UNIT 5

1. Graph Theory
2. Tree
3. Counting Principles → sum rule
   - product rule
   - inclusion - exclusion
   - pigeonhole principle
   - PnC
4. Advance Counting Principles → Recurrence relation
   - generating function

GRAPH THEORY

(G)

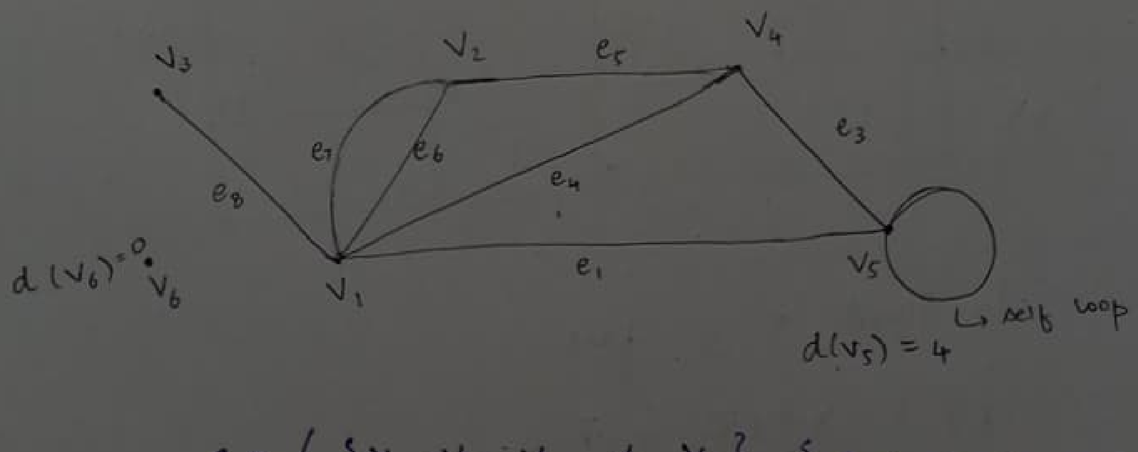
$G = (V, E)$

V : set of vertices

E : set of edges

graph → directed graph (w direction)

→ unidirected graph (w no direction)



$G = (\{V_1, V_2, V_3, V_4, V_5\}, \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\})$

adjacent vertices : having a common edge

Eg : V_3 and V_1 , etc.

adjacent edges : having a common vertex

Eg : e_1 and e_3 , etc.

order of graph O(G) : number of vertices in a graph

$O(G) = 5$

size of graph S(G) : no. of edges in a graph

Types of Edges

1. self Loop : an edge having same end point(s).
2. parallel edges : having same end points for 2 edges.

Types of Graph

1. Simple Graph : having no self loops and no parallel edges.
2. Multi Graph : having parallel edges but no self loops.
3. Pseudo Graph : having both parallel edges and self loops

self loops come under pseudo graphs (only).

Degree of a vertex : no of edges incident on a vertex.

d(V) or deg(V)

$d(V_1) = 5$ $\quad$ $d(V_3) = 1$

$d(V_2) = 3$ $\quad$ $d(V_4) = 3$

$d(V_5) = 4$

Types of Vertices

1. isolated node : a vertex w degree zero

Eg: V_6

2. pendant node : a vertex w degree one Eg: V_3

Handshaking Theorem

If $G=(V,E)$ be an unidirected graph with e edges, then

$$\sum_{v \in V} \deg_G(v) = 2e$$

ie, the sum of degree of the vertices in a unidirectional graph is even.

* each edge contributes to a degree of 2

This theorem is known as handshake theorem because in a handshake there is requirement of 2 people and similarly in this theorem, an edge requires 2 vertices.

Show that the degree of a vertex of a simple graph 'G' on 'n' vertices cannot exceed n-1.

If a graph consists of n vertices and if we take out 1 vertex, then that vertex will be connected maximum to n-1 vertices. Therefore, we can say that the degree of a vertex of a simple graph 'G' on n vertices cannot exceed n-1.

The min. degree for a vertex is 0 ie., isolated node.

degree of any vertex will fall within this range $\longrightarrow$ $\boxed{0 \le \deg_G(v) \le n-1}$ n: no. of vertices

Show that the maximum number of edges in a simple graph with n vertices is $e = \frac{n(n-1)}{2}$

$$\sum_{i=1}^{n} \deg_G(v_i) = 2e \quad \text{(handshaking theorem)}$$

$$\deg(v_1) + \deg(v_2) + \deg(v_3) + \dots + \deg(v_n) = 2e$$

$$n-1 + n-1 + n-1 + \dots + n-1 = 2e$$

$$e = \frac{n(n-1)}{2}$$

degree of any vertex is = n-1

In an undirected graph, the total no. of odd degree vertices is even.

$$\sum_{i=1}^{n} deg_G(v_i) = \underset{\to \text{ even}}{2e}$$

$\underset{\text{no. of vertices w even degree}}{v} + \underset{\text{no. of vertices w odd degree}}{w} = n$

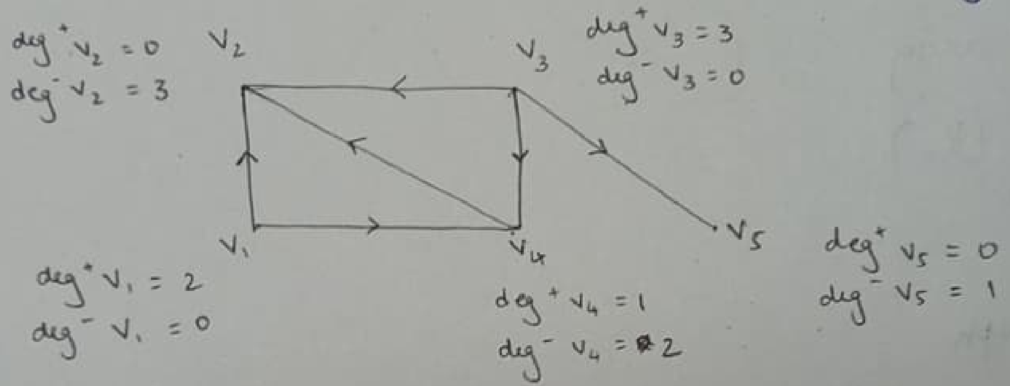
$$\underbrace{\sum_{i=1}^{v} deg_G v(i)}_{\text{even}} + \sum_{i=1}^{w} deg(v_i) = \underbrace{2e}_{\text{even}}$$

$$\sum_{i=1}^{w} deg_G(v_i) = 2e - \sum_{i=1}^{v} deg_G(v_i)$$

$= \text{even} - \text{even}$

$= \text{even}$

Directed Graph : The graph in which the edges have dirn.



outdegree (+) : from where the edge is going to start.
indegree (−) : from where the edge is going to end.

If $G = (V, E)$ be a directed graph with e edges, then

$$\sum deg_G^+ v = \sum deg_G^- v = e$$

Each edge contributes with 2 vertices and also contributes to 1 indegree and 1 outdegree, edge always starts from any point (out degree) and ends at a point (in degree)

Hence, total no. of indegree = total no. of outdegree

$= e$

Types of Graphs

- Null Graph

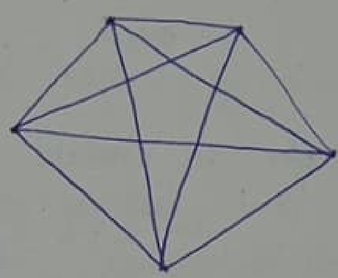
A graph with a collection of single vertices

N_n $\quad$ N_4

- complete graph

K_n $\quad$ K_5

$$\text{edges} = \frac{n(n-1)}{2}$$



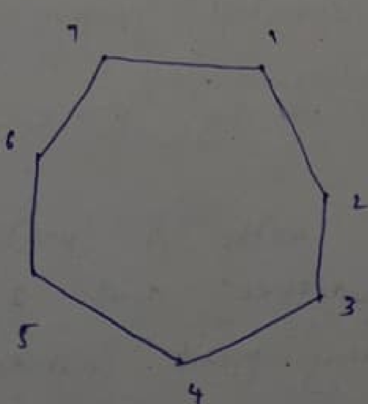
every vertex is adjacent to every other vertex (K_n)

- regular graph

$$\text{edges} = \frac{n \times r}{2}$$

R_n $\quad$ r
↓ $\quad$ ↳ degree
vertices

R_7



null graph
complete graph
are also
regular graphs

all vertices have the same degree

Cycle

C_n

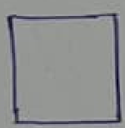
1. $n \geq 3$

2. degree $= 2$

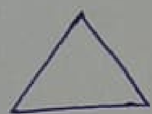
3. every vertex should be adjacent to its neighbour vertices ~~vertex~~

4. no. of edges for $C_n = n$

5. start of node or vertex will be the end of the node or vertex

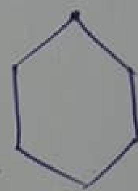


C_4



C_3

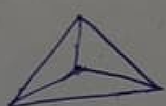
C_5



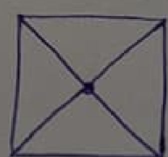
C_6

Wheel

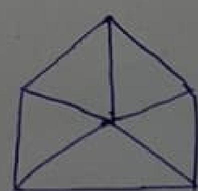
W_n



W_3



W_4



W_5

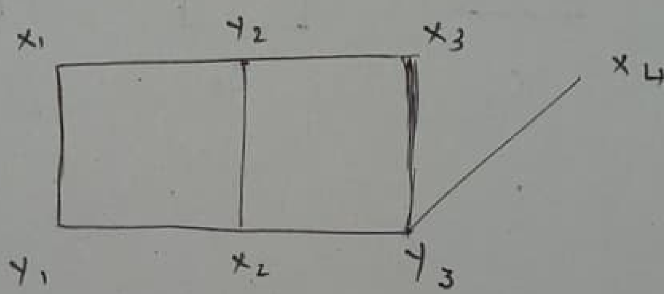
1. degree of centre vertex $= n$
   rest $= 3$

2. we dont consider the centre vertex while naming

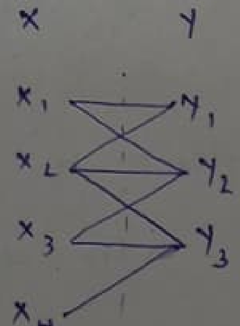
3. no. of edges $= 2n$

Bipartite Graph

A graph $G = (V, E)$ is bipartite if the vertex set V can be partitioned into 2 disjoint subsets V_1 and V_2 such that every edge in E connects a vertex in V_1 and a vertex in V_2 so that no edge in G connects either to ~~two~~ ~~two.~~ vertices in V_1 or to vertices in V_2 is known as a bipartition in G.

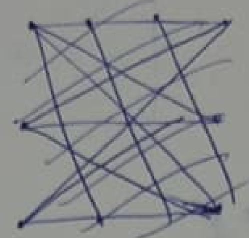


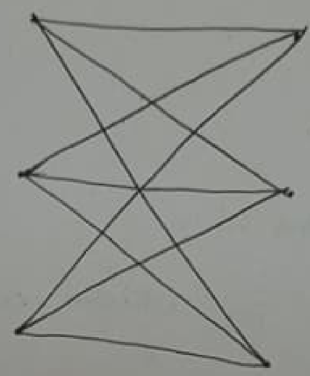
$$V = \{v_1, v_2, \ldots, v_n\}$$



Complete Bipartite Graph ($K_{m,n}$)

$K_{3,3}$





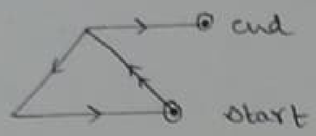
Connected Graph

A unidirected graph G is called connected if there is atleast one path between every pair of vertices of G otherwise G is disconnected.

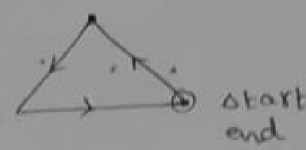
- null graph with more than one vertex is disconnected
- if there's only one vertex then it's a connected graph

Repeated vertices	Repeated edges	open	close	name
Yes	Yes	Yes		open walk
Yes	Yes		Yes	closed walk
Yes	No	Yes		trail
Yes	No		Yes	circuit
No	No	Yes		path
No	No		Yes	cycle

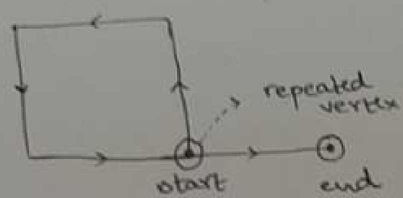
open walk



closed walk

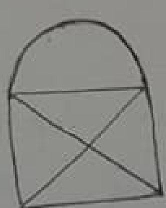


trail



Types of Connected Graphs :

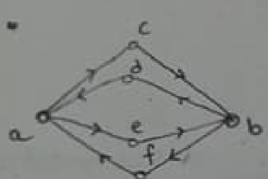
1. Euler Graph : A graph in which every edge is traversed once.



vertices may repeat, but we cannot repeat edges

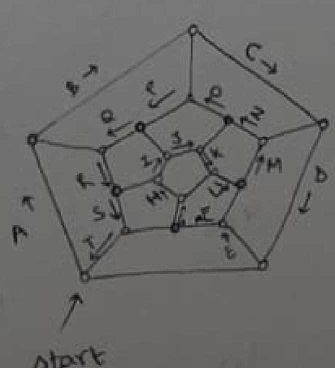
X → since, there is a // edge. we've to take a simple graph.

Eg:



2. Hamiltonian Graph : A graph in which every vertex is traversed exactly once.

Eg :



Q. Does there exist a simple graph with 5 vertices of the following degree

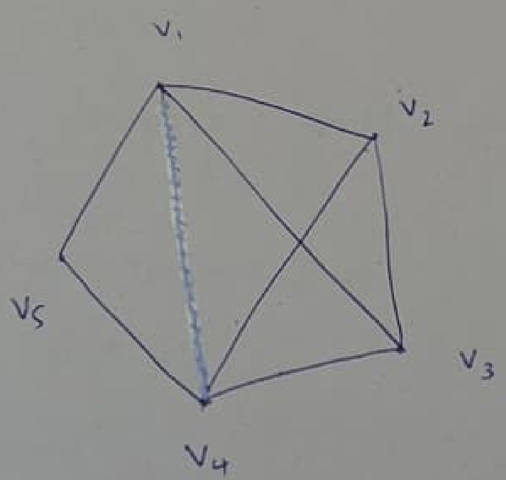
(a) 3, 3, 3, 3, 2
(b) 1, 2, 3, 4, 5
(c) 1, 2, 3, 4, 4
(d) 0, 1, 2, 2, 3
(e) 1, 1, 1, 1, 1

(a) verification of handshaking theorem

$3+3+3+3+2 = 14$ (even)

$14/2 = 7$ = no. of edges

$V = 5$
$E = 7$

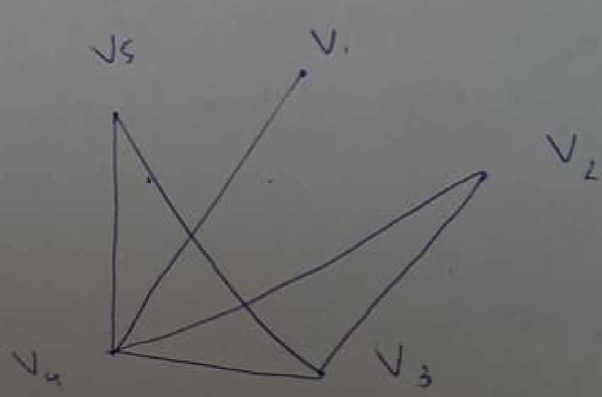


(b) $1+2+3+4+5 = 15$ (odd)

not possible

(c) $1+2+3+4+4 = 14$

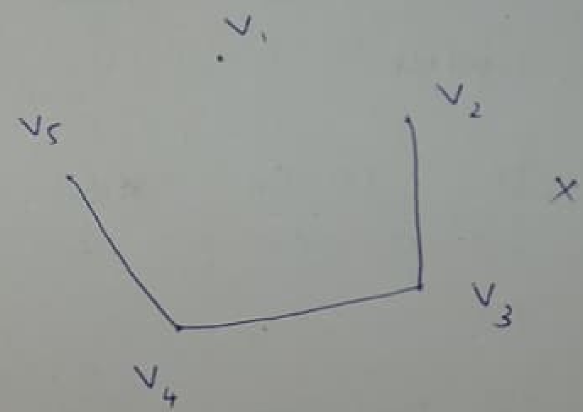
$E = 7$
$V = 5$

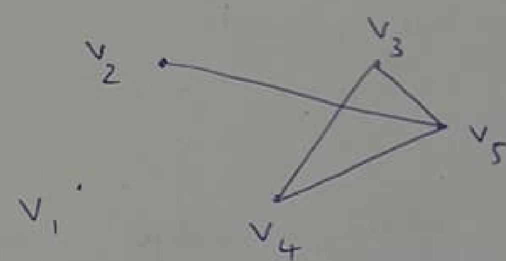


(d) $0+1+2+2+3 = 8$ (even)

$8/2 = 4 = E$

$V = 5$





(e) $1+1+1+1+1 = 5$ (odd)

not possible

Q. Can a simple graph exist with 15 vertices each of degree 5?

A. No. By, Handshaking Theorem, not possible.

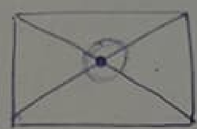
Q. For which value of n are these graphs bipartite?

(a) K_n (b) C_n (c) W_n ~~(d) $K_{m,n}$~~

A. (a) 1, 2

(b) even values (> 3)

(c) not possible

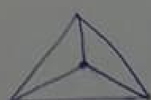


Q. For which value of n are these graphs regular?

(a) K_n From K_2.

(b) C_n $n > 3$

(c) W_n only for $n = 3$



(d) $K_{m,n}$ $m = n$

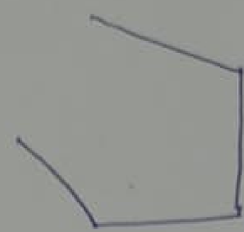
Q. How many vertices does a regular graph of degree 4 with 10 edges have?

$$\frac{n \times r}{2} = \text{edges}$$

$$\frac{n \times 4}{2} = 10 \Rightarrow n = 5$$

Q. Show that every connected graph with n-vertices has n-1 edges. (atleast)

$n = 5$, C_5



Q. For what values of n are these graphs euler?

(a) K_n

(b) C_n

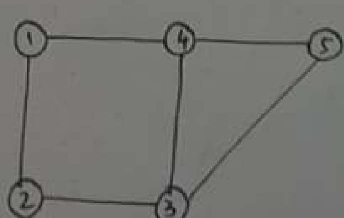
(c) W_n

(d) $K_{m,n}$

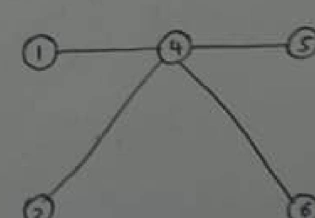
Operations on Graph

$G(V, E)$

union
intersection
difference
set difference
complement

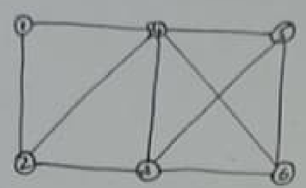


G_1

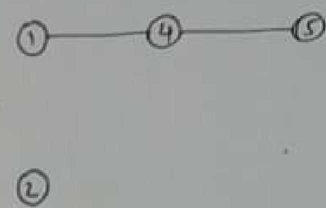


G_2

union $G_1 \cup G_2$

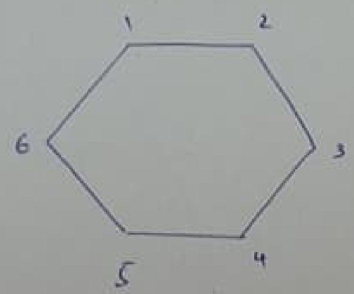


intersection $G_1 \cap G_2$

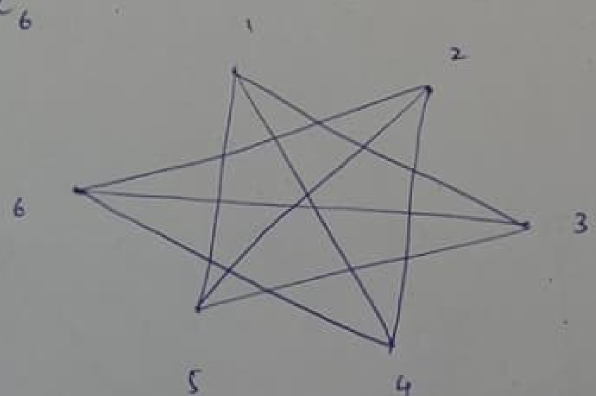


complement

Eg: C_6



$\overline{C_6}$



complement
⇒ edges which were not already present

difference : vertices, edges present only in one graph.

$G_1 \Delta G_2$

Q. If G is a simple graph w 15 edges and $\bar{G}$ has ~~15~~ 13 edges, how many vertices does G have?

A.
$$G + \bar{G} = K_n$$

$$15 + \cancel{15}13 = \frac{n(n-1)}{2}$$

$$(28)2 = n(n-1)$$

$$n(n-1) = 56$$

$$n(n-1) = 8(7)$$

$$n = 8$$

Q. If graph G has v vertices and E edges, how many edges does $\bar{G}$ have?

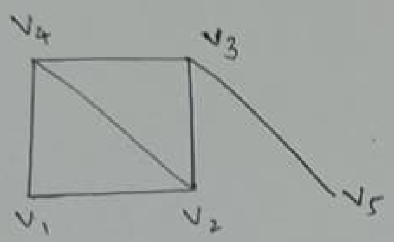
$$\frac{v(v-1)}{2} - e$$

Subgraphs

A part of a graph is known as a subgraph.

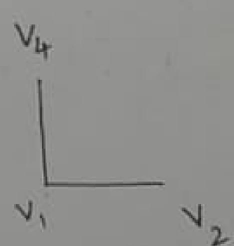
- a single vertex
- a single edge
- the graph itself

subgraphs



edge disjoint subgraphs

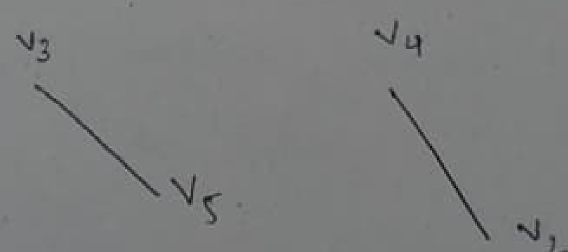
no common edge b/w ~~to~~ two subgraphs



vertex disjoint subgraphs

no common vertex b/w two subgraphs.

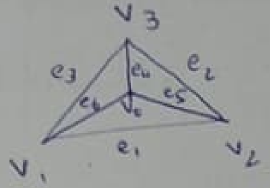
all vertex disjoint subgraphs are edge disjoint s.g.



K_3 , W_3

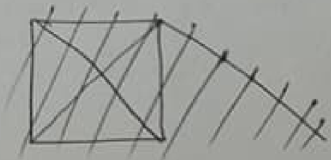
↓

7 subgraphs



Adjacency Matrix

no. of edges



$$\begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 1 & 0 & 1 & 0 \\ v_2 & & & & & \\ v_3 & & & & & \\ v_4 & & & & & \\ v_5 & & & & & \end{array}$$

Isomorphism of Graphs

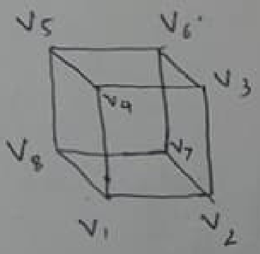
checking if two graphs are mirror images of each other of no.

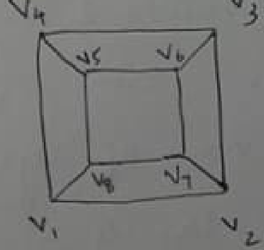
→ degree of the vertices to be compared

→ vertices

→ edges

should be same





Suppose that G and H are isomorphic simple graphs, show that their complementary graphs $\bar{G}$ and $\bar{H}$ are also isomorphic.

no. of vertices in G = no. of vertices in H

G and $\bar{G}$ have the same no. of vertices
H and $\bar{H}$ have the same no of vertices

$\therefore$ no. of vertices in $\bar{G}$ = no. of vertices in $\bar{H}$

$$G + \bar{G} = K_n$$

$$\bar{G} = K_n - G$$
$$= \frac{n(n-1)}{2} - e$$

no. of edges in G = no of edges in H

$$H + \bar{H} = K_n$$

$$\bar{H} = K_n - H$$
$$= \frac{n(n-1)}{2} - e$$

$\therefore$ no. of edges in $\bar{G}$ = no. of edges in $\bar{H}$

edges in

all the edges present in G will be

absent in $\bar{G}$ and all the edges present in H will be absent in $\bar{H}$,

$\therefore$ degree of $\bar{G}$ = degree of $\bar{H}$

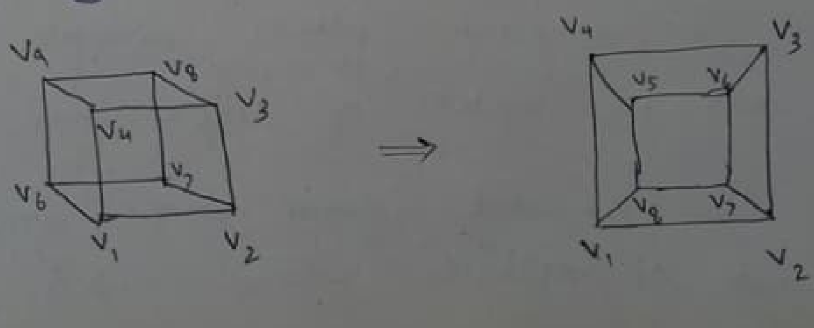
($\because$ degree of G = degree of H)

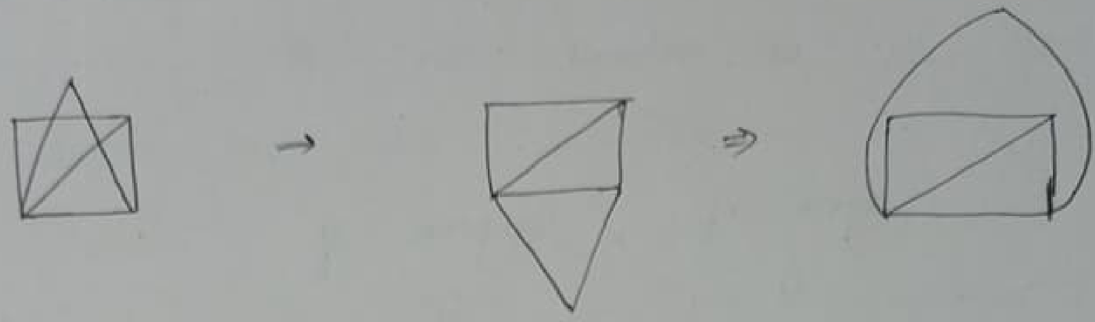
eg:

Planar Graph

A graph G is said to be planar if there exist ~~no~~ some geometric representation of G which can be drawn on a plane such that no two edges of its intersect. The point of intersections are known as crossovers.

A graph that can~~not~~ be drawn on a plane without a crossover b/w its edges is called a planar edges.





Are K_4 and $K_{3,3}$ planar?

K_4

planar

$K_{3,3}$

not planar

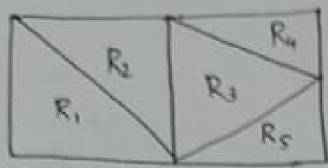
$$\boxed{e \leq 3v - 6}$$

Theorem

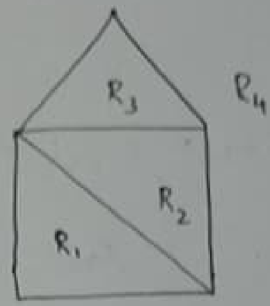
- If G is a connected planar simple graph then G has a vertex of degree not exceeding 5.
- If G is a connected planar simple graph with e edges and v vertices where $v \geq 3$, then $e \leq 3v-6$

Euler's Theorem

Planar Graph



(i)



(ii)

Euler's Formula : Let G be a connected planar simple graph with 'e' edges and 'v' vertices. Let there be 'r' regions.

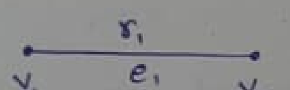
$$r = e - v + 2$$

fig (i) $r = e - v + 2$
$= 11 - 7 + 2$
$= 6$

fig (ii) $r = 7 - 5 + 2$
$= 4$

Proof by Induction

Base Step :



$r = 1 - 2 + 2 = 1$

Inductive Step: Let $r_k = e_k - v_k + 2$ be true

$e_{k+1} = e_k + 1$

$r_{k+1} = r_k + 1$

$r_{k+1} = e_{k+1} - v_{k+1} + 2$

$\therefore$ extending a vertex or edge won't affect the number of regions.

$$r_{k+1} = (e_k + 1) - (v_k + 1) + 2$$

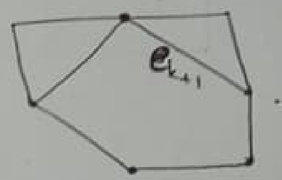
$$r_{k+1} = e_k - v_k + 2 = r_k$$

$$r_{k+1} = e_{k+1} - v_{k+1} + 2$$

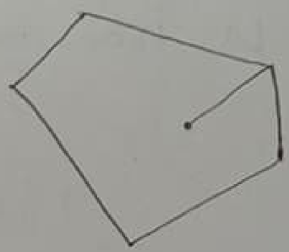
$$= e_{k+1} - v_k + 2$$

$$= (e_k - v_k + 2) + 1$$

$$r_{k+1} = r_k + 1$$



It is not a connected planar graph.

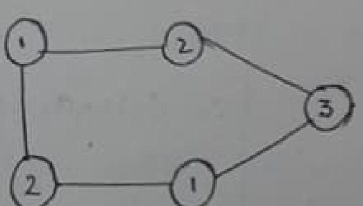


Graph Coloring

- edge coloring (x syllabus)
- vertex coloring ✓

Chromatic number

minimum number of colors required to color the vertex of a graph in such a way that no adjacent vertices have the same color.



$$\chi_k = 3$$

χ_k

$K_n \to n$

$W_n \to 3$ for odd, 4 for even

$K_{m,n} \to 2$

$C_n \to 2$ for even, 3 for odd

TREE

Tree is a simple connected undirected graph w no cycles, self loop and parallel edges. It's a particular type of a graph.

Theorem: Let 'T' be a tree of n-nodes where $n>0$, then it has exactly $(n-1)$ edges

Every node except root node has exactly one parent, so total edges $= n-1$

If we consider $(n-2)$ edges, then one of the node will be disconnected and it will not be a tree.

Theorem : A full m-array tree with 'i' internal vertices contain $(n = mi+1)$ vertices.

Every vertex except leaf vertex is called an internal vertex. Since, each of the 'i' internal vertices has 'm' children, $\therefore$ there are (mi) vertices in the tree other than the root.

$\therefore$ the tree contains $(n = mi+1)$ vertices.

Theorem : A full m-ary tree with

(i) n vertices has $i = (n-1)/m$ internal vertices and $l = [(m-1)n+1]/m$

(ii) 'i' internal vertices have $l = (m-1)i+1$ leaves

(iii) l leaves have $n = (ml-1)/(m-1)$ vertices and $i = (l-1)/(m-1)$ internal leaves.

given $\left[\begin{array}{l} n = mi+1 \\ n = \underbrace{i}_{\text{internal vertices}} + \underbrace{l}_{\text{leaf vertices}} \end{array}\right.$

(i) $n = mi+1$

$n-1 = mi$

$i = \dfrac{n-1}{m}$

as we know, $n = i+l$

$\therefore \quad l = n-i$

$l = n - \dfrac{(n-1)}{m}$

$l = \dfrac{(mn-n+1)}{m}$

$l = \dfrac{(m-1)n+1}{m}$

(ii) $l = n - i$

$l = mi + 1 - i$ $\quad [n = mi + 1]$

$l = i(m-1) + 1$

(iii) $n = mi + 1$

$i = \frac{n-1}{m}$ —— (1)

$n = i + l$

$i = n - l$ —— (2)

from (1) and (2)

$\frac{n-1}{m} = n - l$

$n - 1 = mn - ml$

$n - mn = 1 - ml$

$n(1-m) = 1 - ml$

$n(m-1) = (ml - 1)$

$$\boxed{n = \frac{ml-1}{m-1}}$$

$n = mi + 1$ —— (a)

$n = i + l$ —— (b)

$mi + 1 = i + l$

$mi - i = l - 1$

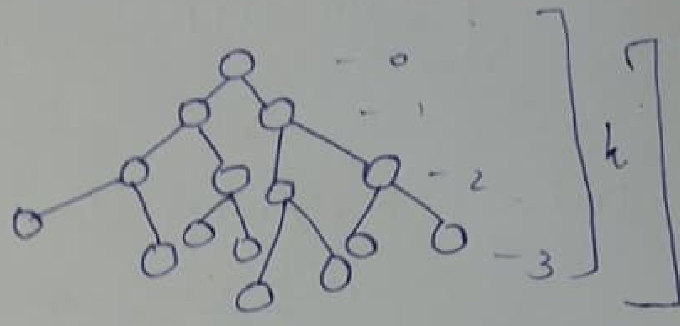
$$\boxed{i = \frac{l-1}{m-1}}$$

There are atmost m^h leaves in an m-arrary tree of hieght h

By induction

Base step $R=0$

$m^0 = 1$



Inductive step : $h=k$

Let m^k leaves are there at $h=k$

for $h=k+1$

$m \cdot m^k$ leaves

If an m-arry tree of hieght h has l leaves, then $h \geq \lceil \log_m l \rceil$

from the previous theorem

$l = m^h$

$h = \log_m l$

(inequality for unbalanced tree)

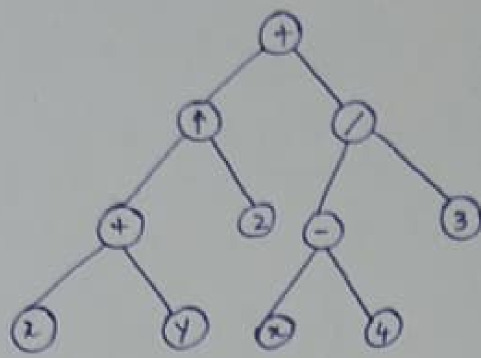
$\therefore$

$h \geq \log_m l$

ceil func. to get integer value from log : $\lceil \log_m l \rceil \leq h$

Q. $((x+y)\uparrow 2) + ((x-4)/3)$

A.



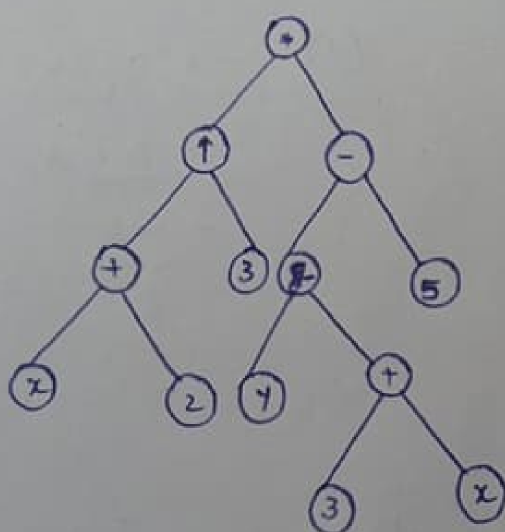
preorder :

$+ \uparrow + x y 2 / - \cancel{8}\, x 4 x$

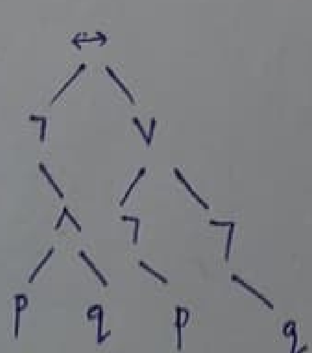
postorder :

$x y + 2 \uparrow x 4 - 3 / +$

Q. $((x+2)\uparrow 3) * ((y-(3+x))-5)$



Q. $\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$



operand towards the right of negation

Q. $(x+xy) + (x|y)$

Q. $x + ((xy+x)|y)$

Q. $(\neg p \wedge (q \leftrightarrow \neg p)) \vee \neg q$

Q. $(A \cap B) - (A \cup (B-A))$

---

prefix exp. - evaluate

L → r

(a) $+ * + \underbrace{- 5\ 3}\ 2\ 1\ 4$

$5-3=2$

$+ * \underbrace{+ 2\ 2}\ 1\ 4$

$+ \underbrace{* 4\ 1}\ 4$

$\underbrace{+ 4\ 4}$

8

(b) $\uparrow \underbrace{+ 2\ 3}\ - 5\ 1$

$\uparrow 5 \underbrace{- 5\ 1}$

$\uparrow 5\ 4$

$5 \uparrow 4$

$(5)^4 \Rightarrow 625$

(c) $* \underbrace{/ 9\ 3} + * 2\ 4 - 7\ 6$

$* 3 + \underbrace{* 2\ 4} - 7\ 6$

$* 3 + 8 \underbrace{- 7\ 6}$

$* 3 \underbrace{+ 8\ 1}$

$\underbrace{* 3\ 9}$

27

(d) $- * 2 \underbrace{/ 8\ 4}\ 3$

$- \underbrace{* 2\ 2}\ 3$

$\underbrace{- 4\ 3}$

1

(e) $\uparrow - \underbrace{* 3\ 3} * 4\ 2\ 5$

$\uparrow - 9 \underbrace{* 4\ 2}\ 5$

$\uparrow \underbrace{- 9\ 8}\ 5$

$\underbrace{\uparrow 1\ 5}$

1

(f) $+ - \underbrace{\uparrow 3\ 2} \uparrow 2\ 3 / 6 - 4\ 2$

$+ - 9 \underbrace{\uparrow 2\ 3} / 6 - 4\ 2$

$+ \underbrace{- 9\ 8} / 6 - 4\ 2$

$+ 1 / 6 \underbrace{- 4\ 2}$

$+ 1 \underbrace{/ 6\ 2}$

$\underbrace{+ 1\ 3}$

4

(g) $* + 3 + 3 \uparrow 3 \underbrace{+ 3 3} 3$

$* + 3 + 3 \underbrace{\uparrow 3 6} 3$

$* + 3 \underbrace{+ 3 (3)^6} 3$

$* \underbrace{+ 3 (3^6 + 3)} 3$

$* (3^6 + 6) 3$

$(3^6 + 6) * 3$

$(729 + 6) * 3$

$(735) * 3$

2205

27

27 14
3 29 14
7 28

postfix evaluation

(a) $5 \underbrace{2 1 -} - 3 1 4 + + *$

$R \to L$

$\underbrace{5 1 -} 3 1 4 + + *$

$4 3 \underbrace{1 4 +} + *$

$4 \underbrace{3 5 +} *$

$\underbrace{4 15 *}$

60

(b) $9 3 / 5 + \underbrace{7 2 -} *$

$\underbrace{9 3 /} 5 + 5 *$

$3 5 + 5 *$

$\underbrace{8 5 *}$

40

(c) $3 2 * 2 \uparrow 5 3 - \underbrace{8 4 /} * -$

$3 2 * 2 \uparrow \underbrace{5 3 -} 2 * -$

$3 2 * 2 \uparrow \underbrace{2 2 *} -$

$\underbrace{3 2 *} 2 \uparrow 4 -$

$\underbrace{6 2 \uparrow} 4 -$

$36 4 -$

32

Construct the ordered rooted tree whose preorder traversal is

$\underset{4}{a}\ \underset{1}{b}\ f\ \underset{3}{c}\ g\ h\ i\ d\ \underset{1}{e}\ \underset{2}{j}\ k\ l$

a = 4 children

c = 3 children

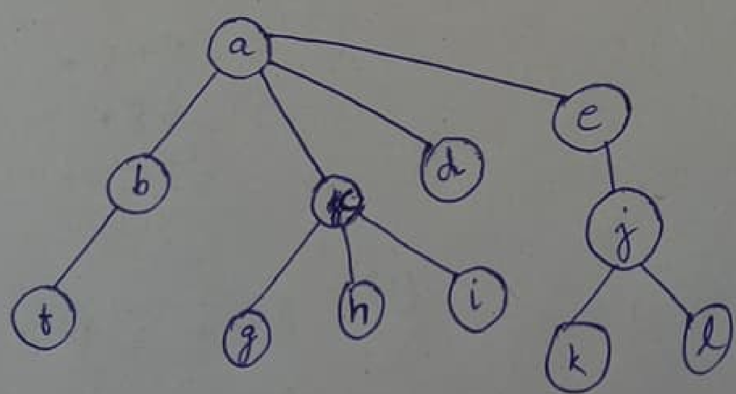
j = 2 children

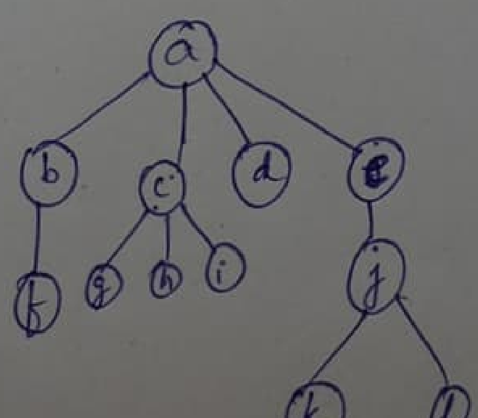
e, b = 1 child

others = leaves

prefix : a b f c g h i d e j k l

R L R





Q. Form a binary search tree for the words

MATHEMATICS
PHYSICS
GEOGRAPHY
ZOOLOGY
METEOROGY
GEOLOGY
PSYCHOLOGY
CHEMISTRY

Basic Counting Principles

* Sum Rule

T

T_1 $\quad$ T_2

n_1 ways $\quad$ n_2 ways $\qquad$ $(n_1 + n_2)$ ways

either T_1 or T_2

Q. Suppose either a faculty or a student is chosen as a representative to a commity, how many diff. choices are there for this representation if there are 37 faculty and 83 students.

A. $37 + 83 = 120$ choices

* Product Rule

T

T_1 $\quad$ T_2

n_1 ways ① $\quad$ ② n_2 ways $\qquad$ $(n_1 \times n_2)$ ways

first T_1 & then T_2

Q. The chairs of an auditorium are to be labelled w an upper case english letter followed by a positive integer not exceeding 100. What is the largest no. of chairs that can be labelled differently?

A. $26 \times 100 = 2600$

Q. How many different bit strings of length 7 are there?

A.

$$\underset{\substack{2\\(0/1)}}{\uparrow}\ \underset{2}{\uparrow}\ \_\ \_\ \_\ \_\ \_ \Rightarrow 2^7 \Rightarrow 128$$

Q. How many strings are there of lower case letter of length ≤ 4 not having the empty string.

A.

4 $\_\ \_\ \_\ \_ \Rightarrow (26)^4$

3 $\_\ \_\ \_ \Rightarrow (26)^3$

2 $\_\ \_ \Rightarrow (26)^2$

1 $\_ \Rightarrow (26)^1$

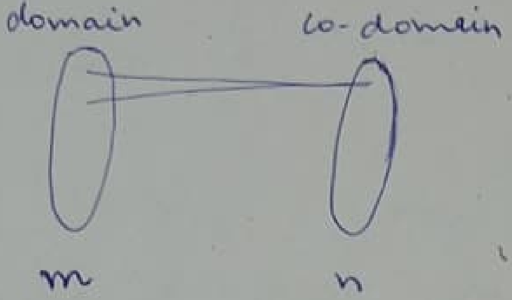
$$(26)^4 + (26)^3 + (26)^2 + (26)^1$$

Q. How many bit strings of length 10 both begin and end w a 1

$(2)^8$

Q. How many functions are there from a set w 'm' elements to a set w 'n' elements.

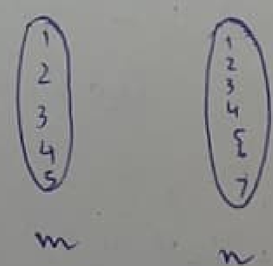
A.



n choices for each element

$\therefore \quad (n)^m$

Q. How many one to one functions are there from a set of m elements to a set of n elements



$m > n$

either no function or it's not one - one

$m = n$

n
$(n-1)$
$(n-2)$
$\vdots$
1

$\Rightarrow n(n-1)(n-2)\ldots 1$

$n!$

$m < n$

$7 \times 6 \times 5 \times 4 \times 3$

n
$(n-1)$
$(n-2)$
$\vdots$
$(n-m+1)$

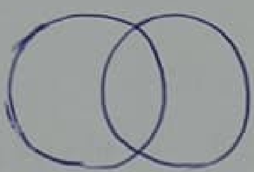
$\Rightarrow n(n-1)(n-2)\ldots(n-m+1)$

Q. An MCQ test contains 10 questions, there are 4 possible ans. for each ques. In how ~~doo~~ many ways can a student ans. the ques if the student answers every question. In how many ways can a student ans. the ques if the student can leave ans. blank.

A.

(i) $(4)^{10}$ (ii) $(5)^{10}$

Subtraction Rule



$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Q. How many bit string w a length 8 ~~ew~~ either start w a 1 or end w double zero.

A.

$\underline{1}\ \_\ \_\ \_\ \_\ \_\ \_\ \_ = 2^7$

$\_\ \_\ \_\ \_\ \_\ \_\ \underline{0}\ \underline{0} = 2^6$

$\underline{1}\ \_\ \_\ \_\ \_\ \underline{\cancel{0}}\ \underline{0}\ \underline{0} = 2^5$

$$\begin{aligned} n(A \cup B) &= 2^6 + 2^7 - 2^5 \\ &= 2^5 (2^2 + 2 - 1) \\ &= 2^5 (4 + 2 - 1) \\ &= 32 (5) \\ &= 160 \end{aligned}$$

Q. How many +ve integers b/w 50 and 100

1. are divisible by 7
2. are divisble by 11
3. are divisible by both 7 & 11

A. 1. $\frac{51}{7} = 7....$

$= 7$

2. $\frac{51}{11} = \cancel{4}\ 4.67 = 5$

3. 1

Q. How many +ve integers b/w 100 & 999 inclusive

1. are divisible by 7
2. are odd
3. have the same 3 decimal digits
4. are not divisible by 4
5. are divisible by 3 or 4
6. are ~~not~~ divisible ~~b~~ by neither 3 nor 4
7. divisible by 3 not 4
8. divisible by 3 & 4

A. 1. $\frac{1000}{7} = 142.85 = 143$

2. $1000 - 500 = 500$

3.

4. $1000 - \frac{1000}{4} = 750$

1. $\frac{900}{7} = 128.57 = 129$

2. 450

3. 9 (111, 222, 333, . . . , 999)

4. $900 - 225 = 675$

5. $900/3 = 300$

$900/4 = 225$

$900/12 = 75$

$525 - 75$

450

6. $900 - 450 = 450$

7. $900/3 = 300$

$900/12 = 75$

$300 - 75$

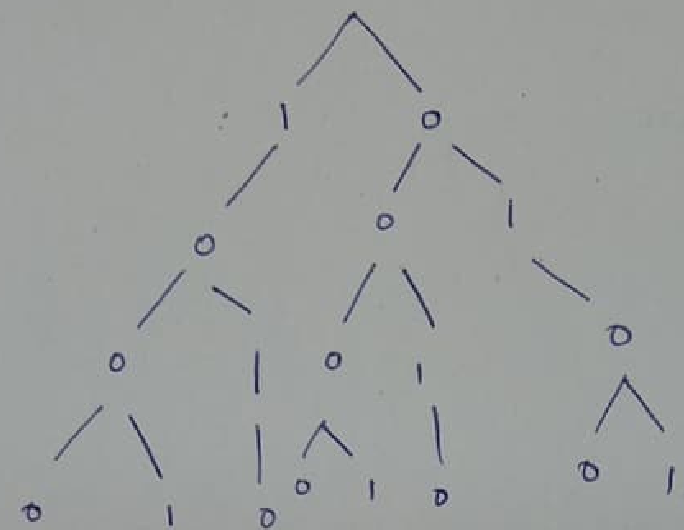
225

8. $900/12 = 75$

Tree Diagram

Q. How many bit strings of length 4 donot have 2 consequetive 1's ?

A.



0001	1000
1001	0100
0101	0010
0000	1010

$\Rightarrow 8$

Q. How many ~~bit~~ decimal strings of length ~~#~~ 3

- donot contain the same decimal digit 3 time
- begin w an odd digit.
- have exactly 2 digits thate are 4

a) $\underline{10}\ \underline{10}\ \underline{10} \Rightarrow 1000$

$\_\ \_\ \_ \Rightarrow 10$ $\Rightarrow 1000 - 10 \Rightarrow 990$

(111, 222, 333)

b $\underset{5}{\underset{\uparrow}{\_}}\ \underset{10}{\_}\ \underset{10}{\_} \Rightarrow 5 \times 10 \times 10 \Rightarrow 500$

(c) $\underline{\ \ }\ \underline{4}\ \underline{4} \Rightarrow 9$

$\underline{4}\ \underline{\ \ }\ \underline{4} \Rightarrow 9$

$\underline{4}\ \underline{4}\ \underline{\ \ } \Rightarrow 9$

27

Q. How many strings of ~~4~~ 3 decimal digits
1. do not contain the same digit twice
2. end with an even digit
3. have exactly 3 digits that are 9

1. $10 \times 10 \times 10 - \{000, 111, 222, 333, 444, 555, 666, 777, 888, 999\}$

990

2. $5 \times 10 \times 10$

3. ${}^{3}C_2 \times 9 = 3 \times 9 = 27$

Q. Each user on a computer system has a password which's 6-8 character long where each char is an uppercase letter or a digit. Each password must contain atleast 1 digit. How many possible passwords are there?

A. $6 \rightarrow$ _ _ _ _ _ _ $\Rightarrow {}^{6}C_1 \cdot {}^{26}C_5 \cdot {}^{9}C_1 = 6 \times 9 \times$

$7 \rightarrow$ _ _ _ _ _ _ _ $\Rightarrow {}^{7}C_1 \cdot {}^{9}C_1 \cdot {}^{26}C_6$

$8 \rightarrow$ _ _ _ _ _ _ _ _ $\Rightarrow {}^{8}C_1 \cdot {}^{9}C_1 \cdot {}^{26}C_7$

$+$

${}^{6}C_1\, {}^{9}C_1\, {}^{26}C_5 + {}^{7}C_1\, {}^{9}C_1\, {}^{26}C_6 + {}^{8}C_1\, {}^{9}C_1\, {}^{26}C_7$

Permutation and Combination

↳ arrangement ↳ selection

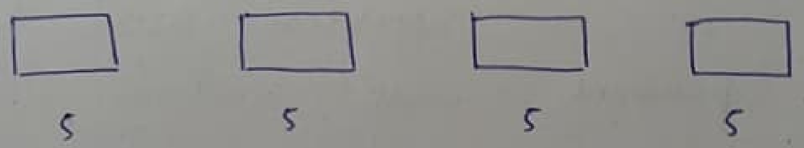
$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

n → total

r → obj to be arranged

Q. Consider a main memory of a system having 4 frames of fixed size 5 kB each. If 4 processes of 3 kB, 2 kB, 4 kB and 1 kB enter the system, then in how many ways the processes can be allocated in the memory?

A.

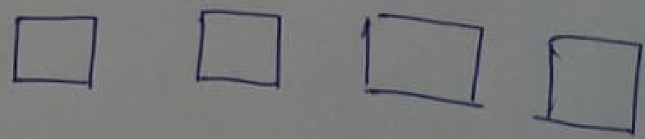


$^{4}P_{4} \rightarrow 4.3.2.1 \Rightarrow 4!$

In how many ways 2 process of 3 kB and 4 kB can enter the system

$^{4}P_{2} \Rightarrow \frac{4!}{2!}$

12



$4 \times 3 \Rightarrow 12$

Q. Let S = a, b, c, d, e, f. How many distinct words of 4 letters can be formed?

A. $\underline{6}\ \underline{5}\ \underline{4}\ \underline{3} \Rightarrow 6 \times 5 \times 4 \times 3$

30×12

360

Q. In how many diff. ways can we award prizes, Ist, IInd & IIIrd among 24 teams if there are no ties.

Ist → 24
IInd → 23
IIIrd → 22

$24 \times 23 \times 22$

Q. How many different strings from the letters of P Q R S T U V W X Y Z can be formed that contains the substring X Y Z.

A. $(8+1)!$

Q. Permutation w repetition

$(6)^4$ (Q1)

Q. How many distinct permutation of the letters in JAGRAN can be formed?

A. $\dfrac{n!}{r_1!\, r_2!\, r_3! \ldots} \Rightarrow \dfrac{6!}{2!} \Rightarrow \dfrac{6 \times 5 \times 4 \times 3 \times 2!}{2!} \Rightarrow 30 \times 12$

$\Rightarrow 360$

Q. How many bitstrings of length n contain exactly r 1's.

^nC_r

Q. In how many ways can a team of 11 players be selected from a pool of 15 players to the play the matches at national level.

A. $^{15}C_{11}$

Q. ~~to~~ In how many ways can a committee of 3g, 4b be formed from a class of ~~20~~

25g, 40b.

A. $^{25}C_3 \cdot {}^{40}C_4$

Q. How many strings of length n can be formed ~~from the alphabet set {0,1}~~ that contains exactly m no. of 0's.

A. ^nC_m

Q. How many bitstring of length 10 contain,

i exactly 4 1's

ii atmost 4 1's

iii atleast 4 1's

iv equal no. of 4's & 1's

(i) _ _ _ _ _ _ _ _ _ _

$^{10}C_4$.

(ii) $^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4$

(iii) $^{10}C_4 + {}^{10}C_5 + {}^{10}C_6 + {}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}$

(iv) $^{10}C_5$*.

Q. How many bitstrings of length 10 have

(i) exactly 3 0's

(ii) more 0's than 1's

(iii) atleast 7 1's

(iv) atleast 3 1's

(i) $^{10}C_3$

(ii) $^{10}C_6 + {}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}$

(iii) $^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}$

(iv) $^{10}C_3 + {}^{10}C_4 + \dots + {}^{10}C_{10}$

Q. How many permutations of the letter abcdefg contain:

(i) string bcd

(ii) string cfga

(iii) strings ba & gf

(iv) strings abc & de

(v) string abc & cde

(vi) string cba and bed

Combination w repetition

$$C(n+r-1, r) = \frac{(n+r-1)!}{(n+r-1-r)!\,r!}$$

total — obj to be selected

$$= \frac{(n+r-1)!}{(n-1)!\,r!}$$

Q. How many solutions does this equation has $x_1 + x_2 + x_3 = 11$ where $x_1\ x_2\ x_3$ are non negative integers?

A.

$n = 3$

$r = 11$

78

$x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$

$n = 3$

$r = 5$

1+2+3 ~~both~~ has already been chosen

21

How many diff outcomes are possible if 10 coins are tossed.

$r = 10$

$n = 2$

In how many ways can 20 similar chairs be chosen by 5 diff. people?

$n = 5$
$r = 20$

10620

How many bitstring contain exactly 8 0's and 10 1's if every 0 must be immidiately followed by a 1

$^{10}C_2$

01 01 01 01 01 01 01 11

How many bitstrings contain exactly 5 0's & 14 1's if every 0 must be immidiately followed by 2 1's?

011 011 011 011 011 1111

$^{14}C_4$

How many bitstrings of length 10 contain atleast 3 1's & 3 0's?

$C(4+2-1, 2)$
$C(n+r-1, r)$

Pigeonhole Principle

If there are k pigeonholes and k+1 pigeons then there's atleast 1 pigeonhole which will keep atleast more than 1 pigeon in it.

$\boxed{}$ $\boxed{}$ $\boxed{}$ $\boxed{}$ $\boxed{}$

best : suppose there are 11 pigeons & 10 ph and we have kept 11 pigeons in 1 pigeonhole

$\boxed{11}$ rest are empty

avg: pigeons are kept randomly.

$\boxed{1}$ $\boxed{2}$ $\boxed{3}$ . . .

worst : $\boxed{1}$ $\boxed{1}$ . . . $\boxed{2}$

last pigeonhde will contain 2 pigeons.

Q. How many students must be in a class to ~~guarant~~ guarantee that atleast 2 students recieve the same score on the final exam if the exam is graded on a scale of 0 - 100 points.

A

0 - 100 $\longrightarrow$ 101 students

worst case : 101 + 1 $\Rightarrow$ 102 students (& 1 student may get any grade)

Q. What is the min. no. of students required in a class to be sure that atleast 6 will recieve the same grade if there are 5 possible grades : A B C D F

A	B	C	D	F	
↑	↑	↑	↑	↑	
↑	↑	↑	↑	↑	
↑	↑	↑	↑	↑	$\longrightarrow$ 25
↑	↑	↑	↑	↑	
↑	↑	↑	↑	↑	

26th will get any of these grades

$\Rightarrow$ 26 students

Q. How many cards must be selected from a standard deck of 52 cards to guarantee
(i) that atleast 3 cards are choosen from the same deck ?
(ii) atleast 3 hearts are selected?

A. (i)

♤	♡	♧	◇	
1	1	1	1	
1	1	1	1	$\Rightarrow$ 8

9th card will be from any of the above choices.

$\Rightarrow$ 9

(ii)

♤	♡	♧	◇	
13	3	13	13	$\Rightarrow$ 42

Q. I've 7 pairs of sock in my drawer. One ~~of~~ for each color of the rainbow. How many socks do I have to draw out in order to ~~gau~~ guarantee that I've grabbed atleast 1 pair?

What if there are likewise colored pair of gloves and I cannot tell the diff b/w gloves & socks, and I ~~wate~~ want a matching set ?

1. — — — — — — — · 7

8th socks will be of any colour.

2.

| | | | | | | |
| | | | | | | |
| | | | | | | $\Rightarrow$ 21

22nd sock | glove $\longrightarrow$ to make a matching pair

Advanced Pigeonhole Principle

If there are m pigeonholes and there are n pigeons, where $n > m$, then there must be atleast 1 pigeonhole in which we can keep atleast ~~tti~~ $\lceil \frac{n}{m} \rceil$

Q. How many different rooms are needed to assign 500 lectures if there are 45 time slots in the university time-table that're availible?

A.

$$\left\lceil \frac{500}{45} \right\rceil \Rightarrow 12$$

Q. There are 5 cargos in a shipyard and a total of 232 containers to be loaded in the cargos. How many containers are min. needed to fill 1 cargo?

A. $\left\lceil \frac{232}{5} \right\rceil = \lceil 46.4 \rceil \doteq 47$

Advanced Counting Technique

Recurrence Relation

- to form
- to solve
  1. iteration method
  2. characteristic root
  3. generating function

Q. The no. of bacterias in ~~our~~ a colony doubles every hour. If a colony begins w 5 bacterias, how many will be present in n hours?

A.

nth hour

$$\boxed{a_n = 2a_{n-1}} \qquad n \geq 1$$

$$\boxed{a_0 = 5}$$

boundary cond^n
initial cond^n

fibonacci series : $f_n = f_{n-1} + f_{n-2}$

$n \geq 2$

factorial $f_n! = f(n-1)! * f(n)$

Q. Find the four terms for each of the following relations

(i) $a_k = 2a_{k-1} + k \quad (k \geq 2,\ a_1 = 1)$

(ii) $a_k = a_{k-1} + 2a_{k-2} \quad (k > 2,\ a_0 = 1,\ a_1 = 2)$

(iii) $a_k = k(a_k - 1)^2 \quad (k \geq 1,\ a_0 = 1)$

(i)

a_2	a_3	a_4	a_5
4	11	26	57

(ii)

a_2	a_3	a_4	a_5
5	11	26	59

(iii)

a_1	a_2	a_3	a_4
1	2	12	576

Q. Show that the sequence $\{2, 3, 4, 5, \ldots, 2+n \ldots\}$ for $n \geq 0$ satisfies the r.r. $a_k = 2a_{k-1} - a_{k-2}$

$k \geq 2$

A.

$a_n = 2+n$

$a_k = 2+k$

$a_{k-1} = 2+(k-1)$

$a_{k-2} = 2+(k-2)$

Q. $a_n = a_{n-1} + 2 \quad , \quad n \geq 2 \; , \; a_1 = 3$

$a_n = a_{n-1} + 2$ —— (1)

put $n = n-1$ in eq (1)

$a_{n-1} = a_{n-2} + 2$ —— (2)

$a_n = a_{n-2} + 2 + 2$

$a_n = a_{n-2} + \underset{(2.2)}{4}$ —— (3)

put $n = n-2$ in eq (1)

$a_{n-2} = a_{n-3} + 2$ —— (4)

$a_n = a_{n-3} + 2 + 4$

$a_n = a_{n-3} + \underset{(2.3)}{6}$ —— (5)

put $n = n-\cancel{4}3$, in eq (1)

$a_{n-\cancel{4}3} = a_{n-\cancel{5}4} + 2$ —— (6)

$a_{n-3} = a_{n-4} + 2$

$a_n = a_{n-4} + 2.4$ —— (7)

$\vdots$

$a_n = a_{n-k} + 2.k$ —— (8)

for $k = n-1$;

$a_n = a_{n-n+1} + 2(n-1)$

$$a_n = a_1 + (n-1)2$$

$$\boxed{a_n = 3 + 2(n-1)}$$ soln.

explicit formula

Q. $a_n = a_{n-1} + 2$ —— (1) , $a_0 = 1$

put $n = n-1$ in eq (1)

$$a_{n-1} = a_{n-2} + 2 \quad \text{—— (2)}$$

from eq (1),

$$a_n = a_{n-2} + 2 \cdot 2 \quad \text{—— (3)}$$

put $n = n-2$ in eq (1),

$$a_{n-2} = a_{n-3} + 2 \quad \text{—— (4)}$$

from eq (3) ;

$$a_n = a_{n-3} + 2 \cdot 3 \quad \text{—— (5)}$$

$$\vdots$$

$$a_n = a_{n-k} + 2 \cdot k$$

for $k = n$;

$$a_n = a_{n-n} + 2 \cdot n$$

$$a_n = a_0 + 2n$$

$$\boxed{a_n = 1 + 2n}$$

Q. $a_n = -(a_{n-1})$, $a_0 = 5$ — (1)

A.

put $n = n-1$ in eq (1),

$a_{n-1} = -(a_{n-2})$ — (2)

from eq (1)

$a_n = -(-(a_{n-2}))$ — (3)

put $n = n-2$ in eq (1),

$a_{n-2} = -(a_{n-3})$ — (4)

from eq (3),

$a_n = -(-(-(a_{n-3})))$

$\vdots$

$a_n = (-1)^k (a_{n-k})$

for $k = n$,

$a_n = (-1)^n (a_{n-n})$

$a_n = (-1)^n (a_0)$

$\boxed{a_n = 5(-1)^n}$

Q. $a_n = 3a_{n-1}$, $a_0 = 2$

— (1)

A. put $n = n-1$ in eq (1)

$a_{n-1} = 3a_{n-2}$ — (2)

from eq (1)

$a_n = 3(3(a_{n-2}))$ — (3)

~~from~~ put $n = n-2$ in eq (1),

$a_{n-2} = 3a_{n-3}$ — (4)

from eq (3),

$a_n = 3(3(3(a_{n-3})))$ — (5)

$\vdots$

$a_n = (3)^k a_{n-k}$

~~put~~ for $k = n$,

$a_n = (3)^n a_{n-n}$

$a_n = (3)^n a_0$

$\boxed{a_n = 2(3)^n}$

Q. $a_n = 2n\, a_{n-1}$, $a_0 = 1$

—— (1)

put $n = n-1$ in eq (1),

$a_{n-1} = 2(n-1)\, a_{n-2}$ —— (2)

from (1),

$$a_n = 2n\,(2(n-1)\, a_{n-2})$$
$$a_n = 2^2\, n(n-1)\, a_{n-2} \quad \text{—— (3)}$$

put $n = n-2$ in eq (1),

$a_{n-2} = 2(n-2)\, a_{n-3}$ —— (4)

from eq (3),

$$a_n = 2^3\, n(n-1)(n-2)\, a_{n-3}$$

$\vdots$

$$a_n = 2^k\, n(n-1)(n-2)\ldots(n-(k-1))\, a_{n-k}$$

for $k = n$,

$$a_n = 2^n\, n(n-1)(n-2)\ldots(n-n+1)\, a_0$$
$$a_n = 2^n\,(n(n-1)(n-2)\ldots 1)\, a_0$$
$$a_n = 2^n\, n! \cdot 1$$
$$\boxed{a_n = 2^n \times n!}$$

Linear Recurrence Relation

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + C_3 a_{n-3} + \dots + C_k a_{n-k} + f(n)$$

$$a_n = a_{n-1} + 2a_{n-2} \quad \text{—(1)}, \quad n \geq 2, \quad a_0 = 0, \; a_1 = 1$$

Let $a_n = r^n$

$$r^n = r^{n-1} + 2r^{n-2} \quad \text{—— (2)}$$

$$r^n - r^{n-1} - 2r^{n-2} = 0$$

$$r^{n-2}(r^2 - r - 2) = 0$$

$$r^{n-2}(r+1)(r-2) = 0$$

$$r = 2, -1$$

characteristic roots

general soln:

$$a_n = b_1 2^n + b_2 (-1)^n$$

→ roots must be real

→ roots are real and distinct

general soln:

$$a_n = b_1 (r_1)^n + b_2 (r_2)^n + \dots + b_k (r_k)^n$$

→ roots are real and non-distinct

$$a_n = (b_1 + b_2 n + b_3 n^2 + \dots b_k n^{k-1})(r_k)^n$$

$$3, 3, 2 \longrightarrow (b_1 + b_2 n) 3^n + b_3 2^n$$

Linear Recurrence Relation

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + C_3 a_{n-3} + \dots + C_k a_{n-k} + f(n)$$

$$a_n = a_{n-1} + 2a_{n-2} \quad \text{— (1)}, \quad n \geq 2, \quad a_0 = 0$$
$$a_1 = 1$$

Let $a_n = r^n$

$$r^n = r^{n-1} + 2r^{n-2} \quad \text{— (2)}$$

$$r^n - r^{n-1} - 2r^{n-2} = 0$$

$$r^{n-2}(r^2 - r - 2) = 0$$

general soln:

$$a_n = b_1 2^n + b_2 (-1)^n$$

$$r^{n-2}(r+1)(r-2) = 0$$

$$r = 2, -1$$

characteristic roots

→ roots must be real

→ roots are real and distinct

general soln:

$$a_n = b_1 (r_1)^n + b_2 (r_2)^n + \dots + b_k (r_k)^n$$

→ roots are real and non-distinct

$$a_n = (b_1 + b_2 n + b_3 n^2 + \dots b_k n^{k-1}) (r_t)^n$$

$$3, 3, 2 \longrightarrow (b_1 + b_2 n) 3^n + b_3 2^n$$

for $n=0$

$$0 = b_1 (2)^0 + b_2(-1)^0$$

for $n=1$,

$$1 = b_1 (2)^1 + b_2 (-1)^1$$

$$b_1 + 1b_2 = 0$$
$$2b_1 - b_2 = 1$$

$$3b_1 = 1$$
$$b_1 = 1/3$$

$$b_1 + b_2 = 0$$
$$b_2 = -b_1$$
$$b_2 = -1/3$$

$\therefore$

$$a_n = 2^n b_1 + (-1)^n b_2$$

$$a_n = 2^n (1/3) + (-1)^n (-1/3)$$

$$\boxed{a_n = \frac{1}{3} (2)^n - \frac{1}{3} (-1)^n}$$

Q9. (a) $a_n = 2a_{n-1}$ $\quad n \geq 1, \; a_0 = 3$

Let $a_n = r^n$

$$r^n = 2r^{n-1}$$

$$r^n - 2r^{n-1} = 0$$

$$r^{n-1}(r-2) = 0$$

$$r = 2$$

general soln : $a_n = b_1 2^n$

for $n = 0$,

$$a_0 = b_1 2^0$$

$$3 = b_1$$

$$b_1 = 3$$

$$\therefore \quad \boxed{a_n = 3 \cdot 2^n}$$

(b) $a_n = a_{n-1}$ for $n \geq 1$, $a_0 = 2$

Let $a_n = r^n$

$$r^n = r^{n-1}$$

$$r^n - r^{n-1} = 0$$

$$r^{n-1}(r-1) = 0$$

$$r = \cancel{2}1$$

g.s: $a_n = b_1 (1)^n$

for $n = 0$

$$a_0 = b_1$$

$$2 = b_1$$

$$\Rightarrow \quad a_n = 2$$

(c) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

let $a_n = r^n$,

$r^n = 5r^{n-1} - 6r^{n-2}$

$r^n - 5r^{n-1} + 6r^{n-2} = 0$

$r^{n-2}(r^2 - 5r + 6) = 0$

$r^{n-2}(r-2)(r-3) = 0$

$r = 2, 3$

g.s : $a_n = (2)^n b_1 + (3)^n b_2$

for $n = 0$,

$(2)^0 b_1 + (3)^0 b_2 = a_0$

$b_1 + b_2 = 1$ —— (1)

$b_2 = 1 - b_1$

for $n = 1$

$2b_1 + 3b_2 = a_1$

$2b_1 + 3b_2 = 0$ —— (2)

$2b_1 + 3(1 - b_1) = 0$

$2b_1 + 3 - 3b_1 = 0$

$\boxed{b_1 = 3}$

$b_2 = 1 - 3$

$\boxed{b_2 = -2}$

$\Rightarrow \quad a_n = (2)^n(3) + (3)^n(-2)$

$$\boxed{a_n = 3(2)^n - 2(3)^n}$$

(d) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$

$a_0 = 6$
$a_1 = 8$

Let $a_n = r^n$

$r^n = 4r^{n-1} - 4r^{n-2}$

$r^n - 4r^{n-1} + 4r^{n-2} = 0$

$r^{n-2}(r^2 - 4r + 4) = 0$

$r^{n-2}(r-2)(r-2) = 0$

$r = 2, 2$

$a_n = (b_1 + b_2 n)2$

for $n = 0$

$a_0 = (b_1 + b_2(0))2$

$a_0 = 2b_1$

$6 = 2b_1$

$\Rightarrow b_1 = 3$

for $n = 1$

$a_1 = (b_1 + b_2(1))2$

$8 = (3 + b_2)2.$

$4 = 3 + b_2$

$\Rightarrow b_2 = 1$

$\therefore \quad a_n = (3 + n) * 2$

$$\boxed{a_n = 2n + 6}$$

(e) $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 1$

Let $a_n = r^n$

$r^n = -4r^{n-1} - 4r^{n-2}$

$r^n + 4r^{n-1} + 4r^{n-2} = 0$

$r^{n-2}(r^2 + 4r + 4) = 0$

$r^{n-2}(r+2)(r+2) = 0$

$r = -2, -2$

$a_n = (b_1 + b_2 n) r$

$a_n = (b_1 + b_2 n)(-2)$ —— (1)

for $n = 0$

$a_0 = (b_1 + b_2(0))(-2)$

$0 = -2b_1$

$b_1 = 0$

for $n = 1$

$a_1 = (b_1 + b_2(1))(-2)$

$1 = (0 + b_2)(-2)$

$-1/2 = b_2$

$\therefore$ from (1)

$a_n = (0 + (-1/2)n)(-2)$

$\boxed{a_n = n}$

$$\boxed{a_n = Ca_{n-1} + C_2 a_{n-2} + C_3 a_{n-3} \cdots + C_n a_{n-k}} + f_n$$

Homogeneous Part

Non-Homogeneous Part

value of $f(n)$ can be
1. $2^n, 3^n, 4^n$
2. n^3, n^2+1, n
3. $n^3 4^n$
4. $(n+1)5^n$
4. $2, 3, 4$

$f(n)$	Assumed Particular Soln.
1. b^n (if b is not the root of the characteristic eq^n)	$\underbrace{Ab^n}_{\text{constant}}$
2. $P(n)$ i.e., polynomial of degree m	$A_0 + A_1 n + A_2 n^2 + \cdots + A_m n^m$
3. $b^n P(n)$ (b is not a characteristic root)	$(A_0 + A_1 n + A_2 n^2 + \cdots + A_m n^m) b^n$
4. b^n (if b is the root of characteristic eq^n of multiplicity s)	$Ab^n n^s$
5. $b^n P(n)$ (if b is the root of characteristic eq^n of multiplicity s)	$(A_0 + A_1 n + A_2 n^2 + A_3 n^3 + \cdots + A_m n^m) \cdot b^n n^s$
6. b	A

Q. $a_{n+2} - 4a_{n+1} + 4a_n = \cancel{\ } 2^n$

A. For homogeneous part,

Let $a_n = r^n$

$$r^{n+2} - 4r^{n+1} + 4r^n = 0$$

$$r^n(r^2 - 4r + 4) = 0$$

$$r^n(r-2)(r-2) = 0$$

$$r = 2, 2$$

$$a_n^{(h)} = (C_1 + C_2 n)\, 2^n$$

For Particular Solution,

$$a_n^{(p)} = A\, 2^n n^2$$

$$a_{n+1} = A\, 2^{n+1} (n+1)^2$$

$$a_{n+2} = A\, 2^{n+2} (n+2)^2$$

$$a_{n+2} - 4a_{n+1} + 4a_n = \cancel{\ }\ 2^n$$

$$A\, 2^{n+2}(n+2)^2 - 4A\, 2^{n+1}(n+1)^2 + 4(A\, 2^n n^2) = 2^n$$

$$\cancel{2^n}\left(A\, 2^2 (n+2)^2 - 4A\, 2(n+1)^2 + 4(An^2)\right) = \cancel{2^n}$$

$$4A(n+2)^2 - 8A(n+1)^2 + 4An^2 = 1$$

$$A = 1/8$$

$$a_n^{(p)} = \frac{1}{8} 2^n n^2$$

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$\boxed{a_n = (c_1 + c_2 n) 2^n + \frac{1}{8} 2^n n^2}$$

boundary conditions, if given, are to be used here

Q. $y_{n+2} - y_{n+1} - 2y_n = n^2$

A. for homogeneous solution,

let $r^n = y_n$

$$r^{n+2} - r^{n+1} - 2r^n = 0$$

$$r^n (r^2 - r - 2) = 0$$

$$r^n (r-1)(r+2) = 0$$

$$r = 1, -2$$

$$a_n^{(h)} = b_1 (1)^n + b_2 (-2)^n$$

$$a_n^{(h)} = b_1 + b_2 (-2)^n$$

for particular solution,

$$y_n^{(p)} = A_0 + A_1 n + A_2 n^2$$

$$y_{n+1} = A_0 + A_1 (n+1) + A_2 (n+1)^2$$

$$y_{n+2} = A_0 + A_1 (n+2) + A_2 (n+2)^2$$

$$y_{n+2} - y_{n+1} - 2y_n = n^2$$

$$A_0 + A_1(n+2) + A_2(n+2)^2 - A_0 - A_1(n+1) - A_2(n+1)^2 -$$

$$2(A_0 + A_1 n + A_2 n^2) = n^2$$

$$\cancel{A_0} + A_1(n+2) + A_2(n^2+4+4n) - \cancel{A_0} - A_1(n+1) - A_2(n^2+1+2n)$$

$$-2A_0 - 2A_1 n + 2A_2 n^2 = n^2$$

$$\text{An}\ \cancel{nA_1} + 2A_1 + \cancel{n^2A_2} + 4A_2 + 4nA_2 - \cancel{nA_1} - A_1 - \cancel{n^2A_2} - A_2 - 2nA_2$$

$$-2A_0 - 2A_1 n + 2n^2 A_2 = n^2$$

$$A_1 + 3A_2 + 2nA_2 - 2A_0 - 2nA_1 + 2n^2A_2 = n^2$$

$$A_1 - 2nA_1 + 3A_2 + 2nA_2 + 2n^2A_2 - 2A_0 = n^2$$

$$\cancel{n^2(2A_2) - 2n(A_1) + (A_1 + 3A_2)}$$

compare coeff. of n^2, n, and const.

Q. What's the general form of a particular soln. gauranteed to exsist of the linear non-homogeneous recurrence relation : $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + f(n)$

if (a) $f(n) = n^2$

(b) $f(n) = 2^n$

(c) $f(n) = n \cdot 2^n$

(d) $f(n) = (-2)^n$

(e) $f(n) = n^2 2^n$

(f) $f(n) = n^3 (-2)^n$

(g) $f(n) = 3$

A. For Homogeneous Part,

$$a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0$$

$$r^n - 6r^{n-1} + 12r^{n-2} - 8r^{n-3} = 0$$

$$r^{n-3}(r^3 - 6r^2 + 12r - 8) = 0$$

$$r^{n-3}(r-2)^3 = 0$$

$$r = 2, 2, 2$$

$$a_n^{(h)} = (b_1 + b_2 n + b_3 n^2)\, 2^n$$

For Particular Solution,

$$a_n^{(p)} = A_0 + A_1 n + A_2 n^2$$

$$a_{n-1} = A_0 + A_1(n-1) + A_2(n-1)^2$$

$$a_{n-2} = A_0 + A_1(n-2) + A_2(n-2)^2$$

$$a_{n-3} = A_0 + A_1(n-3) + A_2(n-3)^2$$

$$a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + f(n)$$

$$A_0 + A_1 n + A_2 n^2 = 6\left(A_0 + A_1(n-1) + A_2(n-1)^2\right) -$$

$$12\left(A_0 + A_1(n-2) + A_2(n-2)^2\right) +$$

$$8\left(A_0 + A_1(n-3) + A_2(n-3)^2\right) + n^2$$

$$A_0 + A_1 n + A_2 n^2 = 6\left(A_0 + nA_1 - n + n^2 A_2 - 2nA_2 + A_2\right) -$$

$$12\left(A_0 + nA_1 - 2A_1 + n^2 A_2 - 4nA_2 + 4A_2\right) +$$

Generating a Function

standard

$$G(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \dots + a_kx^k + \dots$$

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

$\longrightarrow$ coeff. of x

$(1, 1, 1, \dots)$

1. $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^k + \dots$$
$$= \sum_{n=0}^{\infty} x^n$$

$(1, 1, 1, \dots 1)$

2. $$\frac{1-x^{n+1}}{1-x} = x^0 + x^1 + x^2 + \dots + x^n$$
$$= \sum_{k=0}^{n} x^k$$

$(a^0, a^1, a^2, \dots a^k)$

3. $$\frac{1}{1-ax} = a^0x^0 + ax + a^2x^2 + a^3x^3 + \dots + a^kx^k + \dots$$
$$= \sum_{n=0}^{\infty} a^n x^n$$

4. $$\frac{1}{1-x^r} = 1 + x^r + x^{2r} + x^{3r} + \dots + x^{kr} + \dots$$
$$= \sum_{n=0}^{\infty} x^{nr}$$

5. $$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$
$$= \sum_{n=0}^{\infty} (n+1)x^0$$

Shifting Properties of Generating Functions

Standard formula :- $G(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \dots + a_kx^k + \dots$

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

→ multiplication

$$x\,G(x) = 0.x^0 + a_0x^1 + a_1x^2 + a_2x^3 + a_3x^4 + \dots + a_kx^{k+1} + \dots$$

$$(0, a_0, a_1, a_2, \dots)$$

$$x^2 G(x) = 0.x^0 + 0.x^1 + a_0x^2 + a_1x^3 + a_2x^4 + \dots + a_kx^{k+1} + \dots$$

$$(0, 0, a_0, a_1, a_2, \dots)$$

coefficients are shifted by n positions, if G(x) is multiplied by x^n, towards right.

→ division

$$\frac{G(x) - a_0}{x} = a_1 + a_2x + a_3x^2 + \dots$$

$$(a_1, a_2, a_3, \dots)$$

$$\frac{G(x) - a_0 - a_1x}{x^2} = a_2 + a_3x + a_4x^2 + \dots$$

$$(a_2, a_3, a_4, \dots)$$

Coefficients will be shifted by n positions, if ~~G(x)~~ G(x) is divided by x^n, towards left.

Q. 1, 1, 1, 1, 1, 1 $\rightarrow$ find the generating formula

A. $1.x^0 + 1.x^1 + 1.x^2 + 1.x^3 + 1.x^4 + 1.x^5$

$1 + x + x^2 + x^3 + x^4 + x^5$

$\frac{1-x^{5+1}}{1-x} = \frac{1-x^6}{1-x}$ $\left(\frac{1-x^{n+1}}{1-x}\right)$

Q.1 0, 2, 2, 2, 2, 2, 2, 0, 0, 0, ...

A. $0.x^0 + 2.x^1 + 2.x^2 + 2.x^3 + 2.x^4 + 2.x^5 + 2.x^6 + 0.x^7 + 0.x^8 + \ldots$

$2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6$

$2x(1 + x + x^2 + x^3 + x^4 + x^5)$

$2x\left(\frac{1-x^{5+1}}{1-x}\right)$

$2x\left(\frac{1-x^6}{1-x}\right)$

Q2. 0, 0, 0, 1, 1, 1, 1, 1, ...

A. $0.x^0 + 0.x^1 + 0.x^2 + 1.x^3 + 1.x^4 + 1.x^5 + 1.x^6 + 1.x^7 + \ldots$

$x^3 + x^4 + x^5 + x^6 + x^7 + \ldots$

$x^3(1 + x + x^2 + x^3 + x^4 + \ldots)$

$\cancel{x^3\left(\frac{1-x^{4+1}}{1-x}\right)}$ $x^3\left(\frac{1}{1-x}\right)$

$\cancel{x^3\left(\frac{1-x^5}{1-x}\right)}$ $\frac{x^3}{1-x}$

Q3. $0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots$

A. $0.x^0 + 1.x^1 + 0x^2 + 0x^3 + 1x^4 + 0x^5 + 0x^6 + 0x^7 + \ldots$

$x + x^4 + x^7 + x^{10} + \ldots$

$x(1 + x^3 + x^6 + x^9 + \ldots)$

$x\left(\frac{1}{1-x^3}\right)$

Q4. $2, 4, 8, 16, 32, 64, 128, 256, \ldots$

A. ~~2 (1 + 2~~

$2.x^0 + 4x + 8x^2 + 16x^3 + 32x^4 + 64x^5 + 128x^6 + 256x^7 + \ldots$

$2(1 + 2x + 4x^2 + 8x^3 + 16x^4 + 32x^5 + 64x^6 + 128x^7 + \ldots)$

$2\left(\frac{1}{1-2x}\right)$

$\frac{2}{1-2x}$

Q5. $2, -2, 2, -2, 2, \ldots$

A. $2x^0 + (-2x^1) + 2x^2 + (-2x^3) + 2x^4 + \ldots$

$2(x^0 - x^1 + x^2 - x^3 + x^4 - \ldots)$

~~2 (-x^0 + (~~

$2((-1)(x)^0 + (-1)^1x^1 + (-1)^2x^2 + (-1)^3x^3 + \ldots)$

$2\left(\frac{1}{1-(-x)}\right) = \frac{2}{1+x}$

Q6. $1, 1, 0, 1, 1, 1, 1, \ldots$

A. $x^0 + x^1 + x^0 . 0 + x^3 + x^4 + x^5 + x^6 + \ldots$

$1 + x + x^3 + x^4 + x^5 + x^6 + \ldots$

$1 + x + x^3 (1 + x^1 + x^2 + x^3 + \ldots)$

$1 + x + x^3 \left(\frac{1}{1-x}\right)$

$\frac{1 - x^2 + x^3}{1-x} \Rightarrow \cancel{\frac{1+x}{1-x}}$

(or add - subtract x^2)

Q7. $0, 0, 0, 1, 2, 3, 4, \ldots$

$0 . x^0 + 0x^1 + 0x^2 + 1 . x^3 + 2 . x^4 + 3x^5 + 4x^6 + \ldots$

$x^3 + 2x^4 + 3x^5 + 4x^6 + \ldots$

$x^3 (1 + 2x + 3x^2 + 4x^3 + \ldots)$

$x^3 \left(\frac{1}{1-x^2}\right)$

$\frac{x^3}{1-x^2}$

Qa. $a_n = 5$, $\forall n = 0, 1, 2, \ldots$

A. $a_0 = 5$

$a_1 = 5$

$a_2 = 5$

$\vdots$

$5x^0 + 5x^1 + 5x^2 + \ldots \Rightarrow \frac{5}{1-x}$

(b) $a_n = 3^n \quad \forall n = 0, 1, 2, \ldots$

$a_0 = 3^0$

$a_1 = 3^1$

$a_2 = 3^2$

$a_3 = 3^3$

$\vdots$

$a_0 x^0 + a_1 x^1 + a_2 x^2 + \ldots$

$3^0 . x^0 + 3x + 3^2 x^2 + 3^3 x^3 + \ldots$

$1 + 3x + 3^2 x^2 + 3^3 x^3 + \ldots$

$$\frac{1}{1-3x}$$

(c.) $a_n = 2 \quad \forall n = 3, 4, 5, \ldots$

$a_0 = a_1 = a_2 = 0$

$a_3 = 2$

$a_4 = 2$

$a_5 = 2$

$\vdots$

$a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots$

$0 + 0 + 0 + 2x^3 + 2x^4 + 2x^5 + \ldots$

$2x^3 (1 + x + x^2 + \ldots)$

$$\frac{2x^3}{1-x}$$

d. $a_n = 2n+3$, $\forall n = 0, 1, 2, \ldots$

$a_0 = 3$

$a_1 = 5$

$a_2 = 7$

$\vdots$

$a_0 = 1+2$

$a_1 = 1+2\cdot 2$

$a_2 = 1+2\cdot 3$

$\vdots$

$(1\cdot x^0 + 1\cdot x^1 + 1\cdot x^2 + 1\cdot x^3 + \ldots) + 2\,(1x^0 + 2\cdot x + 2x^2 + 2x^3 + \ldots)$

$\frac{1}{1-x} + \frac{2}{1-x^2}$.

Q. Find out the sequence for the following

1. $(3x-4)^3$
2. $(x^3+1)^3$
3. $\frac{1}{1-5x}$
4. $\frac{x^3}{1+3x}$
5. $x^2 + 3x + 7 + \frac{1}{1-x^2}$
6. $\frac{x^4}{1-x^2} - x^3 - x^2 - x + 1$
7. $\frac{x^2}{(1-x)^2}$
8. $2e^{2x}$

A. 1. $(3x-4)^3$

$(3x)^3 - (4)^3 - 3(3x)(4)(3x-4)$

$27x^3 - 64 - 36x(3x-4)$

$27x^3 - 64 - 108x^2 + 144x$

$27x^3 - 108x^2 + 144x - 64$

$\Rightarrow -64 + 144x - 108x^2 + 27x^3$ $\Rightarrow -64, 144, -108, 27$

2. $(x^3+1)^3$

$(x^3)^3 + (1) + 3x^3(x^3+1)$

$x^9 + 1 + 3x^6 + 3x^3$

$1 + 3x^3 + 3x^6 + x^9$

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
coeff :	1,	0	0	3	0	0	3	0	0	1

series : 1, 0, 0, 3, 0; 0, 3, 0, 0, 1

3. $\dfrac{1}{1-5x}$

$\sum_{n=0}^{\infty} (5)^n (x)^n$

coeff : $5^0, 5^1, 5^2, 5^3, \ldots$

4. $\dfrac{x^3}{1+3x}$

$x^3 \cdot \dfrac{1}{1+3x}$

$x^3 \cdot \dfrac{1}{1-(-3x)}$

$x^3 \cdot ((-3)^0$

coeff : 0, 0, 0, $(-3)^0, (-3)^1, (-3)^2, (-3)^3$

0, 0, 0, 1, -3, 9, -27, ...

5. $x^2 + 3x + 7 + \frac{1}{1-x^2}$

$x^2 + 3x + 7 + 1 + x^2 + x^4 + x^6 + \dots$

$8 + 3x + 2x^2 + x^4 + x^6 + x^8 + \dots$

coeff | series: 8, 3, 2, 0, 1, 0, 1, 0, 1, ...

6.

Q.** $a_n = 3a_{n-1} + 2$, $a_0 = 1$

A. $a_n = 3a_{n-1} + 2$ ——— (1)

$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = \sum_{n=0}^{\infty} a_n x^n$

Multiplying eq (1) with x^n and summing it up with 1 to ∞,

$$\sum_{n=1}^{\infty} a_n x^n = 3 \sum_{n=1}^{\infty} a_{n-1} x^n + 2 \sum_{n=1}^{\infty} x^n$$

$$G(x) - a_0 = 3x\, G(x) + \frac{2x}{1-x}$$

$$G(x)(1-3x) = \frac{2x}{1-x} + 1$$

$$G(x) = \frac{1+x}{(1-x)(1-3x)}$$

by partial fraction,

$$\frac{A}{1-x} + \frac{B}{1-3x} = \frac{1+x}{(1-x)(1-3x)}$$

$\Rightarrow$ $A = -1$
$B = 2$

$$G(x) = \frac{-1}{1-x} + \frac{2}{1-3x}$$

$$\sum_{n=0}^{\infty} a_n x^n = -1 \sum_{n=0}^{\infty} x^n + 2 \sum_{n=0}^{\infty} 3^n x^n$$

$$\Rightarrow \boxed{a_n = -1 + 2 \cdot 3^n}$$

Q. $a_n = 3a_{n-1} + 4^{n-1}$, $a_0 = 1$

A.

$$a_n = 3a_{n-1} + 4^{n-1} \quad \text{——— (1)}$$

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots = \sum_{n=0}^{\infty} a_n x^n$$

multiplying eq (1) with x^n and summing it up with 1 to ∞,

$$\sum_{n=1}^{\infty} a_n x^n = 3 \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} 4^{n-1} x^n$$

$$G(x) - a_0 = 3x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + \cancel{4}x \sum_{n=1}^{\infty} 4^{n-1} x^{n-1}$$

$$G(x) - a_0 = 3x\, G(x) + x \sum_{n=1}^{\infty} (\cancel{4}x)^{n-1}\, 4^{n-1}$$

$$G(x) - a_0 = 3x\, G(x) + \frac{x}{1-4x}$$

$$G(x) - 1 = 3x\, G(x) + \frac{x}{1-4x}$$

$$G(x)(1-3x) = \frac{x}{1-4x} + 1$$

$$G(x) = \frac{1-3x}{(1-3x)(1-4x)}$$

$$G(x) = \frac{1}{1-4x}$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (4)^n x^n$$

$$\boxed{a_n = 4^n}$$