UNIT 3 (I)

- POSETS AND LATTICES

- BOOLEAN ALGEBRA
  - ↳ truth table, k-Map, circuit diagrams, circuit minimization

Relations : Partial Order Sets

Set $A \times B$

$R \sim \{(-,-), (-,-), (-,-), (-,-)\}$

reflexive, antisymmetric, transitive

Poset = $\{(-,-), (-,-), (-,-), (-,-)\}$]

collection of sets satisfying relation R

poset → $(Z^+, |)$ → divisibility | divides

$Z^+ \times Z^+$

∴ it's a poset
- $a \in Z^+$
- $a | a$
- $a|b \wedge b|c \Rightarrow a|c$ transitive
- $\forall a \forall b\ ((aRb) \wedge (bRa) \rightarrow a=b)$ antisymmetric

or

- $\forall a. \forall b\ (((a,b) \in R \wedge (b,a) \in R) \rightarrow a = b)$

A relation R on a set A is called partial ordering if it is reflexive, antisymmetric and transitive.

1. $(Z, =) \rightarrow P$
2. $(Z, \neq) \rightarrow$ not a P. $\underbrace{(a \neq b) \wedge (b \neq a)}_{T} \rightarrow \underbrace{a = b}_{F}$ ∴ not AS
3. $(Z, \geq) \rightarrow P$

Comparable Elements

$\{(\ ),(\ ),(\ ),(\ )\}$

$a, b \in A$

exclusive

$a \leqslant b$ or $b \leqslant a$ → b is partially related to

a and b are comparable to each other

$a \nleqslant b$ $b \nleqslant a$

a and b are incomparable

The elements a and b of a poset $(S, \leqslant)$ are called comparable if either

set ↓ partial order

$a \leqslant b$ or $b \leqslant a$ when $a \nleqslant b$

and $b \nleqslant a$ then a and b are called incomparable

Eg: $(Z^+, |)$ $\{(2,4), (5,7), \ldots$

$2|4$ but $4 \nmid 2$ ×

$5|7$ × , $7|5$ ×

Toset : Totally ordered set [every element is comparable]

either $a \leqslant b$ or $b \leqslant a$

$(Z, \leqslant)$

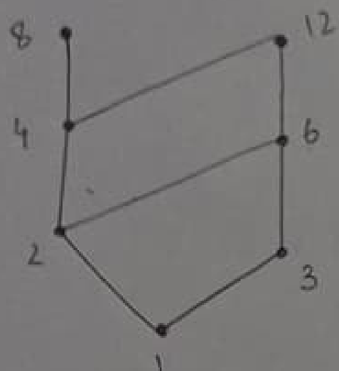
$(Z, \leqslant)$

Hasse Diagram

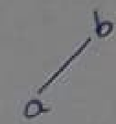
Diagrame to represent POSET and TOSET

$(S, \leq)$ $(Z^+, |)$

1. $(\{1, 2, 3, 4, 6, 8, 12\}, |)$

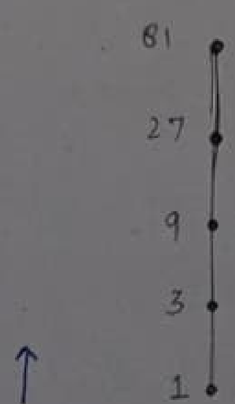


$a \leq b$



no downward travel in Hasse Diagram

2. $(\{1, 3, 9, 27, 81\}, |) \rightarrow$ TOSET

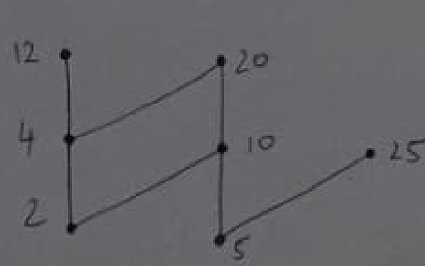


toset : straight line (chain)

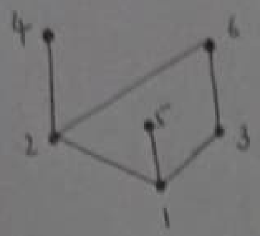
$(3 | 27 \wedge 27 \nmid 3) \rightarrow 3 = 27$?

F

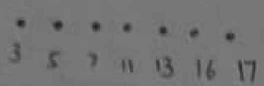
3. $(\{2, 4, 5, 10, 12, 20, 25\}, |)$



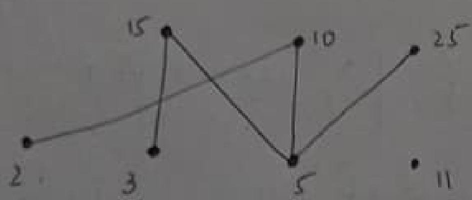
Q1. $(\{1, 2, 3, 4, 5, 6\}, |)$



Q2. $(\{3, 5, 47, 11, 13, 16, 17\}, |)$



Q3. $(\{2, 3, 5, 10, 11, 15, 25\}, |)$



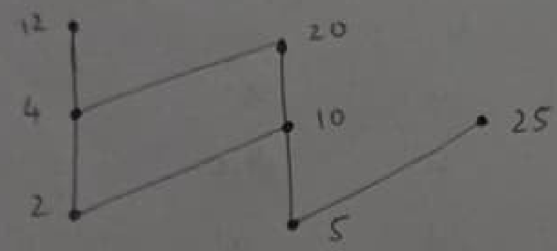
Q4. $(\{1, 3, 9, 27, 81, 243\}, |)$

Special Elements of Hasse Diagram

1. Minimal and Maximum Elements :

'a' is minimal if there is no element $b \in S$ such that $b \not\leq a$ for $a, b \in S$

'a' is maximal if there is no element $b \in S$ such that $a \leq b$ for $a, b \in S$



minimal : 2, 5
maximal : 12, 20, 25
greatest → x
least → x

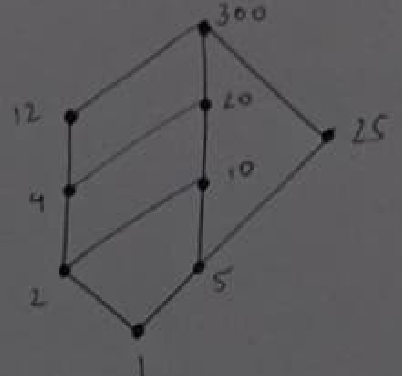
2. Greatest and Least Elements

'a' is the greatest element of poset $(S, \leq)$ if $b \leq a \quad \forall \; b \in S$.

greatest element is unique if it exists.

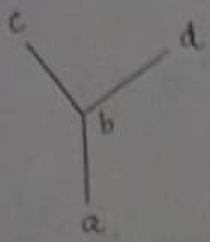
'a' is the least element of poset $(S, \leq)$ if $a \leq b \quad \forall \; a \in S$

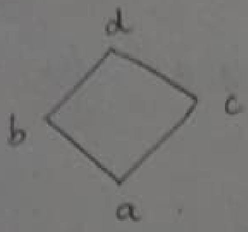
least element is unique if it exists

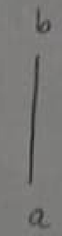


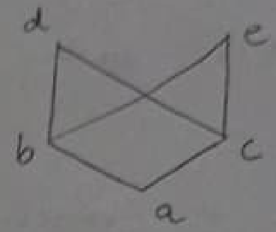
greatest → 300
least → 1

* Trick : If your figure is closed or straight line, then there will be greatest and least element possible





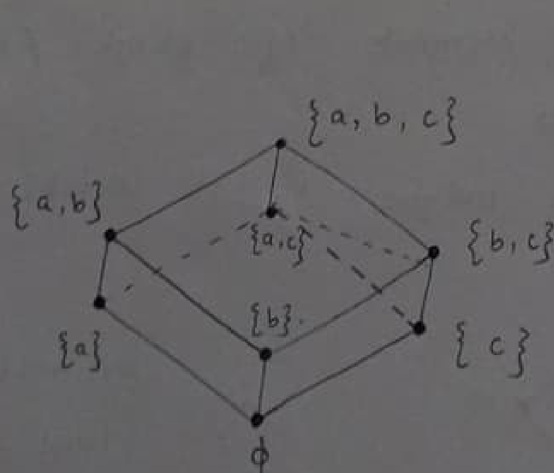




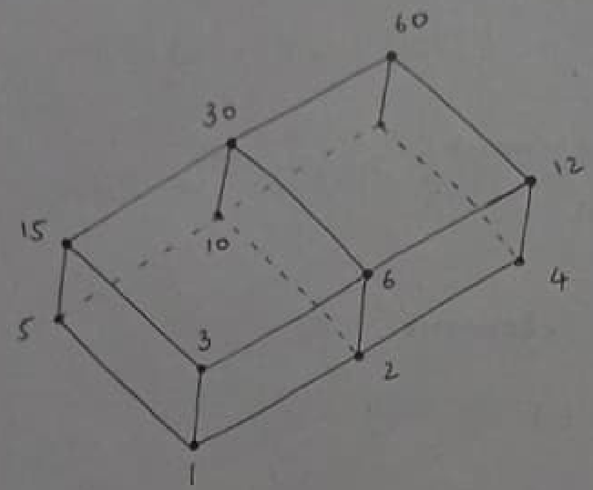
greatest →	×	d	b	×
least →	a	a	a	a
maximal →	c, d	d	b	d, e
minimal →	a	a	a	a

Q1. $(P(S), \subseteq)$, $S = \{a, b, c\}$

$P(S) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\} \}$



$D_{60} \rightarrow$ factors of ~~$\cancel{\ }$~~ 60



Q. Show that there is exactly one greatest element of a poset, if such an element exists.

THEOREM

Let g_1 and g_2 be two greatest elements of poset $(S, \preceq)$

where $g_1, g_2 \in S$

If g_1 is the greatest element

$$g_2 \preceq g_1 \quad \text{—— (1)}$$

If g_2 is the greatest element

$$g_1 \preceq g_2 \quad \text{—— (2)}$$

$$\boxed{g_1 = g_2}$$

$a | b$ and $b | a$

$a = b$

$a \leq b$, $b \leq a$

$a = b$

Q. Show that there is exactly 1 least element of a poset, if such an element exists.

THEOREM

Let l_1 and l_2 be two least element of poset $(S, \preccurlyeq)$

where $l_1, l_2 \in S$

If l_1 is the least element;

$l_1 \preccurlyeq l_2$ —— (1)

If l_2 is the least element;

$l_2 \preccurlyeq l_1$ —— (2)

$\Rightarrow \boxed{l_1 = l_2}$

Q. Give a poset that has:

(i) a minimal element but not a maximal element.
(ii) a maximal element but not a minimum element.
(iii) neither minimal nor maximal element.

(i) $(N, \overset{\geq}{\cancel{\preccurlyeq}})$, $(Z^+, \overset{\geq}{\cancel{\preccurlyeq}})$, $(R^+, \overset{\geq}{\cancel{\preccurlyeq}})$
(ii) $(Z^-, \overset{\geq}{\cancel{\preccurlyeq}})$, $(R^-, \overset{\geq}{\cancel{\preccurlyeq}})$
(iii) $(Z, \overset{\geq}{\cancel{\preccurlyeq}})$, $(R, \overset{\geq}{\cancel{\preccurlyeq}})$

Product Order

$(A, \leq_1)$ and $(B, \leq_2)$ are posets.

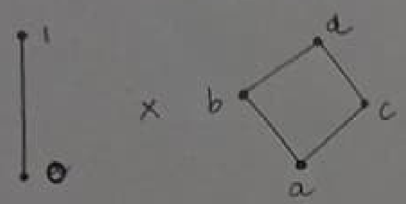
Then $(A \times B, \leq)$ is also a poset.

If $a \leq_1 a'$ in A

and $b \leq_2 b'$ in B

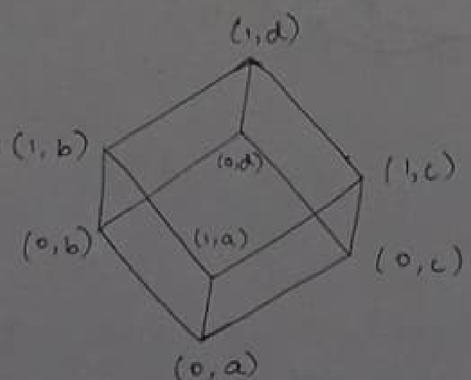
then $(a, b) \leq (a', b')$ in $A \times B$

Q.



$\{0, 1\} \times \{a, b, c, d\}$

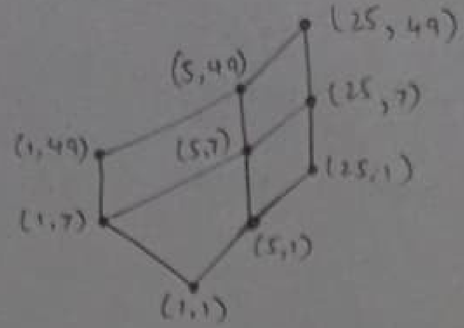
$\{(0,a), (0,b), (0,c), (0,d)$
$(1,a), (1,b), (1,c), (1,d)\}$



Q. $D_{25} \times D_{43}$

$(\{1, 5, 25\}, |) \times (\{1, 7, 49\}, |)$

$\{(1,1), (1,7), (1,49), (5,1), (5,7), (5,49), (25,1),$
$(25,7), (25,49)\}$



THEOREM: If $(A, \leq)$ and $(B, \leq)$ be posets, then $(A \times B, \leq)$ is also a poset.

$(A, \leq_1)$ $\quad$ $(B, \leq_2)$

$a, a' \in A$ $\quad$ $b, b' \in B$

a'' $\quad$ b''

$(A \times B, \leq)$

$\cancel{(a \times b}$ $(a, b) \leq (a, b)$

$a \leq_1 a$

$b \leq_2 b$

Hence, $(A \times B, \leq)$ is reflexive

$(a, b), \; (a', b'), \; (a'', b'')$

If $(a, b) \leq (a', b')$

and $(a', b') \leq (a'', b'')$

then $(a, b) \leq (a'', b'')$

$a, a', a'' \in A$ $\quad$ $b, b', b'' \in B$

$a \preceq_1 a'$ and $\qquad$ $b \preceq_2 b'$ and

$a' \preceq_1 a''$ $\qquad$ $b' \preceq_2 b''$

$\Rightarrow a \preceq_1 a''$ $\qquad$ $\Rightarrow b \preceq_2 b''$

Hence, $(A \times B, \preceq)$ is transitive

If $(a, b) \preceq (a', b')$

and $(a', b') \preceq (a, b)$

then $(a, b) = (a', b')$

$a \preceq_1 a'$ and $a' \preceq_1 a$

$\Rightarrow a = a'$

$b \preceq_2 b'$ and $b' \preceq_2 b$

$\Rightarrow b = b'$

Hence, $(A \times B, \preceq)$ is antisymmetric

$\therefore$ $(A \times B, \preceq)$ is a poset.

Upper Bounds and Lower Bounds

Let B be a subset of a poset $(A, \preceq)$ and element u ~~blong~~ belong to A, this is known as an upper bound of B if u succeeds every element of B.

$(A, \leqslant)$

$(B \subseteq A)$

$U \in A$

$x \leqslant U$

$\forall \; x \in B$

U is upper bound

Let B be a subset of a poset $(A, \leqslant)$ and element l belongs to A, this is known as lower bound of B if l preceeds every element of B.

$(A, \leqslant)$

$(B \subseteq A)$

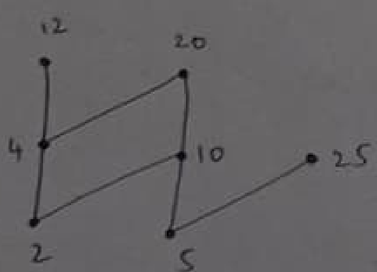
$l \in A$

$l \leqslant x$

$\forall x \in B$

l is lower bound

Eg:



$\{2, 4, 5, 10, 12, 20, 25\}$

ϕ is the lower bound of $\{5, 12\}$

ϕ is the upper bound of $\{5, 12\}$

- upper bound $\to \{5, 12\}$ kisse relation rakh raha he?

Lower bound $\to \{5, 12\}$ se kon relation rakh raha he?

$\{2\}$ is the lower bound of $\{2, 4\}$

$\{4, 12, 20\}$ is the upper bound of $\{2, 4\}$

Least Upper Bound (LUB) : Supremum
Greatest Lower Bound (GLB) : Infimum

$\{u_1, u_2, u_3, u_4\}$ $\{l_1, l_2, l_3, l_4\}$

$u_3 \leq u_1$ $l_2 \leq l_1$
$u_3 \leq u_2$ $l_3 \leq l_1$
$u_3 \leq u_4$ $l_4 \leq l_1$

LUB: Suppose we are having upper bound $\{u_1, u_2, u_3, u_4\}$ and suppose u_3 is the LUB, then :

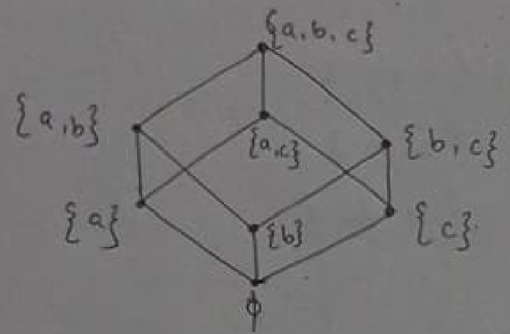
$u_3 \leq u_1$ $u_3 \leq u_2$ $u_3 \leq u_4$	if any of the conditions fail, it will not be the LUB

GLB : Suppose we are having lower bound $\{l_1, l_2, l_3, l_4\}$ and suppose l_3 is the GLB, then :

$l_1 \leq l_3$
$l_2 \leq l_3$
$l_4 \leq l_3$

LATTICES

A poset in which every pair of element has both LUB and GLB is known as lattice.



$\phi \leq \{ \phi, \{a\} \} \leq \{a\}$

$\{ \phi, \{b\} \}$

$\{ \phi, \{c\} \}$

$\{ \phi, \{a, b\} \}$

$\{ \phi, \{b, c\} \}$

$\{ \phi, \{a, c\} \}$

$\{ \phi, \{a, b, c\} \}$

NOTE: $a \vee b$ = a join b
(LUB of a and b)

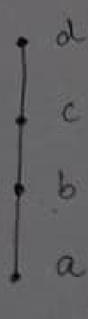
$a \wedge b$ = a meet b
(GLB of a and b)

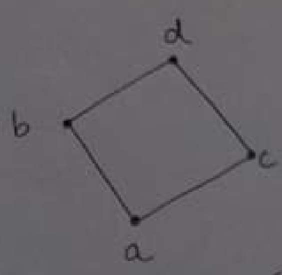
Suppose we are having any toset,

If we take $\{a, b\} \Rightarrow$ LUB $= \{b\}$
GLB $= \{a\}$

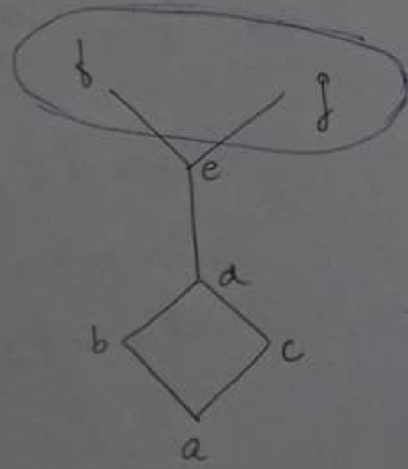
$\{b, c\} \Rightarrow$ LUB $\{c\}$
GLB $= \{b\}$

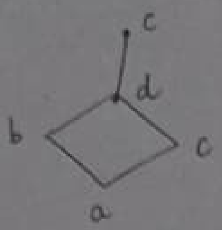
Eg:



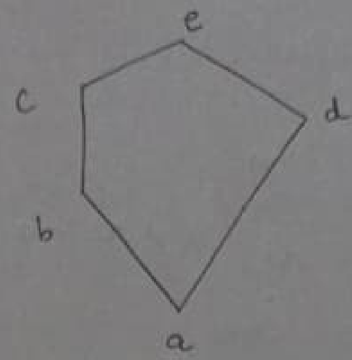


↓
LUB = d
GLB = a

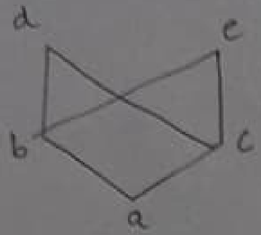




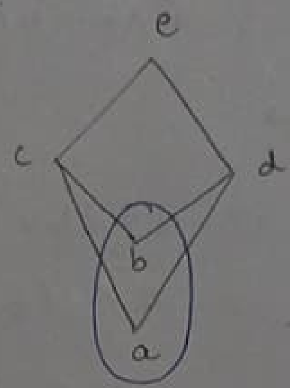
✓



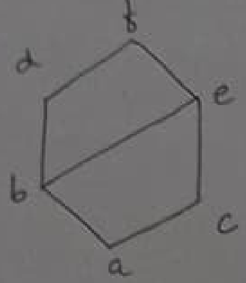
✓



×



×



✓

Sublattices

A non empty subset 'S' of lattice 'L' is called a sub-lattice of lattice of ∀ 'L' if $(L, \preccurlyeq)$

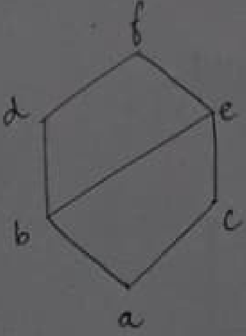
$S \subseteq L$ $(S, \preccurlyeq)$

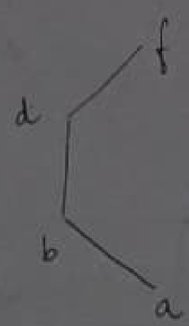
$$a \vee b \in S$$

and

$a \wedge b \in S$ whenever $a, b \in S$

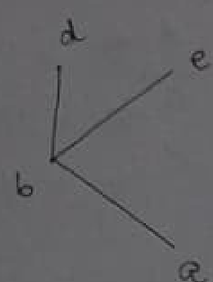
Eg:





✓

is a sub-lattice



×

not a sub-lattice as d and e have no LUB

Properties of Lattices :

1. Idempotent Law

$$a \vee a = a \quad , \quad a \wedge a = a$$

2. Commutative Law

$$a \vee b = b \vee a \quad , \quad a \wedge b = b \wedge a$$

3. Associative Law

$$(a \vee b) \vee c = a \vee (b \vee c)$$
$$(a \wedge b) \wedge c = a \wedge (b \wedge c)$$

4. Absorption Law

$$a \vee (a \wedge b) = a$$
$$a \wedge (a \vee b) = a$$

THEOREM : For any a, b, c, d in a lattice $(L, \leq)$

(i) $a \leq a \vee b$ and $a \wedge b \leq a$

(ii) if $a \leq b$ and $c \leq d$ then $a \vee c \leq b \vee d$ and $a \wedge c \leq d \wedge d$

(ii)

$a \leq b$

$b \leq b \vee d$ $\quad$ $c \leq d$

$d \leq b \vee d$ $\quad$ $d \leq b \vee d$

$a \leq b \vee d$

$c \leq b \vee d$

$$\boxed{a \vee c \leq b \vee d}$$

Types of Lattices :

1. Bounded Lattice :

A lattice is said to be bounded if it has a greatest element and a least element.

$$0, a, 1 \in L$$

$$0 \preceq a \preceq 1$$

* identity propery

$a \vee 1 = 1$, $a \vee 0 = a$

$a \wedge 1 = a$, $a \wedge 0 = 0$

2. Distributive Lattice :

A Lattice $(L, \preceq)$ is said to be distributive if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

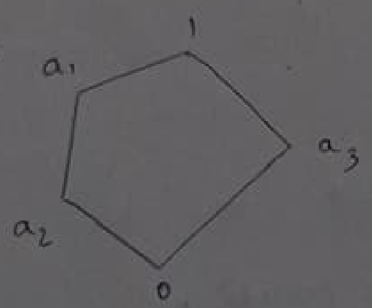
~~ta~~ ~~a v (b~~

$a \vee (b \wedge c)$

$a \wedge (b \vee c)$

$(a \wedge b) \vee c$

Eg:



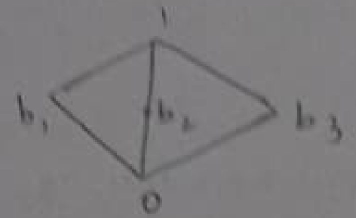
$a_1 \wedge (a_2 \vee a_3) = (a_1 \wedge a_2) \vee (a_1 \wedge a_3)$

$a_1 \wedge 1 = a_2 \vee 0$

$a_1 \neq a_2$

not a distributive lattice.

Eg.



THEOREM : In a distributive lattice $(L, \leq)$, if $a \vee b = a \vee c$ and $b \wedge b = a \wedge c$ imply that $b = c$

$$
\begin{aligned}
b &= b \wedge (b \vee a) && \text{absorption} \\
&= b \wedge (a \vee c) && \text{given} \\
&= (b \wedge a) \vee (b \wedge c) && \text{distributive} \\
&= (a \wedge c) \vee (b \wedge c) && \text{given} \\
&= (a \vee b) \wedge c && \text{distributive} \\
&= (a \vee c) \wedge c && \text{given.} \\
&= c && \text{absorption}
\end{aligned}
$$

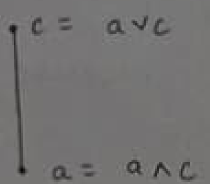
Hence, proved

3. Modular Lattice :

a lattice $(L, \leq)$ is said to be modular if $a \vee (b \wedge c) = (a \vee b) \wedge c$ whenever $a \leq c$

~~THEOREM - In a distributive lattice $(L, \leq)$ if $a \vee b = a \vee c$ and $a \wedge b = a \wedge c$, imply that $b = c$~~

THEOREM : Every distributive lattice is modular



$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$
$$= (a \vee b) \wedge c$$

4. Complimented Lattice

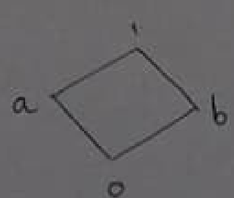
It is a lattice in which every element has one compliment.

$$(L, \wedge, \vee, 0, 1)$$

$b, a \in L$

1. $0 \leq a \leq 1$

2. $a \vee b = 1$
$a \wedge b = 0$



$0 \vee 1 = 1$
$0 \wedge 1 = 0$
$a \vee b = 1$
$a \wedge b = 0$
$a \vee b = b$ ×
$0 \wedge b = 0$
$a \vee 0 = 0$ ×
$a \wedge 0 = 0$

$$\boxed{\begin{array}{ll} (a)' = b & (0)' = 1 \\ (b)' = a & (1)' = 0 \end{array}}$$

$1 \vee 0 = 1$

$1 \wedge 0 = 0$

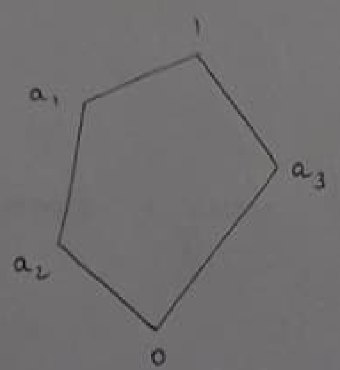
$0 \vee a = a$ ×

$0 \wedge a = 0$

$a \vee 1 = 1$

$a \wedge 1 = a$ ×

not a complimentary lattice because there is no compliment of a.



$a_1 \vee a_3 = 1$

$a_1 \wedge a_3 = 0$

$a_2 \vee a_3 = 1$

$a_2 \wedge a_3 = 0$

$a_2 \vee 0 = a_2$ ×

$a_1 \vee 0 = a_1$ ×

$a_3 \vee 0 = a_3$ ×

$0 \vee 1 = 1$

$0 \wedge 1 = 0$

$(a_3)' = a_2, a_1$

$(a_1)' = a_3$

$(a_2)' = a_3$

$(0)' = 1$

$(1)' = 0$

THEOREM : In a distributive lattice if an element has a compliment, then this compliment is unique.

$a, a_1, a_2 \in L$

$a \vee a_1 = 1$ $\quad$ $a \vee a_2 = 1$

$a \wedge a_1 = 0$ $\quad$ $a \wedge a_2 = 0$

$a_1 = a_1 \wedge 1$ identity

$= a_1 (a \vee a_2)$ given

$= (a_1 \wedge a) \vee (a_1 \wedge a_2)$

$= 0 \vee (a_1 \wedge a_2)$

$= (a \wedge a_2) \vee (a_1 \wedge a_2)$

$= (a \vee a_1) \wedge a_2$

$= 1 \wedge a_2$

$= a_2$

THEOREM : In a complimented and distributive lattice

(i) $a \leq b$

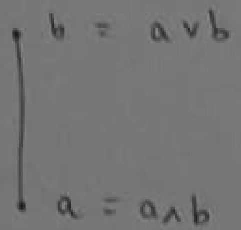
(ii) $a \wedge b' = 0$

(iii) $a' \vee b = 1$

(iv) $b' \leq a'$

* from (i) prove (ii)
(ii) (iii)
(iii) (iv)
(iv) (i)

(ii)



$$a \vee b = b$$
$$(a \vee b) \wedge b' = b \wedge b'$$
$$(a \wedge b') \vee (b \wedge b') = b \wedge b'$$
$$(a \wedge b') \vee 0 = 0$$
$$(a \wedge b') = 0$$

(iii)

$$(a \wedge b') = 0$$
$$(a \wedge b')' = (0)'$$
$$a' \vee b = 1$$

★ $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

$a \leq a \vee b$	$a \leq a \vee c$
$b \wedge c \leq b \leq a \vee b$	$b \wedge c \leq c \leq a \vee c$

$$a \leq (a \vee b) \wedge (a \wedge c)$$
$$b \wedge c \leq (a \vee b) \wedge (a \wedge c)$$
$$\Rightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge \underline{\underline{(a \wedge c)}}$$
$$a \vee (b \wedge c) \leq (a \vee b) \wedge c$$

if $a \leq c$ $\quad$ $a \vee c = c$

UNIT III (IInd)

BOOLEAN ALGEBRA

Duality Law

U_4 , U_3 , boolean

lattice

P, Q	a, b	x, y
$\vee$	$\vee$ join	+ sum
$\wedge$	$\wedge$ meet	. product
T	1 greatest	1
F	0 least	0
$\neg$	'	', —

$$\neg P \wedge (Q \rightarrow R) \equiv \neg P \wedge (\neg Q \vee R) = \neg P \vee (\neg Q \wedge R)$$

$$x \cdot y\bar{z} + \bar{y}z = (x + y + \bar{z}) \cdot (\bar{y} + z)$$

Truth Table

	x	y	z	$\bar{z}$	$\bar{y}$	$xy\bar{z}$	$\bar{y}z$	$xy\bar{z} + \bar{y}z$
0	0	0	0	1	1	0	0	0
1	0	0	1	0	1	0	1	1
2	0	1	0	1	0	0	0	0
3	0	1	1	0	0	0	0	0
4	1	0	0	1	1	0	0	0
5	1	0	1	0	1	0	1	1
6	1	1	0	1	0	1	0	1
7	1	1	1	0	0	0	0	0

$x+y+\bar{z}$	$\bar{y}+z$	$(x+y+\bar{z}).(\bar{y}+z)$
1	1	1
0	1	0
1	0	0
1	1	1
1	1	1
1	1	1
1	0	0
1	1	1

Laws :

1. $x+x = x$ | idempotent
   $x.x = x$

2. $x+y = y+x$ | commutative
   $x.y = y.x$

3. $x+(y+z) = (x+y)+z$ | associative
   $x.(y.z) = (x.y).z$

4. $x.(y+x) = x$ | absorption
   $x+(y.x) = x$

5. $\overline{x+y} = \bar{x}.\bar{y}$ | de morgan's
   $\overline{x.y} = \bar{x}+\bar{y}$

6. $\bar{\bar{x}} = x$ | double complement

7. $x+\bar{x} = 1$ | identity
   $x.\bar{x} = 0$

8. $x + 1 = 1$ | unity property

$x \cdot 0 = 0$ | zero property

9. $x + 0 = x$ | domination law

$x \cdot 1 = x$

10. $x \cdot (y + z) = x \cdot y + x \cdot z$ | distributive

$x + (y \cdot z) = (x + y) \cdot (x + z)$

Canonical SOP and POS :

min $= x \cdot y \cdot \bar{z}$, $x\bar{y}z \rightarrow$ sum $\rightarrow$ SOP

max $= x + y + \bar{z}$, $x + \bar{y} + z \rightarrow$ product $\rightarrow$ POS

Eg: $f(x, y, z) = \bar{x}$

$= \bar{x} + 0$ domination

$= \bar{x} + (y \cdot \bar{y})$ identity

$= (\bar{x} + y) \cdot (\bar{x} + \bar{y})$ distributive

$= (\bar{x} + y + 0) \cdot (\bar{x} + \bar{y} + 0)$

$= (\bar{x} + y + z\bar{z}) \cdot (\bar{x} + \bar{y} + z\bar{z})$

$$= \underbrace{(\bar{x} + y + z) \cdot (\bar{x} + y + \bar{z}) \cdot (\bar{x} + \bar{y} + z) \cdot (\bar{x} + \bar{y} + \bar{z})}_{\text{POS form}}$$

$= \bar{x} \cdot 1$

$= \bar{x} \cdot (y + \bar{y})$

$= \bar{x} \cdot y + \bar{x} \cdot \bar{y}$

$= \bar{x} \cdot y \cdot 1 + \bar{x} \cdot \bar{y} \cdot 1$

$= \bar{x} \cdot y \cdot (z + \bar{z}) + \bar{x} \cdot \bar{y} \cdot (z + \bar{z})$

$$= \underbrace{\bar{x} \cdot y \cdot z + \bar{x} \cdot y \cdot \bar{z} + \bar{x} \cdot \bar{y} \cdot z + \bar{x} \cdot \bar{y} \cdot \bar{z}}_{\text{SOP form}}$$

Eg: $(x + y) \cdot z$

SOP $\rightarrow$ $= (x \cdot 1 + y \cdot 1)\, z$

$= (x \cdot (y + \bar{y}) + y \cdot (x + x))\, z$

$= (x \cdot y + x \cdot \bar{y} + y \cdot x + y \cdot \bar{x})\, z$

$= (xy + x\bar{y} + y\bar{x})\, z$

$= xyz + x\bar{y}z + \bar{x}yz$

POS $\rightarrow$ $= (x + y + 0) \cdot (z + 0 + 0)$

$= (x + y + z\bar{z}) \cdot (z + \cancel{x \cdot \bar{x}}\; x \cdot \bar{x} + y \cdot \bar{y})$

$= (x + y + z)(x + y + \bar{z}) \cdot (z + x + y) \cdot (z + \bar{x} + y)$

$= (z + x + \bar{y}) \cdot (x + z + y)$

Functionally Complete set of connective

$\{+, '\}$ $\{\cdot, '\}$

$\downarrow$ nor $\downarrow$ nor

$\uparrow$ nand $\uparrow$ nand

Q. Show that

(a) $\bar{x} = x \downarrow x = x \uparrow x$

(b) $xy = (x \downarrow x) \downarrow (y \downarrow y) = (x \uparrow y) \uparrow (x \uparrow y)$

(c) $x + y = (x \downarrow y) \downarrow (x \downarrow y) = (x \uparrow x) \uparrow (y \uparrow y)$

(a) $\bar{x} = \overline{x + x}$ $[x = x + x$, idempotent$]$

$\bar{x} = x \downarrow x$

(b) $xy = \overline{\overline{x \cdot y}}$

$= \overline{\bar{x} + \bar{y}}$

$= \bar{x} \downarrow \bar{y}$

$= \overline{x + x} \downarrow \overline{y + y}$

$= (x \downarrow x) \downarrow (y \downarrow y)$

$xy = \overline{\overline{xy}}$

$= \overline{\overline{(xy) + (xy)}}$

$= \overline{(\overline{xy}) + (\overline{xy})}$

$= \overline{xy} \uparrow \overline{xy}$

$= (x \uparrow y) \uparrow (x \uparrow y)$

(c) $x + y = \overline{\overline{x + y}}$

$= \overline{\bar{x} \cdot \bar{y}}$

$= \bar{x} \uparrow \bar{y}$

$= (\overline{x \cdot x}) \uparrow (\overline{y \cdot y})$

$= (x \uparrow x) \uparrow (y \uparrow y)$

$x + y = \overline{\overline{x + y}}$

$= \overline{(x + y) \cdot (x + y)}$

$= \overline{(\overline{x + y}) + (\overline{x + y})}$

$= (\overline{x + y}) \downarrow (\overline{x + y})$

$= (x \downarrow y) \downarrow (x \downarrow y)$

KMAP $\rightarrow$

$f(x, y) =$

	$\bar{y}$	y
$\bar{x}$	00 $_0$	01 $_1$
x	10 $_2$	11 $_3$

$f(x, y, z) =$

	$\bar{y}\bar{z}$	$\bar{y}z$	yz	$y\bar{z}$
$\bar{x}$	000 $_0$	001 $_1$	011 $_3$	010 $_2$
x	100 $_4$	101 $_5$	111 $_7$	110 $_6$

$\Sigma \rightarrow$ SOP

$\pi \rightarrow$ POS

octat
quad
pair
single

$f(x, y, z, w) =$

$xy \backslash zw$	$\bar{z}\bar{w}$	$\bar{z}w$	zw	$z\bar{w}$
$\bar{x}\bar{y}$	0000 0	0001 1	0011 3	0010 2
$\bar{x}y$	0100 4	0101 5	0111 7	0110 6
xy	1100 12	1101 13	1111 15	1110 14
$x\bar{y}$	1000 8	1001 9	1011 11	1010 10

Q. $f(x, y, z, w) = \Sigma\ 0, 1, 2, 3, 7, 8, 11, 12$

	$\bar{z}\bar{w}$	$\bar{z}w$	zw	$z\bar{w}$
$\bar{x}\bar{y}$	1 0	1 1	1 3	1 2
$\bar{x}y$	4	5	1 7	6
xy	1 12	13	15	14
$x\bar{y}$	1 8	9	1 11	10

1 quad + 3 pair

$\bar{x}\bar{y} + x\bar{z}\bar{w} + \bar{x}zw + \bar{y}zw$