > f(y)$ whenever $x < y$
taking reciprocal:-

$$\frac{1}{f(x)} < \frac{1}{f(y)} \quad \text{whenever } x < y$$

$$g(x) < g(y).$$

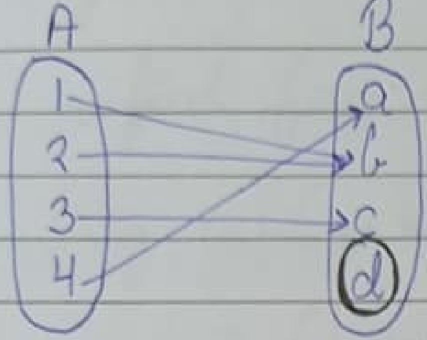
Not onto f^n bcz in codomain d is left.

<u>Q→</u> f

a)
b)
c)
d)

Onto function → other name is surjective function

Not onto f^n bcz in codomain d is left.



Range $\subseteq$ codomain

$\boxed{\text{If Range} = \text{codomain}}$

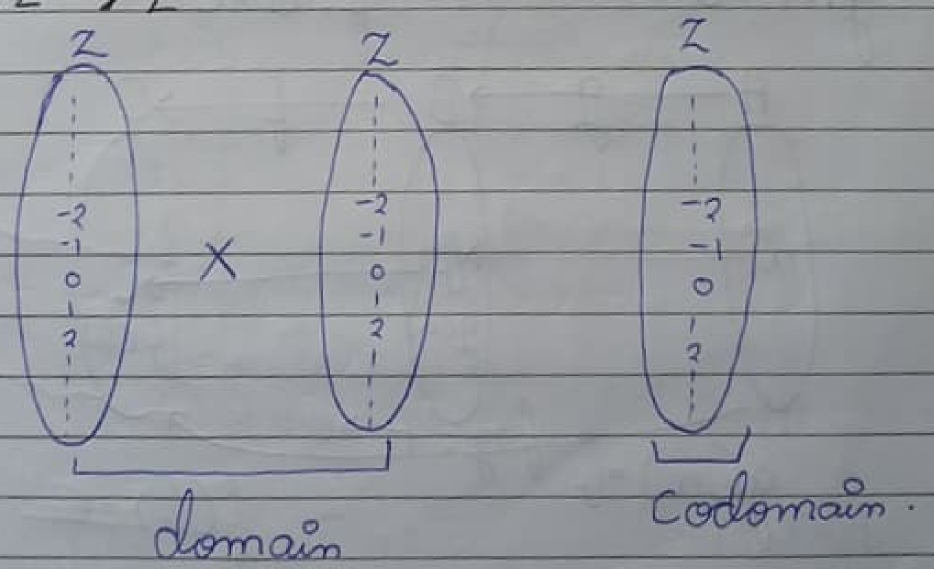
↓ f^n is onto

$f(1) = b$
$f(2) = b$
$f(3) = c$
$f(4) = a$

$\forall y \, \exists x \, (f(x) = y)$
where x is the element of domain and y is the element of co-domain

for every element of codomain & for some element of domain ~~there is~~ that $f(x) = y$

Q→ $f : Z \times Z \to Z$



a) $f(m,n) = m+n$
b) $f(m,n) = m^2 + n^2$
c) $f(m,n) = m$
d) $f(m,n) = |m|$
e) $f(m,n) = m-n$
f) $f(m,n) = m+n+1$
g) $f(m,n) = |n|$
h) $f(m,n) = m^2 - 4$

a) $f(m,n) = m+n$

$f(1,2) = 3$

$f(-1,-1) = -2$

$f(-1,1) = 0$

$\forall y \exists x (f(x) = y)$

Since, $f(m,n) = m+n$

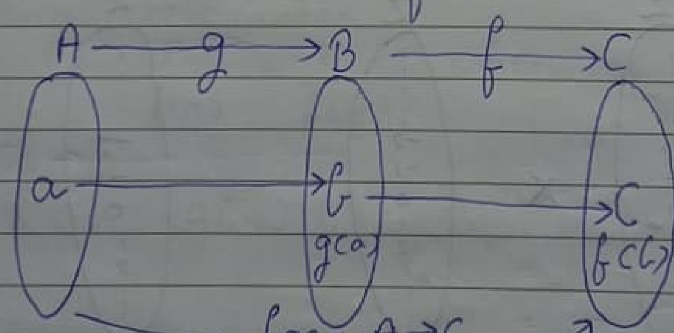
let $m+n = y$

$\forall y \in Z$ there are m & $n \in Z$ such that $m+n = y \in Z$

One-one-onto → other name is bijective or bijection function.

Composition of function →

$g : A \to B$ $\qquad$ $f : B \to C$



$g(a) = b$

$f(b) = c$

$fog(a) = c$

$fog(a) = f[g(a)]$

$= f[b]$

$= c$

Q-

$f(x) = 2x + 3$
$g(x) = 3x + 2$

find fog & gof

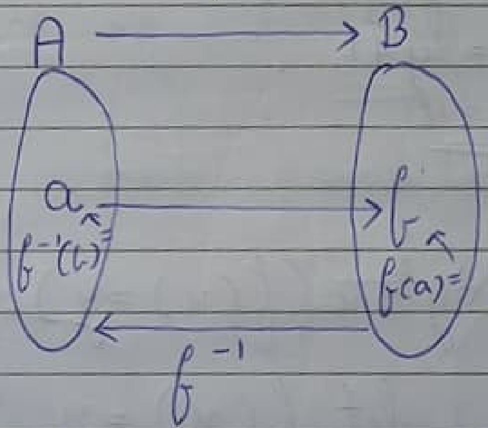
$$fog = f[g(x)]$$
$$= f[3x+2]$$
$$= 2(3x+2) + 3$$
$$= 6x + 7$$

$$gof = g[f(x)]$$
$$= g[2x+3]$$
$$= 3(2x+3)+2$$
$$= 6x + 11$$

$$\boxed{fog \neq gof}$$

Inverse function →

$f : A \to B$



$f : A \to B$ $\qquad$ $f^{-1} : B \to A$

$f(a) = b.$ $\qquad$ $f^{-1}(b) = a.$

NOTE → A one-one-onto correspondence is called invertible.

Q- $f: Z \to Z$

$f(x) = x+1$ is it invertible = ?
what is its inverse = ?

first prove one-one & onto then it is invertible.

$f(x) = y \quad f^{-1}(y) = x$

$$\boxed{f^{-1}(y) = y-1}$$

$x+1 = y \Rightarrow x = y-1$.

one-one →

$\forall x \forall y \, ((f(x) = f(y)) \to x = y)$

let $f(x) = f(y)$

$x+1 = y+1$

$\underline{\underline{x = y}}$

onto → $\forall y \exists x \, (f(x) = y)$

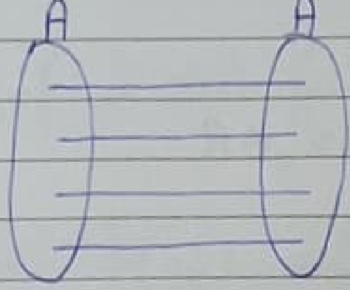
let $f(x) = y$

$x+1 = y$

$\underline{\underline{x = y-1}}$.

Identity function →

$$i_A : A \to A$$

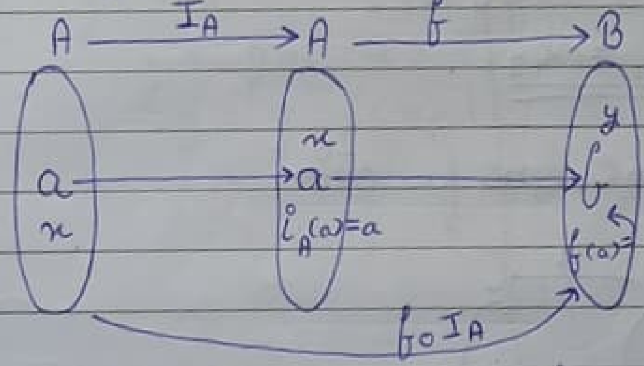


identity function is always one-one onto.

$$i_A(x) = x$$

Theorem: The composition of any function with identity function is the function itself.

i.e., $f \circ I_A(x) = I_B \circ f(x) = f(x)$



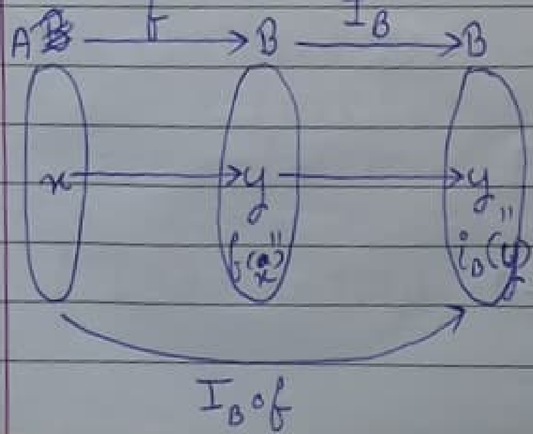
$i_A(x) = x$

$f(x) = y$

$f \circ I_A(x)$

$\Rightarrow f[I_A(x)]$

$\Rightarrow f(x)$



$I_B \circ f(x)$

$\Rightarrow I_B[f(x)]$

$\Rightarrow I_B[y]$

$\Rightarrow y$

Q. If $f : A \to B$ and $g : B \to x$ be one-one onto function; then $g o f$ is also one-one onto and $(g o f)^{-1} = f^{-1} o g^{-1}$.

Since f is one-one

$f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2 \quad \forall x_1, x_2 \in R$

g is one-one

$g(y_1) = g(y_2)$

$\Rightarrow y_1 = y_2$

$g o f(x_1) = g o f(x_2)$

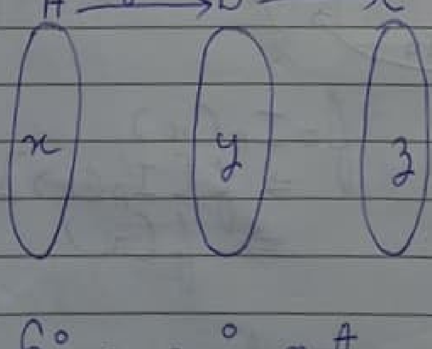
$g[f(x_1)] = g[f(x_2)]$

$f(x_1) = f(x_2)$

$\underline{\underline{x_1 = x_2}}$.

$\therefore$ $g o f$ is one-one function.

$A \xrightarrow{f} B \xrightarrow{g} C$



Since g is onto
for $\forall z \in C$ there is $y \in B$
and since f is onto
for $\forall y \in B$ there is $x \in A$

$z = g(y)$
$z = g[f(x)]$
$z = g \circ f(x)$

$\therefore$ z $g \circ f$ is onto function.

To prove, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

$g \circ f : A \to C$, $(g \circ f)^{-1} : C \to A$
$f : A \to B$, $g : B \to C$
$f^{-1} : B \to A$, $g^{-1} : C \to B$

$f^{-1} \circ g^{-1} = C \to A$

$$\boxed{\begin{aligned} (g \circ f)^{-1}(z) &= x \\ z &= g \circ f(x) \\ z &= g[f(x)] \\ z &= g[y] \\ z &= z \end{aligned}}$$

$y = g^{-1}(z)$
$f^{-1}(y) = x$
$f^{-1}[g^{-1}(z)] = f^{-1} \circ g^{-1}(z)$
$f^{-1} \circ g^{-1}(z) = x$

Q-1 $f : R \to R$, $f(x) = x^2$
find $f^{-1}(4)$ and $f^{-1}(-4)$

Not one-one and onto but still we will find value of inverse because it is not asked about one-one & onto.

$f^{-1}(4) = \{x \in R, f(x) = 4\}$
$= \{x \in R, x^2 = 4\}$
$= \{x \in R, x = \pm 2\}$

$f^{-1}(4) = \{-2, 2\}$

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$f^{-1}(-4) = \{x \in R, f(x) = -4\}$
$= \{x \in R, x^2 = -4\}$
$= \{x \in R, x = \pm 2i\}$
$= \phi$

$f^{-1}(-4) = \phi$

Q-2

$f: R \to R$

$f(x) = \{3x-4, x>0$
$-3x+2, x \le 0\}$

find $f(0), f(2/3), f(-2), f^{-1}(0), f^{-1}(2), f^{-1}(-7)$

$f(0) = -3\times 0+2$
$= 2$

$f(2/3) = 3\times \frac{2}{3} - 4$

$= -2$

$f(-2) = -3\times(-2)+2$
$= 8$

$f^{-1}(0) = \{x \in R \mid f(x) = 0\}$

$= \{x \in R \mid x > 0$ and $3x-4=0\} \cup$
$\{x \in R \mid x \le 0$ and $-3x+2=0\}$

$= \{x \in R \mid x>0$ and $x = 4/3\} \cup$
$\{x \in R \mid x \le 0$ and $x = 2/3\}$

$f^{-1}(0) = \frac{4}{3} \cup \phi = \frac{4}{3}$

$f^{-1}(2) = \{x \in R \mid f(x) = 2\}$

$= \{x \in R \mid x > 0 \text{ and } 3x - 4 = 2\} \cup$
$\{x \in R \mid x \leq 0 \text{ and } -3x + 2 = 2\}$

$= \{x \in R \mid x > 0 \text{ and } x = 2\} \cup$
$\{x \in R \mid x \leq 0 \text{ and } x = 0\}$

$f^{-1}(2) = 2 \cup 0$
$f^{-1}(2) = \{2, 0\}$

$f^{-1}(7) = \{x \in R \mid f(x) = 7\}$ ϕ ??

$= \{x \in R \mid x > 0 \text{ and } 3x - 4 = 7\} \cup$
$\{x \in R \mid x \leq 0 \text{ and } -3x + 2 = 7\}$

$= \{x \in R \mid x > 0 \text{ and } x = 11/3\} \cup$
$\{x \in R \; x \leq 0 \text{ and } x = -5/2\}$

Mathematical Induction →

Piano's Axioms :-

Let $P(n)$ be the formula

Step 1 → ~~face~~ Base step To prove $P(1)$ is true

Step 2 → Inductive step a) let $P(K)$ is true
b) To Prove $P(K+1)$ is true

variety of question

- → Summation
- → Divisibility
- → InEquality
- → Strong MI

Summation →

Q- Use MI to prove that the sum of first 'n' natural numbers is $n(n+1)/2$

Sol^n → Let $P(n) =: 1+2+3 ---- n-1+n$

Base step → To prove $P(1): 1 = \frac{1(1+1)}{2} = 1$

Inductive step → let, $P(K): 1+2+3 ---- (K-1)+K \quad -①$
$= \frac{K(K+1)}{2}$ is true

adding $(K+1)$ in eq ① on both side

$$1+2+3 ---- K+(K+1) = \frac{K(K+1)}{2} + (K+1)$$

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$$= (K+1)\left[\frac{K}{2}+1\right].$$

$$= \frac{(K+1)(K+2)}{2}.$$

Q-1 $1+2+2^2+\cdots+2^n = 2^{n+1}-1$

Q-2 $1.2+2.3+3.4+\cdots+n.(n+1) = \frac{n(n+1)(n+2)}{3}$

Q-3 $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n} = 1-\frac{1}{2^n}$

Q-4 Show that the sum of n terms of AP is $\frac{n}{2}[2a+(n-1)d]$ where $a+(n-1)d$ is the n^{th} term of AP.

Q-5 Prove that for all natural numbers n

$$1+\sum_{K=1}^{n} K.K! = (n+1)!$$

Q-6 Prove that the sum of cubes of the first n natural numbers is

$$\left\{\frac{n(n+1)}{2}\right\}^2, \forall n \geq 1$$

Q-7 Prove that the sum of squares of the first n natural numbers is

$$\frac{n(n+1)(2n+1)}{6}, \forall n \geq 1$$

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Divisibility →

Q- Prove that $6^{n+2} + 7^{2n+1}$ is divisible by 43 for each positive integer using MI.

Sol^n→ $P(n): 6^{n+2} + 7^{2n+1} \;/43$

Base step: for $n=1$

$P(1): 6^{1+2} + 7^{2\times1+1}$

$= 6^3 + 7^3$

$= 216 + 343$

$= 559/43$ divisible by 43

Inductive step: for $n=k$

let $P(k): 6^{k+2} + 7^{2k+1}$ is divisible by 43

$\cancel{: 6^{k+3} + 7^{2k+3}}$

for $n = k+1$

$P(k+1): 6^{(k+1)+2} + 7^{2(k+1)+1}$

$= 6^{k+3} + 7^{2k+3}$

$= 6^{(k+2)+1} + 7^{2k+1+2}$

$= 6^{(k+2)}.6 + 7^{2k+1}.49$

$= 6^{k+2}.6 + 7^{2k+1}.(6+43)$

$= 6\underbrace{(6^{k+2} + 7^{2k+1})}_{P(k)} + 7^{2k+1}.43$

∴ $P(k+1)$ is also divisible by 43.

Q- use MI to show that $11^{n+2} + 12^{2n+1}$ is divisible by 133.

Sol^n→ $P(n): 11^{n+2} + 12^{2n+1} \;/133$

Base step: for $n=1$

$P(1) = 11^{1+2} + 12^{2\times1+1}$

$= 11^3 + 12^3$

ch

$= 1331 + 1728$

$= 3059 \,/\, 133$ divisible by 133

Inductive step → for $n = K$

let $P(K): 11^{K+2} + 12^{2K+1}$ is divisible by 133

for $n = K+1$

$P(K+1): 11^{(K+1)+2} + 12^{2(K+1)+1}$

$= 11^{K+3} + 12^{2K+3}$

$= 11^{(K+2)+1} + 12^{2K+1+2}$

$= 11^{(K+2)} \cdot 11 + 12^{2K+1} \cdot 144$

$= 11^{(K+2)} \cdot 11 + 12^{2K+1} \cdot (11 + 133)$

$= 11 (\underbrace{11^{K+2} + 12^{2K+1}}_{P(K)}) + 12^{2K+1} \cdot 133$

3

∴ $P(K+1)$ is also divisible by 133

Q-1 Show that $x^{n-1} - 1$ is divisible by $x-1$ for all $n \geq 0$.

Q-2 use MI to show that $n^3 + 2n$ is divisible by $3 \;\forall\, n \in \mathbb{Z}^+$

Sol^n 1 → $P(n): x^{n-1} - 1 \,/\, x-1$

Base step: for $n=1$

$P(1) = x^{1-1} - 1$

$= 1 - 1$

43.

$= 0 \,/\, x-1$ divisible by 0

le by 133.

Inductive step: for $n = K$

let $P(K): x^{K-1} - 1$ is divisible by $x-1$

for $n = K+1$

$P(K+1): x^{(K+1)-1} - 1$

$= x^K - 1 + x - x$

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$$= x^{K} - x + x - 1$$
$$= x\left[x^{K-1} - 1\right] + x - 1$$

$\therefore$ $P(K+1)$ is divisible by $x-1$

Sol^n 2→ $P(n): n^3 + 2n \,/\, 3$

Base step: for $n = 1$

$$P(1) = 1^3 + 2$$
$$= 1 + 2 \times 1$$
$$= 3 \,/\, 3$$ divisible by 3

Inductive step: for $n = K$

let $P(K): K^3 + 2K$ is divisible by 3

for $n = K+1$

$$P(K+1): (K+1)^3 + 2(K+1)$$
$$= K^3 + 1 + 3K(K+1) + 2K + 2$$
$$= K^3 + 2K + 3K(K+1) + 3$$
$$= \underbrace{K^3 + 2K}_{P(K)} + 3(K^2 + K + 1)$$

$\therefore$ $P(K+1)$ is divisible by 3

Inequality →

Q- Show that $n! > 2^n$ for any number $n \geq 4$

let $P(n): n! > 2^n$

Base step: $P(4): 4! > 2^4$

$24 > 16$ is true

Inductive step:

let $P(K): K! > 2^K$ is true — ①

for $n = K+1$

$P(K+1):$

Multiply eq ① by 2 ⇒ $2 \cdot K! > 2^K \cdot 2$

$2 \cdot K! > 2^{K+1}$

$5 \cdot K! > 2^{K+1}$

$(4+1)K! > 2^{K+1}$ $\quad\quad K = 4$

$(K+1)\,K! > 2^{K+1}$

$(K+1)! > 2^{K+1}$

Q-1 $2^n > n^2 \quad \forall n \geq 5$

Q-2 $2^n > n! \quad \forall n \geq 10$

Sol^n 1 → let $P(n): 2^n > n^2$

Base step: $P(5): 2^5 > 5^2$

$32 > 25$ is true

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Inductive step: let $P(K): 2^K > K^2$ is true — ①

for $n = K+1$
$P(K+1)$:

Multiply eq ① by $2 \Rightarrow 2 \cdot 2^K > K^2 \cdot 2$
$2^{K+1} > K^2 \cdot 2$
$2^{K+1} > (K+1)^2$ $[2K^2 > (K+1)^2]$

Hence Proved

Q-1

Q- $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$ $n \geq 2$

Soln

$P(2): \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}$

$1 + 0.707 > 1.414$
$1.707 > 1.414$ Proved

$P(K): \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{K}} > \sqrt{K}$ $K \geq 2$ is true

adding $\frac{1}{\sqrt{K+1}}$

$P(K+1): \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{K}} + \frac{1}{\sqrt{K+1}} > \sqrt{K} + \frac{1}{\sqrt{K+1}}$

$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{K}} + \frac{1}{\sqrt{K+1}} > \frac{\sqrt{K}\sqrt{K+1} + 1}{\sqrt{K+1}}$

Q-2

$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{K}} + \frac{1}{\sqrt{K+1}} > \frac{\sqrt{K^2+K} + 1}{\sqrt{K+1}}$

$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{K}} + \frac{1}{\sqrt{K+1}} > \sqrt{K+1}$

Strong MI →

Step 1 → To prove P(1) is true

Step 2 → Let P(2), P(3), P(4) - - - - P(K-1), P(K) all are true

Step 3 → To prove P(K+1) is true

Q-1 Consider the fibonacci sequence of number 1, 1, 2, 3, 5, 8 each term in the sequence from the third term is obtained by adding the previous two terms.

If F_n is the n^{th} term, then the fibonacci series is defined by

$$F_1 = 1,\ F_2 = 1,\ \&\ F_n = F_{n-1} + F_{n-2}\ \forall\ n \geq 3$$

Show that for each +ve integer n, the n^{th} fibonacci in $F_n < 2^n$!

$F_{K+1} < 2^{K+1}$

$F_{K+1} < 2^{K-1} \times 4$

Sol^n → Base step →

$P(3) = 2 < 2^3$

$2 < 8$ is true

Inductive step → let P(4), P(5) - - - P(K-1), P(K) are true

$P(K+1) = F_{K+1} = F_K + F_{K-1} < 2^K + 2^{K-1}$

$F_{K+1} < 2^{K-1}(2+1)$

$F_{K+1} < 2^{K-1} \times 3$

$\Rightarrow F_{K+1} < 2^{K+1}$

Q-2 Show that the set of n elements has exactly 2^n subsets for any non negative integer n.

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Sol^n → Base step → $P(0) = 2^0 = 1$ is true
if set is having zero elements then there is only one subset
Inductive step →
let $P(1), P(2), P(3) --- P(k-1), P(k)$ are true

$$P(k+1) = P($$

Q- Show that if n is an integer greater than 1, then n can be the product of primes.

Base step: $P(2)$: $2 = 2 \times 1$ is true

Inductive step: $P(3), P(4), P(5) --- P(k-1), P(k)$ are true

$P(k+1)$

$k+1$ — Prime no
$k+1$ — composite no.

$$k+1 = a \times b$$
$$2 \leq a \leq b < k+1$$

$2 = 2 \times 1$	$10 = 5 \times 2$
$3 = 3 \times 1$	$11 = 11 \times 1$
$4 = 2 \times 2$	$12 = 2 \times 2 \times 3$
$5 = 5 \times 1$	$13 = 13 \times 1$
$6 = 3 \times 2$	$14 = 2 \times 7$
$7 = 7 \times 1$	
$8 = 2 \times 2 \times 2$	
$9 = 3 \times 3$	

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Q- let $x_1, x_2 \dots x_n$ be n sets. Then prove by MI that

$$\overline{\left(\bigcap_{i=1}^{n} x_i\right)} = \bigcup_{i=1}^{n} \overline{x_i} \quad \forall\, n \geq 1$$

$\overline{A \cap B} = \bar{A} \cup \bar{B}$

$\overline{A \cap B \cap C} = \bar{A} \cup \bar{B} \cup \bar{C}$

Base step: $P(1): \overline{x_1} = \overline{x_1}$ ×

$P(2): \overline{x_1 \cap x_2} = \bar{x}_1 \cup \bar{x}_2$ ✓

Inductive step: let $P(K)$ is true

$$\overline{\left(\bigcap_{i=1}^{K} x_i\right)} = \left(\bigcup_{i=1}^{K} \overline{x_i}\right)$$

for $n = K+1$

To prove $$\overline{\left(\bigcap_{i=1}^{K+1} x_i\right)} = \left(\bigcup_{i=1}^{K+1} \overline{x_i}\right)$$

$$= \overline{\left(\bigcap_{i=1}^{K} x_i \cap x_{K+1}\right)}$$

$$= \bigcup_{i=1}^{K} \overline{x_i} \cup \overline{x_{K+1}}$$

$$= \left(\bigcup_{i=1}^{K+1} \overline{x_i}\right).$$